Algebraic & Geometric Topology 15 (2015) 371–397

The Ptolemy field of 3–manifold representations

STAVROS GAROUFALIDIS
MATTHIAS GOERNER
CHRISTIAN K ZICKERT

The Ptolemy coordinates for boundary-unipotent $\mathrm{SL}(n,\mathbb{C})$–representations of a 3–manifold group were introduced by Garoufalidis, Thurston and Zickert [10] inspired by the $\mathcal{A}$–coordinates on higher Teichmüller space due to Fock and Goncharov. We define the Ptolemy field of a (generic) $\mathrm{PSL}(2,\mathbb{C})$-representation and prove that it coincides with the trace field of the representation. This gives an efficient algorithm to compute the trace field of a cusped hyperbolic manifold.

57N10; 57M27

1 Introduction

1.1 The Ptolemy coordinates

The Ptolemy coordinates for boundary-unipotent representations of a 3–manifold group in $\mathrm{SL}(n,\mathbb{C})$ were introduced by Garoufalidis, Thurston and Zickert [10], inspired by the $\mathcal{A}$–coordinates on higher Teichmüller space due to Fock and Goncharov [7]. In this paper we will focus primarily on representations in $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$.

Given a topological ideal triangulation $\mathcal{T}$ of an oriented compact 3–manifold M, a *Ptolemy assignment* (for $\mathrm{SL}(2,\mathbb{C})$) is an assignment of a non-zero complex number (called a *Ptolemy coordinate*) to each 1–cell of $\mathcal{T}$ such that, for each simplex, the Ptolemy coordinates assigned to the edges ε_{ij} satisfy the *Ptolemy relation*

(1-1) $$c_{03}c_{12}+c_{01}c_{23}=c_{02}c_{13}.$$

The set of Ptolemy assignments is thus an affine variety $P_2(\mathcal{T})$, which is cut out by homogeneous quadratic polynomials.

We define the Ptolemy field of a boundary-unipotent representation and show that it is isomorphic to the trace field. This gives rise to an efficient algorithm for *exact* computation of the trace field of a hyperbolic manifold.

Published: 23 March 2015 DOI: 10.2140/agt.2015.15.371

1.2 Decorated $\mathrm{SL}(2,\mathbb{C})$–representations

The precise relationship between Ptolemy assignments and representations is given by

$$\{\text{Points in } P_2(\mathcal{T})\} \xleftrightarrow{1-1} \{\text{Natural } (\mathrm{SL}(2,\mathbb{C}), P)\text{–cocycles on } M\}$$
$$\xleftrightarrow{1-1} \{\text{Generically decorated } (\mathrm{SL}(2,\mathbb{C}), P)\text{–representations}\}. \tag{1-2}$$

The concepts are briefly described below, and the correspondences are illustrated in the right image in Figure 3 and in Figure 2. We refer to Section 2 for a summary of our notation. The bijections of (1-2) first appeared in Zickert [14] (in a slightly different form), and were generalized to $\mathrm{SL}(n,\mathbb{C})$–representations by Garoufalidis, Thurston and Zickert [10].

- *Natural cocycle* Labeling of the edges of each truncated simplex by elements in $\mathrm{SL}(2,\mathbb{C})$ satisfying the cocycle condition (the product around each face is 1). The long edges are counter-diagonal, ie of the form
$$\begin{pmatrix} 0 & -x^{-1} \\ x & 0 \end{pmatrix}$$
and the short edges are *nontrivial* elements in P. Identified edges are labeled by the same group element.
- *Decorated representation* A *decoration* of a boundary-parabolic representation ρ is an assignment of a coset gP to each vertex of $\widehat{\widetilde{M}}$ which is equivariant with respect to ρ. A decoration is *generic* if for each edge joining two vertices, the two P–cosets gP, hP are distinct as B–cosets. This condition is equivalent to $\det(ge_1, he_1) \neq 0$. Two decorations are considered equal if they differ by left multiplication by a group element g.

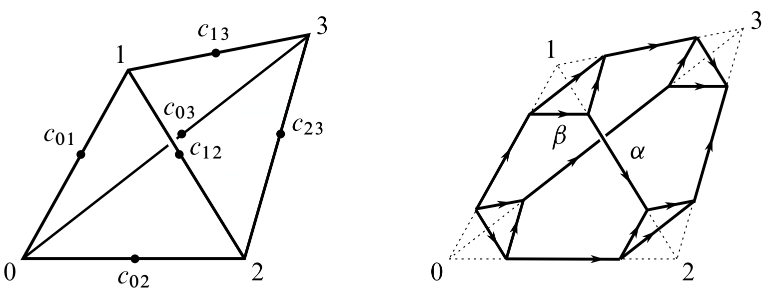

Figure 1: Left: Ptolemy assignment; the Ptolemy relation (1-1) holds. Right: natural cocycle; α is counter-diagonal, $\beta \in P$.

By ignoring the decoration, (1-2) yields a map

$$\mathcal{R}\colon P_2(\mathcal{T}) \to \{(\mathrm{SL}(2,\mathbb{C}), P)\text{–representations}\}/\operatorname{Conj}. \tag{1-3}$$

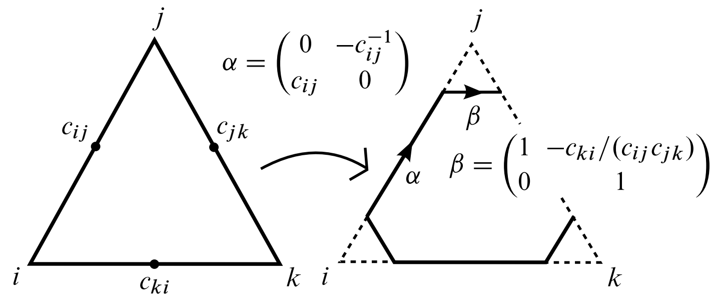

Figure 2: From Ptolemy assignments to natural cocycles

The representation corresponding to a Ptolemy assignment is given explicitly in terms of the natural cocycle.

Remark 1.1 Note that a natural cocycle canonically determines a representation of the edge path groupoid of the triangulation of M by truncated simplices.

Remark 1.2 A decoration of ρ determines a developing map $\widehat{\widetilde{M}} \to \overline{\mathbb{H}}^3$ by straightening the simplices. We shall not need this here. For a discussion of the relationship between decorations and developing maps, see Zickert [14]. For general theory of developing maps, see Dunfield [4].

Remark 1.3 Every boundary-parabolic representation has a decoration, but a representation may have only non-generic decorations. The map $\mathcal{R}$ is thus not surjective in general, and the image depends on the triangulation. However, if the triangulation is sufficiently fine, $\mathcal{R}$ is surjective (see Garoufalidis, Thurston and Zickert [10]). The preimage of a representation depends on the image of the peripheral subgroups (see Proposition 1.10).

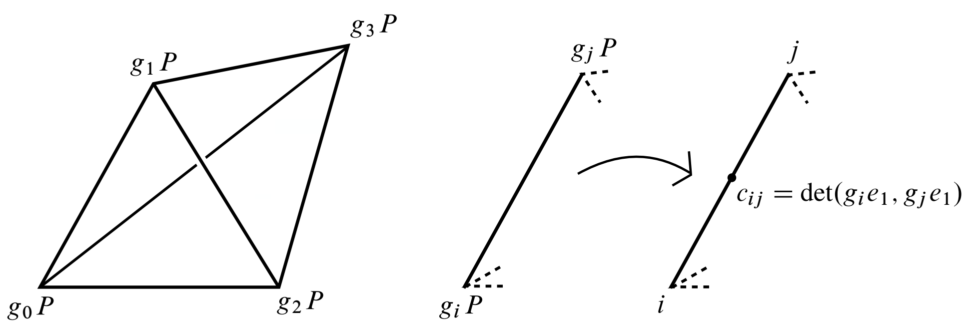

Figure 3: Left: decoration; equivariant assignment of cosets. Right: from decorations to Ptolemy assignments.

1.3 Obstruction classes and $\mathrm{PSL}(2,\mathbb{C})$–representations

There is a subtle distinction between representations in $\mathrm{SL}(2,\mathbb{C})$ versus $\mathrm{PSL}(2,\mathbb{C})$. The geometric representation of a hyperbolic manifold always lifts to an $\mathrm{SL}(2,\mathbb{C})$–representation, but for a one-cusped manifold, no lift is boundary-parabolic (any lift will take a longitude to an element of trace -2; see Calegari [2]).

