Whitney towers, gropes and
Casson–Gordon style invariants of links

MIN HOON KIM

In this paper, we prove a conjecture of Friedl and Powell that their Casson–Gordon type invariant of 2–component links with linking number one is actually an obstruction to being height-3.5 Whitney tower/grope concordant to the Hopf link. The proof employs the notion of solvable cobordism of 3–manifolds with boundary, which was introduced by Cha. We also prove that the Blanchfield form and the Alexander polynomial of links in S^3 give obstructions to height-3 Whitney tower/grope concordance. This generalizes the results of Hillman and Kawauchi.

1 Introduction

In the study of topological knot concordance, various invariants were introduced in seminal papers including Levine [32], Casson and Gordon [3; 4], and Cochran, Orr and Teichner [15]. All of these invariants can be extracted from the 0–surgery manifolds of knots. Influenced by these works, the link slicing problem has been studied extensively using various covers of the 0–surgery manifolds of links. For example, Harvey [26], Cochran, Harvey and Leidy [11] and Horn [29] used Cheeger–Gromov ρ–invariants from PTFA (poly-torsion-free-abelian) covers. In Cha [5; 6], Hirzebruch type invariants from iterated prime power fold covers were defined and used.

In general link concordance problems, it is known that 0–surgery manifolds do not reveal full information. For example, for 2–component links with linking number one, the aforementioned invariants automatically vanish. In fact, those invariants are obtained from solvable covers of 0–surgery manifolds. For 2–component links with linking number one, there are no nontrivial solvable covers of the 0–surgery manifolds (and consequently the aforementioned invariants vanish) because they have perfect fundamental groups. Also, there is an in-depth study which presents related results about link concordance versus 0–surgery homology cobordism; see Cha and Powell [9].

Recently, for 2–component links with linking number one, S Friedl and M Powell [25] introduced a Casson–Gordon style metabelian invariant $\tau(L, \chi)$ by considering another

closed 3–manifold obtained from the link exterior. Also, they found new 2–component links with linking number one which are not concordant to the Hopf link. The aim of this paper is to give a better understanding of $\tau(L, \chi)$ in the context of symmetric Whitney towers and gropes in dimension 4.

Friedl–Powell invariant $\tau(L, \chi)$

To describe our main result, we briefly summarize the construction and main result in [25]. (For more details, see Section 4.)

Let L be an ordered, oriented 2–component link with linking number 1 in S^3 and H be the Hopf link. Define M_L to be the closed 3–manifold obtained by gluing the exteriors of L and H along their boundary, identifying the meridians of corresponding components. For a prime p, choose a homomorphism $\varphi \colon H_1(M_L; \mathbb{Z}) \to \mathbb{Z}/p^i \times \mathbb{Z}/p^j$ which sends two meridians of L to the standard basis $(1, 0)$ and $(0, 1)$, respectively. Let $M_L^\varphi \to M_L$ be the p^{i+j}–fold covering induced by φ. For a prime q and a character $\chi \colon H_1(M_L^\varphi; \mathbb{Z}) \to \mathbb{Z}/q^k$, Friedl and Powell define an invariant

$$\tau(L, \chi) \in L^0(\mathbb{C}(\mathcal{H})) \otimes_{\mathbb{Z}} \mathbb{Z}[1/q]$$

in [25, Section 3.2] (see also our Definition 4.1). Here, $\mathcal{H} = \mathbb{Z}^3$, $\mathbb{C}(\mathcal{H})$ is the quotient field of the group ring $\mathbb{C}[\mathcal{H}]$, and $L^0(\mathbb{C}(\mathcal{H}))$ is the Witt group of finite-dimensional nonsingular sesquilinear forms over $\mathbb{C}(\mathcal{H})$. The main result of [25] essentially says that if L is concordant to H, then $\tau(L, \chi)$ vanishes [25, Theorem 3.5]. For a precise definition of the vanishing of $\tau(L, \chi)$, see Definition 4.2. We omit the precise statement here because we would need to discuss some technicalities including the choice of a metabolizer of the linking form.

Symmetric Whitney tower/grope concordance and $\tau(L, \chi)$

The symmetric Whitney towers and gropes are approximations of embedded surfaces which play a central role in the study of topological 4–manifolds. For example, a special kind of grope with caps gives a topologically embedded disk in the disk embedding theorem of Freedman and Quinn [23]. Also, using symmetric Whitney towers and gropes, T Cochran, K Orr and P Teichner developed a filtration theory of the knot concordance group [15]. It turns out that the structure of this filtration theory is extremely rich (for example, see Cochran, Orr and Teichner [16], Cochran and Teichner [17], Cochran, Harvey and Leidy [12; 13], Cha [7], Davis [21], Franklin [22] and Burke [2]). For links, we are mainly interested in two equivalence relations, *height h Whitney tower concordance* and *height h grope concordance*. (For precise definitions, see Cha [8, Definitions 2.12, 2.15].)

We remark that J Conant, R Schneiderman, and Teichner developed another interesting filtration theory using a coarser notion called order-n Whitney tower concordance (for a survey and references, we refer to [18]). It is not our purpose to study this asymmetric filtration theory of Conant, Schneiderman and Teichner. We focus on the finer equivalence relations, symmetric Whitney tower/grope concordance.

Our main result, Theorem A, says that the Friedl–Powell invariant $\tau(L, \chi)$ can be understood in terms of symmetric Whitney tower/grope concordance as conjectured in [25, Remark 1.3(5)]:

Theorem A *Suppose that L is a 2–component link with linking number 1 and H is the Hopf link. If L and H are height-3.5 Whitney tower (or grope) concordant, then the Friedl–Powell invariant $\tau(L, \chi)$ vanishes for L in the sense of Definition 4.2.*

In the proof, we use the notion of *h–solvable cobordism*, introduced by J C Cha in [8] (for the definition, see Section 3.1). By [8, Theorem 2.13], if two links L and L' are height-$(h + 2)$ Whitney tower/grope concordant, then their exteriors X_L and $X_{L'}$ are h–solvable cobordant for all $h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Actually, we prove Theorem A in Section 4.3 under the (potentially) weaker assumption that there exists a 1.5–solvable cobordism between the exteriors X_L and X_H.

Remark (1) In [15, Theorem 9.11], Cochran, Orr and Teichner proved that if a knot K bounds a Whitney tower/grope of height 3.5 in D^4, or more generally if K is 1.5–solvable, then the Casson–Gordon invariant $\tau(K, \chi)$ vanishes. Our result can be viewed as an analogue for 2–component links with linking number 1.

(2) Theorem A is strictly stronger than [25, Theorem 3.5] by the following fact from [8, Theorem 4.1]: for any integer $n > 2$, there are links which are height-n grope concordant to H but not height-n.5 Whitney tower concordant to H (in particular, not concordant to H).

Symmetric Whitney tower/grope concordance and abelian invariants

In [15, Theorem 1.1], Cochran, Orr and Teichner proved that a Seifert form of a knot K is metabolic if and only if K bounds a height-2.5 grope in D^4. By Schneiderman [35, Corollary 2] and [15, Theorem 8.12], this condition is equivalent to that K bounds a height-2.5 Whitney tower in D^4. Motivated from this result, in Section 5 we prove that Blanchfield form and the multivariable Alexander polynomial are actually obstructions to height-3 Whitney tower/grope concordance.

Abelian link concordance invariants are studied by A Kawauchi [30] and J Hillman [28]. To state our main result, we recall their notation (for details, see Section 5) and main

results. Let L be a μ–component link and let X_L be the exterior of L. Denote $\mathbb{Z}[t_1^{\pm}, \dots, t_{\mu}^{\pm}]$ by Λ_{μ}. The ring Λ_{μ} is endowed with the involution $-: t_i \mapsto t_i^{-1}$. Let S be the multiplicative set generated by $\{t_1 - 1, \dots, t_{\mu} - 1\}$. Denote by $\Lambda_{\mu S}$ the localization of Λ_{μ} with respect to S. Let $\mathcal{K}$ be the quotient field of Λ_{μ}. Using the Hurewicz map $\pi_1(X_L) \to \mathbb{Z}^{\mu}$, we define $H_*(X_L; \Lambda_{\mu})$ and $H_*(X_L; \Lambda_{\mu S})$.

In [28, Chapter 2], Hillman defined the $\mathcal{K}/\Lambda_{\mu S}$–valued localized Blanchfield form b_L on the quotient of the torsion submodule of $H_1(X_L; \Lambda_{\mu S})$ by its maximal pseudonull submodule. Also, he proved that the Witt-class of b_L, denoted by $[b_L]$, in the Witt group $W(\mathcal{K}, \Lambda_{\mu S}, -)$ is a link concordance invariant.

In [30], Kawauchi defined the torsion Alexander polynomial of L which we denote by Δ_L^T. In [30, Theorems A, B], he proved that if two links L_0 and L_1 are concordant, then $\operatorname{rank}_{\Lambda_{\mu}} H_1(X_{L_0}; \Lambda_{\mu}) = \operatorname{rank}_{\Lambda_{\mu}} H_1(X_{L_1}; \Lambda_{\mu})$ and $\Delta_{L_0}^T f_0 \overline{f_0} \doteq \Delta_{L_1}^T f_1 \overline{f_1}$ for some $f_i(t_1, \dots, t_{\mu}) \in \Lambda_{\mu}$, $i = 0, 1$ with $|f_i(1, \dots, 1)| = 1$.

We extend these theorems of Hillman and Kawauchi in terms of symmetric Whitney tower/grope concordance as follows:

Theorem B *Suppose that two links L_0 and L_1 are height-3 Whitney tower/grope concordant. Then $[b_{L_0}] = [b_{L_1}] \in W(\mathcal{K}, \Lambda_{\mu S}, -)$.*

Theorem C *Suppose that two links L_0 and L_1 are height-3 Whitney tower/grope concordant. Then:*

(1) $\operatorname{rank}_{\Lambda_{\mu}} H_1(X_{L_0}; \Lambda_{\mu}) = \operatorname{rank}_{\Lambda_{\mu}} H_1(X_{L_1}; \Lambda_{\mu})$.

(2) $\Delta_{L_0}^T f_0 \overline{f_0} \doteq \Delta_{L_1}^T f_1 \overline{f_1}$ *for some* $f_i(t_1, \dots, t_{\mu}) \in \Lambda_{\mu}$, $i = 0, 1$ *with*

$$|f_i(1, \dots, 1)| = 1.$$

As a special case of Theorems B and C for 2–component links with linking number 1, we have the following special case. This illustrates that the concordance problem between a 2–component link with linking number 1 and the Hopf link is similar to the concordance problem between a knot and the unknot.

Corollary D *Suppose that L is a 2–component link with linking number 1 and H is the Hopf link. If L and H are height-3 Whitney tower/grope concordant, then:*

(1) $[b_L] = 0 \in W(\mathcal{K}, \Lambda_2, -)$.

(2) $\operatorname{rank}_{\Lambda_2} H_1(X_L; \Lambda_2) = 0$.

(3) $\Delta_L^T \doteq f \overline{f}$ *for some* $f(t_1, t_2) \in \Lambda_2$ *such that* $|f(1, 1)| = 1$.

Remark Theorems B and C should be compared to the following equivalent statements for knots about abelian invariants (eg [15, Theorem 1.1] and Kearton [31]).

(1) The knot K bounds a grope of height 2.5 in D^4.

(2) The 0–surgery manifold of K, M_K is 0.5–solvable.

(3) The Seifert form of K is metabolic (or K is algebraically slice).
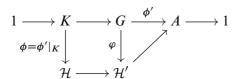
(4) The Blanchfield form of K is Witt-trivial.

Therefore, the most natural assumption for Theorems B and C might be the existence of 0.5–solvable cobordism between link exteriors. The proof for the knot case heavily relies on the existence of Seifert surfaces for K. For general links, as substitutes for Seifert surfaces, there are *immersed* Cooper surfaces studied in Cooper [19] (or its generalization in Cimasoni [10]). However, because of their singularities, a similar approach using Cooper surface seems somewhat difficult.

2 Casson–Gordon type representations

The goal of this section is to prove Theorem 2.3 which will give the key dimension estimate in the proof of Theorem A. Lemma 2.2 and Theorem 2.3 are inspired by [7, Lemma 3.10 and Theorem 3.11]. In the proof of Lemma 2.2, we use the injectivity theorem of Friedl and Powell [24, Theorem 3.1] stated in Lemma 2.1.

We recall the notation used in [24] for the convenience of the reader. Let $\varphi\colon G \to A$ be a surjective group homomorphism, where A is a finite abelian p–group. Assume that $\varphi\colon G \to A$ factors through a surjective homomorphism $\phi'\colon G \to \mathcal{H}'$ to a torsion free abelian group $\mathcal{H}'$. Let $K = \operatorname{Ker}\varphi$, $\mathcal{H} = \operatorname{Im}(\phi'|_K)$ and $\phi\colon K \to \mathcal{H}$ be the restriction of ϕ' to K. Note that $\mathcal{H}$ is a torsion free abelian group. In short, we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & G & \overset{\phi'}{\longrightarrow} & A & \longrightarrow & 1 \\
& & {\scriptstyle \phi=\phi'|_K}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow & \nearrow & & & \\
& & \mathcal{H} & \longrightarrow & \mathcal{H}' & & & &
\end{array}
$$

Suppose that $\alpha\colon K \to \mathrm{GL}(d, Q)$ is a d–dimensional representation to a field Q of characteristic zero such that $\alpha|_{\mathrm{Ker}\,\phi}$ factors through a q–group for some prime q. Let $Q(\mathcal{H})$ be the quotient field of the group ring $Q[\mathcal{H}]$. Note that α and ϕ give a right $\mathbb{Z}K$–module structure on $Q^d \otimes_Q Q(\mathcal{H}) = Q(\mathcal{H})^d$ as follows:

$$(2\text{-}1) \qquad \begin{aligned} Q^d \otimes_Q Q(\mathcal{H}) \times \mathbb{Z}K &\to Q^d \otimes_Q Q(\mathcal{H}), \\ (v \otimes p, g) &\mapsto (v \cdot \alpha(g) \otimes p, \phi(g)). \end{aligned}$$

We write $t = |A|$. $\mathbb{Z}G$ is a left (rank t free) $\mathbb{Z}K$–module. Note that there is a right action of G on dt–dimensional Q–vector space $Q^d \otimes_{\mathbb{Z}K} \mathbb{Z}G$. Equivalently, there is an induced representation $\alpha'\colon G \to \mathrm{GL}(dt, Q)$.

