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Pin(2)–equivariant KO–theory and intersection forms of spin 4–manifolds

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Using the Seiberg–Witten Floer spectrum and Pin(2)–equivariant KO–theory, we prove new Furuta-type inequalities on the intersection forms of spin cobordisms between homology 3–spheres. We then give explicit constrains on the intersection forms of spin 4–manifolds bounded by Brieskorn spheres $\pm\Sigma(2,3,6k\pm 1)$. Along the way, we also give an alternative proof of Furuta's improvement of $\frac{10}{8}$–theorem for closed spin 4–manifolds.

57R58; 57R57

1 Introduction

A natural question in 4–dimensional topology is: Which nontrivial symmetric bilinear forms can be realized as the intersection form of a closed, smooth, spin 4–manifold X? Such a form should be even and unimodular. Therefore, it is indefinite by Donaldson's diagonalizability theorem [8; 9]. After changing the orientation of X if necessary, we can assume that the signature $\sigma(X)$ is non-positive. Then the intersection form can be decomposed as $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ with $p\geq 0, q>0$. Matsumoto's $\frac{11}{8}$ conjecture [22] states that $b_2(X)\geq\frac{11}{8}|\sigma(X)|$, which can be rephrased as $q\geq\frac{3}{2}p$. An important result is the following $\frac{10}{8}$ theorem of Furuta:

Theorem 1.1 [14] *Suppose X is an oriented closed spin 4–manifold with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $p\geq 0, q>0$. Then we have $q\geq p+1$.*

Furuta's proof made use of the finite-dimensional approximation of the Seiberg–Witten equations on closed 4–manifolds and Pin(2)–equivariant K–theory. By doing destabilization and appealing to a result of Stolz [32], Minami and Schmidt independently proved the following improvement:

Theorem 1.2 [23; 29] *Let X be a smooth, oriented, closed spin 4–manifold with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $p\geq 0, q>0$. Then we have*

$$q\geq\begin{cases}p+1 & p\equiv 0,2 \bmod 8,\\ p+2 & p\equiv 4 \bmod 8,\\ p+3 & p\equiv 6 \bmod 8.\end{cases}\tag{1}$$

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Remark 1.3 p is always an even integer by Rokhlin's theorem [27].

An interesting observation is that Schmidt's calculation in [29] about the Adams operations actually implies an alternative proof of the following further improvement, which was first proved by Furuta and Kametani. We will give the proof in Section 3.

Theorem 1.4 [15] *Let X be a smooth, oriented, closed spin 4–manifold with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p, q > 0$. Then $q \geq p + 3$ when $p \equiv 0 \bmod 8$.*

Another direction is to consider the intersection form of a spin 4–manifold with given boundary. Suppose X is not closed but has boundary components, which are homology 3–spheres. The intersection form of X is still even and unimodular but can be definite now. For the definite case, various constraints are found in Frøyshov [10; 11; 12], Ozsváth and Szabó [26], Kronheimer, Mrowka and Ozsváth [17] and Manolescu [18].

For the indefinite case, Furuta and Li and, independently, Manolescu proved the following theorem.[1]

Theorem 1.5 [16; 21] *To each oriented homology 3–sphere Y, we can associate an invariant $\kappa(Y) \in \mathbb{Z}$ with the following properties:*

(a) *Suppose W is a smooth, spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then*

$$\kappa(Y_1) + q \geq \kappa(Y_0) + p - 1.$$

(b) *Suppose W is a smooth, oriented spin manifold with a single boundary Y, with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $q > 0$. Then*

$$\kappa(Y) + q \geq p + 1.$$

Furuta and Li, and, independently, Manolescu proved this theorem by considering $\mathrm{Pin}(2)$–equivariant K–theory on the Seiberg–Witten Floer spectrum. Some new bounds can be obtained from this theorem. For example, the Brieskorn sphere $+\Sigma(2, 3, 12n+1)$ does not bound a spin 4–manifold with intersection form $p(-E_8) \oplus p\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p > 0$.

The main purpose of this paper is to extend Theorem 1.2 to the case of spin cobordisms and get more constrains on the intersection form of a spin 4–manifold with boundary. Here is the first result:

[1] We give Manolescu's statement here. Furuta and Li's statement is slightly different.

Theorem 1.6 *For any $k \in \mathbb{Z}/8$, we can associate an invariant $\kappa o_k(Y)$ to each oriented homology sphere Y, with the following properties:*

(a) *$2\kappa o_k(Y)$ is an integer whose reduction modulo 2 is the Rokhlin invariant $\mu(Y)$.*

(b) *Suppose W is an oriented smooth spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p, q \geq 0$. Let $p = 4l + m$ for $l \in \mathbb{Z}$ and $m = 0, 1, 2, 3$. Then for any $k \in \mathbb{Z}/8$, we have the following inequalities:*

(i) *If $(\mu(Y_0), m) = (0,0), (0,3), (1,0), (1,1)$, then*

$$\kappa o_k(Y_0) + 2l + h(\mu(Y_0), m) \leq \kappa o_{k+q}(Y_1) + \beta^q_{k+q}. \tag{2}$$

(ii) *If $(\mu(Y_0), m) = (0,1), (0,2), (1,2), (1,3)$, then*

$$\kappa o_{k+4}(Y_0) + 2l + h(\mu(Y_0), m) \leq \kappa o_{k+q}(Y_1) + \beta^{4+q}_{k+q}. \tag{3}$$

Here $\beta^j_k = \sum_{i=0}^{j-1} \alpha_{k-i}$, where $\alpha_i = 1$ for $i \equiv 1, 2, 3, 5 \bmod 8$ and $\alpha_i = 0$ for $i \equiv 0, 4, 6, 7 \bmod 8$ (β^0_k is defined to be 0). The constants $h(\mu(Y_0), m)$ are listed below.

	$m = 0$	$m = 1$	$m = 2$	$m = 3$
$\mu(Y_0) = 0$	0	$\frac{5}{2}$	3	$\frac{3}{2}$
$\mu(Y_0) = 1$	0	$\frac{1}{2}$	3	$\frac{7}{2}$

Remark 1.7 When m is even, $\mu(Y_0)=\mu(Y_1)$ and $h(\mu(Y_0), m)$ is an integer. When m is odd, $\mu(Y_0) \neq \mu(Y_1)$ and $h(\mu(Y_0), m)$ is a half-integer.

Setting $p = q = 0$ in Theorem 1.6(b), we get:

Corollary 1.8 *If two homology spheres Y_0, Y_1 are homology cobordant to each other, then $\kappa o_k(Y_0) = \kappa o_k(Y_1)$ for any $k \in \mathbb{Z}/8$.*

The definition of κo_k is similar to that of κ; see Furuta and Li [16] and Manolescu [21]. Roughly, $\kappa o_k(Y)$ is defined as follows. Pick a metric g on Y. By doing finite-dimensional approximation to the Seiberg–Witten equations on (Y, g), we get a topological space I_ν with an action by $G = \mathrm{Pin}(2)$. After changing I_ν by a suitable suspension or desuspension, we consider the following construction: The inclusion of the S^1–fixed point set $I_\nu^{S^1}$ induces a map between the equivariant KO–groups,

$$i^*\colon \widetilde{\mathrm{KO}}_G(I_\nu) \to \widetilde{\mathrm{KO}}_G(I_\nu^{S^1}).$$

We choose a specific reduction $\varphi\colon \widetilde{\mathrm{KO}}_G(I_\nu^{S^1}) \to \mathbb{Z}$. It can be proved that the image of $\varphi \circ i^*$ is an ideal generated by $2^a \in \mathbb{Z}$. We define a as $\kappa o_k(Y)$. Different $k \in \mathbb{Z}/8$ correspond to different suspensions.

In Section 8 we calculate some examples using the results of Manolescu [21] about the Seiberg–Witten Floer spectrum of $\pm\Sigma(2,3,r)$.

Theorem 1.9 (a) *We have $\kappa o_i(S^3) = 0$ for any $i \in \mathbb{Z}/8$.*

(b) *For a positive integer r with $\gcd(r,6) = 1$, let $\Sigma(2,3,r)$ be the Brieskorn spheres oriented as boundaries of negative plumbings and let $-\Sigma(2,3,r)$ be the same Brieskorn spheres with the orientations reversed. The $\kappa o_i(\pm\Sigma(2,3,r))$ are listed below:*

	κo_0	κo_1	κo_2	κo_3	κo_4	κo_5	κo_6	κo_7
$\Sigma(2,3,12n-1)$	1	1	1	0	0	0	0	0
$-\Sigma(2,3,12n-1)$	0	0	-1	-1	0	0	0	0
$\Sigma(2,3,12n-5)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$-\Sigma(2,3,12n-5)$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$\Sigma(2,3,12n+1)$	0	0	0	0	0	0	0	0
$-\Sigma(2,3,12n+1)$	0	0	0	0	0	0	0	0
$\Sigma(2,3,12n+5)$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$-\Sigma(2,3,12n+5)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Remark 1.10 We see that $\kappa o_k(-Y) \neq -\kappa o_k(Y)$ in general, while $\kappa o_k(Y \# (-Y))$ is always 0 by Corollary 1.8. Therefore, κo_k is not additive under connected sum.

If we apply Theorem 1.6(b) to the case $Y_0 = Y_1 = S^3$, the result is weaker than Theorem 1.2. As is the case in K–theory (see Manolescu [21]), we can remedy this by considering the special property of $Y_0 \cong S^3$ called the Floer KO_G–split condition.

Theorem 1.11 *Let W be an oriented, smooth spin cobordism from Y_0 to Y_1, with intersection form $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $p \geq 0, q > 0$. Suppose Y_0 is Floer KO_G–split. Let $p = 4l + m$ for $l \in \mathbb{Z}$ and $m = 0,1,2,3$. Then we have the following inequalities:*

(a) *If $(\mu(Y_0), m) = (0,0), (0,3), (1,0), (1,1)$, then*

$$\kappa o_4(Y_0) + 2l + h(\mu(Y_0), m) + 1 \leq \kappa o_{4+q}(Y_1) + \beta^q_{4+q}. \tag{4}$$

(b) *If $(\mu(Y_0), m) = (0,1), (0,2), (1,2), (1,3)$, then*

$$\kappa o_4(Y_0) + 2l + h(\mu(Y_0), m) + 1 \leq \kappa o_q(Y_1) + \beta^{4+q}_q. \tag{5}$$

Here β^*_* and $h(\mu(Y_0), m)$ are the constants defined in Theorem 1.6.

In particular, S^3 is Floer KO_G–split. Applying $Y_0 = S^3$ to the previous theorem, we get the following useful corollary:

Corollary 1.12 *Let W be an oriented smooth spin 4–manifold whose boundary is a homology sphere Y. Suppose the intersection form of W is $p(-E_8) \oplus q\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with $p \geq 0, q > 0$. Then we have the following inequalities:*

- *If $p = 4l$, then $2l < \kappa o_{4+q}(Y) + \beta^q_{4+q}$.*
- *If $p = 4l+1$, then $2l + \frac{5}{2} < \kappa o_q(Y) + \beta^{4+q}_q$.*
- *If $p = 4l+2$, then $2l + 3 < \kappa o_q(Y) + \beta^{4+q}_q$.*
- *If $p = 4l+3$, then $2l + \frac{3}{2} < \kappa o_{4+q}(Y) + \beta^q_{4+q}$.*

Remark 1.13 If we set $Y = S^3$ in Corollary 1.12, we will recover Theorem 1.2. However, Corollary 1.12 is not enough to prove Theorem 1.4. In order to get the relative version of Theorem 1.4, we have to apply similar constructions on the fixed point set of the Adams operation. This will not be done in the present paper.

Combining the results in Theorem 1.9 with Corollary 1.12, we get some new explicit bounds on the intersection forms of spin 4–manifolds bounded by $\pm\Sigma(2,3,r)$. We give two of them here and refer to Section 8.2 for a complete list.

Example 1.14 We have the following conclusions:

- $-\Sigma(2,3,12n-1)$ does not bound a spin 4–manifold with intersection form $p(-E_8) \oplus (p+1)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p > 0$.
- $-\Sigma(2,3,12n-5)$ does not bound a spin 4–manifold with intersection form $p(-E_8) \oplus p\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $p > 1$.

The paper is organized as follows. In Section 2, we discuss some background material about Pin(2)–equivariant KO–theory. In Section 3, we prove Theorem 1.4 after recalling some basic facts and properties of the Adams operations. In Section 4, we review the basic properties of the Seiberg–Witten Floer spectrum. The numerical invariant κo_k is defined in Section 5 and Theorem 1.6 is proved in Section 6. In Section 7, we introduce the Floer KO_G–split condition and prove Theorem 1.11. In Section 8, we prove Theorem 1.9 and use Corollary 1.12 and Theorem 1.4 to obtain new constraints on the intersection form of a spin 4–manifold with given boundary.

Acknowledgements Many of the constructions are parallel to those in Furuta and Li [16] and Manolescu [21] and are credited throughout. I wish to thank Ciprian Manolescu for suggesting the problem that led to the results in this paper, and for his encouragement and enthusiasm. I am also grateful to the referee for comments on a previous version of this paper.

2 Equivariant KO–theory

2.1 General theory

In this subsection, we review some general facts about equivariant KO–theory, mostly from [30] and [4]. See [2; 3] for basic facts about ordinary K–theory and KO–theory.

Let G be a compact topological group and X be a compact G–space. We denote the Grothendieck group of real G–bundles over X by $\mathrm{KO}_G(X)$.

Fact 2.1 $\mathrm{KO}_G(\mathrm{pt}) = \mathrm{RO}(G)$. Here $\mathrm{RO}(G)$ denotes the real representation ring of G. For a general X, $\mathrm{KO}_G(X)$ is a $\mathrm{RO}(G)$–algebra (with unit).

Remark 2.2 We do not distinguish a representation of G from its representation space.

Fact 2.3 A continuous G–map $f\colon X \to Y$ induces a map $f^*\colon \mathrm{KO}_G(Y) \to \mathrm{KO}_G(X)$.

Fact 2.4 For each subgroup $H \subseteq G$, by restricting the G–action to H, which makes a G–bundle into an H–bundle, we get a functorial restriction map $r\colon \mathrm{KO}_G(X) \to \mathrm{KO}_H(X)$.

Fact 2.5 If G acts freely on X, then the pullback map $\mathrm{KO}(X/G) \to \mathrm{KO}_G(X)$ is a ring isomorphism.

Fact 2.6 For a real irreducible representation space V of G, $\mathrm{End}_G(V)$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\mathbb{Z}\,\mathrm{Ir}_{\mathbb{R}}$, $\mathbb{Z}\,\mathrm{Ir}_{\mathbb{C}}$ and $\mathbb{Z}\,\mathrm{Ir}_{\mathbb{H}}$ denote the free abelian groups generated by irreducible representations of respective types and let $\mathrm{KSp}(X)$ be the Grothendieck group of quaternionic vector bundles over X. Then if G acts trivially on X, we have

$$\mathrm{KO}_G(X) = (\mathrm{KO}(X) \otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{R}}) \oplus (\mathrm{K}(X) \otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{C}}) \oplus (\mathrm{KSp}(X) \otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{H}}). \tag{6}$$

Now suppose X has a distinguished base point p which is fixed by G. Then we define $\widetilde{\mathrm{KO}}_G(X)$ (the reduced KO–group) to be the kernel of the map $\mathrm{KO}_G(X) \to \mathrm{KO}_G(p)$. For a based space X with trivial action, we also have

$$\widetilde{\mathrm{KO}}_G(X) = (\widetilde{\mathrm{KO}}(X) \otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{R}}) \oplus (\widetilde{\mathrm{K}}(X) \otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{C}}) \oplus (\widetilde{\mathrm{KSp}}(X) \otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{H}}). \tag{7}$$

The following fact is proved in [2, Corollary 3.1.6] (which only proved the complex K–theory case, but the proof works without modification in the real case).

