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Algebraic & Geometric Topology 15 (2015) 3009–3022

Constructing thin subgroups commensurable with the figure-eight knot group

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We find infinitely many lattices in $\mathrm{SL}(4,\mathbf{R})$, each of which contains thin subgroups commensurable with the figure-eight knot group.

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1 Introduction

Let Γ be a lattice in a semisimple Lie group G. Then following Sarnak [22], one says that a subgroup Δ of Γ is *thin* if Δ has infinite index in Γ, and is Zariski dense in G. Since it is rather easy to exhibit Zariski dense subgroups of lattices that are free products, the case of most interest is that the thin group Δ is finitely generated and does not decompose as a free product.

In this note, we shall exhibit subgroups of the fundamental group of the figure-eight knot as subgroups of infinitely many incommensurable lattices. A precise statement will be given shortly.

We begin by briefly describing where the lattices in question arise. They were constructed by Morris [18, Theorem 6.55] by a rather general construction which involves L, a real quadratic extension of $\mathbf{Q}$ and D, a central simple division algebra of degree d over L. However in our situation we may assume that $D = L$, and we state only this special case.

In this paper, the field L is a real quadratic extension of $\mathbf{Q}$ and if $A \in \mathrm{SL}(4, L)$, we denote by A^* the matrix obtained by taking the transpose of the matrix obtained from A by applying τ (the nontrivial Galois automorphism) to all its entries. Then one has:

Theorem 1.1 *Suppose that L is a real quadratic extension of $\mathbf{Q}$, with Galois automorphism τ. Suppose that $b_1, \dots, b_4$ are nonzero elements of $\mathbf{Z}$. Set $J = \mathrm{diag}(b_1, \dots, b_4)$. Then the group*

$$\mathrm{SU}(J, \mathcal{O}_L, \tau) = \{A \in \mathrm{SL}(4, \mathcal{O}_L) \mid A^* J A = J\}$$

is a lattice in $\mathrm{SL}(4,\mathbf{R})$.

Published: 10 December 2015 DOI: 10.2140/agt.2015.15.3009

We also note from Morris [18, Proposition 6.55] that, in the case being considered here (when $D = L$), the corresponding forms will represent zero nontrivially, and so the lattices produced are nonuniform.

We shall show:

Theorem 1.2 *Let Γ denote the fundamental group of the figure-eight knot and $L_d = \mathbf{Q}(\sqrt{d})$ for d a positive, square-free integer. Then for every such d, there is a subgroup of finite index in Γ, H_d, and a faithful, Zariski dense representation into a lattice:*

$$r_d\colon H_d \longrightarrow \mathrm{SU}(J_d, \mathcal{O}_{L_d}, \tau_d).$$

The power of this statement lies in the fact that r_d is faithful and Zariksi dense; much of the work of the paper is devoted to this aspect. Once one knows this, it follows immediately that the image group $r_d(H_d)$ has infinite index in the lattice $\mathrm{SU}(J_d, \mathcal{O}_{L_d}, \tau_d)$, since deep results of Margulis (see Morris [18, Chapter 13]) imply that such lattices do not have subgroups which admit homomorphisms onto $\mathbf{Z}$. Moreover, subgroups of finite index in Γ are freely indecomposable, so that it follows from Theorem 1.2 that the image is thin.

This paper is organized as follows. The starting point is a pair of representations of Γ constructed independently by Ballas [3] (which we denote ϕ_t) and Thistlethwaite (private communication) (which we denote ρ_v). These representations are conjugate when $v = 2t$ and while one could work with just one of them, each presents sufficiently interesting features that it seems worthwhile to include both. Explicit generators for both representations are included in the appendix. We begin in Section 2 with the algebraic considerations needed to construct the representations r_d of Theorem 1.2 from the Ballas–Thistlethwaite representation. The main work here is proving that the representation has an integral character in an appropriate sense; the remaining ingredients in constructing r_d are fairly standard once this has been proved.

The results of Section 3 and Section 4 are geometric and lie deeper. Specifically, Section 3 is devoted to the proof that the representations r_d are faithful using results from real projective geometry.

