Algebraic & Geometric Topology 15 (2015) 2755–2785

Reidemeister torsion, peripheral complex and Alexander polynomials of hypersurface complements

YONGQIANG LIU
LAURENŢIU MAXIM

Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial that is transversal (or regular) at infinity. Let $\mathcal{U} = \mathbb{C}^{n+1} \setminus f^{-1}(0)$ be the corresponding affine hypersurface complement. By using the peripheral complex associated to f, we give several estimates for the (infinite cyclic) Alexander polynomials of $\mathcal{U}$ induced by f, and we describe the error terms for such estimates. The obtained polynomial identities can be further refined by using the Reidemeister torsion, generalizing a similar formula proved by Cogolludo and Florens in the case of plane curves. We also show that the above-mentioned peripheral complex underlies an algebraic mixed Hodge module. This fact allows us to construct mixed Hodge structures on the Alexander modules of the boundary manifold of $\mathcal{U}$.

32S25; 32S55, 32S60

1 Introduction

1.1 Background

Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial map, and set $F_0 = f^{-1}(0)$ and $\mathcal{U} = \mathbb{C}^{n+1} \setminus F_0$. The topological study of the hypersurface F_0 and of its complement $\mathcal{U}$ is a classical subject going back to Zariski. Libgober [19; 20; 21] introduced and studied Alexander-type invariants associated to the hypersurface complement $\mathcal{U}$, as induced by f. For F_0 a plane curve [19; 20], or a hypersurface with only isolated singularities, including at infinity [21], Libgober obtained a divisibility result, asserting that the only (possibly) non-trivial global Alexander polynomial of $\mathcal{U}$ divides the product of the local Alexander polynomials associated with each singular point (including at infinity).

More recently, the second author [25] used the intersection homology theory to provide generalizations of these results to the case of hypersurfaces with arbitrary singularities, provided that the defining equation f is transversal at infinity (ie the hyperplane at infinity is generic with respect to the projective completion of F_0). In particular, he proved a general divisibility result (see [25, Theorem 4.2]) relating the global and local Alexander polynomials. Furthermore, Dimca and Libgober [10] showed that for

Published: 10 December 2015 DOI: 10.2140/agt.2015.15.2755

a polynomial transversal at infinity there exist canonical mixed Hodge structures on the (torsion) Alexander invariants of the hypersurface complement. For more results related to Alexander-type invariants for complements of hypersurfaces with non-isolated singularities, see Dimca and Maxim [11] and Liu [23].

A different approach to the study of Alexander polynomials relies on the use of Reidemeister torsion. Milnor [26; 27] showed that the Alexander polynomial of a link coincides with the Reidemeister torsion of the link complement. This approach was further developed by Turaev [33] for the classical Alexander polynomial, and by Lin [22] and Wada [34] for twisted Alexander polynomials. Kirk and Livingston [18] extended this theory to any finite CW complex; in particular, they generalized Milnor's duality theorem for Reidemeister torsion.

Cogolludo and Florens [5] studied twisted Alexander polynomials of plane algebraic curves by using the Reidemeister torsion, and obtained a polynomial identity involving global and local twisted Alexander polynomials. Specializing their result to the classical case (of the trivial representation), one obtains a geometric interpretation of Libgober's divisibility result.

Let us assume from now on that f is transversal at infinity. One of the goals of this paper is to provide a generalization to hypersurfaces with non-isolated singularities of the Cogolludo–Florens identity for Alexander polynomials (see [5, Corollary 5.8]). Our main tool will be the Cappell–Shaneson peripheral complex [4] associated to f. In more detail, we give a new description of the peripheral complex, from which we deduce several error estimates for the Alexander polynomials of the complement. Moreover, by exploiting the relation between the Alexander polynomials and Reidemeister torsion (see Kirk and Livingston [18, Theorem 3.4]), we show how these estimates can be further refined by using the intersection form appearing in the duality for Reidemeister torsion.

Our new description of the peripheral complex can also be used to show that the peripheral complex underlies an algebraic mixed Hodge module. In particular, after explaining the relation between the peripheral complex and the boundary manifold of the complement $\mathcal{U}$, we obtain mixed Hodge structures (MHS) on the Alexander modules of this boundary manifold.

1.2 Main results

Unless otherwise specified, all homology and cohomology groups will be assumed to have $\mathbb{Q}$–coefficients.

Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a degree-d polynomial. We say that f is *transversal* (or *regular*) *at infinity* if f is reduced and the projective closure V of F_0 in $\mathbb{CP}^{n+1}$ is transversal in the stratified sense to the hyperplane at infinity $H = \mathbb{CP}^{n+1} \setminus \mathbb{C}^{n+1}$. Consider the infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$ corresponding to the kernel of the linking number homomorphism

$$f_*\colon \pi_1(\mathcal{U}) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$$

induced by f. Then, under the deck group action, each homology group $H_i(\mathcal{U}^c)$ becomes a $\Gamma := \mathbb{Q}[t, t^{-1}]$–module, called the i^{th} *Alexander module* of the hypersurface complement $\mathcal{U}$. For f transversal at infinity, Maxim [25, Theorem 3.6] showed that $H_i(\mathcal{U}^c)$ is a torsion Γ–module for $i \le n$. We denote by $\delta_i(t)$ the corresponding (global) *Alexander polynomial*.

Let N be an open regular neighbourhood of $V \cup H$ in $\mathbb{CP}^{n+1}$ (see Durfee [14]). Set $\mathcal{U}_0 = \mathbb{CP}^{n+1} \setminus N$. Then $\mathcal{U}_0$ is homotopy equivalent to $\mathcal{U}$, and the boundary $\partial\mathcal{U}_0$ is a $(2n+1)$–dimensional real manifold, called the *boundary manifold of* $\mathcal{U}$. The inclusion $\partial\mathcal{U}_0 \hookrightarrow \mathcal{U}_0$ is an n–homotopy equivalence (see Dimca [8, Proposition (5.2.31)]). Moreover, we have an epimorphism

$$\rho\colon \pi_1(\partial\mathcal{U}_0) \twoheadrightarrow \pi_1(\mathcal{U}_0) = \pi_1(\mathcal{U}) \xrightarrow{f_*} \pi_1(\mathbb{C}^*) = \mathbb{Z},$$

which defines the infinite cyclic cover $(\partial\mathcal{U}_0)^c$ of $\partial\mathcal{U}_0$. The related intersection form $\phi^\rho \in \mathbb{Q}(t)$ for the pair $(\mathcal{U}_0, \partial\mathcal{U}_0)$ is defined on $H_{n+1}(\mathcal{U}_0^c)$; see [18] or Section 5.3 below.

The *peripheral complex* $\mathcal{R}^\bullet$ associated to f (see [4; 25] or Definition 2.5 below) is a torsion Γ–module sheaf complex, which plays a key role in Maxim's generalizations of Libgober's results to the case of hypersurfaces with non-isolated singularities. Our first result is the following (see Proposition 6.1 and Corollaries 3.3 and 6.2):

Theorem 1.1 *Assume that the polynomial $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is transversal at infinity. Then:*

(a) *There are Γ–module isomorphisms*

$$H_i((\partial\mathcal{U}_0)^c) \cong H^{2n+1-i}(\mathbb{CP}^{n+1}; \mathcal{R}^\bullet) \quad \text{for all } i$$

and, in particular, $H_i((\partial\mathcal{U}_0)^c)$ is a torsion Γ–module. Moreover, the zeros of the Alexander polynomial associated to $H_i((\partial\mathcal{U}_0)^c)$ are roots of unity for all i, and have order d except for $i = n$. Finally, $H_i((\partial\mathcal{U}_0)^c)$ is a semi-simple Γ–module for $i \neq n$.

(b) *The peripheral complex $\mathcal{R}^\bullet$ (when regarded as a complex of $\mathbb{Q}$–vector sheaves) is a (shifted) mixed Hodge module, hence the vector spaces $H_i((\partial\mathcal{U}_0)^c)$ inherit mixed Hodge structures from $\mathcal{R}^\bullet$ for all i. Moreover, for $i \neq n$, the mixed Hodge structure on $H_i((\partial\mathcal{U}_0)^c)$ is compatible with the Γ–action, ie $t\colon H_i((\partial\mathcal{U}_0)^c) \to H_i((\partial\mathcal{U}_0)^c)$ is a mixed Hodge structure homomorphism.*

Let $h = f_d$ be the top-degree part of f, with corresponding Milnor fibre $F_h = \{h = 1\}$, and denote by $h_i(t)$ the Alexander polynomial (or order) associated to the torsion Γ–module $H_i(F_h)$. On the other hand, let $\psi_f \mathbb{Q}_{\mathbb{C}^{n+1}}$ be the nearby cycle complex associated to f, and denote by $\psi_i(t)$ the corresponding Alexander polynomial of $H_c^{2n+1-i}(F_0, \psi_f \mathbb{Q}_{\mathbb{C}^{n+1}})$. Liu [23, Theorem 1.1] studied the relation between the polynomials $\psi_i(t)$ and the Alexander polynomials $\delta_i(t)$ of the hypersurface complement $\mathcal{U}$. In particular, he showed that $\psi_i(t) = \delta_i(t)$ for $i < n$, and $\delta_n(t)$ divides $\psi_n(t)$. With the above notations, we have the following result, which establishes a more precise relationship between the polynomials $\delta_n(t)$ and $\psi_n(t)$ (see Theorem 7.1):

Theorem 1.2 *Assume that the degree-d polynomial $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is transversal at infinity. Let ϕ^ρ be the intersection form for $(\mathcal{U}_0, \partial\mathcal{U}_0)$ induced by ρ. Then*

$$h_n(t) \cdot \psi_n(t) = \delta_n^2(t) \cdot \det(\phi^\rho).$$

Moreover, we have the degree estimates[1]

$$\deg(\det(\phi^\rho)) \leq 2d \cdot \mu,$$

where $\mu = |\chi(\mathcal{U})|$ is the absolute value of the Euler characteristic of $\mathcal{U}$.

As an application to the case of polynomials with only isolated singularities, we obtain the following generalization of [5, Corollary 5.8], and a new obstruction on the (degree of the) intersection form:

Corollary 1.3 *Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a degree-d polynomial that is transversal at infinity. Assume that the hypersurface $F_0 = \{f = 0\}$ has only isolated singularities. Then we have the polynomial identity*

$$(t-1)^{(-1)^{n+1}(1+\chi(\mathcal{U}))}(t^d-1)^{\xi} \cdot \prod_{p \in \mathrm{Sing}(F_0)} \Delta_p(t) = \delta_n(t)^2 \cdot \det(\phi^\rho), \tag{1-1}$$

where $\Delta_p(t)$ is the top local Alexander polynomial associated to the point $p \in \mathrm{Sing}(F_0)$ and $\xi = ((d-1)^{n+1} + (-1)^n)/d$. Moreover, the degree of the polynomial $\det(\phi^\rho)$ is even.

