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Euler characteristics of generalized Haken manifolds

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Haken n–manifolds have been defined and studied by B Foozwell and H Rubinstein in analogy with the classical Haken manifolds of dimension 3, based upon the theory of boundary patterns developed by K Johannson. The Euler characteristic of a Haken manifold is analyzed and shown to be equal to the sum of the Charney–Davis invariants of the duals of the boundary complexes of the n–cells at the end of a hierarchy. These dual complexes are shown to be flag complexes. It follows that the Charney–Davis conjecture is equivalent to the Euler characteristic sign conjecture for Haken manifolds. Since the Charney–Davis invariant of a flag simplicial 3–sphere is known to be nonnegative it follows that a closed Haken 4–manifold has nonnegative Euler characteristic. These results hold as well for generalized Haken manifolds whose hierarchies can end with compact contractible manifolds rather than cells.

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1 Introduction

Haken n–manifolds, for $n > 3$, were defined and studied by B Foozwell and H Rubinstein [9; 10; 11; 12] in analogy with the classical Haken manifolds of dimension 3, building on the notion of a boundary pattern, developed in dimension 3 by K Johannson [15; 16]. Foozwell [10; 12] proved that they are aspherical and indeed have universal covering space homeomorphic to euclidean space [9; 10].

These manifolds can be endowed with a hierarchy, that is, a prescription for successively cutting open the manifold until one obtains a disjoint union of n–cells, with a simple regular cell structure on the boundary induced by the cutting submanifolds. In general these Haken cells do not induce a cell complex structure on the original manifold. Nonetheless, we make use of the hierarchy to compute the Euler characteristic of the Haken manifold in terms of the cell structure of the Haken cells at the end of the hierarchy. It turns out that the Euler characteristic is equal to the sum of the Charney–Davis invariants of the simplicial spheres dual to the simple cell structures on the Haken cells.

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A key conceptual observation is that manifolds with boundary patterns may be viewed as right-angled orbifolds with an orbifold Euler characteristic that is invariant under the process of cutting open along a hypersurface.

We also explain how to generalize the notion of Haken manifolds in such a way as to allow arbitrary compact contractible manifolds at the end of a hierarchy, not just cells. Such manifolds are still aspherical but their universal covering need not be euclidean space.

We show that the simplicial spheres dual to the boundary complexes of the associated Haken cells are flag simplicial complexes. Thus the classical Euler characteristic conjecture about even-dimensional closed aspherical manifolds is reduced, for closed generalized Haken manifolds, to the Charney–Davis conjecture for flag generalized simplicial spheres. In particular the Euler characteristic conjecture holds for all closed generalized Haken 4–manifolds. An earlier and more computational proof of the latter result (in the case of ordinary Haken 4–manifolds) appears in Edmonds [8].

Full statements of our results and definitions will be given in subsequent sections.

In Section 2 we analyze the orbifold Euler characteristic that we associate with a manifold with boundary pattern and show that it is invariant under cutting open along a hypersurface. In Section 3 we give a combinatorial interpretation of the notion of a Haken (homotopy) cell, concluding with examples of Haken manifolds arising from CAT(0) cubical manifolds. Finally, in Section 4 we apply the earlier results to the Euler characteristic sign conjecture for even-dimensional aspherical manifolds.

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2 Boundary patterns and orbifolds

We begin with the most basic aspects of Haken n–manifolds, concentrating on manifolds with boundary patterns, deferring the full definitions of Haken cells and Haken manifolds until the next section.

2.1 Boundary patterns

A *boundary pattern* for an n–manifold is a decomposition of the boundary into connected $(n-1)$–manifolds such that the intersection of any k of them is either empty or an $(n-k)$–submanifold. The elements of the boundary pattern are called *facets*. A component of a nonempty intersection of facets is a *stratum*. The relative interior of a stratum is a *pure stratum*. The facets are codimension-one strata. By convention each component of the manifold itself is a codimension-0 stratum.

The boundary pattern is *complete* if the union of its facets is the entire boundary. All boundary patterns considered here will be complete. Notice that each facet inherits an induced boundary pattern. We refer to the entire configuration of facets and their intersections as the *boundary complex*.

The *nerve* of the boundary complex is the abstract simplicial complex L with a vertex for each facet and a $(k-1)$–simplex for each nonempty k–fold intersection. (The empty simplex corresponds to the whole manifold, ie to the codimension-0 stratum.) For each simplex σ of L, let S_σ denote the corresponding union of strata.

