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Stratified obstruction systems for equivariant moduli problems and invariant Euler cycles

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The purpose of this paper is to study finite-dimensional equivariant moduli problems from the viewpoint of stratification theory. We show that there exists a stratified obstruction system for a finite-dimensional equivariant moduli problem. In addition, we define a coindex for a G–vector bundle that is determined by the G–action on the vector bundle and prove that if the coindex of an oriented equivariant moduli problem is bigger than 1, then we obtain an invariant Euler cycle via equivariant perturbation. In particular, we get a localization formula for the stratified transversal intersection of S^1–moduli problems.

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1 Introduction

Let $\pi\colon E \to B$ be an oriented smooth vector bundle of rank k over an n–dimensional closed manifold B and $S\colon B \to E$ a smooth section. The zero locus $S^{-1}(0) \subset B$ contains a lot of topological information about the bundle E when it is transversal to the zero section, and that information is reflected as a cycle in B, called the Euler cycle of E. The Euler cycle represents a homology class which is a topological invariant of E. For an oriented vector bundle, there are two dual viewpoints to the construction of such a topological invariant.

(1) For any smooth section $S\colon B \to E$ (not necessarily transversal), via a slight smooth perturbation we get a smooth section $S + P$ which is transverse to the zero section, so the zero locus $(S + P)^{-1}(0) \subset B$ is an oriented closed submanifold of dimension $n - k$. This submanifold yields a homology class $[(S + P)^{-1}(0)] \in H_{n-k}(B; \mathbb{Z})$, which is independent of the choice of the initial perturbation.

(2) There exists a unique cohomology class on E, denoted by $\Theta \in H^k_{vc}(E)$ (vertical compact cohomology), that restricts to the generator of $H^k_c(F)$ (compact cohomology) on each fiber F. This class is called the Thom class of E, and the cohomological Euler class of E is defined as the pullback of the Thom class by the zero section.

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Now let us consider the equivariant case. Given a finite group Γ, if $\pi\colon E \to B$ is a smooth Γ–equivariant vector bundle and S is a smooth Γ–equivariant section, then the technique of perturbation for the section S is ineffective since in general there is no Γ–equivariant smooth perturbed section of S that is transversal to the zero section. Fukaya and Ono [8] were able to use that Γ is finite to construct a multivalued perturbation such that each branch is transverse to the zero section. The zero locus of such a multisection gives rise to a cycle over the rational numbers $\mathbb{Q}$ and through this technique Fukaya and Ono defined the Euler class of an oriented orbibundle. In particular, for an oriented orbibundle $E \to X$ with a smooth section S having compact locus $S^{-1}(0)$, Lu and Tian [14] gave a very thorough presentation of a construction of a multivalued perturbation of S by gluing together sections that resolve fiber products over the local uniformizing charts of $S^{-1}(0)$. They also showed that the zero locus of the perturbed section yields a rational Euler cycle.

Generally, let G be a connected compact Lie group and B be a smooth manifold on which G acts smoothly and effectively. Suppose that $\pi\colon E \to B$ is an oriented smooth G–vector bundle. Using equivariant cohomology theory, Mathai and Quillen [15] constructed an equivariant Thom class Θ_{eq} of E, which is a compactly supported closed equivariant form such that its integral along the fibers is the constant function 1 on B. Thus the equivariant Euler class of E can be defined as the pullback of the equivariant Thom class by the zero section. For a nontrivial action of a Lie group G on the vector bundle E, it is generally unrealistic to require a smooth section to be equivariant and transversal. In fact, for a G–equivariant smooth section $S\colon B \to E$, there exist some global obstructions (see Costenoble and Waner [4] and Petrie [19]) to deforming S to a G–equivariant smooth transversal section. Therefore transversality is too rigid in the equivariant case and a natural problem is the following.

- *Can we define a new version of transversality in the G–equivariant case such that every G–equivariant smooth section has an equivariant perturbation that satisfies this new transversality?*
- *How do we construct the invariant Euler cycle of an oriented G–vector bundle?*

This is a *finite-dimensional equivariant moduli problem* in the sense of Cieliebak, Riera and Salamon [3]. Independently, Bierstone and Field also discussed the transversality problem in the equivariant case. In [1], Bierstone introduced the notion of general position for smooth G–equivariant maps between smooth G–manifolds. Meanwhile, Field [6] proposed the concept of G–transversality and showed that the two definitions are equivalent (see Field [5]). An infinite-dimensional version of equivariant general position was defined by Hambleton and Lee [12]. They also studied the equivariant perturbation of Yang–Mills moduli space with a compact Lie group action.

For an oriented finite-dimensional G–moduli problem that is regular, ie the isotropy subgroup of the G–action on B is finite, Cieliebak, Riera and Salamon constructed a rational cycle through multi-valued perturbation in the paper [3]. The method of Cieliebak, Riera and Salamon is similar to the one that Fukaya and Ono used in constructing the Euler class of an oriented orbibundle. The main issue is that for an equivariant vector bundle with a finite group action we cannot guarantee that the transversal perturbation of an equivariant section is also equivariant in general, so the perturbed section cannot descend to a single-valued section of the associated orbibundle. Therefore the multi-valued perturbation is necessary to obtain the transversality.

The method we use here is to perturb the equivariant smooth section in the sense of equivariant general position and to represent the fundamental class with a Whitney object rather than a smooth submanifold. Using the geometric chain (cycle) technique introduced by Goresky, we obtain the following main theorem.

Theorem 1.1 *Let (B, E, S) be a finite-dimensional oriented G–moduli problem with* $\dim B = n$ *and* $\operatorname{rank} E = k$. *If* $\operatorname{coind}(B, E) > 1$, *then there exists a smooth equivariant perturbation $P\colon B \to E$ supported in an invariant open neighborhood of $S^{-1}(0)$ such that $S + P$ is in general position with respect to the zero section. The zero locus $(S + P)^{-1}(0) \subset B$ yields a G–invariant $(n-k)$–geometric cycle that represents a homology class*

$$[(S + P)^{-1}(0)] \in H_{n-k}(B; \mathbb{Z}),$$

which is independent of the choice of the perturbation.

This paper is organized as follows. We devote Section 2 to the preliminaries of the definition of Whitney stratified chains and the definition of general position for equivariant smooth maps. In Section 3 we construct the obstruction system of a G–moduli problem. We also define a coindex $\operatorname{coind}(B, E)$ for a G–vector bundle $\pi\colon E \to B$. In Section 4 we give the proof of Theorem 1.1. Finally, in Section 5 we study the transversal intersection of S^1–moduli problems and show that all geometric information of transversal intersection is contained in the fixed submanifold of the S^1–action (Theorem 5.2).

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2 Preliminaries

2.1 Whitney stratification and geometric cycles

The stratified space is motivated by the study of singular manifolds. It naturally arises in the study of algebraic, analytic varieties and singularities of smooth mappings. Intuitively, a stratified space is an object constructed by gluing some smooth manifolds with different dimensions together in a nice way.

Definition 2.1 Let X be a Hausdorff and paracompact topological space, and let $\mathcal{I}$ be a poset with order relations denoted by $\leq$. If X is a locally finite collection of disjoint locally closed manifolds $S_i \subset X$ $(i \in \mathcal{I})$ that satisfies

(1) $X = \bigcup_{i\in\mathcal{I}} S_i$,

(2) $S_i \cap \overline{S}_j \neq \varnothing \Leftrightarrow S_i \subset \overline{S}_j \Leftrightarrow i \leq j$,

then the family $\mathcal{S} = \{S_i \subset X \mid i \in \mathcal{I}\}$ is called a *stratification* of X and $(X, \mathcal{S})$ is called a *stratified space*. A piece $S_i \in \mathcal{S}$ is called a *stratum* of X.

A *stratified subspace* of $(X, \mathcal{S})$ is a subset $Y \subset X$ such that

$$\mathcal{S}_Y = \{S \cap Y \mid S \in \mathcal{S}\}$$

with the induced topology is a stratification of Y.

If $S_i \subset \overline{S}_j$, we write $S_i \leq S_j$. If $S_i \leq S_j$ and $S_i \neq S_j$, we write $S_i < S_j$.

Definition 2.2 Let $\mathcal{S}$ be a stratification of the space X. The *length of a stratum* $S \in \mathcal{S}$ is the integer

$$l_X(S) := \sup\{n \mid S = S_0 < S_1 < \cdots < S_n\}$$

where $S_1, \ldots, S_n$ are strata of X. The *length of stratified space* $(X, \mathcal{S})$ is

$$l(X) := \sup_{i\in\mathcal{I}} l_X(S_i).$$

A stratum $S \in \mathcal{S}$ is called *maximal* (resp. *minimal*) if it is open (resp. closed). The *dimension of a stratified space* $(X, \mathcal{S})$ is defined to be the dimension of its maximal stratum. The stratum $S \in \mathcal{S}$ is called a *regular stratum* if it is open in X; otherwise it is called *singular stratum*. The union of all singular strata, denoted by Σ, is called the *singular part* of X. And the minimal part, denoted by $\Sigma_{\min}$, is the union of minimal

strata. Let $X_i = \bigcup_{j \le i} S_j$, and call X_i a *skeleton* of X. There exists a finite filtration of skeletons

$$X = X_m \supseteq X_{m-1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \varnothing,$$

where m is called the *depth* of X.

Example 2.3 A smooth manifold M is a stratified space with empty singular part.

Example 2.4 A subset V of $\mathbb{R}^n$ is called an *algebraic set* if it is the common zero loci of finitely many real polynomials. The singular set ΣV of all points where V fails to be a smooth manifold is also an algebraic set, hence there is a finite filtration

$$V = V_m \supseteq V_{m-1} \supseteq \cdots \supseteq V_0 \supseteq V_{-1} = \varnothing$$

with $V_{i-1} = \Sigma V_i$. Then V is a stratified space with stratum $S_i = V_i - V_{i-1}$.

Inspired by the ideas of Thom on stratifications, Whitney [22] introduced *Condition A* and *Condition B*. Condition B implies Condition A; this was proved by Mather in his lecture notes [16]. Therefore, in general we only use Condition B to define a Whitney stratification. Let us recall the definition of Whitney's Condition B. Given any $x, y \in \mathbb{R}^n$ such that $x \neq y$, the secant $\widehat{xy}$ is defined to be the line in $\mathbb{R}^n$, which is parallel to the line $\overline{xy}$ (line joining x and y) and passes through the origin.