The obstruction to lifting a boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$–representation to a boundary-parabolic $\mathrm{SL}(2,\mathbb{C})$–representation is a class in $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$. For each such class, there is a Ptolemy variety $P_2^\sigma(\mathcal{T})$, which maps to the set of $\mathrm{PSL}(2,\mathbb{C})$–representations with obstruction class σ. More precisely, $P_2^\sigma(\mathcal{T})$ is defined for each 2–cocycle $\sigma\in Z^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$, and up to canonical isomorphism only depends on the cohomology class of σ. The Ptolemy variety for the trivial cocycle equals $P_2(\mathcal{T})$. The analogue of (1-2) is

$$\{\text{Points in } P_2^\sigma(\mathcal{T})\}\xleftrightarrow{1-1}\left\{\begin{matrix}\text{Lifted natural } (\mathrm{SL}(2,\mathbb{C}),P)\text{–cocycles}\\ \text{with obstruction cocycle } \sigma\end{matrix}\right\} \tag{1-4}$$
$$\twoheadrightarrow\left\{\begin{matrix}\text{Generically decorated } (\mathrm{SL}(2,\mathbb{C}),P)\text{–}\\ \text{representations with obstruction class } \sigma\end{matrix}\right\}.$$

A lifted natural cocycle is defined as above, except that the product along a face is now $\pm I$, where the sign is determined by σ. The right map is no longer a 1–1 correspondence; the preimage of each decorated representation is the choice of lifts, ie parametrized by a cocycle in $Z^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$. We refer to [10] for details. As in (1-3), ignoring the decoration yields a map

$$\mathcal{R}\colon P_2^\sigma(\mathcal{T})\to\left\{\begin{matrix}(\mathrm{PSL}(2,\mathbb{C}),P)\text{–representations}\\ \text{with obstruction class } \sigma\end{matrix}\right\}\Big/\mathrm{Conj}, \tag{1-5}$$

which is explicitly given in terms of the natural cocycle.

Theorem 1.4 (Garoufalidis, Thurston and Zickert [10]) *If M is hyperbolic, and all edges of $\mathcal{T}$ are essential, the geometric representation is in the image of $\mathcal{R}$.*

Remark 1.5 If $\mathcal{T}$ has a non-essential edge, all Ptolemy varieties will be empty. Hence, if $P_2^\sigma(\mathcal{T})$ is non-empty for some σ, and if M is hyperbolic, the geometric representation is detected by the Ptolemy variety of the geometric obstruction class.

1.4 Our results

We view the Ptolemy varieties $P_2^\sigma(\mathcal{T})$ as subsets of an ambient space $\mathbb{C}^e$, with coordinates indexed by the 1–cells of $\mathcal{T}$. Let $T=(\mathbb{C}^*)^v$, with the coordinates indexed by the boundary components of M.

Definition 1.6 The *diagonal action* is the action of T on $P_2^\sigma(\mathcal{T})$, where an element $(x_1,\dots,x_v)\in T$ acts on a Ptolemy assignment by replacing the Ptolemy coordinate c of an edge e with $x_i x_j c$, where x_i and x_j are the coordinates corresponding to the ends of e. Let

$$P_2^\sigma(\mathcal{T})_{\mathrm{red}} = P_2^\sigma(\mathcal{T})/T. \tag{1-6}$$

Definition 1.7 A boundary-parabolic $\mathrm{PSL}(2,\mathbb{C})$–representation is *generic* if it has a generic decoration. It is *boundary-nontrivial* if each peripheral subgroup has nontrivial image.

Remark 1.8 Note that the notion of genericity is with respect to the triangulation. By Theorem 1.4, if all edges of $\mathcal{T}$ are essential (and $\mathcal{T}$ has no interior vertices), the geometric representation of a cusped hyperbolic manifold is always generic and boundary-nontrivial.

Remark 1.9 Note that if M has spherical boundary components (eg if $\mathcal{T}$ is a triangulation of a closed manifold), no representation is boundary-nontrivial.

Proposition 1.10 *The map $\mathcal{R}$ in* (1-5) *factors through $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$, ie we have*

$$\mathcal{R}\colon P_2^\sigma(\mathcal{T})_{\mathrm{red}} \to \left\{\begin{matrix}(\mathrm{PSL}(2,\mathbb{C}),P)\text{–representations}\\ \text{with obstruction class } \sigma\end{matrix}\right\}\Big/\,\mathrm{Conj}. \tag{1-7}$$

The image is the set of generic representations, and the preimage of a generic, boundary-nontrivial representation is finite and parametrized by $H^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$.

Remark 1.11 For the corresponding map from $P_2(\mathcal{T})_{\mathrm{red}}$ to $(\mathrm{SL}(2,\mathbb{C}),P)$–representations, the preimage of a generic boundary-nontrivial representation is a single point.

Remark 1.12 The preimage of a representation which is not boundary-nontrivial is never finite. In fact, its dimension is the number of boundary components that are collapsed. In particular, it follows that if $c\in P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is in a 0–dimensional component (which is not contained in a higher-dimensional component), the image is boundary-nontrivial.

By geometric invariant theory, $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is a variety whose coordinate ring is the ring of invariants $\mathcal{O}^T$ of the coordinate ring $\mathcal{O}$ of $P_2^\sigma(T)$.

Definition 1.13 Let $c\in P_2^\sigma(\mathcal{T})$. The Ptolemy field of c is the field

$$k_c = \mathbb{Q}\big(\{p(c_1,\dots,c_e)\mid p\in\mathcal{O}^T\}\big). \tag{1-8}$$

The Ptolemy field of a generic boundary-nontrivial representation is the Ptolemy field of any preimage under (1-7).

Clearly, the Ptolemy field only depends on the image in $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$. Our main result is the following.

Theorem 1.14 *The Ptolemy field of a boundary-nontrivial, generic, boundary-parabolic representation ρ in $\mathrm{PSL}(2,\mathbb{C})$ or $\mathrm{SL}(2,\mathbb{C})$ is equal to its trace field.*

Remark 1.15 For a cusped hyperbolic 3–manifold the *shape field* is in general smaller than the trace field. The shape field equals the *invariant trace field* (see eg Maclachlan and Reid [12]).

For computations of the Ptolemy field, we need an explicit description of the ring of invariants $\mathcal{O}^T$, or, equivalently, the reduced Ptolemy variety $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$.

Proposition 1.16 *There exist 1–cells $\varepsilon_1, \dots, \varepsilon_v$ of $\mathcal{T}$ such that the reduced Ptolemy variety $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is naturally isomorphic to the subvariety of $P_2^\sigma(\mathcal{T})$ obtained by intersecting with the affine hyperplane $c_{\varepsilon_1} = \cdots = c_{\varepsilon_v} = 1$.*

Corollary 1.17 *Let $c \in P_2^\sigma(\mathcal{T})_{\mathrm{red}}$. Under an isomorphism as in Proposition 1.16, the Ptolemy field of c is the field generated by the Ptolemy coordinates.*

Remark 1.18 A concrete method for selecting 1–cells as in Proposition 1.16 is described in Section 4.3.

Analogues of our results for higher-rank Ptolemy varieties are discussed in Section 6. The analogue of Proposition 1.10 holds for representations that are *boundary-non-degenerate* (see Definition 6.10), and the analogue of Proposition 1.16 leads to a simple computation of the Ptolemy field.

Conjecture 1.19 The Ptolemy field of a boundary-non-degenerate, generic, boundary-unipotent representation ρ in $\mathrm{SL}(n,\mathbb{C})$ or $\mathrm{PSL}(n,\mathbb{C})$ is equal to its trace field.

Remark 1.20 The computation of reduced Ptolemy varieties is remarkably efficient using Magma [1]. For all but a few census manifolds, primary decompositions of the (reduced) Ptolemy varieties $P_2^\sigma(\mathcal{T})$ can be computed in a fraction of a second on a standard laptop. A database can be found at CURVE [5]; see also Falbel, Koseleff and Rouillier [6]. All of our tools have been incorporated into SnapPy [3] by the second author and the Ptolemy fields can be obtained through the command below:

```
>>> from snappy import Manifold
>>> p=Manifold("m019").ptolemy_variety(2,'all')
>>> p.retrieve_solutions().number_field()
... [[x^4 - 2*x^2 - 3*x - 1], [x^4 + x - 1]]
```

The number fields are grouped by obstruction class. In this example, we see that the Ptolemy variety for the nontrivial obstruction class has a component with number field $x^4 + x - 1$, which is the trace field of m019. The above code retrieves a precomputed decomposition of the Ptolemy variety from CURVE [5]. In Sage or SnapPy with Magma installed, you can use `p.compute_solutions().number_field()` to compute the decomposition.

Acknowledgements Stavros Garoufalidis and Christian Zickert were supported in part by NSF grants number DMS-14-06419 and DMS-13-09088, respectively.

2 Notation

2.1 Triangulations

Let M be a compact oriented 3–manifold with (possibly empty) boundary. We refer to the boundary components as *cusps* (although they may not be tori). Let $\widetilde{M}$ be the universal cover of M and let $\widehat{M}$ and $\widehat{\widetilde{M}}$, respectively, be the spaces obtained from M and $\widetilde{M}$ by collapsing each boundary component to a point.

Definition 2.1 A (concrete) *triangulation* of M is an identification of $\widehat{M}$ with a space obtained from a collection of simplices by gluing together pairs of faces by affine homeomorphisms. For each simplex Δ of $\mathcal{T}$ we fix an identification of Δ with a standard simplex.