2.2 Simple cells and homotopy cells

A *simple n–cell* is a compact n–manifold with boundary pattern such that each stratum is homeomorphic to D^{n-k}, where k is its codimension. If each stratum is only required to be a compact contractible manifold, then we have a (simple) *homotopy n–cell*. If c is a simple n–cell, then the nerve L_c of its boundary complex is called its *dual*. It is a triangulation of S^{n-1}. Moreover, since the simplicial complex dual to the boundary complex of S_σ is $\mathrm{Lk}(\sigma)$ (the link of σ in L), we have that $\mathrm{Lk}(\sigma)$ is homeomorphic to $S^{n-\dim\sigma-1}$. Similarly, if M is a homotopy n–cell, then L is an $(n-1)$–dimensional "generalized homology sphere", abbreviated as GHS^{n-1}. (Recall that a simplicial complex L is a *generalized homology $(n-1)$–sphere* if it is a polyhedral homology manifold with the same homology as S^{n-1}.)

Remark 2.1 If a simplicial complex K is a polyhedral homology n–manifold, then the link of each p–simplex σ is a GHS^{n-p-1}. One does not gain much by requiring K to be a manifold; there is no difference for $n \le 3$, and for $n \ge 4$ the only further requirement is that the link of each vertex be simply connected.

For a polyhedral homology n–manifold K and p–simplex $\sigma \in K$, the *dual cone* $D(\sigma)$ to σ is a certain subcomplex of the barycentric subdivision of K which is isomorphic to the cone on the barycentric subdivision of $\mathrm{Lk}(\sigma)$. So, $D(\sigma)$ is a contractible polyhedral homology manifold with boundary. If $\mathrm{Lk}(\sigma)$ is homeomorphic to S^{n-p-1}, then the dual cone of σ is a simple $(n-p)$–cell.

Proposition 2.2 *Suppose a simplicial complex L is a triangulation of S^{n-1} and that for each simplex $\sigma \in L$, $\mathrm{Lk}(\sigma)$ is homeomorphic to $S^{\mathrm{codim}\,\sigma-1}$. Then the space $\mathrm{Cone}(L)$ naturally has the structure of a simple n–cell.*

Sketch of proof If L' denotes the barycentric subdivision of L, then for each simplex $\sigma \in L$ the dual cone $D(\sigma)$ is homeomorphic to a face of a simple cell structure on $\mathrm{Cone}(L')$. (In particular, each facet is the closed star in L' of a vertex of L.) $\square$

Proposition 2.3 *Suppose a simplicial complex L is a GHS^{n-1}. Then there is a simple homotopy n–cell c such that the nerve of its boundary complex is L. Moreover, c is unique up to a strata-preserving homeomorphism.*

Sketch of proof Using different terminology, the proof of this is explained in Davis [6, Theorem 2.2]. The main ingredient in the proof is the fact that every homology m–sphere bounds a contractible $(m+1)$–manifold; see Kervaire [17] (for $m \neq 3$; for $m = 3$ this uses Freedman and Quinn [13]). Using this fact one constructs a "resolution" of $\mathrm{Cone}(L)$ as in Sullivan [19] (also compare Cohen [4]). The result is a homotopy n–cell c, together with a cell-like map $c \to \mathrm{Cone}(L')$ which takes each face of c to the dual cone of its corresponding simplex. The last sentence (uniqueness) follows from the 3–dimensional Poincaré conjecture and the fact that the topological h–cobordism theorem is true in every dimension. $\square$

2.3 Right-angled orbifolds

An orbifold is *right-angled* if it is locally modeled on the action of $(\mathbb{Z}/2)^n$ on $\mathbb{R}^n$ by reflections across the coordinate hyperplanes. A manifold M with boundary pattern naturally has the structure of a right-angled orbifold $O(M)$. Each pure facet has local group $\mathbb{Z}/2$ and each pure stratum of codimension k has local group $(\mathbb{Z}/2)^k$.