[1] Recall that the total degree of a Laurent polynomial in $\mathbb{Q}[t, t^{-1}]$ is defined as the difference between the highest and the lowest power of t (with non-zero coefficients). In particular, unit elements ct^k ($c \in \mathbb{Q}$, $k \in \mathbb{Z}$) of $\mathbb{Q}[t, t^{-1}]$ have total degree zero. The total degree of a product of Laurent polynomials is the sum of the total degrees of the factors. The degree of an element $P/Q \in \mathbb{Q}(t)$ (with P, $Q \in \mathbb{Q}[t, t^{-1}]$) is defined as the difference between the total degrees of P and Q.

1.3 Summary

The paper is organized as follows.

In Section 2, we recall the definition and main results on the Alexander modules, peripheral complex and the Sabbah specialization complex. In Section 3, we give a new description of the peripheral complex associated with a hypersurface. As a byproduct, we show that the peripheral complex underlies a (shifted) algebraic mixed Hodge module. In Section 4, we give several estimates for the Alexander polynomials of the hypersurface complement and study the error terms for such estimates. Section 5 recalls the basic constructions and main results on the Reidemeister torsion of a finite CW complex, and in particular, the duality theorem and the intersection form for the torsion. In Section 6, we introduce the boundary manifold $\partial\mathcal{U}_0$ of the hypersurface complement, and we describe its (linking number) Alexander modules $H_i((\partial\mathcal{U}_0)^c)$ in terms of the peripheral complex. In particular, we endow these Alexander modules $H_i((\partial\mathcal{U}_0)^c)$ with mixed Hodge structures. Finally, Section 7 is devoted to the proof of both Theorem 1.2 and Corollary 1.3.

Acknowledgments

We are grateful to Alex Dimca and Jörg Schürmann for useful discussions. Liu is supported by China Scholarship Council (file No 201206340046). He thanks the Mathematics Department at the University of Wisconsin–Madison for hospitality during the preparation of this work. Maxim is partially supported by grants from NSF (DMS-1304999), NSA (H98230-14-1-0130), Simons Foundation (#277891), and by a grant of the Ministry of National Education, CNCS-UEFISCDI project number PN-II-ID-PCE-2012-4-0156.

2 Preliminaries

2.1 Alexander modules

Let $f = f(x_1, \dots, x_{n+1})\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a reduced degree-d polynomial map, and set $F_0 = f^{-1}(0)$ and $\mathcal{U} = \mathbb{C}^{n+1} \setminus F_0$. We say that f is *transversal at infinity* if the projective closure V of F_0 in $\mathbb{CP}^{n+1}$ is transversal in the stratified sense to the hyperplane at infinity $H = \mathbb{CP}^{n+1} \setminus \mathbb{C}^{n+1} = \{x_0 = 0\}$. If f is transversal at infinity, the affine hypersurface F_0 is homotopy equivalent to a bouquet of n–spheres, ie

$$F_0 \simeq \bigvee_{\mu} S^n, \tag{2-1}$$

where μ denotes the number of spheres in the join (see [13, page 476]). It is shown there that μ can be determined topologically as the degree of the gradient map associated to the homogenization $\widetilde{f}$ of f.

We have a surjective homomorphism: $\pi_1(\mathcal{U}) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$ induced by f, which shall be called the *linking number homomorphism* (see [8, pages 76–77] for a justification of terminology). Let us consider the corresponding infinite cyclic cover $\mathcal{U}^c$ of $\mathcal{U}$. Then, under the deck group action, every homology group $H_i(\mathcal{U}^c, \mathbb{Q})$ becomes a $\Gamma := \mathbb{Q}[t, t^{-1}]$–module.

Definition 2.1 The Γ–module $H_i(\mathcal{U}^c)$ is called the i^{th} *Alexander module* of the hypersurface complement $\mathcal{U}$.

When $H_i(\mathcal{U}^c)$ is a torsion Γ–module, we denote by $\delta_i(t)$ the corresponding *Alexander polynomial* (also called *order* by Milnor [28]). Since $\mathcal{U}$ has the homotopy type of a finite $(n+1)$–dimensional CW complex, $H_i(\mathcal{U}^c) = 0$ for $i > n+1$ and $H_{n+1}(\mathcal{U}^c)$ is a free Γ–module. Hence the only interesting Alexander modules $H_i(\mathcal{U}^c)$ appear in the range $0 \le i \le n$, and the following result holds:

Theorem 2.2 [25, Theorems 3.6 and 4.1] *Assume that $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is a reduced, degree-d polynomial that is transversal at infinity. Then $H_i(\mathcal{U}^c)$ is a finitely generated semi-simple torsion Γ–module for $0 \le i \le n$, and the roots of the corresponding Alexander polynomial $\delta_i(t)$ are roots of unity of order d.*

Remark 2.3 Maxim [25] showed that $H_0(\mathcal{U}^c) \cong \Gamma/(t-1)$, and $H_{n+1}(\mathcal{U}^c)$ is a free Γ–module of rank $|\chi(\mathcal{U})|$. On the other hand, by using the additivity of the Euler characteristic, it is easy to see from (2-1) that $\chi(\mathcal{U}) = (-1)^{n+1}\mu$. Therefore

$$H_{n+1}(\mathcal{U}^c) \cong \Gamma^{\mu}. \tag{2-2}$$

2.2 Linking number local system

Let us consider the local system $\mathcal{L}$ on $\mathcal{U}$ with stalk Γ, and representation of the fundamental group defined by the composition

$$\pi_1(\mathcal{U}) \xrightarrow{f_*} \pi_1(\mathbb{C}^*) \longrightarrow \mathrm{Aut}(\Gamma),$$

with the second map being given by $1_{\mathbb{Z}} \mapsto t$. Here t is the automorphism of Γ given by multiplication by t. $\mathcal{L}$ shall be referred to as the *linking number local system.*

If $A^\bullet$ is a complex of Γ–sheaves, let $\mathcal{D}A^\bullet$ denote its Verdier dual. Then we have that

$$\mathcal{D}\mathcal{L} \cong \mathcal{L}^{\mathrm{op}}[2n+2],$$

where $\mathcal{L}^{\mathrm{op}}$ is the local system obtained from $\mathcal{L}$ by composing all Γ–module structures with the involution $t \mapsto t^{-1}$.

In terms of the local system $\mathcal{L}$, we have the Γ–module isomorphisms [25, Corollary 3.4]

$$H_c^{2n+2-i}(\mathcal{U}, \mathcal{L}) \cong H_i(\mathcal{U}, \mathcal{L}) \cong H_i(\mathcal{U}^c) \tag{2-3}$$

for all i. Similarly,

$$H_i(\mathcal{U}, \mathcal{L}^{\mathrm{op}}) \cong \overline{H_i(\mathcal{U}^c)}, \tag{2-4}$$

where $\bar{*}$ denotes the composition with the involution $t \mapsto t^{-1}$. By using the universal coefficient theorem (eg see Banagl [1, Theorem 3.4.4]), we also obtain

$$H^{i+1}(\mathcal{U}; \mathcal{L}) \cong \mathrm{Free}(H_{i+1}(\mathcal{U}, \mathcal{L}^{\mathrm{op}})) \oplus \mathrm{Torsion}(H_i(\mathcal{U}, \mathcal{L}^{\mathrm{op}})). \tag{2-5}$$

2.3 The peripheral complex

For any complex algebraic variety X and any ring R, we denote by $D_c^b(X, R)$ the derived category of bounded, cohomologically R–constructible complexes of sheaves on X. For a quick introduction to derived categories, the reader is advised to consult Dimca [9].

By choosing a Whitney stratification of V and using the transversal hyperplane at infinity H, we obtain a stratification of the pair $(\mathbb{CP}^{n+1}, V \cup H)$. Then, for any perversity function $\bar{p}$, the intersection homology complex $IC^{\bullet}_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L}) \in D_c^b(\mathbb{CP}^{n+1}, \Gamma)$ is defined by using Deligne's axiomatic construction (see Banagl [1] or Goresky and MacPherson [15]). In this paper, we mainly use the indexing conventions from [15]. In particular, we have the normalization property $IC^{\bullet}_{\bar{p}}(\mathbb{CP}^{n+1}, \mathcal{L})|_{\mathcal{U}} \cong \mathcal{L}[2n+2]$.

Let us recall the following result:

Theorem 2.4 [25, Lemma 3.1] *Assume that the polynomial $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is transversal at infinity. Let j be the inclusion of $\mathcal{U}$ in $\mathbb{CP}^{n+1}$. Then we have quasi-isomorphisms in $D_c^b(\mathbb{CP}^{n+1}, \Gamma)$*

$$IC^{\bullet}_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}) \cong j_!\mathcal{L}[2n+2], \tag{2-6}$$

$$IC^{\bullet}_{\bar{l}}(\mathbb{CP}^{n+1}, \mathcal{L}) \cong Rj_*\mathcal{L}[2n+2], \tag{2-7}$$

where the middle and logarithmic perversities are defined as $\bar{m}(s) = [(s-1)/2]$ and $\bar{l}(s) = [(s+1)/2]$. (Note that $\bar{m}(s) + \bar{l}(s) = s-1$, ie $\bar{m}$ and $\bar{l}$ are superdual perversities, in the sense of [4]*.)*

In the above notations, the Cappell–Shaneson superduality isomorphism can be stated as (see [4, Theorem 3.3])

$$IC^\bullet_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L})^{\mathrm{op}} \cong \mathcal{D}(IC^\bullet_{\bar{l}}(\mathbb{CP}^{n+1}, \mathcal{L}))[2n+2], \tag{2-8}$$

where, if A is a complex of sheaves, A^{op} is the Γ–module obtained from the Γ–module A by composing all module structures with the involution $t \mapsto t^{-1}$.

Definition 2.5 The *peripheral complex* $\mathcal{R}^\bullet \in D^b_c(\mathbb{CP}^{n+1}, \Gamma)$ is defined by the distinguished triangle (see [4])

$$IC^\bullet_{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}) \to IC^\bullet_{\bar{l}}(\mathbb{CP}^{n+1}, \mathcal{L}) \to \mathcal{R}^\bullet[2n+2] \xrightarrow{[1]}, \tag{2-9}$$

or, using Theorem 2.4, by

$$j_!\mathcal{L} \to Rj_*\mathcal{L} \to \mathcal{R}^\bullet \xrightarrow{[1]}. \tag{2-10}$$

Then, up to a shift, $\mathcal{R}^\bullet$ is a self-dual (ie $\mathcal{R}^\bullet \cong \mathcal{D}\mathcal{R}^{\bullet\mathrm{op}}[-2n-1]$), torsion (ie the stalks of its cohomology sheaves are torsion modules), perverse sheaf on $\mathbb{CP}^{n+1}$ (see [25, Section 3.2]). In fact, $\mathcal{R}^\bullet$ has compact support on $V \cup H$ and

$$\mathcal{R}^\bullet|_{V\cup H} \cong (Rj_*\mathcal{L})|_{V\cup H}. \tag{2-11}$$

Remark 2.6 The peripheral complex $\mathcal{R}^\bullet$ as defined here corresponds to $\mathcal{R}^\bullet[-2n-2]$ in the notations of Cappell and Shaneson; see [4] or [25].

2.4 The Sabbah specialization complex

The Sabbah specialization complex [29] (see also its reformulation by Budur [3]) can be regarded as a generalization of Deligne's nearby cycle complex. For a quick introduction to the theory of nearby cycles, the reader is advised to consult Dimca [9] and Massey [24].