Definition 2.5 (Condition B for submanifolds of $\mathbb{R}^n$) Let X and Y be the smooth submanifolds of $\mathbb{R}^n$. Assume that $\dim X = r$. We say that the pair (X, Y) satisfies *Condition B* at a given point $y \in Y$, if the following holds: Let $\{x_i\}$ and $\{y_i\}$ be two sequences of points in X and Y respectively, such that $\{x_i\}$ and $\{y_i\}$ converge to y. Suppose that the tangent space $T_{x_i} X$ converges to some r–plane $\tau \subset \mathbb{R}^n$ and the secants

$$\widehat{x_i y_i} \qquad (x_i \neq y_i)$$

converge to some line $l \subset \mathbb{R}^n$. Then $l \subset \tau$.

This definition can be extended to submanifolds of arbitrary smooth manifolds.

Definition 2.6 Let M be a smooth m–manifold, X and Y be smooth submanifolds. Given $y \in Y$, we say that the pair (X, Y) satisfies *Condition B* at y if for some coordinate chart (φ, U) about y the pair

$$(\varphi(U \cap X), \varphi(U \cap Y))$$

satisfies *Condition B* at $\varphi(y)$ in $\mathbb{R}^m$. This definition is well-defined, as it is independent of the choice of the coordinate chart (see [16]).

Example 2.7 [20, Theorem 4.3.7] Let G be a compact Lie group. If M is a smooth manifold on which G acts smoothly, then the stratification by orbit types of M, denoted by

$$M = \bigsqcup_{H<G} M_{(H)},$$

makes M into a Whitney stratified space, where $M_{(H)}$ is the set of points in M such that the isotropy subgroup of each point is conjugate to H.

Remark 2.8 In general, a stratum $M_{(H)}$ may have connected components with different dimensions. In this case we can refine the decomposition to make each piece of the stratification a submanifold in M. To keep our notation manageable we refine such decomposition and still write it as $M_{(H)}$.

Definition 2.9 Let X be a closed subset of a smooth manifold M. We say that X admits a *Whitney stratification* if there exists a stratification $\mathcal{S}$ on X with a filtration of X by closed subsets

$$X = X_m \supseteq X_{m-1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \varnothing$$

and any pair of strata (S_i, S_j) $(i \leq j)$ satisfies *Condition B*.

The subset X together with the Whitney stratification is called a *Whitney object*. Specially, if $W \subset X$ is a closed subset with a Whitney stratification such that each stratum of W is contained in a single stratum of X, then W is called a *Whitney substratified object* of X.

The idea of representing cocycles by geometric objects was introduced by Whitney [21]. In [9], Goresky gave all technical constructions for a geometric description of homology and cohomology in the context of Whitney stratifications. Let us recall Goresky's method of geometric chains (cycles).

Definition 2.10 (Goresky [9]) A *geometric k–chain* ξ in a fixed Whitney object X consists of a compact k–dimensional Whitney substratified object $|\xi| \subset X$, which is called the support of ξ, together with an orientation of $|\xi|$, which is a choice of an orientation and multiplicity of each k–dimensional stratum. The set of orientations of $|\xi|$ is just the group $H_k(|\xi|, |\xi|_{k-1})$.

Definition 2.11 Let ξ be a geometric k–chain in X. The *reduction* of ξ is the geometric chain whose support is the closure of the union of all components of $|\xi| - |\xi|_{k-1}$ that have been assigned a nonzero multiplicity. In particular, we can identify a geometric chain with its reduction.

Consider the pairs $(|\xi|, |\xi|_{k-1})$ and $(|\xi|_{k-1}, |\xi|_{k-2})$. There exist two exact homology sequences

$$\text{(2-1)}\quad \cdots \to H_k(|\xi|_{k-1}) \xrightarrow{i_*} H_k(|\xi|) \xrightarrow{j_*} H_k(|\xi|, |\xi|_{k-1}) \xrightarrow{\partial_k} H_{k-1}(|\xi|_{k-1}) \to \cdots,$$

$$\text{(2-2)}\quad \cdots \to H_{k-1}(|\xi|_{k-2}) \xrightarrow{i_*} H_{k-1}(|\xi|_{k-1}) \xrightarrow{j_*} H_{k-1}(|\xi|_{k-1}, |\xi|_{k-2}) \xrightarrow{\partial_{k-1}} H_{k-2}(|\xi|_{k-2}) \to \cdots,$$

where ∂_k and ∂_{k-1} are the boundary operators. Given a geometric k–chain ξ, the boundary of ξ, denoted by $\partial\xi$, is defined to be the geometric $(k-1)$–chain with support $|\xi|_{k-1}$ and orientation induced from sequence

$$\text{(2-3)}\qquad H_k(|\xi|, |\xi|_{k-1}) \xrightarrow{\partial_k} H_{k-1}(|\xi|_{k-1}) \xrightarrow{j_*} H_{k-1}(|\xi|_{k-1}, |\xi|_{k-2}).$$

Definition 2.12 We say that ξ is a *geometric k–cycle* if $\partial\xi = 0$.

Note that the orientation of ξ is a homology class $O_\xi \in H_k(|\xi|, |\xi|_{k-1})$, and $\partial\xi = 0$ implies that $\partial_k O_\xi = 0$. Due to the exactness of the homology sequence (2-1), there exists a unique *fundamental class* $\mu_\xi \in H_k(|\xi|)$ such that $j_*(\mu_\xi) = O_\xi$. Let $\iota\colon |\xi| \to X$ be the inclusion. Then ξ represents a homology class $[\xi] = \iota_*\mu_\xi \in H_k(X)$. There is an equivalence relation, called *cobordism*, between geometric k–cycles.

Definition 2.13 Two geometric k–cycles ξ_0 and ξ_1 in X are *cobordant* if there exists a geometric $(k+1)$–chain η in $X \times \mathbb{R}$ and some $\varepsilon > 0$ such that

(1) $|\eta| \subset X \times [0,1]$,

(2) $|\eta| \cap X \times [0,\varepsilon) = |\xi_0| \times [0,\varepsilon)$,

(3) $|\eta| \cap X \times (1-\varepsilon, 1] = |\xi_1| \times (1-\varepsilon, 1]$,

(4) $\partial\eta = \xi_1 \times \{1\} - \xi_0 \times \{0\}$ (modulo reduction).

Denote by $WH_k(X)$ the set of cobordism classes of geometric k–cycles in X. Note that the cobordant cycles represent the same homology class, and therefore we get a *representation map*

$$\text{(2-4)}\qquad R\colon WH_k(X) \to H_k(X).$$

In particular, assume that $Y \subset X$ is an oriented compact Whitney substratified object. If there are no strata of Y with codimension one, then the cycle condition automatically holds, ie Y represents a homology class in X.

2.2 General position of equivariant smooth maps

In this subsection, we recall the definition of general position for a G–equivariant map and state some elementary properties; for more details, we refer to [1].

Let V be a finite-dimensional vector space. Then V is called a G–*space* if there exists a representation of G over V, $\rho_V\colon G \to \mathrm{GL}(V)$. A smooth map $F\colon V \to W$ of two G–spaces is called a G–equivariant map if for any $g \in G$ we have $(\rho_W(g)) \circ F = F \circ (\rho_V(g))$. The set of all smooth G–equivariant maps is denoted by $\mathcal{C}^\infty_G(V, W)$. Let G act trivially on $\mathbb{R}$. A smooth function $f\colon V \to \mathbb{R}$ is G–*invariant* if it satisfies $f \circ (\rho_V(g)) = f$ for all $g \in G$.

Let $\mathcal{C}^\infty_G(V)$ be the set of G–invariant smooth functions on V. Then $\mathcal{C}^\infty_G(V, W)$ has the structure of a $\mathcal{C}^\infty_G(V)$–module with finite polynomial generators [6, Lemma 3.1]. Suppose that $\{F_1, \dots, F_k\}$ is the set of polynomial generators; then for every G–equivariant map $F \in \mathcal{C}^\infty_G(V, W)$ there exist unique G–invariant functions $h_i \in \mathcal{C}^\infty_G(V)$ $(1 \le i \le k)$ such that

$$F(x) = \sum_{i=1}^{k} h_i(x) F_i(x) \quad \text{for all } x \in V.$$

Define a map

$$U\colon V \times \mathbb{R}^k \to W, \quad (x; t_1, \dots, t_k) \mapsto \sum_{i=1}^{k} t_i F_i(x). \tag{2-5}$$

The zero set of U, denoted

$$\mathcal{E} := \{(x, t) \in V \times \mathbb{R}^k \mid U(x, t) = 0\},$$

is called the *universal variety*.

The universal variety $\mathcal{E}$ is a real affine algebraic variety which is uniquely determined (up to a product with an affine space) by V and W; moreover, $\mathcal{E}$ admits a unique minimum Whitney stratification.

Definition 2.14 Define the *graph map* $\Gamma(F)$ of F by

$$\Gamma(F)\colon V \to V \times \mathbb{R}^k, \quad x \mapsto (x, h_1(x), \dots, h_k(x)). \tag{2-6}$$

Clearly we get $F = U \circ \Gamma(F)$ and $F^{-1}(0) = \Gamma(F)^{-1}(\mathcal{E})$. The universal variety $\mathcal{E}$ contains information about all possible zero sets for $F \in \mathcal{C}^\infty_G(V, W)$. Suppose that X is a smooth manifold and $E \subset \mathbb{R}^q$ is an algebraic subvariety. That a smooth map $f\colon X \to \mathbb{R}^q$ is *transverse to* E means that f is transverse to each stratum of the minimum Whitney stratification of E.

Definition 2.15 Let $F \in \mathcal{C}^\infty_G(V,W)$ such that $F(0)=0$. Then F is *in general position with respect to* $0 \in W$ at $0 \in V$ if the graph map $\Gamma(F)$ is transversal to the minimum Whitney stratification of the universal variety $\mathcal{E}$ in $V \times \mathbb{R}^k$.

Definition 2.16 If $W = W_1 \oplus W_2$ is a direct sum decomposition of G–spaces W_1 and W_2, then the G–equivariant map

$$F = (F_1, F_2)\colon V \to W_1 \oplus W_2$$

is in general position with respect to $W_1 \subset W$ at $0 \in V$ if and only if the map

$$F_2\colon V \to W_2$$

is in general position with respect to $0 \in W_2$ at $0 \in V$.

Next we review some basic properties of smooth actions of a compact Lie group on manifolds. Let G be a compact Lie group and M be a smooth manifold. A *smooth G–action* on M is a smooth map

$$l\colon G \times M \to M, \quad (g,x) \mapsto gx \tag{2-7}$$

such that $l(e,x)=x$ and $(g_1g_2)x = g_1(g_2x)$ for any $g_1, g_2 \in G, x \in M$.

