Knots with compressible thin levels

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We produce embeddings of knots in thin position that admit compressible thin levels. We also find the bridge number of tangle sums where each tangle is high distance.

1 Introduction

Thin position has contributed to many advances in the study of knots and 3–manifolds. Gabai defined thin position for knots in S^3 and employed the notion in his proof of Property R [5]. Subsequently, thin position was utilized in the solution to the knot complement problem (see Gordon and Luecke [7]), the recognition problem for S^3 (see Rubinstein [12] and Thompson [14]) and the leveling of unknotting tunnels (see Goda, Scharlemann and Thompson [6]).

The usefulness of thin position for knots in S^3 is grounded in its connections to the topology of the knot exterior. In particular, Thompson showed that a knot in thin position that is not in bridge position contains an essential planar surface in its exterior [15]. Wu strengthened this result to show that every thinnest thin level of a knot in thin position is essential in the knot exterior [18]. Additional results have furthered our understanding of essential surfaces and thin position; see Heath and Kobayashi [8] and Tomova [16]. These results motivate the following natural question.

Question 1 *Given a knot in thin position, are all thin levels essential in the knot exterior?*

This question was originally considered by Thompson during her work on the recognition problem for S^3. In the context of that problem as well as many others, the possibility of a knot or graph in thin position admitting a compressible thin level presents a significant technical challenge. It has long been believed that these challenges are endemic and that there exist knots in thin position that admit a compressible thin level. In fact, candidate examples are well known in the community; see Figure 3. However, demonstrating that one of these candidate embeddings is in thin position has proved

to be a difficult problem. In this paper, we use recent advancements in the study of the distance of bridge surfaces (see the first author, Tomova and Yoshizawa [4] and Johnson and Tomova [10]) to prove the following:

Theorem 1.1 *There are infinitely many knots K that admit a thin position with a compressible thin level.*

Our main result supports the emerging theme that the fundamental properties of thin position are quite subtle. Unlike the closely related invariant of bridge number, width (an integer invariant derived from thin position) is not additive with respect to connected sum; see the first author and Tomova [3]. Additionally, thin position of a knot may not minimize bridge number [3]. Finally, there exist embeddings of the unknot that cannot be isotoped to the standard unknot through embeddings of nonincreasing width; see the second author [19].

Our work also reinforces the notion that knots and 3–manifolds composed of sufficiently complicated pieces have predictable topology; the proof of Theorem 1.1 demonstrates that the obvious embeddings of the candidate knots K coincide with their thin positions. In terms of 3–manifolds, this concept is apparent in work of Kobayashi and Qiu [11], which demonstrates that if sufficiently complicated compact 3–manifolds are glued together along a common boundary component, then the Heegaard genus of the resulting 3–manifold is as expected.

In the setting of knots in S^3, the natural analogue of 3–manifold amalgamation is the gluing together of two collections of arcs each contained in a 3–ball, an operation known as tangle summation. In general, it is difficult to predict the bridge number of a tangle sum; for instance, the tangle sum of two nontrivial tangles can give rise to the unknot. In [1], the first author studies the degeneration of bridge number for tangle sums of 2–strand tangles, and in [2], he gives a lower bound for the bridge number of a tangle sum with some mild restrictions on the tangles being glued together. However, as with 3–manifolds, if we restrict our tangles by requiring that they have sufficiently complicated bridge surfaces, the picture becomes much clearer.

Using the techniques mentioned above, we prove the complete analogue of the main theorem in [11] for the bridge number of knots in S^3 (see Section 3 for relevant definitions):

Theorem 1.2 *Suppose that (B_1, τ_1) and (B_2, τ_2) are n–strand tangles with $n > 1$, and let K be a knot in S^3 that is the tangle sum of (B_1, τ_1) and (B_2, τ_2). In addition, let Σ_i be a β_i–bridge sphere for (B_i, τ_i), where $\beta_i > n$. If $d(\Sigma_i) > 2(\beta_1 + \beta_2 - n)$ for $i = 1, 2$, then*

$$b(K) = \beta_1 + \beta_2 - n.$$

Our paper is structured as follows. In Section 2, we introduce the notions of width and bridge number and we present relevant background. In Section 3, we define distance of a bridge surface and give results connecting distance to the existence of essential surfaces and alternate bridge surfaces. In Section 4, we prove Theorem 1.2. In Section 5, we give an embedding of a knot with a compressible thin level, while in Sections 6, 7 and 8 we show that this embedding is a thin position for the knot. Our proof relies on recent advances in the study of high distance bridge surfaces [10] and the construction of our examples is inspired by the examples of strict subadditivity of width presented in [3].

2 Preliminaries

A *knot* K is an isotopy class of embedded simple closed curves in the 3–sphere. For the remainder of the paper, we fix a Morse function $h\colon S^3 \to \mathbb{R}$ such that h has exactly two critical points, one of index zero and one of index three. Given an embedded simple closed curve k in S^3, we may perturb k (if necessary) so that $h|_k$ is Morse. Let $c_0, \dots, c_n$ denote the critical values of $h|_k$ and choose regular values $r_1, \dots, r_n$ satisfying $c_{i-1} < r_i < c_i$. We say that $h^{-1}(r_i)$ is a *level sphere* having *width* $w(h^{-1}(r_i)) = |k \cap h^{-1}(r_i)|$. The *width* $w(k)$ and *bridge number* $b(k)$ of k are defined to be

$$w(k) = \sum w(h^{-1}(r_i)) \quad \text{and} \quad b(k) = \frac{n+1}{2},$$

respectively. To obtain invariants of the knot K, we minimize these two quantities over all possible embeddings of K. In other words, the *width* $w(K)$ and *bridge number* $b(K)$ of K are defined by

$$w(K) = \min_{k \sim K}\{w(k)\} \quad \text{and} \quad b(K) = \min_{k \sim K}\{b(k)\}.$$

We say that k is a *thin position* of K if $w(k) = w(K)$. More generally, we say k is a *bridge position* if all maxima of $h|_k$ occur above all minima of $h|_k$, and a bridge position k is a *minimal bridge position* if $b(k) = b(K)$. Equivalently, we could define these concepts by fixing an embedding k of K and considering isotopy classes of Morse functions h on K.

Let k and $r_1, \dots, r_n$ be as above. In [13], Scharlemann and Schultens present an alternate formula for calculating width using level spheres which intersect k maximally or minimally. For $1 < i < n$, we call $h^{-1}(r_i)$ a *thick level* if c_{i-1} is a minimum and c_i is a maximum, or a *thin level* if c_{i-1} is a maximum and c_i is a minimum. It is a straightforward exercise to see that if k has m thick levels, then it has $m-1$ thin levels. We let $a_1, \dots, a_m$ (resp. $b_1, \dots, b_{m-1}$) denote the widths of the thick levels (resp.

thin levels), where both sets of numbers are naturally ordered by the height function h. Hence, each k gives rise to a tuple of even integers $(a_1, b_1, a_2, \ldots, b_{m-1}, a_m)$ which we call the *thin-thick* tuple corresponding to k. By [13],

$$(1) \qquad w(k) = \frac{1}{2}\left(\sum a_i^2 - \sum b_i^2\right).$$

In particular, if we select one of the a_i, we have

$$(2) \qquad w(k) \geq \tfrac{1}{2}a_i^2.$$

As demonstrated by Thompson [15] and Wu [18], thin position is related to essential surfaces embedded in the exterior $E(K) = S^3 \setminus \eta(K)$ of K, where $\eta(\cdot)$ denotes an open regular neighborhood. A compact surface S with nonempty boundary properly embedded in $E(K)$ is said to be *meridional* if each curve of ∂S bounds a meridian disk of the solid torus $\overline{\eta(K)}$. As such, meridional surfaces are in one-to-one correspondence with closed surfaces embedded in S^3 and intersecting K transversely. We often blur the distinction between such surfaces; however, for a closed surface S in S^3 intersecting K transversely, we will use S_K to denote $S \cap E(K)$ where appropriate. In an abuse of terminology, we also occasionally refer to S_K as a *punctured* surface despite the fact that it is compact. When we say that two closed surfaces S and T in S^3 which intersect K transversely are isotopic, we will mean that the surfaces are isotopic relative to K (equivalently, S_K is isotopic to T_K in $E(K)$) unless otherwise specified.