Let us recall the relevant definitions. Consider the commutative diagram of spaces and maps

$$\begin{tikzcd} F_0 \arrow[r, "i"] \arrow[d, "f"] & \mathbb{C}^{n+1} \arrow[d, "f"] & \mathcal{U} \arrow[l, "l"'] \arrow[d, "f"] & \mathcal{U}^c \arrow[l, "\pi"'] \arrow[d, "\hat{f}"] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^* \arrow[l] & \widehat{\mathbb{C}}^*, \arrow[l, "\widehat{\pi}"'] \end{tikzcd}$$

where $\widehat{\pi}$ is the universal covering of the punctured disk $\mathbb{C}^*$ and the right-hand square of the diagram is cartesian.

Definition 2.7 The *Sabbah specialization functor* of f is defined by

$$\psi_f^S = i^* Rl_* R\pi_!(l \circ \pi)^*\colon\ D_c^b(\mathbb{C}^{n+1}, \mathbb{Q}) \to D_c^b(F_0, \Gamma),$$

and we call $\psi_f^S \mathbb{Q}_{\mathbb{C}^{n+1}}$ the *Sabbah specialization complex*.

Remark 2.8 The definition of the Sabbah specialization complex is slightly different from that of the nearby cycle complex, where $R\pi_!$ is replaced by $R\pi_*$.

In the following we write $\mathbb{Q}$ for the constant sheaf $\mathbb{Q}_{\mathbb{C}^{n+1}}$ on $\mathbb{C}^{n+1}$.

Consider the natural forgetful functor

$$\mathit{for}\colon\ D_c^b(F_0, \Gamma) \to D_c^b(F_0, \mathbb{Q}),$$

which maps a torsion Γ–module sheaf complex to its underlying $\mathbb{Q}$–complex. Let $\psi_f \mathbb{Q}$ be the Deligne nearby cycle complex associated to f. It is known that one has a non-canonical isomorphism (see [2, page 13])

$$\mathit{for} \circ \psi_f^S(\mathbb{Q}) \cong \psi_f \mathbb{Q}[-1]. \tag{2-12}$$

The next result is a direct consequence of [3, Lemma 3.4(b)].

Lemma 2.9 [23, Section 2.4] *We have a quasi-isomorphism in* $D_c^b(F_0, \Gamma)$,

$$\mathcal{R}^\bullet|_{F_0} \cong \psi_f^S \mathbb{Q}. \tag{2-13}$$

Moreover, we have the distinguished triangle in $D_c^b(\mathbb{C}^{n+1}, \Gamma)$,

$$l_! \mathcal{L} \to Rl_* \mathcal{L} \to i_! \psi_f^S \mathbb{Q} \xrightarrow{[1]}. \tag{2-14}$$

3 Peripheral complex as a mixed Hodge module

In this section, we give a new characterization of the peripheral complex and show that (up to a shift) it underlies a mixed Hodge module. For a quick introduction to the category of mixed Hodge module, the reader is advised to consult Saito [30].

Let $h = f_d$ be the top-degree part of f, with corresponding Milnor fibre $F_h = \{h = 1\}$. Then, it is shown by Maxim [25] that $\mathcal{R}^\bullet|_{H \setminus H \cap V}$ is a local system $\mathcal{L}(h)$ with stalk $\Gamma/(t^d - 1)$ placed in degree 1, ie

$$\mathcal{R}^\bullet|_{H \setminus H \cap V} \cong \mathcal{L}(h)[-1]. \tag{3-1}$$

Theorem 3.1 *Assume that the polynomial $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is transversal at infinity and let V be the projective completion of $F_0 = \{f = 0\}$. Let i_v be the inclusion of F_0 into V and i_h be the inclusion of $H \setminus V \cap H$ into H. Then*

$$\mathcal{R}^\bullet|_V \cong Ri_{v*}\psi_f^S \mathbb{Q}, \tag{3-2}$$

$$\mathcal{R}^\bullet|_H \cong Ri_{h*}\mathcal{L}(h)[-1]. \tag{3-3}$$

Proof Let us only prove (3-2), as (3-3) can be obtained in a similar manner. Consider the following commutative diagram of inclusions:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "l"] \arrow[d, "k'"'] \arrow[dr, "j"] & \mathbb{C}^{n+1} \arrow[d, "k"] \\ \mathbb{CP}^{n+1} \setminus V \arrow[r, "l'"] & \mathbb{CP}^{n+1} \end{tikzcd}$$

Since V intersects H transversally, there exists a base change isomorphism associated with this diagram (see Schürmann [32, Lemma 6.0.5]),

$$l'_! \circ Rk'_* = Rk_* \circ l_!. \tag{3-4}$$

Let v be the inclusion of V into $\mathbb{CP}^{n+1}$. Note that $Rl'_* Rk'_* \mathcal{L} = Rj_*\mathcal{L}$ and, by Section 2.3, we have that $\mathcal{R}^\bullet|_V = (Rj_*\mathcal{L})|_V$. Then we have a distinguished triangle

$$l'_! Rk'_* \mathcal{L} \to Rl'_* k'_* \mathcal{L} \to v_!(\mathcal{R}^\bullet|_V) \xrightarrow{[1]}. \tag{3-5}$$

Using the base change isomorphism (3-4) and the commutativity of the above diagram, the distinguished triangle (3-5) can be written as

$$Rk_* l_! \mathcal{L} \to Rk_* Rl_* \mathcal{L} \to v_!(\mathcal{R}^\bullet|_V) \xrightarrow{[1]}. \tag{3-6}$$

Recall now that there is a distinguished triangle (2-14)

$$l_!\mathcal{L} \to Rl_*\mathcal{L} \to i_!\psi_f^S \mathbb{Q}, \tag{3-7}$$

where i is the inclusion of F_0 into $\mathbb{C}^{n+1}$. By applying the functor Rk_* to this triangle, we obtain the distinguished triangle

$$Rk_* l_! \mathcal{L} \to Rk_* Rl_* \mathcal{L} \to Rk_* i_! \psi_f^S \mathbb{Q} \xrightarrow{[1]}. \tag{3-8}$$

So, by comparing the two triangles (3-6) and (3-8), we get the quasi-isomorphism

$$v_!(\mathcal{R}^\bullet|_V) \cong Rk_* i_! \psi_f^S \mathbb{Q}. \tag{3-9}$$

Since F_0 is closed in $\mathbb{C}^{n+1}$, we have $i_! = Ri_*$. So, for i_v the inclusion of F_0 into V, we have $v \circ i_v = k \circ i$, hence

$$Rk_* i_! \psi_f^S \mathbb{Q} = Rk_* Ri_* \psi_f^S \mathbb{Q} = Rv_* Ri_{v*} \psi_f^S \mathbb{Q}. \tag{3-10}$$

Finally, by applying v^* to (3-9), and using (3-10) and the standard identities $v^* v_! = \mathrm{id}$ and $v^* Rv_* = \mathrm{id}$, we get

$$\mathcal{R}^\bullet|_V \cong Ri_{v*} \psi_f^S \mathbb{Q}. \tag{3-11}$$ □

The result of Theorem 3.1 above can be used to endow the peripheral complex with a mixed Hodge module structure. Recall that there is a natural forgetful functor *for*: $D_c^b(X, \Gamma) \to D_c^b(X, \mathbb{Q})$, which assigns to a torsion complex of Γ–sheaves its underlying $\mathbb{Q}$–complex. In what follows, we will use the same notation for a Γ–complex $\mathcal{A}^\bullet$ and for its underlying $\mathbb{Q}$–complex *for*$(\mathcal{A}^\bullet)$.

After applying the forgetful functor to (3-2) and using (2-12), we obtain the quasi-isomorphism of complexes of $\mathbb{Q}$–sheaves

$$\mathcal{R}^\bullet|_V \cong Ri_{v*} \psi_f \mathbb{Q}[-1]. \tag{3-12}$$

Also note that the local system $\mathcal{L}(h) \in D_c^b(H \setminus H \cap V, \mathbb{Q})$ is induced by the natural d–fold cover map $p\colon F_h \to (H \setminus V \cap H)$ or, more precisely,

$$\mathcal{L}(h) \cong Rp_* \mathbb{Q}_{F_h} \in D_c^b(H \setminus H \cap V, \mathbb{Q}). \tag{3-13}$$

Remark 3.2 Since $h = f_d$ is a degree-d homogeneous polynomial function on $\mathbb{C}^{n+1}$, the hypersurface $V_h = \overline{\{h = 0\}} \subseteq \mathbb{CP}^{n+1}$ is already transversal to the hyperplane at infinity $H = \{x_0 = 0\}$. Let $\mathcal{R}_h^\bullet$ be the peripheral complex associated to h. Then (3-3) implies that $\mathcal{R}^\bullet|_H = \mathcal{R}_h^\bullet|_H$.

We can now prove the following result:

Corollary 3.3 *The peripheral complex $\mathcal{R}^\bullet$ underlies a (shifted) algebraic mixed Hodge module.*

Proof For the purpose of this proof only, we switch to the perverse conventions used in Saito's theory, according to which $\mathcal{R}^\bullet[n+1]$ is a perverse sheaf on $\mathbb{CP}^{n+1}$. All sheaf complexes appearing in this proof will be complexes of $\mathbb{Q}$–sheaves (ie we apply the forgetful functor to all Γ–sheaf complexes).

Consider the inclusions $H \setminus V \cap H \overset{s}{\hookrightarrow} V \cup H \overset{r}{\hookleftarrow} V$, and the associated distinguished triangle in $D_c^b(V \cup H, \mathbb{Q})$

$$s_! \mathcal{R}^\bullet|_{H \setminus V \cap H}[n+1] \to \mathcal{R}^\bullet[n+1] \to r_* \mathcal{R}^\bullet|_V[n+1] \overset{[1]}{\to}. \tag{3-14}$$

Recall that $\mathcal{R}^\bullet|_{H\setminus V\cap H} \cong \mathcal{L}(h)[-1]$, while by (3-12) we have $\mathcal{R}^\bullet|_V \cong Ri_{v*}\psi_f\mathbb{Q}[-1]$. Since $\mathcal{R}^\bullet|_{H\setminus V\cap H}[n+1] \cong \mathcal{L}(h)[n]$ is a perverse sheaf on $H \setminus V\cap H$ and s is a quasi-finite affine map, it follows from [9, Corollary 5.2.17] that $s_!\mathcal{R}^\bullet|_{H\setminus V\cap H}[n+1]$ is a perverse sheaf on $V\cup H$. Moreover, we deduce by (3-13) that $\mathcal{L}(h)[n]$, hence also $s_!\mathcal{R}^\bullet|_{H\setminus V\cap H}[n+1]$, underlie algebraic mixed Hodge modules. Since $\mathbb{Q}[n+1]$ is a perverse sheaf on $\mathbb{C}^{n+1}$ underlying a mixed Hodge module, and the functor $\psi_f[-1]$ preserves perverse sheaves (and mixed Hodge modules), it follows that $(\psi_f[-1])(\mathbb{Q}[n+1])$ is a perverse sheaf on F_0 underlying a mixed Hodge module. Moreover, as i_v is a quasi-finite affine morphism, it follows as above that $\mathcal{R}^\bullet|_V[n+1] \cong Ri_{v*}(\psi_f[-1])(\mathbb{Q}[n+1])$ is a perverse sheaf on V underlying a mixed Hodge module. Finally, since r is proper, $r_! = r_*$ preserves perverse sheaves and mixed Hodge modules, so $r_*\mathcal{R}^\bullet|_V[n+1]$ is a perverse sheaf on $V\cup H$ underlying a mixed Hodge module.