Fact 2.7 Suppose X is a finite, based G–CW complex and the G–action is free away from the base point. Then any element in $\widetilde{\mathrm{KO}}_G(X) \cong \widetilde{\mathrm{KO}}(X/G)$ is nilpotent.

Recall that the augmentation ideal $\mathfrak{a} \subset \mathrm{RO}(G)$ is the kernel of the forgetful map $\mathrm{RO}(G) \cong \mathrm{KO}_G(\mathrm{pt}) \to \mathrm{KO}(\mathrm{pt}) \cong \mathbb{Z}$. Any element in $\mathfrak{a}$ defines an element in $\widetilde{\mathrm{KO}}_G(X)$. By the above fact, we get:

Fact 2.8 Suppose X is a finite, based G–CW complex and the G–action is free away from the base point. Then any element in the augmentation ideal acts on $\widetilde{\mathrm{KO}}^*_G(X)$ nilpotently.

Fact 2.9 For pointed spaces X, Y, there is a natural product map

$$\widetilde{\mathrm{KO}}_G(X) \otimes \widetilde{\mathrm{KO}}_G(Y) \to \widetilde{\mathrm{KO}}_G(X \wedge Y).$$

Fact 2.10 For pointed spaces X, Y, we have $\widetilde{\mathrm{KO}}_G(X \vee Y) \cong \widetilde{\mathrm{KO}}_G(X) \oplus \widetilde{\mathrm{KO}}_G(Y)$.

Let V be a real representation space of G. Denote the reduced suspension $V^+ \wedge X$ by $\Sigma^V X$. The following equivariant version of real Bott periodicity theorem was proved in [4].

Fact 2.11 Suppose the dimension n of V is divisible by 8 and V is a spin representation (which means that the group action $G \to \mathrm{SO}(n) \subset \mathrm{End}(V)$ factors through $\mathrm{Spin}(n)$). Then we have the Bott isomorphism

$$\varphi_V\colon \widetilde{\mathrm{KO}}_G(X) \cong \widetilde{\mathrm{KO}}_G(\Sigma^V X),$$

given by multiplication by the Bott class $b_V \in \widetilde{\mathrm{KO}}_G(V^+)$ under the natural map

$$\widetilde{\mathrm{KO}}_G(V^+) \otimes \widetilde{\mathrm{KO}}_G(X) \to \widetilde{\mathrm{KO}}_G(\Sigma^V X).$$

The Bott isomorphism is functorial under the pointed map $X \to X'$.

Fact 2.12 Bott classes behave well under the restriction map, which means that $i^* b_V = b_{i^*(V)}$. Here i^* is the restriction map (see Fact 2.4) and $i^*(V)$ is the restriction of the representation to the subgroup.

2.2 Pin(2)–equivariant KO–theory

In this section, we will review some important facts about Pin(2)–equivariant KO–theory. Detailed discussions can be found in [29]. See [1] and [5] for general facts about equivariant KO–theory and equivariant stable homotopy theory. From now on, we assume $G \cong \mathrm{Pin}(2)$ unless otherwise noted. Recall that the group Pin(2) can be defined as $S^1 \oplus jS^1 \subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}$. We have

$$\mathrm{RO}(\mathrm{Pin}(2)) \cong \mathbb{Z}[D, K, H]/(D^2-1, DK-K, DH-H, H^2-4(1+D+K)).$$

The representation space of D is $\mathbb{R}$, where the identity component $S^1 \subset \mathrm{Pin}(2)$ acts trivially and $j \in \mathrm{Pin}(2)$ act as multiplication by -1.

The representation space of K is $\mathbb{C} \cong \mathbb{R} \oplus i\mathbb{R}$, where $z \in S^1 \subset \mathrm{Pin}(2)$ acts as multiplication by z^2 (in $\mathbb{C}$) and j acts as reflection along the diagonal.

The representation space of H is $\mathbb{H}$, where the action is given by the left multiplication of $\mathrm{Pin}(2) \subset \mathbb{H}$.

We will also write $\mathbb{R}$ as the trivial one-dimensional representation of G.

Following the notation of [29], we denote $\widetilde{\mathrm{KO}}_G((kD+lH)^+)$ by $\mathrm{KO}_G(kD+lH)$ (we choose ∞ as the base point). Then for $k, l, m, n \in \mathbb{Z}_{\geq 0}$ we have the multiplication map

$$\mathrm{KO}_G(kD+lH) \otimes \mathrm{KO}_G(mD+nH) \to \mathrm{KO}_G((k+m)D+(l+n)H). \tag{8}$$

In order to define this map, we need to fix the identification between $(kD \oplus lH) \oplus (mD \oplus nH)$ and $(k+m)D \oplus (l+n)H$ by sending $(x_1 \oplus y_1) \oplus (x_2 \oplus y_2)$ to $(x_1, x_2) \oplus (y_1, y_2)$. By considering G–equivariant homotopy, it is not hard to see that the multiplication map is commutative when k or l is even. (We will prove that the multiplication map is actually commutative for any k, l, after we give the structure of $\mathrm{KO}_G(kD+lH)$ in Theorem 2.13.)

It is easy to prove (see [29]) that $8D$, $H+4D$ and $2H$ are spin representations. Therefore, we can choose Bott classes $b_{8D} \in \mathrm{KO}_G(8D)$, $b_{2H} \in \mathrm{KO}_G(2H)$ and $b_{H+4D} \in \mathrm{KO}_G(H+4D)$. Multiplication by these classes induces isomorphisms

$$\begin{aligned}\mathrm{KO}_G(kD+lH) \cong \mathrm{KO}_G((k+8)D+lH) &\cong \mathrm{KO}_G((k+4)D+(l+1)H) \\ &\cong \mathrm{KO}_G(kD+(l+2)H).\end{aligned}$$

Since the Bott classes are in the center, it doesn't matter whether we multiply on the left or on the right. Moreover, we can choose the Bott classes to be compatible with each other, which means that $b_{8D}b_{2H} = b_{H+4D}^2$. We fix this choice of Bott classes throughout the paper.

For $k, l \in \mathbb{Z}$, the RO(G)–module $\mathrm{KO}_G(kD+lH)$ is defined to be $\mathrm{KO}_G((k+8a)D+(l+2b)H)$ for any $a, b \in \mathbb{Z}$ which satisfy $k+8a \geq 0$ and $l+2b \geq 0$. Since the Bott classes are chosen to be compatible, the groups defined by different choices of a, b are canonically identified to each other. Again, because the Bott classes are in the center, the multiplication map (8) can now be extended to all $k, l, m, n \in \mathbb{Z}$.

Consider the inclusion $i\colon 7D^+ \to 8D^+$. There is a unique element $\gamma(D) \in \mathrm{KO}_G(-D)$ which satisfies $\gamma(D) b_{8D} = i^*(b_{8D})$. The map

$$\mathrm{KO}_G((k+1)D+lH) \xrightarrow{\cdot\gamma(D)} \mathrm{KO}_G(kD+lH)$$

is just the map induced by the inclusion $kD \oplus lH \to (k+1)D \oplus lH$ for $k, l \geq 0$. Similarly, we can define $\gamma(H) \in \mathrm{KO}_G(-H)$ and $\gamma(H+4D) = \gamma(H)\gamma(D)^4$. Since left multiplication and right multiplication by $\gamma(D)$ or $\gamma(H)$ just correspond to different inclusions of subspaces, which are homotopic to each other, we see that $\gamma(D)$ and $\gamma(H)$ are both in the center.

By Bott periodicity, we only have to compute $\mathrm{KO}_G(lD)$ for $l = -2, -1, 0, \ldots, 5$. This was done by Schmidt, and we list the result here:

Theorem 2.13 [29] *We have the following isomorphisms of $\mathbb{Z}$–modules:*

(a) *Let $A = K - (1+D)$ and $B = H - 2(1+D)$. Then*

$$\begin{aligned}\mathrm{KO}_G(\mathrm{pt}) &\cong \mathrm{RO}(\mathrm{Pin}(2)) \\ &\cong \mathbb{Z}[D, A, B]/(D^2-1, DA-A, DB-B, B^2-4(A-2B)).^2\end{aligned}$$

(b) $\mathrm{KO}_G(-lD) \cong \mathbb{Z} \oplus \bigoplus_{n \geq 1} \mathbb{Z}/2$ *for $l = 1, 2$, generated by $\gamma(D)^{|l|}$ and $\gamma(D)^{|l|}A^n$ for $n \geq 1$.*

(c) $\mathrm{KO}_G(D) \cong \mathbb{Z}$, *generated by $\eta(D)$.*

(d) $\mathrm{KO}_G(lD) \cong \mathbb{Z} \oplus \bigoplus_{m \geq 0} \mathbb{Z}/2$ *for $l = 2, 3$. For $l = 2$, the generators are $\eta(D)^2$ and $\gamma(D)^2 A^m c$, $m \geq 0$; for $l = 3$, they are $\gamma(D)\lambda(D)$ and $\gamma(D)A^m c$, $m \geq 0$.*

(e) $\mathrm{KO}_G(4D)$ *is freely generated by $\lambda(D)$, $D\lambda(D)$, $A^n\lambda(D)$ and $A^m c$ for $m \geq 0$ and $n \geq 1$.*

(f) $\mathrm{KO}_G(5D) \cong \mathbb{Z}$, *generated by $\eta(D)\lambda(D)$.*

Corollary 2.14 *The multiplication map* (8) *is commutative.*

[2] There is a typo in [29], where the relation between A and B is $B^2 - 2(A-2B)$.

Proof We just need to check that $\gamma(D)$, $\eta(D)$, $\lambda(D)$, c commute with each other. This is easy since $\lambda(D)$ and c are in $\mathrm{KO}_G(kD)$ for even k, while $\gamma(D)$ is in the center by our discussion before. □

For our purpose, we don't need to know the explicit constructions of $\eta(D), \lambda(D)$ and c. We just need to know the following properties:

- $\eta(D)$ is the Hurewicz image of an element $\tilde{\eta}(D) \in \pi^0_G(D)$ (G–equivariant stable cohomotopy group of D^+). If we forget about the G–action, $\tilde{\eta}(D)$ is just the Hopf map in $\pi_1^{\mathrm{st}}(\mathrm{pt})$.

- For $\lambda(D)$ and $c \in \mathrm{KO}_G(4D)$, by Bott periodicity and formula (7), we have isomorphisms

$$\begin{aligned}\mathrm{KO}_G(4D) &\cong \mathrm{KO}_G(8D+4)\\ &\cong \mathrm{KO}_G(4)\\ &\cong (\widetilde{\mathrm{KO}}(S^4)\otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{R}}) \oplus (\tilde{\mathrm{K}}(S^4)\otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{C}}) \oplus (\widetilde{\mathrm{KSp}}(S^4)\otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{H}}).\end{aligned}$$

(Here $4 \in \mathrm{RO}(G)$ denotes the trivial 4–dimensional real representation.)

- We can choose suitable Bott classes such that, under these isomorphisms, $\lambda(D)$ corresponds to

$$([V_H]-4\mathbb{R})\otimes 1 \in \widetilde{\mathrm{KO}}(S^4)\otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{R}}$$

and c corresponds to

$$([V_{\mathbb{H}}]-\mathbb{H})\otimes H \in \widetilde{\mathrm{KSp}}(S^4)\otimes \mathbb{Z}\,\mathrm{Ir}_{\mathbb{H}}\,.$$

Here $V_{\mathbb{H}}$ is the quaternion Hopf bundle over $S^4 \cong \mathbb{H}P^2$, $\mathbb{H}$ and $\mathbb{R}$ denote the trivial bundles, and $1, H$ are elements in $\mathrm{RO}(G)$.

- Let $\lambda(H)$ and $c(H)$ be the images of $\lambda(D)$ and c under the Bott isomorphism $\mathrm{KO}_G(4D) \cong \mathrm{KO}_G(8D+H) \cong \mathrm{KO}_G(H)$. Then $\mathrm{KO}_G(H)$ is generated as an $\mathrm{RO}(G)$–algebra by $\lambda(H)$ and $c(H)$.

Remark 2.15 Notice that the element $[V_H]\otimes H \in \mathrm{KSp}\,S^4 \otimes \mathbb{Z}Ir_{\mathbb{H}}$ is represented by the bundle $V_H \otimes_{\mathbb{H}} H$. Hence it is a real bundle of dimension 4 (not 16).

For further discussions, we need to know the multiplicative structures of $\mathrm{KO}_G(lD)$, which are also given by Schmidt. We list some of them that are useful for us:

Theorem 2.16 [29] *The following relations hold:*

(a) $H\lambda(D) = 4c$, $Hc = (A + 2 + 2D)\lambda(D)$, $Dc = c$.

(b) $(D + 1)\gamma(D) = 2A\gamma(D) = B\gamma(D) = 0$.

(c) $(D + 1)\eta(D) = A\eta(D) = B\eta(D) = 0$.

(d) $\gamma(D)\eta(D) = 1 - D$, $\gamma(D)\lambda(D) = \eta(D)^3$.

(e) $\gamma(D)^8 b_{8D} = 8(1 - D)$, $\gamma(H)^2 b_{2H} = K - 2H + D + 5$.

(f) $\gamma(H + 4D)b_{H+4D} = 4(1 - D)$.

(g) $\eta(D)\lambda(D) = \gamma(D)^3 b_{8D}$, $\eta(D)c = 0$.

(h) $\gamma(H)\lambda(H) = 4 - H$ *and* $\gamma(H)c(H) = H - 1 - D - K$.

3 The Adams operations

3.1 Basic properties

In this subsection we give a quick review about the basic properties of the Adams operations. See [2] and [6] for more detailed discussions. Some of the calculations can be found in [29], but we give them here for completeness. For simplicity and concreteness, we only deal with $\psi^k\colon \mathrm{KO}_G(X) \to \mathrm{KO}_G(X)$ for an actual G–space X and we don't do localizations (like [29]).

Let $\mathrm{KO}_G(X)[\![t]\!]$ be the formal power series with coefficients in $\mathrm{KO}_G(X)$. For a bundle E over X, we define $\lambda_t(E) \in \mathrm{KO}_G(X)[\![t]\!]$ to be $\sum_{i=0} t^i[\lambda^i(E)]$. Here $\lambda^i(E)$ is the i^{th} exterior power of E. We let $\psi^0(E) = \operatorname{rank}(E)$, and define

$$\psi_t(E) = \sum_{i=0} t^i \psi^i(E) \in \mathrm{KO}_G(X)[\![t]\!]$$

by

$$\psi_t(E) = \psi^0(E) - t\frac{d}{dt}\log\lambda_{-t}(x). \tag{9}$$

It turns out that, for any $k \in \mathbb{Z}_{\geq 0}$, ψ^k extends to a well-defined operation on $\mathrm{KO}_G(X)$, which satisfies the following nice properties:

(a) ψ^k is functorial with respect to continuous maps $f\colon X \to X'$.

