Connected sum at infinity and 4–manifolds

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We study connected sum at infinity on smooth, open manifolds. This operation requires a choice of proper ray in each manifold summand. In favorable circumstances, the connected sum at infinity operation is independent of ray choices. For each $m \geq 3$, we construct an infinite family of pairs of m–manifolds on which the connected sum at infinity operation yields distinct manifolds for certain ray choices. We use cohomology algebras at infinity to distinguish these manifolds.

1 Introduction

There exist several natural operations for combining manifolds. These include classical connected sum (CS), classical connected sum boundary (CSB), and the less familiar connected sum at infinity (CSI). CSI is roughly what happens to manifold interiors under CSB.

The CSI operation, also called end sum, was introduced by Gompf [7] to study smooth manifolds homeomorphic to $\mathbb{R}^4$. CSI is now a major tool for studying exotic smooth structures on open 4–manifolds (see Gompf and Stipsicz [8, Section 9.4] and Gompf [6]). It was also used by Ancel to study Davis manifolds in unpublished work during the 1980s and by Tinsley and Wright [22] and Myers [17] to study 3–manifolds. Recently, Calcut, King, and Siebenmann [2] gave a general treatment of CSI that yielded a natural proof of the Cantrell–Stallings hyperplane unknotting theorem.

Each of the above operations involves some choices. Under mild restrictions, CS and CSB are independent of these choices [2, Section 2]. CSI requires a choice of proper ray in each manifold summand. As a ray knots in $\mathbb{R}^m$ if and only if $m = 3$, it is not surprising that the result of CSI depends on ray choices in dimension 3. In fact, one may construct such examples where $\mathbb{R}^3$ is summed with itself for various rays (see Myers [17] and the appendix below).

For one-ended manifolds of dimension $m \geq 4$, the binary CSI operation yields a unique manifold up to diffeomorphism provided either:

(1) One summand is smoothly collared at infinity by S^{m-1}.

(2) Both summands satisfy the Mittag–Leffler condition. The Mittag–Leffler condition holds on a manifold M, for instance, if

   (i) M is topologically collared at infinity, or

   (ii) M admits an exhausting Morse function with only finitely many coindex-1 critical points.

Proofs of these two statements will appear in a subsequent paper.

The main purpose of this paper is to prove the following, which answers affirmatively a conjecture of Siebenmann [2].

Main theorem *There exist infinitely many pairs M and N of open, one-ended 4– manifolds such that ray choice alters the proper homotopy type of the CSI of M and N.*

In our explicit examples, one CSI summand is collared at infinity and thus contains a unique ray up to ambient isotopy. So ray choice is relevant even in just one summand. In view of (1) and (2), our examples are, in a sense, the simplest possible.

A question arises: Given a cardinal number c, does there exist an open, one-ended 4–manifold M such that the CSI of M with itself yields at least c manifolds up to proper homotopy? In Section 7, we exhibit an infinite collection of manifolds answering this question in the affirmative for each at most countably infinite c. We conjecture that this question has an affirmative answer when c is uncountable.

In each example used to prove the Main theorem, our ray choices do not alter the homotopy type of the CSI sum. We conjecture that there exist open, one-ended manifolds M and N such that ray choice alters the homeomorphism type but not the proper homotopy type of the CSI of M and N. Further, we conjecture that there exist open, one-ended 4–manifolds M and N such that ray choice alters the diffeomorphism type but not the homeomorphism type of the CSI of M and N. A possible candidate is the CSI of a ladder manifold (as defined in Section 3) and some exotic $\mathbb{R}^4$, although distinguishing the resulting manifolds up to diffeomorphism seems to be beyond present 4–manifold technology.

This paper is organized as follows. Section 2 defines CSI and fixes some notation. Section 3 introduces ladder manifolds and computes their cohomology algebras at

infinity. Section 4 defines stringer sum, an operation related to CSI. Section 5 studies stringer sum for ladder manifolds based on lens spaces. Section 6 proves the Main theorem. Section 7 presents various ways of generalizing our examples, including a proof of the Main theorem in each dimension at least 3. We close with an appendix on various 3–dimensional results.

Acknowledgement   The first author thanks Larry Siebenmann for introducing him to ladder manifolds of dimension $m \geq 7$, with stringers based on products of spheres, during their earlier collaboration [2].

2   Notation and definitions

Throughout, spaces are assumed to be metrizable, separable, and either compact or one-ended ($\mathbb{R}$ excepted). Manifolds are assumed to be smooth, connected, and oriented. Manifold boundaries are oriented by the standard outward normal first convention. A manifold without boundary is *closed* if it is compact and is *open* if it is noncompact. Write $A \approx B$ to mean A is diffeomorphic to B (not necessarily preserving orientation). A map is *proper* provided the inverse image of each compact set is compact. A *ray* is a proper embedding of $[0, \infty)$, where $[0, \infty) \subset \mathbb{R}$ is standardly oriented (see Guillemin and Pollack [9, Chapter 3]).

Definition 2.1   (Connected sum at infinity)   Let M and N be open manifolds of the same dimension $m \geq 2$. Fix rays $r \subset M$ and $r' \subset N$. Form the *connected sum at infinity* (CSI) of (M, r) and (N, r'), denoted $(M, r) \natural (N, r')$, as follows. Let $vr \subset M$ and $vr' \subset N$ be smooth, closed regular neighborhoods of r and r' respectively. Identify $M - \text{Int } vr$ and $N - \text{Int } vr'$ along $\partial vr \approx \mathbb{R}^{m-1}$ and $\partial vr' \approx \mathbb{R}^{m-1}$ via an orientation reversing diffeomorphism ϕ as in Figure 2.1.
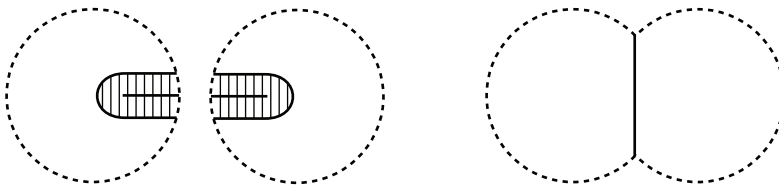

Figure 2.1: CSI summands (M, r) and (N, r') with vr and vr' hatched (left). Result of CSI operation $(M, r) \natural (N, r')$ (right).

Remarks 2.2   (1)   By common abuse, we consider the manifold $(M, r) \natural (N, r')$ to be smooth (see Hirsch [13, page 184]).

(2) Any diffeomorphism of $\mathbb{R}^{m-1}$ is isotopic to a linear mapping (see Milnor [15, page 34]). Together with uniqueness of regular neighborhoods (see Calcut, King and Siebenmann [2, Section 3]), this shows that the diffeomorphism type of $(M, r) \natural (N, r')$ is independent of the choices of vr, vr', and ϕ.

(3) The given definition of CSI is just sufficient for our purposes. It is subsumed by a more general definition [2] that:

   (i) Applies to differentiable, piecewise linear, and topological manifolds.

  (ii) Yields a manifold/ray pair.

 (iii) Is defined on any countable number of summands (see also Gompf [7]).

 (iv) Is commutative and associative.

We will use cohomology algebras at infinity to distinguish manifolds. Just as cohomology is a homotopy invariant of spaces, the cohomology algebra at infinity is a proper homotopy invariant of spaces (see Hughes and Ranicki [14, Chapter 3]). Throughout, let R be a commutative, unital ring. If X is a topological space, then we define the poset $(\mathcal{K}, \leq)$, where $\mathcal{K}$ is the set of compact subsets of X and $K \leq K'$ means $K \subseteq K'$. We have a direct system of graded R–algebras $H^*(X - K; R)$, where $K \in \mathcal{K}$. The morphisms of this direct system are restrictions. Define $H^*_\infty(X; R)$, the *cohomology algebra at infinity*, to be the direct limit of this system. Similarly, we define $\widetilde{H}^*_\infty(X; R)$ using reduced cohomology.