Remark 2.2 By drilling out disjoint balls if necessary (this does not change the fundamental group), we may assume that the triangulation of M is *ideal*, ie that each 0–cell corresponds to a boundary component of M. For example, we regard a triangulation of a closed manifold as an ideal triangulation of a manifold with boundary a union of spheres.

Definition 2.3 A triangulation is *oriented* if the identifications with standard simplices are orientation-preserving.

Remark 2.4 The triangulations in the SnapPy censuses `OrientableCuspedCensus`, `LinkExteriors` and `HTLinkExteriors` [3] are oriented. Unless otherwise specified we shall assume that our triangulations are oriented.

A triangulation gives rise to a triangulation of M by truncated simplices, and to a triangulation of $\widehat{\widetilde{M}}$.

2.2 Miscellaneous

- The number of vertices, edges, faces and simplices, of a triangulation $\mathcal{T}$ are denoted by v, e, f and s, respectively.
- The standard basis vectors in $\mathbb{Z}^k$ are denoted by $e_1, \dots, e_k$.
- The (oriented) edge of simplex k from vertex i to j is denoted by $\varepsilon_{ij,k}$.
- The matrix groups $\left\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right\}$ and $\left\{\left(\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix}\right)\right\}$ are denoted by P and B, respectively. The higher-rank analogue of P is denoted by N.
- A representation is *boundary-parabolic* if it takes each peripheral subgroup to a conjugate of P. Such is also called a (G,P)*–representation* ($G = \mathrm{SL}(2,\mathbb{C})$ or $\mathrm{PSL}(2,\mathbb{C})$). In the higher-rank case, such a representation is called boundary-unipotent.
- A triangulation is *ordered* if $\varepsilon_{ij,k} \sim \varepsilon_{i'j',k'}$ implies that $i < j \iff i' < j'$.

3 The Ptolemy varieties

We define the Ptolemy variety for $n = 2$ following Garoufalidis, Thurston and Zickert [10] (see also Garoufalidis, Goerner and Zickert [8]).

3.1 The $\mathrm{SL}(2,\mathbb{C})$–Ptolemy variety

Assign to each oriented edge $\varepsilon_{ij,k}$ of $\Delta_k \in \mathcal{T}$ a *Ptolemy coordinate* $c_{ij,k}$. Consider the affine algebraic set A defined by the *Ptolemy relations*

$$c_{03,k}c_{12,k} + c_{01,k}c_{23,k} = c_{02,k}c_{13,k}, \quad k = 1,2,\dots,t, \tag{3-1}$$

the *identification relations*

$$c_{ij,k} = c_{i'j',k'} \quad \text{when} \quad \varepsilon_{ij,k} \sim \varepsilon_{i'j',k'}, \tag{3-2}$$

and the *edge orientation relations* $c_{ij,k} = -c_{ji,k}$. By only considering $i < j$, we shall always eliminate the edge orientation relations.

Definition 3.1 The *Ptolemy variety* $P_2(\mathcal{T})$ is the Zariski open subset of A consisting of points with non-zero Ptolemy coordinates.

Remark 3.2 One can concretely obtain $P_2(\mathcal{T})$ from A by adding a dummy variable x and a dummy relation $x \cdot \prod c_{ij,k} = 1$.

Remark 3.3 We can eliminate the identification relations (3-2) by selecting a representative for each edge cycle. This gives an embedding of the Ptolemy variety in an ambient space $\mathbb{C}^e$, where it is cut out by s Ptolemy relations, one for each simplex. Note that when all boundary components are tori, $s = e$.

3.1.1 The figure-8 knot Consider the ideal triangulation of the figure-8 knot complement shown in Figure 4. The Ptolemy variety $P_2(\mathcal{T})$ is given by

$$\begin{aligned} &c_{03,0}c_{12,0}+c_{01,0}c_{23,0}=c_{02,0}c_{13,0},\\ &c_{03,1}c_{12,1}+c_{01,1}c_{23,1}=c_{02,1}c_{13,1},\\ &c_{02,0}=c_{12,0}=c_{13,0}=c_{01,1}=c_{03,1}=c_{23,1},\\ &c_{01,0}=c_{03,0}=c_{23,0}=c_{02,1}=c_{12,1}=c_{13,1}. \end{aligned} \tag{3-3}$$

By selecting representatives $\varepsilon_{23,0}$ and $\varepsilon_{13,0}$ for the two edge cycles, $P_2(\mathcal{T})$ embeds in $\mathbb{C}^2$, where it is given by

$$c_{23,0}c_{13,0}+c_{23,0}^2=c_{13,0}^2,\quad c_{13,0}c_{23,0}+c_{13,0}^2=c_{23,0}^2. \tag{3-4}$$

It follows that $P_2(\mathcal{T})$ is empty, which is no surprise, since the only boundary-parabolic $\mathrm{SL}(2,\mathbb{C})$–representations of the figure-8 knot are abelian. To detect the geometric representation, we need to consider *obstruction classes* (see Section 3.2 below).

3.1.2 The figure-8 knot sister Consider the ideal triangulation of the figure-8 knot sister shown in Figure 5. The Ptolemy variety $P_2(\mathcal{T})$ is given by

$$\begin{aligned} &c_{03,0}c_{12,0}+c_{01,0}c_{23,0}=c_{02,0}c_{13,0},\\ &c_{03,1}c_{12,1}+c_{01,1}c_{23,1}=c_{02,1}c_{13,1},\\ &c_{01,0}=-c_{03,0}=c_{23,0}=-c_{01,1}=c_{03,1}=-c_{23,1},\\ &c_{02,0}=-c_{12,0}=c_{13,0}=-c_{02,1}=c_{12,1}=-c_{13,1}. \end{aligned} \tag{3-5}$$

Selecting representatives $\varepsilon_{23,0}$ and $\varepsilon_{13,0}$ for the two edge cycles, $P_2(\mathcal{T})\in\mathbb{C}^2$ is given by

$$c_{23,0}c_{13,0}+c_{23,0}^2=c_{13,0}^2,\quad c_{23,0}c_{13,0}+c_{23,0}^2=c_{13,0}^2. \tag{3-6}$$

This is equivalent to

$$x^2-x-1=0,\quad x=\frac{c_{13,0}}{c_{23,0}}. \tag{3-7}$$

Remark 3.4 Note that, for ordered triangulations, the identification relations (3-2) do not involve minus signs. The triangulation in Figure 4 is not oriented.

3.2 Obstruction classes

Each class in $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$ can be represented by a $\mathbb{Z}/2\mathbb{Z}$–valued 2–cocycle on $\widehat{M}$, ie an assignment of a sign to each face of $\mathcal{T}$.

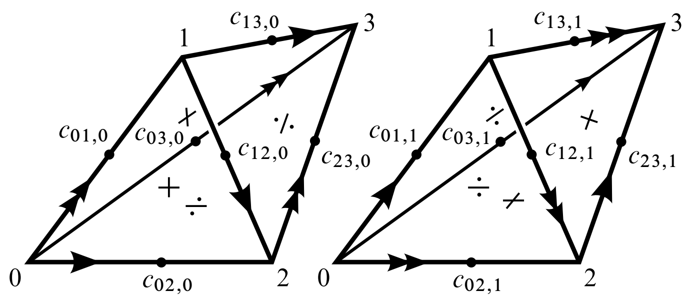

Figure 4: Ordered triangulation of the figure-8 knot. The signs indicate the nontrivial obstruction class.

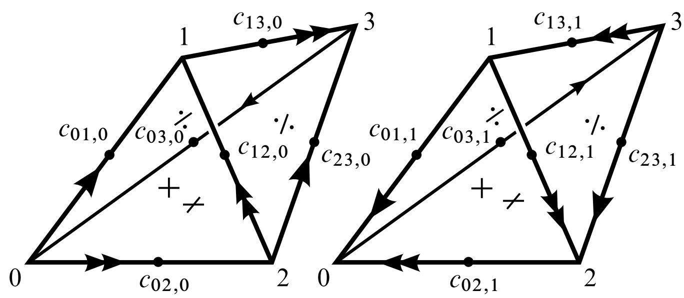

Figure 5: Oriented triangulation of the figure-8 knot sister. The signs indicate the nontrivial obstruction class.

Definition 3.5 Let σ be a $\mathbb{Z}/2\mathbb{Z}$–valued 2–cocycle on $\widehat{M}$. The *Ptolemy variety* for σ is defined as in Definition 3.1, but with the Ptolemy relation replaced by

$$\sigma_{0,k}\sigma_{3,k}c_{03,k}c_{12,k} + \sigma_{0,k}\sigma_{1,k}c_{01,k}c_{23,k} = \sigma_{0,k}\sigma_{2,k}c_{02,k}c_{13,k}, \tag{3-8}$$

where $\sigma_{i,k}$ is the sign of the face of Δ_k opposite vertex i.

