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Algebraic & Geometric Topology 15 (2015) 3653–3705

The algebraic duality resolution at $p = 2$

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The goal of this paper is to develop some of the machinery necessary for doing $K(2)$–local computations in the stable homotopy category using duality resolutions at the prime $p=2$. The Morava stabilizer group $\mathbb{S}_2$ admits a surjective homomorphism to $\mathbb{Z}_2$ whose kernel we denote by $\mathbb{S}_2^1$. The algebraic duality resolution is a finite resolution of the trivial $\mathbb{Z}_2[[\mathbb{S}_2^1]]$–module $\mathbb{Z}_2$ by modules induced from representations of finite subgroups of $\mathbb{S}_2^1$. Its construction is due to Goerss, Henn, Mahowald and Rezk. It is an analogue of their finite resolution of the trivial $\mathbb{Z}_3[[\mathbb{G}_2^1]]$–module $\mathbb{Z}_3$ at the prime $p=3$. The construction was never published and it is the main result in this paper. In the process, we give a detailed description of the structure of Morava stabilizer group $\mathbb{S}_2$ at the prime 2. We also describe the maps in the algebraic duality resolution with the precision necessary for explicit computations.

55Q45; 55T99, 55P60

1 Introduction

Fix a prime p and recall that L_nS is the Bousfield localization of the sphere spectrum S with respect to the wedge $K(0) \vee \cdots \vee K(n)$, where $K(m)$ is the m^{th} Morava K–theory at the prime p. The chromatic convergence theorem of Hopkins and Ravenel [27, Section 8.6] states that the p–local sphere spectrum is the homotopy limit of its localizations L_nS. Further, there is a homotopy pull-back square:

$$\begin{tikzcd} L_nS \arrow[r] \arrow[d] & L_{K(n)}S \arrow[d] \\ L_{n-1}S \arrow[r] & L_{n-1}L_{K(n)}S \end{tikzcd}$$

In theory, the homotopy groups of S can be recovered from those of its Morava K–theory localizations $L_{K(n)}S$. For this reason, computing $\pi_*L_{K(n)}S$ is one of the central problems in stable homotopy theory. A detailed historical account of chromatic homotopy theory can be found in Goerss, Henn, Mahowald and Rezk [14, Section 1].

The difficulty of computing $\pi_*L_{K(n)}S$ varies with p and n. The computation of $\pi_*L_{K(1)}S$ is related to K–theory and is now well understood. For $n \geq 3$, almost

Published: 12 January 2016 DOI: 10.2140/agt.2015.15.3653

nothing is known, which leaves the case $n = 2$. For $p \geq 5$, $\pi_* L_{K(2)} S$ was computed by Shimomura and Yabe in [33]. Behrens gives an illuminating reconstruction of their results in [4]. The case when $p = 3$ proved much more difficult than the problem for $p \geq 5$. It is now largely understood due to the work of Shimomura, Wang, Goerss, Henn, Karamanov, Mahowald and Rezk (see Goerss and Henn [12], Goerss, Henn and Mahowald [13], Goerss, Henn, Mahowald and Rezk [14; 15], Henn, Karamanov and Mahowald [18] and Shimomura and Wang [32]).

The major breakthrough in understanding the case of $n = 2$ and $p = 3$ was the construction by Goerss, Henn, Mahowald and Rezk [14] of a finite resolution of the $K(2)$–local sphere called the *duality resolution*. The duality resolution comes in two flavors. The *algebraic* duality resolution is a finite resolution of the trivial $\mathbb{Z}_3[\![\mathbb{G}_2]\!]$–module $\mathbb{Z}_3$ by permutation modules induced from representations of finite subgroups G of the extended Morava stabilizer group $\mathbb{G}_2$. Its topological counterpart, the *topological* duality resolution, is a finite resolution of $E_2^{h\mathbb{G}_2}$, where E_2 denotes Morava E–theory. It is composed of spectra of the form $\Sigma^k E_2^{hG}$, and realizes the algebraic duality resolution. Both the algebraic duality resolution and the topological duality resolution give rise to spectral sequences which can be used to study $\pi_* L_{K(2)} S$ at $p = 3$.

The existence of a resolution analogous to that of [14, Theorem 4.1] at the prime $p = 2$ was conjectured by Mahowald using the computations of Shimomura [30] and of Shimomura and Wang [31]. The central result of this paper is its construction, which is due to Goerss, Henn, Mahowald and Rezk. The author is grateful for their blessing to record it here.