The above considerations show that $\mathcal{R}^\bullet[n+1]$ can be regarded as an extension of perverse sheaves, both of which underlie mixed Hodge modules. So $\mathcal{R}^\bullet[n+1]$ is an element in the first Yoneda extension group $Y\,\mathrm{Ext}^1(\mathrm{For}(C),\mathrm{For}(A))$ for suitable mixed Hodge modules C and A as described above, where For: $\mathrm{MHM}\to\mathrm{Perv}_{\mathbb{Q}}$ denotes the forgetful functor assigning to a mixed Hodge module the corresponding rational sheaf complex. Since Yoneda Ext groups $Y\,\mathrm{Ext}^i$ agree with the derived category Ext groups $\mathrm{Ext}^i = \mathrm{Hom}(-,-[i])$ for noetherian or artinian abelian categories such as MHM or $\mathrm{Perv}_{\mathbb{Q}}$, and the forgetful functor

$$\mathrm{For}\colon \mathrm{Ext}^i(C,A) \to \mathrm{Ext}^i(\mathrm{For}(C),\mathrm{For}(A))$$

is surjective for all i for given mixed Hodge modules A and C (see [31, Theorem 2.10]), it follows that $\mathcal{R}^\bullet[n+1]$ underlies a mixed Hodge module. □

4 Error estimates for Alexander polynomials

In this section, we give several error estimates for Alexander polynomials of hypersurface complements.

Proposition 4.1 *In our notations, we have Γ–module isomorphisms*

$$H^{2n+1-i}(\mathbb{CP}^{n+1};\mathcal{R}^\bullet) \cong \begin{cases} H_i(\mathcal{U}^c) & i<n, \\ \overline{H_{2n-i}(\mathcal{U}^c)} & i>n, \end{cases} \tag{4-1}$$

and an exact sequence of Γ–modules for $i=n$:

$$0 \to \Gamma^\mu \to \Gamma^\mu \oplus \overline{H_n(\mathcal{U}^c)} \to H^{n+1}(\mathbb{CP}^{n+1};\mathcal{R}^\bullet) \to H_n(\mathcal{U}^c) \to 0. \tag{4-2}$$

Proof Consider the distinguished triangle

$$Rj_!\mathcal{L} \to Rj_*\mathcal{L} \to \mathcal{R}^\bullet \overset{[1]}{\to}$$

of Definition 2.5. By applying the hypercohomology with compact support functor, we have the long exact sequence

$$\cdots \to H^{2n+1-i}(\mathbb{CP}^{n+1}; \mathcal{R}^\bullet) \to H^{2n+2-i}(\mathbb{CP}^{n+1}; Rj_!\mathcal{L}) \\ \to H^{2n+2-i}(\mathbb{CP}^{n+1}; Rj_*\mathcal{L}) \to \cdots .$$

The claim follows from the above sequence together with the following Γ–isomorphisms from Section 2.2:

$$\begin{aligned} H^{2n+2-i}(\mathbb{CP}^{n+1}; Rj_!\mathcal{L}) &\cong H_c^{2n+2-i}(\mathcal{U}; \mathcal{L}) \cong H_i(\mathcal{U}^c), \\ H^{2n+2-i}(\mathbb{CP}^{n+1}; Rj_*\mathcal{L}) &\cong H^{2n+2-i}(\mathcal{U}; \mathcal{L}) \\ &\cong \begin{cases} 0 & i < n+1, \\ \Gamma^\mu \oplus \overline{H_n(\mathcal{U}^c)} & i = n+1, \\ \overline{H_{2n+1-i}(\mathcal{U}^c)} & i > n+1. \end{cases} \end{aligned}$$

□

Recall that $\delta_i(t)$ denotes the Alexander polynomial associated to the Alexander module $H_i(\mathcal{U}^c)$ $(i \leq n)$. Let $r_i(t)$ be the Alexander polynomial of the torsion Γ–module $H^{2n+1-i}(\mathbb{CP}^{n+1}; \mathcal{R}^\bullet)$. The above proposition yields the following relationship between the polynomials r_i and δ_i:

Corollary 4.2 *We have*

$$r_i(t) = \begin{cases} \delta_i(t) & i < n, \\ \overline{\delta_{2n-i}(t)} & i > n, \end{cases} \tag{4-3}$$

and $\delta_n(t) \cdot \overline{\delta_n(t)}$ *divides* $r_n(t)$.

Set

$$\varphi(t) = \frac{r_n(t)}{\delta_n(t) \cdot \overline{\delta_n(t)}}.$$

Let F_h denote as before the Milnor fibre associated to the polynomial $h = f_d$, the top-degree part of the polynomial f. Let $h_i(t)$ be the Alexander polynomial associated to $H_i(F_h)$. Then it was shown in [25, Theorem 4.7] that $h_i(t) = \delta_i(t)$ for $i < n$, and $\delta_n(t)$ divides $h_n(t)$. Set

$$\varphi_1(t) = \frac{h_n(t)}{\delta_n(t)}.$$

Similarly, we let $\psi_i(t)$ denote the Alexander polynomial associated to the torsion Γ–module $H_c^{2n+1-i}(F_0; \psi_f^S \mathbb{Q})$. It was shown in [23, Theorem 1.1] that $\psi_i(t) = \delta_i(t)$ for $i < n$, and $\delta_n(t)$ divides $\psi_n(t)$. Set

$$\varphi_2(t) = \frac{\psi_n(t)}{\delta_n(t)}.$$

As can be seen from their definitions, the polynomials $\varphi_1(t)$ and $\varphi_2(t)$ can be regarded as error estimates for the Alexander polynomial $\delta_n(t)$. The above polynomial invariants are related by the following result:

Theorem 4.3 *Assume that the polynomial $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is transversal at infinity. With the above notations, we have the equalities*

$$r_n(t) = \overline{h_n(t)} \cdot \psi_n(t) = h_n(t) \cdot \overline{\psi_n(t)}, \tag{4-4}$$

$$\varphi(t) = \overline{\varphi_1(t)} \cdot \varphi_2(t) = \varphi_1(t) \cdot \overline{\varphi_2(t)}. \tag{4-5}$$

Remark 4.4 Since the polynomials $h_i(t)$, $\delta_i(t)$ and $\psi_i(t)$ are products of cyclotomic polynomials (eg see [23]), the involution operation $\bar{*}$ keeps these polynomials unchanged.

In the course of proving Theorem 4.3, we need the following technical result:

Lemma 4.5 *We have the Γ–module isomorphisms, for all i,*

$$H_c^{2n+1-i}(H \setminus V \cap H; \mathcal{R}^\bullet) \cong H_i(F_h), \tag{4-6}$$

$$H^{2n+1-i}(H; \mathcal{R}^\bullet) \cong H_i(F_h, \partial F_h), \tag{4-7}$$

$$H^{2n+1-i}(V \cap H; \mathcal{R}^\bullet) \cong H_{i-1}(\partial F_h). \tag{4-8}$$

Proof Choose coordinates $[x_0, \dots, x_{n+1}]$ for $\mathbb{CP}^{n+1}$, so that $H = \{x_0 = 0\}$ is the hyperplane at infinity. Then $\mathbb{O} = [0, \dots, 0, 1]$ corresponds to the origin in $\mathbb{C}^{n+1}$. Define

$$\alpha\colon \mathbb{CP}^{n+1} \to \mathbb{R}_+, \quad \alpha := \frac{|x_0|^2}{\sum_{i=0}^{n+1} |x_i|^2}.$$

Note that α is a well-defined, real analytic and proper function satisfying

$$0 \le \alpha \le 1, \quad \alpha^{-1}(0) = H \quad \text{and} \quad \alpha^{-1}(1) = \mathbb{O}.$$

Since α has only finitely many critical values, there exists η sufficiently small that the interval $(0, \eta]$ contains no critical values. Set

$$U_\eta = \alpha^{-1}[0, \eta).$$

Then U_η is a tubular neighbourhood of H in $\mathbb{CP}^{n+1}$, and note that $\mathbb{CP}^{n+1} \setminus U_\eta$ is a closed large ball of radius $(1-\eta)/\eta$ in $\mathbb{C}^{n+1}$. Set

$$U_\eta^* = \alpha^{-1}(0,\eta) = U_\eta \setminus H.$$

Let us now consider the commutative diagram of inclusions

$$\begin{tikzcd} U_\eta^* \arrow[r, "c"] \arrow[d, "u"] & \mathbb{C}^{n+1} \arrow[d, "k"] & \mathcal{U} \arrow[l, "l"'] \arrow[dl, "j"] \\ U_\eta \arrow[r, "p"] & \mathbb{CP}^{n+1} & \end{tikzcd}$$

and restrict the distinguished triangle (3-6) over U_η. We get a triangle

$$p^* Rk_* l_! \mathcal{L} \to p^* Rk_* Rl_* \mathcal{L} \to p^* v_! (\mathcal{R}^\bullet|_V) \overset{[1]}{\to}, \tag{4-9}$$

where v denotes as before the inclusion of V in $\mathbb{CP}^{n+1}$. Let us first give geometric interpretations to all Γ–modules appearing in the hypercohomology long exact sequence associated to (4-9). First note that

$$\begin{aligned} p^* Rk_* l_! \mathcal{L}[2n+2] &\overset{(1)}{\cong} Ru_* c^* l_! \mathcal{L}[2n+2] \\ &\overset{(2)}{\cong} Ru_* c^* k^* k_! l_! \mathcal{L}[2n+2] \\ &\cong Ru_* c^* k^* j_! \mathcal{L}[2n+2] \\ &\overset{(3)}{\cong} Ru_* u^* p^* IC_{\overline{m}}(\mathbb{CP}^{n+1}, \mathcal{L}) \\ &\cong Ru_* IC_{\overline{m}}(U_\eta^*, \mathcal{L}), \end{aligned}$$

where (1) follows from the base change isomorphism $p^! Rk_* = Ru_* c^!$ (together with $p^! = p^*$ and $c^! = c^*$, as p and c are both open inclusions), for (2) we use the known identity $k^* k_! \cong \mathrm{id}$, and (3) follows from Theorem 2.4.

