Algebraic & Geometric Topology 14 (2014) 3745–3763

Berge–Gabai knots and L–space satellite operations

JENNIFER HOM
TYE LIDMAN
FARAMARZ VAFAEE

Let $P(K)$ be a satellite knot where the pattern P is a Berge–Gabai knot (ie a knot in the solid torus with a nontrivial solid torus Dehn surgery) and the companion K is a nontrivial knot in S^3. We prove that $P(K)$ is an L–space knot if and only if K is an L–space knot and P is sufficiently positively twisted relative to the genus of K. This generalizes the result for cables due to Hedden [13] and Hom [17].

57M25, 57M27, 57R58

1 Introduction

In [27], Ozsváth and Szabó introduced Heegaard Floer theory, producing a set of invariants of three- and four-dimensional manifolds. One example of such invariants is $\widehat{HF}(Y)$, which associates a graded abelian group to a closed 3–manifold Y. When Y is a rational homology three-sphere, rk $\widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$; see [26]. If equality is realized, then Y is called an *L–space*. Examples include lens spaces and, more generally, all connected sums of manifolds with elliptic geometry [28]. L–spaces are of interest for various reasons; for instance, such manifolds do not admit co-orientable taut foliations [24, Theorem 1.4].

A knot $K \subset S^3$ is called an *L–space knot* if it admits a positive L–space surgery. Any knot with a positive lens space surgery is then an L–space knot. In [3], Berge gave a conjecturally complete list of knots that admit lens space surgeries, which includes all torus knots; see Moser [20]. Therefore it is natural to look beyond Berge's list for L–space knots. In [37], Vafaee classifies the twisted $(p, kp \pm 1)$–torus knots admitting L–space surgeries, some of which are known to live outside of Berge's collection. Another related goal is to classify the satellite operations on knots that produce L–space knots. By combining work of Hedden [13] and Hom [17], the (m,n)–cable of a knot $K \subset S^3$ is an L–space knot if and only if K is an L–space knot and $n/m \geq 2g(K)-1$. (Here, m denotes the longitudinal winding.) We generalize this result by introducing a new L–space satellite operation using Berge–Gabai knots (Gabai [11]) as the pattern.

Published: 15 January 2015 DOI: 10.2140/agt.2014.14.3745

Definition 1.1 A knot $P \subset S^1 \times D^2$ is called a *Berge–Gabai knot* if it admits a nontrivial solid torus filling.[1]

To see that this satellite operation is a generalization of cabling, it should be noted that any torus knot with the obvious solid torus embedding is a Berge–Gabai knot; see Seifert [34]. Note also that any Berge–Gabai knot P which is isotopic to a positive braid, when considered as a knot in S^3, admits a positive lens space surgery; for if performing appropriate surgery on P in one of the solid tori in the genus-one Heegaard splitting of S^3 returns a solid torus, then the corresponding surgery on the knot in S^3 will result in a lens space. For positive braids, this surgery is positive by Lemma 2.1 and [20, Proposition 3.2].

Gabai showed in [10] that any Berge–Gabai knot must be either a torus knot or a 1–bridge braid in $S^1 \times D^2$. More precisely, every Berge–Gabai knot $P \subset V = S^1 \times D^2$ is necessarily of the following form. (For a sufficient condition determining when a knot of this form is a Berge–Gabai knot, see [11, Lemma 3.2].) In the braid group B_w, where w is an integer with $w \geq 2$, let σ_i denote the generator of B_w that performs a positive half-twist on strands i and $i+1$. Let $\sigma = \sigma_b \sigma_{b-1} \cdots \sigma_1$ be a braid in B_w with $0 \leq b \leq w-2$ and let t be a nonzero integer. Place σ in a solid cylinder and glue the ends by a $2\pi t/w$ twist, ie form the closure of the braid word $(\sigma_b \sigma_{b-1} \cdots \sigma_1)(\sigma_{w-1}\sigma_{w-2}\cdots\sigma_1)^t$. We only consider the case where this construction produces a knot, rather than a link. This construction forms a torus knot if $b=0$ and a 1–bridge representation of P in V if $1 \leq b \leq w-2$. We call w the *winding number*, b the *bridge width* and t the *twist number* of P. Note that the twist number can be written as $t = t_0 + qw$ for some integers t_0 and q, where t_0 can be chosen so that $1 \leq t_0 \leq w-1$;[2] see Figure 1(a). Also, note that if $b \neq 0$ then the possibility of $t_0 = w-1$ is disallowed as otherwise we would obtain a link with at least two components [11].

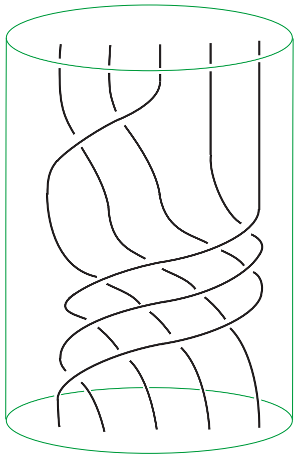
Remark 1.2 Note that if $t<0$, then the braid $\sigma = (\sigma_b \sigma_{b-1} \cdots \sigma_1)(\sigma_{w-1}\sigma_{w-2}\cdots\sigma_1)^t$ is isotopic to a negative braid:

$$\begin{aligned}\sigma &\sim (\sigma_b \sigma_{b-1} \cdots \sigma_1)(\sigma_{w-1}\sigma_{w-2}\cdots\sigma_1)^t \\ &\sim (\sigma_{w-1}\sigma_{w-2}\cdots\sigma_{b+1})^{-1}(\sigma_{w-1}\sigma_{w-2}\cdots\sigma_1)^{t+1}.\end{aligned}$$

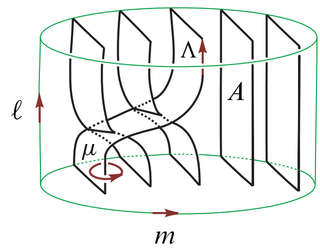
We are now ready to state the main result. Let $P(K)$ denote a satellite knot with pattern P and companion K.

[1] Berge–Gabai knots, in the literature, are defined to be 1–bridge braids in solid tori with nontrivial solid tori fillings. We relax that definition to include torus knots as a proper subfamily.

[2] Our construction of Berge–Gabai knots, which enables us to define them up to isotopy of the knot in $S^1 \times D^2$, is slightly different than that of Gabai [11]. In Gabai's original construction, he always took $q=0$ and considered knots in the solid torus up to homeomorphism of $S^1 \times D^2$ taking one knot to the other.