Remark 3.6 Multiplying σ by a coboundary $\delta(\tau)$ corresponds to multiplying the Ptolemy coordinate of a one-cell e by $\tau(e)$ (see [10] for details). Hence, up to canonical isomorphism, the Ptolemy variety $P_2^\sigma(\mathcal{T})$ only depends on the cohomology class of σ. The Ptolemy variety $P_2(\mathcal{T})$ is the Ptolemy variety for the trivial obstruction class.

3.2.1 Examples In both examples above, $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, and the nontrivial obstruction class σ is indicated in Figures 4 and 5.

For the figure-8 knot, $P_2^\sigma(\mathcal{T})$ is given by

$$-c_{23,0}c_{13,0} + c_{23,0}^2 = -c_{13,0}^2, \quad -c_{13,0}c_{23,0} + c_{13,0}^2 = -c_{23,0}^2, \tag{3-9}$$

which is equivalent to

$$x^2 - x + 1 = 0, \quad x = \frac{c_{13,0}}{c_{23,0}}. \tag{3-10}$$

The corresponding representations are the geometric representation and its conjugate.

For the figure-8 knot sister, the Ptolemy variety becomes

$$-c_{23,0}c_{13,0} - c_{23,0}^2 = c_{13,0}^2, \quad -c_{23,0}c_{13,0} - c_{23,0}^2 = c_{13,0}^2, \tag{3-11}$$

which is equivalent to

$$x^2 + x + 1 = 0, \quad x = \frac{c_{13,0}}{c_{23,0}}. \tag{3-12}$$

4 The diagonal action

Fix an ordering of the 1–cells of $\mathcal{T}$ and of the cusps of M. As mentioned in Remark 3.3, the Ptolemy variety can be regarded as a subset of the ambient space $\mathbb{C}^e$.

Let $T = (\mathbb{C}^*)^v$ be a torus whose coordinates are indexed by the cusps of M. There is a natural action of T on $P_2^\sigma(\mathcal{T})$ defined as follows: for $x = (x_1, \dots, x_v) \in T$ and $c = (c_1, \dots c_e) \in P_2^\sigma(\mathcal{T})$, define a Ptolemy assignment cx by

$$(xc)_i = x_j x_k c_i, \tag{4-1}$$

where j and k (possibly $j = k$) are the cusps joined by the i^{th} edge cycle. The action is thus determined entirely by the 1–skeleton of $\widehat{M}$.

Remark 4.1 There is a more intrinsic definition of this action in terms of decorations: Each vertex of $\widehat{\widehat{M}}$ determines a cusp of M, and if D is a decoration taking a vertex w to gP, the decoration xD takes w to

$$g \begin{pmatrix} x_i & 0 \\ 0 & x_i^{-1} \end{pmatrix} P,$$

where i is the cusp determined by w. The fact that the two definitions agree under the one-to-one correspondence (1-4) is an immediate consequence of the relationship given in the right image in Figure 3.

4.1 The reduced Ptolemy varieties

Definition 4.2 The *reduced Ptolemy variety* $P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is the quotient $P_2^\sigma(\mathcal{T})/T$.

Let $\mathcal{O}$ be the coordinate ring of $P_2^\sigma(\mathcal{T})$, and let $\mathcal{O}^T$ be the ring of invariants. By geometric invariant theory, the reduced Ptolemy variety is a variety whose coordinate ring is isomorphic to $\mathcal{O}^T$.

For $i = 0, 1$, let C_i denote the free abelian group generated by the *unoriented* i–cells of $\widehat{M}$, and consider the maps (first studied by Neumann [13])

$$\alpha\colon C_0 \to C_1, \quad \alpha^*\colon C_1 \to C_0, \tag{4-2}$$

where α takes a 0–cell to the sum of its incident 1–cells, and α^* takes a 1–cell to the sum of its endpoints. The maps α and α^* are dual under the canonical identifications $C_i \cong C_i^*$. Also, α is injective, and α^* has cokernel of order 2 (see [13]).

The following is an elementary consequence of the definition of the diagonal action.

Lemma 4.3 *The diagonal action $P_2^\sigma(\mathcal{T})$ and the induced action on the coordinate ring $\mathcal{O}$ of $P_2^\sigma(\mathcal{T})$ are given, respectively, by*

$$(xc)_i = \Big(\prod_{j=1}^{v} x_j^{\alpha_{ij}}\Big)c_i, \quad x(c^w) = \prod_{j=1}^{v} x_j^{\alpha^*(w)_j} c^w, \tag{4-3}$$

where c^w is the monomial $c_1^{w_1}\cdots c_e^{w_e} \in \mathcal{O}$, $w \in \mathbb{Z}^e$.

Corollary 4.4 *Suppose that $w_1, \dots, w_{e-v}$ form a basis for* $\operatorname{Ker}\alpha^*$*. The monomials $c^{w_1}, \dots, c^{w_{e-v}}$ generate $\mathcal{O}^T$.*

4.1.1 Examples Suppose the 1–skeleton of $\widehat{M}$ looks like the left image in Figure 6 (this is in fact the 1–skeleton of the census triangulation of the Whitehead link complement). We have

$$\alpha^* = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \tag{4-4}$$

and the action of (x_1, x_2) on a Ptolemy assignment c is given in the right image in Figure 6.

The kernel of α^* is generated by $(0, -2, 0, 1)^t$ and $(-1, 1, 1, 0)^t$, so we have

$$\mathcal{O}^T = \langle c_2^{-2}c_4, c_1^{-1}c_2c_3 \rangle. \tag{4-5}$$

Also note that, in each of the examples in Section 3, $x \in \mathcal{O}^T$.

For computations we need a more explicit description of the reduced Ptolemy variety.

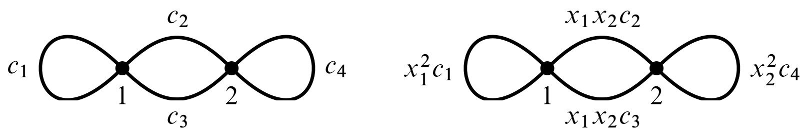

Figure 6: Left: Ptolemy assignment. Right: the diagonal action of (x_1, x_2).

Definition 4.5 Let $T\colon \mathbb{Z}^n \to \mathbb{Z}^m$ be a homomorphism. We say that T is *basic* if there exists a subset J of $\{e_1, \dots, e_n\}$ such that T maps $\mathrm{Span}(J)$ isomorphically onto the image of T. Elements of such a set J are called *basic generators* for T.

We identify C_1 and C_0 with $\mathbb{Z}^e$ and $\mathbb{Z}^v$, respectively.

Proposition 4.6 *The map $\alpha^*\colon C_1 \to C_0$ is basic.*

The proof will be relegated to Section 4.3, where we shall also give explicit basic generators.

Proposition 4.7 *Let $\varepsilon_{i_1}, \dots, \varepsilon_{i_v}$ be basic generators for α^*. The ring of invariants $\mathcal{O}^T$ is isomorphic to $\mathbb{C}[c_1, \dots, c_e]$ modulo the Ptolemy relations and the relations $c_{i_1} = \dots = c_{i_v} = 1$, ie the reduced Ptolemy variety is isomorphic to the subset of $P_2^\sigma(\mathcal{T})$ where the Ptolemy coordinates of the basic generators are 1.*

Proof Let $w_1, \dots, w_{e-v}$ be a basis for $\mathrm{Ker}\,\alpha^*$. Hence, $w_1, \dots, w_{e-v}$ and $\varepsilon_{i_1}, \dots, \varepsilon_{i_v}$ generate C_1. We can thus uniquely express each c_i as a monomial in the w_j and the c_{i_j}. The result now follows from Corollary 4.4. □

Remark 4.8 This is how the Ptolemy varieties are computed in SnapPy.

4.2 Shapes and gluing equations

One can assign to each simplex a *shape*

(4-6)
$$z = \sigma_3\sigma_2 \frac{c_{03}c_{12}}{c_{02}c_{13}} \in \mathbb{C} \setminus \{0, 1\},$$

and one can show (see [10; 8]) that these satisfy Thurston's gluing equations. For the geometric representation of a cusped hyperbolic manifold, the shape field (field generated by the shapes) is equal to the invariant trace field, which is in general smaller than the trace field; see Maclachlan and Reid [12].

Remark 4.9 Note that the shapes are elements in $\mathcal{O}^T$.

4.3 Proof that α^* is basic

Since α^* has cokernel of order 2, it is enough to prove that there is a set of columns of α^* forming a matrix with determinant ± 2. Recall that the columns of α^* correspond to 1–cells of $\mathcal{T}$. We shall thus consider graphs in the 1–skeleton of $\widehat{M}$. We recall some basic results from graph theory. All graphs are assumed to be connected.

Definition 4.10 The *incidence matrix* of a graph G with vertices $v_1, \dots, v_k$ and edges $\varepsilon_1, \dots, \varepsilon_l$ is the $k \times l$ matrix I_G whose (i, j) entry is 1 if v_i is incident to ε_j, and 0 otherwise.