In Section 4 we show that, with the exception of the representation corresponding to the complete hyperbolic structure, ϕ_t has Zariski dense image in $\mathrm{SL}(4, \mathbf{R})$.

2 Constructing r_d

This section is devoted to outlining the computations necessary to exhibit the representations r_d of Theorem 1.2. With a view to a point that arises in the sequel (see Theorem 3.2), we begin by proving the following:

Theorem 2.1 *The representations ρ_v are absolutely irreducible for real $v > 0$.*

Proof If there is an invariant subspace of dimension one, then any commutator must have eigenvalue $= 1$. One can check that the commutator $[x^2, y]$ has such an eigenvalue only when v is a root of $v^2 + v + 1$. The same considerations applied to the transpose representation show there can be no invariant subspace of ρ_v of dimension three.

The case of a two-dimensional invariant subspace is more subtle. One computes that the characteristic polynomial of the longitude is $(-Q + v)^3(-1 + Qv^3)/v^3$, so that if there were such an invariant subspace, either $\rho_v(\lambda)$ or $\rho_v^T(\lambda)$ would have the $1/v^3$ eigenvalue appearing in that subspace. It follows that the orbit of that eigenvector would be two-dimensional. However, one can compute that this happens for neither of these representations. □

Our next claim concerns the traces of the ρ_v.

Lemma 2.2 *The representation of the figure-eight knot given by ρ_v has traces lying in $\mathbf{Z}[v, 1/v]$.*

Proof This involves some computation, as it must. We indicate the mathematics behind the idea, with a sample of its implementation in [6].

Regarding v as a real transcendental, using Burnside's theorem [14, page 648, Corollary 3.4], or by inspection, one can find elements $g_1, \dots, g_{16} \in \Gamma$ which are a basis for the vector space of 4×4 matrices $M(4, \mathbf{R})$; we always choose g_1 to be the identity matrix, which we denote by I.

Let $g_1^*, \dots, g_{16}^*$ be the dual basis with respect to trace, ie $\operatorname{tr}(g_i \cdot g_j^*) = \delta_{ij}$. One can use the action of the figure-eight knot group on this dual basis by left multiplication to obtain a 16–dimensional representation of the group, ie if $\gamma \in \Gamma$, then its action is defined by

$$\gamma \cdot g_i^* = \sum_j \alpha_{ij}(\gamma) g_j^*.$$

Taking traces in this equation, we get

$$\operatorname{tr}(\gamma \cdot g_i^*) = \sum_j \alpha_{ij}(\gamma) \operatorname{tr}(g_j^*).$$

Notice that since we have chosen $g_1 = I$, we have that $\operatorname{tr}(g_j^*) = \operatorname{tr}(g_1 \cdot g_j^*) = \delta_{1j}$, in particular these are all rational integers. Writing $I = \sum \tau_j g_j^*$, notice that $\tau_i = \operatorname{tr}(g_i)$ by duality; one verifies that these traces only have denominators which are powers of v.

Moreover, multiplying by γ and taking traces, we have

$$\operatorname{tr}(\gamma) = \sum_j \operatorname{tr}(g_j)\operatorname{tr}(\gamma \cdot g_j^*) = \sum_{j,k} \operatorname{tr}(g_j)\operatorname{tr}(g_k^*)\alpha_{jk}(\gamma) = \sum_j \operatorname{tr}(g_j)\alpha_{j1}(\gamma).$$

The upshot of these two computations is the following: If with some choice of basis, one computes that the denominators of the entries for the associated 16×16 regular representation, then this collection of denominators contains the denominators for the traces of the original collection of matrices of Γ.

Therefore, if one could find a basis for which this construction gave $\mathbf{Z}[v, 1/v]$–matrices $(\alpha_{ij}(\gamma))$, then this would prove the result claimed by the lemma. However, this appears to be hard. We bypass this difficulty by constructing two representations via two different choices of basis $\{g_i\}$. For the first choice one sees all the matrix entries have denominators v, $(-1+v)$ and $(1+v)$, for the second, one sees denominators v, $(1+3v+v^2)$ and $(1+3v+4v^2)$. Since traces are not dependent on choice of basis, the denominators of the trace of the original representation of Γ must lie in the intersection of these two sets, ie powers of v. □

Corollary 2.3 *If one specializes v to be a unit in any number field, then the resulting representation has integral trace.*

Since when one sets $v = 2t$ the representation ρ_v is conjugate to the representation ϕ_t and this representation has entries lying in $\mathbf{Q}(t) = \mathbf{Q}(v/2)$, we are now in a position to apply the following lemma with $k = \mathbf{Q}(v/2)$.