If $K_1 \subseteq K_2 \subseteq \cdots$ is a compact exhaustion of X, then we may compute $H^*_\infty(X; R)$ using the direct system indexed by the K_j. Namely,

$$(2\text{-}1) \qquad H^*_\infty(X; R) \cong \varinjlim_j H^*(X - K_j; R).$$

We employ the standard explicit model of the direct limit where an element of $H^*_\infty(X; R)$ is represented by an element of $H^*(X - K; R)$ for some compact K. Two representatives $\alpha \in H^*(X - K; R)$ and $\alpha' \in H^*(X - K'; R)$ are equivalent if they have the same restriction in some $H^*(X - K''; R)$, where $K, K' \subseteq K''$.

3 Ladder manifolds

In this section, we define ladder manifolds and compute their cohomology algebras at infinity. Ladder manifolds play a key role in our proof of the Main theorem. Fix closed manifolds X and Y of the same dimension $n \geq 2$.

Definition 3.1 (Ladder manifold) The *ladder manifold* $\mathbb{L}(X, Y)$ of X and Y is the oriented $(n+1)$–manifold obtained from the disjoint union

$$([0, \infty) \times X) \sqcup ([0, \infty) \times Y)$$

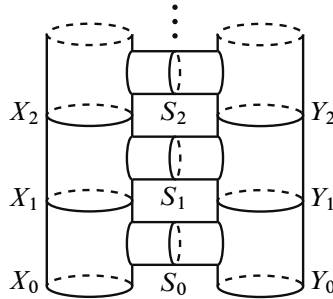by performing countably many oriented 0–surgeries as in Figure 3.1.

Figure 3.1: Ladder manifold $\mathbb{L}(X, Y)$

The manifolds $[0, \infty) \times X$ and $[0, \infty) \times Y$ are the *stringers*. Let $X_t := \{t\} \times X$ and $Y_t := \{t\} \times Y$. The glued-in copies of $D^1 \times S^n$ are the *rungs*, one for each integer $j \geq 0$. Let $S_j := \{0\} \times S^n$ be the central sphere in the jth rung.

More explicitly, fix closed n–balls $B_X \subset X$ and $B'_X \subset \operatorname{Int} B_X$, and similarly for Y. For each integer $j \geq 0$, perform an oriented 0–surgery using $(n + 1)$–disks, one in $\operatorname{Int}([j, j + 1] \times B'_X)$ (see Figure 3.2) and the other in $\operatorname{Int}([j, j + 1] \times B'_Y)$.
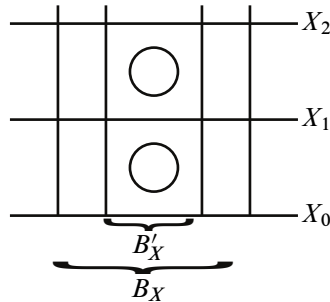

Figure 3.2: Circles indicating disks in $[0, \infty) \times X$ used for 0–surgeries

The ladder manifold $\mathbb{L}(X, Y)$ is canonically oriented. As X and Y are oriented, the stringers $[0, \infty) \times X$ and $[0, \infty) \times Y$ are each given the product orientation. Note that $\mathbb{L}(X, Y)$ has boundary $X_0 \sqcup Y_0$, oriented as $-X_0 - Y_0$ (see Guillemin and Pollack [9, Chapter 3]). Orient each S_j so that the oriented boundary of the first cobordism in Figure 3.3 is $X_{j+1} - X_j + S_j$. It follows that the oriented boundary of the second cobordism in Figure 3.3 is $Y_{j+1} - Y_j - S_j$. This completes our description of the ladder manifold $\mathbb{L}(X, Y)$.

The remainder of this section is devoted to computing the cohomology algebra at infinity of $\mathbb{L}(X, Y)$. For each integer $j \geq 0$, let W_j be the submanifold of $\mathbb{L}(X, Y)$
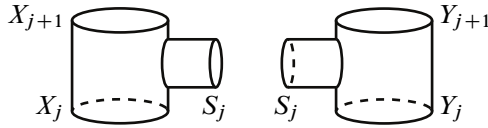
Figure 3.3: Two oriented cobordisms in $\mathbb{L}(X, Y)$

consisting of points of height $t \geq j$ (height is depicted vertically in Figure 3.1). Note that $W_0 = \mathbb{L}(X, Y)$, $W_j \approx \mathbb{L}(X, Y)$ for each j,

$$W_0 \supset W_1 \supset W_2 \supset \cdots$$

and $\bigcap_j W_j = \varnothing$. The inclusions $i_j \colon W_{j+1} \to W_j$ induce the direct system

$$(3\text{-}1) \qquad \widetilde{H}^*(W_0; R) \xrightarrow{i_0^*} \widetilde{H}^*(W_1; R) \xrightarrow{i_1^*} \widetilde{H}^*(W_2; R) \xrightarrow{i_2^*} \cdots.$$

Evidently,

$$(3\text{-}2) \qquad \widetilde{H}_\infty^*(\mathbb{L}(X, Y); R) \cong \varinjlim_j \widetilde{H}^*(W_j; R).$$

Let $J \subset \mathbb{L}(X, Y)$ be the noncompact n–complex shown in Figure 3.4. It is an iterated wedge of n–spheres (the S_j from above), 1–spheres (the T_j shown), and a 1–cell (bottom). The complex J is a variant of Jacob's ladder (see Hughes and Ranicki [14, page 25]).
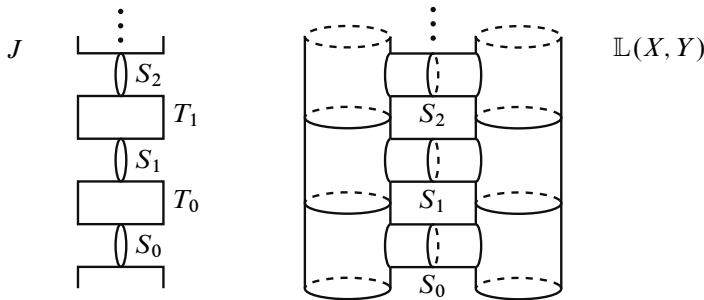

Figure 3.4: One-ended n–complex $J \subset \mathbb{L}(X, Y)$

We remind the reader that R denotes a commutative, unital ring. Recall that $R[x] \cong \bigoplus_{n=0}^\infty R$ and $R[[x]] \cong \prod_{n=0}^\infty R$ as R–modules. In general, $R[x]$ is a free R–module, but $R[[x]]$ need not be. When R is a field, $R[[x]]$ is an R–vector space and hence a free R–module. However, $\mathbb{Z}[[x]]$ is *not* a free $\mathbb{Z}$–module (see Schröer [18]).

The nonzero reduced integer homology groups of J are $\widetilde{H}_n(J) \cong \mathbb{Z}[s]$ and $\widetilde{H}_1(J) \cong \mathbb{Z}[t]$, where s^k corresponds to the fundamental class of S_k and t^k corresponds to the

fundamental class of T_k. By the universal coefficient theorem, the nonzero reduced cohomology groups of J are

$$\tilde{H}^n(J; R) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[s], R) \cong R[\![\sigma]\!],$$

$$\tilde{H}^1(J; R) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[t], R) \cong R[\![\tau]\!],$$

since the Ext terms vanish in all dimensions. All cup products in $\tilde{H}^*(J; R)$ vanish.