Set

$$S_\infty = \alpha^{-1}(\eta')$$

for $0 < \eta' < \eta$. Then we get, as in [25, Theorem 4.7],

$$\begin{aligned} H^{2n+1-i}(U_\eta; p^* Rk_* Rl_! \mathcal{L}) &\cong H^{-i-1}(U_\eta^*; IC_{\overline{m}}(U_\eta^*, \mathcal{L})) \\ &\overset{(1)}{\cong} H^{-i-1}(S_\infty; IC_{\overline{m}}(U_\eta^*, \mathcal{L})|_{S_\infty}) \\ &\cong H_i(S_\infty \setminus S_\infty \cap V; \mathcal{L}) \\ &\overset{(2)}{\cong} H_i(F_h), \end{aligned}$$

where (1) follows from [17, Lemma 8.4.7(c)], and (2) follows from the fact that V intersects H transversally. In fact, the corresponding infinite cyclic cover of $S_\infty \setminus S_\infty \cap V$ is homotopy equivalent to F_h; see [25, Proposition 4.9]. Also note that $IC_{\overline{m}}(U_\eta^*, \mathcal{L})|_{S_\infty} \cong IC_{\overline{m}}(\mathbb{CP}^{n+1}, \mathcal{L})|_{S_\infty}$. Similarly, by using Theorem 2.4 and duality, we have

$$H^{2n+1-i}(U_\eta; p^* Rk_* Rl_* \mathcal{L}) \cong H^{-i-1}(S_\infty; IC_{\overline{l}}(\mathbb{CP}^{n+1}, \mathcal{L})|_{S_\infty}) \cong H_i(F_h, \partial F_h).$$

So, by comparing the hypercohomology long exact sequence associated to (4-9) with the homology long exact sequence induced by the natural inclusion $\partial F_h \to F_h$ and using the above calculations, we get the Γ–module isomorphism

$$H^{2n+1-i}(S_\infty; \mathcal{R}^\bullet|_{S_\infty}) \cong H_{i-1}(\partial F_h). \tag{4-10}$$

Recall that the triangle (3-6) was obtained from (3-5) by a base change isomorphism, so the associated hypercohomology long exact sequences for the restrictions of these triangles over U_η coincide. Note that [17, Lemma 8.4.7(a)] shows that, for any $\mathcal{F}^\cdot \in D_c^b(\mathbb{CP}^{n+1})$, there is an isomorphism

$$H^*(U_\eta; \mathcal{F}^\cdot) \cong H^*(H; \mathcal{F}^\cdot). \tag{4-11}$$

So, by restricting (3-5) over H we get the same hypercohomology long exact sequence as for restricting (3-5) and (3-6) over U_η. Let i_{hv} be the inclusion of $H \cap V$ into H. By using the proper base change isomorphism [9, Theorem 2.3.26] for the diagram

$$\begin{tikzcd}
 & H \setminus H \cap V \arrow[r, "i_h"] \arrow[d] & H \arrow[d] & H \cap V \arrow[l, "i_{hv}"'] \arrow[d] \\
\mathcal{U} \arrow[r, "k'"] & \mathbb{CP}^{n+1} \setminus V \arrow[r, "l'"] & \mathbb{CP}^{n+1} & V, \arrow[l, "v"']
\end{tikzcd}$$

we have (using the notations of Theorem 3.1)

$$(Rl'_! Rk'_* \mathcal{L})|_H = Ri_{h!}((Rk'_* \mathcal{L})|_{H \setminus H \cap V}) = Ri_{h!}(\mathcal{R}^\bullet|_{H \setminus H \cap V}), \tag{4-12}$$

$$(Rv_!(\mathcal{R}^\bullet|_V))|_H = Ri_{hv!}(\mathcal{R}^\bullet|_{H \cap V}). \tag{4-13}$$

So, the hypercohomology long exact sequence associated to the restriction of the triangle (3-5) over H becomes

$$\cdots \to H_c^{2n+1-i}(H \setminus V \cap H; \mathcal{R}^\bullet) \to H^{2n+1-i}(H; \mathcal{R}^\bullet) \to H^{2n+1-i}(V \cap H; \mathcal{R}^\bullet) \to \cdots.$$

Therefore, by the above calculations for the restriction of (3-6) over U_η, we get the Γ–module isomorphisms

$$\begin{aligned} H_c^{2n+1-i}(H \setminus V \cap H; \mathcal{R}^\bullet) &\cong H_i(F_h), \\ H^{2n+1-i}(H; \mathcal{R}^\bullet) &\cong H_i(F_h, \partial F_h), \\ H^{2n+1-i}(V \cap H; \mathcal{R}^\bullet) &\cong H_{i-1}(\partial F_h) \end{aligned}$$

for all i. □

Proof of Theorem 4.3 Let us consider the long exact sequence of hypercohomology with compact supports for the peripheral complex $\mathcal{R}^\bullet$ with respect to the inclusions

$$F_0 \hookrightarrow V \cup H \hookleftarrow H.$$

By using duality and [25, Theorem 4.7], we have that

$$H_i(F_h, \partial F_h) \cong \overline{H_{2n-i}(F_h)} \cong \overline{H_{2n-i}(\mathcal{U}^c)}$$

for $i > n$. On the other hand, [23, Theorem 1.1] yields that

$$H_c^{2n+1-i}(F_0; \psi_f^S \mathbb{Q}) \cong H_i(\mathcal{U}^c)$$

for $i < n$. By using Proposition 4.1 and vanishing results for perverse sheaves on affine spaces, we obtain the Γ–module isomorphisms

$$H_c^{2n+1-i}(F_0; \psi_f^S \mathbb{Q}) \cong H^{2n+1-i}(V \cup H; \mathcal{R}^\bullet) \qquad \text{for } i < n, \tag{4-14}$$

$$H^{2n+1-i}(V \cup H; \mathcal{R}^\bullet) \cong H^{2n+1-i}(H; \mathcal{R}^\bullet) \cong H_i(F_h, \partial F_h) \qquad \text{for } i > n, \tag{4-15}$$

and a short exact sequence for $i = n$,

$$0 \to H_c^{n+1}(F_0; \psi_f^S \mathbb{Q}) \to H^{n+1}(V \cup H; \mathcal{R}^\bullet) \to H^{n+1}(H; \mathcal{R}^\bullet) \to 0. \tag{4-16}$$

Similarly, for the inclusions $(H \setminus V \cap H) \hookrightarrow V \cup H \hookleftarrow V$, there is a short exact sequence

$$0 \to H_c^{n+1}(H \setminus V \cap H; \mathcal{R}^\bullet) \to H^{n+1}(V \cup H; \mathcal{R}^\bullet) \to H^{n+1}(V; \mathcal{R}^\bullet) \to 0. \tag{4-17}$$

By using (4-7) and duality, the Alexander polynomial associated to $H^{n+1}(H; \mathcal{R}^\bullet)$ is $\overline{h_n(t)}$. Moreover, since

$$\mathcal{D}(\psi_f^S \mathbb{Q}) = (\psi_f^S \mathbb{Q})^{\mathrm{op}}[2n+1],$$

we have by Theorem 3.1 that

$$\begin{aligned} H^{n+1}(V; \mathcal{R}^\bullet) \cong H^{n+1}(V; i_{v*}\psi_f^S \mathbb{Q}) &\cong H^{n+1}(F_0; \psi_f^S \mathbb{Q}) \\ &\cong H_c^{n+1}(F_0; (\psi_f^S \mathbb{Q})^{\mathrm{op}}), \end{aligned} \tag{4-18}$$

where the last isomorphism follows from duality and the universal coefficient theorem. So, the Alexander polynomial associated to $H^{n+1}(V;\mathcal{R}^\bullet)$ is $\overline{\psi_n(t)}$.

Since $\mathcal{R}^\bullet$ is supported on $V\cup H$, the result follows now by using the multiplicativity of the Alexander polynomials associated to the short exact sequences (4-16) and (4-17). $\square$

We conclude this section with the following degree estimate:

Proposition 4.6 *Assume that the polynomial $f\colon \mathbb{C}^{n+1}\to\mathbb{C}$ is transversal at infinity.*

(a) *We have the degree estimates*

$$\deg\varphi_2\le\deg\varphi_1\le d\cdot\mu,$$

hence

$$\deg\varphi\le 2d\cdot\mu,$$

where $\mu=|\chi(\mathcal{U})|$ and d is the degree of f.

(b) *If F denotes the generic fibre of f, then F_h and F have isomorphic $\mathbb{Z}$–homology groups.*

(c) *Let $\widetilde{f}$ be the homogenization of f, with corresponding Milnor fibre $\widetilde{F}=\{\widetilde{f}=1\}$. Then, if $\mu=0$, the spaces $\mathcal{U}^c$, F, F_h and $\widetilde{F}$ are all homotopy equivalent to each other.*

Proof (a) Let $\widetilde{F}_t$ be the Milnor fibre of $\widetilde{f}$ defined by $\{\widetilde{f}=t\}$ for small enough $t\in\mathbb{C}^*$. Clearly, $\widetilde{F}_t$ is homotopy equivalent to $\widetilde{F}$. Without loss of generality, t can be chosen so that $F_t=f^{-1}(t)$ is the generic fibre of f, hence F_t is smooth. Since V intersects H transversally, the hyperplane $\{x_0=0\}$ in $\mathbb{C}^{n+2}$ and its parallel hyperplane $\{x_0=1\}$ are both generic for $\widetilde{F}_t$ in the sense of [13]. It follows that, up to homotopy, $\widetilde{F}_t$ is obtained from either the Milnor fibre F_h of $h=f_d$, or from the generic fibre F_t, by attaching $d\cdot\mu$ cells of dimension $n+1$ [13, Proposition 9]. On the other hand, [23, Corollary 6.5] shows that there exists a natural map from $\mathcal{U}^c$ to $\widetilde{F}$, which induces an $(n{+}1)$–homotopy equivalence and, in particular, we have that $H_n(\mathcal{U}^c)\cong H_n(\widetilde{F})$. These two facts together yield that $\deg\varphi_1\le d\cdot\mu$. Note also that by [23, Theorem 1.2] we have that $\dim H^n_c(F_0,\psi_f\mathbb{Q})\le\dim H_n(F_t)$. Therefore, $\deg\varphi_2\le\deg\varphi_1\le d\cdot\mu$, so $\deg\varphi=\deg\varphi_1+\deg\varphi_2\le 2d\cdot\mu$.

(b) The above homotopy argument yields the following isomorphisms for $i\le n-1$:

$$H_i(F_h,\mathbb{Z})\cong H_i(\widetilde{F},\mathbb{Z})\cong H_i(F,\mathbb{Z}).$$

Since F_h and F are n–dimensional affine varieties, both of them have the homotopy type of a finite n–dimensional CW complex. So $H_n(F_h,\mathbb{Z})$ and $H_n(F,\mathbb{Z})$ are free abelian groups, and it remains to show that they have the same rank. This is indeed true, since the above discussion shows that $\chi(F_h) = \chi(F)$.

(c) If $\mu = 0$, then F_h and F are homotopy equivalent to $\widetilde{F}$. In particular, the natural map from $\mathcal{U}^c$ to $\widetilde{F}$ induces isomorphisms on all homology groups with $\mathbb{Z}$–coefficients. Since this map is already an $(n+1)$–homotopy equivalence, it follows by Hurewicz's theorem that $\mathcal{U}^c$ is homotopy equivalent to $\widetilde{F}$. □

Remark 4.7 In fact, we proved the equalities

$$d \cdot \mu - \deg \varphi_1 = \dim H_{n+1}(\widetilde{F}), \tag{4-19}$$

$$\dim H_n(F) - \dim H^n_c(F_0, \psi_f \mathbb{Q}) = \deg \varphi_1 - \deg \varphi_2. \tag{4-20}$$

5 Reidemeister torsion and Alexander polynomials

5.1 Reidemeister torsion of chain complexes

In this section, we recall the definition and main results about the Reidemeister torsion, for more details see Kirk and Livingston [18] and Cogolludo and Florens [5].