Theorem 2.4 *Suppose that $G < \mathrm{SL}(4,k)$ is a finitely generated group with the property that $\operatorname{tr}(\gamma) \in \mathcal{O}_k$ for every $\gamma \in G$. Then G has a subgroup of finite index contained in $\mathrm{SL}(4,\mathcal{O}_k)$.*

Proof Consider

$$\mathcal{O}G = \left\{ \sum a_i \gamma_i \mid a_i \in \mathcal{O}_k, \gamma_i \in G \right\},$$

where the sums are finite. It is shown in [7, Proposition 2.2 and Corollary 2.3] that $\mathcal{O}G$ is an order of a central simple subalgebra $B \subset M(4,k)$ defined over k. Now while $\mathcal{O}G$ need not be an order in $M(4,k)$, it is known that it is contained in some maximal order $\mathcal{D}$ of $M(4,k)$ (see [20, page 131, Exercise 5] and [15, proof of Lemma 2.3]).

Now it is a standard fact that the groups of elements of norm 1 in orders contained in $M(4,k)$ are commensurable (since the intersection of two orders is an order and the unit groups of orders will be irreducible lattices in $\mathrm{SL}(4,\mathbf{R}) \times \mathrm{SL}(4,\mathbf{R})$ [18, Chapter 15I]). In particular, $\mathrm{SL}(4,\mathcal{O}_k)$ and $\mathcal{D}^1$ are commensurable. Let $\Delta = \mathrm{SL}(4,\mathcal{O}_k) \cap \mathcal{D}^1$, which has finite index in both groups. Then $G \leq \mathcal{D}^1$, so that $G \cap \Delta$ has finite index in G and lies inside $\mathrm{SL}(4,\mathcal{O}_k)$ as required. □

What have achieved to this point is that when one specializes v to be any unit of a number field k, there is a representation, namely $\phi_{v/2}$, of a subgroup of finite index in Γ whose image has entries lying in $\mathcal{O}_k$.

We now turn our attention to the unitary aspects required by Theorem 1.1. There is an obvious involution on the matrices in the image of ρ_v given by transposing and mapping $v \to 1/v$. We denote this operation by $A \to A^*$. A routine computation on the generators reveals:

Lemma 2.5 *There is a matrix Q_v with $\det(Q_v) \neq 0$ for which*

$$A^* \cdot Q_v \cdot A = Q_v$$

for all $A \in \phi_{v/2}(\Gamma)$.

Since it's required by the result of Theorem 1.1, for the rest of this paper we assume $k = \mathbf{Q}(\sqrt{d})$ is a real quadratic number field equipped with involution τ_d. Specializing v to be a unit of this field with the property that $\tau_d(v) = 1/v$ (eg one can take the square of a random unit of $\mathbf{Q}(\sqrt{d})$; we call these the *positive units*), we see that Lemma 2.5 and Theorem 2.4 taken together prove:

Theorem 2.6 *For each positive square free integer d, and a positive unit u in the ring of integers of $\mathbf{Q}(\sqrt{d})$, there is a representation of a subgroup of finite index $H_{d,u}$ in Γ:*

$$r_{d,u}\colon H_{d,u} \longrightarrow \mathrm{SU}(Q_v, \mathcal{O}_k, \tau_d).$$

There is one final consideration that must be addressed, namely the form described by Witte is diagonal and Q_v is not. However this concern is addressed as follows. It can easily be shown by using the Gram–Schmidt process for example, that there is a change of basis matrix $M_d \in \mathrm{GL}(\mathbf{Q}(\sqrt{d}))$ for which $M_d^* \cdot Q_v \cdot M_d$ is a diagonal form Δ_v. Then it is a standard argument (see for example [2, Lemma 2.2]) that the groups $\mathrm{SU}(Q_v, \mathcal{O}_k, \tau_d)$ and $\mathrm{SU}(\Delta_v, \mathcal{O}_k, \tau_d)$ are commensurable, so at possibly the expense of passing to a further subgroup of finite index, we obtain a representation

$$r_{d,u}\colon H_{d,u} \longrightarrow \mathrm{SU}(\Delta_v, \mathcal{O}_k, \tau_d)$$

as required by Theorem 1.1.