As in (2-1), α' and ϕ' give a right $\mathbb{Z}G$–module structure on $Q(\mathcal{H}')^{dt} = Q^{dt} \otimes_Q Q(\mathcal{H}')$:

$$(2\text{-}2) \qquad \begin{aligned} Q^{dt} \otimes_Q Q(\mathcal{H}') \times \mathbb{Z}G &\to Q^{dt} \otimes_Q Q(\mathcal{H}'), \\ (v \otimes p, g) &\mapsto (v \cdot \alpha'(g) \otimes p, \phi'(g)). \end{aligned}$$

Regard $\mathbb{Z}/q$ as a $\mathbb{Z}G$–module with the trivial G–action.

Lemma 2.1 [24, Theorem 3.1] *Let $f\colon M \to N$ be a morphism of projective left $\mathbb{Z}G$–modules such that*

$$1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f\colon \mathbb{Z}/q \otimes_{\mathbb{Z}G} M \to \mathbb{Z}/q \otimes_{\mathbb{Z}G} N$$

is injective. Then

$$1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f\colon Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} M \to Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} N$$

is injective.

Using Lemma 2.1, we prove Lemma 2.2 and Theorem 2.3.

Lemma 2.2 *Let $f\colon M \to N$ be a morphism of left $\mathbb{Z}G$–modules.*

(1) *If N is projective, then*

$$dt \cdot \dim_{\mathbb{Z}/q} \mathrm{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) \leq \dim_{Q(\mathcal{H}')} \mathrm{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

(2) *If, in addition, M is finitely generated and free, then*

$$dt \cdot \dim_{\mathbb{Z}/q} \mathrm{Ker}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) \geq \dim_{Q(\mathcal{H}')} \mathrm{Ker}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

Proof (1) Let $k = \dim_{\mathbb{Z}/q} \mathrm{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f)$ (k may be any cardinal number.) Note that $\mathbb{Z}/q \otimes_{\mathbb{Z}G} -$ induces two surjections $(\mathbb{Z}G)^k \to (\mathbb{Z}/q)^k$ and $\mathrm{Im}\, f \to \mathrm{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f)$. Since $(\mathbb{Z}G)^k$ is free, there exists a homomorphism $i\colon (\mathbb{Z}G)^k \to \mathrm{Im}\, f$ such that the following diagram commutes:

$$
\begin{array}{ccc}
(\mathbb{Z}G)^k & \xdashrightarrow{\;\;i\;\;} & \mathrm{Im}\, f \\
\downarrow & & \downarrow \\
(\mathbb{Z}/q)^k & \xrightarrow{\;\cong\;} & \mathrm{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f)
\end{array}
$$

Recall that N is a projective $\mathbb{Z}G$–module. Obviously, $(\mathbb{Z}G)^k$ is a projective $\mathbb{Z}G$–module. Hence, we can apply [Lemma 2.1](#) to $i\colon (\mathbb{Z}G)^k \to \mathrm{Im}\, f \subset N$ and obtain the injection

$$1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} i\colon Q(\mathcal{H}')^{dtk} = Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} (\mathbb{Z}G)^k \hookrightarrow Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} N.$$

Since $\mathrm{Im}\, i \subset \mathrm{Im}\, f$,

$$Q(\mathcal{H}')^{dtk} \cong \mathrm{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} i) \subset \mathrm{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

This implies

$$dt \cdot \dim_{\mathbb{Z}/q} \mathrm{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) = dtk \leq \dim_{Q(\mathcal{H}')}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

(2) Let $M = (\mathbb{Z}G)^n$. Part (1) and the following elementary observation complete the proof:

$$\dim_{\mathbb{Z}/q} \mathrm{Ker}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) + \dim_{\mathbb{Z}/q} \mathrm{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f) = \dim_{\mathbb{Z}/q} \mathbb{Z}/q \otimes_{\mathbb{Z}G} M = n.$$

Similarly,

$$ndt = \dim_{Q(\mathcal{H}')} \mathrm{Ker}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f) + \dim_{Q(\mathcal{H}')} \mathrm{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f). \qquad \square$$

Theorem 2.3 *Suppose C_* is a chain complex of projective left $\mathbb{Z}G$–modules with C_n finitely generated. Let $\{x_i\}_{i\in I}$ be a collection of n–cycles in C_n. Let M be the $(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} \mathbb{Z}G)$–span of $\{[1_{Q(\mathcal{H})^d} \otimes_{\mathbb{Z}K} x_i]\}_{i\in I}$. Let $\bar{M}$ be the $\mathbb{Z}/q$–span of $\{[1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} x_i]\}_{i\in I}$. Then we have*

$$\dim_{Q(\mathcal{H})} H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M \leq dt \cdot \dim_{\mathbb{Z}/q} H_n(\mathbb{Z}/q \otimes_{\mathbb{Z}G} C_*)/\bar{M}.$$

Proof Let $\partial_n\colon C_n \to C_{n-1}$ be the boundary map of C_* and define $f\colon (\mathbb{Z}G)^I \oplus C_{n+1} \to C_n$ by $(e_i, v) \mapsto x_i + \partial_{n+1}(v)$, where $\{e_i\}_{i\in I}$ is the standard basis of $(\mathbb{Z}G)^I$.

Then we have

$$(2\text{-}3) \quad H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M = \operatorname{Ker}(1_{Q(\mathcal{H})^d} \otimes_{\mathbb{Z}K} \partial_n)/\operatorname{Im}(1_{Q(\mathcal{H})^d} \otimes_{\mathbb{Z}K} f),$$
$$H_n(\mathbb{Z}/q \otimes_{\mathbb{Z}G} C_*)/\overline{M} = \operatorname{Ker}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} \partial_n)/\operatorname{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f).$$

From the $\mathbb{Z}G$–module structure on $Q(\mathcal{H}')^{dt}$ in (2-2),

$$Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} \mathbb{Z}G = (Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} \mathbb{Z}G) \otimes_{\mathbb{Z}G} \mathbb{Z}G$$
$$= Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} \mathbb{Z}G.$$

Since C_* is a chain complex of left $\mathbb{Z}G$–modules,

$$Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_* = Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*.$$

Since $\mathcal{H} \hookrightarrow \mathcal{H}'$, $Q(\mathcal{H}')$ is a flat right $Q(\mathcal{H})$–module. Therefore, we have

$$(2\text{-}4) \qquad H_*(Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*) = Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} H_*(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*).$$

Note that the dimension is left unchanged by $Q(\mathcal{H}') \otimes_{Q(\mathcal{H})}$. From (2-4), it follows that

$$\dim_{Q(\mathcal{H})} H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M$$
$$= \dim_{Q(\mathcal{H}')} Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} (H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M)$$
$$= \dim_{Q(\mathcal{H}')} H_n(Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*)/(Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} M).$$

Similarly to (2-3),

$$H_n(Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*)/(Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} M)$$
$$= \operatorname{Ker}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} \partial_n)/\operatorname{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f).$$

From the above observations and the inequality from Lemma 2.2,

$$\dim_{Q(\mathcal{H})} H_n(Q(\mathcal{H})^d \otimes_{\mathbb{Z}K} C_*)/M$$
$$= \dim_{Q(\mathcal{H}')} H_n(Q(\mathcal{H}')^{dt} \otimes_{\mathbb{Z}G} C_*)/(Q(\mathcal{H}') \otimes_{Q(\mathcal{H})} M)$$
$$= \dim_{Q(\mathcal{H}')} \operatorname{Ker}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} \partial_n) - \dim_{Q(\mathcal{H}')} \operatorname{Im}(1_{Q(\mathcal{H}')^{dt}} \otimes_{\mathbb{Z}G} f)$$
$$\leq dt(\dim_{\mathbb{Z}/q} \operatorname{Ker}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} \partial_n) - \dim_{\mathbb{Z}/q} \operatorname{Im}(1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} f))$$
$$= dt \cdot \dim_{\mathbb{Z}/q} H_n(\mathbb{Z}/q \otimes_{\mathbb{Z}G} C_*)/\overline{M}.$$

This completes the proof.   □

We remark that in this paper we use Theorem 2.3 only for a finite collection I.

3   An h–solvable cobordism

In this section, we give the definition of an h–solvable cobordism following [8]. Also, we prove Proposition 3.2 about prime power coverings of 1–solvable cobordisms.

3.1   Definition of h–solvable cobordism

For oriented compact bordered 3–manifolds M and M', a *cobordism W between M and M'* is a 4–dimensional manifold with boundary $\partial W = M \cup_\partial -M'$, where $-M'$ denotes M' with reversed orientation. We often denote a cobordism by $(W; M, M')$. A cobordism $(W; M, M')$ is an *H_1–cobordism* (respectively, a *homology cobordism*) if $H_i(M; \mathbb{Z}) \cong H_i(W; \mathbb{Z}) \cong H_i(M'; \mathbb{Z})$ under the inclusion map for $i \leq 1$ (respectively, for all i). Note that $H_2(W, M)$ is a free abelian group if $(W; M, M')$ is an H_1–cobordism (for example, see [7, Lemma 3.7]). A cobordism $(W; M, M')$ is an *H_1–cobordism with $\mathbb{Q}$–coefficients* if $H_i(M; \mathbb{Q}) \cong H_i(W; \mathbb{Q}) \cong H_i(M'; \mathbb{Q})$ under the inclusion map for $i \leq 1$.

Example   If L is a link in S^3, then the link exterior X_L is a bordered 3–manifold with a canonical homeomorphism between a disjoint union of tori and ∂X_L sending the standard basis to the meridians and 0–framed longitudes of L. If two links L and L' are concordant, then X_L and $X_{L'}$ are homology cobordant bordered 3–manifolds via a concordance exterior and h–solvable cobordant for all $h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ (see the definition of solvable cobordism given below).

We use the following notation for covering maps associated to the derived series.

Convention   (1)   For a space X, there is a sequence of regular covers over X

$$X^{(n+1)} \to X^{(n)} \to \cdots \to X^{(1)} \to X^{(0)} = X$$

which corresponds to the derived series

$$\pi^{(n+1)} \subset \pi^{(n)} \subset \cdots \subset \pi^{(0)} = \pi, \quad \text{where } \pi = \pi_1(X) \text{ and } \pi^{(n+1)} = [\pi^{(n)}, \pi^{(n)}].$$

With this, we can always identify $H_*(X; \mathbb{Z}[\pi/\pi^{(n)}]) = H_*(X^{(n)}; \mathbb{Z})$ as usual.

(2)   For a 4–manifold W with $\pi = \pi_1(W)$, let

$$\lambda_n \colon H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \times H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \to \mathbb{Z}[\pi/\pi^{(n)}]$$

be the intersection form.

(3)   For a covering map $Y \to X$, $\mathrm{Cov}(Y|X)$ denotes its deck transformation group. The action of $\mathrm{Cov}(Y|X)$ on $H_*(Y; \mathbb{Z})$ is a right action.

Definition 3.1 Suppose $(W; M, M')$ is an H_1–cobordism between bordered 3–manifolds M and M' with $\pi = \pi_1(W)$. Let $r = \frac{1}{2} \operatorname{rank} H_2(W, M; \mathbb{Z})$.

(1) A submodule $L \subset H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$ is an *n–Lagrangian* if L projects to a half-rank summand of $H_2(W, M; \mathbb{Z})$ and λ_n vanishes on L.

(2) For an n–Lagrangian L $(k \leq n)$, homology classes

$$d_1, \ldots, d_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(k)}])$$

are *k–duals* if L is generated by $l_1, \ldots, l_r \in L$ whose projections $l'_1, \ldots, l'_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(k)}])$ satisfy $\lambda_k(l'_i, d_j) = \delta_{ij}$.

(3) An H_1–cobordism $(W; M, M')$ is called an *n.5–solvable cobordism* (respectively, *n–solvable cobordism*) if it has an $(n+1)$–Lagrangian (respectively, n–Lagrangian) admitting n–duals. If there exists an h–solvable cobordism from M to M', we say that M is h–solvable cobordant to M' for $h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

3.2 Prime power cover of 1–solvable cobordism

In this subsection, we prove Proposition 3.2 about the (abelian) prime power cover of 1–solvable cobordism for later purpose. For a finitely generated abelian group G, denote the torsion subgroup of G by tG and the free part of G by $FG = G/tG$.