(a)



(b)

Figure 1: Berge–Gabai knots are knots in $S^1 \times D^2$ with nontrivial solid tori fillings. Such knots are always the closure of the braid $(\sigma_b\sigma_{b-1}\cdots\sigma_1)\times(\sigma_{w-1}\sigma_{w-2}\cdots\sigma_1)^t$, where $0 \le b \le w-2$ and $|t| \ge 1$. (a) An example of a braid in a solid cylinder $I \times D^2$ that closes to form a Berge–Gabai knot with $b=2$, $t=3$ and $w=5$. (The fact that the picture depicted above represents a Berge–Gabai knot is verified in [11, Example 3.8].) Recall that we write $t = t_0 + qw$, where here $t_0 = 3$ and $q = 0$. (b) An immersed annulus A that can be arranged to be an embedded surface in $V = S^1 \times D^2$ joining P to $T = \partial V$ by performing oriented cut and paste and adding a $2\pi t/w$ twist. Note that the embedded surface A provides, in the exterior of P, a homology from $w\ell + tm$ in T to Λ in $J = \partial\,\mathrm{nb}(P)$.

Theorem 1.3 *Let P be a Berge–Gabai knot with bridge width b, twist number t and winding number w, and let K be a nontrivial knot in S^3. Then the satellite $P(K)$ is an L–space knot if and only if K is an L–space knot and $(b+tw)/w^2 \ge 2g(K)-1$.*

Note that when $b=0$, we can take $w=m$ and $t=n$, and Theorem 1.3 reduces to the cabling result of [13; 17]. A version of the "if" direction of Theorem 1.3 appears in Motegi [21, Proposition 7.2].

The outline of the proof of Theorem 1.3 is as follows. By applying techniques developed by Gabai [11] and Gordon [12] to carefully explore the framing change of the solid torus surgered along P, we prove the "if" direction of the theorem. More precisely, surgery on $P(K)$ corresponds to first doing surgery on P (namely removing a neighborhood of P from $S^1 \times D^2$ and Dehn filling along the new toroidal boundary component) and then attaching this to the exterior of K. Therefore, if one chooses the filling on P so that the result is a solid torus (using that P is a Berge–Gabai knot), then the overarching surgery on $P(K)$ corresponds to attaching a solid torus to the exterior of K (performing surgery on K). Moreover, note that by positively twisting P by performing a positive Dehn twist on $S^1 \times D^2$ (ie increasing q), we can obtain an infinite family of Berge–Gabai knots. Fixing an L–space knot K, for sufficiently large q the satellite $P(K)$ will admit a positive L–space surgery. Finally, the "only if" direction is proved by methods similar to those used in [17].

In order to prove Theorem 1.3, we establish the following lemma, which may be of independent interest.

Lemma 1.4 *Let $P \subset S^1 \times D^2$ be a negative braid and $K \subset S^3$ be an arbitrary knot. Then the satellite knot $P(K)$ is never an L–space knot.*

We point out that Lemma 1.4 can be extended more generally to the case where P is a homogeneous braid which is not isotopic to a positive braid; see Stallings [35, Theorem 2]. The proof of Lemma 1.4 was inspired by the arguments of Baker and Moore [2].

We have the following corollary concerning the Ozsváth–Szabó concordance invariant τ and the smooth 4–ball genus.

Corollary 1.5 *Let $P \subset S^1 \times D^2$ be a Berge–Gabai knot and $K \subset S^3$ be an L–space knot. If $(b+tw)/w^2 \geq 2g(K)-1$, then*

$$\tau(P(K)) = \tau(P) + w\tau(K),$$
$$g_4(P(K)) = g_4(P) + wg_4(K),$$

where $\tau(P)$ and $g_4(P)$ denote, respectively, the concordance invariant τ and the 4–ball genus of the knot obtained from the standard embedding of $S^1 \times D^2$ into S^3.

Proof If J is an L–space knot, then $\tau(J) = g_4(J) = g(J)$ by [22, Corollary 1.3] and [28, Corollary 1.6]. Furthermore, by Lemma 2.6,

$$g(P(K)) = g(P) + wg(K).$$

By assumption, K is an L–space knot. The result is clear if K is trivial, so assume that K is nontrivial. Since P is a Berge–Gabai knot with a necessarily positive twist number, it follows that P is isotopic to a positive braid. Therefore, by the discussion following Definition 1.1, P has a positive lens space surgery and thus is an L–space knot. Furthermore, by Theorem 1.3, we also have that $P(K)$ is an L–space knot, and the result follows. □

Theorem 1.3 allows one to construct new examples of L–spaces as follows. First, begin with any L–space knot and then satellite with a Berge–Gabai knot satisfying the conditions in Theorem 1.3. Sufficiently large positive surgery will then result in an L–space. Using this technique, we will construct L–spaces with any number of hyperbolic and Seifert fibered pieces in the JSJ decomposition.

Theorem 1.6 *Let r and s be nonnegative integers such that at least one is nonzero. Then there exist infinitely many irreducible L–spaces whose JSJ decompositions consist of exactly r hyperbolic pieces and s Seifert fibered pieces.*

As discussed, an L–space cannot admit a co-orientable taut foliation. Therefore, Theorem 1.6 will yield irreducible rational homology spheres without co-orientable taut foliations whose JSJ decompositions consist of any number of hyperbolic and Seifert fibered pieces. We remark that all rational homology spheres with Sol geometry are L–spaces; see Boyer, Gordon and Watson [6].

It is also natural to ask in what sense Theorem 1.3 can be generalized; in particular, given a satellite knot which is an L–space knot, what must hold for the pattern or the companion? We propose the following conjecture (see also Baker and Moore [2, Question 22]).

Conjecture 1.7 *If $P(K)$ is an L–space knot, then so are K and P.*

Similarly, we conjecture that the converse holds as well, contingent on the pattern being embedded "nicely" in the solid torus (eg as a strongly quasipositive braid closure) and sufficiently "positively twisted" (akin to the condition in Theorem 1.3). We will not attempt to make these notions precise in this paper.

To obtain supporting evidence for Conjecture 1.7, we will study it from the viewpoint of left-orderability. Recall that a nontrivial group G is *left-orderable* if there exists a left-invariant total order on G (see Section 3 for a more detailed discussion). We recall the conjecture of Boyer, Gordon and Watson relating Heegaard Floer homology to the left-orderability of three-manifold groups.