3 Projective considerations: r_d is faithful

This section contains the proof that one can find many representations r_d which are faithful. It is basically geometric in nature and relies upon the fact that these representations are associated to convex real projective structures on the figure-eight knot.

Throughout we denote the complement of the figure-eight knot in $\mathbf{S}^3$ by M and denote its fundamental group by Γ. We begin with some considerations of a fairly general nature. We recall that a group is said to be *nonradical* if there is no infinite normal nilpotent subgroup. The figure-eight knot group (and indeed any finite-volume hyperbolic manifold group) is nonradical. The following is a classical theorem of Zassenhaus.

Theorem 3.1 (Kapovich [13, Theorem 8.4]) *Suppose that G is a finitely generated nonradical group and $\mathcal{L}$ any linear Lie group and that $\{\sigma_n\}$ is a convergent sequence of discrete, faithful representations of G into $\mathcal{L}$, say $\sigma_n \to \sigma_\infty$. Then σ_∞ is discrete and faithful.*

We will also need the following:

Theorem 3.2 (Goldman and Choi [9]) *Let G be a finitely generated nonradical group. Suppose that Ω_n is a sequence of properly convex open domains in $\mathbf{RP}^d$ and $\sigma_n\colon G \to \mathrm{SL}(d+1,\mathbf{R})$ a convergent sequence of discrete faithful representations for which $\sigma_n(G) < \mathrm{Aut}(\Omega_n)$. Denote the limit representation by σ_∞. Then if σ_∞ is irreducible, it preserves some properly convex open subset of $\mathbf{RP}^d$.*

We define a subset of the representation variety $\Omega(G;\mathbf{RP}^d) < \mathrm{Hom}(G,\mathrm{SL}(d+1,\mathbf{R}))$ to be the set of representations $\sigma\colon G \to \mathrm{SL}(d+1,\mathbf{R})$ satisfying:

- σ is discrete and faithful.
- There is a properly convex, open domain $\Omega_\sigma \subset \mathbf{RP}^d$ for which $\sigma(G) < \mathrm{Aut}(\Omega_\sigma)$.

In the case that G is the fundamental group of eg a hyperbolic d–manifold, the set $\Omega(G;\mathbf{RP}^d)$ is nonempty, since it contains the representation corresponding to the complete structure $\rho_1 = \phi_{1/2}$.

Theorem 3.3 *The path component of the set $\mathcal{G} = \{v \mid \rho_v \in \Omega(\Gamma;\mathbf{RP}^3)\}$ which contains 1 is open and closed.*

In particular, once Theorem 3.3 is proven, we have that $\mathcal{G}$ is some interval, and therefore $\mathcal{G} = (0,\infty)$ since the only place the representations ρ_v fail to be defined is at $v = 0$. This implies that any specialization of v in $(0,\infty)$ is discrete and faithful, so that taken in conjunction with Theorem 2.6, we will have proved all of Theorem 1.1, barring the fact the image is Zariski dense.

Proof of Theorem 3.3 The fact that the set is closed follows from a concatenation of results proved above: Section 3 implies that the endpoint of a path of discrete faithful representations is discrete and faithful, then Theorem 3.2 implies the resulting representation is the holonomy of a properly convex structure, since we proved in Theorem 2.1 that the limit representation is irreducible.

Openness is a good deal more subtle. By applying a theorem of Cooper and Long [10, Theorem 0.4] we find that a small deformation of the holonomy of the complete hyperbolic structure of M whose restriction to some (hence any) peripheral subgroup has a common fixed point in $\mathbf{RP}^3$ will itself be the holonomy of a projective structure on M. Furthermore, the cusp of this projective structure is foliated by projective rays with a common endpoint.