For each integer $j \geq 0$, let J_j denote the points in J of height $t \geq j$. Note that $J_0 = J$ and J_j is homeomorphic to J for each j. Define

$$V_j := (X_j \vee J_j) \vee Y_j \subset W_j$$

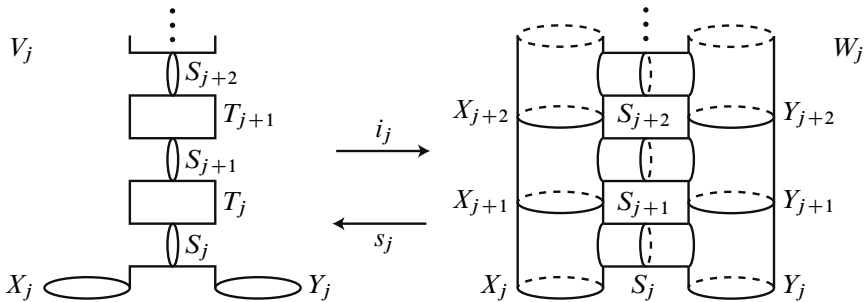as shown in Figure 3.5, where $\iota_j\colon V_j \to W_j$ is inclusion.

Figure 3.5: Strong deformation retract V_j of W_j

Lemma 3.2 *For each j, there is a strong deformation retraction $s_j\colon W_j \to V_j$.*

Proof We begin by retracting the stringer portions of W_j, while fixing the rungs pointwise. Figure 3.6 shows schematically how do this above X_j; the same argument applies above Y_j. Next, simultaneously retract the remaining rung portions as shown in Figure 3.7. This completes our description of s_j. $\qquad\square$

We have the following diagram where the left maps are the obvious inclusions and projections.

(3-3)

$$
\begin{array}{c}
X_j \\
J_j \rightrightarrows V_j \underset{s_j}{\overset{\iota_j}{\rightrightarrows}} W_j \\
Y_j
\end{array}
$$

(a) Region above $X_j - B_X$ retracts to $X_j - B_X$, while region above $B_X - B'_X$ retracts to a hyperboloid

(b) Retraction of region under hyperboloid

(c) Retraction of the region above B'_X

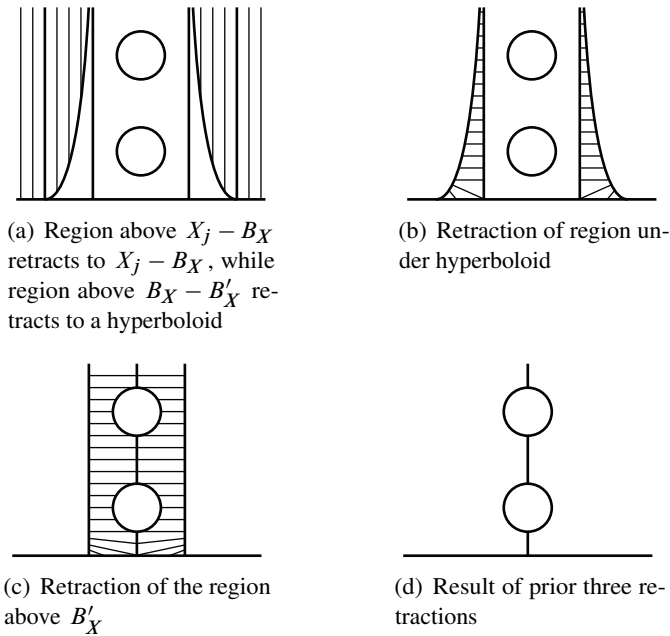(d) Result of prior three retractions

Figure 3.6: Strong deformation retraction of portion of W_j above X_j to the iterated wedge of X_j and an infinite string of n–spheres and intervals. Rungs of W_j are fixed pointwise at all times.
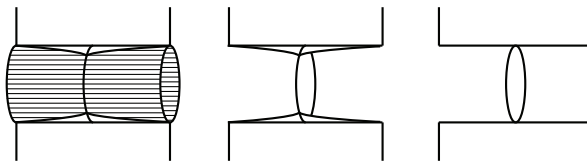

Figure 3.7: Strong deformation retraction of a rung

Corollary 3.3 *For each j, there is an isomorphism of graded R–algebras*

$$\tilde{H}^*(W_j; R) \cong \tilde{H}^*(X; R) \oplus \tilde{H}^*(J_j; R) \oplus \tilde{H}^*(Y; R).$$

The cup product is coordinatewise in the direct sum.

Proof This follows immediately from Lemma 3.2, diagram (3-3), and the computation of the cohomology algebra of a wedge sum (see Hatcher [12, page 215]). □

Recall the direct system (3-1).

Lemma 3.4 *Each i_j^* is surjective.*

Proof For each $j \geq 0$, there is an obvious retraction $r_j\colon W_j \to W_{j+1}$. It sends X_{j+t} to X_{j+2-t} and Y_{j+t} to Y_{j+2-t} for $t \in [0,1]$ and sends the bottom rung of W_j to the bottom rung of W_{j+1}. Thus, $r_j \circ i_j = \mathrm{id}$, and so $(r_j \circ i_j)^* = i_j^* \circ r_j^* = \mathrm{id}^*$. As id^* is an isomorphism on $\widetilde{H}^*(W_{j+1}; R)$, i_j^* is surjective. $\qquad\square$

For $j < k$, let $i_{k,j}^* := i_k^* \circ i_{k-1}^* \circ \cdots \circ i_j^*$. By Lemma 3.4, each element in the direct limit (3-2) has a representative in $\widetilde{H}^*(W_0; R)$. Indeed, if $\alpha \in \widetilde{H}^*(W_j; R)$ represents an element ω in the direct limit, then there exists some $\beta \in \widetilde{H}^*(W_0; R)$ such that $i_{j-1,0}^*(\beta) = \alpha$, so β also represents ω. Thus we can write

$$(3\text{-}4) \qquad \widetilde{H}^*_\infty(\mathbb{L}(X,Y); R) \cong \varinjlim_j \widetilde{H}^*(W_j; R) \cong \widetilde{H}^*(W_0; R)/\sim,$$

where $\alpha \sim \beta$ if and only if there exists j such that $i_{j,0}^*(\alpha) = i_{j,0}^*(\beta)$.

Proposition 3.5 *The cohomology algebra at infinity of* $\mathbb{L}(X,Y)$ *is*

$$\widetilde{H}^k_\infty(\mathbb{L}(X,Y); R) \cong \begin{cases} (\widetilde{H}^n(X; R) \oplus R[\![\sigma]\!] \oplus \widetilde{H}^n(Y; R))/K & \text{if } k = n, \\ \widetilde{H}^k(X; R) \oplus 0 \oplus \widetilde{H}^k(Y; R) & \text{if } 2 \leq k \leq n-1, \\ \widetilde{H}^1(X; R) \oplus R[\![\tau]\!]/R[\tau] \oplus \widetilde{H}^1(Y; R) & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $K := \{(\sum \beta_i, \beta, -\sum \beta_i) \mid \beta = \sum \beta_i \sigma^i \in R[\sigma]\} \cong R[\sigma]$. *The cup product is coordinatewise in the direct sum.*

Proof By the preceding discussion, it remains to describe $\sim$. As

$$\widetilde{H}^*(W_j, W_{j+1}; R) \cong H^*(W_j/W_{j+1}; R),$$

the long exact sequence for the pair (W_j, W_{j+1}) and Lemma 3.4 imply that the homomorphism $i_j^*\colon \widetilde{H}^k(W_j; R) \to \widetilde{H}^k(W_{j+1}; R)$ is an isomorphism for $2 \leq k \leq n-1$. Thus $\widetilde{H}^k_\infty(\mathbb{L}(X,Y); R) \cong \widetilde{H}^k(W_0; R)$ for $2 \leq k \leq n-1$.

For each j, we have the following commutative diagram of spaces.

$(3\text{-}5)$

Here, s_j is the retraction from Lemma 3.2, ι_{j+1} is inclusion, and $d_j := s_j \circ i_j \circ \iota_{j+1}$. The bottom maps are again the obvious inclusions and projections.