Proposition 3.2 *Suppose $(W; M, M')$ is a 1–solvable cobordism with $\varphi \colon \pi_1(W) \to A$ be a surjective group homomorphism to an abelian p–group A and p is prime. We denote the cobordism of the induced coverings by $(W^\varphi; M^\varphi, M'^\varphi)$. Then:*

(1) *$\beta_2(W^\varphi, M^\varphi) = |A|\beta_2(W, M)$ where β_2 is the second Betti number.*

(2) *The inclusion induced map $FH_2(W^\varphi; \mathbb{Z}) \to FH_2(W^\varphi, M^\varphi; \mathbb{Z})$ is surjective.*

(3) *$(W^\varphi; M^\varphi, M'^\varphi)$ is an H_1–cobordism with $\mathbb{Q}$–coefficients.*

Proof **(1)** Fix a (relative) CW–complex structure on (W, M). This induces a (relative) CW–complex structure on (W^φ, M^φ). Let $C_* = C_*(W^\varphi, M^\varphi; \mathbb{Z})$. Then C_* is a chain complex of free $\mathbb{Z}A$–modules and $C_*(W, M; \mathbb{Z}) = C_* \otimes_{\mathbb{Z}A} \mathbb{Z}$. Since $(W; M, M')$ is an H_1–cobordism, $H_i(C_* \otimes_{\mathbb{Z}A} \mathbb{Z}/p) = 0$ for $i = 0, 1$ by the universal coefficient theorem. Since p is prime, Levine's well-known chain homotopy lifting argument [34] shows that $H_i(C_* \otimes_{\mathbb{Z}} \mathbb{Z}/p) = 0$ for $i = 0, 1$. In particular, by the universal coefficient theorem, $H_i(C_*)$ is a torsion abelian group for $i = 0, 1$. By the universal coefficient theorem, $H_i(W^\varphi, M^\varphi; \mathbb{Q}) = H_i(C_*) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $i = 0, 1$.

By taking $C_* = C_*(W^\varphi, M'^\varphi; \mathbb{Z})$, the same argument shows $H_i(W^\varphi, M'^\varphi; \mathbb{Q}) = 0$ for $i = 0, 1$. By Poincaré duality and the universal coefficient theorem,

$$H_i(W^\varphi, M^\varphi; \mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(H_{4-i}(W^\varphi, M'^\varphi; \mathbb{Q}), \mathbb{Q}) = 0 \quad \text{for } i = 3, 4.$$

So, $\beta_2(W^\varphi, M^\varphi) = \chi(W^\varphi, M^\varphi)$ where χ is the Euler characteristic. Similarly, $\chi(W, M) = \beta_2(W, M)$ because $H_i(W, M; \mathbb{Z}) = H_i(W, M'; \mathbb{Z}) = 0$ for $i = 0, 1$. By definition, (W^φ, M^φ) is an A–cover of (W, M) and $\chi(W^\varphi, M^\varphi) = |A|\chi(W, M)$. This completes the proof of (1).

(2) Since $W^\varphi \to W$ is an abelian covering with $\mathrm{Cov}(W^\varphi|W) = A$, $\pi_1(W)^{(1)} \subset \pi_1(W^\varphi)$. The covering map $W^{(1)} \to W^\varphi$ induces $H_2(W^{(1)}; \mathbb{Z}) \to H_2(W^\varphi; \mathbb{Z})$. Let $l_1, \ldots, l_r, d_1, \ldots, d_r$ be the images of the (generators of) a 1–Lagrangian and 1–duals in $H_2(W^\varphi; \mathbb{Z})$. By the definition of 1–solvable cobordism, $\beta_2(W, M) = 2r$. Let $A = \{g_1, \ldots, g_t\}$. From (1), $\beta_2(W^\varphi, M^\varphi) = \beta_2(W, M)|A| = 2rt$.

From the (right) group action of A on $H_2(W^\varphi; \mathbb{Z})$, we can define

$$l_{ij} = l_i \cdot g_j \quad \text{and} \quad d_{kl} = d_k \cdot g_l \quad \text{for } 1 \le i, k \le r \text{ and } 1 \le j, l \le t.$$

By the definition of a 1–Lagrangian and of 1–duals, the intersection pairing

$$\lambda: FH_2(W^\varphi, M^\varphi; \mathbb{Z}) \times FH_2(W^\varphi, M'^\varphi; \mathbb{Z}) \to \mathbb{Z}$$

on the image of $\{l_{ij}, d_{kl}\}$ is given by

$$\begin{pmatrix} 0 & I_{rt \times rt} \\ I_{rt \times rt} & X \end{pmatrix}.$$

Since this matrix has determinant 1 and $\beta_2(W^\varphi, M^\varphi) = 2rt$, the image of $\{l_{ij}, d_{kl}\}$ is a basis of $FH_2(W^\varphi, M^\varphi; \mathbb{Z})$. This proves $\mathrm{inc}_*: FH_2(W^\varphi; \mathbb{Z}) \to FH_2(W^\varphi, M^\varphi; \mathbb{Z})$ is surjective because $\{l_{ij}, d_{kl}\} \subset FH_2(W^\varphi; \mathbb{Z})$.

(3) From (2), $\mathrm{inc}_*: H_2(W^\varphi; \mathbb{Q}) \to H_2(W^\varphi, M^\varphi; \mathbb{Q})$ is surjective. From (1) and the homology long exact sequence of a pair (W^φ, M^φ), $\mathrm{inc}_*: H_i(M^\varphi; \mathbb{Q}) \to H_i(W^\varphi; \mathbb{Q})$ is an isomorphism for $i = 0, 1$. The same argument works for (W, M'). This completes the proof. $\qquad \square$

4 Solvable cobordism and Friedl–Powell invariant

Throughout this section, for any finitely generated abelian group G, tG and FG denote the torsion part of G and the free part of G, respectively. $G^\wedge$ denotes $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. For a finite abelian group G, $G^\wedge = \mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Z})$ since $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}) = \mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Q}) = 0$. $H_*(-)$ denotes homology with integral coefficients.

4.1 Definition of the Friedl–Powell invariant $\tau(L, \chi)$

To define the Friedl–Powell invariant $\tau(L, \chi)$, we set up the notation and conventions used in [25]. Here, L is a 2–component link with linking number 1 and H is the Hopf link. We denote the exterior of J by $X_J = S^3 - \nu(J)$ for $J = H, L$. We can decompose ∂X_L into $Y_a \cup Y_b$ with $Y_a \cong Y_b = S^1 \times D^1 \sqcup S^1 \times D^1$ where both Y_a and Y_b are annuli neighborhood of (parallels of) meridians of L. Define $M_J = X_J \cup_{\partial X_H \times I} -X_H$ for $J = H, L$ where the gluing map respects the ordering of the link components and identifies each of the subsets $Y_a, Y_b \subset \partial X_J$ for $J = L, H$.

For a prime p, we say a group homomorphism $\varphi \colon H_1(M_L) \to \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ is *admissible* if φ sends the two meridians of L to the standard basis $(1, 0), (0, 1)$. (From the Mayer–Vietoris sequence, $H_1(M_L) \cong H_1(X_L) \oplus \mathbb{Z} \cong \mathbb{Z}^3$.) Let $M_L^\varphi \to M_L$ be the p^{i+j}–fold covering space from φ. We denote the Hurewicz map by $\phi' \colon \pi_1(M_L) \to H_1(M_L)$. Define $\phi \colon \pi_1(M_L^\varphi) \to H_1(M_L)$ to be the restriction $\phi'|_{\pi_1(M_L^\varphi)}$ and $\mathcal{H} = \operatorname{Im} \phi$. Choose an isomorphism $\psi \colon \pi_1(T^3) \cong \mathcal{H}$. (Note that $\mathcal{H}$ is isomorphic to $\mathbb{Z}^3$ as a finite-index subgroup of $H_1(M_L) \cong \mathbb{Z}^3$.)

For a prime power character $\chi \colon \pi_1(M_L^\varphi) \to \mathbb{Z}/q^k$, we have the bordism class

$$[(M_L^\varphi, \chi \times \phi) \sqcup -(T^3, \operatorname{tr} \times \psi)] \in \Omega_3(\mathbb{Z}/q^k \times \mathcal{H}),$$

where $\operatorname{tr} \colon \pi_1(T^3) \to \mathbb{Z}/q^k$ is the trivial group homomorphism. From the Atiyah–Hirzebruch spectral sequence calculation given in [25, Section 3.2], $[(M_L^\varphi, \chi \times \phi) \sqcup -(T^3, \operatorname{tr} \times \psi)]$ is q–primary torsion in $\Omega_3(\mathbb{Z}/q^k \times \mathcal{H})$. In other words, there exist a nonnegative integer s, a cobordism W between $q^s M_L^\varphi$ and $q^s T^3$, and $\Phi \colon \pi_1(W) \to \mathbb{Z}/q^k \times \mathcal{H}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
q^s(M_L^\varphi \sqcup -T^3) & \xrightarrow{\chi \times \phi \sqcup \operatorname{tr} \times \psi} & K(\mathbb{Z}/q^k \times \mathcal{H}, 1) \\
{\scriptstyle \partial}\big\downarrow\big\uparrow & \nearrow{\scriptstyle \Phi} & \\
W & &
\end{array}
$$

From the sequence of ring homomorphisms

$$\mathbb{Z}[\pi_1(W)] \xrightarrow{\Phi} \mathbb{Z}[\mathbb{Z}/q^k \times \mathcal{H}] = \mathbb{Z}[\mathbb{Z}/q^k][\mathcal{H}] \to \mathbb{Q}(\xi_{q^k})(\mathcal{H}) \to \mathbb{C}(\mathcal{H}) = \mathcal{K},$$

we can define the twisted intersection form $H_2(W; \mathcal{K}) \times H_2(W; \mathcal{K}) \to \mathcal{K}$. We denote the nonsingular part of the intersection form on $H_2(W; \mathcal{K})$ (respectively, on $H_2(W; \mathbb{Q})$) by $\lambda_{\mathcal{K}}(W)$ (respectively, $\lambda_{\mathbb{Q}}(W)$).

Definition 4.1 (Friedl–Powell invariant) We have

$$\tau(L, \chi) = (\lambda_{\mathcal{K}}(W) - \mathcal{K} \otimes \lambda_{\mathbb{Q}}(W)) \otimes \frac{1}{q^s} \in L^0(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/q],$$

where $L^0(\mathcal{K})$ is the Witt group of finite-dimensional nonsingular sesquilinear forms over $\mathcal{K}$.

In [25, Section 3.2], it is shown that $\tau(L, \chi)$ is well defined. That is, $\tau(L, \chi)$ depends neither on the choice of W nor on the choice of isomorphism $\psi \colon \pi_1(T^3) \to \mathcal{H}$.

In Section 4.2, we will describe the linking form

$$\lambda_L \colon tH_1(X_L^{\varphi}, Y_a^{\varphi}) \times tH_1(X_L^{\varphi}, Y_a^{\varphi}) \to \mathbb{Q}/\mathbb{Z}.$$

Now we give the precise statement that *the Friedl–Powell invariant vanishes*.

Definition 4.2 For 2–component link L with linking number 1, we say *the Friedl–Powell invariant vanishes for L* if for any admissible homomorphism $\varphi \colon H_1(M_L) \to \mathbb{Z}/p^i \otimes \mathbb{Z}/p^j$ and for a prime p, there exists a metabolizer $P = P^{\perp}$ of the linking form

$$\lambda_L \colon tH_1(X_L^{\varphi}, Y_a^{\varphi}) \times tH_1(X_L^{\varphi}, Y_a^{\varphi}) \to \mathbb{Q}/\mathbb{Z}$$

with the following property: For any character of prime-power order $\chi \colon H_1(M_L^{\varphi}) \to \mathbb{Z}/q^k$ which satisfies that $\chi|_{H_1(X_L^{\varphi})}$ factors through

$$
\begin{array}{ccc}
H_1(X_L^{\varphi}) & \xrightarrow{\;\chi|_{H_1(X_L^{\varphi})}\;} & \mathbb{Z}/q^k \\[2pt]
\big\downarrow & \nearrow{\scriptstyle \delta} & \\[2pt]
H_1(X_L^{\varphi}, Y_a^{\varphi}) & &
\end{array}
$$

and that δ vanishes on P, $\tau(L, \chi) = 0 \in L^0(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/q]$.

The following main theorem will be proved in Section 4.3:

Theorem A *Suppose that L is a 2–component link with linking number 1 and H is the Hopf link. If X_L and X_H are 1.5–solvable cobordant, then the Friedl–Powell invariant $\tau(L, \chi)$ vanishes for L in the sense of Definition 4.2. In particular, the conclusion holds if L and H are height-3.5 Whitney tower/grope concordant.*

4.2   1–solvable cobordism and a metabolizer of the linking form

In this subsection, we recall the definition of the linking form

$$\lambda_L \colon tH_1(X_L^\varphi, Y_a^\varphi) \times tH_1(X_L^\varphi, Y_a^\varphi) \to \mathbb{Q}/\mathbb{Z}$$

defined in [25] and prove Proposition 4.3. The adjoint of λ_L, $\mathrm{Ad}(\lambda_L)\colon tH_1(X_L^\varphi, Y_a^\varphi) \to tH_1(X_L^\varphi, Y_a^\varphi)^\wedge$, can be obtained by composing the following isomorphisms:

(a) $tH_1(X_L^\varphi, Y_a^\varphi) \to tH^2(X_L^\varphi, Y_b^\varphi) \to \mathrm{Ext}_\mathbb{Z}(tH_1(X_L^\varphi, Y_b^\varphi), \mathbb{Z}) = tH_1(X_L^\varphi, Y_a^\varphi)^\wedge.$

To see this, we use Poincaré duality, the universal coefficient theorem, $H_1(X_L^\varphi, Y_b^\varphi) \cong H_1(X_L^\varphi, Y_a^\varphi)$ and that $tH_1(X_L^\varphi, Y_a^\varphi)$ is a finite abelian group.

