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Index theory of the de Rham complex on manifolds with periodic ends

TOMASZ MROWKA
DANIEL RUBERMAN
NIKOLAI SAVELIEV

We study the de Rham complex on a smooth manifold with a periodic end modeled on an infinite cyclic cover $\tilde{X} \to X$. The completion of this complex in exponentially weighted L^2 norms is Fredholm for all but finitely many exceptional weights determined by the eigenvalues of the covering translation map $H_*(\tilde{X}) \to H_*(\tilde{X})$. We calculate the index of this weighted de Rham complex for all weights away from the exceptional ones.

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1 Introduction

Let M be a smooth closed orientable manifold of dimension n. The de Rham complex of complex-valued differential forms on M,

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0,$$

is known to be Fredholm in a suitable L^2 completion. This means as usual that the images of d_k are closed and the vector spaces $\ker d_k / \operatorname{im} d_{k-1}$ are finite-dimensional. The alternating sum of the dimensions of these spaces is called the index of the de Rham complex. Since $\ker d_k / \operatorname{im} d_{k-1}$ is isomorphic to the singular cohomology $H^k(M;\mathbb{C})$ by the de Rham theorem, the above index equals $\chi(M)$, the Euler characteristic of M.

This paper extends these classical results to certain noncompact manifolds; those with periodic ends. It builds on the earlier work of Miller [10] and Taubes [16] and can be viewed as a continuation of our research in [12] and [13] on the index theory of elliptic operators on such manifolds.

By a manifold with a periodic end we mean an open Riemannian manifold M whose end, in the sense of Hughes and Ranicki [7], is modeled on an infinite cyclic cover $\tilde{X}$ of a compact manifold X associated with a primitive cohomology class $\gamma \in H^1(X;\mathbb{Z})$;

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the case of several ends can be treated similarly. To be precise, the manifold M is of the form

$$Z_\infty = Z \cup W_0 \cup W_1 \cup W_2 \cup \cdots ,$$

where W_k are isometric copies of the fundamental segment W obtained by cutting X open along an oriented connected submanifold Y Poincaré dual to γ, and Z is a smooth compact manifold with boundary Y.

The de Rham complex of M can be completed in the L^2 norm using (over the end) a Riemannian measure dx lifted from that on X. This completion is, however, not Fredholm; see Remark 2.3. To rectify this problem, we will use L^2_δ norms, which are the L^2 norms on M with respect to the measure $e^{\delta f(x)}\,dx$ over the end. Here δ is a real number and $f\colon \tilde{X} \to \mathbb{R}$ is a smooth function such that $f(\tau(x)) = f(x) + 1$ with respect to the covering translation $\tau\colon \tilde{X} \to \tilde{X}$. The L^2_δ completion of the de Rham complex on M will be denoted by $\Omega^*_\delta(M)$.

Theorem 1.1 *Let M be a smooth Riemannian manifold with a periodic end modeled on $\tilde{X}$, and suppose that $H_*(M;\mathbb{C})$ is finite-dimensional. Then $\Omega^*_\delta(M)$ is Fredholm for all but finitely many δ of the form $\delta = \ln|\lambda|$, where λ is a root of the characteristic polynomial of $\tau_*\colon H_*(\tilde{X};\mathbb{C}) \to H_*(\tilde{X};\mathbb{C})$.*

Conditions on X that guarantee that $H_*(M;\mathbb{C})$ is finite-dimensional can be found in Section 2, together with a proof of Theorem 1.1.

Given a manifold M as in the above theorem, the complex $\Omega^*_\delta(M)$ has a well-defined index $\operatorname{ind}_\delta(M)$. Miller [10] showed that $\operatorname{ind}_\delta(M)$ is an even or odd function of δ according to whether $\dim M = n$ is even or odd, and that $\operatorname{ind}_\delta(M) = (-1)^n \chi(M)$ for sufficiently large $\delta > 0$. We add to this knowledge the following result.