For any $g \in G$ we can construct a smooth map

$$\mu_g\colon M \to M, \quad x \mapsto gx. \tag{2-8}$$

A point $x \in M$ is called a *fixed point* if $\mu_g(x) = x$ for any $g \in G$. For any $x \in M$ the set

$$G(x) = \{gx \mid \text{for all } g \in G\}$$

is called the *orbit of* x, and the closed subgroup

$$G_x = \{g \in G \mid gx = x\}$$

is called the *isotropy subgroup* of x. The action of G on M is *effective* if the map μ_g is the identity mapping on M only for $g=e$, where e is the identity of G. We say that the G–action of G is *free* if for any $x \in M$, $\mu_g(x) = x$ implies that $g = e$.

Definition 2.17 Assume that the Lie group G acts smoothly on manifold M. The action is *proper* if for every compact subset $K \subset M$, the set

$$G_K = \{g \in G \mid (gK) \cap K \neq \varnothing\}$$

is compact.

If G is a compact Lie group, then the smooth action of G on a smooth manifold M is proper. A manifold M is called a *G–manifold* if G acts on M smoothly. In particular, if the action is proper then M is called a *proper G–manifold*. The set of all closed subgroups of G admits an equivalence relation

$$H \sim H' \quad \Leftrightarrow \quad H = gH'g^{-1} \quad \text{for some } g \in G.$$

The equivalence classes, denoted by (H), are called *conjugacy classes*. The set of conjugacy classes bears a partial order: $(H) \leq (H')$ if there exists a $g \in G$ such that $H \subseteq gH'g^{-1}$. For an orbit $G(x)$, the isotropy groups G_{gx} form a conjugacy class (G_x), which is called the *isotropy type* of the orbit $G(x)$.

Given any $x \in M$, the orbit $G(x)$ describes a G–invariant closed submanifold of M. Furthermore, $G(x)$ is isomorphic to G/G_x via the canonical mapping

$$\Phi_x\colon G/G_x \to M, \quad gG_x \mapsto gx. \tag{2-9}$$

In particular, for a proper G–manifold there exists a *G–invariant partition of unity* [20, Theorem 4.2.4] for any covering of the manifold by G–invariant open subsets and a *G–invariant tubular neighborhood theorem* [2, Theorem 2.2] for a closed invariant submanifold.

For each $x \in M$, the normal space $S_x = T_xM/T_x(G(x))$ is called the slice of x. Note that the homogenous space $G \to G/G_x$ is a G_x–principal fiber bundle; furthermore, we get an associated bundle $N_x = G \times_{G_x} S_x$, which is called the *slice bundle* of x. The following *differential slice theorem* shows that every smooth G–manifold M with a proper G–action locally looks like a neighborhood of the zero section in the slice bundle. For the original proof of the slice theorem we refer to Palais [18].

Theorem 2.18 (cf [20, Theorem 4.2.6]) *Let $\Phi\colon G \times M \to M$ be a proper group action, x a point of M and $S_x = T_xM/T_x(G(x))$ the normal space of the orbit of x. Then there exists a G–equivariant diffeomorphism from a G–invariant neighborhood of the zero section of $G \times_{G_x} S_x$ onto a G–invariant neighborhood of $G(x)$ such that the zero section is mapped onto $G(x)$ in a canonical way.*

We summarize what we can get from the differential slice theorem: If G is a compact Lie group, then for each $x \in M$ there exists a G_x–invariant submanifold in M, denoted by S, such that:

(1) $x \in S$.

(2) $gS \cap S \neq \varnothing \Rightarrow g \in G_x$.

(3) For any $y \in S$, $G_y \subseteq G_x$.

(4) $GS = \{gy \mid g \in G, y \in S\}$ is a G–invariant open neighborhood of the orbit $G(x)$ in M.

Definition 2.19 Suppose that M and N are smooth G–manifolds. A smooth map $f \in \mathcal{C}^\infty(M,N)$ is called a *G–equivariant smooth map* if the G–action commutes with f, ie for any $x \in M$ and $g \in G$ we have

$$f(gx) = g(f(x)).$$

Clearly, if f is G–equivariant, then $G_x \subseteq G_{f(x)}$ for any $x \in M$. Let $P \subset N$ be a G–invariant submanifold. For any $x \in f^{-1}(P)$, we can choose a G_x–equivariant diffeomorphism ϕ from a neighborhood V of $f(x)$ in N to a G_x–vector space $W_1 \oplus W_2$ such that $\phi(V \cap P) = W_1$. The pair (V,ϕ) is called a G_x–chart for P at $f(x)$. Choose a slice at x, denoted by S_x. Then f determines a smooth G_x–equivariant map

$$\widetilde{f}\colon S_x \to W_1 \oplus W_2.$$

Definition 2.20 We say that f is *in general position* with respect to P at $x \in M$ if

(1) $f(x) \notin P$; or

(2) $f(x) \in P$ and for some choice of slice S_x and G_x–chart for P at $f(x)$,

$$\widetilde{f}\colon S_x \to W_1 \oplus W_2$$

is in general position with respect to W_1 at x in the sense of Definition 2.16.

If f is in general position with respect to P at every point of M, then we say that f is in general position with respect to P.

Remark 2.21 The above definition is well-defined since the definition is independent of the choice of the slices (cf [1, Proposition 5.6]).

If f is in general position with respect to P at $x \in M$, then the definition implies that f is in general position with respect to P at gx for any $g \in G$; furthermore, it is in general position in a neighborhood of x.

In classical differential topology the set of smooth maps that are in general position is open and dense. Similarly, for the G–manifolds and equivariant maps we have:

Theorem 2.22 (Bierstone [1, Theorem 1.4]) *Suppose that P is an invariant submanifold of N. Then the set of smooth equivariant maps $f \in \mathcal{C}^\infty_G(M,N)$ that are in general position with respect to P is a countable intersection of open dense sets in the Whitney of $\mathcal{C}^\infty$–topology.*

Particularly, when P is a closed invariant submanifold, the set of smooth equivariant maps that are in general position is open and dense in the Whitney topology. Furthermore, Bierstone showed that if $f \in \mathcal{C}^\infty_G(M,N)$ is in general position with respect to P, then $f^{-1}(P) \subset M$ is a Whitney object and every stratum of $f^{-1}(P)$ is a G–invariant submanifold of M. Note that the proper action of a Lie group on a manifold induces a natural stratification; thus a natural approach to the transversality problem in equivariant case is to consider the stratumwise transversality of a smooth equivariant map $f\colon M \to N$ with respect to an invariant submanifold P of N. Given a closed subgroup H of G we can define following three subspaces of M:

$$\begin{aligned} M_{(H)} &:= \{x \in M \mid G_x \sim H\},\\ M^H &:= \{x \in M \mid G_x \supset H\},\\ M_H &:= \{x \in M \mid G_x = H\}. \end{aligned}$$

In general, $M_{(H)}$, M^H and M_H are not connected and each connected component is a submanifold of M. In fact, M^H is just the set of fixed points of H in M. If $H \subset G$ is compact, then these three spaces satisfy

$$M_H = M_{(H)} \cap M^H.$$

Since G is a compact Lie group, the fact that H is an isotropy subgroup of some points in M means H is a closed subgoup of G and hence the compactness of H is automatic. Furthermore, $M_{(H)}$ can be decomposed into

$$M_{(H)} = \bigsqcup_{J \in (H)} M_J.$$

The equivariant map f does not always map $M_{(H)}$ into the set $N_{(H)}$ since in general $G_x \subseteq G_{f(x)}$, however, f maps M^H into N^H. Let $f_H = f|_{M_H}$, then the image of f_H is contained in N^H and the stratumwise transversality of f is defined as follows.

Definition 2.23 Let $f \in \mathcal{C}^\infty_G(M,N)$ and $P \subseteq N$ be a G–invariant submanifold. If for any subgroup H of G the map $f_H\colon M_H \to N^H$ is transverse to the submanifold P^H of N^H, then we say it is a G–equivariant map which admits the *stratumwise transversality* with respect to the G–invariant submanifold P of N.

The concepts of general position and stratumwise transversality are generalizations of classical transversality in differential topology. However, the stratumwise transversality is not open, ie if $f \in \mathcal{C}^\infty_G(M,N)$ admits stratumwise transversality with respect to a G–invariant submanifold P of N, then a small perturbation of f may break the stratumwise transversality. In particular, if a map $f \in \mathcal{C}^\infty_G(M,N)$ is in the general

position with respect to a G–invariant submanifold P of N, then f is stratumwisely transverse to P (see [1, Proposition 6.4]). In conclusion, we have the following relation for equivariant smooth maps:

{classical transversality} $\subset$ {being in general position} $\subset$ {stratumwise transversality}.

3 Stratified obstruction system of a G–moduli problem

In this section we show that for an equivariant vector bundle there exists a family of obstruction bundles and we call it the stratified obstruction system. The existence of the obstruction system implies that there is no equivariant perturbed section that is transversal to the zero section in general.

Let G be a compact Lie group and B be a G–manifold. A G–vector bundle over B is defined as follows.

Definition 3.1 If B is a G–manifold, a *G–vector bundle* on B is a G–space E together with a G–equivariant map $\pi\colon E \to B$ such that

(1) $\pi\colon E \to B$ is a real vector bundle on B;

(2) for any $g \in G$ and $x \in B$, the action $g\colon E_x \to E_{gx}$ is homomorphism of vector spaces.

The above definition implies that $g^{-1}\colon E_{gx} \to E_x$ is also a homomorphism of vector spaces such that $g^{-1} \circ g$ is the identity map on E_x and $g \circ g^{-1}$ is the identity map on E_{gx}. Thus $g\colon E_x \to E_{gx}$ is an isomorphism of vector spaces. Two G–vector bundles over B are called *G–equivalent* if they are equivalent as ordinary vector bundles via a G–equivariant bundle map. Next we describe the appropriate generalization of product bundle in the equivariant case, which gives the local model of G–vector bundles.

Assume that H is a closed subgroup of G and $\rho\colon H \to GL(\mathbb{R}; k)$ a homomorphism. For each H–space V we denote by $\varepsilon^\rho(V)$ the G–vector bundle over $G \times_H V$ with fiber $\mathbb{R}^k$ given by

$$\pi\colon G \times_H (V \times \mathbb{R}^k) \to G \times_H V, \quad \pi([g,(v,e)]) = [g,v], \tag{3-1}$$

where H acts on the fiber $\mathbb{R}^k$ via the homomorphism ρ. Generally, for any G–space X, suppose that H is a closed subgroup of G and $V \subset X$ is a H–invariant subspace; then V is called a *H–slice* provided that the equivariant map

$$\mu\colon G \times_H V \to X, \quad \mu([g,v]) = gv \tag{3-2}$$

is a homomorphism onto an open subset of X.