A priori, this projective structure need not be properly convex. However the main result of [11] shows that if the cusp of the resulting projective structure satisfies slightly stronger hypotheses then the deformed projective structure will be properly convex. Roughly speaking, the additional hypothesis is that the cusp of the deformed projective structure must admit a second foliation by "strictly convex" hypersurfaces that is transverse to the previously mentioned foliation by projective rays.

In [4] it is shown that for all $t \in (0, \infty)$ the representation ρ_t satisfy the hypotheses of the previously mentioned theorems. Since the two families of representations are conjugate this implies that for all $u \in (0, \infty)$, ρ_u also satisfy the same hypotheses. As a result we find that for u sufficiently close to 1 that ρ_u is the holonomy of a properly convex projective structure on M. The holonomy of such a structure is necessarily discrete and faithful and so for u sufficiently close to 1 we find that $\rho_u \in \Omega(\Gamma; \mathbf{RP}^3)$. We thus conclude that $\mathcal{G}$ is open. □

4 Zariski denseness

In this section we analyze the Zariski closure of the groups $\phi_t(\Gamma)$ (which we denote G_t) proving in particular that away from the complete representation, this Zariski closure is all of $\mathrm{SL}(n, \mathbf{R})$. We adopt a largely geometric point of view; other approaches are possible, see the remarks at the conclusion of this section.

We begin by with some background and results which can be found in Benoist [8]. A comprehensive summary of the necessary background in algebraic groups, notions of proximality, and Zariski closures can be found in [16; 17]. We begin by defining *proximality* in the context of groups, group actions, and representations. In all cases proximality is related to the existence of unique attracting fixed points.

Let G be a subgroup of $\mathrm{SL}(n,\mathbf{R})$ and let g be an element of G. We say that g is *proximal* if it has a unique eigenvalue of largest modulus. In this case it is easy to see that this eigenvalue is real. If in addition this eigenvalue of largest modulus is positive then we say that g is *positive proximal*. A group is called *proximal* if it contains a proximal element and *positive proximal* if *every* proximal element is positive proximal.

A group action of G on $\mathbf{RP}^{n-1}$ is *proximal* if for any pair of points $x, y \in \mathbf{RP}^{n-1}$ there is a sequence $\{g_m\}$ of elements in G such that

$$\lim_{m\to\infty} g_m \cdot x = \lim_{m\to\infty} g_m \cdot y.$$

If G is a connected semisimple Lie group then a representation $\rho\colon G \to \mathrm{SL}(n,\mathbf{R})$ is *proximal* if the weight space corresponding to highest restricted weight is 1–dimensional. More specifically, if we let $G = KAN$ be an Iwasawa decomposition of G, where K is a maximal compact subgroup, A is a maximal abelian subgroup, and N is a maximal nilpotent subgroup, then the set

$$\{x \in \mathbf{R}^n \mid n(x) = x \text{ for all } n \in N\}$$

is a line.

We now discuss some relations between these notions. We say that a group $\Gamma \subset \mathrm{SL}(n,\mathbf{R})$ is *strongly irreducible* if every finite-index subgroup of Γ is irreducible. The following two theorems relate the various notions of proximality.

Theorem 4.1 [12, Theorem 2.9] *Let G be a subgroup of $\mathrm{SL}(n,\mathbf{R})$. The following are equivalent:*

(1) *G is strongly irreducible and proximal.*

(2) *G is irreducible and its action on $\mathbf{RP}^{n-1}$ is proximal.*

Theorem 4.2 [1, Theorem 6.3] *Let G be a semisimple Lie group with finite center and let $\rho\colon G \to \mathrm{SL}(n,\mathbf{R})$ be an irreducible representation. The following are equivalent:*

(1) *ρ is proximal.*

(2) *$\rho(G)$ is proximal.*

Given a strongly irreducible and proximal subgroup $G \subset \mathrm{SL}(n,\mathbf{R})$ we can define the *limit set* of G, which we denote by Λ_G as the closure of the set of attractive fixed points in $\mathbf{RP}^{n-1}$ of the proximal elements of G. In [12, Theorem 2.3] it is shown that the action of G on Λ_G is minimal (ie any nonempty closed G–invariant subset of $\mathbf{RP}^{n-1}$ contains Λ_G).