Theorem 1.2 *Let M be as in Theorem 1.1. Then $\operatorname{ind}_\delta(M)$ is a piecewise constant function of δ whose only jumps occur at $\delta = \ln|\lambda|$, where λ is a root of the characteristic polynomial $A_k(t)$ of $\tau_*\colon H_k(\tilde{X};\mathbb{C}) \to H_k(\tilde{X};\mathbb{C})$ for some $k = 0,\dots,n-1$. Every such λ contributes $(-1)^{k+1}$ times its multiplicity as a root of $A_k(t)$ to the jump.*

Together with the results of [10] this completes the calculation of the function $\operatorname{ind}_\delta(M)$. Theorem 1.2 is proved in Section 3. The last section of the paper contains discussion as well as calculations of $\operatorname{ind}_\delta(M)$ for two important classes of examples. The first class consists of manifolds with infinite cylindrical ends studied earlier by Atiyah, Patodi and Singer [1], and the second of manifolds arising in the study of knotted 2–spheres in S^4.

Finally, we will remark that the de Rham complex is a special case of the more general concept of an elliptic complex. The index theory for elliptic complexes on closed manifolds was developed by Atiyah and Singer, whose famous index theorem [2] expresses the index of such a complex in purely topological terms. More generally, Atiyah, Patodi and Singer [1] computed the index for certain elliptic operators (that is, elliptic complexes of length two) on manifolds with cylindrical ends. In our paper [12] we extended their result to general manifolds with periodic ends; our index formula involves a new periodic η–invariant, generalizing the η–invariant of Atiyah, Patodi and Singer from the cylindrical setting.

It would be interesting to compare the above formula for $\operatorname{ind}_\delta(M)$ with the formula in [12] for the index of the operator $d + d^*$ obtained by wrapping up the de Rham complex. The issue is that both the de Rham complex and the operator $d + d^*$ are Fredholm only when completed in L^2_δ norms with $\delta \neq 0$. Since the dual space of $L^2_\delta(M)$ equals $L^2_{-\delta}(M)$ rather than $L^2_\delta(M)$, this completion does not commute with the wrap-up procedure, hence the index of the operator $d + d^*$ completed in L^2_δ norms need not match the index $\operatorname{ind}_\delta(M)$ of the current paper. Atiyah, Patodi and Singer dealt with a similar issue in [1, Proposition 4.9] using a rather precise relation between L^2 harmonic forms and cohomology on manifolds with cylindrical ends; comparing the above indices would likely entail developing such a relation for general manifolds with periodic ends.

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2 The Fredholm property

In this section, we will prove Theorem 1.1 by reducing it to a statement about twisted cohomology of X.

2.1 The Fourier–Laplace transform

The de Rham complex of M is an elliptic complex on the end-periodic manifold M, hence we can use the general theory of such complexes due to Taubes [16]. According to that theory, it is sufficient to check the Fredholm property of the de Rham complex of $\widetilde{X}$ completed in the L^2 norm on $\widetilde{X}$ with respect to the measure $e^{\delta f(x)}\,dx$. The latter complex can be studied using the Fourier–Laplace transform, defined by

$$\hat{\omega}_z = \sum z^k \cdot (\tau^*)^k \omega, \quad z \in \mathbb{C}^*,$$

on compactly supported forms ω on $\widetilde{X}$, and extended by continuity to L^2 forms. The summation in the above formula extends over all integers k, which makes the form $\widehat{\omega}_z$ invariant with respect to τ^*. The form $\widehat{\omega}_z$ then defines a form on X which is denoted by the same symbol. An application of the Fourier–Laplace transform to the de Rham complex on $\widetilde{X}$ results in a family of twisted de Rham complexes

$$\cdots \longrightarrow \Omega^k(X) \xrightarrow{d-\ln z\, df} \Omega^{k+1}(X) \longrightarrow \cdots \tag{1}$$

parameterized by $z \in \mathbb{C}^*$. The following result is proved in Taubes [16, Lemma 4.3].

Proposition 2.1 *Let M be a smooth Riemannian manifold with a periodic end modeled on $\widetilde{X}$. For any given δ, the complex $\Omega^*_\delta(M)$ is Fredholm if and only if the complexes* (1) *are exact for all z such that $|z| = e^\delta$.*

The cohomology of complex (1) is of course the twisted de Rham cohomology $H^*_z(X;\mathbb{C})$ with coefficients in the complex line bundle with flat connection $-\ln z\, df$.