Definition 3.2 (Lashof [13]) A G–vector bundle $\pi\colon E \to B$ of rank k is called *G–locally trivial* if there exists a G–invariant open cover $\{GV_\alpha\}_{\alpha\in I}$ of B, where V_α is an H_α–slice, such that the restriction $E|_{GV_\alpha}$ is G–equivalent to $\varepsilon^{\rho_\alpha}(V_\alpha)$ for some homomorphism $\rho_\alpha\colon H_\alpha \to \mathrm{GL}(\mathbb{R};k)$ (under the identification $\mu\colon G\times_H V_\alpha \to GV_\alpha$).

In particular, every smooth G–vector bundle is G–locally trivial (cf [13, Corollary 1.6]).

According to Definition 3.1(2), for any $x\in B$ there is a representation of the isotropy subgroup G_x over the fiber E_x, therefore E_x is a G_x–vector space. Denote by E_x^f the G_x–fixed subspace of the fiber space E_x, ie

$$E_x^f = \{v\in E_x \mid gv = v \text{ for all } g\in G_x\}.$$

Given a closed subgroup H of G, let $B_{(H)} = \{x\in B \mid G_x \sim H\}$.

Proposition 3.3 *If x and y are contained in the same connected component of $B_{(H)}$, then $\dim E_x^f = \dim E_y^f$.*

To prove the proposition, we need the following lemma (cf [17, Lemma 6.12]):

Lemma 3.4 *Let G be a compact Lie group and H a closed subgroup of G. Then $gHg^{-1}\subset H$ implies that $gHg^{-1} = H$.*

Proof of Proposition 3.3 Without loss of generality, we may assume that $B_{(H)}$ is connected. Observe that $E|_{B_{(H)}}$ is a smooth G–vector bundle; thus it admits a G–local trivialization. Since $B_{(H)}$ is connected, given any smooth path[1]

$$\gamma\colon [0,1]\to B_{(H)}$$

such that $\gamma(0)=x$ and $\gamma(1)=y$, we can find finitely many points on the path γ

$$x_0 = x, x_1, \dots, x_m = y$$

with associated slices V_i $(0\le i\le m)$ such that the G–invariant open subsets $\{GV_i\}_{i=0}^m$ cover the path γ, and $GV_i\cap GV_{i+1}\neq\varnothing$ for $0\le i\le m-1$. Further, we can choose GV_i such that for each GV_i, $\pi^{-1}(GV_i)$ is G–equivalent to

$$\varepsilon^{\rho_i}(V_i) = G\times_{H_i}(V_i\times\mathbb{R}^k),$$

where $H_i = G_{x_i}$ and $\rho_i\colon H_i\to\mathrm{GL}(\mathbb{R};k)$ is a homomorphism. The next thing to do is to verify that for any $z\in GV_i$, we have $\dim E_z^f = \dim E_{x_i}^f$.

[1] Recall that a connected space is not always pathwise connected. However, if a space is connected and locally pathwise connected then it is pathwise connected. A connected manifold is always pathwise connected since the manifold is locally pathwise connected.

We divide the proof into three cases.

Case 1 Assume that $z \in V_i$. Then $G_z \subset H_i$. According to the definition of $B_{(H)}$, we get that G_z and $G_{x_i} = H_i$ are conjugate to H. Since the conjugate relation is an equivalence relation, by transitivity G_z is conjugate to H_i, ie there exists a $g \in G$ such that $G_z = gH_ig^{-1} \subset H_i$. By Lemma 3.4, we get $G_z = H_i$. Furthermore, the representation of G_z over the fiber E_z is equivalent to $\rho_i|_{G_z} = \rho_i$. It follows that the fixed subspaces E_z^f and $E_{x_i}^f$ have the same dimension, ie $\dim E_z^f = \dim E_{x_i}^f$.

Case 2 Assume that $z \in G(x_i)$, ie there exists a $g \in G$ such that $z = gx_i$. Note that $z = gx_i$ implies that $G_z = gG_{x_i}g^{-1}$ and the map

$$g\colon E_{x_i} \to E_z \tag{3-3}$$

is an isomorphism of vector spaces. By definition we have

$$\begin{aligned}
E_{x_i}^f &= \{u \in E_{x_i} \mid hu = u \text{ for all } h \in G_{x_i}\},\\
E_z^f &= \{v \in E_z \mid \widetilde{g}v = v \text{ for all } \widetilde{g} \in G_z\}\\
&= \{v \in E_z \mid \widetilde{g}v = v \text{ for all } \widetilde{g} \in gG_{x_i}g^{-1}\} \qquad (G_z = gG_{x_i}g^{-1})\\
&= \{v \in E_z \mid (ghg^{-1})v = v \text{ for all } h \in G_{x_i}\}.
\end{aligned}$$

For any $v \in E_z^f$, we have $g^{-1}v \in E_{x_i}$ and $h(g^{-1}v) = g^{-1}v$ for all $h \in G_{x_i}$. It follows that the homomorphism of vector spaces

$$g^{-1}\colon E_z \to E_{x_i} \tag{3-4}$$

maps E_z^f onto the subspace $E_{x_i}^f$. It is straightforward to show that the homomorphism (3-3) maps $E_{x_i}^f$ onto the subspace E_z^f. So $E_{x_i}^f \cong E_z^f$ and we obtain $\dim E_z^f = \dim E_{x_i}^f$.

Case 3 Assume that $z \in GV_i$ and $z \notin V_i$. Then there exists a $v \in V_i$ such that $z = gv$ for some $g \in G$. On one hand, due to the result of Case 1 we have

$$\dim E_v^f = \dim E_{x_i}^f. \tag{3-5}$$

On the other hand from the result of Case 2 we get

$$\dim E_v^f = \dim E_z^f. \tag{3-6}$$

At last, according to (3-5) and (3-6) we obtain

$$\dim E_z^f = \dim E_{x_i}^f. \tag{3-7}$$

Note that the path γ is covered by the G–invariant subsets $\{GV_i\}_{i=0}^m$ and for each $0 \le i \le m-1$ $GV_i \cap GV_{i+1} \neq \varnothing$, and therefore we have

$$\dim E_x^f = \dim E_{x_1}^f = \cdots = \dim E_{x_{m-1}}^f = \dim E_y^f.$$

This completes the proof. □

Definition 3.5 (Finite-dimensional G–moduli problem) Let G be an oriented compact Lie group. A *finite-dimensional G–moduli problem* is a triple (B, E, S) with the following properties:

- B is a compact smooth manifold (without boundary) on which G acts smoothly.
- E is a G–vector bundle over B.
- $S\colon B \to E$ is a G–equivariant smooth section.

A G–moduli problem (B, E, S) is *oriented* if B and E are oriented and G acts on B and E by orientation-preserving diffeomorphisms.

In order to make our notation manageable, from now on we assume that $B_{(H)}$ is connected for each closed subgroup H of G. In the general case we may consider it component by component. From Proposition 3.3, for any $x, y \in B_{(H)}$ we have $\dim E_x^f = \dim E_y^f$, ie the dimension of E_x^f is independent of the choice of $x \in B_{(H)}$. Assume that $\operatorname{rank} E = k$ and $\dim E_x^f = l$. Let $\mathcal{F} = \{E_x^f\}_{x \in B_{(H)}}$. Then the collection $\mathcal{F}$ is an G–invariant subspace of $E|_{B_{(H)}}$. Using the local trivialization of $E|_{B_{(H)}}$, we get that for every $x \in B_{(H)}$, there exists a neighborhood U of x in $B_{(H)}$ and a trivialization

$$\phi_U\colon E|_U \to U \times \mathbb{R}^k. \tag{3-8}$$

Moreover, the restriction of ϕ_U on $\mathcal{F}|_U$ induces a map

$$\widetilde{\phi}_U\colon \mathcal{F}|_U \to U \times R^l \subset U \times \mathbb{R}^k, \tag{3-9}$$

which gives a local trivialization of $\mathcal{F}$ over U. For any pair of trivializations ϕ_U and ϕ_V of $E|_{B_{(H)}}$, we have the smooth transition functions

$$g_{UV}\colon U \cap V \to \mathrm{GL}(\mathbb{R}; k) \tag{3-10}$$

given by $g_{UV}(x) = (\phi_U \circ \phi_V^{-1})|_{\{x\} \times \mathbb{R}^k}$ and satisfying the cocycle condition

$$g_{UV} \cdot g_{VW} = g_{UW}, \quad (U \cap V \cap W \neq \varnothing). \tag{3-11}$$

For any $x \in U \cap V$, the fiber E_x is a G_x–vector space. For each $e \in E_x$, under the trivialization ϕ_U we get $\phi_U(e) \in \mathbb{R}^k$, and similarly, under the trivialization ϕ_V,

we have $\phi_V(e) \in \mathbb{R}^k$. Consider the action of $g \in G_x$ on e. Under the different trivializations ϕ_U and ϕ_V we get

$$\phi_U^{-1} \cdot T_U \cdot \phi_U(e) = ge, \tag{3-12}$$

$$\phi_V^{-1} \cdot T_V \cdot \phi_V(e) = ge, \tag{3-13}$$

where $T_U := (\phi_U \cdot g \cdot \phi_U^{-1})|_x$ and $T_V := (\phi_V \cdot g \cdot \phi_V^{-1})|_x$.

According to (3-12) and (3-13), we get

$$\phi_U \circ \phi_V^{-1} \cdot T_V \cdot \phi_V(e) = T_U \cdot \phi_U(e). \tag{3-14}$$

Since $g_{UV} = \phi_U \circ \phi_V^{-1}$ and $\phi_U(e) = g_{UV}(x) \circ \phi_V(e)$, (3-14) is equivalent to

$$g_{UV}(x) \circ T_V \circ \phi_V(e) = T_U \circ g_{UV}(x) \circ \phi_V(e). \tag{3-15}$$

This implies that the action of G_x on E_x is independent of the trivialization. Thus g_{UV} induces a transition function

$$\widetilde{g}_{UV} \colon U \cap V \to \mathrm{GL}(\mathbb{R}; l) \tag{3-16}$$

given by $\widetilde{g}_{UV}(x) = (\widetilde{\phi}_U \circ \widetilde{\phi}_V^{-1})|_{\{x\}\times\mathbb{R}^l}$. The cocycle condition of transition functions $\{\widetilde{g}_{UV}\}$ is determined by (3-11). This implies that the collection $\mathcal{F}$ with the cocycle $\{\widetilde{g}_{UV}\}$ forms a subbundle of $E|_{B_{(H)}}$, denoted by $\mathcal{E}_H$. Since the projection $\pi \colon \mathcal{E}_H \to B_{(H)}$ is G–equivariant and for any $g \in G$ and $x \in B_{(H)}$ the action

$$g \colon E_x^f \to E_{gx}^f \tag{3-17}$$

is a homomorphism of vector spaces, $\mathcal{E}_H$ is a G–equivariant subbundle. Observe that if the section $S \colon B \to E$ is equivariant, for any $x \in B$ and $g \in G_x$ we have

$$g(S(x)) = S(gx) = S(x).$$

Then $S(x)$ is contained in the G_x–fixed subspace $E_x^f \subset E_x$. Therefore $S_H := S|_{B_{(H)}}$ is an equivariant smooth section of $\mathcal{E}_H$; moreover we say that the triple $(B_{(H)}, \mathcal{E}_H, S_H)$ is the *fixed subbundle* of $E|_{B_{(H)}}$ with the induced G–equivariant smooth section.