Conjecture 1.8 (Boyer, Gordon and Watson [6]) *Let Y be an irreducible rational homology sphere. Then Y is an L–space if and only if $\pi_1(Y)$ is not left-orderable.*

We point out that the computational strengths of Heegaard Floer homology and left-orderability tend to be fairly different. One might hope that if Conjecture 1.8 is true then the strengths of each theory could be combined to derive new topological consequences. We utilize this philosophy to establish Conjecture 1.7 under the assumption of Conjecture 1.8.

Proposition 1.9 *Assuming Conjecture 1.8, if $P(K)$ is an L–space knot, then so are P and K.*

Acknowledgements We would like to thank Matthew Hedden for helpful discussions and his interest in our work. We are also grateful to Josh Greene for pointing out Remark 1.2, to Allison Moore, David Shea Vela-Vick and Rachel Roberts for help with the proof of Lemma 1.4, to Ko Honda for a helpful discussion and to Liam Watson for comments on an earlier version of this paper. Hom was partially supported by NSF grant DMS-1307879. Lidman was partially supported by NSF grant DMS-0636643.

2 The main result

In this section, we provide background on 1–bridge braids in solid tori and Dehn surgery on satellite knots; see Berge [4], Gabai [11] and Gordon [12] for further details. Throughout the rest of the paper, we assume that P is a Berge–Gabai knot in $V = S^1 \times D^2$ (ie P admits a nontrivial solid torus surgery) unless otherwise stated. We also consider the standard embedding of $S^1 \times D^2$ into S^3 such that $S^1 \times \{*\}$ bounds an embedded disk in S^3. When it is clear from context, we will not distinguish between the Berge–Gabai knot $P \subset V$ and $P \subset S^3$.

2.1 Berge–Gabai knots

The primary goal of this subsection is to highlight the Dehn surgeries on $P \subset V$ that will return a solid torus. In what follows, we provide a setup similar to that of [11].

An arbitrary knot P in V is called a 1*–bridge braid* if P can be isotoped to be a braid in V that lies in $S^1 \times \partial D^2$ except for one arc that is properly embedded in V, and P is not a torus knot. Gabai [10] showed that any knot in a solid torus with a nontrivial solid torus surgery must be either a torus knot or a 1–bridge braid in $S^1 \times D^2$, and Berge [4] classified all 1–bridge braids in $S^1 \times D^2$ with nontrivial solid tori fillings. We denote the braid index of P by w.

We will consider $\widehat{V}$, the exterior of $P \subset V$. Let $T = \partial V$ and $J = \partial\,\mathrm{nb}(P)$. We equip T with the homological generators (m, ℓ), where ℓ is the longitude $S^1 \times \{*\}$ of T and m is $\{*\} \times \partial D^2$; therefore, ℓ becomes nullhomologous after standardly embedding V in S^3 and removing $\mathrm{nb}(P)$. We equip J with homological generators (μ, Λ) as follows. The generator μ is the meridian of P. Note that m is homologous to $w\mu$ in $\widehat{V}$. To define Λ, consider the immersed annulus A connecting J to T with b arcs of self-intersection in Figure 1(b). By doing oriented cut and paste to the arcs of self-intersection we can arrange A to be an embedded surface in $\widehat{V}$ joining

J to T. Define Λ to be $A \cap J$. Orient m, ℓ, μ and Λ as in Figure 1(b). Note that $A \cap T = w\ell + tm$, and so $w\ell + tm$ is homologous to Λ in $\widehat{V}$.

Let λ be the simple closed curve on J that is homologous to $\Lambda - wt\mu \in H_1(J;\mathbb{Z})$. Thus, we have the following equalities in $H_1(\widehat{V};\mathbb{Z})$:

$$[\lambda] = [\Lambda - wt\mu] = [w\ell + tm - wt\mu] = [w\ell],$$

where the last equality follows from the fact that m is homologous to $w\mu$. In particular, λ becomes nullhomologous after standardly embedding V in S^3 and removing $\mathrm{nb}(P)$. Now the equation $[\lambda] = [\Lambda - wt\mu]$ can be used to switch from (μ, Λ)– to (μ, λ)–coordinates, where (μ, λ) are the usual meridian-longitude coordinates on P when V is standardly embedded in S^3.

We recall that a 1–bridge braid in $S^1 \times D^2$ with winding number w, bridge width b and twist number t can be represented via the braid word

$$\sigma = (\sigma_b \sigma_{b-1} \dots \sigma_1)(\sigma_{w-1}\sigma_{w-2} \dots \sigma_1)^t,$$

where $|t| \geq 1$ and $1 \leq b \leq w-2$. The following lemma is a consequence of [11, Lemma 3.2]:

Lemma 2.1 *Let P be a 1–bridge braid in V and s a positive integer. If filling $\widehat{V}$ along a curve $\alpha = d\mu + s\Lambda$ in J yields $S^1 \times D^2$, then $s = 1$, $d \in \{b, b+1\}$ and* $\gcd(w, d) = 1$.

In (μ, λ)–coordinates these possible exceptional surgeries are $\alpha = (tw + d)\mu + \lambda$, where $d \in \{b, b+1\}$.[3]

Note that when P is an (m, n)–torus knot in V, there are infinitely many surgeries on P that will return a solid torus, including $mn + 1 = tw + b + 1$; this follows, for instance, from the proof of [20, Proposition 3.2].

Let $(P; n_1/n_2)$ denote the result of filling $\widehat{V}$ along the curve $n_1\mu + n_2\lambda$. Lemma 2.1 shows that if P is a Berge–Gabai knot, then $(P; p_d)$ will be homeomorphic to $S^1 \times D^2$ for at least one of the coefficients $p_d = tw + d$, $d \in \{b, b+1\}$.

Note that adding a positive full-twist to all of the w strands of P results in a new knot P' where t changes into $t + w$. Correspondingly, there exists a homeomorphism of the solid torus (doing a positive meridional twist) which takes P to P'. Iterating this process q times, we get the following:

[3] We have stated Lemma 2.1 so that the orientation of (μ, λ) agrees with the standard convention that $\mu \cdot \lambda = 1$. In Gabai's paper [11], μ is oriented opposite to that of Figure 1(b).

Proposition 2.2 *Let P be a Berge–Gabai knot in $S^1 \times D^2$, standardly embedded in S^3, such that $(P; p)$ is homeomorphic to a solid torus. Let P' be the knot obtained from P by adding q positive Dehn twists. Then*

$$(P'; p + qw^2) \cong S^1 \times D^2.$$

Hence if we have a Berge–Gabai knot P with twist number t, adding q full twists to all w strands of P will produce a Berge–Gabai knot with twist number $t + qw$.

