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The Lipschitz metric on deformation spaces of G–trees

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For a finitely generated group G, we introduce an asymmetric pseudometric on projectivized deformation spaces of G–trees, using stretching factors of G–equivariant Lipschitz maps, that generalizes the Lipschitz metric on Outer space and is an analogue of the Thurston metric on Teichmüller space. We show that in the case of irreducible G–trees distances are always realized by minimal stretch maps, can be computed in terms of hyperbolic translation lengths and geodesics exist. We then study displacement functions on projectivized deformation spaces of G–trees and classify automorphisms of G. As an application, we prove the existence of train track representatives for irreducible automorphisms of virtually free groups and nonelementary generalized Baumslag–Solitar groups that contain no solvable Baumslag–Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$.

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1 Introduction

Let G be a finitely generated group. A *G–tree* is a metric simplicial tree on which G acts by simplicial isometries without inversions of edges. A G–tree is *minimal* if it does not contain a proper G–invariant subtree. To a nontrivial minimal G–tree T we associate its *deformation space* $\mathcal{D}$ consisting of G–equivariant isometry classes of nontrivial minimal G–trees T' for which there exist G–equivariant (not necessarily simplicial) maps $T \to T'$ and $T' \to T$. The outer automorphism group $\mathrm{Out}(G)$ contains a subgroup $\mathrm{Out}_{\mathcal{D}}(G)$ (see Definition 3.6) that acts on $\mathcal{D}$ by precomposing the G–actions on the trees. Deformation spaces of G–trees are analogues of Teichmüller spaces of surfaces in the context of group splittings. Important examples include the (unprojectivized) Culler–Vogtmann Outer space of a nonabelian free group (Example 3.3), deformation spaces of, more generally, virtually nonabelian free groups (Example 3.4) and deformation spaces of nonelementary generalized Baumslag–Solitar groups (Example 3.5).

The *topology* of deformation spaces of G–trees has been extensively studied, eg in Clay [7] and Guirardel and Levitt [15]. For instance, the projectivized deformation space $\mathcal{PD} = \mathcal{D}/\mathbb{R}_{>0}$, the space of G–equivariant homothety classes of G–trees in $\mathcal{D}$,

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can be given the structure of a contractible simplicial complex with missing faces. The *geometry* of deformation spaces, however, has only been addressed in the special case of Outer space (see Francaviglia and Martino [14])[1] which admits a description as a space of finite marked metric graphs. Here one studies the asymmetric Lipschitz metric, an analogue of the asymmetric Thurston metric on Teichmüller space. The purpose of this paper is to introduce an asymmetric pseudometric on general projectivized deformation spaces of G–trees that generalizes the asymmetric Lipschitz metric on Outer space. In order to do so, we think of G–trees in $\mathcal{PD}$ as their covolume-1 representatives in $\mathcal{D}$ and for $T, T' \in \mathcal{PD}$ we define

$$d_{\mathrm{Lip}}(T,T') = \log\bigl(\inf_f \sigma(f)\bigr),$$

where f ranges over all G–equivariant Lipschitz maps from T to T' and $\sigma(f)$ denotes the Lipschitz constant of f. Although in general we have $d_{\mathrm{Lip}}(T,T') \neq d_{\mathrm{Lip}}(T',T)$ and $d_{\mathrm{Lip}}(T,T') = 0$ does not imply that T and T' are G–equivariantly isometric (Example 4.4), the Lipschitz metric turns out to have useful properties.

If $\mathcal{PD}$ consists of G–trees that are *irreducible*, ie if G contains a free subgroup of rank 2 acting freely, then the symmetrized Lipschitz metric

$$d_{\mathrm{Lip}}^{\mathrm{sym}}(T,T') = d_{\mathrm{Lip}}(T,T') + d_{\mathrm{Lip}}(T',T)$$

is an actual metric on $\mathcal{PD}$ (Proposition 4.5).

Minimal stretch maps and witnesses A key feature of the Lipschitz metric on Outer space is that the distance between two marked metric graphs is always realized by a map with minimal Lipschitz constant and that the minimum Lipschitz constant equals the maximum ratio of lengths of immersed loops in the corresponding quotient graphs; see [14, Proposition 3.15]. This reflects a theorem of Thurston that the Lipschitz distance between two hyperbolic surfaces in Teichmüller space is always realized by a minimal stretch map and that the extremal Lipschitz constant equals the supremum ratio of lengths of essential simple closed curves [26, Theorem 8.5]. In the same spirit, we will show the following:

Theorems 4.6 and 4.14 *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible G–trees. For all $T, T' \in \mathcal{PD}$ there exists*

(1) *a G–equivariant Lipschitz map $f\colon T \to T'$ such that $d_{\mathrm{Lip}}(T,T') = \log(\sigma(f))$;*

[1] And recently also in the case of the Outer space of a free product; see Francaviglia and Martino [13].

(2) *a hyperbolic group element $\xi \in G$ such that*

$$d_{\mathrm{Lip}}(T,T') = \log\left(\frac{l_{T'}(\xi)}{l_T(\xi)}\right) = \log\left(\sup_g \frac{l_{T'}(g)}{l_T(g)}\right),$$

where g ranges over all hyperbolic group elements of G and by $l_T(g)$ we denote the translation length $\inf_{x\in T} d(x,gx)$ *of g in T; we will call ξ a* witness *for the minimal stretching factor from T to T'.*

Geodesics Francaviglia and Martino [14] showed, making use of a folding technique due to Skora [24], that the asymmetric Lipschitz metric on Outer space is geodesic. We will apply Skora's folding technique in the general context to show:

Theorem 4.23 *If $\mathcal{PD}$ is a projectivized deformation space of irreducible G–trees then for all $T,T' \in \mathcal{PD}$ there exists a d_{Lip}–geodesic (see Definition 4.21) $\gamma\colon [0,1] \to \mathcal{PD}$ with $\gamma(0) = T$ and $\gamma(1) = T'$.*

Train track representatives An automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is *reducible* if there exists a G–tree $T \in \mathcal{PD}$ and a G–equivariant map $f\colon T \to T\Phi$ that leaves an essential proper G–invariant subforest of T invariant, where a subforest $S \subset T$ is *essential* if it contains the hyperbolic axis of some hyperbolic group element. We say that $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is *represented by a train track map* if there exists a G–tree $T \in \mathcal{PD}$ and an extremal G–equivariant Lipschitz map $f\colon T \to T\Phi$ such that, loosely speaking, every iterate of f maps certain immersed paths in T to immersed paths (see Section 5.4 for a precise definition). Bestvina [3] classified free group automorphisms Φ by studying associated displacement functions $T \mapsto d_{\mathrm{Lip}}(T, T\Phi)$ on Outer space. By doing so, he gave an alternative proof of Bestvina–Handel's train track theorem [5, Theorem 1.7] that every irreducible automorphism of a free group is represented by a train track map. Generalizing Bestvina's approach, we will study displacement functions on projectivized deformation spaces of G–trees to classify automorphisms of more general groups and show the following:

Theorem 5.13 *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible G–trees. If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then every irreducible automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is represented by a train track map.*

Throughout, we will pay particular attention to deformation spaces of G–trees for virtually free groups and nonelementary generalized Baumslag–Solitar groups:

Example 5.14 Let G be a finitely generated virtually nonabelian free group, ie G contains a finitely generated nonabelian free subgroup of finite index. Let $\mathcal{PD}$ be the projectivized deformation space of minimal G–trees with finite vertex stabilizers (Example 3.4). Then $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ and every irreducible automorphism $\Phi \in \mathrm{Out}(G)$ is represented by a train track map. This generalizes [5, Theorem 1.7] to virtually free groups.

A *generalized Baumslag–Solitar (GBS) group* is a finitely generated group that acts on a simplicial tree with infinite cyclic vertex and edge stabilizers. Among these groups are the classical *Baumslag–Solitar groups* $\mathrm{BS}(p,q) = \langle x, t \mid tx^p t^{-1} = x^q \rangle$ with $p, q \in \mathbb{Z} \setminus \{0\}$. A GBS group is *nonelementary* if it is not isomorphic to $\mathbb{Z}$, $\mathrm{BS}(1,1) \cong \mathbb{Z}^2$ or the Klein bottle group $\mathrm{BS}(1,-1)$.

Example 5.15 Let G be a nonelementary GBS group that contains no solvable Baumslag–Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$. Let $\mathcal{PD}$ be the projectivized deformation space of minimal G–trees with infinite cyclic vertex and edge stabilizers (Example 3.5). We have $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ and every irreducible automorphism $\Phi \in \mathrm{Out}(G)$ is represented by a train track map.

Remark In an earlier preprint of this paper, the main results were formulated under the additional hypothesis that the G–trees in $\mathcal{PD}$ are not only irreducible but also locally finite. Recently, Francaviglia and Martino [13] independently proved analogous statements for irreducible G–trees with trivial edge stabilizers that are possibly locally infinite (they work in the Outer space of a free product, defined by Guirardel and Levitt in [16]). Making use of an ultralimit argument of Horbez [17], we are now able to remove the local finiteness assumption for arbitrary edge stabilizers. As a special case, this gives an alternative, shorter proof of [13, Theorem 5.12].

Structure of this paper In Sections 2 and 3 we briefly review the notions of G–trees and deformation spaces of G–trees. In Section 4 we introduce the Lipschitz metric on projectivized deformation spaces of G–trees. We show that in the case of irreducible G–trees distances are always realized by minimal stretch maps, can be computed in terms of hyperbolic translation lengths and geodesics exist. In Section 5 we study displacement functions on projectivized deformation spaces of G–trees, classify automorphisms of G, and address the existence of train track representatives for irreducible automorphisms.

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2 G–trees

A *metric simplicial tree* is a contractible 1–dimensional simplicial complex T together with a positive length assigned to every edge. We denote by $V(T)$ the set of vertices and by $E(T)$ the set of edges of T. Every metric simplicial tree T carries a natural path metric $d = d_T$. We equip T with the metric topology, which is generally coarser than the simplicial topology; the two topologies agree if and only if the simplicial complex is locally finite; see Chiswell [6, Lemma 2.2.6]. Any two points $x, y \in T$ are joined by a unique compact geodesic segment $[x, y] \subseteq T$ and between any two disjoint closed connected subsets $A, B \subset T$ there exists a unique compact connecting segment $[a, b] \subseteq T$ such that $A \cap [a, b] = a$ and $B \cap [a, b] = b$. In particular, T is a simplicial $\mathbb{R}$–tree (see [6] for an introduction to $\mathbb{R}$–trees) and, in fact, every simplicial $\mathbb{R}$–tree arises this way [6, Theorem 2.2.10].

Let G be a finitely generated group. A G*–tree* is a metric simplicial tree T on which G acts by simplicial isometries without inversions of edges. Bass–Serre theory gives a correspondence between G–trees and metric graph of groups decompositions of G (see Serre [23]). We will always assume that the simplicial structure on T is not a subdivision of a coarser simplicial structure with respect to which the action of G on T would still be simplicial and without inversions of edges (ie T has no *redundant* vertices). For a vertex or edge $x \in V(T) \cup E(T)$, we denote by $G_x \leq G$ its stabilizer. A group element $g \in G$ is *elliptic in* T if it fixes a point in T and *hyperbolic* if not. The finite-order group elements of G are always elliptic [23, Proposition 19]. A finitely generated group that acts on a simplicial tree by simplicial automorphisms without inversions of edges has a global fixed point if and only if every group element is elliptic [23, Corollary 6.5.3].

A G–tree is *minimal* if it does not contain a proper G–invariant subtree. Minimal G–trees are cocompact, ie their quotient graphs by the action of G are finite (see Bass [2, Proposition 7.9]) and G–equivariant maps between minimal G–trees are always surjective; both properties will be used frequently and without further notice. The *covolume* of a minimal G–tree T is the volume of the finite metric quotient graph $G \backslash T$. There are five types of minimal G–trees (we adopt the naming convention from [15]; see also Culler and Morgan [9]): a minimal G–tree T is *trivial* if it is a point. It is *dihedral* if it is a line and the action of G does not preserve the orientation. The G–tree T is *linear abelian* if it is a line and G acts by translations. It is *genuine abelian* if G fixes an end of T and T is not a line. Lastly, T is *irreducible* if G contains a free subgroup of rank 2 acting freely on T. In the following, we will almost exclusively be concerned with irreducible minimal G–trees, for reasons that will become apparent.

Translation lengths We briefly review well-known facts about translation lengths in G–trees. For further details, see [9] or Paulin [21].

Definition 2.1 Let (T,d) be a G–tree. For a group element $g \in G$, define the *translation length* of g in T by

$$l(g) = l_T(g) := \inf_{x \in T} d(x, gx) \in \mathbb{R}_{\geq 0}$$

and its *characteristic set* in T by $C_g = C_T(g) := \{x \in T \mid d(x,gx) = l_T(g)\} \subseteq T$.

Conjugate group elements have the same translation length, and C_g is always nonempty (ie G acts on T by semisimple isometries) and g–invariant. The translation length function $l_T\colon G \to \mathbb{R}$ defines a point in $\mathbb{R}^{\mathcal{C}(G)}$, where $\mathcal{C}(G)$ denotes the set of conjugacy classes of G. Clearly, G–equivariantly isometric G–trees have the same translation length function. If T has finitely many G–orbits of edges, its translation length function has discrete image in $\mathbb{R}$.

A group element $g \in G$ is elliptic in T if and only if $l(g) = 0$. Its characteristic set is then its fixed point set and for all $x \in T$ the midpoint of the segment $[x, gx]$ is fixed by g. A group element $g \in G$ is hyperbolic in T if and only if $l(g) > 0$. Its characteristic set is then isometric to $\mathbb{R}$, the group element g acts on C_g by translations of length $l(g)$, and for all $k \in \mathbb{Z} \setminus \{0\}$ we have $l(g^k) = |k| \cdot l(g)$ and $C_{g^k} = C_g$. The characteristic set of a hyperbolic group element g is the unique g–invariant line in T. We will instead denote it by A_g and call it the *hyperbolic axis* of g. Every G–tree without a global fixed point contains a unique minimal G–invariant subtree, given by the union of all hyperbolic axes [9, Proposition 3.1].

Proposition 2.2 *Let T be a G–tree and $g, h \in G$.*

(1) *For all $x \in T$ we have $d(x, gx) = l(g) + 2d(x, C_g)$.*

(2) *Suppose that g and h are elliptic. Then $l(gh) = 2d(C_g, C_h)$. In particular, if the fixed point sets of g and h are disjoint then gh and hg are hyperbolic.*

(3) *Suppose that g and h are hyperbolic. If $A_g \cap A_h = \varnothing$ then*

$$l(gh) = l(hg) = l(g) + l(h) + 2d(A_g, A_h)$$

and, in particular, gh and hg are hyperbolic. The hyperbolic axes of gh and hg then both intersect each A_g and A_h.

Proof See for example [9, 1.3; 21, Propositions 1.6 and 1.8]. □

3 Deformation spaces of G–trees

In this section, we will review definitions, facts, and examples from the theory of deformation spaces of G–trees, mainly from [7; 15].

Let $\mathcal{T} = \mathcal{T}(G)$ be the set of G–equivariant isometry classes of nontrivial minimal G–trees. We will always speak of "G–trees" in $\mathcal{T}$ and not of "G–equivariant isometry classes of G–trees".

Definition 3.1 Given a G–tree $T \in \mathcal{T}$, a subgroup $H \leq G$ is an *elliptic subgroup of T* if it fixes a point in T. We associate to T its *deformation space* $\mathcal{D} = \mathcal{D}(T) \subseteq \mathcal{T}$ consisting of all G–trees that have the same elliptic subgroups as T.