(b) ψ^k maps $\widetilde{\mathrm{KO}}_G(X)$ to $\widetilde{\mathrm{KO}}_G(X)$.

(c) For all $x, y \in \mathrm{KO}_G(X)$,
$$\psi^k(x + y) = \psi^k(x) + \psi^k(y) \quad \text{and} \quad \psi^k(xy) = \psi^k(x)\psi^k(y).$$

(d) If x is a line bundle, then $\psi^k(x) = x^k$.

The effect of the Adams operations on the Bott classes can be described by the Bott cannibalistic class. Given a spin G–bundle E over X with rank $n \equiv 0 \bmod 8$, the Bott cannibalistic class $\theta_k^{\mathrm{or}}(E) \in \mathrm{RO}(G)$ is defined by the equation

$$\psi^k(b_E) = \theta_k^{\mathrm{or}}(E) \cdot b_E \quad \text{for } k > 1. \tag{10}$$

When k is odd, this can be explicitly written as (see [6])[3]

$$\theta_k^{\mathrm{or}}(E) = k^{n/2} \prod_{u \in J} \lambda_{-u}(E)(1-u)^{-n}. \tag{11}$$

Here J is a set of k^{th} roots of unity $u \neq 1$ such that J contains exactly one element from each pair $\{u, u^{-1}\}$. Notice that we can define $\theta_k^{\mathrm{or}}(E)$ for any real bundle E of even dimension using formula (11). It can be shown that

$$\theta_k^{\mathrm{or}}(E+F) = \theta_k^{\mathrm{or}}(E)\theta_k^{\mathrm{or}}(F).$$

Now let's specialize to the case $k = 3$. By formula (9), it is easy to check that $\psi^3(x) = x^3 - 3\lambda^2(x)x + 3\lambda^3(x)$. We want to calculate the action of ψ^3 on $\mathrm{RO}(G)$. Since the G–action on H preserves the orientation, we have $\lambda^3(H) = \lambda^1(H) = H$. Using complexification, it is easy to show $\lambda^2(H) = K + D + 3$. Also, we have $\lambda^2(K) = D$. Therefore, we get[4]

$$\psi^3(D) = D, \quad \psi^3(H) = HK - H, \quad \psi^3(K) = K^3 - 3K,$$
$$\psi^3(A) = A^3 + 6A^2 + 9A, \quad \psi^3(B) = AB + B + 4A.$$

Also, applying formula (11), we get

$$\theta_3^{\mathrm{or}}(2) = 3, \quad \theta_3^{\mathrm{or}}(2D) = 1 + 2D, \quad \theta_3^{\mathrm{or}}(H) = A + B + 4D + 5.$$

3.2 Proof of Theorem 1.4

The central part of the proof is the following proposition:

Proposition 3.1 *For any integers $r, a, b \geq 0$ and $l > 0$, there does not exist a G–equivariant map*

$$f\colon (r\mathbb{R} + aD + (4l+b)H)^+ \to (r\mathbb{R} + (a+8l+2)D + bH)^+$$

that induces a homotopy equivalence on the G–fixed point set.

[3] There is a typo in [6, Equation 3.10.4].

[4] There is a typo in [29], where $\psi^3(H) = HK - K$.

Proof Suppose there exists such a map f. After suspension by copies of $\mathbb{R}$, D and H, we can assume $a = 8l' + 6$, $r = 8d$ and $b = 2k$. Let

$$V_1 = 8d\mathbb{R} + 2kH + 8(l + l' + 1)D,$$
$$V_2 = 8d\mathbb{R} + (4l + 2k)H + (8l' + 8)D.$$

Let b_{V_1} and b_{V_2} be the Bott classes of V_1 and V_2, respectively. Consider the element $x = f^*(b_{V_1})$. By the Bott isomorphism and Theorem 2.13(b), we can write x as $b_{V_2}\gamma(D)^2\alpha$ for some $\alpha \in \mathrm{RO}(G)$. Moreover, we can assume $\alpha = p + Ah(A)$ for some integer p and some polynomial $h(A)$ whose coefficients are either 0 or 1.

Claim p is even and $h = 0$.

This is essentially a special case of [29, Proposition 5.21] for $\mathrm{KO}(4l, 8l + 2)$.[5]

By formula (10), we have: $\psi^3(b_{V_1}) = \theta_3^{\mathrm{or}}(V_1) \cdot b_{V_1}$, which implies that

$$\psi^3(x) = f^*(\psi^3(b_{V_1})) = \theta_3^{\mathrm{or}}(V_1) \cdot x. \tag{12}$$

Notice that $x = i^*(b_{V_2} \cdot \alpha)$, where $i\colon (8d\mathbb{R} + (4l + 2k)H + (8l' + 6)D)^+ \to V_2^+$ is the standard inclusion. By formula (10), we have

$$\psi^3(x) = i^*(\psi^3(b_{V_2} \cdot \alpha)) = \theta_3^{\mathrm{or}}(V_2) b_{V_2} \psi^3(\alpha) \cdot \gamma(D)^2. \tag{13}$$

Comparing (12) and (13), we get

$$\big(\theta_3^{\mathrm{or}}(V_2)\psi^3(\alpha) - \theta_3^{\mathrm{or}}(V_1)\alpha\big)\gamma(D)^2 = 0. \tag{14}$$

We can calculate

$$\theta_3^{\mathrm{or}}(V_1) = 3^{4d}(1 + 2D)^{4l+4l'+4}(A + B + 4D + 5)^{2k},$$
$$\theta_3^{\mathrm{or}}(V_2) = 3^{4d}(1 + 2D)^{4l'+4}(A + B + 4D + 5)^{2k+4l}.$$

Notice that $2A\gamma(D) = B\gamma(D) = (1 + D)\gamma(D) = 0$, so we can simplify (14) as

$$3^{4d}((A + 1)^{2k}\alpha - (A + 1)^{4l+2k}\psi^3(\alpha)) \cdot \gamma(D)^2 = 0. \tag{15}$$

Since $\alpha = p + Ah(A)$, we have $\psi^3(\alpha) = p + (A^3 + 6A^2 + 9A)h(A^3 + 6A^2 + 9A)$. Using the relation $2A\gamma(D) = 0$, we can further simplify (15) and get

$$3^{4d} \cdot g(A) \cdot \gamma(D)^2 = 0. \tag{16}$$

Here $g(A) = (A + 1)^{2k}(p + Ah(A)) - (A + 1)^{2k+4l}(p + (A^3 + A)h(A^3 + A))$.

By Theorem 2.13(b), we see that if we expand $g(A)$ as a polynomial in A, the degree-0 coefficient should be 0 and all other coefficients should be even. By our assumption,

[5] There is an error in [29] for $\mathrm{KO}(c, d)$ when $4c - d \equiv -3 \bmod 8$, but we won't consider this case.

the coefficients of h are either 0 or 1. Checking the leading coefficient of $g(A)$, it is easy to see that $h=0$ and $g(A)=p\big((A+1)^{2k}-(A+1)^{2k+4l}\big)$. This implies that p is even. The claim is proved.

Now consider the following commutative diagram:

$$\begin{tikzcd} \widetilde{\mathrm{KO}}_G(V_1^+) \arrow[r, "f^*"] \arrow[d, "\cdot\gamma(H)^{2k}\gamma(D)^{8l+8l'+8}"] & \widetilde{\mathrm{KO}}_G((8d\mathbb{R}+(8l'+6)D+(4l+2k)H)^+) \arrow[d, "\cdot\gamma(H)^{4l+2k}\gamma(D)^{8l'+6}"] \\ \widetilde{\mathrm{KO}}_G((8d\mathbb{R})^+) \arrow[r, "\cong"] & \widetilde{\mathrm{KO}}_G((8d\mathbb{R})^+) \end{tikzcd} \tag{17}$$

The vertical maps are given by the inclusions of subspaces. The bottom map is an isomorphism because f induces a homotopy equivalence on the G–fixed point set. Any automorphism on $\widetilde{\mathrm{KO}}_G((8d\mathbb{R})^+)$ is given by the multiplication of a unit $\tilde{u}\in\mathrm{RO}(G)$. Therefore, we obtain

$$\begin{aligned}\tilde{u}\cdot b_{V_1}\cdot\gamma(H)^{2k}\gamma(D)^{8l+8l'+8} &= x\cdot\gamma(H)^{4l+2k}\gamma(D)^{8l'+6}\\ &= b_{V_2}\cdot\gamma(D)^{8l'+8}\gamma(H)^{4l+2k}\cdot p.\end{aligned}\tag{18}$$

Applying the relations in Theorem 2.16, we simplify this to

$$\begin{aligned}&(K-2H+D+5)^{2l+k}(8(1-D))^{l'+1}\cdot p\\ &\qquad = (K-2H+D+5)^{k}(8(1-D))^{l+l'+1}\cdot\tilde{u}.\end{aligned}\tag{19}$$

Now consider the ring homomorphism $\varphi_0\colon \mathrm{RO}(G)\to\mathbb{Z}$ defined by $\varphi_0(D)=-1$, $\varphi_0(A)=\varphi_0(B)=0$. Notice that $\varphi_0(\tilde{u})=\pm1$ since $\tilde{u}$ is a unit. We get $p=\pm1$, which is a contradiction. This finishes the proof of Proposition 3.1. □

Now suppose X is a closed, oriented, smooth spin 4–manifold with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $p=8l>0$ and $q<p+3$. After doing surgery on loops and taking connect sum with copies of $S^2\times S^2$, we can assume $b_1(W)=0$ and $q=8l+2$. As shown in [14], by doing finite-dimensional approximation of the Seiberg–Witten equations on W, we get a G–equivariant map

$$f\colon (aD+(4l+b)H)^+\to((a+8l+2)D+bH)^+ \quad \text{for some } a,b>0.$$

Moreover, f induces a homotopy equivalence on the G–fixed point set. This is a contradiction to Proposition 3.1. Therefore, Theorem 1.4 is proved.

4 Pin(2)–equivariant Seiberg–Witten Floer theory

Manolescu constructed a Pin(2)–equivariant spectrum class $S(Y,\mathfrak{s})$ for each rational homology sphere Y with a spin structure $\mathfrak{s}$. We will not repeat the constructions here but just collect some useful properties. See [18; 19; 21] for the explicit constructions.

Definition 4.1 Let $s \in \mathbb{Z}_{\geq 0}$. A space of type SWF (at level s) is a pointed, finite G–CW complex X with the following properties:

(a) The S^1–fixed point set X^{S^1} is G–homotopy equivalent to the sphere $(sD)^+$. We define $\mathrm{lev}(X)$ to be s.

(b) The action of G is free on the complement $X - X^{S^1}$.

Definition 4.2 Let X, X' be two spaces of type SWF at level k and k' respectively. A pointed G–map $f\colon X \to X'$ is called admissible if f preserves the base point and satisfies one of the following two conditions:

(a) $k < k'$ and the induced map on the G–fixed point set $f^G\colon X^G \to X'^G$ is a homotopy equivalence.

(b) $k = k'$ and the induced map on the S^1–fixed point set $f^{S^1}\colon X^{S^1} \to X'^{S^1}$ is a homotopy equivalence.

Now consider the set of triples (X,a,b) where X is a space of type SWF and $a \in \mathbb{Z}$, $b \in \mathbb{Q}$.

Definition 4.3 We say that (X,a,b) is stably equivalent to (X',a',b') if $b-b' \in \mathbb{Z}$ and for some $M,N,r > 0$, there exists a G–homotopy equivalence

$$\Sigma^{r\mathbb{R}}\Sigma^{(M-a)D}\Sigma^{(N-b)H}X \cong \Sigma^{r\mathbb{R}}\Sigma^{(M-a')D}\Sigma^{(N-b')H}X'.$$

(Here $\mathbb{R}$ denotes the trivial representation of G.)

Remark 4.4 In [21], Manolescu worked with stable even equivalence, which requires X to be a space of type SWF at even level.

This triple can be thought of the "formal desuspension" of X with a copies of D and b copies of H. We denote $\mathfrak{C}$ to be the set of stable equivalence classes of triples (X,a,b). Informally, we call an element in $\mathfrak{C}$ a spectrum class.

Definition 4.5 For a spectrum class $S = [(X,a,b)] \in \mathfrak{C}$, we let

$$\mathrm{lev}(S) = \mathrm{lev}(X) - a.$$

Remark 4.6 By considering the S^1–fixed point set, we see that two spaces of type SWF at different levels are not G–homotopic to each other. Using this fact, it is easy to prove that $\mathrm{lev}(S)$ is a well-defined quantity.

For $r \in \mathbb{Z}$ and $s \in \mathbb{Q}$, we can define the formal suspension $\Sigma^{rD+sH}\colon \mathfrak{C} \to \mathfrak{C}$ by sending $[(X, a, b)]$ to $[(X, a-r, b-s)]$. It's easy to check that this is a well-defined operation on the set $\mathfrak{C}$.

Now suppose Y is an oriented rational homology 3–sphere with a metric g and a spin structure $\mathfrak{s}$. Let $\mathbb{S}$ be the associated spinor bundle. We consider the global Coulomb splice

$$V = i \ker d^* \oplus \Gamma(\mathbb{S}) \subset i\Omega^1(Y) \oplus \Gamma(\mathbb{S}).$$

Using the quaternionic structure on $\mathbb{S}$, we can define a natural action of G on V: $e^{i\theta} \in G$ takes (α, ϕ) to $(e^{i\theta}\alpha, \phi)$ and $j \in G$ takes (α, ϕ) to $(-\alpha, j\phi)$.

Now we consider the self-adjoint first-order elliptic operator $l\colon V \to V$ defined by $l(\alpha, \phi) = (*d\alpha, \not{D}\phi)$, where $\not{D}$ is the Dirac operator.[6] For any $\tau < \nu$, let V_ν^τ be the subspace spanned by the eigenvectors of l with eigenvalues in the interval $(\tau, \nu]$. Then V_ν^τ is a finite-dimensional G–representation space which is isomorphic to $kD \oplus lH$. We denote $\dim_{\mathbb{R}} V(D)_\tau^\nu$ by k and $\dim_{\mathbb{H}} V(H)_\tau^\nu$ by m.

We pick $-\nu \ll 0 \ll \nu$. By considering the equivariant Conley index of the gradient flow of $\mathrm{CSD}|_{V_{-\nu}^{\nu}}$ (see [19; 21]), we get a G–space I_ν of type SWF at level $\dim_{\mathbb{R}} V(D)_{-\nu}^0$.

Next, we need to recall the definition of $n(Y, \mathfrak{s}, g)$. Choose a compact smooth spin 4–manifold N with $\partial N = Y$. Let $\mathrm{ind}_{\mathbb{C}} \not{D}(N)$ be the index of the Dirac operator on N (with Atiyah–Patodi–Singer boundary conditions). We can define

$$n(Y, \mathfrak{s}, g) := \mathrm{ind}_{\mathbb{C}} \not{D}(N) + \tfrac{1}{8}\sigma(N). \tag{20}$$

Remark 4.7 It can be proved that this definition does not depend on the choice of N. For a rational homology sphere Y, we have $n(Y, \mathfrak{s}, g) \in \frac{1}{8}\mathbb{Z}$. When Y is an integral homology sphere, $n(Y, \mathfrak{s}, g)$ is an integer and has the same parity as the Rokhlin invariant $\mu(Y)$.