Given M a manifold with boundary pattern, we can calculate the orbifold Euler characteristic $\chi^{\mathrm{orb}}(O(M))$ of the associated orbifold by assigning a weight $(\frac{1}{2})^k$ to each pure stratum of codimension k:

$$\chi^{\mathrm{orb}}(O(M)) = \sum \left(\tfrac{1}{2}\right)^{\operatorname{codim} S} \chi(S, \partial S). \tag{1}$$

Here the sum is over all strata S and, as usual, the relative Euler characteristic is given by $\chi(S, \partial S) = \chi(S) - \chi(\partial S)$. By Poincaré duality, $\chi(S, \partial S) = (-1)^{\dim S} \chi(S)$, so (1) can be rewritten as

$$\chi^{\mathrm{orb}}(O(M)) = (-1)^n \sum \left(-\tfrac{1}{2}\right)^{\operatorname{codim} S} \chi(S). \tag{2}$$

For example if M is an n–cube, $n \geq 1$, with its natural boundary pattern, then $\chi^{\mathrm{orb}}(O(M)) = 0$.

When n is odd (and the boundary pattern is complete), $\chi^{\mathrm{orb}}(O(M)) = 0$.

Proposition 2.4 (Manifold doubles) *Suppose O is a right-angled orbifold with l facets. Then there is a closed manifold $\widehat{O}$ and an action of $(\mathbb{Z}/2)^l$ on $\widehat{O}$ with quotient orbifold O.*

Corollary 2.5 *If O is a right-angled orbifold and $\widehat{O}$ is its manifold double, then*

$$\chi^{\text{orb}}(O) = \frac{\chi(\widehat{O})}{2^l}.$$

Remark 2.6 Given a manifold with boundary pattern M, let $\widehat{M}$ $(= \widehat{O}(M))$ denote the manifold double of $O(M)$. One could then take the formula in Corollary 2.5 as the definition of $\chi^{\text{orb}}(O(M))$. More generally if $(\mathbb{Z}/2)^m$ acts by reflections on a manifold $\widehat{M}$ with orbifold quotient O, then

$$\chi^{\text{orb}}(O) = \frac{\chi(\widehat{M})}{2^m}.$$

Examples 2.7 If M has empty boundary, then $\widehat{M} = M$. If M has nonempty connected boundary consisting of a single facet, then $\widehat{M}$ is the ordinary manifold double consisting of two copies of M glued together along their boundary by the identity map. The manifold double of a closed interval is a circle formed out of four closed intervals suitably identified. If M is a 2–simplex (a triangle), then $\widehat{M}$ is a 2–sphere tessellated by 8 right-angled spherical triangles.

Proof of Proposition 2.4 This is essentially a special case of the "basic construction" of Davis [5, Chapter 5], which we outline in the present context. The facets of O give it a mirror structure. We label the facets F_s, $s \in S$, where S is a set of cardinality l, viewed as the standard set of generators of the elementary abelian 2–group G. For each $x \in O$, set

$$S(x) = \{s \in S : x \in F_s\}.$$

For each nonempty subset $T \subset S$, let G_T denote the subgroup of G generated by the involutions in T.

Define an equivalence relation $\sim$ on $G \times O$ by setting

$$(g, x) \sim (h, y) \quad \text{if and only if} \quad x = y \text{ and } gh^{-1} \in G_{S(x)}$$

and then set

$$\widehat{O} = (G \times O)/\sim.$$

The manifold double $\widehat{O}$ in this case is denoted by $\mathcal{U}(G, O)$ in Davis [5], where this object is studied in much greater generality. That $\widehat{O}$ is connected when O is connected follows from [5, Proposition 5.2.4]. That $\widehat{O}$ is an n–manifold follows from [5, Proposition 10.1.10]. The action of G on $G \times O$ with orbit space O clearly descends to an action of G on $\widehat{O}$ with quotient orbifold. □

2.4 Cutting open along a hypersurface

We consider properly embedded, codimension-one submanifolds ("hypersurfaces") that meet the strata transversely. Such manifolds (or, more generally, maps) are called *admissible*. If we cut open along such an admissible hypersurface, the new manifold receives a boundary pattern in which the normal S^0–bundle over the hypersurface becomes a codimension-one stratum. (If the hypersurface is two-sided, the S^0–bundle is trivial and each component of the hypersurface contributes two facets.) The remaining facets are obtained by cutting open the original facets along the boundary of the hypersurface. If M is the manifold with boundary pattern and F is the hypersurface, then denote by $M \odot F$ the result of cutting M open along F.