The finite subgroups of G are elliptic in every G–tree [23, Proposition 19]. If two G–trees $T, T' \in \mathcal{T}$ lie in the same deformation space then for all $g \in G$ we have $l_T(g) = 0$ if and only if $l_{T'}(g) = 0$. The converse, however, is not true, as an infinitely generated subgroup of G all of whose elements fix a point in T need not be elliptic; it then fixes a unique end of T (see Tits [27, Proposition 3.4]).

An edge $e \in E(T)$ of a G–tree $T \in \mathcal{T}$ is *collapsible* if its initial and terminal vertex $\iota(e)$ and $\tau(e)$ lie in distinct G–orbits and either $G_e = G_{\iota(e)}$ or $G_e = G_{\tau(e)}$. Collapsing all edges in the G–orbit of a collapsible edge is an *elementary collapse*. An *elementary expansion* is the reverse of an elementary collapse. A finite sequence of elementary collapses and expansions is an *elementary deformation*. Two G–trees $T, T' \in \mathcal{T}$ have the same elliptic subgroups if and only if their underlying nonmetric G–trees are related by an elementary deformation (see Forester [11, Theorem 4.2]) and if and only if there exist G–equivariant (not necessarily simplicial) maps $T \to T'$ and $T' \to T$ (see [15, Theorem 3.8]).

3.1 Topologies

Let $\mathcal{D}$ be a deformation space of G–trees. We consider three topologies on $\mathcal{D}$: The *equivariant Gromov–Hausdorff topology* (see [21]), the *axes topology*, which is the coarsest topology that makes the assignment of translation length functions

$$l\colon \mathcal{D} \to \mathbb{R}^{\mathcal{C}(G)}, \quad T \mapsto l_T,$$

continuous and the *weak topology*, which describes $\mathcal{D}$ as a union of open cones (see [15]). Here, the *open cone* spanned by a G–tree $T \in \mathcal{D}$ is the set of G–trees obtained by varying the lengths of the finitely many G–orbits of edges of T, while keeping them positive. Equivalently, it is the set of G–trees in $\mathcal{D}$ that are G–equivariantly homeomorphic to T. For a detailed discussion of the three topologies, including contractibility results, see [7; 15].

Definition 3.2 The multiplicative group of positive real numbers $\mathbb{R}_{>0}$ acts on $\mathcal{D}$ by scaling the metrics on the G–trees. The *projectivized deformation space* $\mathcal{PD}$ is the quotient of $\mathcal{D}$ by this action, endowed with the quotient topology.

As a set, we will think of $\mathcal{PD}$ as the covolume–1–section in $\mathcal{D}$. In fact, if we endow $\mathcal{D}$ with the weak topology then the covolume function $\mathcal{D} \to (0,\infty)$, $T \mapsto \operatorname{covol}(T)$ is continuous and the natural projection of the covolume–1–section in $\mathcal{D}$ to the projectivized deformation space $\mathcal{PD}$ is a homeomorphism.

When we equip $\mathcal{D}$ with the weak topology, the quotient $\mathcal{PD}$ inherits the structure of a simplicial complex with missing faces.

Further facts All G–trees in a given deformation space $\mathcal{D}$ have the same type (dihedral, linear abelian, genuine abelian or irreducible; see Section 2). If $\mathcal{D}$ consists of linear abelian or dihedral G–trees then the projectivized deformation space $\mathcal{PD}$ is a point [15, Proposition 3.10]. We say that $\mathcal{D}$ is *genuine abelian* or *irreducible* if the G–trees in $\mathcal{D}$ are genuine abelian or irreducible respectively, which are the only interesting cases.

The weak topology is finer than the equivariant Gromov–Hausdorff topology, which is finer than the axes topology. A weakly converging sequence also converges in the equivariant Gromov–Hausdorff topology and *a fortiori* in the axes topology. The weak topology and the equivariant Gromov–Hausdorff topology agree on any finite union of open cones of $\mathcal{D}$ [15, Proposition 5.2]. The equivariant Gromov–Hausdorff topology and the axes topology agree if $\mathcal{D}$ is irreducible [21].

Two irreducible minimal G–trees T and T' are G–equivariantly isometric if and only if for all $g \in G$ we have $l_T(g) = l_{T'}(g)$ [9, Theorem 3.7]. Therefore, if $\mathcal{D}$ is irreducible, the assignment of translation length functions $l\colon \mathcal{D} \to \mathbb{R}^{\mathcal{C}(G)}$, $T \mapsto l_T$, is injective and the axes topology agrees with the subspace topology defined by this inclusion. In contrast, if $\mathcal{D}$ is a genuine abelian deformation space then all G–trees in $\mathcal{D}$ have the same translation length function up to scaling [15, Proposition 3.10].

If some G–tree in $\mathcal{D}$ is locally finite then all G–trees in $\mathcal{D}$ are locally finite and we say that $\mathcal{D}$ is *locally finite*. All vertex and edge stabilizers of any two G–trees in $\mathcal{D}$ are then commensurable as subgroups of G and $\mathcal{PD}$ is a locally finite complex. If $\mathcal{D}$ consists of locally finite G–trees with finitely generated vertex stabilizers then the weak topology and the equivariant Gromov–Hausdorff topology agree on all of $\mathcal{D}$ [15, Proposition 5.4].

Example 3.3 Let F_n be the free group of rank $n \geq 2$. The deformation space $\mathcal{X}_n$ of minimal F_n–trees that are acted on freely is locally finite and irreducible, and all three topologies agree on $\mathcal{X}_n$. The projectivized deformation space $\mathcal{PX}_n$ is better known as *Culler–Vogtmann Outer space* [10].

Example 3.4 More generally, let G be a finitely generated virtually nonabelian free group, ie G contains a finitely generated nonabelian free subgroup of finite index. It is a standard result that G admits a minimal action on a simplicial tree T with finite vertex (and edge) stabilizers; see Scott and Wall [22, Theorem 7.3]. Since the finite subgroups of G are elliptic in every G–tree, all minimal G–trees with finite vertex stabilizers lie in the same deformation space $\mathcal{D}$. The finite-index nonabelian free subgroup of G must act freely on T, whence $\mathcal{D}$ is irreducible. The deformation space is locally finite and all three topologies agree on $\mathcal{D}$.

Example 3.5 If G is a nonelementary GBS group (as defined in Section 1), all minimal G–trees with infinite cyclic vertex and edge stabilizers belong to the same deformation space $\mathcal{D}$ [11, Corollary 6.10] which is always locally finite. It is genuine abelian if G is a solvable Baumslag–Solitar group $\mathrm{BS}(1,q)$ with $q \neq \pm 1$. In all other cases, it is irreducible and the three topologies agree on $\mathcal{D}$.

3.2 Action of the automorphism group

The automorphism group $\mathrm{Aut}(G)$ acts on $\mathcal{T}$ from the right by precomposing the G–actions on the trees. More precisely, given $T \in \mathcal{T}$ with isometric G–action $\rho\colon G \to \mathrm{Isom}(T)$ and $\Phi \in \mathrm{Aut}(G)$, we let $T\Phi$ be the G–tree with underlying metric simplicial tree T and G–action $\rho \circ \Phi$. One easily sees that the normal subgroup of inner automorphisms $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$ acts trivially on $\mathcal{T}$ and we obtain an induced action of the outer automorphism group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ on $\mathcal{T}$.

If $\phi \in \operatorname{Out}(G)$ leaves the set of elliptic subgroups of $T \in \mathcal{T}$ invariant then $T\phi$ lies in the same deformation space as T. In general, however, the twisted G–tree $T\phi \in \mathcal{T}$ might lie in a different deformation space.

Definition 3.6 For a G–tree $T \in \mathcal{T}$ with associated deformation space $\mathcal{D}$, denote by $\operatorname{Out}_{\mathcal{D}}(G) \leq \operatorname{Out}(G)$ the subgroup of all automorphisms that leave the set of elliptic subgroups of T invariant. The action of $\operatorname{Out}(G)$ on $\mathcal{T}$ restricts to an action of $\operatorname{Out}_{\mathcal{D}}(G)$ on $\mathcal{D}$.

Proposition 3.7 *The group $\operatorname{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{D}$ by mapping open cones to open cones of the same dimension. For every G–tree $T \in \mathcal{D}$ only finitely many G–trees in the $\operatorname{Out}_{\mathcal{D}}(G)$–orbit of T lie in the open cone spanned by T. The action of $\operatorname{Out}_{\mathcal{D}}(G)$ on $\mathcal{D}$ commutes with the action of $\mathbb{R}_{>0}$ and thus descends to an action on $\mathcal{PD}$.*

Proof Let $T, T' \in \mathcal{D}$ and $\phi \in \operatorname{Out}_{\mathcal{D}}(G)$. If T and T' are G–equivariantly homeomorphic then $T\phi$ and $T'\phi$ are G–equivariantly homeomorphic as well, and $\operatorname{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{D}$ by mapping open cones to open cones. Since T and $T\phi$ have the same underlying metric simplicial tree, their open cones have the same dimension and the action of $\operatorname{Out}_{\mathcal{D}}(G)$ on $\mathcal{D}$ commutes with the action of $\mathbb{R}_{>0}$. In order to prove the second statement, suppose that T and $T\phi$ are G–equivariantly homeomorphic. Then $T\phi$ is G–equivariantly isometric to (T, d'), where d' is a metric on T obtained by permuting the lengths of the G–orbits of edges of T, of which there are only finitely many. □

The modular homomorphism If $\mathcal{D}$ is a deformation space of locally finite G–trees, all vertex and edge stabilizers of all G–trees in $\mathcal{D}$ are commensurable as subgroups of G. We then define the *modular homomorphism* $\mu = \mu(\mathcal{D})\colon G \to (\mathbb{Q}_{>0}, \times)$ by

$$\mu(g) = \frac{[H : (H \cap gHg^{-1})]}{[gHg^{-1} : (H \cap gHg^{-1})]},$$

where H is any subgroup of G commensurable with a vertex or edge stabilizer of some G–tree in $\mathcal{D}$. Indeed, μ does not depend on the choice of H. We say that $\mathcal{D}$ has *no nontrivial integral modulus* if $\operatorname{im}(\mu) \cap \mathbb{Z} = \{1\}$.

Lemma 3.8 (Levitt [19, Lemma 2.4]) *Let G be a nonelementary GBS group. The deformation space $\mathcal{D}$ of minimal G–trees with infinite cyclic vertex and edge stabilizers (Example 3.5) has no nontrivial integral modulus if and only if G contains no solvable Baumslag–Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$.*

Remark The group $\mathrm{BS}(1,-n)$ contains a subgroup isomorphic to $\mathrm{BS}(1,n^2)$. Hence, if G contains no solvable Baumslag–Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$ then it contains no solvable Baumslag–Solitar group $\mathrm{BS}(1,q)$ with $q \neq \pm 1$ and, in particular, the deformation space $\mathcal{D}$ is irreducible.

A subgroup $H \leq G$ is *small in* G (as in [15, Section 8]) if there does not exist a G–tree in which the axes of any two hyperbolic group elements of H intersect in a compact set. Being small in G is a commensurability invariant and stable under taking subgroups.

Proposition 3.9 [15, Proposition 8.6] *Let $\mathcal{D}$ be a deformation space of locally finite irreducible G–trees whose vertex and edge stabilizers are all commensurable with a finitely generated subgroup $H \leq G$.*

(1) *If H is small in G then $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$.*
(2) *If every subgroup of G commensurable with H has finite outer automorphism group and $\mathcal{D}$ has no nontrivial integral modulus then $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{D}$ with finitely many orbits of open cones (and on the projectivized deformation space $\mathcal{PD}$ with finitely many orbits of open simplices).*

Example 3.10 The unprojectivized Outer space $\mathcal{X}_n$ (Example 3.3) is locally finite and irreducible, and all vertex and edge stabilizers of the F_n–trees in $\mathcal{X}_n$ are trivial. We clearly have $\mathrm{Out}_{\mathcal{X}_n}(F_n) = \mathrm{Out}(F_n)$, and $\mathrm{Out}(F_n)$ acts on Outer space $\mathcal{P}\mathcal{X}_n$ with finitely many orbits of simplices.

Example 3.11 More generally, let G be a finitely generated virtually nonabelian free group. The deformation space $\mathcal{D}$ of minimal G–trees with finite vertex stabilizers is locally finite and irreducible (Example 3.4). Choosing $H = \{1\}$, we see that $\mu(\mathcal{D}) \equiv 1$. We have $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ and $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices.

Example 3.12 Let G be a nonelementary GBS group that contains no solvable Baumslag–Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$. The deformation space $\mathcal{D}$ of minimal G–trees with infinite cyclic vertex and edge stabilizers is locally finite and irreducible (Example 3.5). Let H be any vertex or edge stabilizer of any G–tree in $\mathcal{D}$. If G acts on a tree such that H acts hyperbolically then all nontrivial elements of H have the same hyperbolic axis (because H is infinite cyclic), whence H is small in G and $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$. Every subgroup of G commensurable with H, being virtually cyclic, has finite outer automorphism group. By Lemma 3.8, $\mathcal{D}$ has no nontrivial integral modulus and hence $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices.

4 The Lipschitz metric

Let $\mathcal{D}$ be a deformation space of G–trees and $T, T' \in \mathcal{D}$. As T and T' lie in the same deformation space, there exists a G–equivariant map $f\colon T \to T'$, which we may choose to be Lipschitz continuous. We denote by $\sigma(f)$ its Lipschitz constant.

Every G–equivariant Lipschitz map $f\colon T \to T'$ is G–equivariantly homotopic relative to the vertices of T to a G–equivariant Lipschitz map $f'\colon T \to T'$ that is linear (ie either constant or an immersion with constant slope) on edges. The Lipschitz constant $\sigma(f')$ is then given by the maximal slope of f' on the finitely many G–orbits of edges of T and we have $\sigma(f') \le \sigma(f)$. We may therefore always assume every G–equivariant Lipschitz map $f\colon T \to T'$ to be linear on edges without increasing its Lipschitz constant.

Definition 4.1 Define $\sigma(T, T') = \inf_f \sigma(f)$, where f ranges over all G–equivariant Lipschitz maps from T to T'.

Recall that, as a set, we think of the projectivized deformation space $\mathcal{PD}$ as the covolume–1–section in $\mathcal{D}$. With this convention, we can assign to each pair of projectivized G–trees $(T, T') \in \mathcal{PD} \times \mathcal{PD}$ the well-defined value $\sigma(T, T')$.

Proposition/Definition 4.2 *The function*

$$d_{\mathrm{Lip}}\colon \mathcal{PD} \times \mathcal{PD} \to \mathbb{R}, \quad (T, T') \mapsto \log(\sigma(T, T'))$$

is an asymmetric pseudometric on $\mathcal{PD}$. That is, for all $T, T', T'' \in \mathcal{PD}$ we have:

(1) $d_{\mathrm{Lip}}(T, T') \ge 0$.

(2) *If T and T' are G–equivariantly isometric then $d_{\mathrm{Lip}}(T, T') = 0$.*

(3) $d_{\mathrm{Lip}}(T, T'') \le d_{\mathrm{Lip}}(T, T') + d_{\mathrm{Lip}}(T', T'')$.

We call d_{Lip} the Lipschitz metric.

Proof To prove (1), let $f\colon T \to T'$ be a G–equivariant Lipschitz map. We will show that $\sigma(f)$ is bounded below by 1. Since the G–trees T and T' are minimal, both f and the induced map on quotient graphs $G \setminus f\colon G \setminus T \to G \setminus T'$ are surjective. We have $\sigma(G \setminus f) = \sigma(f)$ and $\mathrm{vol}(G \setminus T) = \mathrm{vol}(G \setminus T') = 1$. If now $\sigma(f) < 1$ then

$$\mathrm{vol}(\mathrm{im}(G \setminus f)) \le \sigma(f) \cdot \mathrm{vol}(G \setminus T) < 1,$$

contradicting the surjectivity of $G \setminus f$.