Lemma 4.11 *The rank of I_G is $k-1$. If G is a tree, I_G is a $k \times (k-1)$ matrix, and removing any row gives a matrix with determinant ± 1.*

4.3.1 Case 1: a single cusp In this case the result is trivial. The matrix representation for α^* is $(2 \cdots 2)$.

4.3.2 Case 2: multiple cusps, self-edges Suppose $\widehat{M}$ has a self-edge ε_1 (an edge joining a cusp to itself), and consider the graph G consisting of the union of ε_1 with a maximal tree T (see left image in Figure 7). The columns of α^* corresponding to the edges of G then form the matrix

$$B = \left(\begin{array}{c|c} 2 & \\ \hline 0 & I_T \end{array} \right) \tag{4-7}$$

which, by Lemma 4.11, has determinant ± 2.

4.3.3 Case 3: multiple cusps, no self-edges Pick a face with edges $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and add edges to form a graph G such that $G \setminus \varepsilon_1$ is a maximal tree (see right image in Figure 7). The corresponding columns form the matrix

$$C = I_G = \left(\begin{array}{c|c} 1 & \\ 0 & I_T \\ 1 & \\ \hline 0 & \end{array} \right) \tag{4-8}$$

By Lemma 4.11, I_G is invertible and has determinant ± 2. This concludes the proof that α^* is basic.

Note that

$$\det(B) = \det \begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix} = 2, \quad \det(C) = \det \begin{pmatrix} 1 & 1 & \\ 0 & 1 & 1 \\ 1 & & 1 \end{pmatrix} = 2, \tag{4-9}$$

ie only the edges and vertices shown in Figure 7 contribute to the determinant.

Figure 7: Left: tree G with 1–cycle; $G \setminus \varepsilon_1$ is a maximal tree. Right: tree G with 3–cycle; $G \setminus \varepsilon_1$ is a maximal tree.

Remark 4.12 Trees with 1– or 3–cycles are also used in [9, Section 4.6] to study index structures.

5 The Ptolemy field and the trace field

5.1 Explicit description of the Ptolemy field

By Proposition 4.7 any $c \in P_2^\sigma(\mathcal{T})$ is equivalent to a Ptolemy assignment c' whose coordinates for a set of basic generators $\varepsilon_{i_1}, \dots, \varepsilon_{i_v}$ is 1. In particular, it follows that the Ptolemy field (see Definition 1.13) of $c \in P_2^\sigma(\mathcal{T})$ is given by

$$k_c = k_{c'} = \mathbb{Q}(\{c'_{\varepsilon_1}, \dots, c'_{\varepsilon_e}\}). \tag{5-1}$$

Definition 5.1 Let $\rho\colon \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ be a representation. The *trace field* of ρ is the field generated by the traces of elements in the image. We denote it k_ρ.

Our main result is the following. We defer the proof to Section 5.4.

Theorem 5.2 *Let $c \in P_2^\sigma(\mathcal{T})_{\mathrm{red}}$. If the corresponding generic representation ρ of $\pi_1(M)$ in $\mathrm{PSL}(2,\mathbb{C})$ is boundary-nontrivial, the Ptolemy field of c equals the trace field of ρ.*

Remark 5.3 Note that if $c \in P_2^\sigma(\mathcal{T})_{\mathrm{red}}$ is in a degree-0 component, the Ptolemy field is a number field.

5.2 The setup of the proof

Since the natural cocycle is given in terms of the Ptolemy coordinates, it follows that ρ is defined over the Ptolemy field. Hence, the trace field is a subfield of the Ptolemy field.

Fix a maximal tree G with 1– or 3–cycle as in Figure 7. As explained in Section 4.3, the edges of G are basic generators of α^*. We may thus assume without loss of generality that the Ptolemy coordinates c_i of the edges ε_i of G are 1. By (5-1), it is thus enough to show that the Ptolemy coordinates of the remaining 1–cells are in the trace field.

Let γ denote the (lifted) natural cocycle of c. Then γ assigns to each edge path p in $\widehat{M}$ a matrix $\gamma(p) \in \mathrm{SL}(2,\mathbb{C})$. Let

$$\alpha(a) = \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix}, \quad \beta(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \tag{5-2}$$

As shown in Figure 2, γ takes long and short edges to elements of the form $\alpha(a)$ and $\beta(b)$, respectively, where a and b are given in terms of the Ptolemy coordinates.

Since ρ is boundary-nontrivial there exists, for each cusp i of M, a peripheral loop M_i with $\gamma(M_i) \in P$ nontrivial. We shall here refer to such loops as *nontrivial*. Fix such nontrivial loops M_i, once and for all, and let $m_i \neq 0$ be such that $\gamma(M_i) = \beta(m_i)$. For any edge path p with endpoint on a cusp i we can alter M_i by a conjugation if necessary (this does not change m_i) so that p is composable with M_i.

5.3 Proof for one cusp

We first prove Theorem 5.2 in the case where there is only one cusp. In this case, all edges are self-edges, and T consists of a single edge ε_1.

Lemma 5.4 *For any self-edge ε, we have $m_1 c_\varepsilon \in k_\rho$.*

Proof Let X_1 be a peripheral path such that $X_1\varepsilon$ is a loop (see the left image in Figure 8), and let x_1 be such that $\gamma(X_1) = \beta(x_1)$. We have

$$\mathrm{Tr}(\gamma(X_1\varepsilon)) = \mathrm{Tr}(\beta(x_1)\alpha(c_\varepsilon)) = \mathrm{Tr}\left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -c_\varepsilon^{-1} \\ c_\varepsilon & 0 \end{pmatrix}\right) = x_1 c_\varepsilon \in k_\rho. \tag{5-3}$$

Applying the same computation to the loop $X_1 M_1 \varepsilon$ yields

$$\mathrm{Tr}(\beta(x_1)\beta(m_1)\alpha(c_\varepsilon)) = (x_1 + m_1)c_\varepsilon \in k_\rho, \tag{5-4}$$

and the result follows. □

Since the Ptolemy coordinate of ε_1 is 1, it follows that $m_1 \in k_\rho$. Since all edges are self-edges, we have $c_\varepsilon \in k_\rho$ for all 1–cells ε. This concludes the proof in the one-cusped case.

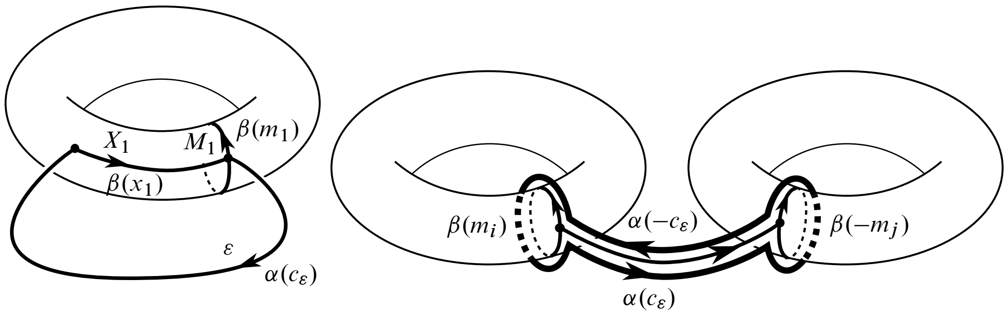

Figure 8: Left: self-edge. Right: edge between cusps.

5.4 The general case

The general case follows the same strategy, but is more complicated since it involves edge paths between multiple cusps.

Lemma 5.5 *If ε is a self-edge from cusp i to itself, $m_i c_\varepsilon \in k_\rho$.*

Proof The proof is identical to that of Lemma 5.4. □

Lemma 5.6 *If two (distinct) cusps i and j are joined by an edge ε in G, we have*

$$m_i m_j \in k_\rho. \tag{5-5}$$

Proof Consider the loop $\varepsilon_j \overline{M}_j \overline{\varepsilon_j} M_i$ shown in the right image in Figure 8. A simple computation shows that

$$\mathrm{Tr}(\alpha(c_\varepsilon)\beta(-m_j)\alpha(-c_\varepsilon)\beta(m_i)) = 2 + m_i m_j c_\varepsilon^2. \tag{5-6}$$

Since $\varepsilon \in T$, $c_\varepsilon = 1$, and the result follows. □

More generally, the following holds.