Suppose now that S is a properly embedded meridional or closed surface in $E(K)$. A *compressing disk* D for S is an embedded disk such that $S \cap D = \partial D$ and ∂D is an essential curve in S. If there is an compressing disk for S, we say S is *compressible*; otherwise, S is *incompressible*. If S is isotopic into $\partial E(K)$, we say S is ∂–*parallel*, and in the case that S is incompressible and not ∂–parallel, we call S *essential*.

There are two other classes of disks we will use, and to define these we consider S as an embedded surface in S^3 intersecting K transversely. A *bridge disk* is an embedded disk Δ such that $\partial\Delta$ is the endpoint union of arcs α and β, where $\Delta \cap S = \alpha$ and $\beta \subset K$. A *cut disk* C is an embedded disk such that $C \cap S = \partial C$, ∂C is essential in S_K, and $C \cap K$ is a single point in the interior of C. If there is a compressing or cut disk (a *c–disk*) for S, we say S is *c–compressible*; otherwise S is *c–incompressible*. If S is *c*–incompressible and not ∂–parallel, S is *c–essential*.

If a surface S is *c*–compressible, then we may surger S along a *c*–disk D to get a new surface S'. We say that S' is obtained by *c*–compressing S. We note that this process has an inverse operation. If D is a compressing disk, we may recover S from S' by performing surgery on S' along an arc α such that $\alpha \cap S' = \partial\alpha$; that is, viewing $\overline{\eta(\alpha)}$

as $\alpha \times \Delta$, where Δ is a small disk, we glue the annulus $\alpha \times \partial \Delta$ to $S' \setminus \eta(\partial \alpha)$. If D is a compressing disk, we recover S using an arc α disjoint from K; when D is a cut disk, we recover S using an arc α contained in K. In either case, we say that S results from *tubing* S'.

The relationship between thin position of a knot K and essential meridional surfaces is made precise with the following theorem:

Theorem 2.1 [18] *Suppose k is a thin position of K which has a thin level. Then, any thinnest thin level $h^{-1}(r_i)$ is an essential meridional surface in $E(K)$.*

A knot K in S^3 is *prime* if its exterior contains no essential meridional annulus. The above result has been strengthened by Tomova, and we will employ the following extension:

Theorem 2.2 [16] *Suppose K is a prime knot in thin position and P is a thin sphere of minimal width. If P' is another thin sphere satisfying $w(P') = w(P) + 2$, then P' is incompressible.*

We note, however, that the preceding theorems do not imply that thin spheres are c–incompressible. In the case of a c–compressible thin sphere, we will be able to use a theorem of Blair and Tomova.

Theorem 2.3 [3] *Let K be a prime knot in thin position and suppose P is a c–compressible thin sphere, with a c–disk D. Then there is a thin sphere P' adjacent to P in the direction of D such that either $D \cap P' \neq \varnothing$ or $w(P') < w(P)$.*

3 Bridge surfaces and distance

In this section, we will define bridge surfaces for knots and tangles and discuss important relationships between bridge surfaces, essential surfaces, and distances between disk sets in the curve complex.

A *punctured* 3–*sphere* is the complement of a disjoint embedded collection of open 3–balls in S^3. We construct our knots by gluing together arcs embedded in punctured 3–spheres, called tangles. A *tangle* (B, τ) consists of a 3–manifold B homeomorphic to a punctured 3–sphere containing a properly embedded 1–manifold τ which has no closed components. In the special case that B is a 3–ball and $n = |\tau|$, we call (B, τ) an n–*strand tangle*. Every embedded 2–sphere Σ in a punctured 3–sphere B separates B into two punctured 3–balls B_1 and B_2. If B_i has $n \geq 2$ boundary components, then

there is a collection of $n-2$ properly embedded arcs Γ_i, called the *spine* of B_i, such that $B_i \setminus \eta(\Gamma_i)$ is homeomorphic to $\Sigma \times I$. If B_i has one boundary component, then Γ_i is a single point in the interior of B_i.

Given (B, τ) such that (B, τ) is a tangle or $B = S^3$ and τ is a knot, we say an embedded 2–sphere Σ in B is a *bridge sphere* if Σ is transverse to τ and if $\tau \cap B_i$ can be simultaneously properly isotoped into a position so that it is the union of I–fibers of $B_i \setminus \eta(\Gamma_i)$, $i = 1, 2$ (such arc components of $\tau \cap B_i$ are called *vertical*) and arcs with both endpoints on Σ that admit mutually disjoint bridge disks (such arc components of $\tau \cap B_i$ are called *∂–parallel*). Letting $\tau_i = \tau \cap B_i$, we say that (B_i, τ_i) is a *trivial tangle* and we express the decomposition of (B, τ) by Σ as a *bridge splitting*,

$$(B, \tau) = (B_1, \tau_1) \cup_\Sigma (B_2, \tau_2).$$

Here we make several other definitions concerning tangles. A tangle (B, τ) is *irreducible* if every 2–sphere embedded in $B \setminus \eta(\tau)$ bounds a ball, and (B, τ) is *prime* if every meridional annulus in $B \setminus \eta(\tau)$ is boundary parallel.

Given an embedding of a knot in S^3 that is Morse with respect to the height function h, every thick level Σ is a bridge sphere for the portion of the knot that lies between the thin level immediately below Σ and the thin level immediately above Σ. If (B, τ) is a knot in S^3 or a tangle in a 3–ball, we say a bridge sphere Σ is an *n–bridge sphere* if (B_i, τ_i), $(i = 1, 2)$ is an n–strand trivial tangle. It is a simple exercise to verify that if k is an embedding of a knot K with one thick level $h^{-1}(r)$, then $h^{-1}(r)$ is an n–bridge sphere, where $n = w(h^{-1}(r))/2$. As above, if S is an embedded surface in B transverse to τ, we will use S_τ to denote $S \setminus \eta(\tau) \subset B \setminus \eta(\tau)$ when we wish to refer to a surface in the exterior of τ in B.

We will utilize "sufficiently complicated" bridge surfaces, where complication is measured via distances between disk sets in the curve complex. For a compact surface S, the (1–skeleton of the) curve complex $\mathcal{C}(S)$ is a graph whose vertices are isotopy classes of essential simple closed curves. Two vertices are connected by an edge if their corresponding curves may be realized disjointly. The vertex set of the curve complex $\mathcal{C}(S)$ has a natural metric constructed by assigning each edge length one and defining the distance between two vertices to be the length of the shortest path between them.

Given a bridge sphere Σ splitting (B, τ) into $(B_1, \tau_1) \cup_\Sigma (B_2, \tau_2)$, we define the disk set $\mathcal{D}_i \subset \mathcal{C}(\Sigma_\tau)$ to be those curves in Σ_τ which bound compressing disks in $B_i \setminus \eta(\tau_i)$, and the distance $d(\Sigma)$ of the splitting to be

$$d(\Sigma) = \min\{d(c_1, c_2) : c_i \in \mathcal{D}_i\}.$$

The utility of this definition is made clear by the following two theorems of Johnson and Tomova (the first is a version of Theorem 4.4 from [10] and the second is implicit in the proof of Theorem 4.2):

Theorem 3.1 [10] *Suppose (N, K) is a tangle or $N = S^3$ and K is a knot, and let (M, T) be a tangle inside of (N, K) such that K is transverse to ∂M and $(\partial M)_K$ is incompressible in $\overline{(N \setminus M)} \setminus \eta(K)$, and $T = K \cap M$. In addition, let Σ be a bridge sphere for (M, T) and let Σ' be a bridge sphere for (N, K). Then one of the following holds:*

- *After isotopy and surgery of $\Sigma'_K \cap M$ along c–disks in M, we obtain a compressed surface Σ''_K such that $\Sigma''_K \cap M$ is parallel to Σ_K.*
- $d(\Sigma) \leq 2 - \chi(\Sigma'_K)$.
- $\chi(\Sigma_K) \geq -3$.