Proposition 2.2 *Let M be a smooth Riemannian manifold with a periodic end modeled on $\widetilde{X}$. Then the following three conditions are equivalent:*

(i) $\Omega^*_\delta(M)$ *is Fredholm for all $\delta \in \mathbb{R}$ away from a discrete set.*

(ii) $H^*_z(X;\mathbb{C})$ *vanishes for all $z \in \mathbb{C}^*$ away from a discrete set.*

(iii) $H^*_z(X;\mathbb{C})$ *vanishes for at least one $z \in \mathbb{C}^*$.*

Proof Observe that the complexes (1) form a holomorphic family of elliptic complexes on $\mathbb{C}^*$; therefore, exactness of (1) at one point is equivalent to exactness away from a discrete set by the analytic Fredholm theorem. The rest of the statement follows from the preceding discussion. □

Remark 2.3 The usual L^2 completion of the de Rham complex on M, that is, the complex $\Omega^*_0(M)$ is not Fredholm because $H^0_z(X;\mathbb{C})$ is not zero when $z = 1$.

2.2 Finite-dimensionality

Fix a finite cell complex structure on X, lift it to $\widetilde{X}$, and consider the chain complex $C_*(\widetilde{X},\mathbb{C})$ and its homology $H_*(\widetilde{X};\mathbb{C})$. The group of integers acts on both by covering translations making them into finitely generated modules over $\mathbb{C}[t,t^{-1}]$. The twisted de Rham theorem tells us that the cohomology $H^*_z(X;\mathbb{C})$ of complex (1) is isomorphic to the cohomology of the complex $\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(C_*(\widetilde{X},\mathbb{C}),\mathbb{C}_z)$, where $\mathbb{C}_z$ denotes a copy of $\mathbb{C}$ viewed as a $\mathbb{C}[t,t^{-1}]$ module with $p(t)$ acting via multiplication by $p(z)$.

Proposition 2.4 *Let M be a smooth Riemannian manifold with a periodic end modeled on $\widetilde{X}$. Then the following two conditions are equivalent:*

(i) *$H_*(M;\mathbb{C})$ is a finite-dimensional vector space.*

(ii) *$H^*_z(X;\mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^*$.*

Proof It is immediate from the Mayer–Vietoris principle that $H^*(M;\mathbb{C})$ is finite-dimensional if and only if $H^*(\widetilde{X};\mathbb{C})$ is finite-dimensional. Since $H_*(\widetilde{X};\mathbb{C})$ is a finitely generated module over the principal ideal domain $\mathbb{C}[t,t^{-1}]$, it admits a primary decomposition

$$H_*(\widetilde{X};\mathbb{C}) = \mathbb{C}[t,t^{-1}]^\ell \oplus \mathbb{C}[t,t^{-1}]/(p_1) \oplus \cdots \oplus \mathbb{C}[t,t^{-1}]/(p_m); \tag{2}$$

therefore, $H^*(\widetilde{X};\mathbb{C})$ is a finite-dimensional vector space if and only if $\ell = 0$ in this decomposition. According to the universal coefficient theorem,

$$H^*_z(X;\mathbb{C}) = \operatorname{Hom}_{\mathbb{C}[t,t^{-1}]}(H_*(\widetilde{X};\mathbb{C}),\mathbb{C}_z) \oplus \operatorname{Ext}_{\mathbb{C}[t,t^{-1}]}(H_*(\widetilde{X};\mathbb{C}),\mathbb{C}_z),$$

hence vanishing of $H^*_z(X;\mathbb{C})$ for at least one z implies that $\ell = 0$. On the other hand, an easy calculation shows that

$$\operatorname{Hom}_{\mathbb{C}[t,t^{-1}]}(V,\mathbb{C}_z) = \operatorname{Ext}_{\mathbb{C}[t,t^{-1}]}(V,\mathbb{C}_z) = 0$$

for any module $V = \mathbb{C}[t,t^{-1}]/(p)$ such that $p(z) \neq 0$. Therefore, $\ell = 0$ implies that $H^*_z(X;\mathbb{C})$ must vanish for all z away from the roots of the polynomials $p_1,\dots,p_m$. □

2.3 Proof of Theorem 1.1

It follows from Propositions 2.2 and 2.4 that if $H^*(\widetilde{X};\mathbb{C})$ is finite-dimensional, the complex $\Omega^*_\delta(M)$ is Fredholm for all δ away from a discrete set. To finish the proof of Theorem 1.1 we just need to identify this discrete set. According to Proposition 2.1, it consists of $\delta = \ln|z|$, where $z \in \mathbb{C}^*$ are the complex numbers for which $H^*_z(X;\mathbb{C})$ fails to be zero. To find them, note that the free part in the prime decomposition (2) vanishes, making $H_*(\widetilde{X};\mathbb{C})$ into a torsion module,

$$H_*(\widetilde{X};\mathbb{C}) = \mathbb{C}[t,t^{-1}]/(p_1) \oplus \cdots \oplus \mathbb{C}[t,t^{-1}]/(p_m). \tag{3}$$