Let C_* be a finite chain complex

$$C_* = C_n \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0$$

with C_i finite-dimensional $\mathbb{F}$–vector spaces and $\partial \circ \partial = 0$. Choose a basis c_i for C_i, $\overline{h_i}$ a basis for the homology $H_i(C_*)$ and h_i a lift of $\overline{h_i}$ to C_i. Let b_i be a basis for $B_i := \mathrm{Image}(\partial : C_{i+1} \to C_i)$ and $\widehat{b}_i$ a lift of b_i in C_{i+1}. If Z_i denotes the i–cycles, by using the inclusions $B_i \subseteq Z_i \subseteq C_i$ together with the isomorphisms $Z_i/B_i \cong H_i(C_*)$ and $C_i/Z_i \cong B_{i-1}$ it follows that $b_i h_i \widehat{b}_{i-1}$ is a basis of C_i. Denote by $[a \mid b]$ the determinant of the transition matrix from the basis a to the basis b.

Definition 5.1 [27] The *torsion* of $(C_*; c, h)$ is defined as

$$\tau(C_*; c, h) = \prod_{i=0}^{n} [b_i h_i \widehat{b}_{i-1} \mid c_i]^{(-1)^i} \in \mathbb{F}^*/\{\pm 1\}. \tag{5-1}$$

The torsion does not depend on the choice of basis b and its lifts. It depends on the choice of c and h as follows:

$$\tau(C_*; c', h') = \tau(C_*; c, h) \prod_i \left(\frac{[h'_i \mid h_i]}{[c'_i \mid c_i]} \right)^{(-1)^i}. \tag{5-2}$$

5.2 Torsion and Alexander polynomials

Let X be a finite connected CW complex, with $\pi = \pi_1(X)$. Fix an epimorphism $\rho\colon \pi \to \mathbb{Z}$ and note that ρ extends naturally to an epimorphism of algebras $\mathbb{Z}[\pi] \to \mathbb{Z}[\mathbb{Z}]$, which we also denote by ρ. We identify $\mathbb{Z}[\mathbb{Z}]$ with the Laurent polynomial ring $\mathbb{Z}[t,t^{-1}]$. If $\widetilde{X} \to X$ is the universal cover of X, then the cellular chain complex $C_*(\widetilde{X};\mathbb{Q})$ is a $\mathbb{Q}[\pi]$–module, freely generated by the lifts of the cells of X. Consider the chain complex of the pair (X,ρ) defined as the complex of $\mathbb{Q}[t,t^{-1}]$–modules

$$C_*^{\rho}(X;\mathbb{Q}[t,t^{-1}]) := \mathbb{Q}[t,t^{-1}] \otimes_{\mathbb{Q}[\pi]} C_*(\widetilde{X};\mathbb{Q}).$$

Let X^c denote the infinite cyclic cover defined by the kernel of ρ. Then, under the action of the deck group $\mathbb{Z}$, the chain complex $C_*(X^c,\mathbb{Q})$ becomes a complex of $\Gamma := \mathbb{Q}[t,t^{-1}]$–modules that is canonically isomorphic to $C_*^{\rho}(X;\mathbb{Q}[t,t^{-1}])$.

Denote by $\mathbb{Q}(t)$ the fraction field of $\mathbb{Q}[t,t^{-1}]$ and define

$$C_*^{\rho}(X,\mathbb{Q}(t)) = C_*(X^c,\mathbb{Q}) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}(t).$$

The i^{th} homology of (X,ρ) (also called the i^{th} *Alexander module*) is defined to be the $\mathbb{Q}[t,t^{-1}]$–module

$$H_i^{\rho}(X,\mathbb{Q}[t,t^{-1}]) := H_i(C_*^{\rho}(X;\mathbb{Q}[t,t^{-1}])) \cong H_i(X^c,\mathbb{Q}),$$

and we extend the definition to $H_i^{\rho}(X,\mathbb{Q}(t)) := H_i(C_*^{\rho}(X,\mathbb{Q}(t)))$. Since $\mathbb{Q}[t,t^{-1}]$ is a principal ideal domain and $\mathbb{Q}(t)$ is flat over $\mathbb{Q}[t,t^{-1}]$, it follows that

$$H_i^{\rho}(X,\mathbb{Q}(t)) = H_i^{\rho}(X,\mathbb{Q}[t,t^{-1}]) \otimes \mathbb{Q}(t).$$

Note that the complex $C_*^{\rho}(X;\mathbb{Q}(t))$ is $\mathbb{Q}(t)$–acyclic if $H_i^{\rho}(X,\mathbb{Q}[t,t^{-1}])$ is a torsion Γ–module for all i.

We now define the Reidemeister torsion for the pair (X,ρ). For this, we first note that the complex $C_*^{\rho}(X;\mathbb{Q}(t))$ is based by construction.

Definition 5.2 Fix a basis for the homology $H_*^{\rho}(X,\mathbb{Q}(t))$. The *Reidemeister torsion* of (X,ρ) with respect to this basis is defined as

$$\tau_{\rho}(X) = \tau\big(C_*^{\rho}(X,\mathbb{Q}(t))\big) \in \mathbb{Q}(t)^*.$$

We next indicate a basis-free definition for the Reidemeister torsion. Since $\mathbb{Q}[t,t^{-1}]$ is a PID, any $\mathbb{Q}[t,t^{-1}]$–module M has a decomposition into a direct sum of cyclic modules. Recall that the order of M is defined as the product of the generators of the torsion part. If the module M is free, the order is 1 by convention.

Definition 5.3 The i^{th} *Alexander polynomial* $\delta_i^\rho(X)$ of (X,ρ) is defined to be the order of the $\mathbb{Q}[t,t^{-1}]$–module $H_i^\rho(X,\mathbb{Q}[t,t^{-1}])$.

The torsion of (X,ρ) can be computed in terms of the Alexander polynomials as follows:

Theorem 5.4 [18, Theorem 3.4] *Let $\tau_\rho(X)$ be the torsion of (X,ρ) with respect to some basis in homology. Then, up to multiplication by ct^k ($c\in\mathbb{Q}^*$ and $k\in\mathbb{Z}$), we have that*

$$\tau_\rho(X)=\prod_i \frac{\delta_{2i+1}^\rho(X)}{\delta_{2i}^\rho(X)}. \tag{5-3}$$

So we can regard the right-hand side of (5-3) as a basis-free definition of the Reidemeister torsion.

5.3 Duality and intersection forms

Let X be a smooth compact $2m$–dimensional manifold with boundary ∂X. Then it is known that X has a PL structure and any two PL–triangulations have a common linear subdivision which is PL. Endow X with the CW decomposition induced by one of these. Since X is compact, the associated CW complex is finite. Note that ∂X inherits the structure of a PL–manifold from X and the triangulations of X can be used to define the chain complex $C_*^\rho(X,\partial X,\mathbb{Q}(t))$. In this setting one can construct non-singular Poincaré duality pairings for $0\le i\le 2m$,

$$H_i^\rho(X,\mathbb{Q}(t))\times H_{2m-i}^\rho(X,\partial X,\mathbb{Q}(t))\to\mathbb{Q}(t). \tag{5-4}$$

The *intersection form* of (X,ρ) is the sesquilinear form

$$\phi^\rho\colon H_m^\rho(X,\mathbb{Q}(t))\times H_m^\rho(X,\mathbb{Q}(t))\to\mathbb{Q}(t) \tag{5-5}$$

defined by the composition

$$H_m^\rho(X,\mathbb{Q}(t))\times H_m^\rho(X,\mathbb{Q}(t))\to H_m^\rho(X,\mathbb{Q}(t))\times H_m^\rho(X,\partial X,\mathbb{Q}(t))\to\mathbb{Q}(t),$$

where the first map is induced by inclusion and the second map is the pairing (5-4).

Let $\bar{*}$ denote the canonical involution on $\mathbb{Q}[t,t^{-1}]$. For each i and for a fixed basis on $H_i^\rho(X,\mathbb{Q}(t))$, we choose the dual basis on $H_{2m-i}^\rho(X,\partial X,\mathbb{Q}(t))$ given by (5-4). Then we have [18; 26] that

$$\tau_\rho(X,\partial X)\cdot\overline{\tau_\rho(X)}=1. \tag{5-6}$$

The following result will be useful later:

Lemma 5.5 [5; 18] *For any (X^{2m}, ρ) as above such that X has the homotopy type of an m–dimensional CW complex and $C_*^{\rho}(\partial X, \mathbb{Q}(t))$ is acyclic,*

$$\tau_\rho(\partial X) = \tau_\rho(X) \cdot \overline{\tau_\rho(X)} \cdot [\det(\phi^\rho)]^{(-1)^m}. \tag{5-7}$$

6 Boundary manifold of a hypersurface complement

In this section, we define the boundary manifold of a hypersurface complement and investigate its topology. We make use of the peripheral complex to study the linking number infinite cyclic cover of the boundary manifold. In particular, by using results from Section 3, we put mixed Hodge structures on the corresponding Alexander modules associated to the boundary manifold.

Choose coordinates $[x_0, \dots, x_{n+1}]$ for $\mathbb{CP}^{n+1}$, and $H = \{x_0 = 0\}$. Define

$$\theta\colon \mathbb{CP}^{n+1} \to \mathbb{R}_+, \quad \theta = \frac{|\tilde{f}|^2 |x_0|^2}{\left(\sum_{i=0}^{n+1} |x_i|^2\right)^{d+1}},$$

with $\tilde{f}$ denoting the homogenization of f (so that V is the zero set of $\tilde{f}$ in $\mathbb{CP}^{n+1}$). Note that θ is well-defined, it is real analytic and proper, and

$$\theta^{-1}(0) = V \cup H.$$

Since θ has only finitely many critical values, there exists a positive real number ε sufficiently small that the interval $(0, \varepsilon]$ contains no critical values. Set

$$\mathcal{U}_0 = \theta^{-1}([\varepsilon, +\infty)).$$

So $\mathcal{U}_0$ is the complement of the regular tubular neighbourhood of $V \cup H$ in $\mathbb{CP}^{n+1}$.

Moreover, $\mathcal{U}_0$ is a manifold with boundary, homotopy equivalent to $\mathcal{U}$ (eg see Dimca [8, page 149]). Note that while $\mathcal{U}_0$ has the homotopy type of a finite $(n+1)$–dimensional CW complex, its boundary $\partial\mathcal{U}_0$ is a smooth, compact, real $(2n+1)$-dimensional manifold. The inclusion $\partial\mathcal{U}_0 \hookrightarrow \mathcal{U}_0$ is an n–homotopy equivalence [8, Proposition (5.2.31)], hence $\pi_i(\partial\mathcal{U}_0) = \pi_i(\mathcal{U}_0)$ for $i < n$, and we have an epimorphism $\pi_n(\partial\mathcal{U}_0) \twoheadrightarrow \pi_n(\mathcal{U}_0)$. Therefore

$$\rho\colon \pi_1(\partial\mathcal{U}_0) \twoheadrightarrow \pi_1(\mathcal{U}_0) = \pi_1(\mathcal{U}) \xrightarrow{f_*} \pi_1(\mathbb{C}^*) = \mathbb{Z}$$

is an epimorphism, which defines the infinite cyclic cover $(\partial\mathcal{U}_0)^c$ of $\partial\mathcal{U}_0$.