Theorem 3.2 [10] *Suppose (N, K) is a tangle or $N = S^3$ and K is a knot, and let (M, T) be a tangle inside of (N, K) such that K is transverse to ∂M and $(\partial M)_K$ is incompressible in $\overline{(N \setminus M)} \setminus \eta(K)$, and $T = K \cap M$. Let Σ be a bridge sphere for (M, T) and F be a surface properly embedded in N such that F is meridional, incompressible and separating. Additionally, suppose $F \cap (\partial M)_K$ is essential in $(\partial M)_K$. Then one of the following holds:*

- $d(\Sigma) \leq 2 - \chi(F_K)$.
- $\chi(\Sigma_K) \geq -3$.
- *Each component of $F \cap (M \setminus \eta(T))$ is boundary parallel in $M \setminus \eta(T)$.*

As an immediate application of Theorem 3.2, we prove the next two lemmas for use later in the paper.

Lemma 3.1 *Suppose (B, τ) is a tangle containing a bridge sphere Σ. Then one of the following holds:*

- $d(\Sigma) \leq 2$.
- $\chi(\Sigma_\tau) \geq -3$.
- (B, τ) *is prime.*

Proof Suppose the second two conclusions do not hold, so that (B, τ) contains an essential meridional annulus A. By Theorem 3.2, $d(\Sigma) \leq 2 - \chi(A) \leq 2$. □

Lemma 3.2 *Suppose K is a prime knot in S^3, and (S^3, K) contains a tangle (B, τ) with bridge sphere Σ. Suppose F is a closed, separating surface in S^3 intersecting K transversely such that F_K is incompressible in $E(K)$. Additionally, suppose that $(\partial B)_K$ is incompressible in the knot complement. Then one of the following holds:*

- $d(\Sigma) \leq 2 - \chi(F_K)$.
- $\chi(\Sigma_K) \geq -3$.
- *After surgery along cut disks and an isotopy of F, we have $F \cap B = \varnothing$; if, in addition, F is isotopic into B, then F_K is parallel to a component of $(\partial B)_K$.*

Proof Since $(\partial B)_K$ and F_K are incompressible meridional surfaces and $E(K)$ is irreducible, we can assume after isotopy that $(\partial B)_K \cap F_K$ is a collection of curves that do not bound disks in either surface. Further, since $E(K)$ is prime, we may remove curves of the intersection of $(\partial B)_K \cap F_K$ which are boundary parallel in $(\partial B)_K$ by either surgery along cut disks or isotopy of F_K, after which we may assume that $(\partial B)_K \cap F_K$ consists of curves essential in $(\partial B)_K$.

By Theorem 3.2, if the first two conclusions do not hold, then each component R_i of $F \cap (B \setminus \eta(\tau))$ is parallel to a subsurface R_i' of $S = \partial(B \setminus \eta(\tau))$. Let $\{R_i'\}$ denote the collection of all such subsurfaces, and isotope F so that the total number of components of intersections of surfaces $R_i' \in \{R_i'\}$ with $\partial\eta(\tau)$ is minimal up to surgery along cut disks and isotopy of F_K. Now, pick an index i so that the corresponding R_i' is innermost in S. If there is an arc $t \in \tau$ such that $\partial\eta(t) \subset R_i'$, then a meridian curve γ' of $\eta(t)$ lifts to a curve $\gamma \subset R_i$. In B, the curve γ bounds a disk C which intersects τ in exactly one point, so that either C is a cut disk, or we could remove the intersection of R_i' and $\partial\eta(t)$ via an isotopy. In either case, we reduce the number of intersections of $\{R_i'\}$ with $\partial\eta(\tau)$.

We conclude that up to surgery along cut disks and isotopy of F_K, $R_i' \cap \eta(\tau) = \varnothing$ for all R_i', so every component R_i of the c–compressed F intersected with $B \setminus \eta(\tau)$ is parallel into ∂B_τ. Thus, we may push all such R_i outside of B. If, in addition, F is isotopic into B, then $\{R_i'\}$ contains exactly one region R_i', and R_i' can be chosen so that $\partial R_i'$ avoids both $\text{int}(\partial(\eta(\tau)))$ and $\text{int}((\partial B)_K)$. Thus, R_i' is a component of $(\partial B)_K$, as desired. $\square$

Lastly, using results of Blair, Tomova and Yoshizawa [4], it is shown in [3] that high distance tangles exist:

Theorem 3.3 *Given positive integers b, D, n with $n < b$ and $b \geq 3$, there exists an n–strand tangle (B, τ) with a bridge sphere Σ such that $|\tau \cap \Sigma| = 2b$ and $d(\Sigma) \geq D$.*

We remark that the existence of high distance tangles also follows from Johnson and Moriah [9], discussed in Section 5.

4 Tangle sum and bridge number

In this section, we show if two n–strand tangles (B, τ) and (B', τ') are glued together to form a knot K in S^3 and both of the tangles have a bridge sphere of sufficiently high distance, then we may determine the bridge number of the resulting knot. In this case, we say K is the *tangle sum* of (B, τ) and (B', τ'). First, we make several definitions about the intersections of various c–disks for a bridge sphere for K.

Suppose Σ is a bridge sphere for a knot K. If there are c–disks D and D' on opposite sides of Σ such that $D \cap D' = \varnothing$, we say that K is c–*weakly reducible*. Otherwise, K is c–*strongly irreducible*.

Lemma 4.1 *Suppose k is an embedding of K such that k has one thick level, a bridge sphere Σ. If Σ is c–weakly reducible, then we may exhibit an embedding k' of K such that $b(k') = b(k)$ and $w(k') < w(k)$.*

Proof Suppose $(S^3, K) = (B_1, \tau_1) \cup_\Sigma (B_2, \tau_2)$, and let D_1 and D_2 be c–disks in B_1 and B_2, respectively, such that $D_1 \cap D_2 = \varnothing$. Since Σ is a 2–sphere, there exist disks C_1 and C_2 in Σ (these disks necessarily intersect K) such that $\partial C_i = \partial D_i$ and $C_1 \cap C_2 = \varnothing$. Since D_i is a c–disk, we may choose an arc $t_i \in \tau_i$ such that $\partial t_i \subset C_i$. Let Δ_i be a bridge disk for t_i that intersects D_i minimally. By a standard cut-and-paste argument, $\Delta_i \cap D_i = \varnothing$; hence $\Delta_i \cap \Sigma \subset C_i$. It follows that $\Delta_1 \cap \Delta_2 = \varnothing$, and there is an isotopy of k supported in $\eta(\Delta_1) \cup \eta(\Delta_2)$ which moves a minimum above a maximum, yielding k' such that $b(k') = b(k)$ and $w(k') = w(k) - 4$. □

Now we connect the distance of tangles to c–weak reducibility. To do this, we need the following lemma due to the second author.

Lemma 4.2 (Zupan [20]) *Suppose Σ is a c–strongly irreducible bridge sphere for a knot K in S^3, where $(S^3, K) = (B_1, \tau_1) \cup_\Sigma (B_2, \tau)$. If c is an essential curve in Σ_K such that c bounds a disk D embedded in $E(K)$, where a collar of c in D is disjoint from Σ_K, then c bounds a compressing disk for Σ_K in $B_i \setminus \eta(\tau_i)$ for $i = 1$ or 2.*

Next, we synthesize Lemma 4.2 and Theorem 3.1 into a single result.

Lemma 4.3 *Suppose K is a prime knot in S^3, (S^3, K) contains a 3–ball M with K transverse to ∂M and $(\partial M)_K$ is incompressible in $E(K)$. Let $T = K \cap M$, Σ be a bridge sphere for (M, T) and Σ' be a bridge sphere for (S^3, K). Then one of the following holds:*

- *After isotopy, $\Sigma' \cap \partial M = \varnothing$.*
- *Σ' is c–weakly reducible.*
- *$d(\Sigma) \leq 2 - \chi(\Sigma'_K)$.*
- *$\chi(\Sigma_K) \geq -3$.*

Proof By Theorem 3.1, if the third and fourth conclusions do not hold, then after isotopy, there exists a collection $\{D_i\}$ of c–compressing disks of $\Sigma'_K \cap M$ in M such that compressing along $\{D_i\}$ yields a compressed surface Σ''_K such that $\Sigma''_K \cap M$ is parallel to Σ_K, and we may choose $\{D_i\}$ so that the collection $\{\partial D_i\}$ is pairwise disjoint. Let $\{\gamma_i\}$ denote the collection of curves in Σ'_K such that $\gamma_i = \partial D_i$. If any γ_i is inessential in Σ'_K, pick an innermost such γ_i. The curve γ_i bounds a disk or cut disk D' contained in Σ'_K and since γ_i is essential in $\Sigma'_K \cap M$, we have $D' \cap \partial M \neq \varnothing$. In this case, since K is prime and $E(K)$ is irreducible, compressing along D_i and discarding a trivial 2–sphere or ∂–parallel annulus yields the same surface as isotoping D' onto D_i, and this isotopy yields a new surface Σ^*_K which intersects ∂M fewer times.