2.2 Surgery on $P(K)$

Let $P(K)$ be a satellite knot with pattern $P \subset V$ and companion K. Let $f\colon V \to \mathrm{nb}(K)$ be a homeomorphism that determines the zero framing of K, ie $[f(S^1 \times \{*\})] = 0 \in H_1(X;\mathbb{Z})$, where $X = S^3 - \mathrm{nb}(K)$. Thus $P(K) = f(P)$.

Recall that $m, \ell \in H_1(T;\mathbb{Z})$ are the natural meridian and longitude coordinates of $T = \partial V$, oriented so that $m \cdot \ell = 1$. Recall also that $\widehat{V} = V - \mathrm{nb}(P)$. Note that $H_1(\widehat{V}) = \mathbb{Z}\langle \ell \rangle \oplus \mathbb{Z}\langle \mu \rangle$, where μ is the class of the meridian of $\mathrm{nb}(P)$. When P is viewed as a knot in S^3, let $\lambda \subset \partial\,\mathrm{nb}(P)$ be the unique curve on $\partial\,\mathrm{nb}(P)$ which is nullhomologous in $S^3 - \mathrm{nb}(P)$ (ie the zero framing of P). That is, if f is as above, then $f(\lambda)$ is the zero framing of $P(K)$. So $S^3_{p_1/p_2}(P(K)) \cong X \cup_f (P; p_1/p_2)$, where the notation means ∂X and $\partial(P; p_1/p_2)$ are identified via the restriction of f to $\partial(P; p_1/p_2) = \partial V$. With the above notation, we have the following lemma.

Lemma 2.3 [12, Lemma 3.3] *For relatively prime integers p_1, p_2, and $P \subset V$ with winding number w:*

(a) $H_1((P; p_1/p_2);\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_{\gcd(w,p_1)}$.

(b) *If $w \neq 0$, the kernel of $H_1(\partial(P; p_1/p_2);\mathbb{Z}) \to H_1((P; p_1/p_2);\mathbb{Z})$ is the cyclic group generated by*

$$\frac{p_1}{\gcd(w, p_1)} m + \frac{p_2 w^2}{\gcd(w, p_1)} \ell.$$

Note that Lemma 2.3 is valid regardless of whether or not P is a Berge–Gabai knot. However, when P is a Berge–Gabai knot, we can use Lemma 2.3 to relate surgeries on K and $P(K)$ in the following sense.

Corollary 2.4 *Let P be a Berge–Gabai knot in V with winding number w and such that $(P; p) \cong S^1 \times D^2$. Then*

$$S^3_p(P(K)) \cong S^3_{p/w^2}(K).$$

Proof The result essentially follows from the fact that

$$S^3_p(P(K)) \cong X \cup_f (P; p).$$

By assumption, $(P; p)$ is homeomorphic to a solid torus. Therefore, in order to find the corresponding surgery coefficient on K, one needs to determine the slope of the meridian of $\partial(P; p)$ under the canonical identification with ∂V, and where it is sent under f.

Note that the slope of the meridian of $(P; p)$ is precisely the generator of

$$\ker\big(H_1(\partial(P; p); \mathbb{Z}) \to H_1((P; p); \mathbb{Z})\big).$$

Using the identification of ∂V and $\partial(P; p)$, we have that the slope of the meridian, in (m, ℓ)–coordinates, is given by (p, w^2) by Lemma 2.3. Since f sends m and ℓ to the meridian and longitude of K, respectively, the result follows. □

Combining Lemma 2.1 with Corollary 2.4, we deduce the following:

Proposition 2.5 *Let P be a Berge–Gabai knot with bridge width $b \neq 0$, winding number w and twist number t, and let K be an arbitrary knot in S^3. Then for at least one $d \in \{b, b+1\}$,*

$$S^3_{d+tw}(P(K)) \cong S^3_{(d+tw)/w^2}(K).$$

Note that $\gcd(d+tw, w^2) = 1$ (see Lemma 2.1). We end this subsection by stating the following lemma, which will be useful in the proof of Theorem 1.3. Let $\Delta_K(T)$ denote the symmetrized Alexander polynomial of K. Recall the behavior of the Alexander polynomial for satellites (see for instance [19]):

$$\Delta_{P(K)}(T) = \Delta_P(T)\Delta_K(T^w). \tag{2.2.1}$$

Lemma 2.6 *Let $P(K)$ be a fibered satellite knot where P has winding number w. Then*

$$g(P(K)) = g(P) + wg(K).$$

Furthermore, if P is a Berge–Gabai knot as above with $t > 0$, then

$$g(P) = \frac{(t-1)(w-1)+b}{2}. \tag{2.2.2}$$

Proof Since $P(K)$ is a fibered knot, we deduce that $\deg \Delta_{P(K)}(T) = g(P(K))$. It also follows that K and P are both fibered [16]. Combining these two facts with (2.2.1), we see that $g(P(K)) = g(P) + wg(K)$.

In order to calculate $g(P)$, notice that P is a positive braid if $t > 0$. Hence, the Seifert surface R obtained from Seifert's algorithm is a minimal genus Seifert surface for P [35]. Then

$$\chi(R) = 1 - 2g(P) \Rightarrow w - b - t(w-1) = 1 - 2g(P). \qquad \square$$

2.3 Input from Heegaard Floer theory

In this subsection we mainly use the notation of [17]. Recall that an *L–space* Y is a rational homology sphere with the simplest possible Heegaard Floer homology, ie $\operatorname{rk} \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$. We say that a knot K in S^3 is an *L–space knot* if it admits a positive L–space surgery.

We let $\tau(K)$ denote the integer-valued concordance invariant from [23]. Let $\mathcal{P}$ denote the set of all knots K for which $g(K) = \tau(K)$. (Recall from [14] that for fibered knots, $g(K) = \tau(K)$ is equivalent to being strongly quasipositive.) If K is an L–space knot, then $K \in \mathcal{P}$. This follows from [28, Corollary 1.6] and the fact that L–space knots are fibered [22, Corollary 1.3].