Let $(W_0; X_L, X_H)$ be a 1–solvable cobordism. Recall $\varphi\colon H_1(M_L) \to \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ is an admissible homomorphism and $H_1(M_L) \cong H_1(X_L) \oplus \mathbb{Z}$. Then, $\varphi|_{H_1(X_L)}$ extends to $H_1(W_0)$ canonically because $H_1(X_L) \cong H_1(W_0)$. In this sense, an admissible homomorphism φ always induces a covering $(W_0^\varphi; X_L^\varphi, X_H^\varphi) \to (W; X_L, X_H)$.

Proposition 4.3   *Suppose $(W_0; X_L, X_H)$ is a 1–solvable cobordism. Let $(W_0^\varphi; X_L^\varphi, X_H^\varphi)$ be a covering induced from an admissible homomorphism $\varphi\colon H_1(M_L) \to \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$, then*

$$P = \mathrm{Ker}(tH_1(X_L^\varphi, Y_a^\varphi) \to tH_1(W_0^\varphi, Y_a^\varphi))$$

is a metabolizer of the linking form λ_L.

Proof   Suppose that we have the diagram (b) with two exact rows. Then,

$$P^\perp = (\mathrm{Ad}(\lambda_L))^{-1}(\mathrm{Ker}\,\partial^\wedge) = \mathrm{Ker}(\mathrm{inc}_*) = P.$$

Hence, it suffices to prove the existence of diagram (b) with two exact rows.

(b)

$$
\begin{array}{ccccc}
tH_2(W_0^\varphi, X_L^\varphi) & \xrightarrow{\partial} & tH_1(X_L^\varphi, Y_a^\varphi) & \xrightarrow{\mathrm{inc}_*} & tH_1(W_0^\varphi, Y_a^\varphi) \\
{\scriptstyle \theta_1} \downarrow {\scriptstyle \cong} & & {\scriptstyle \mathrm{Ad}(\lambda_L)} \downarrow {\scriptstyle \cong} & & {\scriptstyle \theta_2} \downarrow {\scriptstyle \cong} \\
tH_1(W_0^\varphi, Y_a^\varphi)^\wedge & \xrightarrow{\mathrm{inc}_*^\wedge} & tH_1(X_L^\varphi, Y_a^\varphi)^\wedge & \xrightarrow{\partial^\wedge} & tH_2(W_0^\varphi, X_L^\varphi)^\wedge
\end{array}
$$

As in (a), $tH_2(W_0^\varphi, X_L^\varphi) \cong tH_1(W_0^\varphi, X_H^\varphi)^\wedge$ and $tH_1(W_0^\varphi, Y_a^\varphi) \cong tH_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi)^\wedge$ by Poincaré duality and the universal coefficient theorem. (Note that $\partial W_0^\varphi = X_L^\varphi \cup X_H^\varphi$.) So, from the following claim, we can define isomorphisms θ_1 and θ_2.

Claim   *The inclusion maps induce two isomorphisms:*

(1) $tH_1(W_0^\varphi, Y_a^\varphi) \cong tH_1(W_0^\varphi, X_H^\varphi).$

(2) $tH_2(W_0^\varphi, X_L^\varphi) \cong tH_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi).$

Proof of Claim By [Proposition 3.2](3), $(W_0^\varphi; X_L^\varphi, X_H^\varphi)$ is an H_1–cobordism with $\mathbb{Q}$–coefficients. From this and the proofs of [25, Lemmas 2.6, 2.7 and 2.9] (W_0 plays the role of E_C),

$$\text{Coker}(\text{inc}_*\colon H_1(X_H^\varphi, Y_a^\varphi) \to H_1(W_0^\varphi, Y_a^\varphi)) \cong tH_1(W_0^\varphi, Y_a^\varphi).$$

From the homology long exact sequence of a triple $(W_0^\varphi, X_H^\varphi, Y_a^\varphi)$, we have an exact sequence

$$0 \to tH_1(W_0^\varphi, Y_a^\varphi) \to H_1(W_0^\varphi, X_H^\varphi) \to H_0(X_H^\varphi, Y_a^\varphi) = 0,$$

which proves (1).

From the proof of [25, Lemma 2.5], $tH_1(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi) = 0$ and the inclusion map $(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi) \to (\partial W_0^\varphi, X_L^\varphi)$ induces the zero map on H_2. In particular,

$$\text{inc}_*\colon H_2(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi) \to H_2(W_0^\varphi, X_L^\varphi)$$

is also the zero map. From the homology long exact sequence of a triple $(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi, X_L^\varphi)$,

$$H_2(W_0^\varphi, X_L^\varphi) \cong \text{Ker}(\partial\colon H_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi) \to H_1(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi)).$$

By taking torsion subgroups, we obtain (2) via

$$tH_2(W_0^\varphi, X_L^\varphi) \cong \text{Ker}(tH_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi) \to tH_1(\partial W_0^\varphi - Y_a^\varphi, X_L^\varphi) = 0)$$
$$= tH_2(W_0^\varphi, \partial W_0^\varphi - Y_a^\varphi),$$

completing the proof. $\square$

Commutativity of the diagram [(b)] also easily follows. For exactness of the first row of [(b)], we prove the following lemma.

Lemma 4.4 *Suppose $(W_0; X_L, X_H)$ is a 1–solvable cobordism. We have the exact sequence*

$$tH_2(W_0^\varphi, X_L^\varphi) \xrightarrow{\partial} tH_1(X_L^\varphi, Y_a^\varphi) \xrightarrow{\text{inc}_*} tH_1(W_0^\varphi, Y_a^\varphi)$$

which is the restriction of a long exact sequence of triple $(W_0^\varphi, X_L^\varphi, Y_a^\varphi)$ to their torsion subgroups.

Proof of [Lemma 4.4] Since $\text{inc}_* \circ \partial = 0$, we prove that $\text{Ker}(\text{inc}_*) \subset \text{Im} \, \partial$. Let $x \in \text{Ker}(\text{inc}_*)$. By the homology long exact sequence of triple $(W_0^\varphi, X_L^\varphi, Y_a^\varphi)$, there

exists $y \in H_2(W_0^\varphi, X_L^\varphi)$ such that $\partial y = x$. By Proposition 3.2(2), $FH_2(W_0^\varphi) \to FH_2(W_0^\varphi, X_L^\varphi)$ is surjective. Since it is equal to the composition

$$FH_2(W_0^\varphi) \to FH_2(W_0^\varphi, Y_a^\varphi) \xrightarrow{j} FH_2(W_0^\varphi, X_L^\varphi),$$

it follows that j is surjective. We can choose $z \in FH_2(W_0^\varphi, Y_a^\varphi)$ such that $y - j(z) \in tH_2(W_0^\varphi, X_L^\varphi)$. Then, $\partial(y - j(z)) = \partial y = x$ and this shows $\mathrm{Ker}(\mathrm{inc}_*) \subset \mathrm{Im}\,\partial$. $\square$

Note that if $A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence of abelian groups. Since $\mathbb{Q}/\mathbb{Z}$ is a divisible group, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$–module. For any abelian group G, $\mathrm{Ext}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}) = 0$. Hence, $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is an exact functor and we obtain $C^\wedge \to B^\wedge \to A^\wedge$ is exact. This proves that the second row of the diagram (b) is also exact and completes the proof of Proposition 4.3. $\square$

4.3 Proof of Theorem A

In this subsection, we prove Theorem A. Let $(W_0; X_L, X_H)$ be a 1.5–solvable cobordism with $\beta_2(W_0, X_L) = 2r$. Note that $\partial W_0 = X_L \cup \partial X_H \times I \cup -X_H = M_L$. Attach $X_H \times I$ to W_0 along $\partial X_H \times I$ to get

$$W = W_0 \cup_{\partial X_H \times I} X_H \times I$$

with $\partial W = M_L \sqcup -M_H$. Recall that $\varphi \colon H_1(M_L) \to \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$. Applying a Mayer–Vietoris argument to $W = W_0 \cup X_H \times I$, the inclusion induces an isomorphism $H_1(M_L) \cong H_1(W)$. So, φ extends to $H_1(W)$ naturally and denote the induced cobordism of coverings by $(W^\varphi; M_L^\varphi, M_H^\varphi)$.

From Proposition 4.3, we can take a metabolizer

$$P := \mathrm{Ker}(tH_1(X_L^\varphi, Y_a^\varphi) \to tH_1(W_0^\varphi, Y_a^\varphi))$$

of the linking form λ_L. We fix a character $\chi \colon H_1(M_L^\varphi) \to \mathbb{Z}/q^k$ satisfies that $\chi|_{H_1(X_L^\varphi)}$ factors through

$$
\begin{array}{ccc}
H_1(X_L^\varphi) & \xrightarrow{\chi|_{H_1(X_L^\varphi)}} & \mathbb{Z}/q^k \\
\downarrow & \nearrow{\scriptstyle \delta} & \\
H_1(X_L^\varphi, Y_a^\varphi) & &
\end{array}
$$

and δ vanishes on P. It remains to prove that $\tau(L, \chi) = 0$.
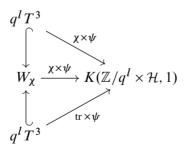
We have the following facts and remarks.

(1) From the arguments of [25, Propositions 2.10, 2.12] (W_0 and W play the role of E_C and W_C, respectively), we have the following: if δ vanishes on P, then there exist an integer $l \geq k$ and a character $H_1(W^\varphi) \to \mathbb{Z}/q^l$, denoted by χ in an abuse of notation, which fits into the following diagram:

$$
\begin{array}{ccccccc}
\pi_1(M_L^\varphi) & \longrightarrow & H_1(M_L^\varphi) & \xrightarrow{\chi} & \mathbb{Z}/q^k & \xrightarrow{q^{l-k}} & \mathbb{Z}/q^l \\
{\scriptstyle \mathrm{inc}_*}\downarrow & & {\scriptstyle \mathrm{inc}_*}\downarrow & & & {\scriptstyle \chi} & \\
\pi_1(W^\varphi) & \longrightarrow & H_1(W^\varphi) & & & &
\end{array}
$$

(2) Let $H_1(M_L) = \mathcal{H}'$ and $\phi': \pi_1(M_L) \to H_1(M_L)$ be the Hurewicz homomorphism. Define $\phi: \pi_1(M_L^\varphi) \to \mathcal{H}'$ to be the restriction of ϕ' to the subgroup $\pi_1(M_L^\varphi)$. Let $\mathcal{H} = \operatorname{Im}\phi$. Since $H_1(M_L) \cong H_1(W)$, ϕ' extends to $\pi_1(W)$. Therefore, we use $\phi': \pi_1(W) \to \mathcal{H}'$ and its restriction $\phi: \pi_1(W^\varphi) \to \mathcal{H}$ as an abuse of notation. Note that $\mathcal{H}'$ is isomorphic to $\mathbb{Z}^3$ and $\mathcal{H}$ is also isomorphic to $\mathbb{Z}^3$ as a finite-index subgroup of $\mathcal{H}'$.

(3) By (1) and (2), we have $\chi \times \phi: \pi_1(W^\varphi) \to \mathbb{Z}/q^l \times \mathcal{H}$. If we write $\mathcal{K} = \mathbb{C}(\mathcal{H})$, then $H_*(M_L^\varphi; \mathcal{K})$, $H_*(W^\varphi; \mathcal{K})$, and $H_*(W^\varphi, M_L^\varphi; \mathcal{K})$ can be defined from

$$
\mathbb{Z}[\pi_1(W^\varphi)] \xrightarrow{\chi \times \phi} \mathbb{Z}[\mathbb{Z}/q^l \times \mathcal{H}] = \mathbb{Z}[\mathbb{Z}/q^l][\mathcal{H}] \to \mathbb{Q}(\xi_{q^l})(\mathcal{H}) \to \mathbb{C}(\mathcal{H}) = \mathcal{K}.
$$

(4) By [25, Lemma 3.4], there is a 4–manifold W_χ bounded by $2q^l$ copies of a 3–torus T^3, which is over $\mathbb{Z}/q^l \times \mathcal{H}$ as follows:

$$
\begin{array}{ccc}
q^l T^3 & & \\
\big\downarrow \quad \searrow^{\chi \times \psi} & & \\
W_\chi & \xrightarrow{\chi \times \psi} & K(\mathbb{Z}/q^l \times \mathcal{H}, 1) \\
\big\uparrow \quad \nearrow^{\mathrm{tr} \times \psi} & & \\
q^l T^3 & &
\end{array}
$$

Here tr denotes the trivial character $\pi_1(T^3) \to \mathbb{Z}/q^l$ and $\psi: \pi_1(T^3) \cong \mathcal{H}$. Furthermore, the intersection forms of W_χ over $\mathbb{Q}$–coefficient and $\mathcal{K}$–coefficients are Witt-trivial.

We can attach $\chi \times \phi: q^l W^\varphi \to \mathbb{Z}/q^l \times \mathcal{H}$ and W_χ in (4) along $\chi \times \psi: q^l T^3 \to \mathbb{Z}/q^l \times \mathcal{H}$ to obtain the cobordism $(q^l W^\varphi \cup W_\chi, \chi \times \phi \cup \chi \times \psi)$ over $\mathbb{Z}/q^l \times \mathcal{H}$

between $q^l(M_L^\varphi, \chi \times \phi)$ and $-q^l(T^3, \mathrm{tr} \times \psi)$. From Definition 4.1,

$$\tau(L, \chi) = (\lambda_\mathcal{K}(q^l W^\varphi \cup W_\chi) - \mathcal{K} \otimes \lambda_\mathbb{Q}(q^l W^\varphi \cup W_\chi)) \otimes 1/q^l \in L^0(\mathcal{K}) \otimes_\mathbb{Z} \mathbb{Z}[1/q].$$

By (4), $[\lambda_\mathbb{Q}(W_\chi)] = 0 \in L^0(\mathbb{Q})$ and $[\lambda_\mathcal{K}(W_\chi)] = 0 \in L^0(\mathcal{K})$. In the following two claims we will prove that $[\lambda_\mathbb{Q}(W^\varphi)] = 0 \in L^0(\mathbb{Q})$ and $[\lambda_\mathcal{K}(W^\varphi)] = 0 \in L^0(\mathcal{K})$. By Novikov additivity, these claims will complete the proof of Theorem A.