Lemma 2.8 *Suppose that $M' = M \odot F$ is the result of cutting M open along a hypersurface. Then*

$$\chi^{\mathrm{orb}}(O(M')) = \chi^{\mathrm{orb}}(O(M)).$$

Proof In the special case where M is closed and F is a closed submanifold, we let F' denote the corresponding S^0–bundle over F. By (1) we have

$$\chi^{\mathrm{orb}}(O(M')) = \chi(M', F') + \tfrac{1}{2}\chi(F') = \chi(M) - \chi(F) + 2 \cdot \tfrac{1}{2}\chi(F) = \chi(M).$$

The general case reduces to this special one by taking a 2^l–fold cover using Corollary 2.5, where l is the number of facets of M. Let $\widehat{M}$ denote the manifold double of $O(M)$. Let $\widehat{F}$ be the preimage of F in $\widehat{M}$. Then $(\mathbb{Z}/2)^l$ acts on the manifold $\widehat{M} \odot \widehat{F}$ with orbifold quotient $O(M')$. Thus

$$\begin{aligned}
\chi^{\mathrm{orb}}(O(M')) &= \frac{1}{2^l}\chi^{\mathrm{orb}}(O(\widehat{M} \odot \widehat{F})) && \text{by Remark 2.6} \\
&= \frac{1}{2^l}\chi(\widehat{M}) && \text{by the special case} \\
&= \frac{1}{2^l} \cdot 2^l \chi^{\mathrm{orb}}(O(M)) && \text{by Corollary 2.5} \\
&= \chi^{\mathrm{orb}}(O(M)).
\end{aligned}$$
□

2.5 Prehierarchies

A *prehierarchy* for a compact n–manifold M with a complete boundary pattern is a sequence of n–manifolds M_k with complete boundary patterns and hypersurfaces F_k,

$$(M_0, F_0), \quad (M_1, F_1), \quad \ldots, \quad (M_m, F_m),$$

where $M_0 = M$, $M_{k+1} = M_k \odot F_k$, and M_{m+1}, the result of cutting M_m open along F_m, is a disjoint union of simple homotopy n–cells.

Theorem 2.9 *Suppose $(M_0, F_0), \ldots, (M_m, F_m)$ is a prehierarchy for $M = M_0$. Then*

$$\chi^{\mathrm{orb}}(O(M)) = \sum_c \chi^{\mathrm{orb}}(O(c)),$$

where the sum is over the homotopy n–cells c in M_{m+1}.

Proof By Lemma 2.8, $\chi^{\mathrm{orb}}(O(M)) = \chi^{\mathrm{orb}}(O(M_{m+1}))$ and $\chi^{\mathrm{orb}}(O(M_{m+1}))$ is additive under disjoint union. □

3 Haken cells and Haken manifolds

Our goal here is to give a combinatorial characterization of a Haken (homotopy) n–cell as having a simplicial flag complex of dimension $n-1$ as its dual nerve. This requires delving somewhat more deeply into some of the intricacies of Haken cells.

3.1 Useful boundary patterns

We need to discuss the somewhat technical notion of a *useful* boundary pattern. A boundary pattern is said to be *useful* if the following hold:

(1) Whenever there is a loop in a single facet that is nullhomotopic in the manifold, then it is nullhomotopic in the facet.

(2) Whenever there is a nullhomotopic loop in the boundary consisting precisely of two arcs, each in distinct facets, then the loop bounds a 2–disk in the boundary meeting the intersection of the two facets in a single arc.

(3) Whenever there is a nullhomotopic loop in the boundary consisting precisely of three arcs, each in distinct facets, then the loop bounds a 2–disk in the boundary meeting the union of the three facets in a single triod.

The slogan here is that "small 2–disks are standard".

Here we mainly need this notion in the case of a simply connected manifold. In this case a boundary pattern is *useful* if and only if:

(1) Each facet is simply connected.

(2) The intersection of any two facets is connected.

(3) If three facets have pairwise nonempty intersections, then all three have nontrivial intersection.

3.2 Essential submanifolds

Let M be an n–manifold with boundary pattern. We consider hypersurfaces $F \subset M$ that meet the facets and their faces transversely.[1] By properly embedded we mean in particular that $F \cap \partial M = \partial F$. If we cut open M along such a submanifold F, the new manifold M' inherits a natural boundary pattern in which the (one or) two components of the boundary of a tubular neighborhood of the hypersurface become facets. The remaining facets are obtained by cutting the original facets open along the boundary of the hypersurface. Note that F inherits a boundary pattern as well.