Let

$$s_K = \sum_{i \in \mathbb{Z}} \bigl(\operatorname{rk} H_*(\hat{A}_i^K) - 1\bigr),$$

where $\hat{A}_i^K$ is the subquotient complex of $CFK^\infty(K)$ defined in [29]. It is proved in [17] that $\operatorname{rk} H_*(\hat{A}_i^K)$ is always odd, and so s_K is always a nonnegative even integer. For a pair of relatively prime nonzero integers m and n, $n > 0$, let

$$t_K^{m/n} = 2\max\bigl(0, n(2\nu(K) - 1) - m\bigr). \tag{2.3.1}$$

Observe that

$$t_K^{m/n} = 0 \quad \text{if and only if} \quad m/n \geq 2\nu(K) - 1. \tag{2.3.2}$$

The term $\nu(K)$ is another integer-valued invariant of K, defined in [30, Definition 9.1], which is bounded below by $\tau(K)$ and above by $g(K)$. In particular, if $K \in \mathcal{P}$, then $\nu(K) = g(K)$.

Let m and n be as above, and suppose that $\nu(K) \geq \nu(\overline{K})$, where $\overline{K}$ denotes the mirror of K. (This condition is automatically satisfied for $K \in \mathcal{P}$.) If $\nu(K) > 0$ or $m > 0$, then

$$\operatorname{rk} \widehat{HF}(S^3_{m/n}(K)) = m + n s_K + t_K^{m/n} \tag{2.3.3}$$

by [30, Proposition 9.6].

By (2.3.3), when $m > 0$ we have that

$$S^3_{m/n}(K) \text{ is an L–space} \quad \text{if and only if} \quad t_K^{m/n} = 0 \text{ and } s_K = 0. \tag{2.3.4}$$

By [25, Theorem 4.4], the group $H_*(\hat{A}_i^K)$ is isomorphic to $\widehat{HF}(S^3_N(K), [i])$ for $N \gg 0$ and $|i| \le N/2$. Thus, we have that

$$K \text{ is an L–space knot} \quad \text{if and only if} \quad s_K = 0. \tag{2.3.5}$$

Actually, if K is a nontrivial L–space knot, $S^3_{m/n}(K)$ is an L–space if and only if $m/n \ge 2g(K) - 1$. This follows from (2.3.2), (2.3.3) and the fact that for K a nontrivial L–space knot, $\nu(K) = g(K) > 0$. (The original argument for the forward direction is given in [18].)

2.4 Proof of Theorem 1.3

This subsection is devoted to the proof of Theorem 1.3. We begin with the proof of Lemma 1.4. We do not review the concept of a quasipositive Seifert surface but instead refer the reader to Hedden [14] and Rudolph [33].

Proof of Lemma 1.4 Suppose for contradiction that $P(K)$ is an L–space knot. Recall that L–space knots are fibered [22; 28]. It is also a well-known fact that a minimal-genus Seifert surface for a negative braid can be expressed as a plumbing of negative Hopf bands [35, Theorem 2]. (See also [1, Theorem 1] for an explicit construction in the case of torus knots.) Since $P(K)$ is fibered, this implies that K is fibered and P is fibered in the solid torus [16], so the fiber for $P(K)$ is constructed by patching the fiber for P in the solid torus to w copies of the fiber for K. As a result, when P is a negative braid, the fiber surface for $P(K)$ contains (at least) as many negative Hopf bands as the one for P.

By the above description of the fiber surface, we can deplumb a negative Hopf band. This means we can decompose the fiber surface for $P(K)$ as a Murasugi sum where one of the summands is not a quasipositive surface. By [33], if a Seifert surface is a Murasugi sum, it is quasipositive if and only if all of the summands are quasipositive. Thus, the fiber surface for $P(K)$ is not a quasipositive surface. However, since $P(K)$ is an L–space knot, it is strongly quasipositive [14], which gives a contradiction. □

We prove Theorem 1.3 only for the cases where $b \ne 0$ (consequently $1 \le t_0 \le w - 2$) and refer the reader to Hedden [13] and Hom [17] for the case $b = 0$.

Proof of Theorem 1.3 ($\Leftarrow$) The proof of this direction follows from Proposition 2.5, which tells us that
$$S^3_{d+tw}(P(K)) \cong S^3_{(d+tw)/w^2}(K).$$
Since K is a nontrivial L–space knot and $(b+tw)/w^2 \geq 2g(K)-1 > 0$, it follows that $S^3_{(d+tw)/w^2}(K)$ is an L–space. Here we are using that $d \geq b$. Therefore, $P(K)$ is an L–space knot.

($\Rightarrow$) For the case that $t < 0$ (see Remark 1.2), we apply Lemma 1.4 to see that $P(K)$ cannot be an L–space knot. Therefore, we can assume that $t > 0$ and $P(K)$ is an L–space knot. For simplicity of notation, we set $m = d + t_0 w + qw^2$, where $d \in \{b, b+1\}$ is such that $(P; m) \cong S^1 \times D^2$. Again from Proposition 2.5 we have
$$\operatorname{rk} \widehat{HF}\big(S^3_m(P(K))\big) = \operatorname{rk} \widehat{HF}(S^3_{m/w^2}(K)). \tag{2.4.1}$$
Since $P(K)$ is an L–space knot, it follows that $g(P(K)) = \tau(P(K))$, and we see that
$$t^m_{P(K)} = 2\max\big(0, 2g(P(K)) - 1 - m\big). \tag{2.4.2}$$
We first suppose that $\nu(K) \geq \nu(\overline{K})$. Since $m > 0$, we may combine (2.3.3), (2.3.5) and (2.4.1) to obtain
$$m + t^m_{P(K)} = m + w^2 s_K + t^{m/w^2}_K,$$
or equivalently
$$t^m_{P(K)} = w^2 s_K + t^{m/w^2}_K. \tag{2.4.3}$$
Note that by Lemma 2.6, (2.2.2) and (2.4.2), we have that
$$t^m_{P(K)} = \max(0, 4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d). \tag{2.4.4}$$

Claim *The equality in* (2.4.3) *does not hold unless both sides are identically zero.*

Proof of the claim If $t^m_{P(K)} \neq 0$ then we have two cases:

Case 1 Suppose $t^{m/w^2}_K = 0$. Using (2.4.4), we see (2.4.3) is equivalent to
$$4wg(K) - 2w - 2t_0 - 2qw + 2b - 2d = w^2 s_K.$$
It follows that w divides $2t_0 + 2d - 2b$. Since $d - b \in \{0, 1\}$ and $1 \leq t_0 \leq w-2$, we conclude that $w = 2t_0 + 2d - 2b$. Since
$$4wg(K) - 2w - w - 2qw = w^2 s_K,$$
then
$$4g(K) - 3 - 2q = ws_K.$$

The number on the right side is even and the one on the left side is odd, which is a contradiction.