Statement (2) is immediate. In order to show (3), observe that for any sequence of G–equivariant Lipschitz maps

$$T \xrightarrow{f} T' \xrightarrow{f'} T''$$

we have $\sigma(T,T'') \le \sigma(f' \circ f)$ and $\sigma(f' \circ f) \le \sigma(f) \cdot \sigma(f')$, whence

$$\begin{aligned}\log(\sigma(T,T'')) \le \inf_{f,f'} \log(\sigma(f' \circ f)) &\le \inf_{f,f'} \log(\sigma(f) \cdot \sigma(f')) \\ &= \inf_{f,f'} (\log(\sigma(f)) + \log(\sigma(f'))) \\ &= \inf_{f} \log(\sigma(f)) + \inf_{f'} \log(\sigma(f')) \\ &= \log(\sigma(T,T')) + \log(\sigma(T',T'')). \qquad \square\end{aligned}$$

Proposition 4.3 *The group* $\mathrm{Out}_{\mathcal{D}}(G)$ *acts on* $(\mathcal{PD}, d_{\mathrm{Lip}})$ *by isometries, ie for all* $T, T' \in \mathcal{PD}$ *and* $\phi \in \mathrm{Out}_{\mathcal{D}}(G)$ *we have* $d_{\mathrm{Lip}}(T\phi, T'\phi) = d_{\mathrm{Lip}}(T,T')$*.*

Proof Every G–equivariant map from T to T' is also G–equivariant with respect to the actions twisted along ϕ, and vice versa. $\square$

The following example demonstrates why we speak of the Lipschitz metric as an "asymmetric pseudometric":

Example 4.4 In general we have $d_{\mathrm{Lip}}(T,T') \neq d_{\mathrm{Lip}}(T',T)$ (see Algom-Kfir and Bestvina [1] for examples in the special case of Outer space; see also the remark after Proposition 4.5). Moreover, $d_{\mathrm{Lip}}(T,T') = 0$ does not generally imply that T and T' are G–equivariantly isometric (see Proposition 4.16 for an exception in the case of Outer space; see also Section 5.1):

Let $G = F_2 * (\mathbb{Z}/2\mathbb{Z})$ and consider the graph of groups decompositions Γ and Γ' of G as in Figure 1, where all edge group inclusions are the obvious ones and all edges have constant length $\frac{1}{3}$.

The corresponding Bass–Serre trees T and T' lie in the same deformation space, as they are related by an elementary collapse followed by an elementary expansion (the intermediate graph of groups is given by Γ_{int}). The vertices of T have valence 3 and 6, whereas the vertices of T' all have valence 5. Hence, T and T' are not homeomorphic and in particular not G–equivariantly isometric. Still, the natural morphism of graphs of groups (in the sense of [2]) from Γ to Γ' lifts to a G–equivariant map from T to T' that is an isometry on edges and thus has Lipschitz constant 1, whence $d_{\mathrm{Lip}}(T,T') = 0$.

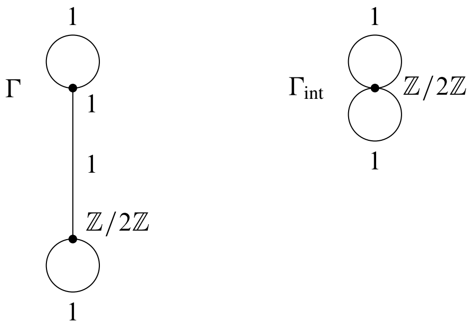

Figure 1: The Bass–Serre trees of the above graphs of groups lie in the same deformation space of G–trees. They are irreducible and locally finite. Since the group G acts on them cocompactly and with finite point stabilizers, it is virtually free.

The symmetrized Lipschitz metric A standard way to overcome these issues is to consider the symmetrized Lipschitz metric

$$d_{\mathrm{Lip}}^{\mathrm{sym}}\colon \mathcal{PD}\times\mathcal{PD}\to\mathbb{R},\quad (T,T')\mapsto d_{\mathrm{Lip}}(T,T')+d_{\mathrm{Lip}}(T',T),$$

which turns out to be an actual metric on projectivized deformation spaces of irreducible G–trees (in Section 4.3 we discuss its convergent sequences):

Proposition 4.5 *If the projectivized deformation space $\mathcal{PD}$ consists of irreducible G–trees then for all $T,T'\in\mathcal{PD}$ we have $d_{\mathrm{Lip}}^{\mathrm{sym}}(T,T')=0$ if and only if T and T' are G–equivariantly isometric.*

Proof By Proposition/Definition 4.2(2) it suffices to show the "only if" direction. Suppose that we have $d_{\mathrm{Lip}}^{\mathrm{sym}}(T,T')=0$, equivalently $d_{\mathrm{Lip}}(T,T')=0$ and $d_{\mathrm{Lip}}(T',T)=0$. Then for all $\varepsilon>0$ there exist G–equivariant $(1+\varepsilon)$–Lipschitz maps $f\colon T\to T'$ and $f'\colon T'\to T$. Let $g\in G$ be a hyperbolic group element in T and $p\in A_g\subset T$ a point in its hyperbolic axis. We have

$$\begin{aligned} l_{T'}(g)\le d(f(p),gf(p)) &= d(f(p),f(gp)) \\ &\le \sigma(f)\cdot d(p,gp)=\sigma(f)\cdot l_T(g)\le(1+\varepsilon)\cdot l_T(g)\end{aligned}$$

and, analogously, $l_T(g)\le(1+\varepsilon)\cdot l_{T'}(g)$. As ε was arbitrary, we conclude that $l_T=l_{T'}$ and hence, by [9, Theorem 3.7], that the irreducible G–trees T and T' are G–equivariantly isometric. □

Remark Thus, for T and T' as in Example 4.4 we have $d_{\mathrm{Lip}}(T',T) > 0$, since $d_{\mathrm{Lip}}(T,T') = 0$ but T and T' are not G–equivariantly isometric.

Nevertheless, the arguments in Section 5 are specific for the asymmetric pseudometric d_{Lip}. Besides, in contrast to d_{Lip}, the symmetrization $d_{\mathrm{Lip}}^{\mathrm{sym}}$ fails to be geodesic, as was shown in [14, Section 6] in the special case of Outer space (see Section 4.4 for the existence of d_{Lip}–geodesics).

4.1 Minimal stretch maps

Theorem 4.6 (Existence of minimal stretch maps) *Let $\mathcal{D}$ be a deformation space of irreducible G–trees. For all $T, T' \in \mathcal{D}$ there exists a G–equivariant Lipschitz map $f\colon T \to T'$ such that $\sigma(f) = \sigma(T,T')$.*

The proof of Theorem 4.6 will involve an argument of Horbez [17] that uses nonprincipal ultrafilters and ultralimits of metric spaces, which are defined as follows:

Definition 4.7 A *nonprincipal ultrafilter* ω on an infinite set I is a finitely additive probability measure with values in $\{0,1\}$ such that all subsets $S \subseteq I$ are ω–measurable and $\omega(S) = 0$ if S is finite.

Existence of nonprincipal ultrafilters follows from the axiom of choice. Given a nonprincipal ultrafilter ω on the set of natural numbers $\mathbb{N}$, for every bounded sequence $(c_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ there exists a unique point $\lim_\omega c_n \in \mathbb{R}$ such that for all open neighborhoods U of $\lim_\omega c_n$ we have $\omega(\{n \in \mathbb{N} \mid c_n \in U\}) = 1$ (see, for instance, Kapovich [18, Section 9.1]). In particular, if the sequence $(c_n)_{n\in\mathbb{N}}$ converges then $\lim_\omega c_n = \lim_{n\to\infty} c_n$.

Definition 4.8 Let ω be a nonprincipal ultrafilter on $\mathbb{N}$. For a sequence of metric spaces $(X_n, d_n)_{n\in\mathbb{N}}$ with basepoints $(p_n)_{n\in\mathbb{N}}$ let X_∞ be the set of all sequences $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$ for which the sequence $(d_n(x_n,p_n))_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded. Let $\sim$ be the equivalence relation on X_∞ defined by

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \quad \text{if} \quad \lim_\omega d_n(x_n,y_n) = 0.$$

Define the *ω–ultralimit* X_ω of $(X_n, d_n, p_n)_{n\in\mathbb{N}}$ as the quotient $X_\infty/\sim$ endowed with the metric $d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \lim_\omega d_n(x_n,y_n)$.

If each (X_n, d_n), $n \in \mathbb{N}$ is a complete $\mathbb{R}$–tree then (X_ω, d_ω) is again a complete $\mathbb{R}$–tree; see Stalder [25, Lemma 4.6]. Moreover, if a group G acts on each (X_n, d_n)

by isometries and for all $g \in G$ the sequence $(d_n(gp_n, p_n))_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded then (X_ω, d_ω) carries a natural isometric G–action: for $g \in G$ and $(x_n)_{n\in\mathbb{N}} \in X_\omega$ we define $g(x_n)_{n\in\mathbb{N}} = (gx_n)_{n\in\mathbb{N}}$. Since for all $g \in G$ and $n \in \mathbb{N}$ we have

$$d_n(gx_n, p_n) \le d_n(gx_n, gp_n) + d_n(gp_n, p_n) = d_n(x_n, p_n) + d_n(gp_n, p_n)$$

and the sequences of real numbers $(d_n(x_n, p_n))_{n\in\mathbb{N}}, (d_n(gp_n, p_n))_{n\in\mathbb{N}}$ are bounded, so is the sequence $(d_n(gx_n, p_n))_{n\in\mathbb{N}}$. For all $g \in G$ we have

$$\begin{aligned} d_\omega(g(x_n)_{n\in\mathbb{N}}, g(y_n)_{n\in\mathbb{N}}) &= \lim_\omega d_n(gx_n, gy_n) \\ &= \lim_\omega d_n(x_n, y_n) = d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) \end{aligned}$$

and the action of G on (X_ω, d_ω) is by isometries.

Proof of Theorem 4.6 Let $T, T' \in \mathcal{D}$ and $C = \sigma(T, T')$. We wish to construct a G–equivariant C–Lipschitz map $f\colon T \to T'$.

Let ω be a nonprincipal ultrafilter on $\mathbb{N}$ and $(f_n\colon T \to T')_{n\in\mathbb{N}}$ a sequence of G–equivariant C_n–Lipschitz maps with $C_n \le 2C$ and $\lim_{n\to\infty} C_n = C$. We will first choose a distinguished basepoint $p \in T$ and show that for all $n \in \mathbb{N}$ the image $f_n(p)$ lies in a bounded subset of T' that does not depend on n. This then implies that for all $g \in G$ the sequence $d'(gf_n(p), f_n(p))_{n\in\mathbb{N}} \subset \mathbb{R}$ is bounded and that the ω–ultralimit $T'_\omega = (T', d', f_n(p))_\omega$ carries a natural isometric G–action. (Evidently, $T_\omega = (T, d, p)_\omega$ carries a natural isometric G–action as well.) Indeed, as the action of G on T is irreducible, G contains a free subgroup of rank 2 acting freely. Suppose that this free subgroup is generated by $g, h \in G$. Since T and T' have the same elliptic subgroups, the free subgroup $\langle g, h\rangle \le G$ also acts freely on T'. If the hyperbolic axes A_g and A_h in T intersect, they must intersect in a compact segment, as we could otherwise find integers $k, l \in \mathbb{Z}\setminus\{0\}$ such that $g^k h^{-l}$ fixes a point in $A_g \cap A_h$. For the following arguments we will assume that they intersect; if they are disjoint, we replace the basis of the free subgroup with $\{g, hg\}$, whose associated axes then intersect by Proposition 2.2(3). Let $p \in A_g \cap A_h$ be a point that lies in both axes and denote the hyperbolic axes of g and h in T' by A'_g and A'_h respectively. By Proposition 2.2(1), and since f_n is G–equivariant and C_n–Lipschitz with $C_n \le 2C$, for all $n \in \mathbb{N}$,

$$\begin{aligned} 2C \cdot l_T(g) = 2C \cdot d(gp, p) &\ge d'(f_n(gp), f_n(p)) \\ &= d'(gf_n(p), f_n(p)) = l_{T'}(g) + 2d'(f_n(p), A'_g) \end{aligned}$$

and hence $d'(f_n(p), A'_g) \leq \frac{1}{2}(2C \cdot l_T(g) - l_{T'}(g)) \leq C \cdot l_T(g)$. Thus, $f_n(p)$ lies within a $(C, l_T(g))$–bounded[2] distance from A'_g and, analogously, within a $(C, l_T(h))$–bounded distance from A'_h. We conclude that $f_n(p)$ lies within a $(C, l_T(g), l_T(h))$–bounded distance from the compact segment $A'_g \cap A'_h$ if the two axes intersect and from the unique compact connecting segment between them if they are disjoint. In particular, $f_n(p)$ lies in a bounded subset of T' that does not depend on n. As remarked above, this implies that the ultralimits $T_\omega = (T, d, p)_\omega$ and $T'_\omega = (T', d', f_n(p))_\omega$ carry natural isometric G–actions.

The G–trees T and T' naturally embed G–equivariantly and isometrically into T_ω and T'_ω respectively: since for all $n \in \mathbb{N}$ the point $f_n(p) \in T'$ lies in a bounded subset that does not depend on n, for all $x \in T'$ the sequence $(d'(x, f_n(p)))_{n\in\mathbb{N}}$ is bounded and hence the constant sequence $(x)_{n\in\mathbb{N}}$ defines a point in T'_ω. One easily verifies that the natural inclusion $T' \hookrightarrow T'_\omega$, $x \mapsto (x)_{n\in\mathbb{N}}$ is indeed G–equivariant and isometric. We analogously obtain a G–equivariant isometric embedding $T \hookrightarrow T_\omega$.