Lemma 5.7 *We have $m_i \in k_\rho$ for all cusps i.*

Proof If G is a tree with 1–cycle, then $c_1 = 1$, so Lemma 5.5 implies that $m_1 \in k_\rho$. Inductively applying Lemma 5.6 for the edge ε_j connecting cusp $i = j-1$ and j

implies the result. If G is a tree with 3–cycle, the Ptolemy coordinates c_1, c_2 and c_3 are 1, so the edges of the face are labeled by $\alpha(1)$ and $\beta(-1)$ only (see Figure 2). Inserting the peripheral loops M_i as in Figure 9, we obtain

$$\operatorname{Tr}\big(\beta(-1)\beta(m_1)\alpha(1)\beta(-1)\beta(m_2)\alpha(1)\beta(-1)\beta(m_3)\alpha(1)\big) \in k_\rho. \tag{5-7}$$

By an elementary computation, the trace equals

$$m_1m_2m_3 - m_1m_2 - m_2m_3 - m_3m_1 + 2 \in k_\rho. \tag{5-8}$$

By Lemma 5.6, $m_im_j \in k_\rho$, so $m_1 \in k_\rho$. The result now follows as above by inductively applying Lemma 5.6. □

Let ε be an arbitrary 1–cell. If ε is a self-edge, Lemmas 5.5 and 5.7 imply that $c_\varepsilon \in k_\rho$. Otherwise, there exists an edge path p in the maximal tree $G \setminus \varepsilon_1$ such that $p * \varepsilon$ is a loop in $\widehat{M}$. By relabeling the cusps and edges if necessary, we may assume that $p = \varepsilon_{i+1} * \varepsilon_{i+2} * \cdots * \varepsilon_j$, where ε_k goes from cusp $k-1$ to cusp k. Pick peripheral paths X_k on cusp k connecting the ends (in M, not $\widehat{M}$) of edges ε_k and ε_{k+1} (see Figure 10). We obtain a loop that can be composed with arbitrary powers of the peripheral loops $M_i, \dots, M_j$. We thus obtain the following traces (where $b_k \in \mathbb{Z}$):

$$\begin{aligned}\operatorname{Tr}\big(\beta(x_i + b_im_i)\alpha(c_{i+1})\beta(x_{i+1} + b_{i+1}m_{i+1})\alpha(c_{i+2})\cdots\\ \cdot\beta(x_j + b_jm_j)\alpha(c_\varepsilon)\big) \in k_\rho.\end{aligned} \tag{5-9}$$

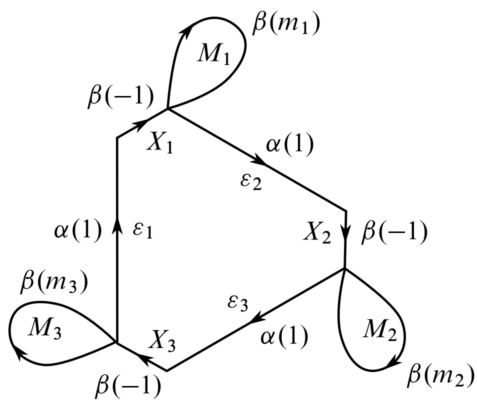

Figure 9: 3–cycle case

It will be convenient to regard $\operatorname{Tr}\big(\beta(x_i)\alpha(c_{i+1})\beta(x_{i+1})\alpha(c_{i+2})\cdots\beta(x_j)\alpha(c_\varepsilon)\big)$ as a function of variables x_i (disregarding that the x_i are fixed expressions of the Ptolemy coordinates).

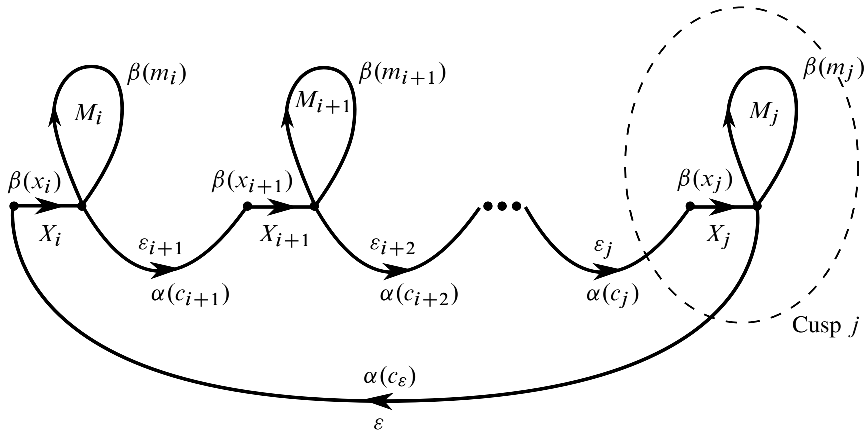

Figure 10: Arbitrary edge ε

Definition 5.8 Given a function $f(x_1,\dots,x_r)$, let $\Delta_i f$ be the function given by

$$\Delta_i f(h) = f(x_1,\dots,x_i+h,\dots,x_r) - f(x_1,\dots,x_i,\dots,x_r). \tag{5-10}$$

The following is elementary.

Lemma 5.9 *If $f(x_1,\dots,x_r)$ is a polynomial where the exponents of all variables x_i are 0 or 1, and where the highest-degree term is $ax_1x_2\cdots x_r$, we have*

$$\Delta_r\big(\cdots\Delta_2\big(\Delta_1 f(h_1)\big)(h_2)\cdots\big) = ah_1h_2\cdots h_r, \tag{5-11}$$

and the left-hand side is thus independent of the x_i.

If, for example, $f(x_1,x_2) = x_1x_2$, we have

$$\begin{aligned} \Delta_1 f(h_1) &= (x_1+h_1)x_2 - x_1x_2 = h_1x_2,\\ \Delta_2\big(\Delta_1 f(h_1)\big)(h_2) &= h_1(x_2+h_2) - h_1x_2 = h_1h_2. \end{aligned} \tag{5-12}$$

Lemma 5.10 *Let $x_1,\dots,x_r$ be variables and $y_1,\dots,y_r$ be constants. The expression*

$$\operatorname{Tr}\big(\beta(x_1)\alpha(y_1)\cdots\beta(x_r)\alpha(y_r)\big) \tag{5-13}$$

is a polynomial in the x_i whose unique highest-degree term is $\prod_{i=1}^r y_i \prod_{i=1}^r x_i$. Moreover, for each monomial term, the exponent of each variable is either 1 or 0.

Proof This follows by induction on r. □

Applying Lemmas 5.10 and 5.9 to the function

$$f(x_i,\dots,x_j)=\operatorname{Tr}\bigl(\beta(x_i)\alpha(c_{i+1})\beta(x_{i+1})\alpha(c_{i+2})\cdots\beta(x_j)\alpha(c_\varepsilon)\bigr), \tag{5-14}$$

we obtain

$$(m_im_{i+1}\cdots m_jc_ic_{i+1}\cdots c_j)c_\varepsilon\in k_\rho. \tag{5-15}$$

Since all m_i are in k_ρ by Lemma 5.7, and all c_i are 1 (since $\varepsilon_i\in T$), it follows that c_ε is in k_ρ. This concludes the proof. □

5.5 Proof of Proposition 1.10

The fact that $\mathcal{R}$ factors follows from the fact that the diagonal action only changes the decoration (by diagonal elements; see Remark 4.1), not the representation. Since the preimage of the right map in (1-4) is parametrized by choices of lifts, ie elements in $Z^1(\widehat{M};\mathbb{Z}/2\mathbb{Z})$, all that remains is to show that the only freedom in the choice of decoration of a boundary-nontrivial representation is the diagonal action. This follows from results in [10]: a decoration is an equivariant map

$$D\colon\ \widehat{\widetilde{M}}^{(0)}\to \mathrm{PSL}(2,\mathbb{C})/P, \tag{5-16}$$

and is thus determined by its image of lifts $\widetilde{e}_1,\dots,\widetilde{e}_v$ of the cusps of M. The freedom in the choice of $D(\widetilde{e}_i)$ is the choice of a coset gP satisfying $g\rho(\mathrm{Stab}(\widetilde{e}_i))g^{-1}\subset P$, where $\mathrm{Stab}(\widetilde{e}_i)\subset\pi_1(M)$ is the stabilizer of $\widetilde{e}_i$, ie a peripheral subgroup corresponding to cusp i. Hence, if $\rho(\mathrm{Stab}(\widetilde{e}_i))$ is nontrivial, the freedom is right-multiplication by a diagonal matrix (if it is trivial, any coset works). Hence, if ρ is boundary-nontrivial, the only freedom in choosing a decoration is the diagonal action.

6 Ptolemy varieties for $n>2$

Many of our results generalize in a straightforward way to the higher-rank Ptolemy varieties $P_n(\mathcal{T})$. We recall the definition of these below, and refer to [10; 8] for details.

We identify all simplices of $\mathcal{T}$ with a standard simplex

$$\Delta^3_n=\bigl\{(x_0,x_1,x_2,x_3)\in\mathbb{R}^4\mid 0\le x_i\le n,\ x_0+x_1+x_2+x_3=n\bigr\} \tag{6-1}$$

and regard $\widehat{M}$ as a quotient of a disjoint union $\coprod_{k=1}^s\Delta^3_{n,k}$, with a copy $\Delta^3_{n,k}$ of Δ^3_n for each simplex k of $\mathcal{T}$. Define

$$\Delta^3_n(\mathbb{Z})=\Delta^3_n\cap\mathbb{Z}^4,$$

and define $\dot{\Delta}^3_n(\mathbb{Z})$ to be $\Delta^3_n(\mathbb{Z})$ with the four vertex points removed. A point in M in the image of $\coprod_{k=1}^s\dot{\Delta}^3_{n,k}(\mathbb{Z})$ is called an *integral point* of M.