Case 2 Suppose $t_K^{m/w^2} \neq 0$. By expanding both sides of (2.4.3) and again using (2.4.4), we see that

$$4wg(K)-2w-2t_0-2qw+2b-2d = w^2 s_K + 4w^2\nu(K)-2w^2-2d-2t_0 w-2qw^2.$$

By rearranging terms, we get

$$4wg(K)-2w+2(b-t_0)-2qw+2t_0 w = w^2(4\nu(K)-2-2q+s_K).$$

Therefore w divides $2(b-t_0)$. Since b and t_0 are both bounded above by $w-2$, we have either $2(b-t_0)=\pm w$ or $b=t_0$.

Recall that we described P as a braid closure in Section 1. Viewing this braid as a mapping class of the disk with w punctures, it is straightforward to verify that if $b=t_0$, the $(t_0+1)^{\text{th}}$ puncture is fixed. Therefore, in this case P has at least two components, which contradicts P being a knot. Thus, we must have $2(b-t_0)=\pm w$.

Substituting and dividing by w gives

$$4g(K)-2\pm 1-2q+2t_0 = w(4\nu(K)-2-2q+s_K).$$

As in Case 1, comparing the parities of each side gives a contradiction. $\square$

Having proved the claim, all the terms in (2.4.3) are identically zero. Since $s_K=0$, (2.3.5) gives that K is an L–space knot. Also, $t^m_{P(K)}=0$ together with (2.4.4) implies

$$\frac{t_0+qw+d-b}{w} \geq 2g(K)-1. \tag{2.4.5}$$

Since $1 \leq t_0 \leq w-2$ and $(d-b)\in\{0,1\}$, we have that $0 \leq t_0+d-b < w$. Note that $2g(K)-1$ is an integer, so we deduce that (2.4.5) holds if and only if

$$q \geq 2g(K)-1,$$

which implies that

$$\frac{b+t_0 w+qw^2}{w^2} \geq 2g(K)-1,$$

as desired.

Now suppose that $\nu(K)<\nu(\overline{K})$. We claim that in this case, $P(K)$ is not an L–space knot, which is a contradiction. Recall from [30, Equation (34)] that $\nu(K)$ is equal to either $\tau(K)$ or $\tau(K)+1$, and from [23, Lemma 3.3] that $\tau(\overline{K})=-\tau(K)$. Thus, when $\nu(K) < \nu(\overline{K})$, it follows that $\nu(\overline{K}) > 0$. By [26, Proposition 2.5], the total rank of $\widehat{HF}(Y)$ for a closed three-manifold Y is independent of the orientation of Y, ie

$$\operatorname{rk} \widehat{HF}(Y) = \operatorname{rk} \widehat{HF}(-Y). \tag{2.4.6}$$

By combining (2.4.6), Proposition 2.5 and the fact that

$$S^3_{m/n}(K) \cong -S^3_{-m/n}(\overline{K}), \tag{2.4.7}$$

we deduce that

$$\operatorname{rk} \widehat{HF}\big(S^3_m(P(K))\big) = \operatorname{rk} \widehat{HF}(S^3_{-m/w^2}(\overline{K})). \tag{2.4.8}$$

By combining (2.3.3), (2.3.5) and (2.4.8), since $P(K)$ is an L–space knot, we have

$$m + t^m_{P(K)} = -m + w^2 s_{\overline{K}} + t^{-m/w^2}_{\overline{K}}. \tag{2.4.9}$$

Using (2.3.1) and the fact that $\nu(\overline{K}) > 0$, we observe that $t^{-m/w^2}_{\overline{K}} \neq 0$.

Claim *The equality in* (2.4.9) *never holds.*

Proof of the claim We prove the claim by considering the following two cases:

Case 1 Suppose $t^m_{P(K)} \neq 0$. Using (2.4.4), by expanding both sides of (2.4.9) we get

$$\begin{aligned} &d + t_0 w + q w^2 + 4 w g(K) - 2w - 2t_0 - 2qw + 2b - 2d \\ &\qquad = -d - t_0 w - q w^2 + w^2 s_{\overline{K}} + 4 w^2 \nu(\overline{K}) - 2w^2 + 2d + 2t_0 w + 2q w^2. \end{aligned}$$

A similar reasoning as in Case 1 of the previous claim shows that this equality gives a contradiction.

Case 2 Suppose $t^m_{P(K)} = 0$. Using (2.4.4), we see that (2.4.9) is equivalent to

$$d + t_0 w + q w^2 = -d - t_0 w - q w^2 + w^2 s_{\overline{K}} + 4 w^2 \nu(\overline{K}) - 2w^2 + 2d + 2t_0 w + 2q w^2.$$

This equation reduces to $2w^2 = w^2 s_{\overline{K}} + 4 w^2 \nu(\overline{K})$. However, this equation has no solutions, since $\nu(\overline{K}) > 0$ and $s_{\overline{K}} \geq 0$. □

Having proved the claim, it follows that if $\nu(K) < \nu(\overline{K})$, then $P(K)$ could not have been an L–space knot. This completes the proof. □

3 Proofs of Theorem 1.6 and Proposition 1.9

Before proving Theorem 1.6 and Proposition 1.9 we remind the reader of a standard fact about geometric structures and Dehn surgery which we will make use of repeatedly without reference; see [15, Proposition 5] and [36, Section 5]. Suppose that M is a compact, orientable, irreducible manifold with incompressible torus boundary (eg the exterior of a nontrivial knot in S^3). Then all but finitely many Dehn fillings of M are irreducible and have the same number of hyperbolic and Seifert fibered pieces in their JSJ decompositions as M.

3.1 JSJ decompositions and L–spaces

Proof of Theorem 1.6 In order to construct the family of manifolds described in the statement of the theorem, we will first construct an L–space satellite knot $K_{s,r}$ with s Seifert fibered pieces and r hyperbolic pieces in the JSJ decomposition. The knot $K_{s,r}$ will be constructed by a sequence of satellite operations using cables and Berge–Gabai knots. As discussed, all but finitely many surgeries on $K_{s,r}$ will then be irreducible rational homology spheres with the desired JSJ decomposition. Since all surgeries with slope at least $2g(K_{s,r})-1$ will result in L–spaces (see Section 2.3), sufficiently large surgeries on $K_{s,r}$ will produce the desired infinite family.