In order to ensure that we are describing an aspherical manifold, the hypersurfaces along which we cut are required to be *essential*. The detailed properties required for the hypersurface to be *essential* will not concern us much here, but these properties include being injective on fundamental group, and a standard relative version of that condition. In particular, any loop in F that bounds a disk in M also bounds a disk in F. In general this property ensures that the induced boundary pattern on $M \odot F$ is useful. See Edmonds [8], Foozwell [9], or Foozwell and Rubinstein [11; 12] for a more complete discussion.

3.3 Haken cells and Haken homotopy cells

A *Haken homotopy n–cell* is defined inductively to be a topological homotopy n–cell with a complete useful boundary pattern in which the facets are themselves Haken homotopy cells. The definition in Foozwell and Rubinstein [12] of a Haken n–cell is the same except, the word "homotopy" is omitted. The inductive definition starts with 0–cells, which are automatically Haken. Any closed interval with the unique complete boundary pattern is Haken. In dimension 2, a p–sided polygon is a Haken 2–cell if and only if $p \geq 4$. It follows that a 2–dimensional face of a general Haken n–cell is a p–gon, with $p \geq 4$.

[1] Foozwell and Rubinstein only consider two-sided hypersurfaces. We allow hypersurfaces to be one-sided.

3.4 Hierarchies

If M is a manifold with useful boundary pattern and $F \subset M$ is an essential codimension-one submanifold, then we say that (M, F) is a *good pair*.

A *hierarchy* for a compact n–manifold M with a complete useful boundary pattern is a prehierarchy

$$(M_0, F_0), \quad (M_1, F_1), \quad \dots, \quad (M_m, F_m)$$

consisting of good pairs, where each M_k has a complete and useful boundary pattern and where M_{m+1} is a disjoint union of Haken homotopy n–cells. By a *generalized Haken n–manifold* we mean a compact n–manifold with a complete useful boundary pattern, which admits a hierarchy.[2]

Proposition 3.1 *A generalized Haken n–manifold is aspherical.*

Proof The proof is modeled on that of Foozwell and Rubinstein [12, Theorem 3.1], with modifications to allow for one-sided hypersurfaces and for generalized Haken cells.

The proof proceeds by induction on the dimension of the manifold and the number of steps in a hierarchy. The cases when $n = 1$ or 2 follow from the classification of manifolds in these dimensions. In addition, in any dimension a Haken manifold with a hierarchy of length 1 is just a collection of contractible manifolds, hence also aspherical.

Inductively, suppose that Haken manifolds of smaller dimension or shorter hierarchy length than M are aspherical. We may assume that M and its cutting hypersurfaces are connected. If M is cut open along the first hypersurface F, then the result is a manifold with boundary pattern M' which is a Haken manifold with shorter hierarchy. By the induction hypothesis M' is aspherical. If F is two-sided, then the hierarchy for M induces one on F, so induction on dimension also shows that F is aspherical. If F is one-sided, then the same argument shows that a suitable connected 2–fold covering $\widetilde{F}$ of F, given by the boundary of a tubular neighborhood N of F, is aspherical. It follows from covering space theory that F itself is aspherical in this case as well.

The Seifert–van Kampen theorem shows that $\pi_1(M)$ is a free product with amalgamation over $\pi_1(F)$ in the case when F is two-sided and separating, $\pi_1(M)$ is an

[2] Foozwell and Rubinstein include a given hierarchy as part of the structure of a Haken manifold. They also require the essential codimension-one submanifolds F_k to be two-sided. In addition these authors require that the end of the hierarchy consist of Haken n–cells, such that they and their faces are homeomorphic to topological cells.

HNN extension over $\pi_1(\widetilde{F})$ in the case when F is one-sided, and $\pi_1(M)$ is an HNN extension over $\pi_1(F)$ in the case when F is two-sided but nonseparating.

Now $\pi_1(F) \to \pi_1(M)$ is injective in the two-sided case, as in Foozwell and Rubinstein [12, Theorem 3.1]. The same argument shows that in the one-sided case we have $\pi_1(\widetilde{F}) \to \pi_1(M)$ injective.

Thus we see that in the nonseparating cases M can be described as the union of two compact aspherical manifolds N and $\overline{M-N}$ intersecting along an aspherical manifold which is π_1–injective into both N and $\overline{M-N}$. In the separating case we may similarly write $M = M_1 \cup_F M_2$.