More precisely, for the norm-one subgroup $\mathbb{S}_2^1$ defined in (2.3.6), we construct a resolution of $\mathbb{Z}_2$ by modules which are induced from representations of finite subgroups of $\mathbb{S}_2^1$. We add a detailed description of the maps in the resolution, which will be used in later computations. However, we do not construct a full algebraic duality resolution of $\mathbb{Z}_2$ by $\mathbb{Z}_2[\![\mathbb{G}_2]\!]$–modules as in [14, Corollary 4.2] (see Remark 1.2.3), nor do we realize the algebraic resolution topologically as in [14, Section 5]. For work on the topological realization of the algebraic duality resolution, we refer the reader to Bobkova's thesis [6].

The algebraic duality resolution has already proved to be a useful tool for computations. We use the results of this paper in [3] to compute an associated graded for $H^*(\mathbb{S}_2^1, (E_2)_* V(0))$, where $V(0)$ is the mod 2 Moore spectrum. The computations of [3] play a crucial role in [2], where we prove that the strongest form of the chromatic splitting conjecture, as stated by Hovey in [21, Conjecture 4.2(v)], cannot hold when $n = p = 2$.

1.1 Background and notation

As in Goerss, Henn, Mahowald and Rezk [14, page 779], "we unapologetically focus on the case [$p = 2$] and $n = 2$ because this is at the edge of our current knowledge."

We let $K(2)$ be Morava K–theory, so that

$$K(2)_* = \mathbb{F}_2[v_2^{\pm 1}]$$

for v_2 of degree 6, and whose formal group law is the Honda formal group law of height two, which we denote by F_2. The Morava stabilizer group $\mathbb{S}_2$ is the group of automorphisms of F_2 over $\mathbb{F}_4$. It admits an action of the Galois group $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. The extended Morava stabilizer group $\mathbb{G}_2$ is

$$\mathbb{G}_2 = \mathbb{S}_2 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

By the Goerss–Hopkins–Miller theorem (see Goerss and Hopkins [16]), the group $\mathbb{G}_2$ acts on Morava E–theory E_2 by maps of E_∞–ring spectra and, for X a finite spectrum,

$$L_{K(2)}X \simeq E_2^{h\mathbb{G}_2} \wedge X.$$

In fact, for any closed subgroup G of $\mathbb{G}_2$, one can form the homotopy fixed point spectrum E_2^{hG}; see the work of Hopkins and Devinatz [10] and of Davis [9]. For a spectrum X, the action of $\mathbb{G}_2$ on $(E_2)_*$ induces an action on

$$(E_2)_*X := \pi_* L_{K(2)}(E_2 \wedge X).$$

For a closed subgroup G of $\mathbb{G}_2$ and a finite spectrum X, there is a convergent descent spectral sequence

$$E_2^{s,t} := H^s(G, (E_2)_t X) \Longrightarrow \pi_{t-s}(E_2^{hG} \wedge X).$$

We describe the most relevant example for this paper here. There is a norm on the group $\mathbb{S}_2$ whose kernel is denoted $\mathbb{S}_2^1$ (see Goerss, Henn, Mahowald and Rezk [14, Section 1.3]). Further,

$$\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2. \tag{1.1.1}$$

Similarly, the norm on $\mathbb{S}_2$ induces a norm on $\mathbb{G}_2$. The kernel is denoted $\mathbb{G}_2^1$ and

$$\mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2. \tag{1.1.2}$$

Let π be a topological generator of $\mathbb{Z}_2$ in $\mathbb{G}_2$ and ϕ_π be its action on E_2. If X is finite, there is a fiber sequence

$$L_{K(2)}X \to E_2^{h\mathbb{G}_2^1} \wedge X \xrightarrow{\phi_\pi - \mathrm{id}} E_2^{h\mathbb{G}_2^1} \wedge X. \tag{1.1.3}$$

For this reason, the spectrum $E_2^{h\mathbb{G}_2^1}$ is often called the *half sphere*. One approach for computing $\pi_* L_{K(2)} X$ is to compute the spectral sequence

$$H^s(\mathbb{G}_2^1, (E_2)_t X) \cong H^s(\mathbb{S}_2^1, (E_2)_t X)^{\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)} \Longrightarrow \pi_{t-s} E_2^{h\mathbb{G}_2^1} \wedge X \tag{1.1.4}$$

and then use the fiber sequence (1.1.3) to pass from $\pi_*(E_2^{h\mathbb{G}_2^1} \wedge X)$ to $\pi_* L_{K(2)} X$.

Computing the E_2–term of (1.1.4) can be difficult. At the prime 3, the algebraic duality resolution of Goerss, Henn, Mahowald and Rezk constructed in [14, Theorem 4.1] gives rise to a first quadrant spectral sequence computing the E_2–term of the analogue of (1.1.4). One of the most important consequences of this paper is the existence of such a spectral sequence at the prime $p = 2$ (Theorem 1.2.4 below).