Recall that if P is a torus knot standardly embedded in the solid torus, then the exterior of P is Seifert fibered over the annulus with a single cone point. We first construct a knot K_s as an s–fold iterated torus knot with appropriately chosen cabling parameters. More specifically, we construct K_s as follows. If s is 0, we simply take K_s to be the unknot. Otherwise, we begin with K_1, the positive (m_1,n_1)–torus knot, for some $m_1,n_1 \geq 2$. Perform the (m_2,n_2)–cable, choosing $n_2/m_2 \geq 2g(K_1)-1$, to obtain the knot K_2. Inductively, we construct K_i to be the (m_i,n_i)–cable of K_{i-1}, where we choose $n_i/m_i \geq 2g(K_{i-1})-1$. The JSJ decomposition of the exterior of K_s now consists of s Seifert pieces. Further, by [13], K_s is an L–space knot.

Let P_1 be a positively twisted hyperbolic Berge–Gabai knot satisfying the condition $(b+t_0w+qw^2)/w^2 \geq 2g(K_s)-1$. We can construct P_1 as follows. Begin with any hyperbolic Berge–Gabai knot (ie hyperbolic in $S^1 \times D^2$; see [4, Theorem 3.2 and page 17] to obtain explicit examples). Now add sufficiently many positive twists until the desired inequality is satisfied (fix b, t_0 and w, and increase q) to obtain P_1. As discussed in Section 2.1, adding positive twists preserves the property of being a Berge–Gabai knot; furthermore, this does not change the type of geometry on the knot exterior, and thus P_1 will still be hyperbolic. If $s \neq 0$, we define $K_{s,1}$ as the satellite knot with companion K_s and pattern P_1. By Theorem 1.3, $K_{s,1}$ is an L–space knot. If $s=0$, take $K_{s,1}$ to be any hyperbolic L–space knot, such as the $(-2,3,7)$–pretzel knot [9]. We now repeat this process r times, ie to obtain $K_{s,i}$, satellite $K_{s,i-1}$ with pattern a hyperbolic Berge–Gabai knot satisfying $(b+t_0w+qw^2)/w^2 \geq 2g(K_{s,i-1})-1$. The process terminates at the knot $K_{s,r}$ whose exterior is irreducible and has s Seifert and r hyperbolic pieces in its JSJ decomposition. We have that $K_{s,r}$ is an L–space knot by repeated application of Theorem 1.3. As discussed above, this completes the proof. □

3.2 Left-orderability

Recall that a nontrivial group G is *left-orderable* if there exists a left-invariant total order on G. Examples of left-orderable groups include $\mathbb{Z}$ and $\mathrm{Homeo}_+(\mathbb{R})$, while any group with torsion (eg a finite group) is not left-orderable. It is natural to ask which three-manifold groups can be left-ordered. Such groups are well-suited for this study due to the following theorem.

Theorem 3.1 (Boyer, Rolfsen and Wiest [7]) *Let Y be a compact, connected, irreducible, P^2–irreducible three-manifold. If there exists a nontrivial homomorphism $f\colon \pi_1(Y) \to G$ where G is left-orderable, then $\pi_1(Y)$ is left-orderable. In particular, if there exists a nonzero degree map from Y to Y' where $\pi_1(Y')$ is left-orderable, then $\pi_1(Y)$ is left-orderable.*

Rather than define *P^2–irreducible*, we simply point out that if Y is orientable, then irreducibility implies P^2–irreducibility. For compact, orientable, irreducible three-manifolds with $b_1 > 0$, it then follows that their fundamental groups are always left-orderable. However, there are more interesting phenomena for rational homology spheres; for example $(+\frac{3}{2})$–surgery on the left-handed trefoil has left-orderable fundamental group, whereas $(-\frac{3}{2})$–surgery has torsion-free, non-left-orderable fundamental group. (This can be deduced for instance from [7, Theorem 1.3].) Surprisingly, the left-orderability of the fundamental groups of three-manifolds is conjecturally characterized by Heegaard Floer homology. The following conjecture was made in [6]:

Conjecture 1.8 (Boyer, Gordon and Watson) *Let Y be an irreducible rational homology sphere. Then Y is an L–space if and only if $\pi_1(Y)$ is not left-orderable.*

There exists a large amount of support for this conjecture, as it is known to be true for manifolds with Seifert or Sol geometry, branched double covers of nonsplit alternating links, graph manifold integer homology spheres and many other families of examples; see for instance Boileau and Boyer [5], Boyer, Gordon and Watson [6] or Peters [31]. We also remark that irreducibility is necessary, as $\Sigma(2,3,7) \# \Sigma(2,3,5)$ has non-left-orderable fundamental group, but is not an L–space.

In the proof of Proposition 1.9 below, we remind the reader that we will be assuming Conjecture 1.8.

Proof of Proposition 1.9 Suppose that $P(K)$ is an L–space knot. Then for all $\alpha \in \mathbb{Q}$ with $\alpha \geq 2g(P(K)) - 1$, we have $S^3_\alpha(P(K))$ is an L–space. For all but finitely many such α, we have that $S^3_\alpha(P(K))$ is irreducible as well. Thus, by Conjecture 1.8, we have that $\pi_1\big(S^3_\alpha(P(K))\big)$ is not left-orderable for $\alpha \gg 2g(P(K)) - 1$.

We first study the pattern P. By [8, Proposition 4.1], for these α, $\pi_1(S^3_\alpha(P))$ is not left-orderable. Furthermore, for all but finitely many α, we have that $S^3_\alpha(P)$ is irreducible. Therefore, we appeal to Conjecture 1.8 to conclude that P is an L–space knot.

We modify the argument of [8, Proposition 4.1] to study the companion K. Recall that w represents the winding number of P in the solid torus V. We also consider the basis (m, ℓ) for $H_1(\partial V; \mathbb{Z})$ as given in Section 2. We choose $n \in \mathbb{Z}$ such that $\gcd(w, n) = 1$ and $n \gg 2g(P(K)) - 1$. As discussed, we have $S^3_n(P(K))$ is irreducible and $\pi_1(S^3_n(P(K)))$ is not left-orderable. We consider the manifold $(P; n)$. We have that the kernel of $i_*\colon H_1(\partial(P; n); \mathbb{Z}) \to H_1((P; n); \mathbb{Z})$ is generated by $nm + w^2\ell$ by Lemma 2.3. Since $\gcd(w, n) = 1$ by assumption, we have that the element $nm + w^2\ell$ is represented by a simple closed curve on $\partial(P; n)$ which bounds in $(P; n)$. It then follows that there exists a degree-one map $\phi\colon (P; n) \to S^1 \times D^2$ which restricts to a homeomorphism on the boundary; see for instance [32, Lemma 2.2]. Since $nm + w^2\ell$ bounds in $(P; n)$, we must have that $\phi(nm + w^2\ell)$ is isotopic to $\{*\} \times D^2$.