There are nine compositions in (3-5) that begin and end on the bottom row. The geometry of these compositions is straightforward given our definition of s_j. Recalling our orientation conventions, these compositions induce a homomorphism on integer homology

$$\psi \colon \widetilde{H}_*(X) \oplus \widetilde{H}_*(J_{j+1}) \oplus \widetilde{H}_*(Y) \to \widetilde{H}_*(X) \oplus \widetilde{H}_*(J_j) \oplus \widetilde{H}_*(Y).$$

In dimensions other than n, $\psi = \mathrm{id} \oplus \mathrm{inclusion} \oplus \mathrm{id}$. In dimension n, ψ is given by $(a, b, c) \mapsto (a, (c-a)s^j + b, c)$ for $b \in s^{j+1}\mathbb{Z}[s]$.

As X and Y are closed and oriented n–manifolds, the group

$$\widetilde{H}_{n-1}(W_j; \mathbb{Z}) \cong \begin{cases} \widetilde{H}_{n-1}(X; \mathbb{Z}) \oplus 0 \oplus \widetilde{H}_{n-1}(Y; \mathbb{Z}) & \text{if } n > 2 \\ \widetilde{H}_{n-1}(X; \mathbb{Z}) \oplus t^j \mathbb{Z}[t] \oplus \widetilde{H}_{n-1}(Y; \mathbb{Z}) & \text{if } n = 2 \end{cases}$$

is free abelian. Thus $\mathrm{Ext}(\widetilde{H}_{m-1}(W_j; \mathbb{Z}), R) = 0$ for $m = 1$ and $m = n - 1$. The universal coefficient theorem implies that i_j^* is the dual homomorphism of ψ in these dimensions. Therefore, i_j^* is clear in dimension 1. In dimension n, i_j^* sends (α, β, γ) to $(\alpha - \beta_j, \beta - \beta_j \sigma^j, \gamma + \beta_j)$, where $\beta \in \sigma^j R[\![\sigma]\!]$. To determine $\sim$ in the remaining dimensions 1 and n, it suffices to describe the subgroups of $\widetilde{H}^1(W_0; R)$ and $\widetilde{H}^n(W_0; R)$ consisting of elements that are sent to 0 by some $i_{j,0}^*$. By our description of i_j^* as the dual of ψ, these subgroups are exactly $0 \oplus R[\tau] \oplus 0$ in dimension 1 and K in dimension n.

The cup product structure of the algebra $\widetilde{H}_\infty^*(\mathbb{L}(X, Y); R)$ can be summarized as $[\alpha] \cup [\beta] = [\alpha \cup \beta]$ for $\alpha, \beta \in \widetilde{H}^*(W_0; R)$. This is a direct consequence of the definition of a direct limit of algebras and the fact that every element of $\widetilde{H}_\infty^*(\mathbb{L}(X, Y); R)$ has a representative in $\widetilde{H}^*(W_0; R)$. Combining these facts yields the claim about cup products in the statement of this proposition. □

Remark 3.6 Observe that the canonical map

$$\pi \colon \widetilde{H}^n(X; R) \oplus 0 \oplus \widetilde{H}^n(Y; R) \to (\widetilde{H}^n(X; R) \oplus R[\![\sigma]\!] \oplus \widetilde{H}^n(Y; R))/K$$

is injective. In particular, if R is a field, then the image of π is a 2–dimensional R–vector subspace of an uncountably-infinite dimensional R–vector space.

4 Stringer sum

We now define stringer sum, an operation on a ladder manifold and a disjoint stringer that yields a new ladder manifold. Recall that a manifold A is *neatly* embedded in a manifold B if $A \cap \partial B = \partial A$ and this intersection is transverse. A *straight* ray in $[0, \infty) \times A$ has the form $[0, \infty) \times \{a\}$.

Definition 4.1 (Stringer sum) Fix a ladder manifold $\mathbb{L}(X, Y)$ and a disjoint stringer $[0, \infty) \times Z$ of the same dimension $n + 1 \geq 3$. Fix neatly embedded, straight rays $r \subset [0, \infty) \times Z$ and $r' \subset \mathbb{L}(X, Y)$, where r' lies in one of the stringers of $\mathbb{L}(X, Y)$ and avoids B_X and B_Y (see Figure 4.1). Define the *stringer sum* of $(\mathbb{L}(X, Y), r')$ and $([0, \infty) \times Z, r)$, denoted

$$(\mathbb{L}(X, Y), r') \cdot ([0, \infty) \times Z, r),$$

as follows. Let $vr \subset [0, \infty) \times Z$ and $vr' \subset \mathbb{L}(X, Y)$ be normal, closed tubular neighborhoods of r and r' respectively. Identify $([0, \infty) \times Z) - \operatorname{Int} vr$ and $\mathbb{L}(X, Y) - \operatorname{Int} vr'$ along ∂vr and $\partial vr'$ via an orientation reversing, fiber respecting diffeomorphism so that connected sum is achieved at each height.
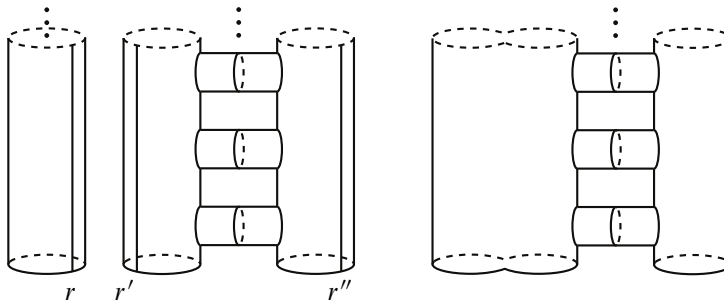

Figure 4.1: Stringer $[0, \infty) \times Z$ with ray r and ladder manifold $\mathbb{L}(X, Y)$ with rays r' and r'' (left). Stringer sum $(\mathbb{L}(X, Y), r') \cdot ([0, \infty) \times Z, r)$ (right).

With notation as in Figure 4.1, observe that

$$(\mathbb{L}(X, Y), r') \cdot ([0, \infty) \times Z, r) \approx \mathbb{L}(X \# Z, Y),$$
$$(\mathbb{L}(X, Y), r'') \cdot ([0, \infty) \times Z, r) \approx \mathbb{L}(X, Y \# Z),$$

where # denotes oriented connected sum CS.

5 Examples: Lens spaces

For each positive integer k, let L_k denote the 3–dimensional lens space $L(k, 1)$, which is obtained by performing $-k$–surgery on the unknot in S^3 (see Gompf and Stipsicz [8, page 158]). Recall that

$$\widetilde{H}_i(L_k; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 3, \\ \mathbb{Z}_k & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \widetilde{H}^i(L_k; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 3, \\ \mathbb{Z}_k & \text{if } i = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and $\widetilde{H}^i(L_k; \mathbb{Z}_k) \cong \mathbb{Z}_k$ for $i = 1, 2, 3$.

For computability, we now restrict to field coefficients. Fix a prime $p > 0$. By Poincaré duality, if M is a closed m–manifold and $0 \neq \alpha \in \tilde{H}^i(M; \mathbb{Z}_p)$ where $i < m$, then there exists $\beta \in \tilde{H}^{m-i}(M; \mathbb{Z}_p)$ such that $\alpha \cup \beta \in \tilde{H}^m(M; \mathbb{Z}_p)$ is a generator (see Hatcher [12, page 250]).

If M is an $(n+1)$–manifold, then define

$$\Gamma_p(M) := \langle \alpha \cup \beta \mid \deg \alpha, \deg \beta < n \rangle \leq \tilde{H}^n_\infty(M; \mathbb{Z}_p)$$

to be the vector subspace of $\tilde{H}^n_\infty(M; \mathbb{Z}_p)$ generated by products of classes of degree less than n in $\tilde{H}^*_\infty(M; \mathbb{Z}_p)$. The dimension of Γ_p as a $\mathbb{Z}_p$–vector space, denoted $\dim_{\mathbb{Z}_p} \Gamma_p$, is a graded algebra invariant since an isomorphism of graded algebras respects products and gradings.

By Proposition 3.5 and Remark 3.6, we have

$$\dim_{\mathbb{Z}_p} \Gamma_p(\mathbb{L}(X, Y)) \leq 2$$

for any ladder manifold.