According to Milnor [11, Assertion 4], the order ideal $(p_1 \cdots p_m)$ of this module is spanned by the characteristic polynomial of $\tau_*\colon H_*(\widetilde{X};\mathbb{C}) \to H_*(\widetilde{X};\mathbb{C})$. The calculation with the universal coefficient theorem as in the proof of Proposition 2.4 now completes the proof. □

2.4 A sufficient condition

Let M be a smooth orientable manifold with a periodic end modeled on $\widetilde{X}$. Vanishing of $\chi(X)$ is obviously a necessary condition for the vector space $H_*(M;\mathbb{C})$ to be finite-dimensional. To come up with a sufficient condition, observe that the derivative df defines a closed 1–form on X, and let $\xi = [df] \in H^1(X;\mathbb{C})$ be its cohomology class. The cup product with ξ gives rise to the chain complex

$$H^0(X;\mathbb{C}) \xrightarrow{\cup\xi} H^1(X;\mathbb{C}) \xrightarrow{\cup\xi} \cdots \xrightarrow{\cup\xi} H^n(X;\mathbb{C}). \tag{4}$$

Proposition 2.5 *Suppose the chain complex* (4) *is exact. Then $H_*(M;\mathbb{C})$ is a finite-dimensional vector space for any smooth orientable manifold with periodic end modeled on $\widetilde{X}$.*

This proposition can be derived as a special case of Taubes [16, Theorem 3.1]. That proof is analytic in nature; here is another proof which is purely topological.

According to Proposition 2.4, it is sufficient to prove that the twisted cohomology $H^*_z(X;\mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^*$. We will show that it does so for all $z \neq 1$ in a sufficiently small neighborhood of 1. For such z, there is a spectral sequence (E^*_r, d_r) which starts at $E^*_1 = H^*(X;\mathbb{C})$, converges to $H^*_z(X;\mathbb{C})$ and whose differentials are given by the Massey products with the class ξ; see Farber [5, Section 10.9] and Pajitnov [14]. The convergence of this spectral sequence to zero is therefore a necessary and sufficient condition for the finite-dimensionality of $H_*(M;\mathbb{C})$. The chain complex (4) is the term (E^*_1, d_1) of that spectral sequence, hence its exactness is sufficient for the finite-dimensionality of $H^*(X;\mathbb{C})$.

Note that the vanishing of the Euler characteristic of X is not a sufficient condition for $H_*(M;\mathbb{C})$ to be finite-dimensional. An example is provided by the connected sum of $S^1 \times S^{n-1}$ with any manifold that is not a rational homology sphere but has Euler characteristic 2.

3 The index calculation

Let M be a smooth Riemannian manifold with periodic end modeled on the infinite cyclic cover $\widetilde{X}$, and assume that $H_*(M;\mathbb{C})$ is finite-dimensional. Let $\tau\colon \widetilde{X} \to \widetilde{X}$ be a covering translation, and denote by $A_k(t)$ the characteristic polynomial of $\tau_*\colon H_k(\widetilde{X};\mathbb{C}) \to H_k(\widetilde{X};\mathbb{C})$. (The polynomial $A_1(t)$ is known as the Alexander polynomial of the fundamental group of X.) Denote by Δ the set of all δ of the form $\delta = \ln|\lambda|$, where λ is a root of the product polynomial $A_0(t)\cdots A_{n-1}(t)$.

According to Theorem 1.1, the complex $\Omega^*_\delta(M)$ is Fredholm for all δ away from Δ. Its index $\mathrm{ind}_\delta(M)$ is a piecewise constant function away from Δ, where it may jump. We wish to calculate the size of these jumps.

3.1 Excision principle

Let δ_1 and δ_2 be two weights in $\mathbb{R}-\Delta$, and complete the de Rham complex of $\widetilde{X}$ in the L^2 norm with respect to the measures $e^{\delta_1 f(x)}\,dx$ on the negative end of $\widetilde{X}$ and $e^{\delta_2 f(x)}\,dx$ on the positive end. This complex will be denoted by $\Omega^*_{\delta_1\delta_2}(\widetilde{X})$. This is a Fredholm complex, whose index will be denoted by $\mathrm{ind}_{\delta_1\delta_2}(\widetilde{X})$.