Definition 3.6 (Partition of G–moduli problem) The family of the fixed subbundles

$$\{(B_{(H)}, \mathcal{E}_H, S_H) \mid H < G\}$$

is called the *partition of* (B, E, S). Also we write

$$(B, E, S) = \bigsqcup_{(H)} (B_{(H)}, \mathcal{E}_H, S_H),$$

where (H) runs over the all isotropy classes.

Define $\mathcal{O}_H$ be the quotient bundle of $\mathcal{E}_H$: it is also a G–equivariant bundle and we have a direct sum decomposition of G–vector bundles

$$E|_{B_{(H)}} = \mathcal{E}_H \oplus \mathcal{O}_H.$$

In particular, we say that the vector bundle $\mathfrak{o}_H\colon \mathcal{O}_H \to B_{(H)}$ is the *obstruction bundle* of $E|_{B_{(H)}}$.

Definition 3.7 (Obstruction system of G–moduli problem) The family of the obstruction bundles

$$\{(B_{(H)}, \mathcal{O}_H, \mathfrak{o}_H) \mid H < G\}$$

is called the *obstruction system* of (B, E, S).

Definition 3.8 (Coindex of G–vector bundle) The *coindex* of G–vector bundle $\pi\colon E \to B$ is defined to be the integer

$$\operatorname{coind}(B, E) = \max_{H<G, H\neq e} \{\operatorname{codim} B_{(H)} - \operatorname{rank} \mathcal{O}_H\},$$

where e is the identity of G. The coindex is uniquely determined by the G–actions on B and E.

Suppose that the G–moduli problem (B, E, S) is oriented. Then the orientation on E determines an orientation on each fiber E_x. In particular, the induced orientation on E_x is preserved by the G_x–action on E_x. Let $E_x^m = E_x/E_x^f$ be the quotient space of E_x^f, ie the moving part under the G_x–action. Then each fiber E_x can be decomposed into the direct sum of G_x–subspaces as

$$E_x = E_x^f \oplus E_x^m. \tag{3-18}$$

In fact, the moving subspace E_x^m is the fiber of obstruction bundle $\mathcal{O}_H$ at $x \in B_{(H)}$. The orientation on E_x induces the orientations on E_x^f and E_x^m, which are preserved by the G_x–action. Therefore we assign to each fiber of fixed subbundle $\mathcal{E}_H$ an induced orientation. If this induced orientation is smooth, then $\mathcal{E}_H$ is *oriented*. In particular, the obstruction bundle

$$\mathfrak{o}_H\colon \mathcal{O}_H \to B_{(H)}$$

is *oriented* if and only if $\mathcal{E}_H$ is oriented. We say that the obstruction system of an oriented G–moduli problem is *oriented* if each obstruction bundle is oriented. Next we consider the transversality of the partition of G–moduli problem. We say that the partition $\{(B_{(H)}, \mathcal{E}_H, S_H) \mid H < G\}$ is *transversal* if for each H of G the section S_H is transverse to the zero section of $\mathcal{E}_H$.

Proposition 3.9 *Let (B, E, S) be a G–moduli problem. If $S\colon B \to E$ is in general position with respect to the zero section over B, then the partition*

$$\{(B_{(H)}, \mathcal{E}_H, S_H) \mid H < G\}$$

is transversal.

Proof Notice that E is a G–vector bundle over B, and therefore B can be considered as an embedded G–invariant closed submanifold in E. Given any closed subgroup H of G, let $J \in (H)$. For any $b \in B$ assume that $J \subset G_b$ and define $(E_b)^J$ as the J–fixed subspace of E_b, ie

$$(E_b)^J = \{e \in E_b \mid ge = e \text{ for all } g \in J\}.$$

Consider the J–fixed submanifold of E:

$$\begin{aligned} E^J &= \{(b,e) \in E \mid g(b,e) = (b,e) \text{ for all } g \in J\} \\ &= \{(b,e) \in E \mid gb = b, ge = e \text{ for all } g \in J\} \\ &= \{(b,e) \in E \mid b \in B^J, e \in (E_b)^J\} \\ &= \bigsqcup_{b \in B^J} (E_b)^J. \end{aligned}$$

Observe that the section $S\colon B \to E$ is in general position with respect to the zero section over B, and hence it admits stratumwise transversality, ie

$$S|_{B_J}\colon B_J \to E^J \tag{3-19}$$

is transverse to the J–fixed submanifold $B^J \subset E^J$. For each $b \in S^{-1}(0) \cap B_J$ we have $G_b = J$ and

$$\begin{aligned} dS(b)(T_b(B_J)) + T_b(B^J) = T_b(E^J) &= T_b(B^J) \oplus (E_b)^J \\ &= T_b(B^J) \oplus (E_b)^{G_b} \qquad (G_b = J) \\ &= T_b(B^J) \oplus E_b^f. \qquad ((E_b)^{G_b} = E_b^f) \end{aligned}$$

It follows that the linear map

$$\delta S(b)\colon T_b(B_J) \xrightarrow{dS(b)} T_b(B^J) \oplus E_b^f \xrightarrow{\text{proj}} E_b^f \tag{3-20}$$

is surjective. Consider the H–fixed subbundle $(B_{(H)}, \mathcal{E}_H, S_H)$. To verify that the section

$$S_H\colon B_{(H)} \to \mathcal{E}_H \tag{3-21}$$

is transversal to the zero section, we only need to show that for each $b \in S^{-1}(0) \cap B_{(H)}$ the vertical differential

$$\delta S(b)\colon T_b B_{(H)} \xrightarrow{dS(b)} T_b(\mathcal{E}_H) \xrightarrow{\text{proj}} E_b^f \tag{3-22}$$

is surjective. Note that $B_{(H)} = \bigsqcup_{J \in (H)} B_J$. For any $b \in S^{-1}(0) \cap B_{(H)}$ there exists a B_J such that $b \in S^{-1}(0) \cap B_J$. The stratumwise transversality of S implies that the map (3-20) is surjective. Observe that $T_b(B_J)$ is a tangent subspace of $T_b B_{(H)}$, and therefore the vertical differential

$$\delta S(b)\colon T_b B_{(H)} \to E_b^f$$

is surjective. It follows that for any $H \subset G$ the section $S_H\colon B_{(H)} \to \mathcal{E}_H$ is transverse to the zero section; moreover, the partition $\{(B_{(H)}, \mathcal{E}_H, S_H) \mid H < G\}$ is transversal. □

4 Invariant Euler cycle of a G–moduli problem

Proof of Theorem 1.1 Our first goal is to show that there exists a G–equivariant perturbation $P\colon B \to E$ supported in a G–invariant open neighborhood of $S^{-1}(0)$ such that $S + P$ is in general position with respect to the zero section over B. The idea of the proof is canonical. We can reduce the problem to the local situation and construct a local equivariant perturbation. Then using the G–invariant partition of unity we can glue those local perturbations to get a global one.

For any $x \in S^{-1}(0)$, let S_x be the slice at x. From the differential slice theorem, there exists a triple (U, ϕ, G_x) satisfying the following properties:

(1) $U \subset S_x$ is a G_x–invariant open neighborhood of zero in the G_x–vector space S_x.

(2) $\phi\colon U \to B$ is a G_x–equivariant embedding such that $x = \phi(0)$.

(3) ϕ induces a G–equivariant diffeomorphism form $G \times_{G_x} U$ onto a G–invariant open neighborhood of x in B, denoted by W, via

$$\Phi\colon [g, y] \mapsto g \cdot \phi(y),$$

where $[g, y] \in G \times_{G_x} U$ is the equivalence class determined by equivalence relation

$$(g, y) \sim (h^{-1} \cdot g, h \cdot y) \quad \text{for all } h \in G_x.$$

Since $S^{-1}(0) \subset B$ is compact we can choose finitely many points $x_i \in S^{-1}(0)$ $(0 \le i \le q)$ with triples (U_i, ϕ_i, H_i) and induced maps Φ_i such that

$$S^{-1}(0) \subset \bigcup_{i=0}^{q} W_i,$$

where H_i is the isotropy subgroup of x_i and W_i is the G–invariant open neighborhood of x_i in B determined by the image of Φ_i. Assume that E_i is the fiber of E at x_i. Since U_i is a contractible neighborhood of zero in S_i there exists an H_i–equivariant trivialization of the pullback bundle $\phi_i^* E = U_i \times E_i$.