We can consider the following element in $\mathfrak{C}$:[7]

$$S(Y, \mathfrak{s}) := \big[(I_\nu, \dim_{\mathbb{R}} V(D)_{-\nu}^0, \dim_{\mathbb{H}} V(H)_{-\nu}^0 + \tfrac{1}{2} n(Y, \mathfrak{s}, g))\big]. \tag{21}$$

Notice that the level of $S(Y, \mathfrak{s})$ is always 0.

[6] Since Y is a rational homology sphere, there is a unique flat spin connection on $\mathbb{S}$; we choose it as the base connection and use it to define $\not{D}$.

[7] Our convention is different from [19] and [21] , where the second component in the triple denotes the complex dimension of the G–representation.

Theorem 4.8 (Manolescu [19; 21]) *The element $S(Y,\mathfrak{s}) \in \mathfrak{C}$ is independent of the metric g, the cut-off ν and the other choices in the construction. Thus $S(Y,\mathfrak{s})$ is an invariant of the pair $(Y,\mathfrak{s})$.*

Remark 4.9 In this paper, since we only use the numerical invariants, we don't need to make $\mathfrak{C}$ a category and $S(Y,\mathfrak{s})$ a functor. Therefore, we don't define $S(Y,\mathfrak{s})$ as a natural spectrum invariant. See Section 3.4 of [18] for a discussion about naturality.

Suppose W is a smooth spin cobordism between rational homology 3–spheres Y_0 and Y_1 with $b_1(W)=0$. Further, we assume that W is equipped with a metric g and a spin structure $\mathfrak{t}$ such that $g|_{Y_i}=g_i$ and $\mathfrak{t}|_{Y_i}=\mathfrak{s}_i$.

The following theorem is important for our constructions:

Theorem 4.10 (Manolescu [19; 21]) *By doing finite-dimensional approximation for the Seiberg–Witten equations on W, we obtain an admissible map*

$$f\colon \Sigma^{a_0 D}\Sigma^{b_0 H}(I_0)_\nu \to \Sigma^{a_1 D}\Sigma^{b_1 H}(I_1)_\nu. \tag{22}$$

Here $(I_0)_\nu$ and $(I_1)_\nu$ are the Conley indices for the approximated Seiberg–Witten flow. Let V_i denotes the Coulomb slice on Y_i, for $i=0,1$. The differences in the suspension indices are

$$a_0-a_1=\dim_{\mathbb{R}} V_1(D)^0_{-\nu}-\dim_{\mathbb{R}} V_0(D)^0_{-\nu}-b_2^+(W), \tag{23}$$

$$\begin{aligned} b_0-b_1=\dim_{\mathbb{H}} V_1(H)^0_{-\nu}-\dim_{\mathbb{H}} V_0(H)^0_{-\nu}& \\ +\tfrac{1}{2}n(Y_1,\mathfrak{s}_1,g_1)-\tfrac{1}{2}n(Y_0,\mathfrak{s}_0,g_0)-\tfrac{1}{16}\sigma(W).& \end{aligned} \tag{24}$$

5 Numerical invariants

Let Y be a rational homology sphere and $\mathfrak{s}$ be a spin structure on Y. In the previous section, we defined an invariant $S(Y,s)\in\mathfrak{C}$. In this section, we will extract a set of numerical invariants $\kappa o_i(Y,s)$ from $S(Y,s)$ for $i\in\mathbb{Z}/8$.

Definition 5.1 For $l=-2,-1,0,\dots,5$, we define the group homomorphisms

$$\varphi_l\colon \mathrm{KO}(lD)\to\mathbb{Z}$$

as follows (see Theorem 2.13):

(a) For $l=0$, $\varphi_l(D)=-1$ and $\varphi_l(A)=\varphi_l(B)=0$, then extend φ_l by the multiplicative structure on $\mathrm{RO}(G)$.

(b) For $l=-1,-2$, $\varphi_l(\gamma(D)^{|l|})=1$ and $\varphi_l(\gamma(D)^{|l|}A^n)=0$ for $n\geq 1$.

(c) For $l=1$, $\varphi_l(\eta(D))=1$.

(d) For $l=2$, $\varphi_l(\eta(D)^2)=1$ and $\varphi_l(\gamma(D)^2A^mc)=0$.

(e) For $l=3$, $\varphi_l(\gamma(D)\lambda(D))=1$ and $\varphi_l(\gamma(D)A^mc)=0$.

(f) For $l=4$, $\varphi_l(\lambda(D))=1$, $\varphi_l(D\lambda(D))=-1$, and $\varphi_l(A^n\lambda(D))=\varphi_l(A^mc)=0$.

(g) For $l=5$, $\varphi_l(\eta(D)\lambda(D))=1$.

For other values of $l\in\mathbb{Z}$, we use the Bott isomorphism to identify $\mathrm{KO}(lD)$ with $\mathrm{KO}((l-8k)D)$ for $-2\leq l-8k\leq 5$ and apply the above definition.

Lemma 5.2 *For any $a\in\mathrm{KO}_G(\mathrm{pt})$ and $b\in\mathrm{KO}_G(kD)$, we have $\varphi_0(a)\varphi_k(b)=\varphi_k(a\cdot b)$.*

Proof This is a straightforward calculation using Theorem 2.13 and Theorem 2.16. □

Remark 5.3 The map φ_0 just takes the trace of $j\in\mathrm{Pin}(2)$; the other φ_l are defined such that the torsion elements are killed and Lemma 5.2 holds.

We consider the map $\tau\colon D^+\to D^+$ which maps x to $-x$. By suspension with copies of D, we get an admissible involution $\tau\colon(kD)^+\to(kD)^+$ for $k>0$.

The following lemma is a straightforward corollary of the equivariant Hopf theorem (see [7]).

Lemma 5.4 *When $0\leq k<l$, any admissible map $f\colon(kD)^+\to(lD)^+$ is G–homotopic to the standard inclusion. For $0\leq k=l$, any admissible map $f\colon(kD)^+\to(kD)^+$ is either homotopic to τ or to the identity map, depending on $\deg(f)$.*

The map τ induces the involution $\tau^*\colon\mathrm{KO}_G(kD)\to\mathrm{KO}_G(kD)$. For $k,l>0$ and any $a\in\mathrm{KO}_G(kD)$, $b\in\mathrm{KO}_G(lD)$, the following equalities are easy to check by Lemma 5.4:

$$\tau^*(a)\cdot b=a\cdot\tau^*(b)=\tau^*(a\cdot b)\quad\text{and}\quad\tau^*(a)\cdot\tau^*(b)=a\cdot b. \tag{25}$$

Using this fact, we can define $\tau^*\colon\mathrm{KO}_G(kD)\to\mathrm{KO}_G(kD)$ for any $k\in\mathbb{Z}$ by identifying $\mathrm{KO}_G(kD)$ with $\mathrm{KO}_G(k'D)$ for any $0<k'\equiv k$ mod 8 using Bott periodicity. Moreover, formula (25) now holds for all $k,l\in\mathbb{Z}$.

Now consider the element $u\in\mathrm{RO}(G)$ defined by $\tau^*(b_{8D})=u\cdot b_{8D}$. Then for $l\in\mathbb{Z}$ and any element $\alpha\in\mathrm{KO}_G(lD)$, we have $\tau^*(\alpha)\cdot b_{8D}=\alpha\cdot\tau^*(b_{8D})=(u\alpha)\cdot b_{8D}$, which implies $\tau^*(\alpha)=u\alpha$.

Lemma 5.5 *We have the following properties about τ^* and u:*

(a) *τ^* acts as identity on $\mathrm{KO}_G(lD)$ for $l \neq 0, 4 \bmod 8$.*

(b) *u is a unit with $\varphi_0(u) = 1$.*

(c) *$\varphi_l \circ \tau^* = \varphi_l$ for any $l \in \mathbb{Z}$.*

Proof (a) We have $\gamma(D)b_{8D} = i^*(b_{8D})$, where i^* is the inclusion $(7D)^+ \to (8D)^+$. Therefore, we get $\tau^*(\gamma(D)b_{8D}) = (\tau \circ i)^*(b_{8D})$. By Lemma 5.4, $\tau \circ i$ is G–homotopic to i, thus $\tau^*(\gamma(D)b_{8D}) = i^*(b_{8D}) = \gamma(D)b_{8D}$, which implies that $\tau^*(\gamma(D)) = \gamma(D)$.

Since τ^* induces an involution on $\mathrm{KO}_G(D) \cong \mathbb{Z}$, we have $\tau^*(\eta(D)) = \pm\eta(D)$. But since

$$\tau^*(\eta(D)) \cdot \gamma(D) = \eta(D) \cdot \tau^*(\gamma(D)) = \eta(D)\gamma(D) = 1 - D \neq -\eta(D)\gamma(D),$$

we get $\tau^*(\eta(D)) = \eta(D)$.

By formula (25), $\tau^*(a) = a$ implies $\tau^*(ab) = ab$ for any b. Therefore, we see that τ^* acts as the identity map on $\mathrm{KO}_G(kD)$ for $k \neq 0, 4 \bmod 8$.

(b) We have $u^2 = 1$ because $\tau^2 = \mathrm{id}$. Since

$$u \cdot (1 - D) = \tau^*(1 - D) = \tau^*(\gamma(D) \cdot \eta(D)) = \gamma(D) \cdot \eta(D) = 1 - D,$$

we see that $(u - 1)(1 - D) = 0$. We get $\varphi_0(u) = 1$ by Lemma 5.2.

(c) This is straightforward from (b) and Lemma 5.2. □

Now suppose X is a space of type SWF at level l. A choice of G–homotopy equivalence $X^{S^1} \cong (lD)^+$ gives us an inclusion map $i\colon (lD)^+ \to X$, which we call a trivialization; this induces the map $i^*\colon \widetilde{\mathrm{KO}}_G(X) \to \mathrm{KO}_G(lD)$. Consider the map $\varphi_l \circ i^*\colon \widetilde{\mathrm{KO}}_G(X) \to \mathbb{Z}$.

Proposition 5.6 *The submodule $\mathrm{Im}(i^*)$ and the map $\varphi_l \circ i^*$ are both independent of the choice of the trivialization. Moreover, we have $\mathrm{Im}(\varphi_l \circ i^*) = (2^k)$ for some $k \in \mathbb{Z}_{\geq 0}$.*

Proof By Lemma 5.4, there are two possible trivializations, i and $i \circ \tau$. We have $\mathrm{Im}(i \circ \tau)^* = \tau^*(\mathrm{Im}\, i^*) = u \cdot \mathrm{Im}(i^*)$. Since u is a unit, the multiplication by u does not change the submodule $\mathrm{Im}(i^*)$. Moreover, we have $\varphi_l \circ (i \circ \tau)^* = \varphi_l \circ \tau^* \circ i^* = \varphi_l \circ i^*$ by Lemma 5.5(c).

For the second statement, we consider the exact sequence

$$\cdots \to \widetilde{\mathrm{KO}}_G(X) \xrightarrow{i^*} \mathrm{KO}_G(lD) \xrightarrow{\delta} \widetilde{\mathrm{KO}}{}^1_G(X/X^{S^1}) \to \cdots$$

Since the G–action is free away from the basepoint and $(1-D) \in \mathrm{RO}(G)$ is in the augmentation ideal, $(1-D)$ acts on $\widetilde{\mathrm{KO}}{}^1_G(X/X^{S^1})$ nilpotently by Fact 2.8. Therefore, we can find $m \gg 0$ such that $(1-D)^m\,\mathrm{KO}_G(lD) \subset \ker(\delta) = \mathrm{Im}(i^*)$. It follows that $2^m \in \mathrm{Im}(\varphi_l \circ i^*)$ and $\mathrm{Im}(\varphi_l \circ i^*) = (2^k)$ for some $0 \le k \le m$. □

Proposition 5.6 justifies the following definition:

Definition 5.7 For a G–space X of type SWF at level l, we define $\mathcal{J}(X)$ to be the image of i^* for any trivialization i, and let $\kappa o(X)$ be the integer k such that $\varphi_l(\mathcal{J}(X)) = (2^k)$.

Let's study the properties of $\mathcal{J}(X)$ and $\kappa o(X)$. First, recall that we defined the constants $\beta_k^0 = 0$ and $\beta_k^j = \sum_{i=0}^{j-1} \alpha_{k-i}$ for $j \ge 1$, where $\alpha_i = 1$ for $i \equiv 1,2,3,5 \bmod 8$ and $\alpha_i = 0$ for $i \equiv 0,4,6,7 \bmod 8$. It's easy to see that $\beta_j^k = \beta_{j'}^k$ for $j \equiv j' \pmod 8$. The integers β_j^k are important because of the following proposition:

Proposition 5.8 *For integers $0 \le j \le k$ and an admissible map $i\colon ((k-j)D)^+ \to (kD)^+$, we have the following commutative diagram, where the map $m_k^j\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by $2^{\beta_k^j}$:*

$$\begin{tikzcd}
\mathrm{KO}_G(kD) \arrow[r, "i^*"] \arrow[d, "\varphi_k"] & \mathrm{KO}_G((k-j)D) \arrow[d, "\varphi_{k-j}"] \\
\mathbb{Z} \arrow[r, "m_k^j"] & \mathbb{Z}
\end{tikzcd} \tag{26}$$

Proof The case $j=0$ follows from Lemma 5.5. When $j>0$, by Lemma 5.4, the map i is G–homotopic to the standard inclusion. Because of the associativity of i^* and m_l^k, we only need to prove the case $j=1$. In this case, the map i^* is just the multiplication by $\gamma(D)$ and m_k^1 is the multiplication by 2^{α_k}. Since both φ_k and i^* are compatible with the Bott isomorphism, we only need to check the cases $k = 1, 2, \dots, 8$. This can be proved by straightforward calculations using Definition 5.1, Theorem 2.16 and Theorem 2.13. □

The following proposition studies the behavior of $\mathcal{J}(X)$ and $\kappa o(X)$ under the Bott isomorphism:

Proposition 5.9 *Let X be a space of type* SWF *at level k. We have the following:*

(a) $\mathcal{J}(X)\cdot b_{8D} = \mathcal{J}(\Sigma^{8D}X)$ *and* $\kappa o(\Sigma^{8D}X) = \kappa o(X)$.

(b) $\mathcal{J}(X)\cdot(K-2H+D+5) = \mathcal{J}(\Sigma^{2H}X)$ *and* $\kappa o(\Sigma^{2H}X) = \kappa o(X)+2$.

(c) $\kappa o(\Sigma^{H+4D}X) = \kappa o(X) + 3 - \beta^4_{k+4}$.

Proof (a) Since $(\Sigma^{8D}X)^{S^1} = \Sigma^{8D}(X^{S^1})$, statement (a) follows from the functoriality of the Bott isomorphism.