Proposition 5.1 *The stringer sums*

$$(\mathbb{L}(L_p, S^3), r') \cdot ([0, \infty) \times L_p, r) \approx \mathbb{L}(L_p \,\#\, L_p, S^3),$$
$$(\mathbb{L}(L_p, S^3), r'') \cdot ([0, \infty) \times L_p, r) \approx \mathbb{L}(L_p, L_p)$$

have nonisomorphic $\mathbb{Z}_p$–cohomology algebras at infinity.

Proof Consider the algebras $\tilde{H}^*_\infty(\mathbb{L}(L_p \,\#\, L_p, S^3); \mathbb{Z}_p)$ and $\tilde{H}^*_\infty(\mathbb{L}(L_p, L_p); \mathbb{Z}_p)$, computed as in Proposition 3.5. Notice that

$$\dim_{\mathbb{Z}_p} \Gamma_p(\mathbb{L}(L_p \,\#\, L_p, S^3)) = 1 \qquad \text{and} \qquad \dim_{\mathbb{Z}_p} \Gamma_p(\mathbb{L}(L_p, L_p)) = 2. \qquad \square$$

6 Proof of the main theorem

For each integer $k \geq 1$, let E_k be the D^2 bundle over S^2 with Euler number $-k$, which is a 0–handle union a 2–handle attached along a $-k$ framed unknot (see Gompf and Stipsicz [8, pages 119–120]). Note that $\partial E_k = L_k$. Define

$$Y_k := \mathbb{L}(L_k, S^3) \cup_\partial E_k \cup_\partial D^4,$$
$$Z_k := ([0, \infty) \times L_k) \cup_\partial E_k \approx \text{Int } E_k.$$

Both Y_k and Z_k are smooth, open, one-ended 4–manifolds. We refer to Y_k as a *capped ladder* and Z_k as a *capped stringer* (see Figure 6.1).
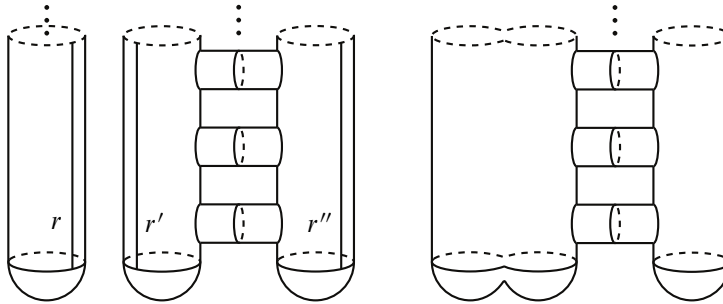
Figure 6.1: Capped stringer Z_k, capped ladder Y_k, and result of CSI operation $(Y_k, r') \natural (Z_k, r)$

Let $r', r'' \subset \mathbb{L}(L_k, S^3) \subset Y_k$ and $r \subset [0, \infty) \times L_k \subset Z_k$ be rays as in Proposition 5.1 and Figure 6.1.

Theorem 6.1 *Let $p > 0$ be prime. The manifolds*

$$M_1 := (Y_p, r') \natural (Z_p, r) \quad and \quad M_2 := (Y_p, r'') \natural (Z_p, r)$$

are not proper homotopy equivalent.

Proof First, note that

$$M_1 \approx \mathbb{L}(L_p \# L_p, S^3) \cup_\partial (E_p \#_\partial E_p) \cup_\partial D^4,$$
$$M_2 \approx \mathbb{L}(L_p, L_p) \cup_\partial E_p \cup_\partial E_p,$$

where $\#_\partial$ denotes oriented connected sum boundary CSB. Thus

$$\widetilde{H}^*_\infty(M_1; \mathbb{Z}_p) \cong \widetilde{H}^*_\infty(\mathbb{L}(L_p \# L_p, S^3); \mathbb{Z}_p),$$
$$\widetilde{H}^*_\infty(M_2; \mathbb{Z}_p) \cong \widetilde{H}^*_\infty(\mathbb{L}(L_p, L_p); \mathbb{Z}_p).$$

By Proposition 5.1, $\widetilde{H}^*_\infty(M_1; \mathbb{Z}_p)$ and $\widetilde{H}^*_\infty(M_2; \mathbb{Z}_p)$ are not isomorphic. Therefore M_1 and M_2 are not proper homotopy equivalent. $\square$

Remark 6.2 Observe that $Y_j \not\approx Y_k$ and $Z_j \not\approx Z_k$ for positive integers $j \neq k$, and $Y_j \not\approx Z_k$ for any positive integers j and k. These observations hold by the following facts:

(i) $\widetilde{H}^2_\infty(Y_j; \mathbb{Z}) \cong \mathbb{Z}_j$.

(ii) $\widetilde{H}^2_\infty(Z_j; \mathbb{Z}) \cong \mathbb{Z}_j$.

(iii) $\widetilde{H}^1_\infty(Y_j; \mathbb{Z}) \cong \mathbb{Z}[[\tau]]/\mathbb{Z}[\tau]$.

(iv) $\widetilde{H}^1_\infty(Z_k; \mathbb{Z}) = 0$.

Facts (i) and (iii) follow from Proposition 3.5, while (ii) and (iv) follow from the basic property that $\widetilde{H}^*_\infty([0, \infty) \times X; R) \cong \widetilde{H}^*(X; R)$ for each closed manifold X. Hence Theorem 6.1 implies the Main theorem as there are infinitely many primes.

Remark 6.3 As noted by the referee, the Main theorem may be deduced from any single example from Theorem 6.1 by using an infinite variety of caps. For instance, fix a prime $p > 0$ and let $Y_{p,q}$ denote the result of replacing the D^4 cap in Y_p with a cap C_q such that $q \geq 2$ and $\pi_1(C_q) \cong \mathbb{Z}_q$. Although the resulting pairs of manifolds Z_p and $Y_{p,q}$ (with p fixed and for various q) are not homotopy equivalent by π_1, they have diffeomorphic ends.

7 Generalizations of the main examples

Our main examples from Section 6 are readily modified to produce more 4–dimensional examples as well as others of all dimensions at least 3. Define

$$T^k := \underbrace{S^1 \times \cdots \times S^1}_{k}.$$

(1) In Y_p, S^3 can be replaced with any $\mathbb{Z}_p$–homology 3–sphere Σ^3 and D^4 with any smooth null-cobordism of Σ^3.

(2) For any $j \geq 1$, Z_p can be replaced with Z_{jp} since $\widetilde{H}^i(L_{jp}; \mathbb{Z}_p) \cong \mathbb{Z}_p$ for $i = 1, 2, 3$.

(3) To obtain examples in all dimensions $n + 1 \geq 4$, replace L_p with $L_p \times T^{n-3}$ in both Y_p and Z_p, and replace S^3 with S^n. Cap with $E_p \times T^{n-3}$ and D^{n+1}. Crossing with S^1 does not affect $\dim_{\mathbb{Z}_p} \Gamma_p$. We obtain infinitely many examples this way by the following observations derived from Remark 6.2 and the Künneth formula:

$$\text{torsion } \widetilde{H}^2_\infty(Y_p; \mathbb{Z}) \cong \mathbb{Z}_p, \qquad\qquad \text{torsion } \widetilde{H}^2_\infty(Z_p; \mathbb{Z}) \cong \mathbb{Z}_p,$$

$$\widetilde{H}^1_\infty(Y_p; \mathbb{Z}) \cong \mathbb{Z}^{n-3} \oplus \mathbb{Z}[\![\tau]\!]/\mathbb{Z}[\tau], \qquad \widetilde{H}^1_\infty(Z_p; \mathbb{Z}) \cong \mathbb{Z}^{n-3}.$$

(4) Let Σ_g be the closed surface of genus g. Let H_g be the 3–dimensional handlebody with $\partial H_g = \Sigma_g$. Define

$$Y_g := \mathbb{L}(\Sigma_g, S^2) \cup_\partial H_g \cup_\partial D^3,$$

$$Z_g := ([0, \infty) \times \Sigma_g) \cup_\partial H_g.$$

Let $r', r'' \subset Y_g$ and $r \subset Z_g$ be straight rays as in Figure 6.1. For $g, h \in \mathbb{Z}^+$, define

$$M_1(g, h) := (Y_g, r') \natural (Z_h, r),$$

$$M_2(g, h) := (Y_g, r'') \natural (Z_h, r).$$

Fix any prime $p > 0$. By Proposition 3.5,

$$\dim_{\mathbb{Z}_p} \Gamma_p(M_1(g,h)) = 1,$$
$$\dim_{\mathbb{Z}_p} \Gamma_p(M_2(g,h)) = 2.$$

Thus, $M_1(g,h)$ and $M_2(g,h)$ are not proper homotopy equivalent.