Claim 1 *We have* $[\lambda_\mathbb{Q}(W^\varphi)] = 0 \in L^0(\mathbb{Q})$.

Proof of Claim 1 Applying the relative Mayer–Vietoris sequence (see for example Hatcher [27, page 152]), the sequence

$$\to H_i(\partial X_H \times I, \partial X_H) \to H_i(W_0, X_J) \oplus H_i(X_H \times I, X_H) \to H_i(W, M_J) \to$$

is exact. Since $H_*(\partial X_H \times I, \partial X_H)$ and $H_*(X_H \times I, X_H)$ vanish, $H_i(W_0, X_J) \cong H_i(W, M_J)$. Similarly, $H_i(W_0^\varphi, X_J^\varphi) \cong H_i(W^\varphi, M_J^\varphi)$.

For brevity, let $A = \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$ and write $A = \{g_1, \ldots, g_t\}$. Since $(W_0; X_L, X_H)$ is a 1.5–solvable cobordism, by Proposition 3.2(1) and (2),

$$\beta_2(W^\varphi, M_L^\varphi) = \beta_2(W_0^\varphi, X_L^\varphi) = |A| \cdot \beta_2(W_0, X_L) = 2rt$$

and $\mathrm{inc}_*\colon H_2(W_0^\varphi; \mathbb{Q}) \to H_2(W_0^\varphi, X_J^\varphi; \mathbb{Q})$ is surjective for $J = L, H$. Since we have $H_2(W_0^\varphi, X_J^\varphi; \mathbb{Q}) \cong H_2(W^\varphi, M_J^\varphi; \mathbb{Q})$, $\mathrm{inc}_*\colon H_2(W^\varphi; \mathbb{Q}) \to H_2(W^\varphi, M_J^\varphi; \mathbb{Q})$ is surjective, too. Applying Proposition 3.2(3), $H_i(W_0^\varphi, M_J^\varphi; \mathbb{Q}) = 0$ for $i = 0, 1$. From the homology long exact sequence of a pair (W^φ, M_J^φ), this proves that $(W^\varphi, M_L^\varphi, M_H^\varphi)$ is an H_1–cobordism over $\mathbb{Q}$–coefficients.

Recall $\partial W = M_L \sqcup -M_H$. For $X = \partial W^\varphi, M_L^\varphi$, and M_H^φ, let

$$I_X = \mathrm{Im}(\mathrm{inc}_*\colon H_2(X; \mathbb{Q}) \to H_2(W^\varphi; \mathbb{Q})).$$

For $J = L, H$, using the homology long exact sequences of pairs,

$$H_2(W^\varphi; \mathbb{Q})/I_{\partial W^\varphi} \cong H_2(W^\varphi; \mathbb{Q})/I_{M_J^\varphi} \cong H_2(W^\varphi, M_J^\varphi; \mathbb{Q})$$

whose rank is $2rt$. (A similar argument was used by Cochran and Kim in the proof of [14, Proposition 2.6].) We remark that to prove the last isomorphism, we used the fact that $\mathrm{inc}_*\colon H_1(M_J^\varphi; \mathbb{Q}) \to H_1(W^\varphi; \mathbb{Q})$ is an isomorphism for $J = L, H$.

Let $l_1, \ldots, l_r, d_1, \ldots, d_r$ be (generators of) a 2–Lagrangian and 1–duals in $H_2(W^\varphi; \mathbb{Z})$. From the (right) group action of A on $H_2(W^\varphi; \mathbb{Z})$, we define

$$l_{ij} = l_i \cdot g_j \quad \text{and} \quad d_{kl} = d_k \cdot g_l \quad \text{for } 1 \le i, k \le r \text{ and } 1 \le j, l \le t.$$

The intersection pairing $\lambda_{\mathbb{Q}}(W^{\varphi})$: $H_2(W^{\varphi};\mathbb{Q})/I_{\partial W^{\varphi}} \times H_2(W^{\varphi};\mathbb{Q})/I_{\partial W^{\varphi}} \to \mathbb{Q}$ with respect to (the image of) $\{l_{ij}, d_{kl}\}$ is

$$\begin{pmatrix} 0 & I_{rt \times rt} \\ I_{rt \times rt} & X \end{pmatrix}$$

because $l_i \cdot d_k$ is the Kronecker delta δ_{ik}.

Let $L(\mathbb{Q}) \subset H_2(W^{\varphi};\mathbb{Q})/I_{\partial W^{\varphi}}$ be the $\mathbb{Q}$–span of the image of $l_{ij} \otimes 1_{\mathbb{Q}}$. Then, $\lambda_{\mathbb{Q}}(W^{\varphi})$ vanishes on $L(\mathbb{Q}) \times L(\mathbb{Q})$ and $\dim_{\mathbb{Q}} L(\mathbb{Q}) = \frac{1}{2} \dim_{\mathbb{Q}}(H_2(W^{\varphi};\mathbb{Q})/I_{\partial W^{\varphi}}) = rt$. So, $[\lambda_{\mathbb{Q}}(W^{\varphi})] = 0 \in L^0(\mathbb{Q})$. □

By [25, Lemma 3.2], $H_*(M_J^{\varphi};\mathcal{K}) = 0$ for $J = H$ or L. Therefore, the twisted intersection form

$$\lambda_{\mathcal{K}}(W^{\varphi}): H_2(W^{\varphi};\mathcal{K}) \times H_2(W^{\varphi};\mathcal{K}) \to \mathcal{K}$$

is nonsingular.

Claim 2 *We have $[\lambda_{\mathcal{K}}(W^{\varphi})] = 0 \in L^0(\mathcal{K})$.*

Proof of Claim 2 Let α: $\pi_1(W^{\varphi}) \xrightarrow{\chi} \mathbb{Z}/q^l \hookrightarrow \mathbb{C}^{\times} = \mathrm{GL}(1,\mathbb{C})$ and α': $\pi_1(W) \to \mathrm{GL}(t,\mathbb{C})$ be the induced representation of α. Recall the maps ϕ': $\pi_1(W) \to \mathcal{H}'$ and ϕ: $\pi_1(W^{\varphi}) \to \mathcal{H}$ in (2). Define $\Gamma := \mathrm{Im}(\alpha \times \phi)$. There is a corresponding cover $(W^{\Gamma}, M_L^{\Gamma}) \to (W^{\varphi}, M_L^{\varphi})$, where $\pi_1(W^{\Gamma}) = \mathrm{Ker}(\alpha \times \phi)$. Recall $W^{\varphi} \to W$ is $\mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$–cover and $\alpha \times \phi$: $\pi_1(W^{\varphi}) \to \mathbb{C}^{\times} \times \mathcal{H}$. Since $\mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$, $\mathbb{C}^{\times}$ and $\mathcal{H}$ are abelian,

$$\pi_1(W)^{(2)} \le \pi_1(W^{\varphi})^{(1)} \le \mathrm{Ker}(\alpha \times \phi) = \pi_1(W^{\Gamma}).$$

Equivalently, there is a sequence of coverings

$$W^{(2)} \to W^{\Gamma} \to W^{\varphi} \to W.$$

Since $\mathbb{Z}/q^l \hookrightarrow \mathbb{C}^{\times}$ is injective, $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,\chi$, where α: $\pi_1(W^{\varphi}) \xrightarrow{\chi} \mathbb{Z}/q^l \hookrightarrow \mathbb{C}^{\times}$. From this, $\Gamma \overset{\mathrm{def}}{=} \mathrm{Im}(\alpha \times \phi) = \mathrm{Im}(\chi \times \phi)$. In particular, the ring homomorphism $\mathbb{Z}[\pi_1(W^{\varphi})] \to \mathcal{K}$ in (3) factors through $\mathbb{Z}\Gamma$ and

$$C_*(W^{\varphi};\mathcal{K}) \overset{\mathrm{def}}{=} \mathcal{K} \otimes_{\mathbb{Z}[\pi_1(W^{\varphi})]} C_*(W^{\varphi};\mathbb{Z}[\pi_1 W^{\varphi}]) = \mathcal{K} \otimes_{\mathbb{Z}\Gamma} C_*(W^{\Gamma};\mathbb{Z}).$$

Choose 2–cycles $\{\tilde{l}_1, \dots, \tilde{l}_r\} \subset C_2(W^{\Gamma};\mathbb{Z})$ which represent the image of (generators of) the 2–Lagrangian under the map induced by $W_0^{(2)} \to W^{(2)} \to W^{\Gamma}$. The covering map $W^{\Gamma} \to W^{\varphi}$ induces a surjection $\mathrm{Cov}(W^{\Gamma}|W) \to \mathrm{Cov}(W^{\varphi}|W) = \{g_1, \dots, g_t\}$.

Choose a lift $\widetilde{g}_j \in \mathrm{Cov}(W^\Gamma | W)$ of g_j for each $j = 1, \dots, t$. From the right action of $\mathrm{Cov}(W^\Gamma | W)$ on $C_2(W^\Gamma; \mathbb{Z})$, define

$$\widetilde{l}_{ij} = \widetilde{l}_i \cdot \widetilde{g}_j \quad \text{for } 1 \le i \le r \text{ and } 1 \le j \le t.$$

Let

$$L(\mathcal{K}) \subset H_2(W^\varphi; \mathcal{K}) = H_2(\mathcal{K} \otimes_{\mathbb{Z}\Gamma} C_*(W^\Gamma; \mathbb{Z}))$$

be the $\mathcal{K}$–span of $\{[1_\mathcal{K} \otimes \widetilde{l}_{ij}] \mid 1 \le i \le r, 1 \le j \le t\}$ in $H_2(W^\varphi; \mathcal{K})$. We remark that $L(\mathcal{K})$ does not depend on the choice of $\widetilde{g}_i$. We claim that $L(\mathcal{K})$ is a Lagrangian for the nonsingular twisted intersection form $\lambda_\mathcal{K}(W^\varphi)$.

First, we prove $\lambda_\mathcal{K}$ vanishes on $L(\mathcal{K}) \times L(\mathcal{K})$. Since $\lambda_\mathcal{K}$ is $\mathcal{K}$–sesquilinear, the following is enough:

$$\lambda_\mathcal{K}([1_\mathcal{K} \otimes_{\mathbb{Z}\Gamma} \widetilde{l}_{ik}], [1_\mathcal{K} \otimes_{\mathbb{Z}\Gamma} \widetilde{l}_{jl}]) = \sum_{g \in \mathrm{Cov}(W^\Gamma | W)} \lambda_{W^\Gamma}(\widetilde{l}_i, \widetilde{l}_j) \widetilde{g}_j g \widetilde{g}_k^{-1} = 0.$$

Now, we prove $\dim_\mathcal{K} L(\mathcal{K}) = \frac{1}{2} \dim_\mathcal{K} H_2(W^\varphi; \mathcal{K})$. Recall that $H_*(M_L; \mathcal{K}) = 0$ by [25, Lemma 3.2]. Therefore, $\mathrm{inc}_*: H_2(W^\varphi; \mathcal{K}) \to H_2(W^\varphi, M_L^\varphi; \mathcal{K})$ is an isomorphism. Now, for simplicity, we abuse notation by regarding $\widetilde{l}_{ij}$ as an element in $C_2(W^\Gamma, M_L^\Gamma; \mathbb{Z})$ and $L(\mathcal{K})$ as a subspace of $H_2(W^\varphi, M_L^\varphi; \mathcal{K})$.

Recall that $\{l_1, \dots, l_r\}$ are the chosen generators of the 2–Lagrangian in $H_2(W^\varphi; \mathbb{Z})$. Since the covering $W^\Gamma \to W$ sends $\widetilde{g}_j$ to 1, the image of $\{[\widetilde{l}_{ij}] \in H_2(W^\Gamma; \mathbb{Z}) \mid 1 \le i \le r, 1 \le j \le t\}$ in $H_2(W, M_L; \mathbb{Z})$ (via covering induced map) is exactly $\{\pi(l_1), \dots, \pi(l_r)\}$ where $\pi: H_2(W^\varphi) \to H_2(W) \to H_2(W, M_L)$.

Since $(W_0; X_L, X_H)$ is a 1.5–solvable cobordism with $\beta_2(W_0, X_L) = 2r$, we have that $H_2(W, M_L) \cong H_2(W_0, X_L)$ is a free abelian group of rank $2r$. Let

$$L(\mathbb{Z}/q) \subset H_2(W, M_L; \mathbb{Z}/q) \cong (\mathbb{Z}/q)^{2r}$$

be the $\mathbb{Z}/q$–span of $\{\pi(l_i) \otimes_{\mathbb{Z}} 1_{\mathbb{Z}/q}\}_{i=1}^r$. By the definition of a 2–Lagrangian, $\{\pi(l_1), \dots, \pi(l_r)\}$ generates a rank r–summand of $H_2(W, M_L) \cong \mathbb{Z}^{2r}$. In particular, from the universal coefficient theorem, $\dim_{\mathbb{Z}/q} L(\mathbb{Z}/q) = r$.