A classical theorem of J H C Whitehead then implies that M is aspherical. □

3.5 Characterization of Haken homotopy cells

As we saw in Section 2.2, the boundary complex of a simple n–cell may be viewed as the dual complex of a simplicial $(n-1)$–sphere and that of a simple homotopy n–cell as the (resolved) dual complex of a GHS^{n-1}. We now look more closely at the consequences of the Haken condition.

Proposition 3.2 *If X is a Haken homotopy n–cell, then for each k–face S_σ of X, $k \le n$, the 1–skeleton of its dual simplicial $(k-1)$–sphere contains no empty triangle. (The dual simplicial generalized $(k-1)$–sphere to S_σ is identified with the link of the $(n-k-1)$–simplex σ corresponding to S_σ in the simplicial dual.)*

Proof In the case $\sigma = X$ the assertion is clear from the definition if $n \le 2$. In general it is an interpretation of being a "useful" boundary pattern. Since all faces of a Haken homotopy cell are themselves Haken homotopy cells, the general result follows. □

Recall that a simplicial complex in which any collection of $k+1$ pairwise adjacent vertices spans a k–simplex is called a *flag simplicial complex*. Suggestively we think of a *nonflag complex* as having a minimal *empty simplex* of some dimension k greater than 1, ie a subcomplex equivalent to the boundary of a k–simplex that does not actually span a k–simplex.

Lemma 3.3 *If L is a flag simplicial complex and $\sigma \in L$, then $\mathrm{Lk}(\sigma)$ is flag.*

Proof Let $\eta \subset \mathrm{Lk}(\sigma)$ be a minimal empty simplex. Since L is flag there is a simplex $\tau \in L$ such that $\eta = \partial\tau$. We need to show that $\tau \in \mathrm{Lk}(\sigma)$. Now $\sigma * \partial\tau \cup \tau = \partial\rho$ for some $\rho \in L$ since L is flag. But then $\rho = \sigma * \tau$, implying that $\tau \in \mathrm{Lk}(\sigma)$. □

Lemma 3.4 *A simplicial complex L is flag if and only if for each simplex σ in L (including the empty simplex) its link* $\mathrm{Lk}(\sigma)$ *contains no empty triangle.*

Proof If L is flag and $\sigma \in L$, then $\mathrm{Lk}(\sigma)$ is flag by the preceding result, and hence contains no empty triangle.

For the converse assume that neither L nor any link $\mathrm{Lk}(\sigma)$ contains an empty triangle. We must show that L is flag. To this end we proceed by induction (on dimension, say). Let $v_0, \dots, v_n$ be vertices spanning a minimal nonsimplex. By hypothesis we may assume that $n \geq 3$. Consider $\mathrm{Lk}(v_0)$. Note that $v_1, \dots, v_n \in \mathrm{Lk}(v_0)$. Also all the edges $v_i v_j$ $(1 \leq i, j \leq n)$ lie in $\mathrm{Lk}(v_0)$, since $v_0 v_i v_j$ is a 2–simplex of L by the minimality hypothesis. By hypothesis $\mathrm{Lk}(v_0)$ contains no empty triangles, hence by induction $\mathrm{Lk}(v_0)$ is flag. Thus $v_1 \cdots v_n$ is an n–simplex of $\mathrm{Lk}(v_0)$. But then $v_0 v_1 \cdots v_n$ is a simplex of L, as required. □

Theorem 3.5 *A simple homotopy n–cell is a Haken homotopy n–cell if and only if the dual simplicial* GHS^{n-1} *is flag.*

Proof First suppose M is a Haken homotopy n–cell. We need to argue that the simplicial $(n-1)$–sphere L dual to the boundary complex of M is flag. It is part of the definition of a Haken homotopy n–cell that the simplicial dual of the boundary complex contains no empty triangle in its 1–skeleton. Since all faces of a Haken homotopy cell are themselves Haken homotopy cells there are no empty triangles in $\mathrm{Lk}(\sigma, L)$ for any simplex σ in L. By Lemma 3.4 this implies L is flag, as required.

Second suppose M is a simple, homotopy n–cell with a simple regular homotopy cell complex structure on its boundary, for which the dual simplicial generalized sphere L is flag. We may assume that $n \geq 3$.