Given that U_i and E_i are H_i–vector spaces, the space of H_i–equivariant smooth maps $\mathcal{C}^\infty_{H_i}(U_i, E_i)$ is a $\mathcal{C}^\infty_{H_i}(U_i)$–module with finitely many polynomial generators. Suppose that $F_1, F_2, \dots, F_{r_i}$ are the generators of $\mathcal{C}^\infty_{H_i}(U_i, E_i)$. Since a G–equivariant map on W_i is uniquely determined by its restriction to U_i, the local section $S|_{W_i}$ is uniquely determined by an H_i–equivariant map $\widetilde{S}_i \in \mathcal{C}^\infty_{H_i}(U_i, E_i)$. There exists a unique set of H_i–invariant smooth functions $\boldsymbol{h} = (h_1, \dots, h_{r_i}) \in \mathcal{C}^\infty_{H_i}(U_i)^{r_i}$ such that

$$\widetilde{S}_i = \sum_{j=1}^{r_i} h_j F_j. \tag{4-1}$$

The graph map of $\widetilde{S}_i$ is

$$\Gamma(\widetilde{S}_i)\colon U_i \to U_i \times \mathbb{R}^{r_i}, \quad x \mapsto (x, \boldsymbol{h}(x)) \tag{4-2}$$

and the universal variety is

$$\mathcal{E}_i = \left\{ (x,t) \in U_i \times \mathbb{R}^{r_i} \;\middle|\; \sum_{j=1}^{r_i} t_j F_j(x) = 0 \right\}.$$

From Definition 2.15, $\widetilde{S}_i$ is in general position if and only if (4-2) is transverse to $\mathcal{E}_i$ (every stratum of $\mathcal{E}_i$) in $U_i \times \mathbb{R}^{r_i}$. Given any $\boldsymbol{c} = (c_1, \dots, c_{r_i}) \in \mathbb{R}^{r_i}$ we can make a perturbation of the graph map (4-2) by setting

$$x \mapsto (x, h_1(x) + c_1, \dots, h_{r_i}(x) + c_{r_i}). \tag{4-3}$$

Since the set of points $\boldsymbol{c} \in \mathbb{R}^{r_i}$ such that the map (4-3) is transversal to $\mathcal{E}_i$ is dense in $\mathbb{R}^{r_i}$ we can choose H_i–invariant functions $\boldsymbol{l}_i = (l_1, \dots, l_{r_i}) \in \mathcal{C}^\infty_{H_i}(U_i)^{r_i}$ such that the map

$$x \mapsto (x, h_1(x) + l_1(x), \dots, h_{r_i}(x) + l_{r_i}(x)) \tag{4-4}$$

is transverse to $\mathcal{E}_i$ in $U_i \times \mathbb{R}^{r_i}$. Let

$$\sigma_i = \sum_{j=1}^{r_i} l_j F_j. \tag{4-5}$$

Then $\widetilde{S}_i + \sigma_i$ is in general position over U_i. Furthermore, σ_i determines a unique G–equivariant local section $P_i\colon W_i \to E$ such that $S|_{W_i} + P_i$ is in general position. Let $W_{q+1} = B - S^{-1}(0)$; then W_{q+1} is a G–invariant open subset since $S^{-1}(0)$ is closed and G–invariant. Notice that $W_0, W_1, \ldots, W_{q+1}$ form a G–invariant open covering of B, there exists a G–invariant partition of unity on B, ie there are G–invariant smooth functions $\chi_j\colon B \to [0,1]$, $0 \le j \le q+1$ such that

$$\operatorname{supp}(\chi_j) \subset W_j, \quad \sum_{j=1}^{q+1} \chi_j(x) = 1 \quad \text{for all } x \in B.$$

Let $P = \sum_{j=0}^{q} \chi_j P_j$; then P is supported in $\bigcup_{i=0}^{q} W_i$, which is a G–invariant open neighborhood of $S^{-1}(0)$. According to the openness and the density of the set of the smooth equivariant sections that are in general position, via the choice of

$$(\boldsymbol{l}_0, \ldots, \boldsymbol{l}_q) \in \prod_{i=0}^{q} \mathcal{C}^{\infty}_{H_i}(U_i)^{r_i}$$

we can make $S + P$ be in general position with respect to the zero section over B.

We are now in a position to verify that the zero locus $(S+P)^{-1}(0) \subset B$ represents a homology class in $H_{n-k}(B;\mathbb{Z})$. Note that E is a G–vector bundle over B; therefore B can be considered as an embedded G–invariant submanifold of E. For simplicity, let $\widehat{S} = S + P$. Note that $\widehat{S}\colon B \to E$ is a G–equivariant smooth section, which is in general position with respect to the zero section. Hence, from the result of Bierstone [1, Proposition 6.5], the zero locus $X = \widehat{S}^{-1}(0) \subset B$ is a compact Whitney object with G–invariant submanifolds as its strata. From Proposition 3.9, for each closed subgroup H of G the section $\widehat{S}_H\colon B_{(H)} \to \mathcal{E}_H$ is transverse to the zero section of $\mathcal{E}_H$. Thus we see that $\widehat{S}_H^{-1}(0) = X \cap B_{(H)}$ is a G–invariant submanifold of dimension $r_H := \dim B_{(H)} - \operatorname{rank} \mathcal{E}_H$. Let $X_H = \widehat{S}_H^{-1}(0)$; then

$$X = \bigsqcup_{H<G} X_H.$$

In particular, if $H = e$ is trivial subgroup of G, then $B_{(e)} = B_e$ is an open subset of B so that B_e is oriented and $\dim B_{(e)} = \dim B$. Furthermore, the orientations on $B_{(e)}$ and E determines an orientation on X_e, ie X_e is an oriented submanifold with dimension $r_e = n - k$. Note that the coindex of (B, E) satisfies $\operatorname{coind}(B, E) > 1$:

we obtain $r_H \leq n-k-2$ when $H \neq e$. At last we get that $X \subset B$ is an oriented compact G–invariant Whitney object with dimension $n-k$; especially, there is no codimension-one stratum, thus the cycle condition is automatic. Therefore X yields a G–invariant $(n-k)$–geometric cycle ξ_X; moreover, through the representation map (2-4) we get a homology class $[\xi_X] \in H_{n-k}(B;\mathbb{Z})$.

Finally, we have to show that homology class $[\xi_X]$ is independent of the choice of P. To prove such independence, we only need to verify that different equivariant perturbations of section S yield the G–invariant $(n-k)$–geometric cycles that are cobordant. In this step we need the following lemma which is a relative version of Theorem 2.22 and we give its proof at the end of this section for the completeness.

Lemma 4.1 *Let $\pi\colon E \to B$ be a G–vector bundle, $S\colon B \to E$ be a G–equivariant smooth section and $K \subset B$ be a G–invariant closed compact subset. If S is in general position with respect to the zero section over K, then there exists a G–equivariant smooth section $\widetilde{S}$ such that $\widetilde{S}$ is in general position with respect to the zero section over B, and the restriction of $\widetilde{S}$ on K is equivalent to S, ie $\widetilde{S}|_K = S|_K$.*

Suppose that S_0 and S_1 are two G–equivariant smooth sections that are in general position. Let $X_0 = S_0^{-1}(0)$ and $X_1 = S_1^{-1}(0)$. Then we get two G–invariant $(n-k)$–geometric cycles, denoted by ξ_0 and ξ_1, such that $|\xi_0| = X_0$ and $|\xi_1| = X_1$. Let G act trivially on $\mathbb{R}$; then $E \times \mathbb{R}$ and $B \times \mathbb{R}$ are two G–spaces and we can construct a new G–vector bundle of rank k as follows:

$$\begin{array}{ccc} \mathbb{R}^k & \longrightarrow & E \times \mathbb{R} \\ & & \big\downarrow \pi \\ & & B \times \mathbb{R} \end{array}$$

Define a section of the above G–vector bundle,

$$S\colon B \times \mathbb{R} \to E \times \mathbb{R}, \quad (x,t) \mapsto (1-t)S_0(x) + tS_1(x). \tag{4-6}$$

Clearly S is a G–equivariant smooth section and $S(x,0) = S_0(x)$, $S(x,1) = S_1(x)$. Note that S_0 and S_1 are in general position over B, hence S is in general position over the compact and closed subset $K = K_0 \cup K_1$ of $B \times \mathbb{R}$, where $K_0 = X_0 \times \{0\}$ and $K_1 = X_1 \times \{1\}$. By the above lemma we can construct a G–equivariant smooth section of $E \times \mathbb{R}$, denoted by $\widetilde{S}$, which is in general position and $\widetilde{S}|_K = S|_K$. Let $X = \widetilde{S}^{-1}(0) \cap (B \times [0,1])$. Then X is an oriented $(n-k+1)$–dimensional compact Whitney object; moreover, X yields a $(n-k+1)$–geometric chain η such that $|\eta| = X \subset B \times [0,1]$. From the equivariant isotopy theorem (cf [1, Theorem 1.5]) there

exists a $\delta > 0$ and an equivariant homeomorphism

$$\Upsilon_0\colon B \times (-\delta, \delta) \to B \times (-\delta, \delta) \tag{4-7}$$

covering the identity map, such that the restriction $\Upsilon_0|_{B\times\{0\}}$ is the identity map and

$$\Upsilon_0((\widetilde{S}|_{B\times(-\delta,\delta)})^{-1}(0)) = S_0^{-1}(0) \times (-\delta, \delta) = |\xi_0| \times (-\delta, \delta).$$

Similarly, there exists a $\epsilon > 0$ and an equivariant homeomorphism

$$\Upsilon_1\colon B \times (1-\epsilon, 1+\epsilon) \to B \times (1-\epsilon, 1+\epsilon) \tag{4-8}$$

such that $\Upsilon_1|_{B\times\{1\}}$ is the identity map and

$$\Upsilon_1((\widetilde{S}|_{B\times(1-\epsilon,1+\epsilon)})^{-1}(0)) = S_1^{-1}(0) \times (1-\epsilon, 1+\epsilon) = |\xi_1| \times (1-\epsilon, 1+\epsilon).$$

Let $\theta = \min\{\delta, \epsilon\}$. Then we have

(1) $|\eta| \cap B \times [0, \theta) = |\xi_0| \times [0, \theta)$,

(2) $|\eta| \cap B \times (1-\theta, 1] = |\xi_1| \times (1-\theta, 1]$,

(3) $\partial\eta = \xi_1 \times \{1\} - \xi_0 \times \{0\}$.