To apply Theorem 2.3, we fit our notation with that used in Section 2. Define $A = \mathbb{Z}/p^i \oplus \mathbb{Z}/p^j$, $G = \pi_1(W)$, $K = \pi_1(W^\varphi)$, $C_* = C_*(W, M_L; \mathbb{Z}[\pi_1(W)])$, $Q = \mathbb{C}$, $Q(\mathcal{H}) = \mathcal{K}$, $d = 1$, $\alpha \times \phi: \pi_1(W^\varphi) \to \mathbb{C}^\times \times \mathcal{H}$ and $\alpha' \times \phi': \pi_1(W) \to \mathrm{GL}(t, \mathbb{C}) \times \mathcal{H}'$. (As a $\mathbb{Z}K$–module, C_* is isomorphic to $C_*(W^\varphi, M_L^\varphi; \mathbb{Z}[\pi_1(W^\varphi)])$.) We remark that we assumed in Section 2 that $\alpha|_{\mathrm{Ker}\,\phi}$ factors through a q–group for some prime q. This is automatically satisfied for $\alpha: \pi_1(W^\varphi) \xrightarrow{\chi} \mathbb{Z}/q^l \hookrightarrow \mathbb{C}^\times$.

With this notation, apply Theorem 2.3 for the case $I = \varnothing$ (that is, $M = \overline{M} = 0$) and $n = 0, 1$ to obtain

$$\dim_{\mathcal{K}} H_n(W^\varphi, M_J^\varphi; \mathcal{K}) \leq \dim_{\mathbb{Z}/q} H_n(W, M_J; \mathbb{Z}/q) = 0$$

for $n = 0, 1$ and $J = L$ or H. By duality and universal coefficient spectral sequence, $H_i(W^\varphi, M_L^\varphi; \mathcal{K}) = 0$ for $i = 3, 4$. From this,

$$\dim_{\mathcal{K}} H_2(W^\varphi, M_L^\varphi; \mathcal{K}) = \chi^{\mathcal{K}}(W^\varphi, M_L^\varphi) = \chi^{\mathbb{Q}}(W^\varphi, M_L^\varphi) = 2rt.$$

The last equality is from $\beta_2(W^\varphi, M_L^\varphi) = 2rt$ and the fact that $(W^\varphi; M_L^\varphi, M_H^\varphi)$ is an H_1–cobordism over $\mathbb{Q}$–coefficients. These are proved in the proof of Claim 1.

Now, we apply Theorem 2.3 for the case $n = 2$, $I = \{i \mid 1 \leq i \leq r\}$ and x_i is a 2–cycle in C_* such that

$$[1_{\mathcal{K}} \otimes_{\mathbb{Z}\Gamma} \widetilde{l}_i] = [1_{\mathcal{K}} \otimes_{\mathbb{Z}K} x_i] \in H_2(\mathcal{K} \otimes_{\mathbb{Z}K} C_*) = H_2(W^\varphi, M_L^\varphi; \mathcal{K}) \quad \text{for } i = 1, \dots, r.$$

Recall $\widetilde{l}_{ij} = \widetilde{l}_i \cdot \widetilde{g}_j$, $\widetilde{g}_j \in \mathrm{Cov}(W^\Gamma | W)$ is a lifting of $g_j \in \mathrm{Cov}(W^\varphi | W)$. Since $\mathrm{Cov}(W^\varphi | W)$ can be identified with the set of cosets of K in G, by the definition in Theorem 2.3,

$$M = \text{the } \mathcal{K}\text{–span of } \{[1_{\mathcal{K}} \otimes_{\mathbb{Z}\Gamma} \widetilde{l}_{ij}] \mid 1 \leq i \leq r, 1 \leq j \leq t\} = L(\mathcal{K}).$$

Similarly, by the definition in Theorem 2.3, $\overline{M}$ is the $\mathbb{Z}/q$–span of $\{[1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} x_i]\}_{i=1}^r$. Since $\{[1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}G} x_i]\}_{i=1}^r = \{1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}} \pi(l_i)\}_{i=1}^r$,

$$\overline{M} = \text{the } \mathbb{Z}/q\text{–span of } \{1_{\mathbb{Z}/q} \otimes_{\mathbb{Z}} \pi(l_i) \mid 1 \leq i \leq r\} = L(\mathbb{Z}/q).$$

From the conclusion of Theorem 2.3 for the above case, we have the inequality

$$\dim_{\mathcal{K}} H_2(W^\varphi, M_L^\varphi; \mathcal{K}) - \dim_{\mathcal{K}} L(\mathcal{K})$$
$$\leq t \cdot (\dim_{\mathbb{Z}/q} H_2(W, M_L; \mathbb{Z}/q) - \dim_{\mathbb{Z}/q} L(\mathbb{Z}/q));$$

that is,

$$\dim_{\mathcal{K}} L(\mathcal{K}) \geq \dim_{\mathcal{K}} H_2(W, M_L; \mathcal{K}) - t \cdot (\dim_{\mathbb{Z}/q} H_2(W, M_L; \mathbb{Z}/q) - \dim_{\mathbb{Z}/q} L(\mathbb{Z}/q))$$
$$= 2rt - rt + rt = rt.$$

On the other hand, $\dim_{\mathcal{K}} L(\mathcal{K}) \leq rt$ because $L(\mathcal{K})$ is the $\mathcal{K}$–span of rt elements. So, $\dim_{\mathcal{K}} L(\mathcal{K}) = rt = \frac{1}{2} \dim_{\mathcal{K}} H_2(W^\varphi; \mathcal{K})$ and $L(\mathcal{K})$ is a Lagrangian of $\lambda_{\mathcal{K}}(W^\varphi)$. Therefore, $[\lambda_{\mathcal{K}}(W^\varphi)] = 0 \in L^0(\mathcal{K})$. □

5 Solvable cobordism and abelian invariants of links

In this section, we study the abelian invariants of links (studied in [28; 30]) in the context of Whitney tower/grope concordance using h–solvable cobordism. Throughout this section, μ is the fixed natural number. Denote $\mathbb{Z}[t_1^\pm, \ldots, t_\mu^\pm]$ by Λ_μ. The ring Λ_μ is endowed with the involution $-: t_i \mapsto t_i^{-1}$. Let S be the multiplicative set generated by $\{t_1 - 1, \ldots, t_\mu - 1\}$. Denote the localization of Λ_μ with respect to S by $\Lambda_{\mu S}$. Let $\mathcal{K}$ be the quotient field of Λ_μ.

5.1 Blanchfield form of μ–component links

Let R be a unique factorization domain with an involution $-$ and quotient field K (our case is $R = \Lambda_{\mu S}, K = \mathcal{K}$). We recall the definition of the Witt group $W(K, R, -)$.

A *linking pairing over R* is an R–module M with a sesquilinear pairing

$$b: M \times M \to K/R$$

such that for all $x, y, z \in M$ and $r \in R$,

  (1) $b(x, y + z) = b(x, y) + b(x, z)$,

  (2) $b(rx, y) = rb(x, y) = b(x, \bar{r}y)$,

  (3) $b(x, y) = \overline{b(y, x)}$.

(Here, the involution $-$ on K/R is induced from the involution on R.) We denote it by (M, b) or just b when M is clearly understood. A linking pairing (M, b) is *primitive* (*nonsingular*) if the adjoint of b,

$$\mathrm{Ad}(b): M \to \mathrm{Hom}_R(M, K/R)$$

is an injection (an R–module isomorphism), respectively. The sum of linking pairings (M, b) and (M', b') is $(M \oplus M', b \oplus b')$. A pairing (M, b) is *neutral* if there is a submodule N of M such that

$$N = N^\perp = \{m \in M \mid b(n, m) = 0 \text{ for all } n \in N\}.$$

Two pairings (M, b) and (M', b') are *Witt equivalent* if there are neutral pairings (N, c) and (N', c') such that $(M, b) \oplus (N, c) \cong (M', b') \oplus (N', c')$. Then, the set of Witt equivalence classes of linking pairings over R with an involution $-$ is an abelian group, denoted by $W(K, R, -)$.

For an R–module M, following [28, Chapter 3], we define the R–torsion submodule of M,

$$tM = \{m \in M \mid rm = 0 \text{ for some } r \neq 0 \in R\} = \mathrm{Ker}(M \to M \otimes_R K),$$

the maximal pseudonull submodule of M,

$$zM = \mathrm{Ker}(tM \to \mathrm{Ext}_R^1(\mathrm{Ext}_R^1(tM, R), R)),$$
$$\hat{t}M = tM/zM.$$

A torsion R–module M is *pseudonull* if $M_\mathfrak{p} = 0$ for every height-1 prime ideal $\mathfrak{p}$ of R where $M_\mathfrak{p}$ is the localization of M with respect to the multiplicative set $M - \mathfrak{p}$. Note that a submodule and a quotient module of a pseudonull module are pseudonull.

The following observation will be useful.

Lemma 5.1 *Suppose that*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is an exact sequence of R–modules and $A = tA$ and $D = tD$. Then,

$$tA \xrightarrow{f|} tB \xrightarrow{g|} tC \xrightarrow{h|} tD$$

is also exact. (Here, $f|$, $g|$, and $h|$ are the restriction of f, g and h to their R–torsion submodules, respectively.)

Proof Since $tA = A$ and $A \xrightarrow{f} B \xrightarrow{g} C$ is exact, $\mathrm{Im}\, f| = \mathrm{Im}\, f = \mathrm{Ker}\, g$. Then, $\mathrm{Ker}\, g| = \mathrm{Ker}\, g \cap tB = \mathrm{Im}\, f| \cap tB = \mathrm{Im}\, f|$ since $\mathrm{Im}\, f| \subset tB$.

Since $B \xrightarrow{g} C \xrightarrow{h} D$ is exact, $\mathrm{Ker}\, h| = \mathrm{Ker}\, h \cap tC = \mathrm{Im}\, g \cap tC$. Suppose that $x \in B$ and $g(x) \in tC$. Then, there exists $r \in R - \{0\}$ such that $g(rx) = rg(x) = 0$. Since $A = tA$ and $D = tD$, we have the commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\ \ g\ \ } & C \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\
B \otimes_R K & \xrightarrow[g \otimes 1_K]{\cong} & C \otimes_R K
\end{array}
$$

where i and j are natural maps. Then, $g \otimes 1_K \circ i\,(rx) = j \circ g(rx) = 0$. Since $g \otimes 1_K$ is injective, $rx \in \mathrm{Ker}\, i = tB$. There exists $s \in R - \{0\}$ such that $srx = 0$. Therefore, $g(x) \in \mathrm{Im}\, g|$ and $\mathrm{Ker}\, h| = \mathrm{Im}\, g \cap tC \subset \mathrm{Im}\, g|$. Since $h \circ g = 0$, $\mathrm{Im}\, g| \subset \mathrm{Ker}\, h|$. This completes the proof. $\qquad\square$

Let L be a μ–component link and X_L be the link exterior of L. From Alexander duality, the Hurewicz map becomes $\pi_1(X_L) \to H_1(X_L) = \mathbb{Z}^\mu$. We have the exact sequence

$$H_1(\partial X_L; \Lambda_\mu) \to H_1(X_L; \Lambda_\mu) \to H_1(X_L, \partial X_L; \Lambda_\mu) \to H_0(\partial X_L; \Lambda_\mu)$$

whose extremal terms are $\prod_{i=1}^{\mu}(t_i - 1)$–torsion (in particular, S–torsion) because $\mathbb{Z}^{\mu}$–cover of ∂X_L is a disjoint union of $S^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$. From this observation, by localizing the above sequence with respect to S, we obtain $H_1(X_L; \Lambda_{\mu}S) \cong H_1(X_L, \partial X_L; \Lambda_{\mu}S)$. It follows from the (localized) Blanchfield duality [1] (as in [28, page 36]) that we have the following primitive linking pairing:

$$b_L \colon \hat{t}H_1(X_L; \Lambda_{\mu}S) \times \hat{t}H_1(X_L; \Lambda_{\mu}S) \to \mathcal{K}/\Lambda_{\mu}S.$$

Here, to define b_L, we need the fact that $\mathcal{K}/\Lambda_{\mu}S$ contains no nontrivial pseudonull submodule, [28, Theorem 3.9(2)].

In this setting, Hillman [28, Theorem 2.4] proved $[b_L] \in W(\mathcal{K}, \Lambda_{\mu}S, -)$ is a concordance invariant of L. Here is our theorem which generalizes [28, Theorem 2.4].

Theorem B *Suppose L_0 and L_1 are μ–component links. If two link exteriors X_{L_0} and X_{L_1} are 1–solvable cobordant, then $[b_{L_0}] = [b_{L_1}] \in W(\mathcal{K}, \Lambda_{\mu}S, -)$. In particular, the conclusion holds if L_0 and L_1 are height-3 Whitney tower/grope concordant.*

Proof We proceed similarly to Hillman's proof of [28, Theorem 2.4], using a 1–solvable cobordism instead of a concordance exterior. For this purpose, we need the following fact from [8]. Suppose W is a 1–solvable cobordism between X_{L_0} and X_{L_1}.

Lemma 5.2 (A special case of [8, Corollary 4.14]) *The sequence*

$$tH_2(W, \partial W; \Lambda_{\mu}) \xrightarrow{\partial} tH_1(\partial W; \Lambda_{\mu}) \xrightarrow{i_*} tH_1(W; \Lambda_{\mu})$$

is exact.

In fact, Lemma 5.2 is obtained from [8, Corollary 4.14] by setting $n = 1$, $G = \mathbb{Z}^{\mu}$, $\phi \colon \pi_1(W) \to H_1(W) = \mathbb{Z}^{\mu}$, and $R = \mathbb{Z}$.