When G is a Zariski closed semisimple subgroup of $\mathrm{SL}(n,\mathbf{R})$ we can describe Λ_G more explicitly. Since G is proximal, Λ_G is nonempty and so let $x \in \Lambda_G$. Consider the orbit $G\cdot x$ of x. First, observe that $G\cdot x \subset \Lambda_G$ by G–invariance. Furthermore, since $G\cdot x$ is the orbit of an algebraic group acting algebraically on a variety we have that $G\cdot x$ is a smooth subvariety of $\mathbf{RP}^{n-1}$. If we let H be the Zariski closure of $G\cdot x$ then we see that $G\cdot x$ is open in H and so $H\backslash(G\cdot x)$ is a Zariski closed subset of $\mathbf{RP}^{n-1}$. Since Zariski closed sets are closed in the standard topology, minimality implies that $H\backslash(G\cdot x)$ is either empty or Λ_G. However, $x \notin H\backslash(G\cdot x)$ and so $H\backslash(G\cdot x)$ is empty. As a result we see that $G\cdot x$ is a nonempty closed G–invariant subset and thus $G\cdot x = \Lambda_G$. By minimality of the action on the limit set we also find that Λ_G is the *unique* closed orbit for the action of G on $\mathbf{RP}^{n-1}$.

We can say even more. Let $G = KAN$ be an Iwasawa decomposition and let x_N be the (unique) point in $\mathbf{RP}^{n-1}$ corresponding to a highest-weight vector. By our choice of x_N we see that the groups A and N both fix x_N and so $G\cdot x_N = K\cdot x_N$. Thus the orbit of x_N is closed and so $G\cdot x_N = \Lambda_G$ (ie the limit set of G is the orbit of a compact group.)

We can now identify the Zariski closure of $\phi_t(\Gamma)$, which we denote G_t. The representation $\phi_{1/2}$ corresponds to the holonomy of the complete finite-volume hyperbolic structure on the figure-eight knot and so by the Borel density theorem we have $G_{1/2} = \mathrm{SO}(3,1)$. The main goal of this section is to show that when $t \neq \frac{1}{2}$ that $G_t = \mathrm{SL}(4,\mathbf{R})$. The proof is based on the following heuristic:

A Lie subgroup of $\mathrm{SL}(4,\mathbf{R})$ *with a large orbit in* $\mathbf{RP}^3$ *must be large.*

Theorem 4.3 *If $t \neq \frac{1}{2}$ then $G_t = \mathrm{SL}(4,\mathbf{R})$. In particular, for $t \neq \frac{1}{2}$, $\phi_t(\Gamma)$ is Zariski dense.*

Proof Let $t \neq \frac{1}{2}$. We have previously seen that ρ_t is an absolutely irreducible representation whose image preserves a properly convex open subset of $\mathbf{RP}^3$. We first show that $\rho_t(\Gamma)$ is strongly irreducible. Suppose for contradiction that $\rho_t(\Gamma)$ is not strongly irreducible. By work of Benoist [8, Lemma 2.9] we can find a finite-index subgroup of Γ that splits as a nontrivial direct sum. However, any finite-index subgroup of Γ is the fundamental group of a finite-volume hyperbolic 3–manifold and such groups never admit nontrivial direct sum decompositions.

Let G_t^0 be the connected component of G_t containing the identity. We claim that G_t^0 is semisimple. To see this observe that by [12, Lemma 2.6] we find that G_t^0 acts irreducibly on $\mathbf{R}^4$. As a result the action of G_t^0 turns $\mathbf{R}^4$ into a simple $\mathbf{R}[G_t^0]$ module. Let R_t be the unipotent radical of G_t^0. Since R_t is unipotent and solvable