If every curve in $\{\gamma_i\}$ is inessential in Σ'_K, we can repeat the above process to show that after isotopy, $\Sigma'_K = \Sigma''_K$, and as such Σ'_K may be made disjoint from ∂M. Otherwise, some γ_i is essential in Σ'_K. Let D_j be the first c–disk in the sequence of surgeries along c–disks such that γ_j is essential in Σ'_K. Hence γ_j bounds a c–disk for Σ'. As Σ'' is a bridge sphere for (M, T), we can find compressing disks C and C' for Σ''_K which are contained in the boundary of regular neighborhoods of bridge disks Δ and Δ' on opposite sides of Σ'' such that Δ and Δ' meet in a point p contained in K. Let c and c' be the boundary curves of C and C' in Σ''_K. Since γ_j bounds a c–compressing disk taking Σ'_K to Σ''_K and c and c' are contained in Σ''_K, both c and c' are disjoint from γ_j.

To form a contradiction, we assume that Σ' is c–strongly irreducible. We may recover Σ'_K by tubing Σ''_K to a collection of meridional surfaces some number of times. These tubes may be chosen to be disjoint from c and c'; hence, we may view c and c' as essential curves in Σ'_K that bound disks meeting the hypothesis of Lemma 4.2. Since we have assumed that Σ' is strongly irreducible, then, by Lemma 4.2, both c and c' bound compressing disks E and E' for Σ'_K. If E and E' are on opposite sides of Σ'_K, then D_j and one of E and E' are c–disks for Σ'_K embedded on opposite sides of Σ'_K, a contradiction to our assumption. Hence, we can assume that E and E' are embedded on the same side of Σ'.

By construction, c bounds a twice-punctured disk $F \subset \Sigma''_K$ and c' bounds a twice-punctured disk $F' \subset \Sigma''_K$, where $F \cap F'$ is a collection of unpunctured disks in Σ''_K

together with a single disk component F'' containing the single puncture p named above. Let α and α' denote the arcs in c and c' which cobound F''.

After the tubing operations, $\alpha \cup \alpha'$ bounds a punctured disk G'' in Σ'_K which is the intersection of punctured disks G and G' bounded by c and c' in Σ'_K. Moreover, by construction every puncture contained in $G \cap G'$ is contained in G'' since we can assume that all components of $F \cap F'$ other than F'' are disjoint from the tubes in the tubing operations. It is possible that in Σ'_K, G'' contains more than a single puncture; however, $|K \cap G''| \equiv |K \cap F''|$ (mod 2), since the tubing operation preserves this parity. On each side of Σ', there is a natural pairing of $\Sigma' \cap K$ given by the collection of trivial arcs (each puncture is connected to exactly one other puncture). However, since c bounds a compressing disk for Σ'_K, all punctures of G must be paired with punctures in G, and since c' bounds a compressing disk, all punctures of G' must be paired with punctures in G'. It follows that there is a pairing on the punctures in G'', a contradiction to the fact that G'' meets K in an odd number of points. Thus, Σ' must be c–weakly reducible. □

We will need a result regarding c–essential surfaces in trivial tangles.

Lemma 4.4 *Suppose B is $S^2 \times I$ or a 3–ball, (B, τ) is a trivial tangle, and S is a properly embedded essential surface in $B \setminus \eta(\tau)$. Then $S \cap \partial B \neq \varnothing$.*

Proof If B is a 3–ball, then the lemma follows from Tomova [17, Lemma 2.9]. To form a contradiction, suppose that S is a properly embedded essential surface in $B \setminus \eta(\tau)$ such that $S \cap \partial B = \varnothing$ and $B = S^2 \times I$. Let $\{\Delta_i\}$ be a collection of bridge disks for the boundary parallel arcs of τ.

By irreducibility of $B \setminus \eta(\tau)$, we can assume that there are no closed curves of intersection between S and $\bigcup_i \Delta_i$. Since $S \cap \partial B = \varnothing$, any arc of $S \cap \bigcup_i \Delta_i$ has both boundary components in K. Let S' be the component of S that meets some Δ_j in an outermost arc β in Δ_j. The arc β together with a subarc of K cobound a disk in Δ_j that is disjoint from S in its interior. The boundary of a regular neighborhood of this disk contains a compressing disk for S, unless S' is a boundary parallel annulus. In each case, we contradict the fact that S is essential. Hence, we can assume that S is disjoint from $\bigcup_i \Delta_i$.

Since S is disjoint from $\bigcup_i \Delta_i$, S is a properly embedded essential surface in $B \setminus \eta(\tau')$ where τ' is the vertical arc in τ. By [17, Lemma 2.10], S is inessential, which is a contradiction. □

To prove the first main theorem, we need to slightly adapt the notion of width. For an embedding k of a knot K, define the *bridge-width* $bw(k)$ to be $bw(k) = (b(k), w(k))$, and let the *bridge-width* $bw(K)$ of K be defined as

$$bw(K) = \min_{k \sim K}\{bw(k)\},$$

where this minimum is taken using the dictionary ordering. Any k satisfying $bw(k) = bw(K)$ is called a *bridge-thin* position, and it is clear from the definition that in this case $b(k) = b(K)$. We note that the following theorem is implicit in the arguments of Wu in [18]:

Theorem 4.1 [18] *Suppose k is a bridge-thin position of K which has a thin level. Then any thinnest thin level $h^{-1}(r_i)$ is an essential meridional surface in $E(K)$.*

Proof In [18], Wu shows that if a thinnest thin level of an embedding k of K is compressible, then we may exhibit an embedding k' of K such that $b(k') \leq b(k)$ and $w(k') < w(k)$. The desired statement easily follows. □

Finally, we may prove the main theorem from this section.

Proof of Theorem 1.2 First, we claim that K is prime. If not, there is an essential meridional annulus A contained in $E(K)$. Let $S = \partial B_1 = \partial B_2$ and choose A so that $|A \cap S_K|$ is minimal. If S_K is c–compressible, then by Theorem 3.2, with the c–disk playing the role of the essential surface F, $d(\Sigma_1) \leq 2$ or $d(\Sigma_2) \leq 2$, a contradiction (note that when we apply Theorem 3.2 $M = N = B_1$ or $M = N = B_2$). Hence, we can assume that S_K is c–essential.

If $A \cap S_K \neq \varnothing$, then a curve $\gamma \subset A \cap S_K$ which is essential in S_K gives rise to a cut disk for S_K, so γ must be boundary parallel in S_K. Choose such a γ which is outermost, so that γ bounds a once-punctured disk D which avoids A. However, now we may surger A along D to get a new essential annulus A' which intersects S_K fewer times. It follows that $A \cap S_K = \varnothing$, so $A \subset B_1$ or $A \subset B_2$, which contradicts Lemma 3.1; hence K is prime.

Observe that there is an embedding k' of K with two thick levels parallel to Σ_1 and Σ_2 and one thin level parallel to $S = \partial B_1 = \partial B_2$, where k' has bridge number $\beta_1 + \beta_2 - n$. Suppose F is any c–essential surface in $E(K)$. By Lemma 3.2 applied to (B_1, τ_1), either $-\chi(F_K) > 2(\beta_1 + \beta_2 - n) - 2$ or $F \cap B_1 = \varnothing$ after isotopy. In the second case, we may apply Lemma 3.2 to (B_2, τ_2) to conclude that again $-\chi(F_K) > 2(\beta_1 + \beta_2 - n) - 2$, or else F_K is isotopic to S_K.