Using Lemma 5.2 in place of the exact sequence in [28, page 39], we can carry out the arguments in [28, page 39–40] for a 1–solvable cobordism W. For convenience of the readers, we present details below.

Note that

$$\partial W = X_{L_0} \cup \mu(S^1 \times S^1 \times I) \cup -X_{L_1}$$

and $\mathbb{Z}^{\mu} = H_1(X_{L_i}) \xrightarrow{\text{inc}_*} H_1(W)$ is an isomorphism for $i = 0, 1$.

By the ($\Lambda_{\mu}S$–coefficient) Mayer–Vietoris sequence of the triple $(\partial W, X_{L_0}, X_{L_1})$,

$$H_1(\partial W; \Lambda_{\mu}S) \cong H_1(X_{L_0}; \Lambda_{\mu}S) \oplus H_1(X_{L_1}; \Lambda_{\mu}S),$$

since $H_i(\mu(S^1 \times S^1 \times I); \Lambda_\mu)$ is S–torsion for $i = 0, 1$. From this, the (localized) Blanchfield form

$$b_{\partial W} \colon \hat{t} H_1(\partial W; \Lambda_{\mu S}) \times \hat{t} H_1(\partial W; \Lambda_{\mu S}) \to \mathcal{K}/\Lambda_{\mu S}$$

is the direct sum $b_{L_0} \oplus (-b_{L_1})$. If we show $b_{\partial W}$ is neutral, then $[b_{L_0}] = [b_{L_1}] \in W(\mathcal{K}, \Lambda_{\mu S}, -)$ because $b_{L_0} \oplus (b_{L_1} \oplus (-b_{L_1})) = b_{L_1} \oplus b_{\partial W}$. Therefore, it suffices to find a submodule P_0 of $\hat{t} H_1(\partial W; \Lambda_{\mu S})$ such that $P_0 = P_0^\perp$.

Let $I_{\partial W}$ and I_W be $\Lambda_{\mu S}$–coefficient intersection forms of ∂W and W, respectively. We have Blanchfield form,

$$b_W \colon \hat{t} H_1(W; \Lambda_{\mu S}) \times \hat{t} H_2(W, \partial W; \Lambda_{\mu S}) \to \mathcal{K}/\Lambda_{\mu S}.$$

Let $\pi \colon t H_1(\partial W; \Lambda_{\mu S}) \to \hat{t} H_1(\partial W; \Lambda_{\mu S})$ be the quotient map. Let

$$i_{\#} \colon C_*(\partial W; \Lambda_{\mu S}) \to C_*(W; \Lambda_{\mu S}) \quad \text{and} \quad i_* \colon t H_1(\partial W; \Lambda_{\mu S}) \to t H_1(W; \Lambda_{\mu S})$$

be the inclusion induced maps. Let

$$P = \mathrm{Im}(t H_2(W, \partial W; \Lambda_{\mu S}) \xrightarrow{\partial} t H_1(\partial W; \Lambda_{\mu S}) \xrightarrow{\pi} \hat{t} H_1(\partial W; \Lambda_{\mu S})).$$

By [Lemma 5.2](#), $P = \pi(\mathrm{Ker}\, i_*)$. We will prove that $P^\perp = P^{\perp\perp}$.

First, we prove $P \leq P^\perp$. Choose relative 2–cycles Q and R in $C_2(W, \partial W; \Lambda_{\mu S})$ representing the classes in $t H_2(W, \partial W; \Lambda_{\mu S})$. Denote the boundaries of Q and R by $q, r \in C_1(\partial W; \Lambda_{\mu S})$, respectively. The corresponding classes $[q], [r]$ in $\hat{t} H_1(\partial W; \Lambda_{\mu S})$ are actually in P. There exists $a \in \Lambda_{\mu S} - \{0\}$ such that aq is the boundary of a 2–chain $u \in C_2(\partial W; \Lambda_{\mu S})$. Then

$$b_{\partial W}([q], [r]) = a^{-1} I_{\partial W}(u, r) = -a^{-1} I_W(i_{\#}(u), R) = -b_W([i_{\#}(q)], [R]) \pmod{\Lambda_{\mu S}}.$$

Since q is the boundary of the relative 2–cycle Q, $[i_{\#}(q)] = 0 \in \hat{t} H_1(W; \Lambda_{\mu S})$. Therefore

$$b_{\partial W}([q], [r]) = 0 \quad \text{for all } [q], [r] \in P.$$

This shows that $P \leq P^\perp$.

Suppose that $x \in C_1(\partial W; \Lambda_{\mu S})$ represents a class in $t H_1(\partial W; \Lambda_{\mu S})$ and $[x] \in P^\perp$. That is,

$$b_{\partial W}([x], [y]) = 0$$

for all y which is the boundary of a relative 2–cycle $Y \in C_2(W, \partial W; \Lambda_{\mu S})$. Therefore

$$b_W([i_{\#}(x)], [Y]) = -b_{\partial W}([x], [y]) = 0 \quad \text{for all } [Y] \in \hat{t} H_2(W, \partial W; \Lambda_{\mu S}).$$

By Blanchfield duality for $(W, \partial W)$, the adjoint of b_W,

$$\text{Ad}(b_W)\colon \hat{t}H_1(W; \Lambda_{\mu S}) \to \text{Hom}_{\Lambda_{\mu S}}(\hat{t}H_2(W, \partial W; \Lambda_{\mu S}), K/\Lambda_{\mu S})$$

is injective. Therefore, $[i_\#(x)] = 0 \in \hat{t}H_1(W; \Lambda_{\mu S})$ or $i_\#(x)$ represents a homology class in $zH_1(W; \Lambda_{\mu S})$. The above argument shows $i_*(\pi^{-1}(P^\perp))$ is pseudonull as a submodule of $zH_1(W; \Lambda_{\mu S})$. Since $P = \pi(\text{Ker}\, i_*) \leq P^\perp$, $\text{Ker}\, i_* \subset \pi^{-1}(P^\perp)$. Thus, $\pi^{-1}(P^\perp)/\text{Ker}\, i_* \cong i_*(\pi^{-1}(P^\perp))$ is pseudonull, and therefore, $P^\perp/P = \pi(\pi^{-1}(P^\perp))/\pi(\text{Ker}\, i_*)$ is also pseudonull. Since $\mathcal{K}/\Lambda_{\mu S}$ has no nontrivial pseudonull submodule and $P^\perp/P$ is pseudonull,

$$P^\perp = \{x \mid b_{\partial W}(x, y) = 0 \text{ for all } y \in P\} = \{x \mid b_{\partial W}(x, y) \text{ for all } y \in P^\perp\} = P^{\perp\perp}.$$

This completes the proof. $\square$

5.2 Multivariable Alexander polynomial of links

In this subsection, we prove Theorem C which generalizes [30, Theorems A, B] concerning the Fox–Milnor condition for the Alexander polynomial of links.

First, we recall some definitions of [30]. Since Λ_μ is Nötherian, for a finitely generated Λ_μ–module M, we can choose a presentation matrix P of M from an exact sequence

$$\Lambda_\mu^m \xrightarrow{P} \Lambda_\mu^n \to M \to 0.$$

Moreover, for all k, one can choose a $m \times n$ presentation matrix P with $n > k$ and $m \geq n - k$. In this situation, define *the kth Alexander polynomial of M*, denoted by $\Delta_k(M)$, to be the greatest common divisor of the size $(n-k) \times (n-k)$ minors of P. (It is well known that $\Delta_k(M)$ is well defined up to a unit of Λ_μ which is proved by Crowell and Fox in [20].)

Remark 5.3 (1) From [1, Theorem 4.10], if $d = \text{rank}_{\Lambda_\mu} M$, then $\Delta_d(M) = \Delta_0(tM)$.

(2) From [30, Lemma 2.4], if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of Λ_μ–torsion modules, then $\Delta_0(M) = \Delta_0(M')\Delta_0(M'')$. More generally, if

$$0 \to M_1 \to M_2 \to \cdots \to M_r \to 0$$

is exact, then $\prod_{i=1}^r \Delta_0(M_i)^{(-1)^r} = 1$.

(3) Suppose that $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4$ is an exact sequence of Λ_μ–torsion modules. Then $0 \to \text{Im}\, f_1 \to M_2 \to M_3 \to \text{Im}\, f_3 \to 0$ is exact. By (2),

$$\Delta_0(M_2)\Delta_0(\text{Im}\, f_3) = \Delta_0(M_3)\Delta_0(\text{Im}\, f_1).$$

Here $\Delta_0(\text{Im}\, f_1)$ and $\Delta_0(\text{Im}\, f_3)$ divide $\Delta_0(M_1)$ and $\Delta_0(M_4)$, respectively.

Recall that L is a μ–component link in S^3 and the meridian map is $\pi_1(X_L) \to \mathbb{Z}^\mu$. We define the torsion Alexander polynomial of L by $\Delta_L^T := \Delta_0(tH_1(X_L; \Lambda_\mu))$. Now we state our theorem.

Theorem C *Suppose L_0 and L_1 are μ–component links. If two link exteriors X_{L_0} and X_{L_1} are 1–solvable cobordant, then:*

(1) $\mathrm{rank}_{\Lambda_\mu} H_1(X_{L_0}.\Lambda_\mu) = \mathrm{rank}_{\Lambda_\mu} H_1(X_{L_1}; \Lambda_\mu)$.

(2) $$\Delta_{L_0}^T f_0 \bar{f}_0 \doteq \Delta_{L_1}^T f_1 \bar{f}_1$$

  for some $f_i(t_1, \dots, t_\mu) \in \Lambda_\mu$, with $i = 0, 1$ and with $|f_i(1, \dots, 1)| = 1$.

In particular, the conclusion holds if L_0 and L_1 are height-3 Whitney tower/grope concordant.

To prove Theorem C, we need to prove the following generalization of [30, Lemma 2.1].

Lemma 5.4 *Let X be a finite connected CW–complex with an epimorphism*

$$\gamma \colon \pi_1(X) \to \mathbb{Z}^\mu.$$

Let X_0 be a subcomplex of X. For some fixed k, if $H_k(X, X_0; \mathbb{Z}) = \mathbb{Z}^l$ and

$$\mathrm{rank}_{\Lambda_\mu} H_k(X, X_0; \Lambda_\mu) = l$$

then the lth Alexander polynomial $A = \Delta_l(H_k(X, X_0; \Lambda_\mu)) = \Delta_0(tH_k(X, X_0; \Lambda_\mu))$ satisfies $|A(1, \dots, 1)| = 1$.

Remark For the $l = 0$ case (see [30, Lemma 2.1]), we only need to assume that $H_k(X, X_0; \mathbb{Z}) = 0$ because from our proof, we can deduce

$$\mathrm{rank}_{\Lambda_\mu} H_k(X, X_0; \Lambda_\mu) \le 0.$$

In this sense, Lemma 5.4 is a generalization of [30, Lemma 2.1].

Proof of Lemma 5.4 Since X_0 is a subcomplex of X, for all q, we fix a basis for the qth (cellular) chain complex $C_q(X, X_0; \mathbb{Z}) \cong \mathbb{Z}^{s_q}$. By lifting each element in the chosen bases, we also fix a Λ_μ–basis for the $C_q(X, X_0; \Lambda_\mu)$ for all q. With these chosen bases, we can write $\partial_q \colon C_q(X, X_0; \Lambda_\mu) \to C_{q-1}(X, X_0; \Lambda_\mu)$ as a matrix (α_{ij}^q), $\alpha_{ij}^q \in \Lambda_\mu$.

With respect to the chosen basis of $C_*(X, X_0; \mathbb{Z})$, $\partial_q \colon C_q(X, X_0; \mathbb{Z}) \to C_{q-1}(X, X_0; \mathbb{Z})$ is represented by the integral matrix $(\alpha_{ij}^q(1, \dots, 1))$. Let $\tilde{r}_q = \mathrm{rank}(\alpha_{ij}^q)$, $r_q = \mathrm{rank}(\alpha_{ij}^q(1, \dots, 1))$. Then $r_q \le \tilde{r}_q$. Since $H_k(X, X_0; \mathbb{Z}) = \mathbb{Z}^l$,

$$l = \mathrm{rank}_{\mathbb{Z}} \mathrm{Ker}\, \partial_k - \mathrm{rank}_{\mathbb{Z}} \mathrm{Im}\, \partial_{k+1} = s_k - r_k - r_{k+1}.$$

Similarly, from $\operatorname{rank}_{\Lambda_\mu} H_k(X, X_0; \Lambda_\mu) = l$,

$$l = s_k - \tilde{r}_k - \tilde{r}_{k+1}.$$

Since $r_q \leq \tilde{r}_q$ for all q,

$$l = s_k - \tilde{r}_k - \tilde{r}_{k+1} \leq s_k - r_k - r_{k+1} = l$$

which implies that $r_k = \tilde{r}_k, r_{k+1} = \tilde{r}_{k+1}$.