Proposition 3.1 $\quad \mathrm{ind}_{\delta_2}(M)-\mathrm{ind}_{\delta_1}(M)=\mathrm{ind}_{\delta_1\delta_2}(\widetilde{X}).$

Proof Let $c\in\mathbb{R}$ be a regular value of $f\colon \widetilde{X}\to\mathbb{R}$; then $Y=f^{-1}(c)$ is a submanifold of $\widetilde{X}$ separating it as $\widetilde{X}=\widetilde{X}_-\cup\widetilde{X}_+$. Write $M=Z\cup\widetilde{X}_+$ for some smooth compact manifold Z with boundary Y. An application of the excision principle to these two splittings yields

$$\mathrm{ind}_{\delta_1}(M)+\mathrm{ind}_{\delta_1\delta_2}(\widetilde{X})=\mathrm{ind}_{\delta_2}(M)+\mathrm{ind}_{\delta_1\delta_1}(\widetilde{X}).$$

Note that the complex $\Omega^*_{\delta_1\delta_1}(\widetilde{X})$ is exact because the complexes (1) obtained from it by the Fourier–Laplace transform are exact for all z with $|z|=e^{\delta_1}$. Therefore, $\mathrm{ind}_{\delta_1\delta_1}(\widetilde{X})=0$ and the proof is complete. □

3.2 Computing the cohomology of $\Omega^*_{\delta_1\delta_2}(\widetilde{X})$

We will proceed by several reductions, the first being from weighted forms to weighted cellular cochains. To be precise, fix a finite cell complex structure on X and lift it to $\widetilde{X}$. Also, introduce the Hilbert space $\ell^2_{\delta_1\delta_2}$ of the sequences $\{x_k\mid k\in\mathbb{Z}\}$ of complex numbers such that

$$\sum_{k<0} e^{2\delta_1 k}|x_k|^2<\infty \quad\text{and}\quad \sum_{k>0} e^{2\delta_2 k}|x_k|^2<\infty.$$

Theorem 2.17 of Miller [10], which is a weighted version of the L^2 de Rham theorem [4; 9] establishes an isomorphism between the cohomology of $\Omega^*_{\delta_1\delta_2}(\widetilde{X})$ and the cellular cohomology of $\widetilde{X}$ with $\ell^2_{\delta_1\delta_2}$ coefficients. Miller actually uses weighted *simplicial* cohomology, but this is readily seen to be isomorphic to the more standard and convenient cellular version.

Proposition 3.2 *View $\ell^2_{\delta_1\delta_2}$ as a $\mathbb{C}[t,t^{-1}]$ module with t acting as the right shift operator, $t(x_k)=x_{k+1}$. Then for all but finitely many δ_1 and δ_2, the cohomology of $\widetilde{X}$ with $\ell^2_{\delta_1\delta_2}$ coefficients equals the homology of the complex*

$$\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}\big(C_*(\widetilde{X},\mathbb{C}),\ell^2_{\delta_1\delta_2}\big). \tag{5}$$

Proof This will follow as soon as we show that the images of the boundary operators ∂ in complex (5) are closed. These boundary operators

$$\partial\colon \big(\ell^2_{\delta_1\delta_2}\big)^k \to \big(\ell^2_{\delta_1\delta_2}\big)^\ell$$

are matrices whose entries are Laurent polynomials in t. Since $\mathbb{C}[t,t^{-1}]$ is a principal ideal domain, each ∂ will have a diagonal matrix in properly chosen bases. The statement now follows from the fact that the operator $t-\lambda\colon \ell^2_{\delta_1\delta_2}\to\ell^2_{\delta_1\delta_2}$ is Fredholm for all λ with $|\lambda|$ different from e^{δ_1} and e^{δ_2}; see for instance Conway [3, Proposition 27.7 (c)]. □

The universal coefficient theorem now tells us that the $\ell^2_{\delta_1\delta_2}$ cohomology of $\widetilde{X}$ is isomorphic to

$$\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}\big(H_*(\widetilde{X};\mathbb{C}),\ell^2_{\delta_1\delta_2}\big)\oplus \mathrm{Ext}_{\mathbb{C}[t,t^{-1}]}\big(H_*(\widetilde{X};\mathbb{C}),\ell^2_{\delta_1\delta_2}\big). \tag{6}$$

Recall from the proof of Theorem 1.1 that $H_*(\widetilde{X};\mathbb{C})$ is a torsion module (3) whose order ideal is spanned by the characteristic polynomial of $\tau_*\colon H_*(\widetilde{X};\mathbb{C})\to H_*(\widetilde{X};\mathbb{C})$. Therefore, our next step will be to compute (6), one cyclic module at a time.