The facets of M are simple cells whose boundaries are duals of links of vertices, hence also flag by Lemma 3.3. Therefore by induction the facets are Haken homotopy cells.

It remains to check that the boundary pattern is useful, ie that it satisfies conditions (1)–(3) in Section 3.1. Condition (1) is immediate from the fact that the facets are homotopy cells and condition (2) follows from the fact that the dual complex is simplicial (hence has no digons). Condition (3) is immediate from the fact that the dual complex has no empty triangles. □

3.6 Some examples

We describe a wide class of locally CAT(0) manifolds in all dimensions that are Haken. Related discussion appears in Foozwell and Rubinstein [12, Section 5]. In contrast we point out examples of Haken manifolds that do not support locally CAT(0) metrics. Finally, we indicate some standard examples of closed aspherical manifolds in higher dimensions that are not generalized Haken, even virtually.

3.6.1 Locally CAT(0) manifolds that are Haken We outline a general process for imposing a Haken or generalized Haken structure on a closed manifold M with a locally CAT(0) cubical structure. The process always succeeds when the cubical structure on M arises from the action of a right-angled Coxeter group W associated with the dual Haken homotopy n–cell X (also called a "mirrored space") corresponding to a flag triangulation L of a GHS^{n-1}. As in Davis [5], there is a cubical CAT(0) structure on a manifold $\mathcal{U}(W, X)$ with a free, cocompact action of W. Choosing a normal, torsion-free, finite-index subgroup $\Gamma < W$, one obtains a closed aspherical, locally CAT(0) manifold $M = \mathcal{U}(W, X)/\Gamma$. Such a manifold M can be seen to be Haken, as we now explain in somewhat greater generality.

Suppose M is a closed n–manifold with a locally CAT(0), cubical structure. Since M is a polyhedral homology manifold, the link of each vertex is a GHS^{n-1}, and since M is an actual manifold, the link of each vertex is simply connected (assuming $n \geq 3$). The universal cover $\widetilde{M}$ is a CAT(0) cube complex. The coordinate hyperplanes in each cube extend to "hyperplanes" in the universal cover $\widetilde{M}$. The hyperplanes, and the intersections of hyperplanes, inherit a CAT(0) cubical structure from $\widetilde{M}$. In general, these hyperplanes need only be homology submanifolds of codimension one; however, if the link of each cubical face is a simplicial sphere, then any hyperplane (as well as any intersection of hyperplanes) is an actual locally flat submanifold. The image of a hyperplane in M need not be an embedded homology submanifold (a "hypersurface"); however, in many cases hypersurfaces are embedded. For example, if the cubical structure comes from a cocompact action of a right-angled Coxeter group W on $\widetilde{M}$ and if $\Gamma = \pi_1(M)$ is a normal, torsion-free subgroup of finite index in W, then the hypersurfaces are embedded by a lemma of Millson and Jaffee; see [5, Lemma 14.1.8]. When the hypersurfaces are embedded, they can be used to define a hierarchy for M (in a generalized sense where the hypersurfaces are only required to be homology submanifolds). The "cells" at the end of the hierarchy are stars of vertices in the barycentric subdivision of the cubical complex, ie they are dual cones. (When the links of vertices are simplicial spheres, these dual cones are actual simple cells.) In the general case, one can replace each dual cone by its resolution by a homotopy cell; see Cohen [4] or Sullivan [19]. The result is a manifold which is homeomorphic to M, together with a collection of embedded hypersurfaces which are actual submanifolds. The end of the hierarchy is the collection of homotopy cells obtained by resolving the dual cones.

3.6.2 Manifolds that are Haken but not locally CAT(0) Many examples of Haken manifolds are not related to any locally CAT(0) cubical structure. If $\pi\colon M \to B$ is the projection map of a fiber bundle with fiber Σ and if the base and fiber are both Haken manifolds or generalized Haken manifolds, then so is M. One easily constructs a hierarchy for M from hierarchies for B and Σ. To see this, note that if F is a hypersurface in B, then $\pi^{-1}(F)$ is a hypersurface in M. Hence the inverse image of a hierarchy for B yields the beginning of a hierarchy for M which cuts M into a disjoint union of manifolds of the form $\Sigma \times c$, where c is a homotopy cell from the end of the hierarchy for B. A hierarchy for Σ then gives a hierarchy for M.