We refer to $\partial\mathcal{U}_0$ as the *boundary manifold* of the hypersurface complement $\mathcal{U}$. For a study of topological properties of such boundary manifolds, see Cohen and Suciu [6; 7].

Proposition 6.1 *We have the Γ–module isomorphisms*

$$H_i^{\rho}(\partial\mathcal{U}_0, \mathbb{Q}[t,t^{-1}]) \cong H_i((\partial\mathcal{U}_0)^c) \cong H^{2n+1-i}(V \cup H; \mathcal{R}^\bullet). \tag{6-1}$$

In particular, $H_i((\partial\mathcal{U}_0)^c)$ is a torsion Γ–module and $C_*^{\rho}(\partial\mathcal{U}_0, \mathbb{Q}(t))$ is $\mathbb{Q}(t)$–acyclic. Moreover, the zeros of the Alexander polynomial associated to $H_i((\partial\mathcal{U}_0)^c)$ are roots of unity for all i, and have order d except for $i = n$. Finally, $H_i((\partial\mathcal{U}_0)^c)$ is a semi-simple Γ–module for $i \neq n$.

Proof We have the isomorphisms of Γ–modules (see [25, Corollary 3.4, Lemma 3.5])

$$IH_i^{\bar{m}}(\mathbb{CP}^{n+1}, \mathcal{L}) \cong H_i(\mathcal{U}, \mathcal{L}) \cong H_i(\mathcal{U}_0, \mathcal{L}) \cong H_i^{\rho}(\,U_0, \mathbb{Q}[t,t^{-1}]), \tag{6-2}$$

$$IH_i^{\bar{l}}(\mathbb{CP}^{n+1}; \mathcal{L}) \cong H_i^{\mathrm{BM}}(\mathcal{U}; \mathcal{L}) \cong H_i^{\rho}(\mathcal{U}_0, \partial\mathcal{U}_0; \mathbb{Q}[t,t^{-1}]), \tag{6-3}$$

where H_*^{BM} denotes the Borel–Moore homology and the last isomorphism in (6-3) follows by Poincaré duality and homotopy equivalence. So, by comparing the homology exact sequence (with $\mathbb{Q}[t,t^{-1}]$–coefficients) of the pair $(\mathcal{U}_0, \partial\mathcal{U}_0)$ with the hypercohomology long exact sequence of the distinguished triangle defining the peripheral complex, we obtain the isomorphism of Γ–modules

$$H_i^{\rho}(\partial\mathcal{U}_0, \mathbb{Q}[t,t^{-1}]) \cong H^{2n+1-i}(\mathbb{CP}^{n+1}; \mathcal{R}^\bullet). \tag{6-4}$$

And, since $\mathcal{R}^\bullet$ is supported on $V \cup H$, we obtain the isomorphisms in (6-1). Moreover, as $\mathcal{R}^\bullet$ is a Γ–torsion sheaf complex, (6-4) gives that $H_i((\partial\mathcal{U}_0)^c) \cong H_i^{\rho}(\partial\mathcal{U}_0, \mathbb{Q}[t,t^{-1}])$ is a Γ–torsion module. In particular, the chain complex $C_*^{\rho}(\partial\mathcal{U}_0, \mathbb{Q}(t))$ is $\mathbb{Q}(t)$–acyclic. The zeros of the corresponding Alexander polynomials are roots of unity since this is the case for the Alexander polynomials associated to the modules $H^{2n+1-i}(V \cup H; \mathcal{R}^\bullet)$ (eg see [25]). The remaining claims follow from Theorem 2.2 and Proposition 4.1. □

The following result is a direct consequence of Proposition 6.1 and Corollary 3.3.

Corollary 6.2 *The Alexander modules $H_i((\partial\mathcal{U}_0)^c)$ of the boundary manifold $\partial\mathcal{U}_0$ are endowed with mixed Hodge structures induced from the peripheral complex for all i. Moreover, for $i \neq n$, this mixed Hodge structure is compatible with the Γ–action, ie $t\colon H_i((\partial\mathcal{U}_0)^c) \to H_i((\partial\mathcal{U}_0)^c)$ is a mixed Hodge structure homomorphism for $i \neq n$.*

Proof By Corollary 3.3, the peripheral complex $\mathcal{R}^\bullet$ underlies a (shifted) mixed Hodge module. Hence, the $\mathbb{Q}$–vector space isomorphism (underlying the Γ–module isomorphism of Proposition 6.1)

$$H_i((\partial\mathcal{U}_0)^c) \cong H^{2n+1-i}(V \cup H; \mathcal{R}^\bullet) \tag{6-5}$$

defines a mixed Hodge structure on $H_i((\partial\mathcal{U}_0)^c)$ for all i.

In order to prove the second claim, note that by [23] the mixed Hodge structure on

$$H_c^{2n-i}(F_0;\psi_f\mathbb{Q}) \cong H_i(F_h) \quad \text{for } i<n$$

is compatible with the Γ–action. Then the isomorphism (4-14) shows that the resulting mixed Hodge structure on $H_i((\partial\mathcal{U}_0)^c)$ has the same property for $i<n$.

By Alexander duality on $H_*(F_h)$, the mixed Hodge structure on $H_i(F_h,\partial F_h)$ is compatible with the Γ–action for $i>n$. Then, by using the isomorphism (4-15), the resulting mixed Hodge structure on $H_i((\partial\mathcal{U}_0)^c)$ satisfies the same property for $i>n$. □

Our next result gives a geometric interpretation of the homology of the boundary manifold.

Let $g=x_0\tilde{f}$ be the homogeneous polynomial of degree $d+1$ whose zero locus is the divisor $V\cup H$. Consider the associated Milnor fibre $F_g=\{g=1\}$ and its boundary manifold ∂F_g. Then there exists a natural $(d+1)$–fold covering map (see [8, page 149])

$$\partial F_g \to \partial\mathcal{U}_0. \tag{6-6}$$

Proposition 6.3 *The covering map* (6-6) *induces isomorphisms of* $\mathbb{Q}$*–vector spaces*

$$H_i(\partial F_g)\cong H_i(\partial\mathcal{U}_0) \quad \text{for } i\neq n,\ n+1. \tag{6-7}$$

Moreover, if the complex numbers $\lambda_\alpha=\exp(2\pi i\alpha/(d+1))$*, with* $\alpha=1,2,\dots,d$*, are not among the roots of* $\psi_n(t)$*, then the isomorphism* (6-7) *holds for all* i*. In particular, this is the case if* $\mu=0$ *(eg* f *is homogeneous).*

Proof Let $N(\lambda,i)$ be the number of direct summands in the $(t-\lambda)$–torsion part of $H_i((\partial\mathcal{U}_0)^c;\mathbb{C})$, ie the number of the Jordan blocks with eigenvalue λ for the automorphism on $H_i((\partial\mathcal{U}_0)^c;\mathbb{C})$ induced by the Γ–action. Define a rank-one local system $\mathcal{L}_\lambda$ on $\partial\mathcal{U}_0$ by the composed map

$$\pi_1(\partial\mathcal{U}_0)\xrightarrow{\rho}\mathbb{Z}\longrightarrow\mathbb{C}^*,$$

where the last map is defined by $1_{\mathbb{Z}}\mapsto\lambda$. If $\lambda=1$, then $\mathcal{L}_\lambda=\mathbb{C}$. The Wang exact sequence

$$\cdots\longrightarrow H_i((\partial\mathcal{U}_0)^c;\mathbb{C})\xrightarrow{t-\lambda}H_i((\partial\mathcal{U}_0)^c;\mathbb{C})\longrightarrow H_i(\partial\mathcal{U}_0;\mathcal{L}_\lambda)\longrightarrow H_{i-1}((\partial\mathcal{U}_0)^c;\mathbb{C})\longrightarrow\cdots$$

yields (eg see [12, Theorem 4.2]) that

$$\dim H_i(\partial\mathcal{U}_0;\mathcal{L}_\lambda)=N(\lambda,i)+N(\lambda,i-1) \tag{6-8}$$

for all i. On the other hand, by using the $(d+1)$–fold covering map (6-6), we have that

$$H_i(\partial F_g;\mathbb{C}) \cong \bigoplus_{\lambda^{d+1}=1} H_i(\partial\mathcal{U}_0;\mathcal{L}_\lambda). \tag{6-9}$$

If $\lambda^d \neq 1$, then it follows from Proposition 6.1 that $N(\lambda,i)=0$ for $i\neq n$. (Note that $\gcd(d,d+1)=1$.) So, by using (6-8), we get the isomorphism (6-7) for $i\neq n,\ n+1$.

Moreover, if the complex numbers $\lambda_\alpha=\exp(2\pi i\alpha/(d+1))$, with $\alpha=1,2,\dots,d$, are not among the roots of $\psi_n(t)$, then the short exact sequence (4-16) shows that $N(\lambda,n)=0$ for $\lambda\in\{\lambda_\alpha \mid \alpha=1,2,\dots,d\}$. So, in view of (6-8), the isomorphism (6-7) holds also for $i=n,\ n+1$. In particular, this is the case when $\mu=0$, since by [23, Proposition 4.2] it follows that $\psi_n(t)=h_n(t)$ has only roots of unity of order d. □

Remark 6.4 The natural inclusion $\partial\mathcal{U}_0\hookrightarrow\mathcal{U}_0$ is an n–homotopy equivalence, and so is the inclusion $\partial F_g\hookrightarrow F_g$ (see [8, Proposition (3.2.4)]). Then Proposition 6.3 yields that $H_i(F_g)\cong H_i(\mathcal{U})$ for $i<n$ (compare with [10, Theorem 1.4]).

Example 6.5 Let $V\cup H$ be the hypersurface in $\mathbb{CP}^{n+1}$ defined by $g=x_0x_1\cdots x_{n+1}$. Then both ∂F_g and $\partial\mathcal{U}_0$ are homotopy equivalent to $S^n\times(S^1)^{n+1}$ (see [8, Examples (5.2.29)]).

7 Alexander polynomial estimates via Reidemeister torsion

In this section, we refine the error estimates for Alexander polynomials given in Section 4 by making use of Reidemeister torsion and the intersection form.

Proposition 6.1 can be used to prove the following refinement of Theorem 4.3:

Theorem 7.1 *Assume that the degree-d polynomial $f\colon\mathbb{C}^{n+1}\to\mathbb{C}$ is transversal at infinity. Let ϕ^ρ be the intersection form for $(\mathcal{U}_0,\partial\mathcal{U}_0)$ associated to ρ. Then, with the notations from Section 4, we have*

$$\det(\phi^\rho)=\varphi(t), \tag{7-1}$$

$$h_n(t)\cdot\psi_n(t)=\delta_n^2(t)\cdot\det(\phi^\rho). \tag{7-2}$$

Moreover, $\deg(\det(\phi^\rho))=\deg\varphi(t)\le 2d\cdot\mu$, *where* $\mu=|\chi(\mathcal{U})|$.