(b) We have the commutative diagram induced by the inclusions of subspaces:

$$\begin{tikzcd} \widetilde{\mathrm{KO}}_G(\Sigma^{2H}X) \arrow[r] \arrow[d] & \widetilde{\mathrm{KO}}_G(X) \arrow[d] \\ \widetilde{\mathrm{KO}}_G((\Sigma^{2H}X)^{S^1}) \arrow[r, "\cong"] & \widetilde{\mathrm{KO}}_G(X^{S^1}) \end{tikzcd} \tag{27}$$

Since $(\Sigma^{2H}X)^{S^1} = \Sigma^{2H}(X^{S^1})$, the map in the bottom row is the identity. If we identify $\widetilde{\mathrm{KO}}_G(\Sigma^{2H}X)$ with $\widetilde{\mathrm{KO}}_G(X)$ using the Bott isomorphism, then the top horizontal map is the multiplication by $\gamma(H)^2 b_{2H} = K-2H+D+5$ (by Theorem 2.16). This implies that $\mathcal{J}(\Sigma^{2H}X) = (K-2H+D+5)\mathcal{J}(X)$. We also have $\kappa o(\Sigma^{2H}X) = \kappa o(X)+2$ since $\varphi_0(K-2H+D+5) = 4$.

(c) Again, by inclusions of subspaces, we have

$$\begin{tikzcd} \widetilde{\mathrm{KO}}_G(\Sigma^{H+4D}X) \arrow[r] \arrow[d] & \widetilde{\mathrm{KO}}_G(X) \arrow[d] \\ \mathrm{KO}_G((\Sigma^{H+4D}X)^{S^1}) \arrow[r, "\cdot\gamma(D)^4"] & \mathrm{KO}_G(X^{S^1}) \end{tikzcd}$$

Since $(\Sigma^{H+4D}X)^{S^1} \cong \Sigma^{4D}(X^{S^1})$, the bottom horizontal map is the multiplication by $\gamma(D)^4$. If we identify $\widetilde{\mathrm{KO}}_G(\Sigma^{H+4D}X)$ with $\widetilde{\mathrm{KO}}_G(X)$ using the Bott isomorphism, the top horizontal map is the multiplication by $\gamma(H+4D)b_{H+4D} = 4(1-D)$ (by Theorem 2.16). Therefore, under appropriate trivializations, we see that the maps

$$i_1^*\colon \widetilde{\mathrm{KO}}_G(X) \cong \widetilde{\mathrm{KO}}_G(\Sigma^{H+4D}X) \to \mathrm{KO}_G((k+4)D)$$

and

$$i_2^*\colon \widetilde{\mathrm{KO}}_G(X) \to \mathrm{KO}_G(kD)$$

are related by $\gamma(D)^4\cdot i_1^*(x) = 4(1-D)\cdot i_2^*(x)$. Since $\varphi_0(4(1-D)) = 8$, statement (c) follows from Proposition 5.8 (for $j=4$) and Lemma 5.2. □

We have the following proposition, which is the analogue of [21, Lemma 3.8].

Proposition 5.10 *Let X_1 and X_2 be spaces of type* SWF. *Suppose there is a based G–equivariant homotopy equivalence f from $\Sigma^{r\mathbb{R}} X_1$ to $\Sigma^{r\mathbb{R}} X_2$ for some $r \geq 0$. Then we have $\mathcal{J}(X_1) = \mathcal{J}(X_2)$ and $\kappa o(X_1) = \kappa o(X_2)$.*

Proof The proof in [21] works with some modifications. Suppose X_1, X_2 are both at level k. By Proposition 5.9(a), we can replace X_i by $\Sigma^{8D} X_i$ and assume $k > 1$. Also, we can suspend some more copies of $\mathbb{R}$ and assume that $8 \mid r$. Choose trivializations i_1 and i_2 of X_1 and X_2, respectively. They give homotopy equivalences

$$(r\mathbb{R} + kD)^+ \cong (\Sigma^{r\mathbb{R}} X_1)^{S^1} \quad \text{and} \quad (r\mathbb{R} + kD)^+ \cong (\Sigma^{r\mathbb{R}} X_2)^{S^1}.$$

Composing them with $f^{S^1}\colon (\Sigma^{r\mathbb{R}} X_1)^{S^1} \to (\Sigma^{r\mathbb{R}} X_2)^{S^1}$, we get the equivariant homotopy equivalence $h\colon (r\mathbb{R} + kD)^+ \to (r\mathbb{R} + kD)^+$. Since $k > 1$, by the equivariant Hopf theorem, h is based-homotopic to $\tau_1 \wedge \tau_2$. The map $\tau_1\colon (r\mathbb{R})^+ \to (r\mathbb{R})^+$ is either the identity or a map with degree -1. Therefore, $\tau_1^*(b_{r\mathbb{R}}) = a \cdot b_{r\mathbb{R}}$, where $b_{r\mathbb{R}}$ is the Bott class and $a \in \mathrm{RO}(G)$ is a unit. Also, $\tau_2\colon (kD)^+ \to (kD)^+$ is either the identity or the map τ we defined before. Therefore, $\tau_2^*(x)$ is either x or ux (see Lemma 5.5). We have shown that the map $h^*\colon \widetilde{\mathrm{KO}}_G((r\mathbb{R} + kD)^+) \to \widetilde{\mathrm{KO}}_G((r\mathbb{R} + kD)^+)$ is just multiplication by some unit in $\mathrm{RO}(G)$, which does not change any submodule.

Now consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{KO}}_G(X_2) \arrow[r, "\cong"] \arrow[d, "i_2^*"] & \widetilde{\mathrm{KO}}_G(\Sigma^{r\mathbb{R}} X_2) \arrow[r, "f^*"] \arrow[d, "(\Sigma^{r\mathbb{R}} i_2)^*"] & \widetilde{\mathrm{KO}}_G(\Sigma^{r\mathbb{R}} X_1) \arrow[r, "\cong"] \arrow[d, "(\Sigma^{r\mathbb{R}} i_1)^*"] & \widetilde{\mathrm{KO}}_G(X_1) \arrow[d, "i_1^*"] \\
\mathrm{KO}_G(kD) \arrow[r, "\cong"] & \widetilde{\mathrm{KO}}_G((r\mathbb{R}+kD)^+) \arrow[r, "h^*"] & \widetilde{\mathrm{KO}}_G((r\mathbb{R}+kD)^+) \arrow[r, "\cong"] & \mathrm{KO}_G(kD)
\end{tikzcd}$$

In each row, the first map is a Bott isomorphism and the third map is the inverse to a Bott isomorphism. We see that $b_{r\mathbb{R}} \cdot \mathrm{Im}(i_2^*) = h^*(b_{r\mathbb{R}} \cdot \mathrm{Im}(i_2^*)) = b_{r\mathbb{R}} \cdot \mathrm{Im}(i_1^*)$. Therefore, we have $\mathrm{Im}(i_1^*) = \mathrm{Im}(i_2^*)$, which implies $\kappa o(X_1) = \kappa o(X_2)$. $\square$

Definition 5.11 For a spectrum class $S = [(X, a, b)] \in \mathfrak{C}$, we let

$$\kappa o(S) = \kappa o(\Sigma^{(8M-a)D} \Sigma^{(2N-b')H} X) - 2N - s \tag{28}$$

for any $M, N, b' \in \mathbb{Z}$ and $s \in [0, 1)$ making $8M - a \geq 0$, $2N - b' \geq 0$ and $b = b' + s$.

Proposition 5.12 *$\kappa o(S)$ is well-defined.*

Proof By Proposition 5.9(a)–(b), it's easy to prove that the right-hand side of formula (28) is independent of the choice of M, N. By choosing $M, N \gg 0$, we see that changing the representative of S from (X, a, b) to $(\Sigma^D X, a+1, b)$ or $(\Sigma^H X, a, b+1)$ does not change the value of $\kappa o(S)$. By Definition 4.3 and Proposition 5.10, we proved that $\kappa o(S)$ does not change when we change the representative of the spectrum class. □

By definition of the suspension of a spectrum class and Proposition 5.9, it is easy to prove:

Proposition 5.13 *For any spectrum class $S \in \mathfrak{C}$ at level k, we have:*

- $\kappa o(\Sigma^{8D} S) = \kappa o(S)$.
- $\kappa o(\Sigma^{2H} S) = \kappa o(S) + 2$.
- $\kappa o(\Sigma^{H+4D} S) = \kappa o(S) + 3 - \beta^4_{k+4}$.

With these discussions, we can now define the invariants for 3–manifolds.

Definition 5.14 For an oriented rational homology sphere Y and a spin structure $\mathfrak{s}$ on Y, we define $\kappa o_i(Y, \mathfrak{s}) = \kappa o(\Sigma^{iD} S(Y, \mathfrak{s}))$ for any $i \in \mathbb{Z}_{\geq 0}$. Then $\kappa o_i(Y, \mathfrak{s}) = \kappa o_{i+8}(Y, \mathfrak{s})$, which allow us to define $\kappa o_i(Y, \mathfrak{s})$ for $i \in \mathbb{Z}/8$.

6 Proof of Theorem 1.6

In this section, we will prove Theorem 1.6.

Let X_0, X_1 be two spaces of type SWF at level k_0 and k_1, respectively. Suppose there is an admissible map $f\colon X_0 \to X_1$ (which implies $k_0 \leq k_1$). By Proposition 5.8, we can choose suitable trivializations such that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{\mathrm{KO}}_G(X_1) \arrow[r, "f^*"] \arrow[d, "i_1^*"] & \widetilde{\mathrm{KO}}_G(X_0) \arrow[d, "i_0^*"] \\
\mathrm{KO}_G(k_1 D) \arrow[r, "(f^{S^1})^*"] \arrow[d, "\varphi_{k_1}"] & \mathrm{KO}_G(k_0 D) \arrow[d, "\varphi_{k_0}"] \\
\mathbb{Z} \arrow[r, "m_{k_1}^{k_1-k_0}"] & \mathbb{Z}
\end{tikzcd}$$

Therefore, we get $m_{k_1}^{k_1-k_0}(\mathrm{Im}(\varphi_{k_1} \circ i_1^*)) \subset \mathrm{Im}(\varphi_{k_0} \circ i_0^*)$. This implies that

$$(2^{\kappa o(X_1)+\beta_{k_1}^{k_1-k_0}}) \subset (2^{\kappa o(X_0)}) \subset \mathbb{Z}.$$

Therefore, we get the following proposition:

Proposition 6.1 *Let X_0, X_1 be two spaces of type* SWF *at levels k_0 and k_1, respectively. Suppose there is an admissible map $f\colon X_0 \to X_1$. Then we have*

$$\kappa o(X_0) \le \kappa o(X_1) + \beta_{k_1}^{k_1-k_0}. \tag{29}$$

Next we generalize this inequality to the spectrum classes:

Definition 6.2 Let $S_0, S_1 \in \mathfrak{C}$ be two spectrum classes. We say that S_0 dominates S_1 if we can find representatives $S_i = [(X_i, a, b)]$ for $i = 1, 2$ and an admissible map f from X_0 to X_1.

Proposition 6.3 *Let $S_0, S_1 \in \mathfrak{C}$ be two spectrum classes at levels k_0 and k_1, respectively. Suppose S_0 dominates S_1. Then we have*

$$\kappa o(S_0) \le \kappa o(S_1) + \beta_{k_1}^{k_1-k_0}. \tag{30}$$

Proof Since an admissible map $f\colon X_0 \to X_1$ gives an admissible map

$$\Sigma^{aH+bD} f\colon \Sigma^{aH+bD} X_0 \to \Sigma^{aH+bD} X_1$$

for any $a, b \in \mathbb{Z}_{\ge 0}$, this proposition is a straightforward corollary of Proposition 6.1 and Definition 5.11. □

By considering the natural inclusion $X \to \Sigma^D X$, it is easy to see that S always dominates $\Sigma^D S$. Therefore, we get the following corollary, which will be useful in Section 8.

Corollary 6.4 *For any spectrum class $S \in \mathfrak{C}$ at level k, we have*

$$\kappa o(S) \le \kappa o(\Sigma^D S) + \alpha_{k+1}.$$

Now let Y_0, Y_1 be two rational homology 3–spheres and $\mathfrak{s}_i$ be spin structures on them, respectively. Suppose $(W, \mathfrak{s})$ is a smooth oriented spin cobordism from $(Y_0, \mathfrak{s}_0)$ to $(Y_1, \mathfrak{s}_1)$. After doing surgery along loops in W, we can assume that $b_1(W) = 0$ without loss of generality. Then by Theorem 4.10, we see that

$$\Sigma^{-\frac{\sigma(W)}{16}H} S(Y_0, \mathfrak{s}_0) \quad \text{dominates} \quad \Sigma^{b_2^+(W)D} S(Y_1, \mathfrak{s}_1).$$

We can do suspensions and prove that

$$\Sigma^{-\frac{\sigma(W)}{16}H}(\Sigma^{kD} S(Y_0, \mathfrak{s}_0)) \quad \text{dominates} \quad \Sigma^{(b_2^+(W)+k)D} S(Y_1, \mathfrak{s}_1)$$

for any $k \in \mathbb{Z}$. Applying Proposition 6.3, we get:

Theorem 6.5 *Suppose $(W,\mathfrak{s})$ is a smooth, oriented spin cobordism from $(Y_0,\mathfrak{s}_0)$ to $(Y_1,\mathfrak{s}_1)$. Then for any $k\in\mathbb{Z}$, we have the inequality*

$$\kappa o_{k+b_2^+(W)}(Y_1,\mathfrak{s}_1)+\beta_{k+b_2^+(W)}^{b_2^+(W)}\geq \kappa o\Big(\Sigma^{-\frac{\sigma(W)}{16}H}(\Sigma^{kD}S(Y_0,\mathfrak{s}_0))\Big). \tag{31}$$

In general, $\kappa o(\Sigma^{-\frac{\sigma(W)}{16}H}(\Sigma^{kD}S(Y_0,\mathfrak{s}_0)))$ can be expressed in terms of $\kappa o_k(Y_0,\mathfrak{s}_0)$ or $\kappa o_{k+4}(Y_0,\mathfrak{s}_0)$, but the explicit formula is messy. For simplicity, we now focus on the integral homology sphere case.

Remark 6.6 Suppose Y is an oriented integral homology 3–sphere. There is a unique spin structure $\mathfrak{s}$ on Y, and we simply write $S(Y,\mathfrak{s})$ and $\kappa o_i(Y,\mathfrak{s})$ as $S(Y)$ and $\kappa o_i(Y)$, respectively.

Suppose both Y_i are integral homology spheres, then the intersection form of W is a unimodular, even form. Let's assume that the intersection from can be decomposed as

$$p(-E_8)\oplus q\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad\text{for } p,q\geq 0.$$

In this case, we have $\frac{1}{16}\sigma(W)=-\frac{1}{2}p$ and $b_2^+(W)=q$. Recall that the spectrum class invariant $S(Y_0)$ is defined by

$$\big[(I_\nu,\dim_{\mathbb{R}}V(D)^0_{-\nu},\dim_{\mathbb{H}}V(H)^0_{-\nu}+\tfrac{1}{2}n(Y_0,\mathfrak{s},g))\big].$$

The third component of this triple may be an integer or a half-integer, depending on the Rokhlin invariant $\mu(Y_0)$.