Observe next that if $(d(x_n, p))_{n\in\mathbb{N}}$ is bounded then $(d'(f_n(x_n), f_n(p)))_{n\in\mathbb{N}}$ is bounded as well, since for all $n \in \mathbb{N}$ we have $d'(f_n(x_n), f_n(p)) \leq 2C \cdot d(x_n, p)$. Thus, the maps $(f_n)_{n\in\mathbb{N}}$ induce a natural map

$$f_\omega\colon T_\omega \to T'_\omega, \quad (x_n)_{n\in\mathbb{N}} \mapsto (f_n(x_n))_{n\in\mathbb{N}}.$$

The map f_ω is easily seen to be G–equivariant, since for all $g \in G$ we have

$$\begin{aligned} f_\omega(g(x_n)_{n\in\mathbb{N}}) = f_\omega((gx_n)_{n\in\mathbb{N}}) = (f_n(gx_n))_{n\in\mathbb{N}} \\ = (gf_n(x_n))_{n\in\mathbb{N}} = g(f_n(x_n))_{n\in\mathbb{N}} = gf_\omega((x_n)_{n\in\mathbb{N}}). \end{aligned}$$

Moreover, f_ω is C–Lipschitz, since for all $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in T_\omega$ we have

$$\begin{aligned} d'_\omega(f_\omega((x_n)_{n\in\mathbb{N}}), f_\omega((y_n)_{n\in\mathbb{N}})) &= \lim_\omega d'(f_n(x_n), f_n(y_n)) \\ &\leq \lim_\omega (C_n \cdot d(x_n, y_n)) \\ &= \lim_\omega C_n \cdot \lim_\omega d(x_n, y_n) \\ &= C \cdot d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}). \end{aligned}$$

Finally, T'_ω is a complete $\mathbb{R}$–tree, being the ω–ultralimit of complete $\mathbb{R}$–trees (namely, metric simplicial trees). In particular, the metric simplicial tree T' embeds into T'_ω as a closed subspace, as complete subspaces of complete metric spaces are closed. By the nature of $\mathbb{R}$–trees, there exists a continuous nearest point projection of T'_ω onto the closed G–invariant subtree T', which is easily seen to be G–equivariant and 1–Lipschitz. We define $f\colon T \to T'$ as the composition of the G–equivariant isometric

[2] We mean bounded in terms of C and $l_T(g)$.

embedding $T \hookrightarrow T_\omega$ with $f_\omega\colon T_\omega \to T'_\omega$ and the nearest point projection $T'_\omega \to T'$, and we obtain a G–equivariant C–Lipschitz map from T to T'. □

Train tracks and optimal maps In fact, we will be interested in particularly nice G–equivariant Lipschitz maps realizing $\sigma(T,T')$, so-called *optimal maps*. In order to define and construct optimal maps, we involve the concept of *train tracks*:

Definition 4.9 Let $\mathcal{D}$ be a deformation space of G–trees and $T \in \mathcal{D}$. A *direction* at a point $x \in T$ is a germ of isometric embeddings $\gamma\colon [0,\varepsilon) \to T$, $\varepsilon > 0$ with $\gamma(0) = x$. Given $g \in G$ with $gx \neq x$, we will denote the unique direction at x pointing towards gx by $\delta_{x,gx}$. Denote the set of directions at x by $D_x T$. A *train track structure* on T is a collection of equivalence relations, one on $D_v T$ for each vertex $v \in V(T)$, such that two directions $\delta_1, \delta_2 \in D_v T$ are equivalent (denoted $\delta_1 \sim \delta_2$) if and only if for all $g \in G$ the directions $g\delta_1, g\delta_2 \in D_{gv} T$ are equivalent as well. Equivalence classes of directions at a vertex $v \in V(T)$ are called *gates* at v. A *turn* at a vertex $v \in V(T)$ is a pair of directions at v. Given a train track structure on T, we say that a turn at a vertex is *illegal* if the two directions are equivalent, ie if they represent the same gate, and *legal* if not. Whenever a nondegenerate immersed path γ in T passes through a vertex v of T, we may locally reparametrize γ to an isometric embedding so that the incoming direction (with opposite orientation) and the outgoing direction of γ at v define a turn at v. A nondegenerate immersed path in T is *legal* if it only makes legal turns and *illegal* otherwise.

Definition 4.10 Let $\mathcal{D}$ be a deformation space of G–trees and $T, T' \in \mathcal{D}$. Let $f\colon T \to T'$ be a G–equivariant map that is linear on edges. We denote the union of all (closed) edges of T on which f attains its maximal slope by $\Delta(f) \subset T$ and we call it the *tension forest* of f. The tension forest $\Delta(f) \subset T$ is a G–invariant subforest.

Every G–equivariant map $f\colon T \to T'$ that is linear on edges defines a natural train track structure on its tension forest $\Delta = \Delta(f) \subset T$ as follows: for each vertex $v \in V(\Delta)$ we have a map $D_v f\colon D_v\Delta \to D_{f(v)}T'$ that maps the direction of $\gamma\colon [0,\varepsilon) \to \Delta$ with $\gamma(0) = v$ to the direction of the unique isometric embedding in the reparametrization class of $f \circ \gamma$ (since f does not collapse any edges in its tension forest, it has nonzero slope on the image of γ). We define two directions $\delta_1, \delta_2 \in D_v\Delta$ to be equivalent if $D_v f(\delta_1) = D_v f(\delta_2)$. By the G–equivariance of f, this collection of equivalence relations is indeed a train track structure on Δ.

The tension forest $\Delta(f)$ endowed with the train track structure defined by f might have vertices of valence 1 and, more generally, there might be vertices with only one gate.

Definition 4.11 A G–equivariant Lipschitz map $f\colon T \to T'$ that realizes $\sigma(T,T')$ and is linear on edges is an *optimal map* if its tension forest $\Delta(f)$ has at least 2 gates at every vertex.

Optimality of f implies that any legal path in $\Delta(f)$ may be extended in both directions to a longer legal path and, inductively, that there exists a legal line in $\Delta(f)$. This will be made use of in the proof of Theorem 4.14.

Proposition 4.12 *Let $\mathcal{D}$ be a deformation space of G–trees and $T,T' \in \mathcal{D}$. Every G–equivariant Lipschitz map $f\colon T \to T'$ that realizes $\sigma(T,T')$ and is linear on edges is G–equivariantly homotopic to an optimal map $f'\colon T \to T'$ with $\Delta(f') \subseteq \Delta(f)$. If f is not optimal to begin with then we have $\Delta(f') \neq \Delta(f)$.*

Theorem 4.6 and Proposition 4.12 imply that if $\mathcal{D}$ is an irreducible deformation space then for all $T,T' \in \mathcal{D}$ there exists an optimal map $f\colon T \to T'$.

Proof Let $\Delta = \Delta(f)$. If a vertex $v \in V(\Delta)$ has only one gate $\delta \in D_v\Delta$, slightly move $f(v)$ in the direction of $D_v f(\delta) \in D_{f(v)}T'$ (see Figure 2).

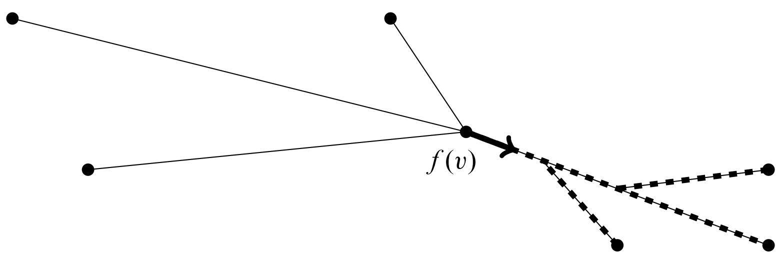

Figure 2: The image of Δ under f (dashed) and the direction in which we slightly move $f(v)$ (arrow)

Perform this perturbation G–equivariantly and keep the homotopy fixed on all other G–orbits of vertices of T. This decreases the slope of f on the G–orbits of all edges of Δ adjacent to v and we obtain a G–equivariant Lipschitz map $f'\colon T \to T'$ with $\Delta(f') \subset \Delta$ but $\Delta(f') \neq \Delta$. Keeping the perturbation small enough ensures that the (finitely many) G–orbits of the edges of $T \setminus \Delta$ adjacent to v, on which the slope is increased, do not become part of the new tension forest. As f is assumed to have minimal Lipschitz constant among all G–equivariant Lipschitz maps from T to T', we will not have removed all edges of Δ and started over with a new tension forest that corresponds to a strictly smaller maximal stretching factor. This process eventually terminates by the cocompactness of T. □

4.2 Witnesses

The results in this section will imply that the Lipschitz metric on projectivized deformation space of irreducible G–trees can be computed in terms of hyperbolic translation lengths. We begin with an easy observation:

Lemma 4.13 *Let $\mathcal{D}$ be a deformation space of G–trees and $T, T' \in \mathcal{D}$. For any G–equivariant Lipschitz map $f\colon T \to T'$ and any hyperbolic group element $g \in G$ we have $\sigma(f) \geq l_{T'}(g)/l_T(g)$. In particular, we have $\sigma(T,T') \geq \sup_g l_{T'}(g)/l_T(g)$, where g ranges over all hyperbolic group elements of G.*

Proof Let $p \in A_g$. We have $l_{T'}(g) \leq d(gf(p), f(p)) \leq \sigma(f) \cdot d(gp, p) = \sigma(f) \cdot l_T(g)$, whence the claim. □

Theorem 4.14 (Existence of witnesses) *Let $\mathcal{D}$ be a deformation space of irreducible G–trees. For all $T, T' \in \mathcal{D}$ there exists a hyperbolic group element $\xi \in G$ such that*

$$\sigma(T,T') = \frac{l_{T'}(\xi)}{l_T(\xi)} = \sup_g \frac{l_{T'}(g)}{l_T(g)},$$

where g ranges over all hyperbolic group elements of G. In fact, we can always arrange that some (and hence any) fundamental domain for the action of ξ on its hyperbolic axis $A_\xi \subset T$ meets each G–orbit of vertices of T at most 10 times.

We will call a hyperbolic group element $\xi \in G$ (or, depending on the context, its hyperbolic axis $A_\xi \subset T$) satisfying $\sigma(T,T') = l_{T'}(\xi)/l_T(\xi)$ a *witness* for the minimal stretching factor from T to T'. A hyperbolic group element $g \in G$ such that some (and hence any) fundamental domain for the action of g on its axis $A_g \subset T$ meets each G–orbit of vertices of T at most 10 times will be called a *candidate of T*. Theorem 4.14 asserts that there always exists a witness which is a candidate (our notion of candidates is nonstandard, as remarked below). We will denote by $\operatorname{cand}(T) \subset G$ the set of candidates of T.

If we choose for each $g \in \operatorname{cand}(T)$ a fundamental domain for the action of g on its axis $A_g \subset T$, these fundamental domains project to only finitely many different edge loops in the quotient graph $G \setminus T$. In particular, the set of translation lengths $\{l_T(g) \mid g \in \operatorname{cand}(T)\} \subset \mathbb{R}$ is finite. At the same time, for any $T' \in \mathcal{D}$ the set $\{l_{T'}(g) \mid g \in \operatorname{cand}(T)\} \subset \mathbb{R}$ is finite as well, since the image of $l_{T'}$ in $\mathbb{R}$ is discrete and we have $l_{T'}(g) \leq \sigma(T,T') \cdot l_T(g)$ for all $g \in G$. Clearly, if $T, T' \in \mathcal{D}$ are G–equivariantly homeomorphic then $\operatorname{cand}(T) = \operatorname{cand}(T')$.

Remark With significantly more effort, one can further show that there always exists a witness whose hyperbolic axis projects to a loop in $G \setminus T$ with certain topological properties, as was done in [14, Proposition 3.15] for free F_n–trees and in [13, Theorem 9.10] in the special case of irreducible G–trees with trivial edge stabilizers. However, the weaker finiteness properties of candidates discussed above will suffice for all our applications.

In the proof of Theorem 4.14 we will use the following characterization of witnesses:

Lemma 4.15 *Let $\mathcal{D}$ be a deformation space of G–trees and $T, T' \in \mathcal{D}$. For an optimal map $f\colon T \to T'$ and a hyperbolic group element $g \in G$, the following are equivalent:*

(1) $\sigma(f) = l_{T'}(g)/l_T(g)$.

(2) *The hyperbolic axis $A_g \subseteq T$ is contained in the tension forest $\Delta(f)$ and it is legal with respect to the train track structure defined by f.*

(3) *The hyperbolic axis $A_g \subseteq T$ is contained in the tension forest $\Delta(f)$ and $f(A_g) \subseteq T'$ equals A'_g, the hyperbolic axis of g in T'.*

Proof (2)⇒(3)⇒(1) Since f is G–equivariant and A_g is legal, the image $f(A_g)$ is a g–invariant line and thus equals A'_g. Consequently, and since A_g is assumed to lie in the tension forest $\Delta(f)$, for $p \in A_g$ we have

$$l_{T'}(g) = d(gf(p), f(p)) = d(f(gp), f(p)) = \sigma(f) \cdot d(gp, p) = \sigma(f) \cdot l_T(g)$$

and we conclude that $\sigma(f) = l_{T'}(g)/l_T(g)$.

(1)⇒(2) We will argue by contradiction. First, suppose that A_g is not contained in the tension forest $\Delta(f)$. Then for any $p \in A_g$ the segment $[p, gp]$ is stretched by strictly less than $\sigma(f)$ and we have

$$\begin{aligned} l_{T'}(g) \le d(gf(p), f(p)) &= d(f(gp), f(p)) \\ &< \sigma(f) \cdot d(gp, p) = \sigma(f) \cdot l_T(g) \end{aligned}$$

whence $\sigma(f) > l_{T'}(g)/l_T(g)$. On the other hand, if A_g is contained in $\Delta(f)$ but not legal with respect to the train track structure defined by f then there exists a vertex $v \in V(A_g)$ at which the two directions of A_g define the same gate. The images of $[g^{-1}v, v]$ and $[v, gv]$ under f then overlap in a segment of positive length and $l_{T'}(g)$ is strictly smaller than $\sigma(f) \cdot l_T(g)$, whence $l_{T'}(g)/l_T(g) < \sigma(f)$. □

Proof of Theorem 4.14 Since T and T' are irreducible, there exists an optimal map $f\colon T \to T'$ (this is the only step in the proof that uses irreducibility). By Lemma 4.15, in order to prove the claim, it suffices to find a hyperbolic group element $\xi \in G$ whose axis $A_\xi \subseteq T$ is contained in $\Delta = \Delta(f)$ and legal with respect to the train track structure defined by f. It will be clear from our construction of ξ that a fundamental domain for the action of ξ on A_ξ meets each G–orbit of vertices of T at most 10 times, ie ξ is a candidate.

Since Δ has at least 2 gates at every vertex, we can find a legal ray $R \subset \Delta$ based at some vertex $v_0 \in V(\Delta)$. There always exists a vertex $x \in V(R)$ such that $x = gx_0$ for some $x_0 \in [v_0, x)$ and some hyperbolic group element $g \in G$, which can be seen as follows: Since T is minimal and therefore cocompact, there are only finitely many G–orbits of vertices in T. We can thus find pairwise distinct vertices $x_0, x_1, x_2 \in V(R)$ and $g_1, g_2 \in G$ such that $x_1 = g_1 x_0$ and $x_2 = g_2 x_1$. If either g_1 or g_2 is hyperbolic, we are done. If both are elliptic, each g_i fixes only the midpoint of the segment $[x_{i-1}, x_i]$ and the product $g = g_2 g_1$ maps x_0 to x_2. The fixed point sets of g_1 and g_2 being disjoint, g is hyperbolic by Proposition 2.2(2).

We choose x to be the first vertex of R with this property, for which the segment $[x_0, x]$ meets the G–orbit of x_0 at most 3 times (there could lie an elliptic translate of x_0 in between x_0 and gx_0) and each G–orbit of vertices other than that of x_0 at most 2 times. The segment $[x_0, x] \subset R$ is then a closed fundamental domain for the action of g on A_g and the stretching factor of f on any subsegment of A_g equals that of f on $[x_0, x] \subset \Delta$, whence $A_g \subseteq \Delta$. If A_g is legal, we are done. If not, since all turns of A_g in between x_0 and x are legal but A_g is assumed illegal, the turns at x_0 and x must be illegal. We then have $A_g \cap R = [x_0, x]$ and we continue moving along the legal ray R until we reach the first vertex $y \in V(R)$ with $y = hy_0$ for some $y_0 \in (x, y)$ and some hyperbolic group element $h \in G$. Analogously, the segment $[y_0, y]$ meets the G–orbit of y_0 at most 3 times and each G–orbit of vertices other than that of y_0 at most 2 times. Note that the open segment (x, y_0) meets each G–orbit of vertices of T at most 2 times.

If $A_h \subseteq \Delta$ is legal, we are done. If not, we have $A_g \cap A_h = \varnothing$ and the product hg is hyperbolic by Proposition 2.2(3). A closed fundamental domain for the action of hg on its axis A_{hg} is given by $[x_0, hx] = [x_0, x] \cup [x, y_0] \cup [y_0, y] \cup h[y_0, x] \subset \Delta$, since we have $hg[x_0, hx] \cap [x_0, hx] = \{hx\}$ (see Figure 3). We conclude that $A_{hg} \subseteq \Delta$. In particular, the fundamental domain $[x_0, hx)$ meets each G–orbit of vertices of T at most $3+2+3+2 = 10$ times and hg is a candidate of T.