Proof By Theorem 5.4, we have

$$\tau_\rho(\mathcal{U}_0) = \tau_\rho(\mathcal{U}) = \prod_{i=0}^{n} \delta_i(t)^{(-1)^{i+1}}. \tag{7-3}$$

Since $\mathcal{U}_0$ has the homotopy type of a finite $(n+1)$–dimensional CW complex and the complex $C_*^\rho(\partial\mathcal{U}_0, \mathbb{Q}(t))$ is $\mathbb{Q}(t)$–acyclic, Lemma 5.5 yields the Alexander polynomial identity

$$\tau_\rho(\partial\mathcal{U}_0) = \prod_{i=0}^{n} \{\delta_i(t) \cdot \overline{\delta_i(t)}\}^{(-1)^{i+1}} \cdot [\det(\phi^\rho)]^{(-1)^{n+1}}. \tag{7-4}$$

On the other hand, by using Theorem 5.4 and Proposition 6.1, we have

$$\tau_\rho(\partial\mathcal{U}_0) = \prod_{i=0}^{2n} r_i(t)^{(-1)^{i+1}}. \tag{7-5}$$

Recall that, by (4-3), the polynomials $r_i(t)$ and $\delta_i(t)$ are related by

$$r_i(t) = \begin{cases} \delta_i(t) & i < n, \\ \overline{\delta_{2n-i}(t)} & i > n. \end{cases} \tag{7-6}$$

So, by plugging (7-5) and (7-6) into formula (7-4), we obtain

$$r_n(t) = \delta_n(t) \cdot \overline{\delta_n(t)} \cdot \det(\phi^\rho). \tag{7-7}$$

Therefore,

$$\det(\phi^\rho) = \frac{r_n(t)}{\delta_n(t) \cdot \overline{\delta_n(t)}} = \varphi(t) \tag{7-8}$$

and, by using Theorem 4.3 and Remark 4.4, we get the identity

$$h_n(t) \cdot \psi_n(t) = \delta_n^2(t) \cdot \det(\phi^\rho). \tag{7-9}$$

The degree estimate $\deg(\det(\phi^\rho)) = \deg\varphi \le 2d \cdot \mu$ follows from Proposition 4.6. □

Remark 7.2 Since V intersects H transversally, we can take the (closed) regular neighbourhood N of $V \cup H$ to be the union of a regular neighbourhood $N(V)$ of V with a tubular neighbourhood of the hyperplane at infinity (after rounding corners). Then

$$\partial\mathcal{U}_0 = \big(S_R^{2n+1} \setminus (S_R^{2n+1} \cap N^\circ(V))\big) \cup (B_R \cap \partial N(V)), \tag{7-10}$$

where B_R is a closed large ball of radius R in $\mathbb{C}^{n+1}$ with boundary sphere S_R^{2n+1}. In [25, Proposition 4.9] it is shown that the infinite cyclic cover of $S_R^{2n+1} \setminus (S_R^{2n+1} \cap N(V))$ is homotopy equivalent to the Milnor fibre F_h. Moreover, if $(B_R \cap \partial N(V))^c$ denotes the corresponding infinite cyclic cover of $B_R \cap \partial N(V)$, it follows as in Lemma 4.5 that

$$H_i((B_R \cap \partial N(V))^c) \cong H_c^{2n+1-i}(F_0, \psi_f^S \mathbb{Q}) \tag{7-11}$$

for all i. These two facts together give a geometric interpretation of Theorem 7.1, which is also consistent with the proof of [5, Theorem 5.6].

Remark 7.3 If $\mu = 0$, then the chain complex $C_*^{\rho}(\mathcal{U}_0, \mathbb{Q}(t))$ is $\mathbb{Q}(t)$–acyclic, thus the intersection pairing of Section 5.3 is trivial. In this case, we have $\det(\phi^{\rho}) = 1$. This fact, coupled with the previous theorem, gives another proof of the result obtained in [23], asserting that

$$\mu = 0 \implies \varphi = \varphi_1 = \varphi_2 = 1. \tag{7-12}$$

Example 7.4 If $\mu = 0$ (eg f is homogeneous), then $\delta_i(t) = h_i(t)$ for all i. Then it is known (see [8, (4.1.21)]) that

$$\prod_{i=0}^{n} h_i(t)^{(-1)^{i+1}} = (t^d - 1)^{-\chi(F_h)/d}. \tag{7-13}$$

So,

$$\tau_{\rho}(\partial \mathcal{U}_0) = (t^d - 1)^{-2\chi(F_h)/d}. \tag{7-14}$$

As an application of Theorem 7.1, we have the following:

Corollary 7.5 *Assume that the polynomial $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is transversal at infinity and the hypersurface F_0 has only isolated singularities. Then*

$$(t-1)^{\mu + (-1)^{n+1}} (t^d - 1)^{\xi} \prod_{p \in \mathrm{Sing}(F_0)} \Delta_p(t) = \delta_n(t)^2 \cdot \det(\phi^{\rho}), \tag{7-15}$$

where $\Delta_p(t)$ is the (top) local Alexander polynomial associated to the singular point $p \in \mathrm{Sing}(F_0)$ and $\xi = ((d-1)^{n+1} + (-1)^n)/d$. In particular, $\deg(\det(\phi^{\rho})) = \deg \varphi$ is even.

Proof We only need to compute the polynomials $\psi_n(t)$ and $h_n(t)$. For the case of isolated singularities, we have by [23, Section 5.2.1] that

$$\psi_n(t) = (t-1)^{\mu} \prod_{p \in \mathrm{Sing}(F_0)} \Delta_p(t), \tag{7-16}$$

while [8, Example (4.1.23)] provides another equality,

$$h_n(t) = (t-1)^{(-1)^{n+1}} (t^d - 1)^{\xi}, \tag{7-17}$$

where $\xi = ((d-1)^{n+1} + (-1)^n)/d$. Then (7-15) follows from Theorem 7.1.

Since F_0 has only isolated singularities, $V \cap H$ is a smooth hypersurface in H. Then $\{h + x_0^d = 0\}$ is a smooth degree-d hypersurface in $\mathbb{CP}^{n+1}$. By [8, Corollary (5.4.4)], $\chi(F_0, \psi_f \mathbb{Q}) + \chi(H \cap V)$ equals the Euler characteristic number of any smooth degree-d hypersurface in $\mathbb{CP}^{n+1}$, so, in particular,

$$\chi(F_0, \psi_f) + \chi(H \cap V) = \chi(\{h + x_0^d = 0\}). \tag{7-18}$$

Note also that

$$\chi(F_h) + \chi(H \cap V) = \chi(\{h + x_0^d = 0\}). \tag{7-19}$$

By the last two identities, we get $\chi(F_h) = \chi(F_0, \psi_f \mathbb{Q})$, which shows that $\deg \varphi_1 = \deg \varphi_2$. So $\deg \varphi = \deg \varphi_1 + \deg \varphi_2 \equiv 0 \pmod 2$. □

Remark 7.6 (a) When $n = 1$, Corollary 7.5 also follows from [5, Corollary 5.8].

(b) It would be interesting to see if the above property of (the degree of determinant of) the intersection pairing remains valid if f has arbitrary singularities.

(c) For the case of isolated singularities, μ is given by the formula

$$\mu = (d-1)^{n+1} - \sum_{p \in \mathrm{Sing}(F_0)} \mu_p, \tag{7-20}$$

where μ_p is the Milnor number of f at p (see [13]). Then the degree estimates of Theorem 7.1, together with Corollary 7.5, yield that

$$2(d-1)^{n+1} = 2 \deg \delta_n(t) + \deg(\det(\phi^\rho)) \leq 2 \deg \delta_n(t) + 2d \cdot \mu.$$

Therefore,

$$\deg \delta_n(t) \geq (d-1)^{n+1} - d \cdot \mu. \tag{7-21}$$

In particular, we obtain a non-vanishing result for $H_n(\mathcal{U}^c)$ for small μ. Such examples (with $\mu = 1, 2$) are given in [16].

Example 7.7 If F_0 is smooth, then $\delta_n(t) = 1$ (see [21, Lemma 1.5]) and $\mu = (d-1)^{n+1}$. So, by Corollary 7.5, we conclude that

$$\det(\phi^\rho) = (t-1)^{(d-1)^{n+1} + (-1)^{n+1}} (t^d - 1)^\xi,$$

where $\xi = ((d-1)^{n+1} + (-1)^n)/d$.

Example 7.8 In relation to Remark 7.6(c), consider the hypersurface in $\mathbb{CP}^3$ defined by $V = \{x_0x_1x_2 + x_3^3 = 0\}$. Then V has only isolated singularities and it is known that $\mu = 2$ for any generic hyperplane at infinity (see [16, Conjecture 20]). So, $\deg \delta_2(t) \geq (3-1)^3 - 3 \cdot 2 = 2$.

Example 7.9 Consider the hypersurface in $\mathbb{CP}^2$ defined by $V = \{x_0x_1^3 + x_1^4 + x_2^4 = 0\}$. The singular locus of V is just one point, $p = [1, 0, 0]$. Let $H = \{x_0 = 0\}$ be the hyperplane at infinity. Note that $V \cap H$ is a smooth hypersurface in H, hence V is transversal to H. The link pair of the point p in $(\mathbb{CP}^2, V)$ is obtained by intersecting the affine variety $x_1^3 + x_1^4 + x_2^4 = 0$ in $\mathbb{C}^2$ with a small sphere about the origin. Since we work in a neighbourhood of the origin, by an analytic change of coordinates this is the same as the link pair of the origin in the variety $x_1^3 + x_2^4 = 0$. The polynomial $x_1^3 + x_2^4$ is weighted homogeneous with weighted degree 12 for the weight $(4, 3)$, and the characteristic polynomial of the monodromy homeomorphism of the associated Milnor fibration is $(t^4 - t^2 + 1)(t^2 - t + 1)$. So the link of the singular point is a rational homology sphere, which in turn yields that F_0 is a rational homology manifold. In particular, $\delta_1(1) \neq 0$ (eg see [23, Corollary 5.4]). Also note that the local Alexander polynomial of the link of the singularity has prime divisors, none of which divides $t^4 - 1$. Thus, they cannot be among the prime divisors of $\delta_1(t)$ (see [21]), hence $\delta_1(t) = 1$. Equation (7-20) yields that $\mu = 3$. By Corollary 7.5, we conclude that

$$\det(\phi^\rho) = (t-1)^4(t^4-1)^2(t^4-t^2+1)(t^2-t+1).$$

Note also that $\gcd(12, 5) = 1$, so it follows from by Proposition 6.3 that ∂F_g and $\partial \mathcal{U}_0$ have the same rational homology groups, where $g = x_0(x_0x_1^3 + x_1^4 + x_2^4)$.

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School of Mathematical Sciences, University of Science and Technology of China
No 96, JinZhai Road, Hefei, 230026, China

Department of Mathematics, University of Wisconsin–Madison
480 Lincoln Drive, Office 713, Madison, WI 53706-1388, USA

`liuyq@mail.ustc.edu.cn`, `maxim@math.wisc.edu`

`http://home.ustc.edu.cn/~liuyq/`, `http://www.math.wisc.edu/~maxim/`

Received: 13 June 2014