Proposition 6.7 *Let Y_0 be an integral homology 3–sphere and $p\in\mathbb{Z}_{\geq 0}$. Then we have the following relations:*

(a) *Suppose $\mu(Y_0)=0\in\mathbb{Z}_2$. We have*

$$\begin{aligned}
\kappa o\big(\Sigma^{\frac{p}{2}H}(\Sigma^{kD}S(Y_0))\big)&=\kappa o_k(Y_0)+2l && \text{for } p=4l,\\
\kappa o\big(\Sigma^{\frac{p}{2}H}(\Sigma^{kD}S(Y_0))\big)&=\kappa o_{k+4}(Y_0)+\tfrac{5}{2}+2l-\beta_k^4 && \text{for } p=4l+1,\\
\kappa o\big(\Sigma^{\frac{p}{2}H}(\Sigma^{kD}S(Y_0))\big)&=\kappa o_{k+4}(Y_0)+3+2l-\beta_k^4 && \text{for } p=4l+2,\\
\kappa o\big(\Sigma^{\frac{p}{2}H}(\Sigma^{kD}S(Y_0))\big)&=\kappa o_k(Y_0)+2l+\tfrac{3}{2} && \text{for } p=4l+3.
\end{aligned}$$

(b) *Suppose $\mu(Y_0)=1\in\mathbb{Z}_2$. We have*

$$\begin{aligned}
\kappa o\big(\Sigma^{\frac{p}{2}H}(\Sigma^{kD}S(Y_0))\big)&=\kappa o_k(Y_0)+2l && \text{for } p=4l,\\
\kappa o\big(\Sigma^{\frac{p}{2}H}(\Sigma^{kD}S(Y_0))\big)&=\kappa o_k(Y_0)+2l+\tfrac{1}{2} && \text{for } p=4l+1,\\
\kappa o\big(\Sigma^{\frac{p}{2}H}(\Sigma^{kD}S(Y_0))\big)&=\kappa o_{k+4}(Y_0)+3+2l-\beta_k^4 && \text{for } p=4l+2,\\
\kappa o\big(\Sigma^{\frac{p}{2}H}(\Sigma^{kD}S(Y_0))\big)&=\kappa o_{k+4}(Y_0)+\tfrac{7}{2}+2l-\beta_k^4 && \text{for } p=4l+3.
\end{aligned}$$

Proof Let's denote $(I_\nu, \dim_{\mathbb{R}} V(D)^0_{-\nu}, \dim_{\mathbb{H}} V(H)^0_{-\nu} + \frac{1}{2}n(Y_0, \mathfrak{s}, g))$ by (X, a, b). For $\mu(Y_0) = 0$ and $p = 4l$, we have $b \in \mathbb{Z}$. Take $M, N \gg 0$ and let $N' = N + l$. Then by Definition 5.11 we have

$$\begin{aligned}\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) &= \kappa o(\Sigma^{(8M+k-a)D}\Sigma^{(2N+2l-b)H} X) - 2N \\ &= \kappa o(\Sigma^{(8M+k-a)D}\Sigma^{(2N'-b)H} X) - 2N' + 2l \\ &= \kappa o_k(Y) + 2l.\end{aligned} \tag{32}$$

For $p = 4l + 1$, take $M, N \gg 0$ and let $N' = N + l$. Then we have

$$\begin{aligned}\kappa o(\Sigma^{\frac{p}{2}H}(\Sigma^{kD} S(Y_0))) &= \kappa o(\Sigma^{(8M+k-a)D}\Sigma^{(2N+2l+1-b)H} X) - 2N - \tfrac{1}{2} \\ &= \kappa o(\Sigma^{H}(\Sigma^{kD}(X, a, b))) + 2l - \tfrac{1}{2} \\ &= \kappa o_{k+4}(Y_0) + \tfrac{5}{2} + 2l - \beta^4_k.\end{aligned} \tag{33}$$

The other cases can be proved similarly. □

Now, combining the above proposition and Theorem 6.5, we obtain Theorem 1.6.

7 The KO_G–split condition

Now consider the space $X = (8kD + (2l+1)H)^+$ for $k, l \in \mathbb{Z}_{\geq 0}$. We have the map

$$i^*\colon \widetilde{\mathrm{KO}}_G(X) \to \mathrm{KO}_G(8kD)$$

induced by the inclusion. By Theorem 2.13, we see that $\mathrm{KO}_G(8kD + (2l+1)H)$ is generated by $(b_{2H})^l(b_{8D})^k\lambda(H)$ and $(b_{2H})^l(b_{8D})^k c(H)$ as an $\mathrm{RO}(G)$–module, and the map i^* is multiplication by $\gamma(H)^{2l+1}$. Using Theorem 2.16, we get

$$\begin{aligned}i^*((b_{2H})^l(b_{8D})^k\lambda(H)) &= (2 + A - 2D - 2B)^l(2 - 2D - B)\cdot(b_{8D})^k, \\ i^*((b_{2H})^l(b_{8D})^k c(H)) &= (A - 2B)^l(B - A)\cdot(b_{8D})^k.\end{aligned} \tag{34}$$

This discussion motivates the following definition:

Definition 7.1 Let X be a space of type SWF at level $8k$. X is called even KO_G–split if $\mathcal{J}(X)$ is the submodule generated by $(2+A-2D-2B)^l(2-2D-B)\cdot(b_{8D})^k$ and $(A-2B)^l(B-A)\cdot(b_{8D})^k$ for some $l \in \mathbb{Z}_{\geq 0}$.

Next, we consider the space $X = ((8k+4)D + 2lH)^+$. The map

$$i^*\colon \widetilde{\mathrm{KO}}_G(X) \to \mathrm{KO}_G((8k+4)D)$$

is just multiplication by $\gamma(H)^{2l}$. We know $\widetilde{\mathrm{KO}}_G(X) = \mathrm{KO}_G((8k+4)D) \cdot (b_{2H})^l$ by the Bott isomorphism. Since

$$\gamma(H)^{2l}(b_{2H})^l = (K - 2H + D + 5)^l = (A + 2D + 6 - 2H)^l$$

(see Theorem 2.16), we have

$$\mathrm{Im}(i^*) = (A + 2D + 6 - 2H)^l \cdot \mathrm{KO}_G((8k+4)D) \subset \mathrm{KO}_G((8k+4)D).$$

This motivates the following definition:

Definition 7.2 Let X be a space of type SWF at level $8k+4$. X is called odd KO_G–split if $\mathcal{J}(X) = (A + 2D + 6 - 2H)^l \cdot \mathrm{KO}_G((8k+4)D)$ for some $l \in \mathbb{Z}_{\geq 0}$.

KO_G–split spaces are special because of the following proposition (cf Proposition 6.1).

Proposition 7.3 *Let X_0, X_1 be two spaces of type* SWF *at levels k_0, k_1, respectively, and let f be an admissible map from X_0 to X_1. Suppose that $k_0 < k_1$ and X_0 is odd or even KO_G–split (which implies that $k_0 \equiv 0$ or $4 \bmod 8$). Then we have*

$$\kappa o(X_0) < \kappa o(X_1) + \beta_{k_1}^{k_1 - k_0}. \tag{35}$$

Before proving this proposition, we need to make a digression into the general properties of $\mathrm{KO}_G(4D)$ and $\mathrm{RO}(G)$.

Lemma 7.4 *The following properties hold:*

(a) *Any element in $\mathrm{RO}(G)$ can be uniquely written as $bD + f(A) + Bg(A)$ for some polynomials f, g and integer b.*

(b) *Any element in $\mathrm{RO}(G)$ can be uniquely written as $bD + f(A) + Hg(A)$ for some polynomials f, g and integer b.*

(c) *Any element in $\mathrm{KO}_G(4D)$ can be uniquely written as $bD\lambda(D) + f(A)\lambda(D) + g(A)c$ for some polynomials f, g and integer b.*

(d) *The map $\mathrm{RO}(G) \to \mathrm{KO}_G(4D)$ defined by multiplication by $\lambda(D)$ is injective.*

(e) *An element $\omega = bD\lambda(D) + f(A)\lambda(D) + g(A)c$ belongs to $\mathrm{RO}(G)\lambda(D)$ if and only if $4 \mid g(A)$. Moreover, if $(A + 2D + 6 - 2H)^l\omega \in \mathrm{RO}(G)\cdot\lambda(D)$ for some l, then $\omega \in \mathrm{RO}(G)\cdot\lambda(D)$.*

(f) *Suppose $(A - 2B)^l h(A, B) = 0 \in \mathrm{RO}(G)$ for some two-variable polynomial h in A, B. Then we have $h(A, B) = 0$ in $\mathrm{RO}(G)$.*

(g) *Suppose $f(D) = h(A, B)$ for some 2–variable polynomial h without degree-0 term and some polynomial f. Then $h(A, B) = 0$.*

Proof (a)–(d) can be proved by straightforward calculation using Theorem 2.13. The first statement of (e) is a corollary of (b), (c) and the relation $H\lambda(D) = 4c$. Let's prove the second statement of (e). We have $Hc = (1 + D + K)\lambda(D)$ and $(2D+6)c = 8c = 2H\lambda(D)$. Therefore, $(A+2D+6-2H)^l\omega \in \mathrm{RO}(G)\lambda(D)$ implies $A^l\omega \in \mathrm{RO}(G)\lambda(D)$. It follows that $4 \mid A^l g(A)$, which implies that $4 \mid g(A)$ and $\omega \in \mathrm{RO}(G)\lambda(D)$.

For (f), we can assume that $h(A, B) = f(A) + Bg(A)$ for some polynomials f, g. Consider the map $\psi\colon \mathrm{RO}(G) \to \mathbb{Q}[x]$ defined by $\psi(D) = 1, \psi(B) = x$ and $\psi(A) = x^2/4 + 2x$. Then

$$0 = \psi((A-2B)^l(f(A) + Bg(A))) = \Big(\frac{x^2}{4}\Big)^l\Big(f\Big(\frac{x^2}{4} + 2x\Big) + xg\Big(\frac{x^2}{4} + 2x\Big)\Big),$$

which implies that $0 = f(x^2/4+2x) + xg(x^2/4+2x)$. Considering the leading term in x, we see that $f(x) = g(x) = 0$.

For (g), we can simplify $h(A, B)$ as $Ag_1(A) + Bg_2(A)$ for some polynomials g_1, g_2 by the relation $B^2 - 4(A-2B) = 0$. Then the conclusion follows from (a). □

Lemma 7.5 *Suppose $a(1-D)\lambda(D) \in (A+2D+6-2H)^l\,\mathrm{KO}_G(4D)$ for some $a \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$. Then we have $2^{2l+1} \mid \varphi_4(a(1-D)\lambda(D))$.*

Proof Since $\varphi_4(a(1-D)\lambda(D)) = 2a$, the conclusion is trivial when $l = 0$. Now suppose $l > 0$. Let $a(1-D)\lambda(D) = (A+2D+6-2H)^l\cdot\omega$ for some $\omega \in \mathrm{KO}_G(4D)$. By Lemma 7.4(e), we see that $\omega \in \mathrm{RO}(G)\lambda(D)$. Write ω as $(bD + f(A) + Bg(A))\lambda(D)$. By Lemma 7.4(d), we get $a(1-D) = (A-2B-2D+2)^l(bD + f(A) + Bg(A))$. Using the relation $(1-D)A = (1-D)B = 0$, we can simplify this equality as $a(1-D)-(f(0)+bD)(2-2D)^l = (A-2B)^l(b+f(A)+Bg(A))$. By Lemma 7.4(g), we get that $(A-2B)^l(b + f(A) + Bg(A)) = 0 \in \mathrm{RO}(G)$. By Lemma 7.4(f), we have $b+f(A)+Bg(A) = 0$. This implies that $\omega = b(D-1)\lambda(D)$ and $\varphi_4(a(1-D)\lambda(D)) = -2^{2l+1}b$ for some $b \in \mathbb{Z}$. □

Lemma 7.6 *Suppose $a(1-D)$ is in the ideal of $\mathrm{RO}(G)$ generated by*

$$(2 + A - 2D - 2B)^l(2-2D-B) \quad \text{and} \quad (A-2B)^l(B-A)$$

for some $l \in \mathbb{Z}_{\geq 0}$. Then we have $2^{2l+3} \mid \varphi_0(a(1-D))$.

Proof We assume $l > 0$ first. By Lemma 7.4(a) and the relation $A(1-D) = B(1-D) = 0$, we can express $a(1-D)$ as

$$(2-2D-B)(2-2D+A-2B)^l(b(1-D) + f_1(A) + Bg_1(A)) \\ + (A-2B)^l(B-A)(f_2(A) + Bg_2(A)) \tag{36}$$

for some integer b and polynomials f_1, f_2, g_1, g_2.

As in the proof of Lemma 7.5, we can simplify this formula and use Lemma 7.4(g) to get

$$\begin{aligned}(37)\quad & -B(A-2B)^l(f_1(A)+Bg_1(A)) \\ & \qquad + (A-2B)^l(B-A)(f_2(A)+Bg_2(A)) = 0 \in \mathrm{RO}(G).\end{aligned}$$

We have $-B(f_1(A)+Bg_1(A)) + (B-A)(f_2(A)+Bg_2(A)) = 0$ by Lemma 7.4(f). Simplifying this, we obtain

$$\begin{aligned}(38)\quad & -4Ag_1(A) - Af_2(A) + 4Ag_2(A) \\ & \qquad + B\big(-f_1(A)+f_2(A)+8g_1(A)-Ag_2(A)-8g_2(A)\big) = 0.\end{aligned}$$

This implies that

$$-4Ag_1(A) - Af_2(A) + 4Ag_2(A) = 0,$$
$$-f_1(A) + 8g_1(A) + f_2(A) - Ag_2(A) - 8g_2(A) = 0.$$

Considering the degree-1 term of the first identity, we get $4 \mid f_2(0)$. Also, we have $8 \mid -f_1(0) + f_2(0)$ by checking the degree-0 term of the second identity. Therefore, we have $4 \mid f_1(0)$, which implies that $\varphi_0(a(1-D)) = 2^{2l+2}(2b + f_1(0))$ can be divided by 2^{2l+3}.