In order to show that A_{hg} is legal, it suffices to show that $[x_0, hx]$ does not make any illegal turns and that the directions $hg\delta_{x_0,x}$ and $\delta_{hx,y}$ are not equivalent. By the legality of R, it is clear that all turns of the subsegment $[x_0, y]$ are legal. The turn of $[x_0, hx]$ at y is legal if and only if $\delta_{y,y_0} \nsim \delta_{y,hx}$. This is equivalent to $\delta_{y_0,h^{-1}y_0} \nsim \delta_{y_0,x}$

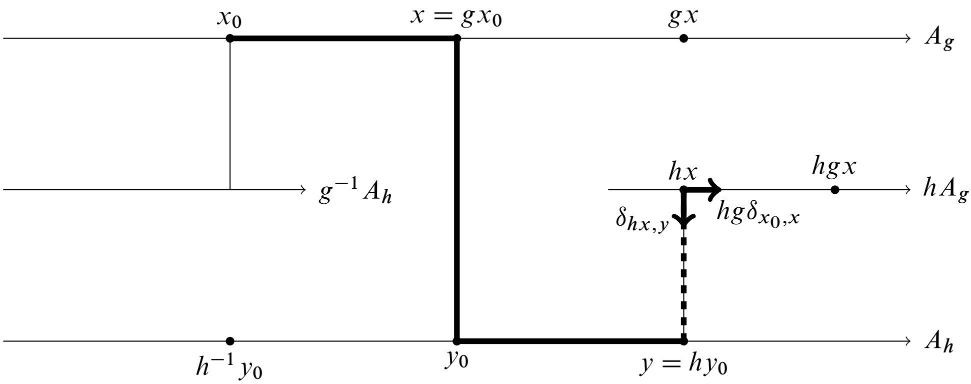

Figure 3: The segment $[x_0, hx]$ (both bold and dashed, where we know that the bold part lies in R) and the directions $hg\delta_{x_0,x}$ and $\delta_{hx,y}$ (arrows)

but which is true since A_h is assumed illegal (ie $\delta_{y_0,h^{-1}y_0} \sim \delta_{y_0,y}$) and $\delta_{y_0,y} \nsim \delta_{y_0,x}$ by the legality of R. Lastly, we need to show that $hg\delta_{x_0,x} \nsim \delta_{hx,y}$. We analogously observe that this is the case if and only if $\delta_{x,gx} \nsim \delta_{x,y_0}$ but which is true as A_g is illegal (ie $\delta_{x,gx} \sim \delta_{x,x_0}$) and $\delta_{x,x_0} \nsim \delta_{x,y_0}$ by the legality of R. □

We give a proof of the following well-known fact in the language of trees:

Proposition 4.16 *The Lipschitz metric on Outer space $\mathcal{PX}_n$ (Example 3.3) is an asymmetric metric. That is, if two F_n–trees $T, T' \in \mathcal{PX}_n$ satisfy $d_{\mathrm{Lip}}(T, T') = 0$ then they are F_n–equivariantly isometric.*

Proof Let $f: T \to T'$ be an optimal map with $\sigma(f) = 1$. For $e \in E(T)$ we denote by $\sigma_e(f)$ the slope of f on e. Since f is surjective, the induced map on metric quotient graphs $F_n \backslash f: F_n \backslash T \to F_n \backslash T'$ is surjective as well and we have

$$1 = \mathrm{vol}(\mathrm{im}(F_n \backslash f)) = \left(\sum_{e \in E(F_n \backslash T)} \sigma_e(f) \cdot \mathrm{length}(e) \right) - C,$$

where $C \geq 0$ measures overlaps of images of edges. Since T has covolume 1 and $\sigma_e(f) \leq \sigma(f) = 1$ for all $e \in E(T)$, we conclude that $1 \leq \sigma(f) - C = 1 - C$, whence $C = 0$. Consequently, we have $\sigma_e(f) = 1$ for all $e \in E(T)$ and hence $\Delta(f) = T$.

The F_n–trees in $\mathcal{PX}_n$ are irreducible, and in order to prove the claim it suffices to show that for all hyperbolic (here, nontrivial) group elements $g \in F_n$ we have $l_T(g) = l_{T'}(g)$.

On the one hand, if $g \in F_n$ is hyperbolic and $p \in A_g \subset T$ a point in its hyperbolic axis, we have

$$l_{T'}(g) \leq d(f(p), gf(p)) = d(f(p), f(gp)) \leq \sigma(f) \cdot d(p, gp) = l_T(g).$$

On the other hand, suppose that there exists a hyperbolic group element $g \in F_n$ such that $l_{T'}(g)$ is strictly smaller than $l_T(g)$. Since the tension forest of f is all of T, by Lemma 4.15 the hyperbolic axis $A_g \subset T$ cannot be legal with respect to the train track structure defined by f. Hence, we can find a vertex $v \in V(A_g)$ at which the turn defined by A_g is not legal, ie at which the germs of two adjacent edges are mapped to the same germ under f. Since F_n acts on T freely, the two germs are not F_n–equivalent and we can find a fundamental domain $X \subset T$ for the action of F_n on T that contains the two germs and has volume 1. Its image $f(X) \subset T'$ is a fundamental domain for the action of F_n on T' whose volume is strictly smaller than 1, contradicting the fact that T' has covolume 1. □

Remark The proof of Proposition 4.16 is specific for free F_n–trees, as the two germs may otherwise be G–equivalent (their common vertex may be stabilized by a nontrivial group element that swaps the two adjacent edges). In that case, we can no longer find a fundamental domain of volume 1 that contains both germs.

4.3 Convergent sequences

In this section we relate topological convergence in projectivized deformation spaces of G–trees with convergence with respect to the (symmetrized) Lipschitz metric.

Proposition 4.17 *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible G–trees and $(T_k)_{k \in \mathbb{N}}$ a sequence of G–trees in $\mathcal{PD}$ that converges to $T \in \mathcal{PD}$ in the weak topology. Then $\lim_{k \to \infty} d_{\mathrm{Lip}}^{\mathrm{sym}}(T_k, T) = 0$.*

In the weak topology, $\mathcal{PD}$ is homeomorphic to the covolume–1–section in the unprojectivized deformation space $\mathcal{D}$. Thus, the sequence $(T_k)_{k \in \mathbb{N}}$ weakly converges to T also as covolume–1–representatives in $\mathcal{D}$. The weak topology being the finest of the three topologies, $(T_k)_{k \in \mathbb{N}}$ converges to T in all three topologies, where convergence in the unprojectivized axes topology means that for all $g \in G$ we have $\lim_{k \to \infty} l_{T_k}(g) = l_T(g)$ (pointwise convergence of translation length functions).

Proof We will first show that $\lim_{k \to \infty} d_{\mathrm{Lip}}(T_k, T) = 0$. Let $(f_k\colon T_k \to T)_{k \in \mathbb{N}}$ be a sequence of optimal maps. By Theorem 4.14, for all $k \in \mathbb{N}$ there exists a candidate $\xi_k \in \mathrm{cand}(T_k) \subset G$ such that

$$d_{\mathrm{Lip}}(T_k, T) = \log\left(\frac{l_T(\xi_k)}{l_{T_k}(\xi_k)}\right).$$

Since the sequence $(T_k)_{k\in\mathbb{N}}$ converges weakly, it meets only finitely many open simplices of $\mathcal{PD}$ and the G–trees $(T_k)_{k\in\mathbb{N}}$ are of only finitely many G–equivariant homeomorphism types. After decomposing the sequence into subsequences (for each of which we will obtain the same result), we may assume that the G–trees are in fact all G–equivariantly homeomorphic, or even equal as nonmetric G–trees. The set of candidates $\operatorname{cand}(T_k) \subset G$ is then independent of k and $(l_T(\xi_k))_{k\in\mathbb{N}}$ takes only finitely many values. After decomposing $(T_k)_{k\in\mathbb{N}}$ into subsequences once more, we may assume that $(l_T(\xi_k))_{k\in\mathbb{N}}$ is constant, say $l_T(\xi_k) = C$ for all $k \in \mathbb{N}$.

By the remarks made above, the sequence $(T_k)_{k\in\mathbb{N}}$ converges also as covolume–1–representatives in the unprojectivized axes topology. Thus, for all $K \in \mathbb{N}$ we have $\lim_{k\to\infty} l_{T_k}(\xi_K) = l_T(\xi_K) = C$. Recall that the candidates $(\xi_K)_{K\in\mathbb{N}} \subset G$ give rise to only finitely many different edge loops in the quotient graph $G \setminus T_1$. In fact, if two candidates ξ_{K_1} and ξ_{K_2} give rise to the same edge loop in $G \setminus T_1$, they give rise to the same edge loop in $G \setminus T_k$ for all $k \in \mathbb{N}$ (because the G–trees $(T_k)_{k\in\mathbb{N}}$ are equal as nonmetric G–trees). Thus, the family of sequences $\{(l_{T_k}(\xi_K))_{k\in\mathbb{N}} \mid K \in \mathbb{N}\}$ is finite and for all $\varepsilon > 0$ there exists $N > 0$ such that for all $K \in \mathbb{N}$ we have

$$|C - l_{T_k}(\xi_K)| < \varepsilon$$

whenever $k \geq N$. In particular, we have $|C - l_{T_k}(\xi_k)| < \varepsilon$ whenever $k \geq N$ and we conclude that $\lim_{k\to\infty} l_{T_k}(\xi_k) = C$. Consequently,

$$\lim_{k\to\infty} d_{\mathrm{Lip}}(T_k, T) = \log\left(\lim_{k\to\infty} \frac{l_T(\xi_k)}{l_{T_k}(\xi_k)}\right) = \log\left(\frac{C}{\lim_{k\to\infty} l_{T_k}(\xi_k)}\right) = \log(1) = 0.$$

Showing that $\lim_{k\to\infty} d_{\mathrm{Lip}}(T, T_k) = 0$ is similar but easier, because it does not require that the sequence $(T_k)_{k\in\mathbb{N}}$ meets only finitely many open simplices of $\mathcal{PD}$. □

As for the converse of Proposition 4.17, we have the following:

Proposition 4.18 *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible G–trees and $(T_k)_{k\in\mathbb{N}}$ a sequence of G–trees in $\mathcal{PD}$ such that for some $T \in \mathcal{PD}$ we have $\lim_{k\to\infty} d_{\mathrm{Lip}}^{\mathrm{sym}}(T_k, T) = 0$. Then $(T_k)_{k\in\mathbb{N}}$ converges to T in the axes topology.*

In contrast to convergence in the unprojectivized axes topology, the sequence $(T_k)_{k\in\mathbb{N}}$ converges to T in the projectivized axes topology if there exist positive real numbers $(C_k)_{k\in\mathbb{N}}$ such that for all $g \in G$ we have $\lim_{k\to\infty} C_k \cdot l_{T_k}(g) = l_T(g)$ (pointwise convergence of projectivized translation length functions).

Proof We will argue as in the proof of [14, Theorem 4.11]. For any positive real-valued function f satisfying $\sup \frac{1}{f(x)} < \infty$ we have $\sup \frac{1}{f(x)} = \frac{1}{\inf f(x)}$. Therefore,

since
$$\frac{1}{\frac{l_{T_k}(g)}{l_T(g)}} = \frac{l_T(g)}{l_{T_k}(g)} \leq \sigma(T_k, T) < \infty$$
for all hyperbolic group elements $g \in G$, we have
$$\lim_{k\to\infty} d_{\mathrm{Lip}}^{\mathrm{sym}}(T_k, T) = 0 \quad \Leftrightarrow \quad \lim_{k\to\infty} \frac{\sup_g \frac{l_{T_k}(g)}{l_T(g)}}{\inf_g \frac{l_{T_k}(g)}{l_T(g)}} = 1.$$
Assuming that $\lim_{k\to\infty} d_{\mathrm{Lip}}^{\mathrm{sym}}(T_k, T) = 0$, we conclude that for all $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for all $k \geq K$ we have

(1) $$\inf_g \frac{l_{T_k}(g)}{l_T(g)} \leq \sup_g \frac{l_{T_k}(g)}{l_T(g)} \leq \inf_g \frac{l_{T_k}(g)}{l_T(g)} \cdot (1+\varepsilon).$$

Clearly, for all hyperbolic group elements $\xi \in G$ we have
$$\inf_g \frac{l_{T_k}(g)}{l_T(g)} \leq \frac{l_{T_k}(\xi)}{l_T(\xi)} \leq \sup_g \frac{l_{T_k}(g)}{l_T(g)}.$$
Setting $I_k = \inf_g l_{T_k}(g)/l_T(g)$, inequality (1) implies that $I_k \leq l_{T_k}(\xi)/l_T(\xi) \leq I_k \cdot (1+\varepsilon)$ whenever $k \geq K$. In particular, the unprojectivized translation length functions $((1/I_k) l_{T_k})_{k\in\mathbb{N}}$ converge to l_T uniformly and *a fortiori* pointwise. Thus, the G–trees $(T_k)_{k\in\mathbb{N}}$ converge to T in the projectivized axes topology. □

Recall from Section 3.1 that if $\mathcal{PD}$ is a projectivized deformation space of locally finite irreducible G–trees with finitely generated vertex stabilizers then the equivariant Gromov–Hausdorff topology, the axes topology, and the weak topology agree on $\mathcal{PD}$.

Corollary 4.19 *If $\mathcal{PD}$ consists of locally finite irreducible G–trees with finitely generated vertex stabilizers then the symmetrized Lipschitz metric $d_{\mathrm{Lip}}^{\mathrm{sym}}$ induces the standard topology on $\mathcal{PD}$.*

Proof Since the locally finite complex $\mathcal{PD}$ is metrizable, it suffices to show that the two topologies have the same convergent sequences. This immediately follows from Propositions 4.17 and 4.18 and the fact that the three topologies agree on $\mathcal{PD}$. □

Example 4.20 The symmetrized Lipschitz metric induces the standard topology on the projectivized deformation spaces discussed in Examples 3.3, 3.4 and 3.5.

4.4 Folding paths and geodesics

Let $\mathcal{D}$ be a deformation space of G–trees and $T, T' \in \mathcal{D}$. A G–equivariant map $f\colon T \to T'$ is *simplicial* if it maps each edge of T isometrically to an edge of T'. A G–equivariant map $f\colon T \to T'$ is a *morphism* if it is an isometry on edges (but not necessarily simplicial) or, equivalently, if the simplicial structures on T and T' may be subdivided such that f becomes simplicial.

Suppose we are given two G–trees $T, T' \in \mathcal{D}$ and a morphism $f\colon T \to T'$. Skora [24] has described a technique of "folding T along f" to obtain a 1–parameter family of G–trees $(T_t)_{t\in[0,\infty]}$ together with morphisms $\phi_t\colon T \to T_t$ and $\psi_t\colon T_t \to T'$ such that

- $T_0 = T$ and $T_\infty = T'$,
- $\phi_0 = \mathrm{id}_T$, $\phi_\infty = \psi_0 = f$ and $\psi_\infty = \mathrm{id}_{T'}$,
- for all $t \in [0,\infty]$ the following diagram commutes:

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "\phi_t"'] & & T' \\ & T_t \arrow[ur, "\psi_t"'] & \end{tikzcd}$$

Explicitly, for $t \in [0,\infty]$ we let $\sim_t$ be the equivalence relation on T generated by $x \sim_t y$ if $f(x) = f(y)$ and $f([x,y]) \subseteq D_t(f(x))$, where $D_t(f(x))$ denotes the closed ball of radius t around $f(x) \in T'$. As a set, we define T_t as the quotient $T/\sim_t$. We let $\phi_t\colon T \to T_t$ be the G–equivariant quotient map and $\psi_t\colon T_t \to T'$ the unique induced G–equivariant map, and we equip T_t with the maximal metric making both ϕ_t and ψ_t 1–Lipschitz.