Lemma 3.3 *Let λ be a complex number such that $|\lambda|$ is different from e^{δ_1} and e^{δ_2}. Then, for any cyclic module $V=\mathbb{C}[t,t^{-1}]/(t-\lambda)^m$, we have*

$$\mathrm{Ext}_{\mathbb{C}[t,t^{-1}]}\big(V,\ell^2_{\delta_1\delta_2}\big)=0.$$

Proof We know from the proof of Proposition 3.2 that the operator $t-\lambda\colon \ell^2_{\delta_1\delta_2}\to\ell^2_{\delta_1\delta_2}$ is Fredholm. In addition, one can easily check that all finite sequences belong to its image. Since such sequences are dense in $\ell^2_{\delta_1\delta_2}$, the operator $t-\lambda$ is surjective, and so are the operators $(t-\lambda)^m$ for all m. The result is now immediate from the definition of Ext. □

Lemma 3.4 *Let λ be a complex number such that $|\lambda|$ is different from e^{δ_1} and e^{δ_2}. Assume that $\delta_2<\delta_1$. Then, for any cyclic module $V=\mathbb{C}[t,t^{-1}]/(t-\lambda)^m$, the dimension of* $\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(V,\ell^2_{\delta_1\delta_2})$ *is m if $e^{\delta_2}<|\lambda|<e^{\delta_1}$ and zero otherwise.*

Proof For such a module, $\mathrm{Hom}_{\mathbb{C}[t,t^{-1}]}(V,\ell^2_{\delta_1\delta_2})$ equals the kernel of the operator $(t-\lambda)^m\colon \ell^2_{\delta_1\delta_2}\to\ell^2_{\delta_1\delta_2}$. And computing this kernel is a straightforward exercise with infinite series. □

3.3 Proof of Theorem 1.2

Let λ be a root of the product polynomial $A_0(t)\cdots A_{n-1}(t)$ of multiplicity $m = m_0+\cdots+m_{n-1}$, where m_k is the multiplicity of λ as a root of $A_k(t)$. Choose generic δ_1 and δ_2 so that $e^{\delta_2} < |\lambda| < e^{\delta_1}$ and there are no other roots of $A_0(t)\cdots A_{n-1}(t)$ whose absolute values fit in this interval. It follows from Proposition 3.1 and the cohomology calculation in the previous section that

$$\mathrm{ind}_{\delta_1}(M) = \mathrm{ind}_{\delta_2}(M) - \sum(-1)^k m_k,$$

which is exactly the formula claimed in Theorem 1.2. □

4 Discussion and examples

Let M be a smooth Riemannian manifold of dimension n with a periodic end modeled on $\widetilde{X}$, and suppose that $H^*(\widetilde{X};\mathbb{C})$ is finite-dimensional. Then, for any $\delta \in \mathbb{R}-\Delta$, the de Rham complex $\Omega^*_\delta(M)$ is Fredholm and its index is

$$\mathrm{ind}_\delta(M) = (-1)^n\chi(M) + \sum(-1)^k\,\#\{\lambda \mid A_k(\lambda)=0,\, |\lambda| > e^\delta\},$$

where the roots λ of $A_k(t)$ are counted with their multiplicities. This formula is obtained by combining Theorem 1.2 with a theorem by Miller [10] according to which $\mathrm{ind}_\delta(M) = (-1)^n\chi(M)$ for sufficiently large $\delta > 0$. In that paper Miller also shows that the function $\mathrm{ind}_\delta(M)$ is even or odd depending on whether n is even or odd. This is consistent with the above formula because of Blanchfield duality, which says that $A_k(\lambda)=0$ if and only if $A_{n-k-1}(1/\lambda)=0$ with matching multiplicities.