The case $l = 0$ is similar. We also get the identity (38). □

Proof of Proposition 7.3 Consider the commutative diagram

$$\begin{tikzcd} \widetilde{\mathrm{KO}}_G(X_1) \arrow[r, "f^*"] \arrow[d, "i_1^*"] & \widetilde{\mathrm{KO}}_G(X_0) \arrow[d, "i_0^*"] \\ \mathrm{KO}_G(k_1 D) \arrow[r, "(f^{S^1})^*"] \arrow[d, "\varphi_{k_1}"] & \mathrm{KO}_G(k_0 D) \arrow[d, "\varphi_{k_0}"] \\ \mathbb{Z} \arrow[r, "m_{k_1}^{k_1-k_0}"] & \mathbb{Z} \end{tikzcd}$$

(a) Suppose X_0 is odd KO_G–split. Then $k_0 = 8k+4$ for some integer k and $\mathrm{KO}_G(k_0 D) = \mathrm{KO}_G(4D) \cdot (b_{8D})^k$ by the Bott isomorphism. Moreover,

$$\mathrm{Im}(i_0^*) = (A + 2D + 6 - 2H)^l \cdot \mathrm{KO}_G(4D) \cdot (b_{8D})^k$$

for some $l \in \mathbb{Z}_{\geq 0}$. A simple calculation shows that $\kappa o(X_0) = 2l$. Suppose $\kappa o(X_1) = r$. Then we can find an element $z \in \widetilde{\mathrm{KO}}_G(X_1)$ such that $\varphi_{k_1} i_1^*(z) = 2^r$. Therefore

$$\varphi_{k_0}(\omega) = 2^{r+\beta_{k_1}^{k_1-k_0}}, \quad \text{where } \omega = (f^{S^1})^*(i_1^*(z)).$$

Since $k_1 > k_0$, the map $(f^{S^1})^*$ factors through $\mathrm{KO}_G((k_0+1)D) \to \mathrm{KO}_G(k_0 D)$. Therefore, we see that $\omega = \gamma(D) \cdot (a\eta(D)\lambda(D)) \cdot (b_{8D})^k = a(1-D)\lambda(D) \cdot (b_{8D})^k$ for some $a \in \mathbb{Z}$. Because of the commutative diagram, we have $\omega \in \mathrm{Im}(i_0^*)$. By Lemma 7.5, we get $2^{2l+1} \mid \varphi_{k_0}(\omega)$. This implies that

$$2l + 1 \leq r + \beta_{k_1}^{k_1-k_0}.$$

(b) Suppose X_0 is even KO_G–split with $k_0 = 8k$. Notice that $\kappa o(X) = 2l + 2$ if $\mathcal{J}(X)$ is the submodule generated by $(2 + A - 2D - 2B)^l (2 - 2D - B)(b_{8D})^k$ and $(A - 2B)^l (B - A)(b_{8D})^k$. Using Lemma 7.6, the proof is almost the same as the previous case. □

By Proposition 5.9, we see that $\Sigma^{2H} X$ and $\Sigma^{8D} X$ are even (odd) KO_G–split if X is even (odd) KO_G–split. Therefore, Proposition 5.10 justifies the following definition:

Definition 7.7 A spectrum class $S = [(X, a, b + r)]$ with $a, b \in \mathbb{Z}, r \in [0, 1)$ is called even (odd) KO_G–split if, for integers $M, N \gg 0$, $\Sigma^{(8M-a)D} \Sigma^{(2N-b)H} X$ is even (odd) KO_G–split.

Example 7.8 For any $a, b \in \mathbb{Z}$ and $r \in [0, 1)$, $[(S^0, 8a, 2b+1+r)]$ is even KO_G–split and $[(S^0, 8a+4, 2b+r)]$ is odd KO_G–split.

The following proposition is easy to prove using Proposition 7.3

Proposition 7.9 *Let $S_0, S_1 \in \mathfrak{C}$ be two spectrum classes at levels k_0, k_1 respectively, with $k_0 < k_1$. Suppose S_0 is even or odd KO_G–split and S_0 dominates S_1. Then we have*

$$\kappa o(S_0) < \kappa o(S_1) + \beta_{k_1}^{k_1-k_0}. \tag{39}$$

Now let Y be a homology sphere. Recall that we have a spectrum class invariant $S(Y)$ at level 0.

Definition 7.10 Y is called Floer KO_G–split if $\Sigma^H S(Y)$ is even KO_G–split and $\Sigma^{4D} S(Y)$ is odd KO_G–split.

Remark 7.11 For simple examples like $Y = \pm\Sigma(2,3,12n+1)$ or $\pm\Sigma(2,3,12n+5)$, the two conditions in the above definition are either both true or both false. We expect that this fails in more complicated examples. If we only assume one of these two conditions, only half of the cases in Theorem 1.11 are still true.

Remark 7.12 We will see in Section 8 below that S^3, $\pm\Sigma(2,3,12n+1)$ and $-\Sigma(2,3,12n+5)$ are Floer KO_G–split, while $+\Sigma(2,3,12n+5)$ is not.

Proof of Theorem 1.11 When $\mu(Y_0) = 0$, $S(Y_0) = [(X,a,b)]$ for some space X and some integers a, b. For large integers M, N, we have the following:

(i) The space $\Sigma^{(8M-a)D}\Sigma^{(2N-b+1)H}X$ is even KO_G–split.

(ii) The space $\Sigma^{(8M-a+4)D}\Sigma^{(2N-b)H}X$ is odd KO_G–split.

Now consider $p = 4l + m$ for $m = 0, 1, 2, 3$:

- For $p = 4l$, $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0) = [(\Sigma^{4D}X, a, b-2l)]$ is odd KO_G–split by (ii).
- For $p = 4l+1$, $\Sigma^{\frac{p}{2}H}S(Y_0) = [(\Sigma^{H}X, a, b-2l+\frac{1}{2})]$ is even KO_G–split by (i).
- For $p = 4l+2$, $\Sigma^{\frac{p}{2}H}S(Y_0) = [(\Sigma^{H}X, a, b-2l)]$ is even KO_G–split by (i).
- For $p = 4l+3$, $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0) = [(\Sigma^{4D}X, a, b-2l-2+\frac{1}{2})]$ is odd KO_G–split by (ii).

Similarly, we can prove that, when $\mu(Y_0) = 1$, $\Sigma^{\frac{p}{2}H}S(Y_0)$ is even KO_G–split for $p = 4l+2$ and $4l+3$ while $\Sigma^{\frac{p}{2}H}\Sigma^{4D}S(Y_0)$ is odd KO_G–split for $p = 4l$ and $4l+1$.

Now repeat the proof of Theorem 1.6 for $k = 0$ or 4, using Proposition 7.9 instead of Proposition 6.3. Notice that the two sides of the same inequalities are either both integers or both half-integers. The inequalities are proved. □

8 Examples and explicit bounds

In this section, we will prove Theorem 1.9 describing the values of $\kappa o_i(S^3)$ and $\kappa o_i(\pm\Sigma(2,3,r))$ with $\gcd(r,6) = 1$. We will also use Corollary 1.12 to give some new bounds for the intersection forms of spin 4–manifolds with given boundaries.

8.1 Basic examples

If Y is a rational homology sphere admitting a metric g with positive scalar curvature, then by the arguments in [19], we obtain

$$S(Y,\mathfrak{s}) = [(S^0, 0, n(Y,\mathfrak{s})/2)].$$

In particular, S^3 is Floer KO_G–split and $\kappa o_i(S^3) = 0$ for any $i \in \mathbb{Z}/8$.

Manolescu [21] gave two examples of spaces of type SWF that are related to the spectrum class invariants of the Brieskorn spheres $\pm\Sigma(2,3,r)$. We recall the construction here.

Suppose that G acts freely on a finite G–CW complex Z, with the quotient space $Q = Z/G$. Let $\widetilde{Z} = ([0,1]\times Z)/\sim$, where

$$(0,z) \sim (0,z') \quad \text{and} \quad (1,z) \sim (1,z') \quad \text{for all } z, z' \in Z,$$

denote the unreduced suspension of Z, where G acts trivially on the $[0,1]$ factor. We can take one of the two cone points (say $(0,z) \in \widetilde{Z}$) as the base point and view $\widetilde{Z}$ as a pointed G–space. It's easy to see that $\widetilde{Z}$ is of type SWF at level 0.

We want to compute $\kappa o(\Sigma^{kD}\widetilde{Z})$ for $k = 0, 1, \dots, 7$. It turns out that the method in [21] also works here. Namely, the inclusion $i\colon (\Sigma^{kD}\widetilde{Z})^{S^1} = \Sigma^{kD}S^0 \to \Sigma^{kD}\widetilde{Z}$ gives the long exact sequence

$$(40) \qquad \cdots \to \widetilde{\mathrm{KO}}_G(\Sigma^{kD}\widetilde{Z}) \xrightarrow{i^*} \mathrm{KO}_G(kD) \xrightarrow{p^*} \mathrm{KO}^1_G(\Sigma^{kD}\widetilde{Z}, (kD)^+) \to \cdots .$$

By exactness of the sequence, we have $\mathrm{Im}(i^*) = \ker(p^*)$. By definition, we have

$$\mathrm{KO}^1_G(\Sigma^{kD}\widetilde{Z}, (kD)^+) \cong \widetilde{\mathrm{KO}}^1_G(\Sigma^{kD}\Sigma Z_+) \cong \widetilde{\mathrm{KO}}_G(\Sigma^{kD} Z_+).$$

By abuse of notation, we still use p^* to represent the map between $\mathrm{KO}_G(kD)$ and $\widetilde{\mathrm{KO}}_G(\Sigma^{kD}Z_+)$. Checking the maps in the exact sequence, one can see that p^* is induced by the natural projection $p\colon \Sigma^{kD}Z_+ \to (kD)^+$. Since G acts freely on $\Sigma^{kD}Z_+$ away from the base point, we see that $\widetilde{\mathrm{KO}}_G(\Sigma^{kD}Z_+) \cong \widetilde{\mathrm{KO}}((\Sigma^{kD}Z_+)/G)$. Notice that $(Z\times kD)/G$ is a vector bundle over Q and $(\Sigma^{kD}Z_+)/G$ is the Thom space of this bundle. We are interested in two cases:

- $Z \cong G$, acting on itself via left multiplication.
- $Z \cong T \cong S^1 \times jS^1 \subset \mathbb{C}\times j\mathbb{C} \subset \mathbb{H}$, with G acting on T by left multiplication in $\mathbb{H}$.

The first case is easy since the isomorphism

$$\widetilde{\mathrm{KO}}_G(\Sigma^{kD}Z_+) \cong \widetilde{\mathrm{KO}}(S^k)$$

is given by $i_1^* \circ r_0$, where $i_1\colon S^k \to \Sigma^{k\mathbb{R}} Z_+$ is the standard inclusion and

$$r_0\colon \widetilde{\mathrm{KO}}_G(\Sigma^{kD} Z_+) \to \widetilde{\mathrm{KO}}(\Sigma^{k\mathbb{R}} Z_+)$$

is the restriction map (see Fact 2.4 in Section 2). It follows that $\mathrm{Im}(i^*) = \ker(p^*) = \ker(i_1^* \circ r_0 \circ p^*) = \ker(r)$, where $r\colon \mathrm{KO}_G(kD) \to \widetilde{\mathrm{KO}}(S^k)$ is the restriction map.

We know the structure of $\widetilde{\mathrm{KO}}(S^k)$:

- $\widetilde{\mathrm{KO}}(S^0) \cong \mathrm{KO}(\mathrm{pt}) \cong \mathbb{Z}$.
- $\widetilde{\mathrm{KO}}(S^1) \cong \mathbb{Z}_2$, generated by the Hurewicz image of the Hopf map in $\pi_3(S^2)$.
- $\widetilde{\mathrm{KO}}(S^2) \cong \mathbb{Z}_2$, generated by the Hurewicz image of the square of the Hopf map.
- $\widetilde{\mathrm{KO}}(S^4) \cong \mathbb{Z}$, generated by $V_{\mathbb{H}} - 4$, where $V_{\mathbb{H}}$ is the quaternion Hopf bundle.
- $\widetilde{\mathrm{KO}}(S^k) \cong 0$ for $k = 3, 5, 6, 7$.

Therefore, by the explicit description of $\eta(D)$, $\lambda(D)$, c after Theorem 2.13, we get the following results about the kernel of $r\colon \mathrm{KO}_G(kD) \to \widetilde{\mathrm{KO}}(S^k)$:

- For $k = 0$, $\ker(r)$ is the submodule generated by $1 - D, A, B$.
- For $k = 1$, $\ker(r)$ is generated by $2\eta(D)$.
- For $k = 2$, $\ker(r)$ is generated by $2\eta(D)^2$ and $\gamma(D)^2 c$.
- For $k = 4$, $\ker(r)$ is generated by $\lambda(D) - c, (1 - D)\lambda(D), A\lambda(D)$ and Ac.
- For $k = 3, 5, 6, 7$, $\ker(r) \cong \mathrm{KO}_G(kD)$.

From this, we get:

Proposition 8.1 *We have $\kappa o(\Sigma^{kD} \widetilde{G}) = 0$ for $k = 3, 4, 5, 6, 7$ and $\kappa o(\Sigma^{kD} \widetilde{G}) = 1$ for $k = 0, 1, 2$.*

Now let's consider the case $Z \cong T$. We want to find $\ker(p^*)$ for $p^*\colon \mathrm{KO}_G(kD) \to \mathrm{KO}_G(\Sigma^{kD} T_+)$. Notice that $S^1 \subset G$ acts trivially on $(kD)^+$ and freely on T, with $T/S^1 = S^1$. We have $\widetilde{\mathrm{KO}}_G(\Sigma^{kD} T_+) = \widetilde{\mathrm{KO}}((\Sigma^{kD} S^1_+)/\mathbb{Z}_2)$. The space $(\Sigma^{kD} S^1_+)/\mathbb{Z}_2$ can be identified with $[0,1] \times (kD)^+/\sim$, where

$$(0, x) \sim (1, -x) \quad \text{and} \quad (t_1, \infty) \sim (t_2, \infty) \quad \text{for any } x \in (kD)^+ \text{ and } t_1, t_2 \in [0,1].$$

Consider the inclusion $i_2\colon \{0\} \times (kD)^+ \to (\Sigma^{kD} S^1_+)/\mathbb{Z}_2$. Notice that

$$((\Sigma^{kD} S^1_+)/\mathbb{Z}_2)/(kD)^+ \cong S^{k+1}.$$

We get the long exact sequence

$$\cdots \to \widetilde{\mathrm{KO}}(S^{k+1}) \xrightarrow{\delta} \widetilde{\mathrm{KO}}(S^{k+1}) \to \widetilde{\mathrm{KO}}((\Sigma^{kD} S^1_+)/\mathbb{Z}_2) \xrightarrow{i_2^*} \widetilde{\mathrm{KO}}(S^k) \to \cdots. \tag{41}$$

By checking the iterated mapping cone construction, which gives us this long exact sequence, it is not hard to prove that δ is induced by the map $f\colon S^{k+1} \to S^{k+1}$ with $\deg(f) = 0$ for even k and $\deg(f) = 2$ for odd k.

When $k = 2, 4, 5, 6$, we have $\widetilde{\mathrm{KO}}(S^{k+1}) = 0$. Therefore, i_2^* is injective, which implies that $i_1^* \circ r_0\colon \widetilde{\mathrm{KO}}_G(\Sigma^{kD} T_+) \to \widetilde{\mathrm{KO}}((kD)^+)$ is injective (i_1^* and r_0 are defined as in the case $Z \cong G$). We see that when $k = 2, 4, 5, 6$, just like the case $Z \cong G$, the kernel of p^* is the kernel of the restriction map $r\colon \mathrm{KO}_G(kD) \to \widetilde{\mathrm{KO}}(S^k)$. Thus, we get $\kappa o(\Sigma^{kD} \widetilde{T}) = \kappa o(\Sigma^{kD} \widetilde{G})$ for these values of k.

For $k = 0$, consider $[0,1]$ as the subset $\{1 + je^{i\theta} \mid \theta \in [0, \pi]\} \subset T$. The left endpoint is mapped to the right endpoint under the action of $-j \in G$. This embedding of $[0,1]$ gives us the following explicit description of the map

$$p^*\colon \mathrm{RO}(G) \cong \widetilde{\mathrm{KO}}_G(S^0) \to \widetilde{\mathrm{KO}}_G(T_+) \cong \mathrm{KO}_G(T) \cong \mathrm{KO}(T/G) = \mathrm{KO}(S^1).$$

Starting from a representation space V of G, we get an trivial bundle $V \times [0,1]$ over $[0,1]$. Identifying $(x, 0)$ with $((-j) \circ x, 1)$ for any $x \in V$, we get a bundle E over S^1. Then $[E] \in \mathrm{KO}(S^1)$ is the image of $[V] \in \mathrm{RO}(G)$ under p^*.