Let k be a bridge-thin position of K. We note that if k has a thick level of width at least $2(\beta_1 + \beta_2 - n)$, then k has at least $\beta_1 + \beta_2 - n$ critical points and the theorem holds. If k is a bridge position with bridge sphere Σ, then by Lemma 4.3 either $\Sigma \cap S = \varnothing$, Σ is c–weakly reducible, or $-\chi(\Sigma) > 2(\beta + \beta' - n) - 2$. However, the first case contradicts Lemma 4.4, and the second case implies that k is not bridge-thin by Lemma 4.1. In the third case, k has a thick level of width greater than $2(\beta_1 + \beta_2 - n)$.

Suppose now that k has a thin level. By Theorem 4.1, a thinnest thin level R of k is essential. Suppose that R is c–compressible. Let $\{D_i\}_{i=1}^m$ be a maximal sequence of cut disks taking R to a c–incompressible surface R'. Since the properties of being incompressible and nonboundary parallel are preserved under surgery along cut disks, all components of R' are c–essential. By c–compressing R along the $m-1$ cut disks $\{D_i\}_{i=1}^{m-1}$, we get a surface of some number of components, one of which, call it R'', contains ∂D_m. Let R^* be the result of c–compressing R'' along D_m. Since all essential curves in planar surfaces are separating, R^* is a c–essential, planar, meridional surface of two components.

Since the components of R^* are c–essential, either $-\chi(R_K) \geq -\chi(R_K^*) > 2(\beta_1 + \beta_2 - n) - 2$, or each component of R_K^* is isotopic to S_K. In the first case, k has a thick level of width greater than $2(\beta_1 + \beta_2 - n)$. In the second case, since R^* is obtained by c–compressing R'', then R'' is obtained by tubing one component of R^* to the other component of R^* along a subarc of K. However, tubing together parallel surfaces results in a compressible surface. This contradicts the incompressibility of R''. Thus, we conclude that R must be c–essential and as such $-\chi(R_K) > 2(\beta_1 + \beta_2 - n) - 2$ (completing the proof), or R_K is parallel to S_K.

If R_K is parallel to S_K, then S is a level thin sphere with respect to k, and we may construct a new embedding k^* from k by pushing all maxima above all minima in both B_1 and B_2. Note that $b(k^*) = b(k)$, and k^* has exactly two thick spheres $\Sigma_1' \subset B_1$ and $\Sigma_2' \subset B_2$. By Theorem 3.1, either Σ_i' is isotopic to Σ_i or $-\chi((\Sigma_i)'_K) > -\chi((\Sigma_i)_K)$. In the second case, $b(k) > b(k')$, a contradiction. It follows that Σ_i' is isotopic to Σ_i for each $i \in \{1, 2\}$ and $b(k) = b(k')$, as desired. $\square$

5 A high distance tangle

In this section we use a result of Johnson and Moriah to construct a tangle with a high-distance bridge sphere that stays high distance even after many crossing changes. Their construction uses plat presentations of knots.

Let $\mathcal{B}_n$ denote the n–strand braid group. Given $\alpha \in \mathcal{B}_{2k}$ we can visualize α as a regular projection of $2k$ arcs in the plane such that each arc has one endpoint on

the line $y = 0$, has the other endpoint on the line $y = 1$, is contained in the region between $y = 0$ and $y = 1$ and increases monotonically in y. By connecting each pair of consecutive endpoints of α on the line $y = 1$ by an arc and similarly connecting pairs of consecutive endpoints on the line $y = 0$, we form the projection of a knot $\hat{\alpha}$. Such a presentation of a knot is known as a $2k$–*plat*.

There is a natural bridge sphere associated to a $2k$–plat. We can view the knot as being embedded in a neighborhood of its projection in the xy–plane in $\mathbb{R}^3$. Let $h: \mathbb{R}^3 \to \mathbb{R}$ be projection onto the y–axis. After taking the one point compactification of each xz–plane intersecting the knot and gluing in 3–balls above and below, h extends to a Morse function on S^3, and the 2–sphere $h^{-1}(\frac{1}{2})$ is a bridge sphere for the $2k$–plat. We call this the *induced bridge sphere*.
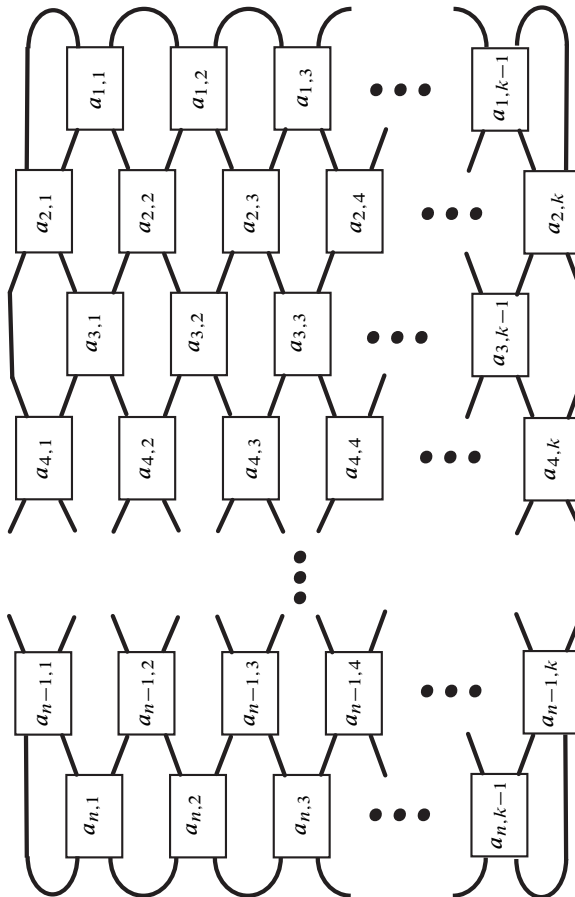

Figure 1: The $2k$–plats studied by Johnson and Moriah

In [9], Johnson and Moriah study $2k$–plats of the form depicted in Figure 1. Recall that $\mathcal{B}_2$ is isomorphic to the integers, hence, the labels of $a_{i,j}$ in Figure 1 dictate which element of $\mathcal{B}_2$ lies inside the corresponding braid box. A knot with a $2k$–plat presentation of the form depicted in Figure 1 will be called a *highly twisted plat*. Let n be the number of rows associated to a highly twisted plat. Let $\lceil x \rceil$ be the ceiling function, which is equal to the smallest integer greater than or equal to x. Johnson and Moriah showed the following:

Theorem 5.1 [9] *If $K \subset S^3$ is a highly twisted $2k$–plat with $k \geq 3$ and $|a_{i,j}| \geq 3$ for all i, j, then $d(\Sigma) = \lceil n/(2(k-2)) \rceil$, where Σ is the induced bridge sphere.*
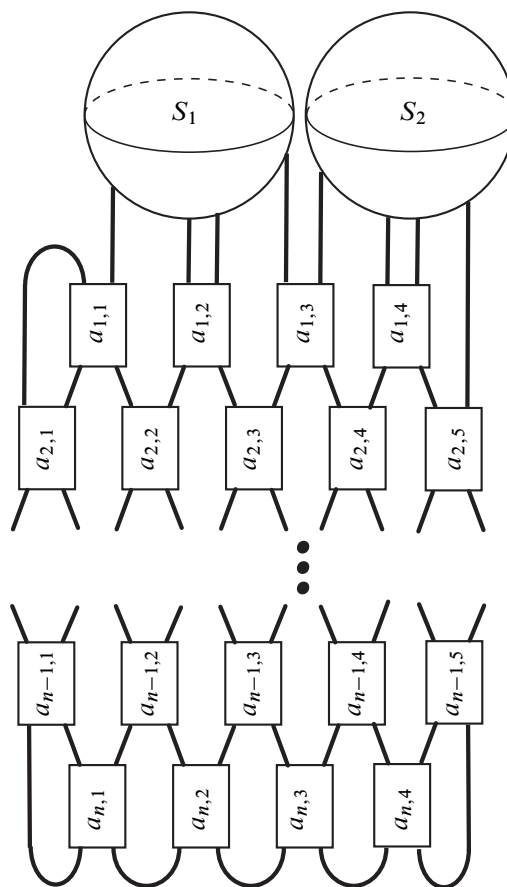