6.1 Definition of the Ptolemy variety

Assign to each $(t,k) \in \Delta^3_{n,k}(\mathbb{Z})$ a *Ptolemy coordinate* $c_{t,k}$. For each simplex k, we have $|\Delta_{n-2}(\mathbb{Z})| = \binom{n+1}{3}$ *Ptolemy relations*

$$c_{\alpha+1001,k}c_{\alpha+0110,k} + c_{\alpha+1100,k}c_{\alpha+0011,k} = c_{\alpha+1010,k}c_{\alpha+0101,k}, \quad \alpha \in \Delta_{n-2}(\mathbb{Z}), \tag{6-2}$$

as well as *identification relations*

$$c_{t,k} = \pm c_{t',k'} \quad \text{when } (t,k) \sim (t',k'). \tag{6-3}$$

Remark 6.1 The signs in (6-3) depend in a nontrivial way on the face pairings (see [8]). For ordered triangulations the signs are always positive. As in Remark 3.3 we can eliminate the identification relations by selecting a representative of each integral point of M.

Definition 6.2 The *Ptolemy variety* $P_n(\mathcal{T})$ is the subset of the affine algebraic set defined by the Ptolemy and identification relations, consisting of the points where all Ptolemy coordinates are non-zero.

For general n we denote the group of upper-triangular matrices with 1 on the diagonal by N (instead of P). As in (1-2) we have

$$\begin{aligned}\{\text{Points in } P_n(\mathcal{T})\} &\overset{1-1}{\longleftrightarrow} \{\text{Natural } (\mathrm{SL}(n,\mathbb{C}),N)\text{–cocycles on } M\} \\ &\overset{1-1}{\longleftrightarrow} \{\text{Generically decorated } (\mathrm{SL}(n,\mathbb{C}),N)\text{–representations}\}.\end{aligned} \tag{6-4}$$

6.2 The diagonal action

Let D be the group of diagonal matrices in $\mathrm{SL}(n,\mathbb{C})$. We identify D with the torus $(\mathbb{C}^*)^{n-1}$ via the identification

$$(\mathbb{C}^*)^{n-1} \to D, \quad (a_1,\dots,a_{n-1}) \mapsto \mathrm{diag}(a_1, a_2/a_1, \dots, a_{n-1}/a_{n-2}, 1/a_{n-1}). \tag{6-5}$$

As in Remark 4.1, we have a diagonal action of the torus $T = D^v$ on the set of decorated representations, where $(D_1,\dots,D_v) \in T$ acts by replacing the coset gN assigned to a vertex w by gD_iN, where i is the cusp corresponding to w. The corresponding action on $P_n(\mathcal{T})$ is described in Lemma 6.4 below.

Let C_1^n be the group generated by the integral points of M, and let $C_0^n = C_0 \otimes \mathbb{Z}^{n-1}$. In Garoufalidis and Zickert [11] we defined maps

$$\alpha\colon C_0^n \to C_1^n, \quad \alpha^*\colon C_1^n \to C_0^n, \tag{6-6}$$

generalizing (4-2). The map α^* takes an integral point (t,k) to $\sum x_i \otimes e_{t_i}$, where x_i is the cusp determined by vertex i of simplex k. We shall not need the definition of α.

Lemma 6.3 [11] *The map α^* is surjective with cokernel $\mathbb{Z}/n\mathbb{Z}$.*

By selecting an ordering of the natural generators of C_0^n and C_1^n, we regard α and α^* as matrices. The following is an elementary consequence of (6-4).

Lemma 6.4 *The diagonal action of $T=(\mathbb{C}^*)^{v(n-1)}$ on $P_n(\mathcal{T})$ and the corresponding action on the coordinate ring $\mathcal{O}$ of $P_n(\mathcal{T})$ are given, respectively, by*

$$(xc)_t = \left(\prod_{j=1}^{v(n-1)} x_j^{\alpha_{tj}}\right) c_t, \quad x(c^w) = \prod_{j=1}^{v(n-1)} x_j^{\alpha^*(w)_j} c^w. \tag{6-7}$$

Corollary 6.5 *The ring of invariants $\mathcal{O}^T$ is generated by $c^{w_1},\dots,c^{w_r}$, where $r=$ $\operatorname{rank}(C_1^n)-\operatorname{rank}(C_0^n)$ and $w_1,\dots,w_r$ are a basis for $\operatorname{Ker}\alpha^*$.*

Definition 6.6 The *Ptolemy field* of a Ptolemy assignment $c\in P_n(\mathcal{T})$ is defined as

$$k_c = \mathbb{Q}(c^{w_1},\dots,c^{w_r}), \tag{6-8}$$

where $w_1,\dots,w_r$ are (integral) generators of $\operatorname{Ker}\alpha^*$.

The following is proved in Section 6.4.

Proposition 6.7 *The map $\alpha^*\colon C_1^n \to C_0^n$ is basic.*

Corollary 6.8 *Let $p_1,\dots,p_{(n-1)v}$ be integral points that are basic generators of C_1^n. The ring $\mathcal{O}^T$ is generated by the Ptolemy relations together with the relations $c_{p_1} = \dots = c_{p_{(n-1)v}} = 1$. Equivalently, the reduced Ptolemy variety is isomorphic to the subvariety of $P_n(\mathcal{T})$ consisting of Ptolemy assignments with $c_{p_i}=1$.*

Proof This follows the proof of Proposition 4.7 word by word. □

Remark 6.9 This is how the Ptolemy varieties and Ptolemy fields at [5] are computed.

6.3 Representations

Definition 6.10 Let ρ be an $(\mathrm{SL}(n,\mathbb{C}),N)$–representation, and let I_i denote the image of the peripheral subgroup corresponding to cusp i. We say that ρ is *boundary-non-degenerate* if each I_i has an element whose Jordan canonical form has a single (maximal) Jordan block.

Proposition 6.11 *The map*

$$\mathcal{R}\colon\ P_n(\mathcal{T})_{\mathrm{red}} \to \{(\mathrm{SL}(n,\mathbb{C}),N)\text{–representations}\}/\,\mathrm{Conj} \tag{6-9}$$

maps onto the generic representations, and the preimage of a generic boundary-non-degenerate representation consists of a single point.

Proof The proof is identical to the proof in Section 5.5 for $n=2$. □

Conjecture 6.12 The Ptolemy field of a generic, boundary-non-degenerate representation is equal to its trace field.

Remark 6.13 Much of the theory also works for $\mathrm{PSL}(n,\mathbb{C})$–representations by means of obstruction classes in $H^2(\widehat{M};\mathbb{Z}/n\mathbb{Z})$. When n is even, obstruction classes in $H^2(\widehat{M};\mathbb{Z}/2\mathbb{Z})$ were defined in [10] for representations in $p\mathrm{SL}(n,\mathbb{C})=\mathrm{SL}(n,\mathbb{C})/\pm I$. For $\mathrm{PSL}(n,\mathbb{C})$–representations, both the Ptolemy field and the trace field are only defined up to n^{th} roots of unity. The generalized obstruction classes are used on the website [5] and will be explained in a forthcoming publication.

6.4 Proof that α^* is basic

By Lemma 6.3, we need to prove the existence of integral points such that the corresponding columns of α^* form a matrix of determinant $\pm n$. As in Section 4.3 we split the proof into three cases.

6.4.1 Basic matrix algebra Let I_k be the identity matrix, R_k the sparse matrix whose first row contains entirely of 1's, S_k the sparse matrix whose lower diagonal consists of 1's ($S_1=0$), and T_k the sparse matrix whose lower right entry is 1. The index k denotes that the matrices are $k\times k$. For $k=3$, we have

$$R_3=\begin{pmatrix}1&1&1\\&&\\&&\end{pmatrix},\quad S_3=\begin{pmatrix}&&\\1&&\\&1&\end{pmatrix},\quad T_3=\begin{pmatrix}&&\\&&\\&&1\end{pmatrix}. \tag{6-10}$$

Lemma 6.14 *We have*

$$\det(I_k+R_k-S_k)=k+1,\quad \det(I_k+R_k+T_k-S_k)=2k+1. \tag{6-11}$$

Proof This follows, for example, by expanding the determinant using the last column. The matrices $I_k + R_k - S_k$ are shown below for $k = 1, 2, 3$ and 4:

$$(2), \quad \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{pmatrix}. \tag{6-12}$$

For $I_k + R_k + T_k - S_k$, the only difference is that the lower right entry is now 2. □

Lemma 6.15 *Let A, B, C, D be $k \times k$, $k \times l$, $l \times k$, and $l \times l$ matrices, respectively, and let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. If D is invertible, we have*

$$\det(M) = \det(D)\det(A - BD^{-1}C). \tag{6-13}$$

Proof This follows from the identity

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{pmatrix}.$$ □

6.4.2 One cusp Pick any face of $\mathcal{T}$ and consider the integral points shown in Figure 11. Let A_n be the $(n-1) \times (n-1)$ matrix formed by the corresponding columns of α^*. The columns are ordered as shown in the figures, and the rows, ie the generators $x \otimes e_i$ of C_0^n, are ordered in the natural way (increasing in i). The following is an immediate consequence of the definition of α^*.