We know that $\mathrm{KO}(S^1)$ is generated by the one-dimensional trivial bundle $[1]$ and the one-dimensional nontrivial bundle $[m]$, subject to the relation $2([1] - [m]) = 0$. Using the explicit description of p^*, we see that $p^*(1) = [1]$, $p^*(D) = [m]$ and $p^*(A) = p^*(B) = 0$. Therefore, we get $\kappa o(\widetilde{T}) = 2$.

Applying Corollary 6.4 for $S = \Sigma^{2D} \widetilde{T}$, we get $\kappa o(\Sigma^{3D} \widetilde{T}) + 1 \geq \kappa o(\Sigma^{2D} \widetilde{T}) = 1$. Applying Corollary 6.4 for $S = \Sigma^{3D} \widetilde{T}$, we get $0 = \kappa o(\Sigma^{4D} \widetilde{T}) + 0 \geq \kappa o(\Sigma^{3D} \widetilde{T})$. Therefore, we see that $\kappa o(\Sigma^{3D} \widetilde{T}) = 0$.

Applying Corollary 6.4 for $S = \Sigma^{2D} \widetilde{T}$ and $S = \Sigma^{D} \widetilde{T}$, we get $\kappa o(\Sigma^{D} \widetilde{T}) = 1$ or 2.

For $k = 7$, the map $\delta\colon \widetilde{\mathrm{KO}}(S^8) \to \widetilde{\mathrm{KO}}(S^8)$ is multiplication by 2. Since $\widetilde{\mathrm{KO}}(S^7) = 0$, we get $\widetilde{\mathrm{KO}}((\Sigma^{kD} S^1_+)/\mathbb{Z}_2) = \mathbb{Z}_2$. This implies that

$$p^*(2b_{8D} \cdot \gamma(D)) = 2p^*(b_{8D} \cdot \gamma(D)) = 0.$$

Therefore, $2b_{8D} \cdot \gamma(D) \in \ker(p^*)$ and $\kappa o(\Sigma^{7D} \widetilde{T}) = 0$ or 1.

Lemma 8.2 *We have $\kappa o(\Sigma^{D} \widetilde{T}) = 2$ and $\kappa o(\Sigma^{7D} \widetilde{T}) = 1$.*

Proof This can be proved directly using the Gysin sequence, but here we use a different approach. Manolescu [21; 20] proved that $S(-\Sigma(2,3,11)) = [(\widetilde{T}, 0, 1)]$, where $-\Sigma(2,3,11)$ is a negatively oriented Brieskorn sphere. Therefore, by Definition 5.11 and Proposition 5.13, we get

$$\kappa o_i(-\Sigma(2,3,11)) = \kappa o(\Sigma^{(i+4)D} \widetilde{T}) + 1 - \beta_i^4.$$

In particular,

$$\kappa o_3(-\Sigma(2,3,11)) = \kappa o(\Sigma^{7D}\widetilde{T}) - 2 \quad \text{and} \quad \kappa o_5(-\Sigma(2,3,11)) = \kappa o(\Sigma^{D}\widetilde{T}) - 2.$$

Since $-\Sigma(2,3,11)$ bounds a smooth spin 4–manifold with intersection form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (see [21]), we can apply Corollary 1.12 for $p=0$, $q=1$ and get $\kappa o_5(-\Sigma(2,3,11)) \geq 0$, which implies $\kappa o(\Sigma^{D}\widetilde{T}) \geq 2$. We get $\kappa o(\Sigma^{D}\widetilde{T}) = 2$ by our discussion before the lemma.

We can also apply Theorem 1.6 for $Y_0 = S^3$, $Y_1 = -\Sigma(2,3,11)$, $p = 0$, $q = 1$ and $k = 2$. We have $\kappa o_3(-\Sigma(2,3,11)) \geq -1$ and $\kappa o(\Sigma^{7D}\widetilde{T}) \geq 1$. Therefore, $\kappa o(\Sigma^{7D}\widetilde{T}) = 1$ by our discussions before. □

We summarize our results in the following proposition.

Proposition 8.3 *We have $\kappa o(\Sigma^{kD}\widetilde{T}) = 2$ for $k = 0, 1$, $\kappa o(\Sigma^{kD}\widetilde{T}) = 1$ for $k = 2, 7$ and $\kappa o(\Sigma^{kD}\widetilde{T}) = 0$ for $k = 3, 4, 5, 6$.*

Now we calculate $\kappa o_i(\pm\Sigma(2,3,r))$ with $\gcd(6,r) = 1$. The spectrum class invariants $S(\pm\Sigma(2,3,r))$ are given in [21].

Proposition 8.4 (Manolescu) *We have the following results for $S(\pm\Sigma(2,3,r))$:*

$$S(\Sigma(2,3,12n-1)) = \Big[\Big(\widetilde{G} \vee \bigvee_1^{n-1} \Sigma G_+, 0, 0\Big)\Big].$$

$$S(-\Sigma(2,3,12n-1)) = \Big[\Big(\widetilde{T} \vee \bigvee_1^{n-1} \Sigma^2 G_+, 0, 1\Big)\Big].$$

$$S(\Sigma(2,3,12n-5)) = \Big[\Big(\widetilde{G} \vee \bigvee_1^{n-1} \Sigma G_+, 0, \tfrac{1}{2}\Big)\Big].$$

$$S(-\Sigma(2,3,12n-5)) = \Big[\Big(\widetilde{T} \vee \bigvee_1^{n-1} \Sigma^2 G_+, 0, \tfrac{1}{2}\Big)\Big].$$

$$S(\Sigma(2,3,12n+1)) = \Big[\Big(S^0 \vee \bigvee_1^{n} \Sigma^{-1} G_+, 0, 0\Big)\Big].^8$$

$$S(-\Sigma(2,3,12n+1)) = \Big[\Big(S^0 \vee \bigvee_1^{n} G_+, 0, 0\Big)\Big].$$

$$S(\Sigma(2,3,12n+5)) = \Big[\Big(S^0 \vee \bigvee_1^{n} \Sigma^{-1} G_+, 0, -\tfrac{1}{2}\Big)\Big].$$

$$S(-\Sigma(2,3,12n+5)) = \Big[\Big(S^0 \vee \bigvee_1^{n} G_+, 0, \tfrac{1}{2}\Big)\Big].$$

As we mentioned in Remark 7.12, $\pm\Sigma(2,3,12n+1)$ and $-\Sigma(2,3,12n+5)$ are KO_G–split because of Example 7.8. Using the relations in Theorem 2.13 and Theorem 2.16, it is not hard to prove that the space $(8MD\oplus(2N+2)H)^+$ is not even KO_G–split for integers $M,N\gg 0$. This implies that $+\Sigma(2,3,12n+5)$ is not KO_G–split.

Since it's easy to see that wedging with a free G–space does not change the κo invariants, we don't need to consider those $\Sigma^l G_+$ factors. By Definition 5.11 and Proposition 5.13, we can use Propositions 8.1 and 8.3 to prove the results in Theorem 1.9 easily.

8.2 Explicit bounds

Now we use Corollary 1.12 and Proposition 3.1 to get explicit bounds on the intersection forms of spin 4–manifolds with boundary $\pm\Sigma(2,3,r)$.

Theorem 8.5 *Let W be an oriented, smooth spin 4–manifold with $\partial W=\pm\Sigma(2,3,r)$. Assume that the intersection form of W is $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for $p>1, q>0$.*[9] *If the reduction of p modulo 8 is m, then we have $q-p\geq c_m$, where the c_m are the constants listed below. (Recall that the reduction of p modulo 2 is the Rokhlin invariant of the boundary.)*

	$m=0$	$m=2$	$m=4$	$m=6$
$\Sigma(2,3,12n-1)$	2	0	1	2
$-\Sigma(2,3,12n-1)$	3	(2)	(3)	3
$\Sigma(2,3,12n+1)$	(3)	1	(2)	(3)
$-\Sigma(2,3,12n+1)$	3	1	2	3

	$m=1$	$m=3$	$m=5$	$m=7$
$\Sigma(2,3,12n-5)$	1	2	3	3
$-\Sigma(2,3,12n-5)$	2	(1)	(2)	2
$\Sigma(2,3,12n+5)$	(2)	0	(1)	(2)
$-\Sigma(2,3,12n+5)$	2	3	4	4

Remark 8.6 Some of the bounds in Theorem 8.5 can also be obtained by other methods. For example, the case $m=2$ for $\Sigma(2,3,12n+1)$ can be obtained using the κ–invariant (see [21]). Also, some bounds can be obtained by the filling method for small n. For

[8] Strictly speaking, by this we mean the spectrum class of $(\mathbb{H}^+\vee\bigvee_1^n\Sigma^3G_+,0,1)$.

[9] It is easy to see that the conclusions are not true for $p=0,1$. For example, $\pm\Sigma(2,3,12n-1)$ bounds a spin manifold with intersection form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

example, the case $m = 2, 4$ for $-\Sigma(2,3,11)$ can be deduced from Theorem 1.2, using the fact that $\Sigma(2,3,11)$ bounds a spin 4–manifold with intersection form $2(-E_8) \oplus 2\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. However, the bounds that we put in the parentheses in Theorem 8.5 appear to be new for general n.

Proof Since we can do surgeries on loops without changing intersection forms, we will always assume $b_1(W) = 0$.

(a) Suppose $\Sigma(2,3,12n+1)$ bounds a spin 4–manifold with intersection form

$$8l(-E_8) \oplus (8l+2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $l > 0$. Then we get a spin cobordism from $-\Sigma(2,3,12n+1)$ to S^3 with the same intersection form. By Theorem 4.10, $\Sigma^{4lH} S(-\Sigma(2,3,12n+1))$ dominates $\Sigma^{8l+2} S(S^3)$. Since $S(-\Sigma(2,3,12n+1)) = [(S^0 \vee G_+ \vee \cdots \vee G_+, 0, 0)]$ and $S(S^3) = [(S^0, 0, 0)]$, by Definition 4.3, we get a map

$$f\colon \Sigma^{r\mathbb{R}+(4l+M)H+ND}(S^0 \vee G_+ \vee \cdots \vee G_+) \to \Sigma^{r\mathbb{R}+MH+(8l+2+N)D} S^0$$

for some $M, N \in \mathbb{Z}$. Restricting to the first factor of $S^0 \vee G_+ \vee \cdots \vee G_+$, we obtain

$$g\colon \Sigma^{r\mathbb{R}+(4l+M)H+ND} S^0 \to \Sigma^{r\mathbb{R}+MH+(8l+2+N)D} S^0,$$

which induces a homotopy equivalence between the G–fixed point sets. This a contradiction with Proposition 3.1. The case $m = 0$ for $\Sigma(2,3,12n+1)$ is proved.

(b) Suppose $\Sigma(2,3,12n+5)$ bounds a smooth spin manifold with intersection form

$$(8l+1)(-E_8) \oplus (8l+2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $l > 0$. Then we get a spin cobordism from $-\Sigma(2,3,12n+5)$ to S^3. As the previous case, this implies that $\Sigma^{(4l+1/2)H} S(-\Sigma(2,3,12n+5))$ dominates $\Sigma^{(8l+2)D} S(S^3)$. Since $\Sigma^{(4l+1/2)H} S(-\Sigma(2,3,12n+5)) = [(\Sigma^{4lH} S^0, 0, 0)]$, we get the contradiction as before. This proves the case $m = 1$ for $\Sigma(2,3,12n+5)$.

(c) Suppose $-\Sigma(2,3,12n-1)$ bounds a spin 4–manifold with intersection form

$$(8l+2)(-E_8) \oplus (8l+3)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $l \geq 0$. By Corollary 1.12, we get

$$4l + 3 < \kappa o_{3+8l}(-\Sigma(2,3,12n-1)) + \beta^{8l+7}_{8l+3} = -1 + 4 + 4l,$$

which is a contradiction. This proves the case $m = 2$ for $-\Sigma(2,3,12n-1)$.

Using similar methods as in (c), we can prove all the other cases except

- $m=0$ for $\pm\Sigma(2,3,12n-1)$ and $-\Sigma(2,3,12n+1)$,
- $m=7$ for $\Sigma(2,3,12n-5)$ and $-\Sigma(2,3,12n+5)$,
- $m=1$ for $-\Sigma(2,3,12n-5)$.

(d) We need to introduce another approach in order to prove the rest of the cases. Consider the orbifold D^2–bundle over $S^2(2,3,r)$. This gives us an orbifold X' with boundary $+\Sigma(2,3,r)$. We have $b_2^+(X')=0$, $b_2^-(X)=1$ and X' has a unique spin structure $\mathfrak{t}$. Now suppose $-\Sigma(2,3,r)$ bounds a spin manifold X with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then we can glue X and X' together to get an oriented closed spin 4–orbifold. We have

$$\mathrm{ind}_{\mathbb{C}}\,\not{D}(X\cup X')=p+\omega(\Sigma(2,3,r),X',\mathfrak{t}).$$

Here $\omega(\Sigma(2,3,r),X',\mathfrak{t})$ is the Fukumoto–Furuta invariant defined in [13]. Saveliev [28] proved that $\omega(\Sigma(2,3,r),X',\mathfrak{t})=-\bar{\mu}(\Sigma(2,3,r))=\bar{\mu}(-\Sigma(2,3,r))$, where $\bar{\mu}$ is the Neumann–Siebenmann invariant [24; 25]. In [13], Fukumoto and Furuta considered the finite-dimensional approximation of the Seiberg–Witten equations on the orbifold $X\cup X'$ and constructed a stable Pin(2)–equivariant map

$$\left(\tfrac{1}{2}\,\mathrm{ind}_{\mathbb{C}}\,\not{D}(X\cup X')H\right)^+\to(b_2^+(X\cup X')D)^+$$

which induces a homotopy equivalence on the Pin(2)–fixed point set. (Recall that H and D are Pin(2)–representations defined in Section 2). Since $b_2^+(X\cup X')=q$ and $\mathrm{ind}_{\mathbb{C}}\,\not{D}(X\cup X')=p+\bar{\mu}(-\Sigma(2,3,r))$, we can apply Proposition 3.1 to get

$$q-p\geq 3+\bar{\mu}(-\Sigma(2,3,r))\quad\text{if } 0<p+\bar{\mu}(-\Sigma(2,3,r))\text{ can be divided by 8.}$$

Similarly, suppose $\Sigma(2,3,r)$ bounds a spin 4–manifold X' with intersection form $p(-E_8)\oplus q\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. We can consider $X'\cup(-X)$ and repeat the argument above. We get

$$q-p\geq 2+\bar{\mu}(\Sigma(2,3,r))\quad\text{if } 0<p+\bar{\mu}(\Sigma(2,3,r))\text{ can be divided by 8.}$$

The invariants $\bar{\mu}(\pm\Sigma(2,3,r))$ were computed in [24; 25; 31]:

$$\bar{\mu}(\pm\Sigma(2,3,12n-1))=\bar{\mu}(\pm\Sigma(2,3,12n+1))=0,$$
$$\bar{\mu}(\Sigma(2,3,12n-5))=\bar{\mu}(-\Sigma(2,3,12n+5)=1,$$
$$\bar{\mu}(-\Sigma(2,3,12n-5))=\bar{\mu}(\Sigma(2,3,12n+5))=-1.$$

Therefore, simple calculations prove the rest of the cases. □

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