Example 4.1 A manifold with product end is a smooth Riemannian manifold whose end is modeled on $\widetilde{X} = \mathbb{R}\times Y$, where Y is a closed Riemannian manifold. The metric on $\mathbb{R}\times Y$ is presumed to be the product metric. The index theory on such manifolds has been studied by Atiyah, Patodi and Singer [1]. The covering translation induces an identity map τ_* on the homology of $\mathbb{R}\times Y$. Since $\lambda = 1$ is the only root of the characteristic polynomial of τ_*, the complex $\Omega^*_\delta(M)$ is Fredholm for all $\delta \neq 0$. Its index $\mathrm{ind}_\delta(M)$ equals $\chi(M)$ if the dimension of M is even, and $\mathrm{sign}(-\delta)\cdot\chi(M)$ if the dimension of M is odd. Note that the same is true for any manifold whose periodic end is modeled on $\widetilde{X}$ such that the characteristic polynomial of $\tau_*\colon H_*(\widetilde{X};\mathbb{C}) \to H_*(\widetilde{X};\mathbb{C})$ only has unitary roots.

Example 4.2 This example originates in Fox's "Quick trip" [6, Example 11]. Fox constructs a 2–knot in the 4–sphere with the property that the infinite cyclic cover of its exterior has first homology isomorphic to the additive group of dyadic rationals. A nice plumbing construction of this knot described in Rolfsen's book [15, Section 7.F] shows that it has a Seifert surface diffeomorphic to $S^1 \times S^2 - D^3$. This is shown in Figure 1, where the red circle indicates a standardly embedded 2–sphere in the 4–sphere.

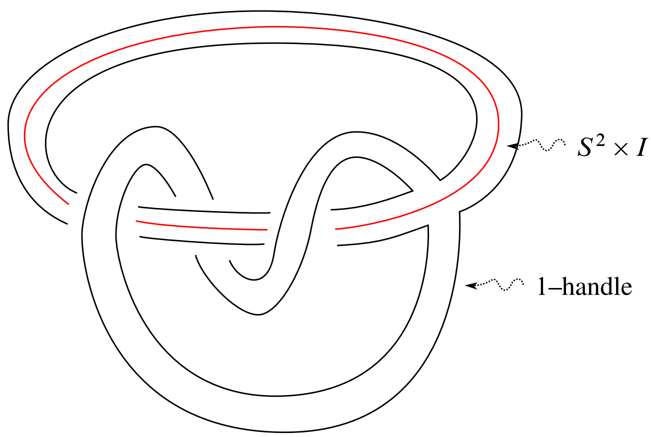

Figure 1: Plumbed 3–manifold bounding a knot

The Seifert surface is obtained from $S^2 \times I$ by adjoining a 3–dimensional 1–handle that links the 2–sphere twice, as indicated. Perform a surgery on the knot so that the Seifert surface is capped off by the core 3–disk of the surgery. The resulting manifold X has the integral homology of $S^1 \times S^3$. It follows from a calculation in [15, Section 7.F] that the characteristic polynomials $A_k(t)$ of the covering translation $\tau_*\colon H_k(\widetilde{X};\mathbb{C}) \to H_k(\widetilde{X};\mathbb{C})$ (which are the same as the Alexander polynomials) are as follows: $A_0(t) = t-1$, $A_1(t) = t-2$, $A_2(t) = t^{-1}-2$ and $A_3(t) = t-1$. Cut $\widetilde{X}$ along a copy of $S^1 \times S^2$ and fill it in by $D^2 \times S^2$ to obtain an end-periodic manifold M. A straightforward calculation shows that $\chi(M) = 2$. The complex $\Omega^*_\delta(M)$ is Fredholm away from $\delta = 0$ and $\delta = \pm\ln 2$. Its index is equal to 1 if $0 < |\delta| < \ln 2$, and is equal to 2 otherwise.

One can construct many more such examples; for instance it is known [8] that any integer polynomial $A(t)$ satisfying $A(1) = \pm 1$ is the first Alexander polynomial of a knot in the 4–sphere, with $A(t^{-1})$ the second polynomial (describing H_2 of the infinite cyclic cover). As in Example 4.2, such knots can be constructed (see [15, Section 7.F, Exercise 6]) as the boundary of a once-punctured connected sum of copies of $S^2 \times S^1$.

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Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA 02139, USA

Department of Mathematics, MS 050, Brandeis University
Waltham, MA 02454, USA

Department of Mathematics, University of Miami
PO Box 249085, Coral Gables, FL 33124, USA

mrowka@mit.edu, ruberman@brandeis.edu, saveliev@math.miami.edu

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