the Lie–Kolchin theorem implies that there is a nontrivial $\mathbf{C}[R_t]$–submodule, $E_{\mathbf{C}}$, of simultaneous 1–eigenvectors of R_t in $\mathbf{C}^4$. A simple computation shows that $E_{\mathbf{C}}$ is conjugation invariant and so there is a nontrivial $\mathbf{R}[R_t]$–submodule, $E_{\mathbf{R}} \subset \mathbf{R}^4$, of simultaneous 1–eigenvectors of R_t whose complexification is $E_{\mathbf{C}}$. However, since R_t is normal in G_t^0 we see that $E_{\mathbf{R}}$ is a nontrivial $\mathbf{R}[G_t^0]$–submodule. However, simplicity implies that this submodule must be all of $\mathbf{R}^4$. Therefore R_t acts trivially on $\mathbf{R}^4$ and is thus trivial. We conclude that G_t^0 is reductive. Furthermore, since G_t^0 is proximal and irreducible we see that it has trivial center and is thus semisimple.

The group $\phi_t(\Gamma)$ is proximal and so Theorem 4.2 implies that the representation induced by the inclusion of G_t^0 into $\mathrm{SL}(4,\mathbf{R})$ is also proximal.

Next we show that $\Lambda_{G_t^0}$ contains a codimension-1 submanifold. Let Γ_p be a peripheral subgroup of Γ. By work of [4, Section 6] it is possible to conjugate so that $\phi_t(\Gamma_p)$ to a lattice in the 2–dimensional abelian Lie group H of matrices of the form

$$\begin{pmatrix} 1 & 0 & s & s^2/2-t \\ 0 & e^t & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus we see that the Zariski closure of $\phi_t(\Gamma_p)$ (and hence G_t) contains H. A generic orbit of H can be written in homogenous coordinates as

$$\{[-\log(|x|)+y^2/2+c : \epsilon x : y : 1] \mid x>0\},$$

where $\epsilon \in \{\pm 1\}$ and $c \in \mathbf{R}$ (see Figure 1). Since G_t^0 is irreducible we see that $\Lambda_{G_t^0}$ contains a point, z of one of these orbits and so $H\cdot z \subset G\cdot z \subset \Lambda_{G_t^0}$. Furthermore, since $\Lambda_{G_t^0}$ is closed we see that it contains the closure of this orbit, which is the boundary of a properly convex domain $\Omega \subset \mathbf{RP}^3$.

There are now two cases to consider: either $\Lambda_{G_t^0}$ has codimension 0 and is thus equal to $\mathbf{RP}^3$ or $\Lambda_{G_t^0}$ is a codimension-1 submanifold of $\mathbf{RP}^3$ and is thus equal to $\partial\Omega$. In the first case work of Benoist [8, Lemma 3.9 and Corollary 3.5] shows that $G_t^0 = \mathrm{SL}(4,\mathbf{R})$, in which case we are done.

We now rule out the second case. In this case we see that there is a point $w \in \mathbf{RP}^3$ such that $\Lambda_{G_t^0} = K\cdot w = \partial\Omega$, where K is a maximal compact subgroup of G_t^0. As a result we see that (up to a subgroup of index 2) K preserves Ω. Since K is compact we see that there is a point $x_0 \in \Omega$ such that both x_0 and its dual point $x_0 \in \Omega^*$ are both fixed by K.[1] The hyperplane dual to x_0^* provides us an affine patch containing

[1] See [17, Section 9] for a definition of dual point of properly convex domains.