We can prove the Main theorem for 3–manifolds using the collection of pairs Y_1 and Z_g for $g \in \mathbb{Z}^+$. These manifolds are distinguished by the following facts:

$$\tilde{H}^1_\infty(Y_1; \mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}[\![\tau]\!]/\mathbb{Z}[\tau],$$
$$\tilde{H}^1_\infty(Z_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

Interestingly, we cannot distinguish Y_g and $Y_{g'}$ for any $g \neq g'$.

We obtain infinitely many new examples in all dimensions $n + 1 \geq 4$ by considering ladders and stringers based on $\Sigma_g \times T^{n-2}$ and S^n. In these dimensions, one can distinguish all of the summands. Details are left to the interested reader.

(5)   Let $p, q > 0$ be distinct primes. Define

$$M(p,q) := \mathbb{L}(L_p, L_q) \cup_\partial E_p \cup_\partial E_q.$$

Fix neat, straight rays $r', r'' \subset \mathbb{L}(L_p, L_q)$, one in each stringer. The same technique used to prove Proposition 3.5 yields the following table where $M := M(p,q)$.

	$\dim_{\mathbb{Z}_p} \Gamma_p$	$\dim_{\mathbb{Z}_q} \Gamma_q$
$(M, r') \natural (M, r')$	1	2
$(M, r'') \natural (M, r'')$	2	1
$(M, r') \natural (M, r'')$	2	2

Thus the CSI of M with itself yields at least 3 distinct manifolds up to proper homotopy. We obtain infinitely many M with this property since

$$\tilde{H}^2_\infty(M(p,q); \mathbb{Z}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_q.$$

(6)   Given 3–dimensional lens spaces $L_{p_1}, L_{p_2}, \ldots, L_{p_{m+1}}$, we define the *generalized capped ladder manifold* $M(p_1, p_2, \ldots, p_{m+1})$ inductively as

$$M(p_1, p_2, \ldots, p_{m+1})$$
$$:= \left(M(p_1, p_2, \ldots, p_m), r\right) \natural \left(\mathbb{L}(S^3, L_{p_{m+1}}) \cup_\partial (D^4 \sqcup E_{p_{m+1}}), r'\right)$$

where r and r' are neat, straight rays in the L_{p_m} and S^3 stringers respectively. Now, fix $p_1, p_2, \ldots, p_m > 0$ to be distinct primes. A similar calculation to the one in the

previous item shows that the CSI of $M(p_1, p_2, \ldots, p_m)$ with itself yields at least $m + 1$ distinct manifolds up to proper homotopy. We obtain infinitely many M with this property since

$$\widetilde{H}^2_\infty(M(p_1, p_2, \ldots, p_m); \mathbb{Z}) \cong \bigoplus_{i=1}^m \mathbb{Z}_{p_i}.$$

(7) Fix $p_1, p_2, \ldots$ to be distinct positive primes. Define $M(p_1, p_2, \ldots)$ in analogy with the previous item. We will show that the CSI of $M(p_1, p_2, \ldots)$ with itself yields countably infinitely many distinct manifolds up to proper homotopy and that we obtain infinitely many M with this property since

$$\widetilde{H}^2_\infty(M(p_1, p_2, \ldots); \mathbb{Z}) \cong \prod_{i=1}^\infty \mathbb{Z}_{p_i}.$$

One cannot compute $\widetilde{H}^*_\infty(M(p_1, p_2, \ldots); R)$ by taking W_j to be the points of height at least j nor can W_j be the complement of the first j stringers since W_j must come from a compact exhaustion of $M(p_1, p_2, \ldots)$.
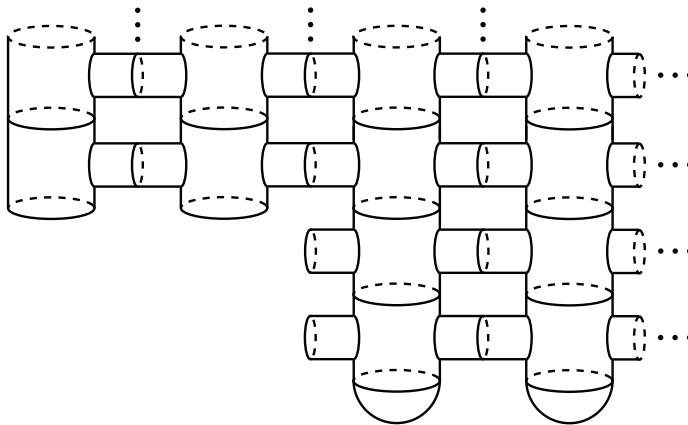

Figure 7.1: Submanifold W_j ($j = 2$ depicted) of an infinite ladder manifold. Points in the first j stringers of height less than j have been removed

Instead, define W_j as in Figure 7.1. In particular, $W_0 = M(p_1, p_2, \ldots)$, $W_{j+1} \subset W_j$ for all $j \geq 0$, and $\bigcap_j W_j = \varnothing$. Let $i_j \colon W_{j+1} \to W_j$ be inclusion. Unlike our previous examples, there is no retraction $W_j \to W_{j+1}$.

Similar to Lemma 3.2 above, there is a strong deformation retraction $s_j \colon W_j \to V_j$ where V_j appears as in Figure 7.2. In particular, V_j is an iterated wedge of lens spaces, 3–spheres (the doubly indexed family S_i^k), 1–spheres (the doubly indexed family T_i^k),
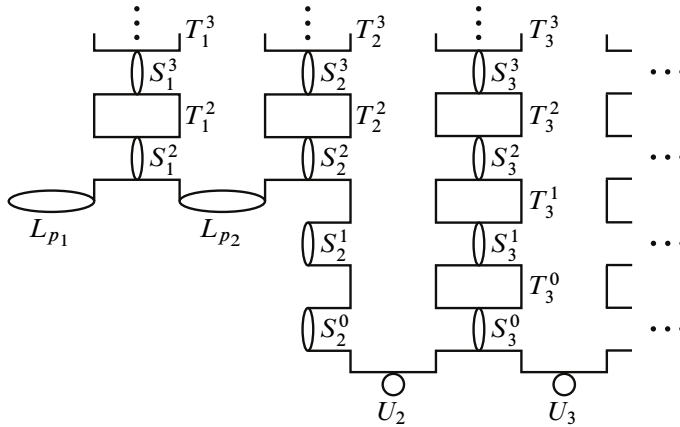
Figure 7.2: Strong deformation retract V_j of W_j ($j = 2$ depicted)