Since $\operatorname{Coker} \partial_{k+1} = C_k(X, X_0; \Lambda_\mu) / \operatorname{Im} \partial_{k+1}$ and $\operatorname{Im} \partial_k \cong C_k(X, X_0; \Lambda_\mu) / \operatorname{Ker} \partial_k$, we have the short exact sequence

$$0 \to H_k(X, X_0; \Lambda_\mu) \to \operatorname{Coker} \partial_{k+1} \to \operatorname{Im} \partial_k \to 0.$$

As a submodule of a free module, $\operatorname{Im} \partial_k$ is a Λ_μ–torsion free module of rank $\tilde{r}_k = r_k$. Then $tH_k(X, X_0; \Lambda_\mu) = t \operatorname{Coker} \partial_{k+1}$ and $\dim_{\mathcal{K}} \operatorname{Coker} \partial_{k+1} \otimes_{\Lambda_\mu} \mathcal{K} = l + r_k$:

$$\Delta_l(H_k(X, X_0; \Lambda_\mu)) = \Delta_0(tH_k(X, X_0; \Lambda_\mu)) = \Delta_0(t \operatorname{Coker} \partial_{k+1})$$
$$= \Delta_{l+r_k}(\operatorname{Coker} \partial_{k+1}).$$

The first and last equalities follow from Remark 5.3(1). Similarly, we have the short exact sequence

$$0 \to H_k(X, X_0; \mathbb{Z}) \to \operatorname{Coker} \partial_{k+1}^{\mathbb{Z}} \to \operatorname{Im} \partial_k^{\mathbb{Z}} \to 0.$$

(Here, to avoid confusion, we denote the differential on $C_*(X, X_0; \mathbb{Z})$ by $\partial_*^{\mathbb{Z}}$.) As a subgroup of a finitely generated free abelian group, $\operatorname{Im} \partial_k^{\mathbb{Z}}$ is a free abelian group of rank r_k. Therefore

$$\operatorname{Coker} \partial_{k+1}^{\mathbb{Z}} = H_k(X, X_0; \mathbb{Z}) \oplus \mathbb{Z}^{r_k} = \mathbb{Z}^{l+r_k}.$$

(Here we used the assumption that $H_k(X, X_0; \mathbb{Z}) = \mathbb{Z}^l$.) Note that the matrices (α_{ij}^{k+1}) and $(\alpha_{ij}^{k+1}(1, \ldots, 1))$ are presentation matrices of $\operatorname{Coker} \partial_{k+1}$ and $\operatorname{Coker} \partial_{k+1}^{\mathbb{Z}}$, respectively. Therefore

$$|\Delta_l(H_k(X, X_0; \Lambda_\mu))(1, \ldots, 1)| = |\Delta_{l+r_k}(\operatorname{Coker} \partial_{k+1})(1, \ldots, 1)| = 1.$$

This completes the proof.   □

Proof of Theorem C (1) Let W be a 1–solvable cobordism between X_{L_0} and X_{L_1}. In particular, the inclusion induces $\mathbb{Z}^\mu = H_1(X_{L_0}) \cong H_1(W)$ and $H_1(W, X_{L_0}) = H_1(W, X_{L_1}) = 0$. By Poincaré duality and the universal coefficient theorem,

$$H_2(W, X_{L_0}) \cong H^2(W, X_{L_1}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_2(W, X_{L_1}), \mathbb{Z}) = \mathbb{Z}^{2r}.$$

(Since W is a 1–solvable cobordism between X_{L_0} and X_{L_1}, $\mathrm{rank}_{\mathbb{Z}}\, H_2(W, X_{L_1})$ is even.) Let $C_* = C_*(W, X_{L_0}; \Lambda_\mu)$. Then

$$H_i(C_* \otimes_{\Lambda_\mu} \mathbb{Z}) = H_i(W, X_{L_0}; \mathbb{Z}) = 0 \quad \text{for } i = 0, 1.$$

Since $\Lambda_\mu = \mathbb{Z}[\mathbb{Z}^\mu]$ and $\mathbb{Z}^\mu$ is a poly-torsion-free-abelian group, then by [15, Proposition 2.10],

$$H_i(C_* \otimes_{\Lambda_\mu} \mathcal{K}) = H_i(W, X_{L_0}; \Lambda_\mu) \otimes_{\Lambda_\mu} \mathcal{K} = 0 \quad \text{for } i = 0, 1.$$

Similarly, $H_i(W, X_{L_1}; \Lambda_\mu) \otimes_{\Lambda_\mu} \mathcal{K} = 0$ for $i = 0, 1$. From duality and the universal coefficient theorem, $H_i(W, X_{L_0}; \Lambda_\mu) \otimes_{\Lambda_\mu} \mathcal{K} = 0$ for $i = 3, 4$. So

$$\mathrm{rank}_{\Lambda_\mu}\, H_2(W, X_{L_i}; \Lambda_\mu) = \chi(C_*) = \chi(C_*(W, X_{L_i}; \mathbb{Z})) = \mathrm{rank}_{\mathbb{Z}}\, H_2(W, X_{L_i}; \mathbb{Z}) = 2r$$

for $i = 0, 1$. As in Lemma 5.2, the existence of a 1–Lagrangian and 1–duals implies that the following is exact for $i = 0, 1$:

$$tH_2(W, X_{L_i}; \Lambda_\mu) \to H_1(X_{L_i}; \Lambda_\mu) \to H_1(W; \Lambda_\mu) \to tH_1(W, X_{L_i}; \Lambda_\mu).$$

(Note that $H_1(W, X_{L_i}; \Lambda_\mu) = tH_1(W, X_{L_i}; \Lambda_\mu)$ for $i = 0, 1$.) In particular, (1) is proved because

$$\mathrm{rank}_{\Lambda_\mu}\, H_1(X_{L_0}; \Lambda_\mu) = \mathrm{rank}_{\Lambda_\mu}\, H_1(W; \Lambda_\mu) = \mathrm{rank}_{\Lambda_\mu}\, H_1(X_{L_1}; \Lambda_\mu).$$

(2) For $i = 0, 1$, applying Lemma 5.1 to the exact sequence

$$tH_2(W, X_{L_i}; \Lambda_\mu) \to H_1(X_{L_i}; \Lambda_\mu) \to H_1(W; \Lambda_\mu) \to tH_1(W, X_{L_i}; \Lambda_\mu),$$

the following is also exact:

$$tH_2(W, X_{L_i}; \Lambda_\mu) \to tH_1(X_{L_i}; \Lambda_\mu) \to tH_1(W; \Lambda_\mu) \to tH_1(W, X_{L_i}; \Lambda_\mu).$$

Denote the 0th Alexander polynomial of these modules and $tH_1(\partial W; \Lambda_\mu)$ by

$$\Delta_{2,i}, \quad \Delta_{L_i}^T, \quad \Delta_W, \quad \Delta_{1,i} \quad \text{and} \quad \Delta_{\partial W}$$

respectively.

In (1), we proved $\mathrm{rank}_{\Lambda_\mu}\, H_1(W, X_{L_i}; \Lambda_\mu) = 0$ using the fact that $H_1(W, X_{L_i}) = 0$. Using an Euler characteristic argument, we also proved $\mathrm{rank}_{\Lambda_\mu}\, H_2(W, X_{L_i}; \Lambda_\mu) = 2r$ where $H_2(W, X_{L_i}) = \mathbb{Z}^{2r}$. Applying Lemma 5.4 to $(X, X_0) = (W, X_{L_i})$ for the cases $(k, l) = (2, 2r)$ and $(1, 0)$, we obtain $|\Delta_{2,i}(1, \ldots, 1)| = |\Delta_{1,i}(1, \ldots, 1)| = 1$. Using Remark 5.3(3), there exist g_i and g_i' in Λ_μ which divide $\Delta_{2,i}$ and $\Delta_{1,i}$, respectively and they satisfy $\Delta_{L_i}^T g_i \doteq \Delta_W g_i'$. Since $|g_i(1, \ldots, 1)|$ and $|g_i'(1, \ldots, 1)|$ are nonnegative integers which divide 1, they are 1. Let $g = g_0 g_1'$ and $g' = g_0' g_1$. Then, $\Delta_{L_0}^T g \doteq \Delta_W g_0' g_1' \doteq \Delta_{L_1}^T g'$ with $|g(1, \ldots, 1)| = |g'(1, \ldots, 1)| = 1$.

Since Λ_μ is a unique factorization domain, we can split $\Delta^T_{L_i} = u_i v_i$ and $\Delta_{\partial W} = uv$ uniquely (up to units of Λ_μ) so that v_0, v_1, v consist of all irreducible factors $f \in \Lambda_\mu$ with $|f(1,\dots,1)| \neq 1$ in $\Delta^T_{L_0}, \Delta^T_{L_1}, \Delta_{\partial W}$. Now, $\Delta^T_{L_0} g \doteq \Delta^T_{L_1} g'$ becomes $u_0 v_0 g \doteq u_1 v_1 g'$. Since g, g', u_0 and u_1 are factors with $|f(1,\dots,1)| = 1$, $v_0 \doteq v_1$.

From the Mayer–Vietoris sequence,

$$H_1(\partial X_{L_0}; \Lambda_\mu) \to H_1(X_{L_0}; \Lambda_\mu) \oplus H_1(X_{L_1}; \Lambda_\mu) \to H_1(\partial W; \Lambda_\mu) \to H_0(\partial X_{L_0}; \Lambda_\mu)$$

is exact. Since the extremal terms are $\prod_{i=1}^\mu (t_i - 1)$–torsion, we can apply Lemma 5.1 to get the following exact sequence:

$$tH_1(\partial X_{L_0}; \Lambda_\mu) \to tH_1(X_{L_0}; \Lambda_\mu) \oplus tH_1(X_{L_1}; \Lambda_\mu) \to tH_1(\partial W; \Lambda_\mu)$$
$$\to tH_0(\partial X_{L_0}; \Lambda_\mu).$$

By Remark 5.3(3), $\Delta_{\partial W} \lambda \doteq \Delta^T_{L_0} \Delta^T_{L_1} \lambda'$ for some factors λ, λ' of $\prod_{i=1}^\mu (t_i - 1)$. By the reciprocity of Blanchfield pairing [1],

$$\Delta^T_{L_i} \doteq \overline{\Delta^T_{L_i}} \quad \text{for } i = 0, 1.$$

In particular, $u_0 = \bar{u}_0$. Now, we have

$$u \doteq u_0 u_1 \doteq \bar{u}_0 u_1.$$

Recall that a linking pairing (M, b) is neutral if M has a submodule N such that $N = N^\perp$. By Theorem B, we proved that the Blanchfield form of ∂W is neutral. This implies that $\Delta_{\partial W} = h\bar{h}$ for some $h \in \Lambda_\mu$ by [28, Lemma 3.26]. In particular,

$$u \doteq f\bar{f} \quad \text{for some } f \in \Lambda_\mu \text{ with } |f(1,\dots,1)| = 1.$$

Combining all these observations,

$$\Delta^T_{L_0} f\bar{f} \doteq u_0 v_0 u = u_0 \bar{u}_0 u_1 v_0 \doteq u_0 \bar{u}_0 u_1 v_1 \doteq \Delta^T_{L_1} u_0 \bar{u}_0.$$

Here, f and u_0 satisfy the conditions $|f(1,\dots,1)| = 1$, $|u_0(1,\dots,1)| = 1$. This completes the proof. $\qquad\square$

Remark It should be noted that Theorem C is not a direct consequence of Theorem B. From Theorem B without Lemma 5.4, one may deduce that if X_{L_0} and X_{L_1} are 1–solvable cobordant, then

(1) $\operatorname{rank}_{\Lambda_\mu} H_1(X_{L_0}; \Lambda_\mu) = \operatorname{rank}_{\Lambda_\mu} H_1(X_{L_1}; \Lambda_\mu)$,

(2) $\Delta^T_{L_0} f_0 \bar{f}_0 \doteq \Delta^T_{L_1} f_1 \bar{f}_1$

for some $f_0, f_1 \in \Lambda_\mu S - \{0\}$. Lemma 5.4 is crucial to obtain the stronger conclusion that we can choose $f_0, f_1 \in \Lambda_\mu$ such that $|f_0(1,\dots,1)| = |f_1(1,\dots,1)| = 1$.

Finally, we mention what can be deduced from Theorems B and C for the special case of 2–component links with linking number 1. Note that by the work of Levine [33], the Blanchfield form (without localization) $b_L \colon tH_1(X_L; \Lambda_2) \times tH_1(X_L; \Lambda_2) \to \mathcal{K}/\Lambda_2$ is nonsingular.

Corollary D *Suppose L is a 2–component link with linking number 1. If X_L and X_H are 1–solvable cobordant, then:*

(1) $[b_L] = 0 \in W(\mathcal{K}, \Lambda_2, -)$.

(2) $\beta(L) = 0$.

(3) $\Delta_0(L) \doteq f\,\overline{f}$ *for some $f \in \Lambda_2$ such that $|f(1,1)| = 1$.*

In particular, the conclusion holds if L and H are height-3 Whitney tower/grope concordant.

Proof Let L be a 2–component link with linking number 1. Assume X_L and X_H are 1–solvable cobordant. Since $X_H = S^1 \times S^1 \times I$ and the $\mathbb{Z} \oplus \mathbb{Z}$ cover of X_H is $\mathbb{R} \times \mathbb{R} \times I$, $[b_H] = 0$, $\beta(H) = 0$ and $\Delta_0(H) = 1$. This shows (1) and (2). With the notation in the proof of Theorem C (applied to $L_0 = H$ and $L_1 = L$), $u_0 = 1$ and

$$\Delta_0(H) f\,\overline{f} \doteq \Delta_0(L) u_0 \overline{u}_0$$

for some $f \in \Lambda_2$ such that $|f(1,1)| = 1$. Since $\Delta_0(H) = 1$ and $u_0 = 1$, $\Delta_0(L) = f\,\overline{f}$ for some $f \in \Lambda_2$ such that $|f(1,1)| = 1$. This completes the proof of (3). □

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Department of Mathematics, Pohang University of Science and Technology
Gyungbuk 790-784, South Korea

kminhoon@gmail.com