For all $t \in [0,\infty]$, after possibly restricting to the unique minimal G–invariant subtree, the G–tree T_t lies in the deformation space $\mathcal{D}$. The map $[0,\infty] \to \mathcal{D}$, $t \to T_t$ is continuous in the equivariant Gromov–Hausdorff topology (see [24; 7; 16]) and therefore also in the axes topology. The intermediate G–trees $(T_t)_{t\in[0,\infty]}$ are contained in a finite union of open cones [15, Lemma 6.5], because of which the map is also continuous in the weak topology. We obtain a path $[0,\infty] \to \mathcal{PD}$, $t \to T_t$ that is continuous in all three topologies.

As in [14] in the special case of Outer space, one can make use of folding paths to construct geodesics in projectivized deformation spaces of irreducible G–trees:

Definition 4.21 Let $\mathcal{PD}$ be a projectivized deformation space of G–trees. A path $\gamma\colon [a,b] \to \mathcal{PD}$, $t \mapsto \gamma(t)$ with $a < b \in \mathbb{R}$ is d_{Lip}*–continuous* if for all convergent sequences $(x_n)_{n\in\mathbb{N}} \subset [a,b]$ with $\lim_{n\to\infty} x_n = x$ we have

$$\lim_{n\to\infty} d_{\mathrm{Lip}}(\gamma(x_n),\gamma(x)) = 0 \quad \text{and} \quad \lim_{n\to\infty} d_{\mathrm{Lip}}(\gamma(x),\gamma(x_n)) = 0.$$

We say that a d_{Lip}–continuous path $\gamma\colon [a,b] \to \mathcal{PD}$, $t \mapsto \gamma(t)$ with $a < b \in \mathbb{R}$ is a d_{Lip}–*geodesic* if for all $x < y < z \in [a,b]$ we have

$$d_{\mathrm{Lip}}(\gamma(x),\gamma(y)) + d_{\mathrm{Lip}}(\gamma(y),\gamma(z)) = d_{\mathrm{Lip}}(\gamma(x),\gamma(z)).$$

Remark In metric spaces, geodesics in the above sense can be reparametrized to have unit speed. However, since d_{Lip} is an asymmetric pseudometric, unit speed reparametrizations in $\mathcal{PD}$ need not always exist.

Lemma 4.22 *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible G–trees and $\gamma\colon [a,b] \to \mathcal{PD}$ be a d_{Lip}–continuous path with $a < b \in \mathbb{R}$. If for all $x < y < z \in [a,b]$ there exists a hyperbolic group element $\xi \in G$ such that*

$$\sigma(\gamma(x),\gamma(y)) = \frac{l_{\gamma(y)}(\xi)}{l_{\gamma(x)}(\xi)} \quad \text{and} \quad \sigma(\gamma(y),\gamma(z)) = \frac{l_{\gamma(z)}(\xi)}{l_{\gamma(y)}(\xi)} \tag{2}$$

then γ is a d_{Lip}–geodesic.

Proof We have

$$\sup_g \frac{l_{\gamma(z)}(g)}{l_{\gamma(x)}(g)} \geq \frac{l_{\gamma(z)}(\xi)}{l_{\gamma(x)}(\xi)} = \frac{l_{\gamma(y)}(\xi)}{l_{\gamma(x)}(\xi)} \cdot \frac{l_{\gamma(z)}(\xi)}{l_{\gamma(y)}(\xi)} = \sup_g\left(\frac{l_{\gamma(y)}(g)}{l_{\gamma(x)}(g)}\right) \cdot \sup_g\left(\frac{l_{\gamma(z)}(g)}{l_{\gamma(y)}(g)}\right)$$

and hence $d_{\mathrm{Lip}}(\gamma(x),\gamma(z)) \geq d_{\mathrm{Lip}}(\gamma(x),\gamma(y)) + d_{\mathrm{Lip}}(\gamma(y),\gamma(z))$, from which we conclude that $d_{\mathrm{Lip}}(\gamma(x),\gamma(z)) = d_{\mathrm{Lip}}(\gamma(x),\gamma(y)) + d_{\mathrm{Lip}}(\gamma(y),\gamma(z))$. □

By Proposition 4.17, if $\mathcal{PD}$ is irreducible then any path in $\mathcal{PD}$ that is continuous in the weak topology — such as the folding path $[0,\infty] \to \mathcal{PD}$, $t \to T_t$ described above — is d_{Lip}–continuous.

Theorem 4.23 (Existence of d_{Lip}–geodesics) *If $\mathcal{PD}$ is a projectivized deformation space of irreducible G–trees then for all $T, T' \in \mathcal{PD}$ there exists a d_{Lip}–geodesic $\gamma\colon [0,1] \to \mathcal{PD}$ with $\gamma(0) = T$ and $\gamma(1) = T'$.*

Proof Let $f\colon T \to T'$ be an optimal map and $\xi \in G$ a witness for the distance from T to T'. By Lemma 4.22, it suffices to construct a path $\gamma\colon [0,1] \to \mathcal{PD}$ from T to T' such that for all $x < y < z \in [0,1]$ we have (2). We will construct such a path in the unprojectivized deformation space $\mathcal{D}$, and since any witness for the minimal stretching factor between two G–trees remains a witness after scaling the metrics on the trees, the projection of the path to $\mathcal{PD}$ will still satisfy (2). In order to do so, we again regard T and T' as their covolume-1 representatives in $\mathcal{D}$. Let

$$C = \exp(d_{\mathrm{Lip}}(T, T')) = \frac{l_{T'}(\xi)}{l_T(\xi)}$$

and let $\overline{T}$ be the G–tree obtained from T by G–equivariantly shrinking each edge of T that is mapped to a point under f to length 0 (collapsing these edges does not create any new elliptic subgroups, as the G–equivariant map $f\colon T \to T'$ factors through the quotient) and G–equivariantly shrinking all other edges so that they are stretched by the factor C under f. Note that we only shrink edges in the complement of the tension forest $\Delta(f)$. Then, homothete $\overline{T}$ to $C\overline{T}$ such that $f\colon C\overline{T} \to T'$ becomes an isometry on edges, ie a morphism. We may now fold $C\overline{T}$ along f to obtain a family of G–trees $(T_t)_{t\in[0,\infty]}$ that interpolate between $C\overline{T}$ and T' as explained above (see Figure 4 for a structural sketch).

$$\begin{tikzcd}
C\overline{T} \arrow[d, squiggly, no head] \arrow[ddrr, "f"] & & \\
\overline{T} \arrow[d, squiggly, no head] \arrow[drr, "f"] & & \\
T \arrow[rr, "f"] & & T'
\end{tikzcd}$$

Figure 4: The path from T to $\overline{T}$ to $C\overline{T}$ to T' in $\mathcal{D}$ projects to a geodesic from T to T' in $\mathcal{PD}$.

This produces a path $\gamma\colon [0,1] \to \mathcal{D}$ from T to T' that is continuous in all three topologies and also with respect to d_{Lip}.

We claim that for every G–tree S in between T and $C\overline{T}$ we have

$$\sigma(T, S) = \frac{l_S(\xi)}{l_T(\xi)} \quad \text{and} \quad \sigma(S, T') = \frac{l_{T'}(\xi)}{l_S(\xi)}.$$

Analogously, we claim that for every intermediate G–tree T_t in between $C\overline{T}$ and T' we have

$$\sigma(T, T_t) = \frac{l_{T_t}(\xi)}{l_T(\xi)} \quad \text{and} \quad \sigma(T_t, T') = \frac{l_{T'}(\xi)}{l_{T_t}(\xi)}.$$

As for any $a \le s \le t \le b$ the same construction of a path from $\gamma(s)$ to $\gamma(t)$ yields precisely the restriction of γ to $[s,t]$, this then proves that γ satisfies (2).

First, consider a G–tree S that lies in between T and $\overline{T}$. As S is obtained from T by shrinking edges of T, we have $\sigma(T,S) \le 1$. However, as we only shrink edges outside of $\Delta(f)$, the hyperbolic axis $A_\xi \subset \Delta(f)$ is not touched and we have $l_S(\xi)/l_T(\xi) = 1$. We may immediately deduce from this that $\sigma(T,S) = l_S(\xi)/l_T(\xi)$, as for all hyperbolic group elements $g \in G$ we have $\sigma(T,S) \ge l_S(g)/l_T(g)$ (see Lemma 4.13). Likewise, the map $f\colon S \to T'$ still has Lipschitz constant C so that $\sigma(S,T') \le C$. The axis $A_\xi \subset \Delta(f) \subset S$ remains legal and is stretched by the factor C, whence $l_{T'}(\xi)/l_S(\xi) = C$ and therefore $\sigma(S,T') = l_{T'}(\xi)/l_S(\xi)$.

Analogously, if S lies in between $\overline{T}$ and $C\overline{T}$, say $S = C'\overline{T}$ with $C' \in [1, C]$, then $\sigma(T,S) \le C'$ and $l_S(\xi)/l_T(\xi) = C'$, whence $\sigma(T,S) = l_S(\xi)/l_T(\xi)$. The map $f\colon S \to T'$ has Lipschitz constant C/C' and the axis $A_\xi \subset \Delta(f) \subset S$ is stretched by C/C'. We conclude that $\sigma(S,T') = l_{T'}(\xi)/l_S(\xi)$.

Consider now an intermediate G–tree T_t in between $C\overline{T}$ and T'. As the quotient map $\phi_t\colon C\overline{T} \to T_t$ is 1–Lipschitz, the composition

$$T \xrightarrow{\mathrm{id}} C\overline{T} \xrightarrow{\phi_t} T_t$$

is C–Lipschitz. The hyperbolic axis $A_\xi \subset \Delta(f) \subset T$ is legal with respect to f and hence does not get folded in $T_t = C\overline{T}/\sim_t$. We therefore have $l_{T_t}(\xi)/l_T(\xi) = C$, whence $\sigma(T,T_t) = l_{T_t}(\xi)/l_T(\xi)$. Analogously, the induced map $\psi_t\colon T_t \to T'$ is 1–Lipschitz and the hyperbolic axis $A_\xi \subset \Delta(\psi_t) \subset T_t$ is legal with respect to ψ_t. We conclude that $l_{T'}(\xi)/l_{T_t}(\xi) = 1$ and hence that $\sigma(T_t,T') = l_{T'}(\xi)/l_{T_t}(\xi)$. $\square$

5 Displacement functions

Let $\mathcal{PD}$ be a projectivized deformation space of G–trees and $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$. We equip $\mathcal{PD}$ with the Lipschitz metric d_{Lip} and define the *displacement function* associated to Φ as the function

$$\widetilde{\Phi}\colon \mathcal{PD} \to \mathbb{R}_{\ge 0}, \quad T \mapsto d_{\mathrm{Lip}}(T, T\Phi).$$

We call Φ *elliptic* if $\inf \widetilde{\Phi} = 0$ and the infimum is realized. We say Φ is *hyperbolic* if $\inf \widetilde{\Phi} > 0$ and the infimum is realized. Lastly, we say Φ is *parabolic* if $\inf \widetilde{\Phi}$ is not realized.

5.1 Elliptic automorphisms

If $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is elliptic then, by definition, there exists a G–tree $T \in \mathcal{PD}$ such that $d_{\mathrm{Lip}}(T, T\Phi) = 0$. One would like to conclude that T lies in the fixed point set of Φ, but from Example 4.4 we know that the asymmetric pseudometric d_{Lip} fails to be an asymmetric metric, ie $d_{\mathrm{Lip}}(T,T')=0$ does generally not imply that T and T' are G–equivariantly isometric. However, the G–trees T and T' in the counterexample are not homeomorphic and thus they do not lie in the same $\mathrm{Out}_{\mathcal{D}}(G)$–orbit, for they would otherwise have the same underlying metric simplicial tree. Therefore, one may still ask whether d_{Lip} is an asymmetric metric on $\mathrm{Out}_{\mathcal{D}}(G)$–orbits. As we will see, the general answer is "no" (Example 5.6) but it is "yes" in certain cases (Proposition 5.1). This answers a question in an earlier preprint of this paper. The arguments in this section arose out of discussions with Camille Horbez and Gilbert Levitt.

The separation property of d_{Lip} on $\mathrm{Out}_{\mathcal{D}}(G)$–orbits Let $T\in\mathcal{PD}$, $\Phi\in\mathrm{Out}_{\mathcal{D}}(G)$ such that $d_{\mathrm{Lip}}(T,T\Phi)=0$. If T is irreducible then there exists an optimal map $f\colon T\to T\Phi$ with $\sigma(f)=1$, and one easily shows (as in the proof of Proposition 4.16) that f has stretching factor 1 on all edges of T. After subdividing the simplicial structures on T and $T\Phi$ (independently of each other) by G–equivariantly adding redundant vertices, f becomes simplicial (as defined in Section 4.4). We will denote the subdivided G–trees again by T and $T\Phi$.

If all edge stabilizers of T are finitely generated then by Bestvina and Feighn [4, Section 2] the simplicial map f factors as a finite composition of G–equivariant simplicial quotient maps, so-called *folds*, which can be classified into types IA-IIIA, IB-IIIB and IIIC (we refer the reader to [4] for definitions). All folds other than type IIA and IIB folds irreversibly decrease the metric covolume, so they cannot occur. After subdividing the simplicial structure on T once more, a type IIB fold is a composition of two type IIA folds (these subdivisions add only a finite number of G–orbits of vertices), so we may assume that f factors as a finite composition of type IIA folds. Explicitly, a type IIA fold is a simplicial quotient map $T\to T/\!\sim$, where $\sim$ is a G–equivariant equivalence relation on T that is of the following form: there are distinct edges $e_1,e_2\in E(T)$ with $\iota(e_1)=\iota(e_2)\in V(T)$ and a group element $g\in G_{\iota(e_1)}$ such that $ge_1=e_2$, and $\sim$ is the equivalence relation generated by $he_1\sim he_2$ for all $h\in G$. Intuitively, on the level of quotient graphs of groups, performing a type IIA fold corresponds to pulling an element of a vertex stabilizer along an edge (see Figure 5).

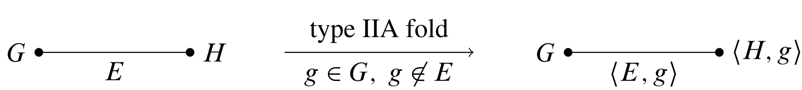

Figure 5: The effect of a type IIA fold on the quotient graph of groups

A type IIA fold always enlarges but never reduces an edge group. We will make use of this behavior to confirm the separation property of d_{Lip} on $\mathrm{Out}_{\mathcal{D}}(G)$–orbits in the following special case:

Proposition 5.1 (Levitt) *Let $\mathcal{PD}$ be a projectivized deformation space of locally finite irreducible G–trees with finitely generated edge stabilizers. If $\mathcal{PD}$ has no nontrivial integral modulus (see Section 3.2) and if $T \in \mathcal{PD}$ and $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ satisfy $d_{\mathrm{Lip}}(T, T\Phi) = 0$ then T and $T\Phi$ are G–equivariantly isometric.*

Before we turn to the proof of Proposition 5.1, we discuss the existence of *maximal elliptic subgroups*, ie elliptic subgroups that are not properly contained in any other elliptic subgroup. A maximal elliptic subgroup is always a vertex stabilizer.