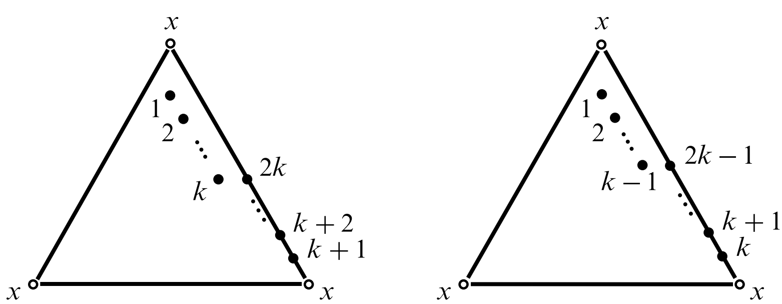

Figure 11: Left: basic generators, $n = 2k+1$. Right: basic generators, $n = 2k$.

Lemma 6.16 *The matrix A_n is given by*

$$A_{2k+1} = \begin{pmatrix} I_k + R_k + T_k & I_k \\ S_k & I_k \end{pmatrix}, \quad A_{2k} = \begin{pmatrix} 2 & 0 \cdots 0\,1 & 0 \\ 0 & I_{k-1} + R_{k-1} & I_{k-1} \\ 0 & S_{k-1} & I_{k-1} \end{pmatrix}. \tag{6-14}$$

Corollary 6.17 *The determinant of A_n is $\pm n$.*

Proof This follows from Lemma 6.15 and Lemma 6.14. □

6.4.3 Multiple cusps, self-edges Pick a face with a self-edge, and extend to a maximal tree with 1–cycle G as in the left image in Figure 7. Let $T = G \setminus \varepsilon_1$, and let B_n denote the matrix formed by the columns of α^* corresponding to the face points shown in the left image in Figure 12 together with the edge points on T. We order the generators $x_i \otimes e_j$ of C_0^n as

$$x_1 \otimes e_1, \dots, x_1 \otimes e_{n-1}, \quad x_2 \otimes e_{n-1}, \dots, x_2 \otimes e_1, \tag{6-15}$$

with a similar scheme for the other vertices. The following is an immediate consequence of the definition of α^*.

Lemma 6.18 *The matrix B_n is given by*

$$B_n = \left(\begin{array}{c|c} I_{n-1} + R_{n-1} & \\ S_{n-1} & I_T \otimes \mathbb{Z}^{n-1} \\ \hline 0 & \end{array}\right) \tag{6-16}$$

where $I_T \otimes \mathbb{Z}^{n-1}$ is the matrix obtained from I_T by replacing each non-zero entry by I_{n-1}.

Corollary 6.19 *The determinant of B_n is $\pm n$.*

Proof This follows from

$$\det(B_n) = \pm \det \begin{pmatrix} I_{n-1} + R_{n-1} & I_{n-1} \\ S_{n-1} & I_{n-1} \end{pmatrix} = \pm n, \tag{6-17}$$

where the second equality follows from Lemmas 6.15 and 6.14. □

6.4.4 Multiple cusps, no self-edge Pick a maximal tree with 3–cycle G, and let C_n be the matrix formed by the columns of α^* corresponding to the face points in the right image in Figure 12 together with the edge points on $T = G \setminus \varepsilon_1$.

Lemma 6.20 *The matrix C_n is given by*

$$C_n = \left(\begin{array}{c|c} I_{n-1} & \\ S_{n-1} & I_T \otimes \mathbb{Z}^{n-1} \\ R_{n-1} & \\ \hline 0 & \end{array}\right). \tag{6-18}$$

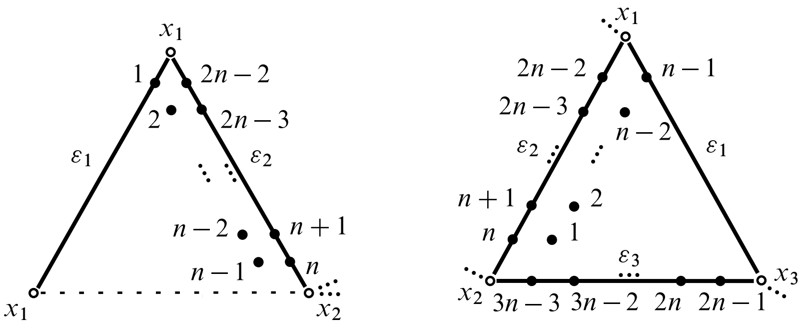

Figure 12: Left: basic generators, tree with 1–cycle. Right: basic generators, tree with 3–cycle.

Corollary 6.21 *The determinant of C_n is $\pm n$.*

Proof We have

$$\det(C_n) = \pm\det(M), \quad M = \begin{pmatrix} I_{n-1} & I_{n-1} & \\ S_{n-1} & I_{n-1} & I_{n-1} \\ R_{n-1} & & I_{n-1} \end{pmatrix}. \tag{6-19}$$

Using Lemma 6.15 with

$$A = \begin{pmatrix} I_{n-1} & I_{n-1} \\ S_{n-1} & I_{n-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I_{n-1} \end{pmatrix}, \quad C = \begin{pmatrix} R_{n-1} & 0 \end{pmatrix}, \quad D = I_{n-1},$$

we have

$$\det(M) = \det\begin{pmatrix} I_{n-1} & I_{n-1} \\ S_{n-1} - R_{n-1} & I_{n-1} \end{pmatrix} = \det(I_{n-1} + R_{n-1} - S_{n-1}) = n; \tag{6-20}$$

the second equation follows from Lemma 6.15 and the third from Lemma 6.14. □

This concludes the proof that α^* is basic.

References

[1] **W Bosma**, **J Cannon**, **C Playoust**, *The Magma algebra system, I: The user language*, J. Symbolic Comput. 24 (1997) 235–265

[2] **D Calegari**, *Real places and torus bundles*, Geom. Dedicata 118 (2006) 209–227 MR2239457

[3] **M Culler**, **N M Dunfield**, **J R Weeks**, *SnapPy: A computer program for studying the geometry and topology of 3–manifolds* `http://snappy.computop.org/`

[4] **N M Dunfield**, *Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds*, Invent. Math. 136 (1999) 623–657 MR1695208

[5] **E Falbel**, **S Garoufalidis**, **A Guilloux**, **M Görner**, **P-V Koseleff**, **F Rouillier**, **C Zickert**, *CURVE database* http://curve.unhyperbolic.org/database.html

[6] **E Falbel**, **P V Koseleff**, **F Rouillier**, *Representations of fundamental groups of* 3*–manifolds into* $\mathrm{PGL}(3,\mathbb{C})$*: Exact computations in low complexity* arXiv:1307.6697

[7] **V Fock**, **A Goncharov**, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. (2006) 1–211 MR2233852

[8] **S Garoufalidis**, **M Goerner**, **C K Zickert**, *Gluing equations for* $\mathrm{PGL}(n,\mathbb{C})$*–representations of* 3*–manifolds* arXiv:1207.6711 To appear in Alg. & Geom. Topol.

[9] **S Garoufalidis**, **C D Hodgson**, **H Rubinstein**, **H Segerman**, 1*–efficient triangulations and the index of a cusped hyperbolic* 3*–manifold* arXiv:1303.5278 To appear in Geom. & Topol.

[10] **S Garoufalidis**, **D P Thurston**, **C K Zickert**, *The complex volume of* $\mathrm{SL}(n,\mathbb{C})$*–representations of* 3*–manifolds* arXiv:1111.2828 To appear in Duke Math. J.

[11] **S Garoufalidis**, **C K Zickert**, *The symplectic properties of the* $\mathrm{PGL}(n,\mathbb{C})$*–gluing equations* arXiv:1310.2497 To appear in J. Quantum Topol.

[12] **C Maclachlan**, **A W Reid**, *The arithmetic of hyperbolic* 3*–manifolds*, Graduate Texts in Mathematics 219, Springer, New York (2003) MR1937957

[13] **W D Neumann**, *Combinatorics of triangulations and the Chern–Simons invariant for hyperbolic* 3*–manifolds*, from: "Topology '90", Ohio State Univ. Math. Res. Inst. Publ. 1, de Gruyter, Berlin (1992) 243–271 MR1184415

[14] **C K Zickert**, *The volume and Chern–Simons invariant of a representation*, Duke Math. J. 150 (2009) 489–532

School of Mathematics, Georgia Institute of Technology
686 Cherry Street, Atlanta, GA 30332-0160, USA

Pixar Animation Studios
1200 Park Avenue, Emeryville, CA 94608, USA

Department of Mathematics, University of Maryland
College Park, MD 20742-4015, United States

stavros@math.gatech.edu, enischte@gmail.com, zickert@umd.edu

http://www.math.gatech.edu/~stavros, http://www.unhyperbolic.org/, http://www2.math.umd.edu/~zickert

Received: 21 January 2014 Revised: 9 May 2014