Figure 2: The tangle (B_3, τ_3)

We form the tangle (B_3, τ_3) depicted in Figure 2 in the following way. Let L be a highly twisted 10–plat. Considering the plat in $\mathbb{R}^3$ for the moment, let γ_1 and γ_2

be two arcs which are contained in the xy–plane at height $y = 1$, such that γ_1 has endpoints in the second and third maxima of L and γ_2 has endpoints in the fourth and fifth maxima of L. Then $B_3 = S^3 \setminus (\eta(\gamma_1) \cup \eta(\gamma_2))$ and $\tau_3 = L \cap B_3$. The induced bridge sphere Σ for L persists as a bridge sphere for (B_3, τ_3). Recall that the distance of Σ as a bridge sphere for L is the distance in $\mathcal{C}(\Sigma_L)$ between two disk sets D_1 and D_2. After drilling out γ_1 and γ_2 the distance of Σ as a bridge sphere for (B_3, τ_3) is the distance between disk sets D_1' and D_2' where $D_1' \subset D_1$ and $D_2' = D_2$. Hence, the distance of Σ as a bridge surface of (B_3, τ_3) is greater than or equal to the distance of Σ as a bridge surface for L. The following result immediately follows from Theorem 5.1.

Corollary 5.1 *Let Σ be the induced bridge sphere for (B_3, τ_3) and suppose $|a_{i,j}| \geq 3$ for all i, j. Then $d(\Sigma) \geq n/6$.*

6 A knot with a compressible thin level

We construct a knot K in S^3 with a compressible thin level from (B_3, τ_3) in the following way. Let S_1 and S_2 be the two 2–spheres that make up ∂B_3. Let (B_1, τ_1) and (B_2, τ_2) be two 2–strand tangles with 3–bridge spheres Σ_1 and Σ_2. Glue B_1 to B_3 by identifying ∂B_1 with S_1 and glue B_2 to the result by identifying ∂B_2 with S_2, so that the result is a knot K in S^3. Additionally, choose τ_1 and τ_2 such that $d(\Sigma_1) > 12$ and $d(\Sigma_2) > 12$. Similarly, choose τ_3 such that $|a_{i,j}| \geq 3$ for all i, j and $n > 72$. By Corollary 5.1, these restrictions imply that $d(\Sigma_3) > 12$ where Σ_3 is the induced bridge sphere for (B_3, τ_3). Later in the paper we will need to put additional restrictions on the values of the $a_{i,j}$'s.

In the remainder of this section, we will establish several topological properties of $E(K)$ which will allow us to find thin position of K in Sections 7 and 8. From this point forward, we will suppress the notation $(S_1)_K$ and $(S_2)_K$ and consider S_1 and S_2 to be 4–punctured spheres.

Lemma 6.1 *The 4–punctured spheres S_1 and S_2 are c–essential in $E(K)$.*

Proof Suppose by way of contradiction that S_i is c–compressible for $i = 1$ or 2, and choose a c–disk D such that $|D \cap (S_1 \cup S_2)|$ is minimal. If $\text{int}(D) \cap (S_1 \cup S_2) \neq \varnothing$, let γ be a curve of intersection which is innermost in D, so that γ bounds a c–disk $D' \subset D$ which avoids $S_1 \cup S_2$. If γ is essential in S_i, then $|D' \cap (S_1 \cup S_2)| < |D \cap (S_1 \cup S_2)|$, a contradiction. If γ is inessential in S_i, then either we can remove γ with an isotopy of D, which contradicts the minimality of $|D \cap (S_1 \cup S_2)|$, or γ is boundary parallel in S_i and surgery of D along γ yields an essential annulus $A \subset B_i$, which contradicts Lemma 3.1.

Thus, suppose that $\text{int}(D) \cap (S_1 \cup S_2) \neq \varnothing$. Then D is an essential surface in (B_i, τ_i) and ∂D is essential in S_i, so by Theorem 3.2, we have $d(\Sigma_i) \leq 2$, a contradiction. $\square$

Lemma 6.2 *The knot K is prime.*

Proof As in the proof of Theorem 1.2, we suppose by way of contradiction that $E(K)$ contains an essential meridional annulus A, chosen to intersect S_1 and S_2 minimally. First, suppose $A \cap (S_1 \cup S_2) \neq \varnothing$. Any curve of intersection γ which is essential in S_i gives rise to a cut disk, so we may assume that such curves are boundary parallel in S_i. Choosing such a γ which is outermost in S_i yields a once-punctured disk D which avoids A, and performing surgery on A along D yields a new essential annulus A' such that $|A' \cap (S_1 \cup S_2)| < |A \cap (S_1 \cup S_2)|$, a contradiction.

On the other hand, if $A \cap (S_1 \cup S_2) = \varnothing$, then $A \subset B_i$ for $i = 1, 2$, or 3, which contradicts Lemma 3.1. $\square$

Lemma 6.3 *Suppose that F is a c–essential planar meridional surface in $E(K)$. Then either F_K is isotopic to S_1 or S_2, or $-\chi(F_K) > 10$.*

Proof By Lemma 6.2, K is prime, so by Lemma 3.2, we have that either $-\chi(F_K) > 10$ or after isotopy $F \cap B_i = \varnothing$ for $i = 1, 2$. Hence, F is a c–essential planar meridional surface embedded in the interior of B_3. By Lemma 3.2 applied to B_3, $-\chi(F_K) > 10$ or F_K is isotopic to S_1 or S_2. $\square$

Lemma 6.4 *Suppose that F is an essential meridional 6–punctured sphere in $E(K)$. Then F is isotopic to S_1 tubed to S_2 along an arc of τ_3.*

Proof By Lemma 6.3, such an F must be c–compressible. Since K is prime, c–compressing F yields two 4–punctured spheres F_1 and F_2. As incompressibility is preserved surgery along cut disks and K is prime, both F_1 and F_2 are c–essential. Hence, F may be constructed by tubing F_1 to F_2 along an arc in K. If F_1 is isotopic to F_2, then F is compressible, and so F_1 and F_2 are nonisotopic. By Lemma 6.3 (possibly after relabeling), F_1 is isotopic to S_1 and F_2 is isotopic to S_2, as desired. $\square$

7 Thin position of K

First, we will describe our candidate k' for thin position of K, and then we will eliminate all other possibilities via an exhaustive argument, proving that k' is width minimizing. The position k' is depicted in Figure 3. By construction k' has exactly

three maxima and exactly one minimum in each of B_1 and B_2, with all of the maxima above the minimum in each of these 3–balls. Similarly, k' has exactly one maximum and exactly five minima in B_3, with the maximum above all of the minima. Thus, k' has three thick spheres, Σ_1, Σ_3 and a third which we will call Σ_2'. Note that Σ_2' intersects B_2 in a surface which is isotopic to Σ_2 after a single surgery along a compressing disk. The two thin spheres of k' are S_1 and an 8–punctured sphere we will denote by S_3. Note that after a single surgery along a compressing disk, S_3 is isotopic to $S_1 \cup S_2$. The thin-thick tuple for k' is $(10, 8, 10, 4, 6)$, which gives $w(k') = 78$ by (1).
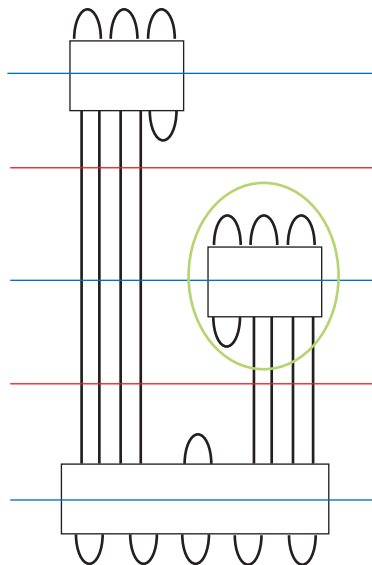

Figure 3: This embedding is k', the candidate thin position of K.