Lemma 5.2 *Let $\mathcal{PD}$ be a projectivized deformation space of locally finite G–trees. If $\mathcal{PD}$ has no nontrivial integral modulus then for any G–tree $T \in \mathcal{PD}$ and any edge $e \in E(T)$ the edge group G_e is contained in a maximal elliptic subgroup of T.*

Proof We first observe that, under these assumptions, for any vertex $v \in V(T)$ the vertex group $G_v \leq G$ is not properly contained in a conjugate of itself. Suppose to the contrary that there exists a vertex $v \in V(T)$ such that G_v is a proper subgroup of gG_vg^{-1} for some $g \in G$. We then have

$$\mu(g) = \frac{[G_v : (G_v \cap gG_vg^{-1})]}{[gG_vg^{-1} : (G_v \cap gG_vg^{-1})]} = \frac{1}{[gG_vg^{-1} : G_v]}$$

with $[gG_vg^{-1} : G_v] > 1$, in which case $\mu(g^{-1}) = 1/\mu(g)$ is a nontrivial integral modulus, contradicting our assumptions. To prove the lemma, we again argue by contradiction. Suppose that the edge group G_e is not contained in a maximal elliptic subgroup. Each vertex group adjacent to e is then properly contained in another vertex group, which is again properly contained in yet another vertex group. Inductively, we obtain an infinite properly ascending chain of vertex groups that lie in only finitely many conjugacy classes by the cocompactness of T. We conclude that there exists a vertex $v \in V(T)$ and a group element $g \in G$ such that G_v is a proper subgroup of gG_vg^{-1}, which contradicts the first part of the proof. □

Proof of Proposition 5.1 Since T is irreducible and has finitely generated edge stabilizers, after subdividing the simplicial structures on T and $T\Phi$ there exists a G–equivariant simplicial map $f\colon T \to T\Phi$ that factors as a finite composition of type IIA folds. We claim that $T\Phi$ cannot be obtained from T by nontrivial type IIA folds, whence T and $T\Phi$ are G–equivariantly isometric:

By Lemma 5.2, the stabilizer G_e of any edge $e \in E(T)$ is contained in a maximal elliptic subgroup of T, which is always a vertex stabilizer. Let $M_i \le G$, $i \in I$ be the maximal elliptic subgroups of T that contain G_e. Since T has only finitely many G–orbits of vertices, the vertex groups M_i, $i \in I$ fall into only finitely many conjugacy classes, and we assume for a moment that they are in fact all conjugate. Then, for a distinguished maximal elliptic subgroup M containing G_e, the image of the modular homomorphism $\mu\colon G \to (\mathbb{Q}_{>0}, \times)$ defined by

$$\mu(g) = \frac{[M : (M \cap gMg^{-1})]}{[gMg^{-1} : (M \cap gMg^{-1})]}$$

contains the values

$$\frac{[M_i : (M \cap M_i)]}{[M : (M \cap M_i)]} = \frac{[M_i : (M \cap M_i)]}{[M : (M \cap M_i)]} \cdot \frac{[(M \cap M_i) : G_e]}{[(M \cap M_i) : G_e]} = \frac{[M_i : G_e]}{[M : G_e]}, \quad i \in I.$$

Since $\mathcal{PD}$ has no nontrivial integral modulus, Forester [12, Lemma 8.1] implies the indices $[M_i : G_e]$, $i \in I$ can take only finitely many values. Consequently, there exists a maximum index $\mathcal{I}(G_e)$ of G_e in the maximal elliptic subgroups M_i, $i \in I$. If the maximal elliptic subgroups containing G_e are not all conjugate, we associate to each of their finitely many conjugacy classes the maximum index of G_e and define $\mathcal{I}(G_e)$ as the sum of these. One readily sees that for all $g \in G$ we have $\mathcal{I}(gG_eg^{-1}) = \mathcal{I}(G_e)$, so $\mathcal{I}(G_e) = \mathcal{I}(G_{e'})$ if e and e' lie in the same G–orbit of edges of T. Finally, let

$$\mathcal{I}(T) = \sum_{e \in G\backslash T} \mathrm{length}(e) \cdot \mathcal{I}(G_e),$$

where e ranges over the finitely many edges in the metric quotient graph of groups of T. The value $\mathcal{I}(T)$ is insensitive to simplicial subdivisions of T and for all $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ we have $\mathcal{I}(T\Phi) = \mathcal{I}(T)$. On the other hand, after performing a type IIA fold, for the enlarged edge group $\langle E, g\rangle$ we have $\mathcal{I}(\langle E, g\rangle) < \mathcal{I}(E)$, whereas all other edge groups are left invariant. Thus, if $T' \in \mathcal{PD}$ is obtained from T by a nontrivial sequence of type IIA folds then $\mathcal{I}(T') < \mathcal{I}(T)$, whence the claim. □

Example 5.3 Let G be a finitely generated virtually nonabelian free group and $\mathcal{PD}$ the projectivized deformation space of minimal G–trees with finite vertex stabilizers (Example 3.4). We know from Example 3.11 that $\mathcal{PD}$ has no nontrivial integral modulus. Hence, if for $T \in \mathcal{PD}$ and $\Phi \in \mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ we have $d_{\mathrm{Lip}}(T, T\Phi) = 0$ then T and $T\Phi$ are G–equivariantly isometric.

Corollary 5.4 *Let G be a finitely generated virtually nonabelian free group and $\mathcal{PD}$ the projectivized deformation space of minimal G–trees with finite vertex stabilizers. An automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ is elliptic with respect to d_{Lip} if and only if it has finite order.*

Proof It follows from [15, Proposition 8.6] that $\mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finite point stabilizers. Thus, and by Example 5.3, every elliptic automorphism $\Phi \in \mathrm{Out}(G)$ has finite order. Conversely, by Clay [8] every finite-order automorphism of G has a fixed point in $\mathcal{PD}$ and thus is elliptic. □

Example 5.5 Let G be a nonelementary GBS group which contains no solvable Baumslag–Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$. Let $\mathcal{PD}$ be the projectivized deformation space of minimal G–trees with infinite cyclic vertex and edge stabilizers (Example 3.5). By Lemma 3.8, $\mathcal{PD}$ has no nontrivial integral modulus. Thus, if for $T \in \mathcal{PD}$ and $\Phi \in \mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ we have $d_{\mathrm{Lip}}(T, T\Phi) = 0$ then T and $T\Phi$ are G–equivariantly isometric.

Remark For G and $\mathcal{PD}$ as in Example 5.5, let b be the first Betti number of the topological space $G \backslash T$ for any $T \in \mathcal{PD}$; this number is an invariant of $\mathcal{PD}$, as it is not affected by elementary deformations. The stabilizer of each $T \in \mathcal{PD}$ under the action of $\mathrm{Out}(G)$ on $\mathcal{PD}$ is virtually isomorphic to $\mathbb{Z}^k$, where $k = b$ or $b-1$ depending on G [19, Theorem 3.10].

However, the asymmetric Lipschitz pseudometric d_{Lip} does not restrict to an asymmetric metric on $\mathrm{Out}_{\mathcal{D}}(G)$–orbits in general:

Example 5.6 (Horbez) Let $G = \mathrm{BS}(1,6) * F_2 = \langle x, t \mid txt^{-1} = x^6 \rangle * F_2$ and consider the graph of groups decompositions Γ and Γ' of G shown in Figure 6, where all edge group inclusions are the obvious ones and all edges have length $\frac{1}{3}$.

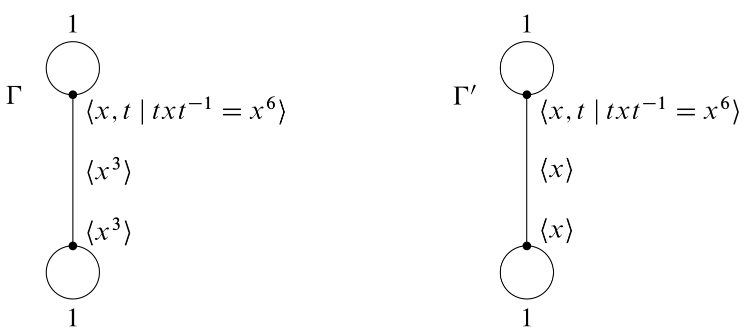

Figure 6: The Bass–Serre trees of the graphs of groups shown above lie in the same $\mathrm{Out}_{\mathcal{D}}(G)$–orbit. They are irreducible but not locally finite.

Let T and T' be the corresponding G–trees. The automorphism

$$\varphi\colon \mathrm{BS}(1,6) \xrightarrow{\cong} \mathrm{BS}(1,6), \quad x \mapsto x^3, \quad t \mapsto t,$$

induces an automorphism $\Phi = \varphi * \mathrm{id}_{F_2} \in \mathrm{Aut}(G)$ and we have $T' = T\Phi$. Similarly as in Example 4.4, the natural morphism of graphs of groups from Γ to Γ' lifts to a G–equivariant map from T to $T\Phi$ (namely, a type IIA fold) that is an isometry on edges and thus has Lipschitz constant 1, whence $d_{\mathrm{Lip}}(T, T\Phi) = 0$. However, T and $T\Phi$ are not G–equivariantly isometric, as the group element $x \in G$ stabilizes an edge in $T\Phi$ but not in T (x is not a conjugate of x^3).

5.2 Nonparabolic automorphisms

Let $\mathcal{PD}$ be a projectivized deformation space of irreducible G–trees and $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ a nonparabolic automorphism, ie $\inf \widetilde{\Phi}$ is realized. Let $T \in \mathcal{PD}$ such that $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$ and let $f\colon T \to T\Phi$ be an optimal map with tension forest $\Delta = \Delta(f) \subset T$. The following observation will be used in the proof of Theorem 5.13:

Proposition 5.7 *After a small perturbation of the metric on T, preserving the condition that $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$, the map $f\colon T \to T\Phi$ is G–equivariantly homotopic to an optimal map $f'\colon T \to T\Phi$ with $\Delta(f') \subseteq \Delta$ such that*

$$f'(\Delta(f')) \subseteq \Delta(f').$$

Proof Suppose that $f(\Delta)$ is not contained in Δ and let $e \in E(\Delta)$ be an edge such that $f(e) \nsubseteq \Delta$. Slightly scale up the metric on Δ and down on $T \setminus \Delta$ while maintaining covolume 1. This lowers the stretching factor on e and produces a new tension forest, of the original map f made linear on edges, that is properly contained in the old one. Since $d_{\mathrm{Lip}}(T, T\Phi)$ is minimal among all translation distances of Φ, we will not have removed all edges of Δ and started over with a new tension forest that corresponds to a strictly smaller maximal stretching factor. In particular, there always exists an edge $e' \in E(\Delta)$ such that $f(e') \subseteq \Delta$. The stretching factor of f on e' remains unchanged and we preserve the condition that $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$. As T has only finitely many G–orbits of edges, after finitely many repetitions we have $f(\Delta) \subseteq \Delta$. If at this point Δ has a vertex with only one gate, we perturb f to an optimal map f' as in the proof of Proposition 4.12. □

5.3 Parabolic automorphisms

Let $\mathcal{PD}$ be a projectivized deformation space of G–trees and $T \in \mathcal{PD}$. We say a G–invariant subforest $S \subseteq T$ is *essential* if it contains the hyperbolic axis of some hyperbolic group element. The notion of essential G–invariant subforests generalizes the notion of homotopically nontrivial subgraphs of marked metric graphs in Outer space.

Definition 5.8 An automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is *reducible* if there exists a G–tree $T \in \mathcal{PD}$ and a G–equivariant map $f\colon T \to T\Phi$ that leaves an essential proper G–invariant subforest of T invariant. If Φ is not reducible, it is *irreducible*.

As we will see, parabolic automorphisms are often reducible (Corollary 5.10). For this, let $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ be a parabolic automorphism (ie $\inf \widetilde{\Phi}$ is not realized) and $(T_k)_{k\in\mathbb{N}}$ a sequence of G–trees in $\mathcal{PD}$ such that $\lim_{k\to\infty} d_{\mathrm{Lip}}(T_k, T_k\Phi) = \inf \widetilde{\Phi}$. For $\Theta > 0$ we denote by $\mathcal{PD}(\Theta)$ the $\mathrm{Out}_{\mathcal{D}}(G)$–invariant subspace of $\mathcal{PD}$ consisting of all G–trees $T \in \mathcal{PD}$ that satisfy $l_T(g) \geq \Theta$ for all hyperbolic group elements $g \in G$. We call $\mathcal{PD}(\Theta)$ the Θ*–thick part* of $\mathcal{PD}$.

Proposition 5.9 *If the projectivized deformation space $\mathcal{PD}$ consists of irreducible G–trees and $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then for only finitely many $k \in \mathbb{N}$ we have $T_k \in \mathcal{PD}(\Theta)$.*

Proof We will argue as in the proof of [20, Claim 72] by Meucci. Suppose that the proposition is false and that, after passing to a subsequence, we have $T_k \in \mathcal{PD}(\Theta)$ for all $k \in \mathbb{N}$. This will lead to a contradiction.

Since $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbit of simplices, it acts on the thick part $\mathcal{PD}(\Theta)$ cocompactly in all three topologies. In particular, the image of $(T_k)_{k\in\mathbb{N}}$ in the quotient $\mathcal{PD}(\Theta)/\mathrm{Out}_{\mathcal{D}}(G)$ has a weakly convergent subsequence. We can thus find a sequence of outer automorphisms $(\psi_k)_{k\in\mathbb{N}} \subset \mathrm{Out}_{\mathcal{D}}(G)$ such that, after passing to a subsequence, $(T_k\psi_k)_{k\in\mathbb{N}}$ weakly converges in $\mathcal{PD}(\Theta)$ to some $T \in \mathcal{PD}(\Theta)$. We have

$$\begin{aligned} d_{\mathrm{Lip}}(T\psi_k^{-1}, T\psi_k^{-1}\Phi) &\leq d_{\mathrm{Lip}}(T\psi_k^{-1}, T_k) + d_{\mathrm{Lip}}(T_k, T_k\Phi) + d_{\mathrm{Lip}}(T_k\Phi, T\psi_k^{-1}\Phi) \\ &= d_{\mathrm{Lip}}(T, T_k\psi_k) + d_{\mathrm{Lip}}(T_k, T_k\Phi) + d_{\mathrm{Lip}}(T_k\psi_k, T), \end{aligned}$$

where $\lim_{k\to\infty} d_{\mathrm{Lip}}(T, T_k\psi_k) = \lim_{k\to\infty} d_{\mathrm{Lip}}(T_k\psi_k, T) = 0$ by Proposition 4.17. Hence, $\lim_{k\to\infty} d_{\mathrm{Lip}}(T, T\psi_k^{-1}\Phi\psi_k) = \lim_{k\to\infty} d_{\mathrm{Lip}}(T_k, T_k\Phi) = \inf \widetilde{\Phi}$.

By Theorem 4.14, for all $k \in \mathbb{N}$ there exists a candidate $\xi_k \in \mathrm{cand}(T)$ such that

$$\sigma(T, T\psi_k^{-1}\Phi\psi_k) = \frac{l_{T\psi_k^{-1}\Phi\psi_k}(\xi_k)}{l_T(\xi_k)} = \frac{l_T(\psi_k^{-1}\Phi\psi_k(\xi_k))}{l_T(\xi_k)}.$$

The translation length function of T has discrete image in $\mathbb{R}$ and hence the numerator takes discrete values. Since the candidates of T have only finitely many different translation lengths, the denominator takes only finitely many values and we conclude that the sequence $(\sigma(T, T\psi_k^{-1}\Phi\psi_k))_{k\in\mathbb{N}}$ is discrete. For large k we thus have

$$d_{\mathrm{Lip}}(T\psi_k^{-1}, T\psi_k^{-1}\Phi) = d_{\mathrm{Lip}}(T, T\psi_k^{-1}\Phi\psi_k) = \inf \widetilde{\Phi}$$

contradicting the assumption that Φ is parabolic. □

Corollary 5.10 *Under the assumptions of Proposition 5.9, for large k any optimal map $f\colon T_k \to T_k\Phi$ leaves an essential proper G–invariant subforest of T_k invariant up to G–equivariant homotopy. In particular, every parabolic automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is reducible.*

If T is a minimal G–tree then a subforest $S \subseteq T$ with no trivial components is a *core subforest* if it does not have any vertices of valence 1. Every G–invariant subforest $S \subseteq T$ with no trivial components contains a unique (possibly empty) maximal G–invariant core subforest $\mathrm{core}(S) \subseteq S \subseteq T$, obtained by inductively removing G–orbits of edges whose terminal or initial vertex has valence 1. The process of removing G–orbits of edges terminates after finitely many steps by the cocompactness of T.