Thus ξ_0 and ξ_1 are cobordant, and represent the same homology class in $H_{n-k}(B; \mathbb{Z})$. This completes the proof. □

Proof of Lemma 4.1 Since B is a proper G–manifold, for every point $x \in B$ there exists a G–invariant open neighborhood of x, denoted by $U_x \subset B$. Clearly K has a G–invariant open covering

$$K \subset \bigcup_{x\in K} U_x.$$

Note that K is compact: there exist finitely many points $x_1, \dots, x_l$ of K such that $K \subset \bigcup_{i=1}^{l} U_i$, where $U_i = U_{x_i}$. Let $U_K = \bigcup_{i=1}^{l} U_i$. Then U_K is a G–invariant open neighborhood of K. As K is G–invariant and closed, $U_0 := B - K$ is a G–invariant open subset of B. It follows that $U_0, U_1, \dots, U_l$ forms a finite open covering of B with G–invariant open subsets. Using the G–invariant partition of unity on a proper G–manifold, we have the G–invariant smooth functions

$$\chi_i\colon B \to [0, 1], \quad 0 \le i \le l$$

such that $\operatorname{supp}(\chi_i) \subset U_i$ and $\sum_{i=0}^{l} \chi_i(x) = 1$ for any x of B. Furthermore, we get

$$\chi_0(x) = \begin{cases} 0 & x \in K, \\ 1 & x \in B - U_K, \end{cases} \qquad \sum_{i=1}^{l} \chi_i(x) = \begin{cases} 1 & x \in K, \\ 0 & x \in B - U_K. \end{cases}$$

Assume that $\rho_0 = \chi_0$ and $\rho_1 = \sum_{i=1}^{l} \chi_i$. For any G–equivariant smooth section S' that is in general position with respect to the zero section over B, let $\widetilde{S} = \rho_0 S' + \rho_1 S$. Then $\widetilde{S}$ is a G–equivariant smooth section. In particular, $\widetilde{S}$ is in general position over the invariant closed subset $K \cup (B - U_K)$ and $\widetilde{S}|_K = S|_K$. In fact, it is in general position over an invariant neighborhood of $K \cup (B - U_K)$ since a smooth equivariant map in general position at a point implies that it is in general position over an invariant neighborhood of this point (cf [1, Lemma 6.2 and Proposition 6.3]). By density we can choose S' such that $\widetilde{S}$ is in general position over $U_K - K$. □

Example 4.2 Given three coprime integers p_0, p_1 and p_2, consider

$$B = S^5 = \left\{ (z_0, z_1, z_2) \in \mathbb{C}^3 \;\middle|\; \sum_{i=0}^{2} |z_i|^2 = 1 \right\}.$$

Let $G = S^1$ act on B by

$$\lambda(z_0, z_1, z_2) = (\lambda^{p_0} z_0, \lambda^{p_1} z_1, \lambda^{p_2} z_2) \quad \text{for all } \lambda \in S^1.$$

Then the singular strata of the orbit type stratification of B are

$$B_0 := \{b \in B \mid G_b = \mathbb{Z}_{p_0}\} \cong S^1/\mathbb{Z}_{p_0},$$
$$B_1 := \{b \in B \mid G_b = \mathbb{Z}_{p_1}\} \cong S^1/\mathbb{Z}_{p_1},$$
$$B_2 := \{b \in B \mid G_b = \mathbb{Z}_{p_2}\} \cong S^1/\mathbb{Z}_{p_2}.$$

Let $\pi\colon E \to B$ be a S^1–equivariant plane bundle. Note that the codimensions of B_0, B_1 and B_2 are 4, and moreover, the rank of obstruction bundle over each singular stratum is smaller than or equivalent to 2, and by the above theorem we may obtain the invariant Euler cycle via equivariant perturbation.

Example 4.3 (An application in symplectic geometry) Theorem 1.1 can be applied to the study of symplectic geometry. Let (M, ω, J) be a compact spherically positive symplectic manifold, $L \subset M$ a relatively spin Lagrangian submanifold and let $\beta \in H_2(M, L; \mathbb{Z})$. Fukaya, Oh, Ohta and Ono [7] studied the moduli space of stable $(k+1)$–marked pseudo-holomorphic discs with respect to L and β. They proved that there exists an oriented Kuranishi structure on the moduli space $\mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \ldots, P_k)$[2] and they developed a Lagrangian Floer theory over $\mathbb{Z}$ coefficients [7, Theorem 1.1].

To develop a Lagrangian Floer theory over $\mathbb{Z}$, the main technique is to construct a single-valued perturbation of the moduli space, which can give rise to a virtual moduli

[2]For the details of the definition of Lagrangian Floer moduli space and its Kuranishi structure refer to the paper [7, Sections 2 and 11].

cycle over $\mathbb{Z}$. Using the notion of the sheaf of groups and the notion of normal bundles in the sense of stacks the authors constructed a suitable single-valued perturbation of the Kuranishi structure (see [7, Theorem 3.1]). Applying this single-valued perturbation to the concrete Lagrangian Floer moduli space of the spherically positive symplectic manifold, they constructed a Lagrangian Floer theory over the integers.

In fact, for any point $p \in \mathcal{M}^{\text{main}}_{k+1}(\beta; P_1, \dots, P_k)$ the Kuranishi chart associated to p is an oriented Γ_p–equivariant moduli problem (V_p, E_p, S_p) with coindex greater than 1, where Γ_p is a finite group (see [7, Proposition 12.1]). More precisely, E_p is a Γ_p–equivariant vector bundle over V_p and S_p: $V_p \to E_p$ is a smooth Γ_p–equivariant section. Note that $\text{coind}(V_p, E_p) > 1$; therefore by Theorem 1.1, we may construct a Γ_p–equivariant perturbation of S_p such that the perturbed section, denoted by S'_p, is in general position with respect to the zero section of E_p over V_p. Furthermore, we can construct a global perturbation of the moduli space by gluing together those local equivariant perturbations over Kuranishi charts in a compatible way. In particular, this global perturbation may yield a geometric cycle with dimension equal to the virtual dimension of the moduli space.

5 Transversal intersection of S^1–moduli problems

In this section we study the intersection problem of S^1–moduli problems.

Definition 5.1 (Goresky [9]) Let X be a fixed Whitney object. Assume that V and W are two substratified objects in X. We say V is *transverse* to W provided that for every stratum $R \subset V$ and every stratum $S \subset W$ satisfy

(1) $R \cap S = \varnothing$, or

(2) R is transverse to S in the stratum $X_i \subset X$ that contains R and S.

For a compact smooth n–manifold B on which $G = S^1$ acts, there exists a canonical Whitney stratification on B determined by orbit types. For simplicity we assume that the G–action is semi-free and the G–fixed loci is connected. With this assumption there exist only two orbit types. Let

$$B_0 = \{x \in B \mid G_x = e\}, \quad B_1 = \{x \in B \mid G_x = G\}.$$

Then $B = B_0 \sqcup B_1$. Let (B, E_α, S_α) and (B, E_β, S_β) be two oriented G–moduli problems such that $\text{rank } E_\alpha = k$ and $\text{rank } E_\beta = n - k$. Assume that S_α and S_β are in general position. Then the associated moduli spaces

$$M_\alpha = \{x \in B \mid S_\alpha(x) = 0\}, \quad M_\beta = \{x \in B \mid S_\beta(x) = 0\}$$

are Whitney substratified objects with G–invariant strata in B. Let

$$\begin{aligned} M_{\alpha,0} &= M_\alpha \cap B_0, \quad M_{\alpha,1} = M_\alpha \cap B_1, \\ M_{\beta,0} &= M_\beta \cap B_0, \quad M_{\beta,1} = M_\beta \cap B_1 \end{aligned}.$$

Then the Whitney stratifications induced by orbit types on M_α and M_β are

$$M_\alpha = M_{\alpha,0} \sqcup M_{\alpha,1}, \qquad M_\beta = M_{\beta,0} \sqcup M_{\beta,1}.$$

Denote the partitions of (B, E_α, S_α) and (B, E_β, S_β) by

$$\begin{aligned} (B, E_\alpha, S_\alpha) &= (B_0, \mathcal{E}_{\alpha,0}, S_{\alpha,0}) \sqcup (B_1, \mathcal{E}_{\alpha,G}, S_{\alpha,G}), \\ (B, E_\beta, S_\beta) &= (B_0, \mathcal{E}_{\beta,0}, S_{\beta,0}) \sqcup (B_1, \mathcal{E}_{\beta,G}, S_{\beta,G}). \end{aligned}$$

Suppose that M_α is transverse to M_β. By Definition 5.1 we get that $M_{\alpha,0}$ is transverse to $M_{\beta,0}$ in B_0 and $M_{\alpha,1}$ is transverse to $M_{\beta,1}$ in B_1. Since $B_0 \subset B$ is an open subset, $\dim B_0 = n$. Observe that $M_{\alpha,0}$ is $(n-k)$–dimensional and $M_{\beta,0}$ is k–dimensional. If the transversal intersection $M_{\alpha,0} \cap M_{\beta,0}$ is nontrivial, ie $M_{\alpha,0} \cap M_{\beta,0} \neq \varnothing$, then $M_{\alpha,0} \cap M_{\beta,0}$ is an invariant submanifold of dimension 0. From another aspect, for any $z \in M_{\alpha,0} \cap M_{\beta,0}$ the orbit $G(z)$ belongs to $M_{\alpha,0} \cap M_{\beta,0}$ since $M_{\alpha,0} \cap M_{\beta,0}$ is G–invariant. Note that the isotropy subgroup $G_z = e$; $G(z)$ is isomorphic to $G = S^1$. So

$$\dim(M_{\alpha,0} \cap M_{\beta,0}) \geq \dim G(z) = 1$$

and this leads to a contradiction with $\dim(M_{\alpha,0} \cap M_{\beta,0}) = 0$. Hence $M_{\alpha,0} \cap M_{\beta,0} = \varnothing$ and we have that

$$Z := M_\alpha \cap M_\beta = M_{\alpha,1} \cap M_{\beta,1}$$

is a submanifold of B_1.

From now on, we assume that the G–fixed subbundles $\mathcal{E}_{\alpha,G} \to B_1$ and $\mathcal{E}_{\beta,G} \to B_1$ are oriented. Consider the obstruction bundles $\mathfrak{o}_\alpha\colon \mathcal{O}_{\alpha,1} \to B_1$ and $\mathfrak{o}_\beta\colon \mathcal{O}_{\beta,1} \to B_1$. Suppose that

$$\operatorname{rank} \mathcal{O}_{\alpha,1} = k - n_\alpha \quad \text{and} \quad \operatorname{rank} \mathcal{O}_{\beta,1} = n - k - n_\beta,$$

where n_α and n_β are the ranks of the G–fixed subbundles $\mathcal{E}_{\alpha,G}$ and $\mathcal{E}_{\beta,G}$ respectively. The orientations of $\mathcal{E}_{\alpha,G}$ and $\mathcal{E}_{\beta,G}$ induce the orientations of the obstruction bundles.

Let $\dim B_1 = n_1$. Observe that $S_\alpha\colon B \to E_\alpha$ is in general position, the section

$$S_{\alpha,G} = S_\alpha|_{B_1}\colon B_1 \to \mathcal{E}_{\alpha,G}$$

is transverse to the zero section of the G–fixed bundle $\mathcal{E}_{\alpha,G}$ and the zero locus

$$M_{\alpha,1} = S_{\alpha,G}^{-1}(0) \subset B_1$$

is an oriented submanifold with dimension $n_1 - n_\alpha$. Similarly,

$$M_{\beta,1} = S_{\beta,G}^{-1}(0) \subset B_1$$

is an oriented $(n_1 - n_\beta)$–dimensional submanifold. Note that M_α and M_β are Whitney substratified objects; however, on the level of sets the intersection set $Z = M_\alpha \cap M_\beta$ is a submanifold of fixed loci B_1 with dimension $n_1 - n_\alpha - n_\beta$.