If the bundle is not trivial, then even when the base and fiber have locally CAT(0) cubical structures one cannot expect M to have such a structure. For example, if M is an oriented S^1–bundle over B and its Euler class in $H^2(B;\mathbb{Z})$ does not have finite order, then M does not admit a locally CAT(0)–metric; see Bridson and Haefliger [2, Theorem II.6.12] and Frigerio, Lafont and Sisto [14, Lemma 12.1].

Another class of such examples arises from solvmanifolds. Since any solvmanifold can be constructed via an iterated sequence of torus bundles, starting from a torus, solvmanifolds are Haken. However, if the fundamental group of the solvmanifold is not virtually free abelian, then it does not admit a locally CAT(0) metric; see the solvable subgroup theorem [2, Theorem II.7.8].

3.6.3 Non-Haken aspherical manifolds Examples include irreducible, locally symmetric spaces of rank greater than 1. On the one hand, the fundamental group of an irreducible, locally symmetric space of rank greater than 1 has Kazhdan's property T. For a general recent reference see the book of Bekka, de la Harpe and Valette [1]. On the other hand, the fundamental group of a Haken manifold splits as a nontrivial free product with amalgamation or as a nontrivial HNN extension. Such a splitting leads to an action of the group without a fixed point on a tree, which implies that the group does not have property T.

4 The Euler characteristic conjecture

We apply the preceding work to the following fundamental conjecture about aspherical manifolds in the case of generalized Haken n–manifolds.

Euler characteristic sign conjecture 4.1 *If M is a closed, aspherical manifold of even dimension $n = 2m$, then the Euler characteristic of M satisfies $(-1)^m \chi(M) \geq 0$.*

The conjectured sign corresponds to the sign of the Euler characteristic of a product of m surfaces of genus $g \geq 1$. This conjecture was first proposed as a question by W Thurston in the 1970s; see the Kirby problem set [18]. The first interesting and, in general, still unresolved case is in dimension 4.

Recall that we gave a formula (1) for the orbifold Euler characteristic as follows:

$$\chi^{\mathrm{orb}}(O(M)) = \sum \left(\tfrac{1}{2}\right)^{\operatorname{codim} S} \chi(S, \partial S).$$

In terms of the nerve L, the orbifold Euler characteristic of a Haken n–manifold can be rewritten as

$$\chi^{\mathrm{orb}}(O(M)) = (-1)^n \sum_{\sigma} \left(-\tfrac{1}{2}\right)^{\dim \sigma + 1} \chi(S_\sigma),$$

where the sum is over all simplices in L, including the empty simplex. If each stratum has Euler characteristic equal to 1 (eg if M is a Haken homotopy n–cell) and n is even, then this formula reads $\chi^{\mathrm{orb}}(O(M)) = \lambda(L)$, where $\lambda(L)$ is the *Charney–Davis quantity*, defined as

$$\lambda(L) := 1 + \sum_{\sigma \in L} \left(-\tfrac{1}{2}\right)^{\dim \sigma + 1}.$$

So, Theorem 2.9 can be restated in the following form.

Theorem 4.2 *If M is a closed generalized Haken n–manifold, $n = 2m$, then*

$$\chi(M) = \sum_{c} \lambda(L_c),$$

where c ranges over the Haken (homotopy) n–cells at the end of a hierarchy for M and L_c denotes the simplicial nerve associated with c.

By Theorem 3.5 the dual nerve of the boundary complex of a Haken n–cell or Haken homotopy n–cell is a flag complex.

The Charney–Davis conjecture may be stated as follows.

Conjecture 4.3 (Charney and Davis [3]) *Let L be a flag triangulated $(2k-1)$–dimensional sphere (or generalized homology sphere). Then $(-1)^k \lambda(L) \geq 0$.*

An immediate corollary of Theorems 2.9 and 3.5 is the following.

Corollary 4.4 *The Charney–Davis conjecture for generalized homology $(2k-1)$–spheres implies the Euler characteristic sign conjecture for closed generalized Haken manifolds of dimension $2k$.*

The Charney–Davis conjecture is only known in the trivial case $k=1$ and the case $k=2$.

Theorem 4.5 (Davis and Okun [7]) *Let L be a flag triangulated 3–sphere (or homology 3–sphere). Then $\lambda(L) \geq 0$.*

Thus we have proved the following result.

Corollary 4.6 *If M is a closed generalized Haken 4–manifold, then $\chi(M) \geq 0$.*

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