Proof For $T \in \mathcal{PD}$ and $\varepsilon > 0$, let $T^\varepsilon \subseteq T$ be the union of all subsets of the form $\bigcup_{k\in\mathbb{Z}} g^k[x, gx]$ with $g \in G$ hyperbolic and $x \in T$ such that $d(x, gx) \le \varepsilon$. In particular, T^ε contains the axes of all hyperbolic group elements $g \in G$ with $l_T(g) \le \varepsilon$. Although $T^\varepsilon \subseteq T$ is generally not a simplicial subcomplex of T, we will still speak of T^ε as a (nonsimplicial) subforest, as it becomes a subcomplex after subdividing the simplicial structure on T. In fact, T^ε has no trivial components and its maximal G–invariant core subforest $\mathrm{core}(T^\varepsilon) \subseteq T^\varepsilon$ will be a genuine simplicial subforest of T. Since G acts on T by isometries, if $\bigcup_{k\in\mathbb{Z}} g^k[x, gx]$ is contained in T^ε then for all $h \in G$ the translate

$$h\Big(\bigcup_{k\in\mathbb{Z}} g^k[x, gx]\Big) = \bigcup_{k\in\mathbb{Z}} (hgh^{-1})^k[hx, hgx]$$

is contained in T^ε as well. Thus, $T^\varepsilon \subseteq T$ is G–invariant.

Since $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices, the complex $\mathcal{PD}$ must be finite-dimensional, say of dimension $d \in \mathbb{N}$, and the number of G–orbits of edges of any $T \in \mathcal{PD}$ is bounded above by $d+1$. Because the G–trees in $\mathcal{PD}$ have covolume 1, in any G–tree $T \in \mathcal{PD}$ there exists an orbit of edges with associated edge length greater than or equal to $\frac{1}{d+1}$. Therefore, for $\varepsilon < \frac{1}{d+1}$ the subforest $T^\varepsilon \subseteq T$ is a proper subforest. Given G–invariant simplicial subforests $S' \subseteq S$ of T with no trivial components, the subforest S' is a proper subforest of S if and only if $G \backslash S - G \backslash S'$ consists of at least one edge. Hence, as the G–trees in $\mathcal{PD}$ have at most $d+1$ G–orbits of edges, the number $d+1$ is a uniform bound for the length of any chain of proper G–invariant simplicial subforests with no trivial components of any G–tree in $\mathcal{PD}$.

Let $D = \inf \widetilde{\Phi}$. Also, let $\varepsilon < 1/(d+1)$ and $\Theta = \varepsilon / e^{(D+1)(d+1)}$. By Proposition 5.9, we can choose k so large that $T_k \notin \mathcal{PD}(\Theta)$ and $d_{\mathrm{Lip}}(T_k, T_k\Phi) < D+1$. For

$i = 0, \dots, d+1$, define $\delta_i = \varepsilon / e^{(D+1)i}$ and consider the chain of G–invariant subforests

$$T_k^{\varepsilon} = T_k^{\delta_0} \supseteq T_k^{\delta_1} \supseteq \cdots \supseteq T_k^{\delta_{d+1}} = T_k^{\Theta}$$

all of which are proper subforests of T_k. Note that $T_k^{\Theta} \neq \varnothing$ since $T_k \notin \mathcal{PD}(\Theta)$ and thus there exists a hyperbolic group element $g \in G$ with $l_{T_k}(g) < \Theta$ whose axis lies in T_k^{Θ}. The associated chain of core subforests is a chain of G–invariant simplicial subforests of T_k whose number of proper inclusions is bounded by d by the arguments given above. Thus, there exists $i \in \{0, \dots, d\}$ for which we have $\operatorname{core}(T_k^{\delta_{i+1}}) = \operatorname{core}(T_k^{\delta_i})$. Since $d_{\mathrm{Lip}}(T_k, T_k\Phi) < D+1$, the Lipschitz constant of the optimal map $f\colon T_k \to T_k\Phi$ is smaller than e^{D+1} and we have

$$f(\operatorname{core}(T_k^{\delta_{i+1}})) \subseteq f(T_k^{\delta_{i+1}}) \subseteq T_k^{\delta_i}.$$

The subforest $\operatorname{core}(T_k^{\delta_i}) \subseteq T_k^{\delta_i}$ is a G–equivariant deformation retract of $T_k^{\delta_i} \subseteq T_k$ and the obvious deformation retraction extends to a G–equivariant self homotopy equivalence h of T_k (this is easily seen on the level of quotient graphs of groups). Now f is G–equivariantly homotopic to the G–equivariant map

$$T_k \xrightarrow{f} T_k\Phi \xrightarrow{h} T_k\Phi$$

that leaves the proper G–invariant simplicial subforest $\operatorname{core}(T_k^{\delta_{i+1}}) = \operatorname{core}(T_k^{\delta_i}) \subset T_k$ invariant. As we remarked above, there exists a hyperbolic group element $g \in G$ whose axis lies in T_k^{Θ} and thus also in $\operatorname{core}(T_k^{\delta_i})$, and we conclude that $\operatorname{core}(T_k^{\delta_i})$ is essential. $\square$

5.4 Train track representatives

Let $\mathcal{PD}$ be a projectivized deformation space of G–trees and $T \in \mathcal{PD}$. Moreover, let $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$.

Definition 5.11 An optimal map $f\colon T \to T\Phi$ is a *train track map* if it satisfies the following three conditions:

(1) $\Delta(f) = T$.

(2) f maps edges to legal paths (see Definition 4.9).

(3) If f maps a vertex $v \in V(T)$ to a vertex $f(v) \in V(T\Phi)$ then it maps legal turns at v to legal turns at $f(v)$. (If v has 2 gates then $f(v)$ could alternatively lie in the interior of an edge. Since f is linear on edges, it then maps inequivalent directions at v to inequivalent directions at $f(v)$.)

If f is a train track map then for any legal line $L \subset T$ and every $k \in \mathbb{N}$ the image $f^k(L) \subset T\Phi^k$ is again a legal line. We say that an automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is *represented by a train track map* if there exists a G–tree $T \in \mathcal{PD}$ and an optimal map $f\colon T \to T\Phi$ that is a train track map.

Proposition 5.12 *Let $\mathcal{PD}$ be a projectivized deformation space of G–trees. If an automorphism $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is represented by a train track map $f\colon T \to T\Phi$ then $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$ and, in particular, Φ is nonparabolic.*

Proof Our argument is a generalization of [3, Remark 8]. Suppose that $f\colon T \to T\Phi$ is an optimal map that is a train track map. By Theorem 4.14 and Lemma 4.15, there exists a hyperbolic group element $\xi \in G$ whose axis $A_\xi \subset T$ lies in $\Delta(f)$ and is legal with respect to the train track structure defined by f (once we know that there exists an optimal map $f\colon T \to T\Phi$, Theorem 4.14 no longer requires T to be irreducible). Since f is a train track map, for all $k \in \mathbb{N}$ the image $f^k(A_\xi) \subset T\Phi^k$ is a ξ–invariant line—and thus equals the hyperbolic axis of ξ in $T\Phi^k$—that lies in $\Delta(f)$ and is legal. We therefore have

$$\begin{aligned}\sigma(T, T\Phi^k) = \sup_g \frac{l_{T\Phi^k}(g)}{l_T(g)} \geq \frac{l_{T\Phi^k}(\xi)}{l_T(\xi)} &= \frac{l_{T\Phi}(\xi)}{l_T(\xi)} \cdots \frac{l_{T\Phi^k}(\xi)}{l_{T\Phi^{k-1}}(\xi)} \\ &= \sigma(T, T\Phi) \cdots \sigma(T\Phi^{k-1}, T\Phi^k) \\ &= \sigma(T, T\Phi)^k\end{aligned}$$

from which we conclude that $\sigma(T, T\Phi^k) = \sigma(T, T\Phi)^k$. In order to show that $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$, let $T' \in \mathcal{PD}$ be any other G–tree. We have

$$\begin{aligned}k \cdot d_{\mathrm{Lip}}(T, T\Phi) &= d_{\mathrm{Lip}}(T, T\Phi^k) \\ &\leq d_{\mathrm{Lip}}(T, T') + d_{\mathrm{Lip}}(T', T'\Phi^k) + d_{\mathrm{Lip}}(T'\Phi^k, T\Phi^k) \\ &\leq d_{\mathrm{Lip}}^{\mathrm{sym}}(T, T') + k \cdot d_{\mathrm{Lip}}(T', T'\Phi)\end{aligned}$$

and hence $d_{\mathrm{Lip}}(T, T\Phi) \leq \frac{1}{k} \cdot d_{\mathrm{Lip}}^{\mathrm{sym}}(T, T') + d_{\mathrm{Lip}}(T', T'\Phi)$. Letting k go to infinity, we see that $d_{\mathrm{Lip}}(T, T\Phi) \leq d_{\mathrm{Lip}}(T', T'\Phi)$. □

As for existence of train track representatives, we have the following:

Theorem 5.13 (Existence of train track representatives) *Let $\mathcal{PD}$ be a projectivized deformation space of irreducible G–trees. If $\mathrm{Out}_{\mathcal{D}}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices then every irreducible automorphism (see Definition 5.8) $\Phi \in \mathrm{Out}_{\mathcal{D}}(G)$ is represented by a train track map.*

Proof Since the automorphism Φ is irreducible, by Corollary 5.10 it is nonparabolic, ie $\inf \widetilde{\Phi}$ is realized. Let $T \in \mathcal{PD}$ such that $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$ and let $f\colon T \to T\Phi$ be an optimal map, which exists by the irreducibility of $\mathcal{PD}$. We claim that f already satisfies (1) and (2) of Definition 5.11:

Assertion (1) immediately follows from Proposition 5.7, as we could otherwise slightly perturb the metric on T and find an optimal map $T \to T\Phi$ that leaves an essential proper G–invariant subforest of T invariant (the tension forest of an optimal map is always essential by Theorem 4.14 and Lemma 4.15), contradicting the assumption that Φ is irreducible.

As for (2), suppose that an edge $e \in E(T)$ is mapped over an illegal turn. Slightly fold the illegal turn G–equivariantly and scale the metric on T back to covolume 1. The optimal map $f\colon T \to T\Phi$ naturally induces a G–equivariant map that we make linear on edges relative to the vertices of T. The performed perturbation lowers the stretching factor of f on the edge induced by e, which therefore drops out of the tension forest. Each witness $A_\xi \subset \Delta(f) = T$ is legal with respect to f and does not get folded, whence the stretching factor of f on A_ξ does not increase.[3] Hence, we preserve the condition that $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$ and the Lipschitz constant of f remains minimal among all G–equivariant Lipschitz maps from T to $T\Phi$. After perturbing f to an optimal map as in the proof of Proposition 4.12, we obtain an optimal map $T \to T\Phi$ whose tension forest is a proper subforest of T. By Proposition 5.7, this again contradicts the assumption that Φ is irreducible.

Finally, we may perturb T and f by an arbitrarily small amount, preserving the condition that $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$ and that $f\colon T \to T\Phi$ is an optimal map — and therefore also preserving conditions (1) and (2) — such that (3) of Definition 5.11 is satisfied as well: if f maps a legal turn at a vertex $v \in V(T)$ to an illegal turn, slightly fold the illegal turn G–equivariantly (see Figure 7).

Again, each witness $A_\xi \subset T$ is legal with respect to f and does not get folded so that the stretching factor of f on A_ξ does not increase. Thus, we preserve the property that $d_{\mathrm{Lip}}(T, T\Phi) = \inf \widetilde{\Phi}$ and that f is a minimal stretch map. The perturbation makes the legal turn at v illegal, but the induced map f made linear on edges is still optimal and v still has at least two gates, for f would otherwise give rise to an optimal map whose tension forest is a proper subforest of T. The folding decreases the number $G(T) = \sum_w \max\{0, G(w) - 2\}$, where w ranges over the finitely many G–orbits of

[3] In fact, there also exists a witness $A_\xi \subset T$ whose f–image $A_{\Phi(\xi)} \subset T$ does not get folded either, as the induced map would otherwise have strictly smaller Lipschitz constant, contradicting the fact that $d_{\mathrm{Lip}}(T, T\Phi)$ is minimal among all translation distances of Φ.

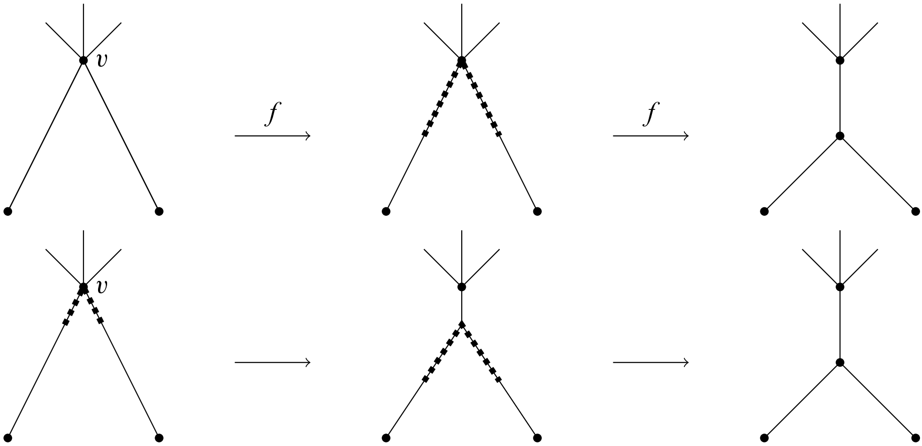

Figure 7: Legal and illegal (dashed) turns: the upper row shows turns before folding, the bottom row after folding. The number $G(T)$ in the upper row is $3+2+(2+1)=8$, whereas in the bottom row it has decreased to $2+(2+0)+(2+1)=7$.

vertices of T and $G(w)$ denotes the number of gates at w. After finitely many steps, we obtain an optimal map $f\colon T \to T\Phi$ that also satisfies condition (3). □

Example 5.14 Let G be a finitely generated virtually nonabelian free group and $\mathcal{PD}$ the projectivized deformation space of minimal G–trees with finite vertex stabilizers (Example 3.4); it is irreducible and $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices (Example 3.11). We conclude that every irreducible automorphism of G is represented by a train track map. This generalizes [5, Theorem 1.7] to virtually free groups.

Example 5.15 Let G be a nonelementary GBS group that contains no solvable Baumslag–Solitar group $\mathrm{BS}(1,n)$ with $n \geq 2$. The projectivized deformation space $\mathcal{PD}$ of minimal G–trees with infinite cyclic vertex and edge stabilizers is irreducible (Example 3.5) and $\mathrm{Out}_{\mathcal{D}}(G) = \mathrm{Out}(G)$ acts on $\mathcal{PD}$ with finitely many orbits of simplices (Example 3.12). Hence, every irreducible automorphism of G is represented by a train track map.

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