Let k denote any other embedding of K. We will use a case-by-case analysis to show $w(k) \geq 78$.

Case 1 *The embedding k is a bridge position.*

Proof Suppose that k is a bridge position with bridge sphere Σ. By Lemma 4.3, either Σ can be made disjoint from S_1 via an isotopy transverse to K, Σ is c–weakly reducible, or $-\chi(\Sigma_K) > 10$. In the first case, S_1 is an essential surface for one of the trivial tangles bounded by Σ, a contradiction to Lemma 4.4. In the second case, k is not thin by Lemma 4.1. Finally, if $-\chi(\Sigma_K) > 10$, then $w(\Sigma) \geq 14$ and thus $w(k) \geq 98$ by inequality (2).   □

It follows that k must have a thin level.

Case 2 *The embedding k has exactly one thin level.*

Proof Let S be the unique thin level for k. By Theorem 2.3, we can assume that S is c–essential. By Lemma 6.3, either S_K is isotopic to S_1 or S_2, or $-\chi(S_K) > 10$. In the latter case, k has a thick level with at least 16 punctures, and using inequality (2) we get $w(k) \geq 128$. Hence, we can assume that S_K is isotopic to S_1 or S_2.

Without loss of generality, suppose that S_K is isotopic to S_1 and that B_1 is above S_1. Let Σ'_1 denote the thick level above S_1. We could easily show that $\Sigma'_1 = \Sigma_1$, but for this case we need only that $w(\Sigma'_1) \geq 6$, which follows immediately from the fact that K meets B_1 in a nontrivial tangle. Let Σ denote the thick level below S_1. It follows that Σ is a bridge sphere for $(S^3 \setminus \eta(B_1), K \setminus \eta(\tau_1))$, and so by Theorem 3.1, we have that $-\chi(\Sigma_K) \geq d(\Sigma_3) - 2 > 10$ or else Σ is isotopic to Σ_3 after surgery along some number of c–disks of $\Sigma \cap B_3$. In the first case, $w(\Sigma) \geq 14$ and $w(k) \geq 98$. In the second case, if no surgery is necessary, then Σ is isotopic to Σ_3. In this case, S_2 is an essential surface embedded in the trivial tangle cobounded by Σ and S_1; this contradicts Lemma 4.4. If it is necessary to c–compress Σ to produce a surface isotopic to Σ_3, then $w(\Sigma) > w(\Sigma_3) = 10$, so $w(\Sigma) \geq 12$, and by (1),

$$w(k) \geq \tfrac{1}{2}(6^2 + 12^2 - 4^2) = 82. \qquad \square$$

Case 3 *The embedding k has two or more thin levels, and one has width at least 8.*

Proof If S is a thin level of k and $w(S) = 8$, then k has two thick levels adjacent to S with width at least 10. Since K is prime, any other thin level must have width at least 4 and any thick level not adjacent to S must have width at least 6. Hence,

$$w(k) \geq \tfrac{1}{2}(6^2 + 10^2 + 10^2 - 4^2 - 8^2) = 78. \qquad \square$$

Case 4 *The embedding k has exactly two thin levels, both of width 4.*

Proof As in Case 2, we may assume one of the thin levels is isotopic to S_1. If S is the other thin level with width 4, then S is a thinnest thin level since K is prime. By Theorem 2.1, S is essential and, since K is prime, must also be c–essential. By Lemma 6.3, S must be isotopic to S_1 or S_2, or k is not thin. But if S is isotopic to S_1, then by the isotopy extension theorem we may replace the embedded arcs of k between S_1 and S with vertical arcs, reducing $w(k)$. Thus, S is isotopic to S_2. Let Σ denote the thick level between S_1 and S_2, and note that the region between S_1 and S_2 is B_3.

By Theorem 3.1, either $-\chi(\Sigma_K) \geq d(\Sigma_3) - 2 > 10$, or after isotopy, Σ c–compresses to a surface isotopic to Σ_3. In the first case $w(\Sigma) \geq 14$, and in the second case, we note that Σ separates S_1 and S_2, while Σ_3 does not, so we must c–compress Σ at least once before it is isotopic to Σ_3, implying that $w(\Sigma) > w(\Sigma_3) = 10$. In either case, $w(\Sigma) \geq 12$, and by (1),

$$w(k) \geq \tfrac{1}{2}(6^2 + 12^2 + 6^2 - 4^2 - 4^2) = 92. \qquad \square$$

Case 5 *The embedding k has three or more thin levels.*

Proof Note that since K is prime, all thin levels of k have width at least 4 or k is not thin. If at least one thin level of k has width at least 8, then

$$w(k) \geq \tfrac{1}{2}(6^2 + 6^2 + 10^2 + 10^2 - 4^2 - 4^2 - 8^2) = 88.$$

Hence, we can assume that all thin levels for k have width 4 or 6.

First, consider the case when k has at least two thin levels of width 6. Call these thin levels S and S'. By Theorems 2.1 and 2.2, S and S' are incompressible, which implies that both S and S' are constructed by tubing S_1 to S_2 along some arc of τ_3, by Lemma 6.4. Since S and S' are disjoint, they are constructed by tubing S_1 to S_2 along a common arc of τ_3. Hence, S is isotopic to S'. By the isotopy extension theorem we may replace the embedded arcs of k between S and S' with vertical arcs, reducing $w(k)$.

Next, consider the remaining cases when k has two or more thin levels of width 4. Call two such thin levels S and S'. As argued in Case 4, each of S and S' are isotopic to S_1 or S_2. Without loss of generality, suppose S is isotopic to S_1. If S' is isotopic to S_1, then by the isotopy extension theorem we may replace the embedded arcs of k between S and S' with vertical arcs, reducing $w(k)$. Thus, S' is isotopic to S_2. Let S'' be a third thin level of k. Since $4 \leq w(S'') \leq 6$, and S'' cannot be isotopic to S_1 or S_2, then $w(S'') = 6$. By Theorem 2.2, S' is incompressible, which implies that S'' may be constructed by tubing S_1 to S_2 by Lemma 6.4.

However, any three level spheres cut S^3 into four components (two components that are homeomorphic to B^3 and two components that are homeomorphic to $S^2 \times I$), where any component is adjacent to one or two other components. On the other hand, $S^3 \setminus \eta(S_1 \cup S_2 \cup S'')$ has four components, one of which (the component containing the tube) is adjacent to all three other components, a contradiction. $\qquad \square$

In summary, assuming k is thin, we have ruled out Cases 1, 2 and 5; hence we can assume that k has exactly two thin levels. We have also ruled out Cases 3 and 4, so

we can assume that one thin level has width 4 and the other has width 6. As previously argued, any thin level of width 4 must be isotopic to S_1 or S_2. Without loss of generality, we will assume that S_1 is a thin level. This one remaining case (the case in which k has two thin levels, one isotopic to S_1 and the other having width 6) is more complicated than the others, and we examine it in the next section.

8 The final remaining case

To complete the proof of the main theorem, we suppose that k has two thin levels, one of which is S_1, and the other of which we denote by S^*. By Theorem 2.2, S^* is incompressible, which implies that S^* may be constructed by tubing S_1 to S_2 along an arc of τ_3, by Lemma 6.4.

By Corollary 5.1, if (B_3, τ_3) is altered in a way that increases $|a_{ij}|$ for any i and j, then the distance bound on Σ_3 is preserved and all of the properties of K and k that were established in Sections 6 and 7 are preserved. Denote the strands of τ_3 by t_1, t_2, t_3 and t_4. To motivate our choices of a_{ij}, we need a clear picture of the surface S^*.