2–spheres (the singly indexed family U_i; recall that each cap E_{p_i} is a D^2 bundle over S^2) and 1–cells. Thus

$$\widetilde{H}_1(W_j; R) \cong \bigoplus_{i=1}^{j} \widetilde{H}_1(L_{p_i}; R) \oplus \bigoplus_{i=1}^{j} t_i^j R[t_i] \oplus \bigoplus_{i=j+1}^{\infty} R[t_i],$$

$$\widetilde{H}_2(W_j; R) \cong \bigoplus_{i=1}^{j} \widetilde{H}_2(L_{p_i}; R) \oplus u^j R[u]$$

$$, \widetilde{H}_3(W_j; R) \cong \bigoplus_{i=1}^{j} \widetilde{H}_3(L_{p_i}; R) \oplus \bigoplus_{i=1}^{j-1} s_i^j R[s_i] \oplus \bigoplus_{i=\max\{j,1\}}^{\infty} R[s_i],$$

and

$$\widetilde{H}^1(W_j; R) \cong \prod_{i=1}^{j} \widetilde{H}^1(L_{p_i}; R) \times \prod_{i=1}^{j} \tau_i^j R[\![\tau_i]\!] \times \prod_{i=j+1}^{\infty} R[\![\tau_i]\!],$$

$$\widetilde{H}^2(W_j; R) \cong \prod_{i=1}^{j} \widetilde{H}^2(L_{p_i}; R) \times \mu^j R[\![\mu]\!],$$

$$\widetilde{H}^3(W_j; R) \cong \prod_{i=1}^{j} \widetilde{H}^3(L_{p_i}; R) \times \prod_{i=1}^{j-1} \sigma_i^j R[\![\sigma_i]\!] \times \prod_{i=\max\{j,1\}}^{\infty} R[\![\sigma_i]\!].$$

To compute the maps i_j^*, we write i_j as a composition of inclusions. Let W_j' denote W_j minus the interior of the cap $E_{p_{j+1}}$, so $W_j = W_j' \cup_\partial E_{p_{j+1}}$. Let $f_j\colon W_{j+1} \to W_j'$ and $g_j\colon W_j' \to W_j$ be inclusions, so $i_j = g_j \circ f_j$. There is a retract $r_j\colon W_j' \to W_{j+1}$,

namely apply the process in Lemma 3.2 *upwards* to the region $W_j' - W_{j+1}$ of W_j' and then flip the exposed 3–spheres and 1–cells up or to the right. Thus, with R coefficients, each f_j^* is surjective and, by the long exact sequence of the pair (W_j', W_{j+1}), f_j^* is an isomorphism in dimension 2.

Excision and Thom's isomorphism theorem yield isomorphisms

$$H^{k+2}(W_j, W_j'; R) \to H^{k+2}(E_{p_{j+1}}, L_{p_{j+1}}; R) \leftarrow H^k(S^2; R)$$

for each k. So, with R coefficients, the long exact sequence of the pair (W_j, W_j') implies that g_j^* is surjective in dimension 2.

The Mayer–Vietoris sequence with $W_j = W_j' \cup_\partial E_{p_{j+1}}$ and integer coefficients implies that g_j^* is an isomorphism in dimension 1 and is not surjective in dimension 3. The universal coefficient theorem implies that each of i_j^*, f_j^*, and g_j^* (with R coefficients) in dimensions 1 and 3 is dual to the appropriate homomorphism on integer homology since the relevant Ext terms vanish. In particular,

$$\widetilde{H}_\infty^1(M(p_1, p_2, \ldots); \mathbb{Z}) \cong \prod_{i=1}^\infty \mathbb{Z}[\![\tau_i]\!] / \mathbb{Z}[\tau_i].$$

In dimension 2, the map $i_j^* \colon \widetilde{H}^2(W_j; \mathbb{Z}) \to \widetilde{H}^2(W_{j+1}; \mathbb{Z})$ is

$$\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_j} \times \mathbb{Z} \times \mu^{j+1} \mathbb{Z}[\![\mu]\!] \to \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_j} \times \mathbb{Z}_{p_{j+1}} \times \mu^{j+1} \mathbb{Z}[\![\mu]\!],$$

where we have written $\mathbb{Z} \times \mu^{j+1} \mathbb{Z}[\![\mu]\!]$ for $\mu^j \mathbb{Z}[\![\mu]\!]$. This map is the identity on each factor except $\mathbb{Z}$ maps to $\mathbb{Z}_{p_{j+1}}$ surjectively. Thus $\widetilde{H}_\infty^2(M(p_1, p_2, \ldots); \mathbb{Z})$ is isomorphic to

$$\widetilde{H}^2(W_0; \mathbb{Z}) \cong \mathbb{Z}[\![\mu]\!] \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$$

modulo the subgroup of elements eventually sent to zero, namely $p_1\mathbb{Z} \times p_2\mathbb{Z} \times p_3\mathbb{Z} \times \cdots$. Hence

$$\widetilde{H}_\infty^2(M(p_1, p_2, \ldots); \mathbb{Z}) \cong \prod_{i=1}^\infty \mathbb{Z}_{p_i}.$$

One may write down explicitly the maps i_j^* in dimension 3 as well. Note that we have a natural embedding

$$\prod_{i=1}^\infty \widetilde{H}^3(L_{p_i}; R) \to \widetilde{H}_\infty^3(M(p_1, p_2, \ldots); R)$$

induced by

$$(\alpha, \beta, \gamma, \ldots) \mapsto -(\alpha\sigma_1^0, \alpha\sigma_2^0 + \beta\sigma_2^1, \alpha\sigma_3^0 + \beta\sigma_3^1 + \gamma\sigma_3^2, \ldots) \in \widetilde{H}^3(W_0; R).$$

Let $M = M(p_1, p_2, \ldots)$ and let r_i be a neat, straight ray in the ith stringer of M. For each $i \geq 1$, we have

$$\dim_{\mathbb{Z}_{p_i}} \Gamma_{p_i}(M) = 1.$$

If i and j are distinct positive integers, then

$$\dim_{\mathbb{Z}_{p_i}} \Gamma_{p_i}((M, r_i) \natural (M, r_i)) = 1 \quad \text{and} \quad \dim_{\mathbb{Z}_{p_j}} \Gamma_{p_j}((M, r_i) \natural (M, r_i)) = 2.$$

Hence the manifolds $(M, r_i) \natural (M, r_i)$, $i \geq 1$, are pairwise distinct up to proper homotopy.

Appendix: CSI and $\mathbb{R}^3$

The purpose of this appendix is to give a simple proof in the smooth category that a CSI of $\mathbb{R}^3$ with itself need not yield $\mathbb{R}^3$.

Fix a smooth, proper embedding $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. Let $H := \operatorname{Im} f$ be a hyperplane in $\mathbb{R}^3$. Let A and B denote the closures in $\mathbb{R}^3$ of the two components of $\mathbb{R}^3 - H$. So $\partial A = \partial B = H$, $A \cap B = H$ and $A \cup B = \mathbb{R}^3$. As we are interested in H up to ambient isotopy of $\mathbb{R}^3$, we assume $f(0) = 0$.

Definition A.1 (Nice 2–disk) A 2–disk $D \subset \mathbb{R}^3$ is *nice* provided: (i) D is neatly embedded in A or in B, and (ii) ∂D is essential in $H - \{0\}$.

Lemma A.2 *Let $K \subset \mathbb{R}^3$ be compact. Then, there exists a nice 2–disk $D \subset \mathbb{R}^3 - K$.*

Proof Let $D^3 \subset \mathbb{R}^3$ be a 3–disk centered at 0 and containing K. By replacing K with D^3, we may assume K is connected. Let $B^2 \subset \mathbb{R}^2$ be a disk centered at 0 containing $f^{-1}(K)$. Let $K' := K \cup f(B^2)$, which is compact and connected. Let $S \subset \mathbb{R}^3$ be a 2–sphere such that K' lies inside S, and S meets H transversely. So, $S \cap H$ is a finite disjoint union of circles disjoint from K', at least one of which is essential in $H - \{0\}$. If there exist components of $S \cap H$ that are inessential in $H - \{0\}$, then let C be one that is innermost in $H - \{0\}$. Then, C bounds 2–disks $\Delta \subset H - \{0\}$ and $D_1, D_2 \subset S$ (see Figure A.3). Note that Δ is disjoint from K'.