By extending ϕ to be the identity on the exterior of K, one obtains a degree-one map from $S^3_n(P(K))$ to $S^3_{n/w^2}(K)$. Since $S^3_n(P(K))$ is irreducible and $\pi_1(S^3_n(P(K)))$ is not left-orderable, we have $\pi_1(S^3_{n/w^2}(K))$ is not left-orderable by Theorem 3.1. Since w is fixed, by choosing a sufficiently large n with $\gcd(w, n) = 1$, we can arrange that $S^3_{n/w^2}(K)$ is irreducible as well. Again, by Conjecture 1.8, K is an L–space knot. □

References

[1] **S Akbulut**, **B Ozbagci**, *Lefschetz fibrations on compact Stein surfaces*, Geom. Topol. 5 (2001) 319–334 MR1825664

[2] **K L Baker**, **A H Moore**, *Montesinos knots, Hopf plumbings and L–space surgeries* `arXiv:1404.7585`

[3] **J Berge**, *Some knots with surgeries yielding lens spaces*, unpublished manuscript

[4] **J Berge**, *The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$*, Topology Appl. 38 (1991) 1–19 MR1093862

[5] **M Boileau**, **S Boyer**, *Graph manifolds $\mathbb{Z}$–homology 3–spheres and taut foliations* arXiv:1303.5264

[6] **S Boyer**, **C M Gordon**, **L Watson**, *On L–spaces and left-orderable fundamental groups*, Math. Ann. 356 (2013) 1213–1245 MR3072799

[7] **S Boyer**, **D Rolfsen**, **B Wiest**, *Orderable 3–manifold groups*, Ann. Inst. Fourier (Grenoble) 55 (2005) 243–288 MR2141698

[8] **A Clay**, **L Watson**, *On cabled knots, Dehn surgery, and left-orderable fundamental groups*, Math. Res. Lett. 18 (2011) 1085–1095 MR2915469

[9] **R Fintushel**, **R J Stern**, *Constructing lens spaces by surgery on knots*, Math. Z. 175 (1980) 33–51 MR595630

[10] **D Gabai**, *Surgery on knots in solid tori*, Topology 28 (1989) 1–6 MR991095

[11] **D Gabai**, *1–bridge braids in solid tori*, Topology Appl. 37 (1990) 221–235 MR1082933

[12] **C M Gordon**, *Dehn surgery and satellite knots*, Trans. Amer. Math. Soc. 275 (1983) 687–708 MR682725

[13] **M Hedden**, *On knot Floer homology and cabling, II*, Int. Math. Res. Not. 2009 (2009) 2248–2274 MR2511910

[14] **M Hedden**, *Notions of positivity and the Ozsváth–Szabó concordance invariant*, J. Knot Theory Ramifications 19 (2010) 617–629 MR2646650

[15] **W Heil**, *Elementary surgery on Seifert fiber spaces*, Yokohama Math. J. 22 (1974) 135–139 MR0375320

[16] **M Hirasawa**, **K Murasugi**, **D S Silver**, *When does a satellite knot fiber?*, Hiroshima Math. J. 38 (2008) 411–423 MR2477750

[17] **J Hom**, *A note on cabling and L–space surgeries*, Algebr. Geom. Topol. 11 (2011) 219–223 MR2764041

[18] **P Kronheimer**, **T Mrowka**, **P Ozsváth**, **Z Szabó**, *Monopoles and lens space surgeries*, Ann. of Math. 165 (2007) 457–546 MR2299739

[19] **W B R Lickorish**, *An introduction to knot theory*, Graduate Texts in Mathematics 175, Springer, New York (1997) MR1472978

[20] **L Moser**, *Elementary surgery along a torus knot*, Pacific J. Math. 38 (1971) 737–745 MR0383406

[21] **K Motegi**, *L–space surgery and twisting operation* arXiv:1405.6487

[22] **Y Ni**, *Knot Floer homology detects fibred knots*, Invent. Math. 170 (2007) 577–608 MR2357503

[23] **P Ozsváth**, **Z Szabó**, *Knot Floer homology and the four-ball genus*, Geom. Topol. 7 (2003) 615–639 MR2026543

[24] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and genus bounds*, Geom. Topol. 8 (2004) 311–334 MR2023281

[25] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR2065507

[26] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. 159 (2004) 1159–1245 MR2113020

[27] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR2113019

[28] **P Ozsváth**, **Z Szabó**, *On knot Floer homology and lens space surgeries*, Topology 44 (2005) 1281–1300 MR2168576

[29] **P S Ozsváth**, **Z Szabó**, *Knot Floer homology and integer surgeries*, Algebr. Geom. Topol. 8 (2008) 101–153 MR2377279

[30] **P S Ozsváth**, **Z Szabó**, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. 11 (2011) 1–68 MR2764036

[31] **T Peters**, *On L–spaces and non left-orderable* 3*–manifold groups* `arXiv:0903.4495`

[32] **Y W Rong**, *Degree one maps of Seifert manifolds and a note on Seifert volume*, Topology Appl. 64 (1995) 191–200 MR1340870

[33] **L Rudolph**, *Quasipositive plumbing (Constructions of quasipositive knots and links, V)*, Proc. Amer. Math. Soc. 126 (1998) 257–267 MR1452826

[34] **H Seifert**, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. 60 (1933) 147–238 MR1555366

[35] **J R Stallings**, *Constructions of fibred knots and links*, from: "Algebraic and geometric topology, part 2", (R J Milgram, editor), Proc. Sympos. Pure Math. 32, Amer. Math. Soc. (1978) 55–60 MR520522

[36] **W P Thurston**, *The geometry and topology of* 3*–manifolds*, Lecture notes, Princeton University (1980) Available at `http://library.msri.org/nonmsri/gt3m`

[37] **F Vafaee**, *On the knot Floer homology of twisted torus knots*, Int. Math. Res. Not. 2014 (2014) `arXiv:1311.3711`

Department of Mathematics, Columbia University
New York, NY 10027, USA

Department of Mathematics, University of Texas at Austin
Austin, TX 78712, USA

Mathematics Department, California Institute of Technology
Pasadena, CA 91125, USA

`hom@math.columbia.edu, tlid@math.utexas.edu, vafaee@caltech.edu`

Received: 26 June 2014