Suppose for the moment that S^* is S_1 tubed to S_2 along t_1, the leftmost arc in Figure 4. Let B^* denote that ball bounded by S^* which is disjoint from $S_1 \cup S_2$, and let $\tau^* = k \cap B^*$. Then (B^*, τ^*) is a 3–strand tangle. Now, we may construct an isotopy of S^* which drags S_1 from one endpoint of t_1 to the other. An appropriate analogy is to picture a cord being pulled back into a vacuum cleaner, where in this setting S_2 takes the place of the vacuum cleaner. As such, each of the three arcs of (B^*, τ^*) could be constructed by taking one of t_2, t_3 or t_4 and attaching its endpoint to t_1 in S_1; see Figure 4. We wish to show that, under certain assumptions on the parameters a_{ij}, each of these arcs is knotted in B^*.
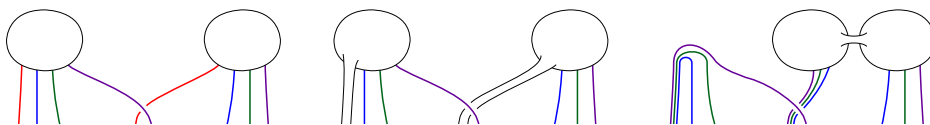

Figure 4: On the left, we have a portion of the knot K, shown with S_1 and S_2. In the middle, we have a tube connecting S_1 to S_2 along a, which yields S^*. On the right, we have an alternate picture of S^* created by dragging S_1 along a until it meets S_2.

We determine the parameters a_{ij} in the following way: $a_{1,1}$, $a_{1,3}$, $a_{2,3}$, $a_{2,4}$, $a_{3,2}$, $a_{3,3}$ and $a_{3,4}$ are odd positive integers greater than or equal to 3, $a_{1,2}$, $a_{1,4}$, $a_{2,1}$, $a_{2,2}$, $a_{2,5}$

and $a_{3,1}$ are even integers with absolute value greater than 3. Moreover, we require $a_{2,1}$ and $a_{2,2}$ to be negative and $a_{3,1}$ to be positive. If i is even and greater than 3, then $a_{i,j}$ is a negative even integer less than -3 for all j. If i is odd and greater than 4, then $a_{i,j}$ is a positive even integer greater than or equal to 3 for all j. These restrictions are depicted in Figure 5. In that figure, a label of E indicates that $a_{i,j}$ is even and a label of O indicates $a_{i,j}$ is odd. Similarly, the superscripts in the figure indicate the sign of $a_{i,j}$. The absence of a superscript indicates that choosing a sign is unnecessary for our construction.
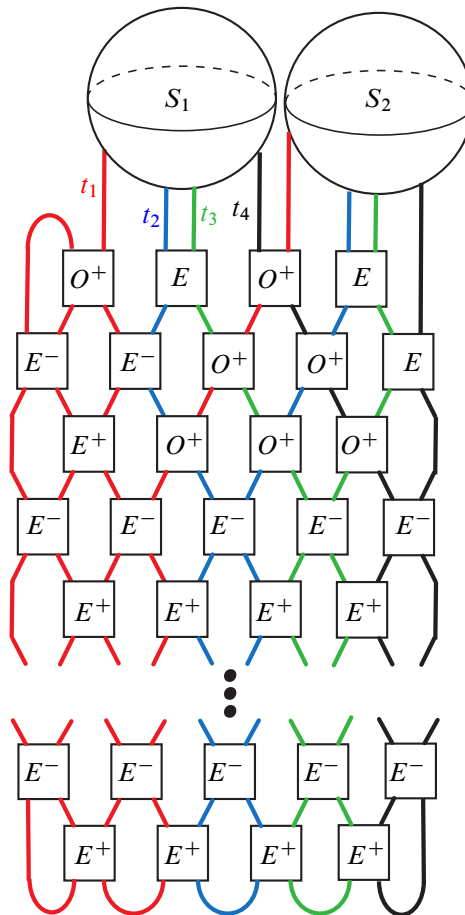

Figure 5: Restrictions on the $a_{i,j}$ for the tangle in B_3

Note that there are unique arcs s^1_{12} and s^2_{12} connecting the endpoints of t_1 and t_2 in S_1 and S_2 when we ignore the endpoints of the strands of τ_3 that are not t_1 or t_2. Let K_{12} denote the knot $t_1 \cup s^1_{12} \cup t_2 \cup s^2_{12}$. We can similarly define the knots K_{ij}

where $i < j$. Using Figure 5, we can easily obtain diagrams for each of these knots. These diagrams are depicted in Figure 6. Note that each of these diagrams is reduced and alternating. Hence, each of the knots K_{12}, K_{13}, K_{14}, K_{23}, K_{24} and K_{34} is nontrivial.
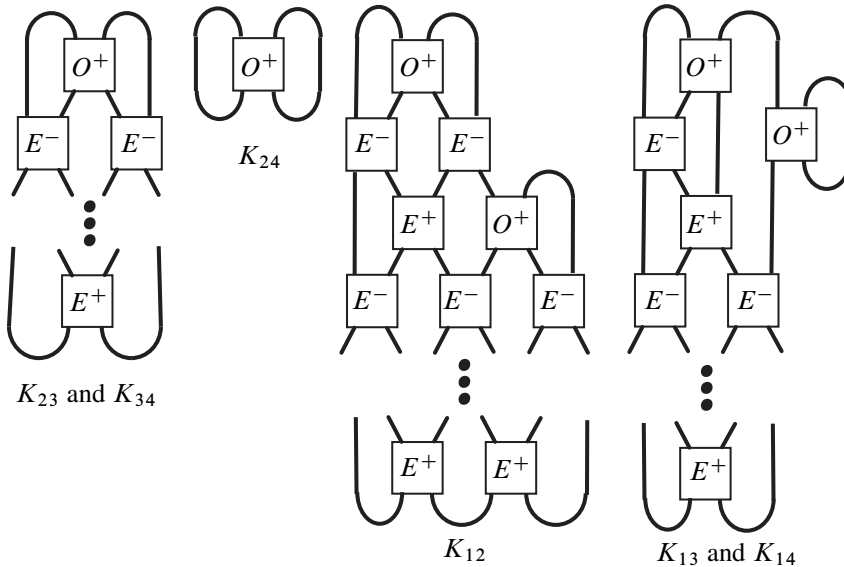

Figure 6: Diagrams for each of the knots K_{12}, K_{13}, K_{14}, K_{23}, K_{24} and K_{34}

Case 6 *The embedding k has two thin levels, one of width 4 and the other of width 6.*

Proof As above, we may assume without loss of generality that one of the thin levels is S_1 and the other, S^*, is the result of tubing S_1 to S_2 along some strand t_i of τ_3. Note that S^* bounds two 3–balls in S^3; let B^* denote the one which is disjoint from $B_1 \cup B_2$. Suppose without loss of generality that S_1 is above S^* with respect to the height function h. Hence, k has a thick level Σ^* below S^*, which implies that $\Sigma^* \subset B^*$.

Let $\tau^* = k \cap B^*$, and consider the 3–strand tangle (B^*, τ^*), noting that Σ^* is a bridge surface for (B^*, τ^*). We will show that each arc t of τ^* intersects Σ^* at least four times. If not, then (B^*, t) is a trivial 1–strand tangle. However, we may view t as the union of t_i and t_j with the unique arc s_{ij}^2 connecting ∂t_i to ∂t_j in S_2. If $|t \cap \Sigma^*| = 2$, then (ignoring the other strands of τ^*) t is isotopic into S^*, and it follows that one of the knots K_{12}, K_{13}, K_{14}, K_{23}, K_{24} or K_{34} is the unknot, a contradiction.

We conclude that $w(\Sigma^*) \geq 12$. Since k has two other thick levels of widths at least 6 and 8, we have

$$w(k) \geq \tfrac{1}{2}(6^2 + 8^2 + 12^2 - 4^2 - 6^2) = 96. \qquad \square$$

In summary, we have the following:

Proof of Theorem 1.1 Let k' be the embedding of K depicted in Figure 3. Noting that S_3 is a compressible thin level of k', we need only show that $w(K) = 78$, so that k' is thin. Let k be a thin position of K. By Cases 1, 2 and 5, we can assume that k has exactly two thin levels. By Cases 3 and 4, we can assume that one thin level has width 4 and the other has width at least 6. If we assume that the thin levels have width exactly 4 and 6, then $w(k) \geq 96$ by Case 6, so k is not thin. It follows that the three thick levels of k have widths at least 6, 10, and 10; hence

$$w(k) \geq \tfrac{1}{2}(6^2 + 10^2 + 10^2 - 4^2 - 8^2) = 78. \qquad \square$$

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