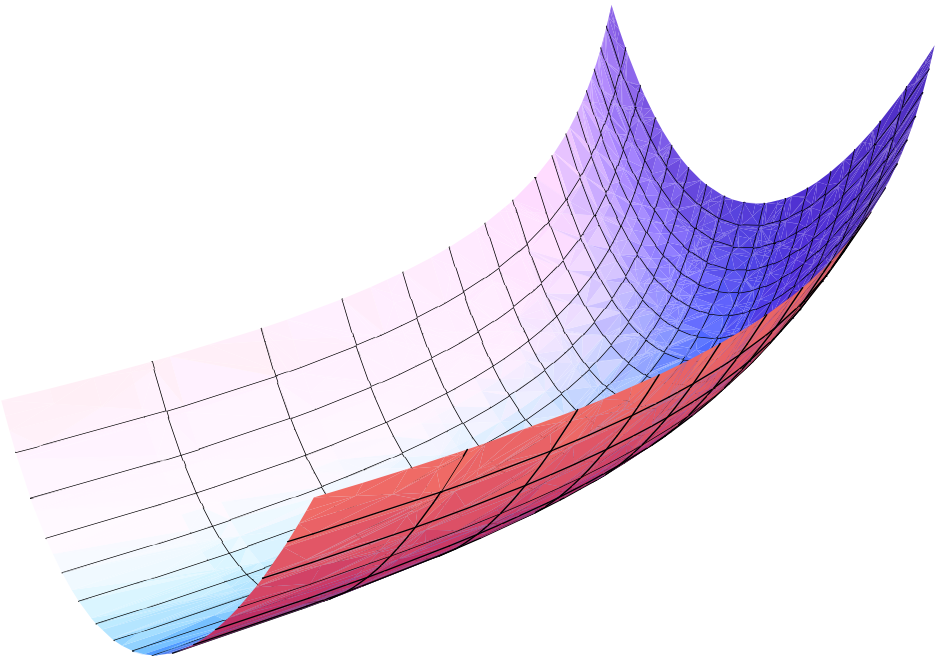

Figure 1: A generic orbit of H

Ω and for which x_0 is the center of mass of Ω. In these affine coordinates K acts by affine transformations fixing x_0.

If we let $\mathcal{E}$ be a John ellipsoid[2] for Ω centered at x_0 in this affine patch, then we see that K preserves $\mathcal{E}$. Since $\mathcal{E}$ is has maximal volume $\partial\mathcal{E}$ has nonempty intersection with $\partial\Omega$. Since K acts transitively on $\partial\Omega$ we see that $\partial\Omega \subset \partial\mathcal{E}$.

For dimensional reasons we see that $\partial\Omega$ has nonempty interior in $\partial\mathcal{E}$. Since $\partial\Omega$ is a K–orbit we see that it is an open subset of $\partial\mathcal{E}$. We conclude that $\partial\Omega = \partial\mathcal{E}$ and thus Ω is an ellipsoid. Since $\phi_t(\Gamma)$ preserves this ellipsoid we see that it is conjugate to a subgroup of $\mathrm{SO}(3,1)$. This is a contradiction since, for example the image of the longitude has an eigenvalue whose inverse is not also an eigenvalue. □

Remark Given the explicit nature of the subgroups in this paper there are alternative, more algebraic ways of proving Zariski denseness. For example, it suffices to show that the adjoint action of $\rho_t(\Gamma)$ on $\mathfrak{sl}_4$ is irreducible (a Mathematica notebook with irreducibility calculations for our groups can be found at [5]). The anonymous referee

[2] A John ellipsoid for a convex subset Ω of affine space is an ellipsoid of maximal Euclidean volume contained in Ω with the same center of mass.

also pointed out other techniques that could be employed to show that such explicit subgroups are Zariski dense (see [21] and [19, Theorem 9.10]). We chose to use the above proof because of its geometric nature which highlights the relationship between convex projective structures and the Zariski closure of their holonomy representations. We plan to pursue this relationship in more detail in future work.

Appendix: The matrices

The discrete faithful representation corresponding to the hyperbolic structure occurs at $v = 1$:

$$\rho_v(x) = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2}(1+1/v) & \frac{1}{\sqrt{12}}(1-1/v) \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}(1+1/v) & -\frac{1}{\sqrt{12}}(1-1/v) \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\rho_v(y) = \begin{pmatrix} v+\frac{1}{2} & -v+\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{12}}(7-4v) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{12}}(4/v-1) \\ \frac{1}{2}v & -\frac{1}{2}v & 1 & \frac{1}{\sqrt{3}}(1-v) \\ \frac{\sqrt{3}}{2}v & -\frac{\sqrt{3}}{2}v & 0 & 2-v \end{pmatrix}.$$

The discrete faithful representation corresponding to the hyperbolic structure occurs at $t = \frac{1}{2}$:

$$\phi_t(x) = \begin{pmatrix} 1 & 0 & 1 & t-1 \\ 0 & 1 & 1 & t \\ 0 & 0 & 1 & t+\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \phi_t(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2+1/t & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

The representations become conjugate for $v = 2t$.

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Received: 29 October 2014 Revised: 24 December 2014