Each of $\Delta \cup D_1$ and $\Delta \cup D_2$ is a (piecewise smooth) embedded 2–sphere in $\mathbb{R}^3$. Let B_1 and B_2 be the 3–disks in $\mathbb{R}^3$ with boundaries $\Delta \cup D_1$ and $\Delta \cup D_2$ respectively. As K' is connected, $K' \subset \operatorname{Int} B_1$ or $K' \subset \operatorname{Int} B_2$, but not both. Without loss of generality, assume K' lies in $\operatorname{Int} B_1$. Using B_2, isotop D_2 past Δ to a parallel copy of Δ. The hyperplane H is fixed in the background during this isotopy of S. The isotoped sphere is again called S. Note that K' remains inside S and C has been eliminated from
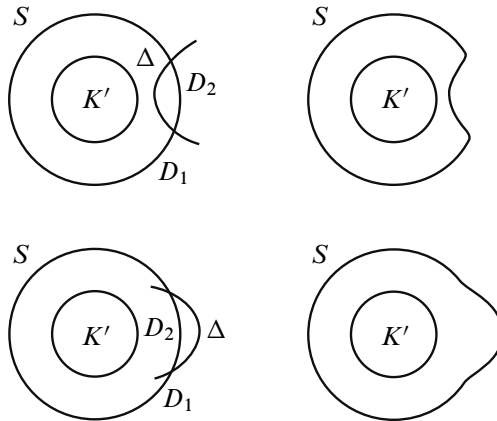
Figure A.3: Two possibilities: Δ inside or outside S (left) and resulting 2–sphere S after isotopy across B_2 (right)

$S \cap H$. Repeat this procedure until all components of $S \cap H$ are essential in $H - \{0\}$. Now, let C be a component of $S \cap H$ that bounds a disk $D \subset S$ disjoint from any other components of $S \cap H$. The disk D is nice. □

Let $\mathbb{R}^3_+$ denote closed upper half-space.

Lemma A.3 *Either $A \approx \mathbb{R}^3_+$ or $B \approx \mathbb{R}^3_+$.*

Proof Use Lemma A.2 to obtain a proper, disjoint collection D_k, $k \in \mathbb{Z}^+$, of nice 2–disks. As each D_k lies in A or in B, we may assume, without loss of generality, that infinitely many D_k lie in A.
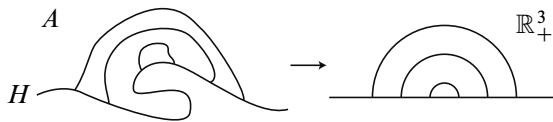

Figure A.4: Diffeomorphism $A \to \mathbb{R}^3_+$

To build a diffeomorphism $A \to \mathbb{R}^3_+$, proceed as indicated in Figure A.4 using repeatedly these tools: (i) the 2– and 3–dimensional smooth Schoenflies theorems (see Hatcher [11, Theorem 1.1] and Cerf [3, Chapter III]) and (ii) the fact that every diffeomorphism of S^2 extends to the 3–disk (see Munkres [16] and Smale [20], and also Thurston [21, Theorem 3.10.11]). □

Lemma A.4 *Let $r \subset H$ be a ray. Then A or B is ambient isotopic to a smooth regular neighborhood of r in $\mathbb{R}^3$. In particular, the ambient isotopy class of H in $\mathbb{R}^3$ is determined by the ambient isotopy class of r in $\mathbb{R}^3$ and conversely.*

Proof By Lemma A.3, either A or B is diffeomorphic to $\mathbb{R}^3_+$. Without loss of generality, let $g\colon A \to \mathbb{R}^3_+$ be a diffeomorphism. Let $s \subset \operatorname{Int}\mathbb{R}^3_+$ be any straight ray, so $\mathbb{R}^3_+$ is a smooth regular neighborhood of s in $\mathbb{R}^3$. As $g(r) \subset \mathbb{R}^2 \times \{0\}$ is necessarily unknotted (see Calcut, King and Siebenmann [2, page 1845]), s is ambient isotopic to $g(r)$ in $\mathbb{R}^3$. It follows that A is a smooth regular neighborhood of $g^{-1}(s)$ in $\mathbb{R}^3$ and r is ambient isotopic to $g^{-1}(s)$ in $\mathbb{R}^3$. This proves the first claim in the lemma. The second claim now follows by uniqueness of regular neighborhoods [2, Section 3]. $\square$

Proposition A.5 *Let $r \subset H$ be any ray. The following are equivalent:*

(1) H *is unknotted in* $\mathbb{R}^3$.

(2) r *is unknotted in* $\mathbb{R}^3$.

(3) $A \approx \mathbb{R}^3_+$ *and* $B \approx \mathbb{R}^3_+$.

Proof (1) $\Leftrightarrow$ (2) by Lemma A.4. (1) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (1), the hypotheses give orientation preserving diffeomorphisms $g\colon A \to \mathbb{R}^3_+$ and $h\colon B \to \mathbb{R}^3_-$. Identify $\mathbb{R}^2 \times \{0\}$ with $\mathbb{R}^2$. Define the diffeomorphism

$$\psi := g \circ h^{-1}|\colon \mathbb{R}^2 \to \mathbb{R}^2.$$

So the diffeomorphism

$$k := (\psi \times \operatorname{id}) \circ h\colon B \to \mathbb{R}^3_-$$

satisfies $k|H = g|H$. Thus we have a homeomorphism $\mu\colon \mathbb{R}^3 \to \mathbb{R}^3$, where $\mu|A = g$ and $\mu|B = k$ are diffeomorphisms. By standard collaring results in Hirsch [13, page 184], we may assume μ is a diffeomorphism and $\mu(H) = \mathbb{R}^2$. The result follows since μ is isotopic to the identity by Milnor [15, page 34]. $\square$

Example A.6 Let $s \subset \mathbb{R}^3$ be a straight ray and let $r \subset \mathbb{R}^3$ be a knotted ray (see Fox and Artin [5, page 983]). Clearly $(\mathbb{R}^3, s) \natural (\mathbb{R}^3, s) \approx \mathbb{R}^3$. Consider

$$M := (\mathbb{R}^3, r) \natural (\mathbb{R}^3, r).$$

As r is knotted, $\mathbb{R}^3 - \operatorname{Int}\nu r \not\approx \mathbb{R}^3_+$ by Proposition A.5. Let $H \subset M$ be the hyperplane determined by $\partial \nu r$. By Lemma A.3, $M \not\approx \mathbb{R}^3$.

Remarks A.7 (1) Lemma A.3 was proved by Harrold and Moise [10] in the piecewise linear category (see also Sikkema [19]). This lemma can also be deduced in the topological (locally flat) category as follows. Consider the 2–sphere $H \cup \{\infty\}$ with at most one singular point embedded in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. If neither side of the 2–sphere is a 3–ball, then their union cannot be S^3. This can be deduced from Eaton's Mismatch Theorem [4] and a result of Bing on tame surfaces [1].

(2)   In the piecewise linear category, Myers [17] showed that the CSI of $\mathbb{R}^3$ with itself yields uncountably many distinct 3–manifolds.

(3)   While $\mathbb{R}^3 - \operatorname{Int} \nu r \not\approx \mathbb{R}^3_+$ in Example A.6, its interior *is* diffeomorphic to $\mathbb{R}^3$. In particular, M is contractible. More generally, if $L \subset \mathbb{R}^3$ is a smooth proper multiray with at most countably many components, then $\mathbb{R}^3 - L \approx \mathbb{R}^3$. To see this, it suffices to prove that each compact $K \subset \mathbb{R}^3 - L$ is contained in a ball. So, let $K \subset \mathbb{R}^3 - L$ be compact. Let $B \subset \mathbb{R}^3$ be a ball containing K. Let $F \colon \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ be an ambient isotopy such that:

  (i) $F_0 = \mathrm{id}$.

  (ii) $F_t(L) \subset L$ for each $t \in [0, 1]$.

  (iii) $F_t | K = \mathrm{id}$ for each $t \in [0, 1]$.

  (iv) $F_1(L)$ is disjoint from B.

Such an F is obtained by integrating a suitable vector field tangent to L and vanishing on K. The required ball is $F_1^{-1}(B)$.

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