Consider the direct sum of (B, E_α, S_α) and (B, E_β, S_β). Let $E = E_\alpha \oplus E_\beta$ and $S = S_\alpha \oplus S_\beta$. Then we get a new oriented G–moduli problem (B, E, S) with $\dim B = \operatorname{rank} E = n$. The associated partition of (B, E, S) is $(B, E, S) = (B_0, \mathcal{E}_0, S_0) \sqcup (B_1, \mathcal{E}_G, S_G)$, ie

$$\begin{array}{ccc} G \curvearrowright \mathcal{E}_0 & & G \curvearrowright \mathcal{E}_G \\ \downarrow \uparrow s_0 & \qquad & \downarrow \uparrow s_G \\ G \curvearrowright B_0 & & G \curvearrowright B_1 \end{array}$$

where

$$\begin{aligned} \mathcal{E}_0 &= \mathcal{E}_{\alpha,0} \oplus \mathcal{E}_{\beta,0}, \quad & S_0 &= S_{\alpha,0} \oplus S_{\beta,0} \\ \mathcal{E}_G &= \mathcal{E}_{\alpha,G} \oplus \mathcal{E}_{\beta,G}, \quad & S_G &= S_{\alpha,G} \oplus S_{\beta,G}. \end{aligned}$$

The obstruction bundle over B_1 is

$$\mathfrak{o}\colon \mathcal{O}_1 \to B_1,$$

where $\mathcal{O}_1 = \mathcal{O}_{\alpha,1} \oplus \mathcal{O}_{\beta,1}$ and $\mathfrak{o} = \mathfrak{o}_\alpha \oplus \mathfrak{o}_\beta$.

Note that $S_{\alpha,0}$ and $S_{\beta,0}$ are transverse to the zero sections of $\mathcal{E}_{\alpha,0}$ and $\mathcal{E}_{\beta,0}$ respectively and $M_{\alpha,0} = S_{\alpha,0}^{-1}(0)$ intersects with $M_{\beta,0} = S_{\beta,0}^{-1}(0)$ in B_0 transversally. This implies that S_0 is transverse to the zero section of $\mathcal{E}_0$ and we get

$$S_0^{-1}(0) = M_{\alpha,0} \cap M_{\beta,0} = \varnothing.$$

Similarly, we obtain that S_G is transverse to the zero section of $\mathcal{E}_G$ and

$$S_G^{-1}(0) = M_{\alpha,1} \cap M_{\beta,1} = Z.$$

Define

$$\Psi(E_\alpha, E_\beta) = \int_B e(E_\alpha \oplus E_\beta).$$

We call $\Psi(E_\alpha, E_\beta)$ the *intersection number* of G–moduli problems (B, E_α, S_α) and (B, E_β, S_β). Firstly, we consider the nondegenerate case, ie S_α and S_β are transverse to the zero sections and the moduli spaces M_α intersect with M_β in B transversally. In this case $M_\alpha \subset B$ is an oriented G–invariant submanifold of dimension $n - k$, and

$M_\beta \subset B$ is a k–dimensional invariant submanifold. Since M_α is transverse to M_β in B the intersection number is

$$\#(M_\alpha \cdot M_\beta) = \int_B \mathrm{PD}(M_\alpha) \wedge \mathrm{PD}(M_\beta),$$

where $\mathrm{PD}(\cdot)$ is the Poincaré dual. Note that $\mathrm{PD}(M_\alpha) = e(E_\alpha)$ and $\mathrm{PD}(M_\beta) = e(E_\beta)$, hence we get

$$\int_B \mathrm{PD}(M_\alpha) \wedge \mathrm{PD}(M_\beta) = \int_B e(E_\alpha) \wedge e(E_\beta) = \int_B e(E_\alpha \oplus E_\beta).$$

Therefore, in the nondegenerate case the intersection number of G–moduli problems is equivalent to the intersection number of the associated moduli spaces, ie

$$\Psi(E_\alpha, E_\beta) = \#(M_\alpha \cdot M_\beta).$$

In general, the existence of the obstruction system of G–moduli problems implies that the equivariant smooth section that is transverse to the zero section does not always exist. However, the equivariant sections which are in general position are generic, and particularly we have the following.

Theorem 5.2 *Assume that $S_\alpha\colon B \to E_\alpha$ and $S_\beta\colon B \to E_\beta$ are in general position with respect to the zero sections respectively. If the G–moduli space M_α is transverse to the G–moduli space M_β in the sense of Definition 5.1 then*

$$\Psi(E_\alpha, E_\beta) = \int_Z i^*\left(\frac{e_G(\mathcal{O}_1)}{e_G(\mathcal{N}_{B_1/B})}\right),$$

where $Z = M_\alpha \cap M_\beta$, $\mathcal{N}_{B_1/B}$ is the normal bundle of B_1 in B and i^* is the map induced by the inclusion $i\colon Z \hookrightarrow B_1$.

Proof Let $\Theta_G \in \Omega^n_{G,vc}(E)$ be the equivariant Thom form of E. By the definition of equivariant Thom form (cf [15, Theorem 6.4]), the leading component of Θ_G, denoted by

$$\Theta = (\Theta_G)_{[n]} \in \Omega^n_{vc}(E),$$

is a nonequivariant Thom form of E. Denote by $i_0\colon B \to E$ the embedding of B in E as the zero section, then the equivariant Euler class of E is $e_G(E) = i_0^*(\Theta_G)$ and the ordinary one is $e(E) = i_0^*(\Theta)$. Observe that $\dim B = n$. According to the definition of equivariant integral we have

$$\int_B e_G(E) = \int_B i_0^*(\Theta_G) = \int_B i_0^*\big((\Theta_G)_{[n]}\big) = \int_B e(E),$$

and therefore we get

$$\Psi(E_\alpha, E_\beta) = \int_B e_G(E). \tag{5-1}$$

Let $\mathcal{N}_{B_1/B}$ be the normal bundle of B_1 in B. The G–action on $\mathcal{N}_{B_1/B}$ only fixes the zero section B_1. This implies that the normal bundle $\mathcal{N}_{B_1/B}$ has even rank and is orientable. In particular, with a fixed orientation the equivariant Euler class of the normal bundle $e_G(\mathcal{N}_{B_1/B})$ is invertible. Using the Atiyah–Bott–Berline–Vergne localization formula (cf [10, Theorem C.53]) we have

$$\int_B e_G(E) = \int_{B_1} \frac{j^* e_G(E)}{e_G(\mathcal{N}_{B_1/B})}, \tag{5-2}$$

where j^* is the map induced by the inclusion $j\colon B_1 \hookrightarrow B$. Note that $j^* e_G(E) = e_G(j^* E)$ and the pullback bundle $j^* E$ is equivalent to $E|_{B_1}$. From another aspect, $E|_{B_1}$ can split into the direct sum of a G–fixed subbundle and obstruction bundle that are all G–equivariant, ie $E|_{B_1} = \mathcal{E}_G \oplus \mathcal{O}_1$. Using equivariant Chern–Weil theory (cf [11, Chapter 8]), and following the proof of Whitney product formula for the Euler class, we get

$$j^* e_G(E) = e_G(\mathcal{E}_G) \wedge e_G(\mathcal{O}_1). \tag{5-3}$$

Consider the equivariant Euler class $e_G(\mathcal{E}_G)$. Note that the equivariant section

$$S_G\colon B_1 \to \mathcal{E}_G$$

is transverse to the zero section; the zero locus $Z = S_G^{-1}(0) \subset B_1$ is an invariant submanifold. Moreover, the normal bundle of Z in B_1, denoted by $\mathcal{N}_{Z/B_1}$, is isomorphic to $\mathcal{E}_G|_Z$. Assume that $\iota\colon B_1 \hookrightarrow \mathcal{E}_G$ is the embedding of B_1 into $\mathcal{E}_G$ as the zero section. Without loss of generality, we may choose an equivariant Thom form of $\mathcal{E}_G$, denoted by $\Phi_G \in \Omega^{n_\alpha+n_\beta}_{G,vc}(\mathcal{E}_G)$, such that the support of the pullback by S_G

$$S_G^*(\Phi_G) \in \Omega^{n_\alpha+n_\beta}_G(B_1)$$

is contained in an invariant tubular neighborhood of Z in B_1. Let

$$\Phi = (\Phi_G)_{[n_\alpha+n_\beta]} \in \Omega^{n_\alpha+n_\beta}_{vc}(\mathcal{E}_G)$$

be the leading component of Φ_G. Then Φ is a nonequivariant Thom form of $\mathcal{E}_G$. Given any $z \in Z$, let N_z be the fiber of $\mathcal{N}_{Z/B_1}$ at z and $E_{G,z}$ be the fiber of $\mathcal{E}_G$ at z. Because the image of a fiber of $\mathcal{N}_{Z/B_1}$ under S_G is homotopic to a fiber of $\mathcal{E}_G$ we

have

$$\begin{aligned}\int_{N_z} S_G^*(\Phi_G) &= \int_{N_z} S_G^*(\Phi) \quad (\dim N_z = n_\alpha + n_\beta)\\ &= \int_{E_{G,z}} \Phi \qquad (\Phi \text{ is a Thom form})\\ &= 1.\end{aligned}$$

Note that $e_G(\mathcal{E}_G) = \iota^*(\Phi_G)$. The next thing to do is to verify that $\iota^*(\Phi_G)$ is an equivariant Thom form of $\mathcal{N}_{Z/B_1}$. The proof is straightforward since we have

$$\begin{aligned}\int_{N_z} \iota^*(\Phi_G) &= \int_{N_z} \iota^*(\Phi) \qquad (\dim N_z = n_\alpha + n_\beta)\\ &= \int_{N_z} S_G^*(\Phi) \qquad (\iota^*(\Phi) - S_G^*(\Phi) \text{ is } d\text{–exact})\\ &= 1.\end{aligned}$$

Denote by $\mathrm{PD}_G(Z)$ the equivariant Poincaré dual of Z in B_1, which is defined as an equivariant Thom form of the normal bundle $\mathcal{N}_{Z/B_1}$. It follows that

$$e_G(\mathcal{E}_G) = \mathrm{PD}_G(Z). \tag{5-4}$$

Let $i\colon Z \hookrightarrow B_1$ be the inclusion. Combining (5-2), (5-3) and (5-4), we get

$$\int_B e_G(E) = \int_{B_1} \frac{\mathrm{PD}_G(Z) \wedge e_G(\mathcal{O}_1)}{e_G(\mathcal{N}_{B_1/B})} = \int_Z i^*\left(\frac{e_G(\mathcal{O}_1)}{e_G(\mathcal{N}_{B_1/B})}\right). \qquad \square$$

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