To state the main results and describe this spectral sequence, we must introduce some subgroups of $\mathbb{S}_2$. The group $\mathbb{S}_2$ has a unique conjugacy class of maximal finite subgroups of order 24. Fix a representative and call it G_{24}. The group G_{24} is isomorphic to the semidirect product of a quaternion group denoted Q_8 and a cyclic group of order 3 denoted C_3 (see Section 2.4); that is,

$$G_{24} \cong Q_8 \rtimes C_3.$$

However, there are two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$. If π is as above (1.1.3), the groups G_{24} and

$$G'_{24} := \pi G_{24} \pi^{-1}$$

are representatives for the distinct conjugacy classes. The group $\mathbb{S}_2$ also contains a central subgroup C_2 of order 2 generated by the automorphism $[-1](x)$ of the formal group law F_2 of $K(2)$. Therefore, $\mathbb{S}_2^1$ contains a cyclic subgroup

$$C_6 := C_2 \times C_3.$$

Choose a generator ω of C_3 and an element i in G_{24} such that G_{24} is generated by i and ω. That is,

$$G_{24} = \langle i, \omega \rangle.$$

Let $j = \omega i \omega^2$ and $k = \omega^2 j \omega$. The group $\mathbb{S}_2$ can be decomposed as a semidirect product

$$\mathbb{S}_2 \cong K \rtimes G_{24}$$

for a Poincaré duality group K of dimension 4. Similarly,

$$\mathbb{S}_2^1 \cong K^1 \rtimes G_{24}$$

for a Poincaré duality group K^1 of dimension 3; see Section 2.5. The homology of the groups K and K^1 play a central role in the construction of the duality resolution; see Section 3.1.

The group $\mathbb{S}_2^1$ is a profinite group and one can define the completed group ring

$$\mathbb{Z}_2[\![\mathbb{S}_2^1]\!] = \lim_{i,j} \mathbb{Z}/(2^i)[\mathbb{S}_2^1/U_j]$$

where $\{U_j\}$ forms a system of open subgroups such that $\bigcap_j U_j = \{e\}$. For any closed subgroup G of $\mathbb{S}_2^1$, we let

$$\mathbb{Z}_2[\![\mathbb{S}_2^1/G]\!] := \mathbb{Z}_2[\![\mathbb{S}_2^1]\!] \otimes_{\mathbb{Z}_2[\![G]\!]} \mathbb{Z}_2.$$

1.2 Statement of the results

The main result of this paper is the following theorem.

Theorem 1.2.1 (Goerss, Henn, Mahowald and Rezk, unpublished) *Let $\mathbb{Z}_2$ be the trivial $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module. There is an exact sequence of complete left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \to \mathbb{Z}_2 \to 0$$

where

$$\mathscr{C}_p = \begin{cases} \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}]\!] & \text{if } p = 0, \\ \mathbb{Z}_2[\![\mathbb{S}_2^1/C_6]\!] & \text{if } p = 1, 2, \\ \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}']\!] & \text{if } p = 3. \end{cases}$$

The resolution of Theorem 1.2.1 is called the *algebraic duality resolution*. The name is motivated by the fact that the exact sequence of Theorem 1.2.1 exhibits a certain twisted duality. To make this precise, let $\mathrm{Mod}(\mathbb{S}_2^1)$ denote the category of finitely generated left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules. As above, let π be a topological generator of $\mathbb{Z}_2$ in $\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2$. For a module M in $\mathrm{Mod}(\mathbb{S}_2^1)$, let $c_\pi(M)$ denote the left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module whose underlying $\mathbb{Z}_2$–module is M, but whose $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module structure is twisted by the element π.

Theorem 1.2.2 (Henn, Karamanov and Mahowald, unpublished) *Let*

$$\mathscr{C}_p^* = \mathrm{Hom}_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathscr{C}_p, \mathbb{Z}_2[\![\mathbb{S}_2^1]\!]).$$

There is an isomorphism of complexes of left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \mathscr{C}_0 \arrow[r] \arrow[d, "f_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_\pi(\mathscr{C}_0^*) \arrow[r, "c_\pi(\partial_1^*)"] & c_\pi(\mathscr{C}_1^*) \arrow[r, "c_\pi(\partial_2^*)"] & c_\pi(\mathscr{C}_2^*) \arrow[r, "c_\pi(\partial_3^*)"] & c_\pi(\mathscr{C}_3^*) \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

Remark 1.2.3 The resolution of Theorem 1.2.1 has the following shortcoming: it does not extend to a resolution of the group $\mathbb{G}_2$ or of the group $\mathbb{S}_2$ as in [14, Corollary 4.2]. This is due to the fact that (1.1.1) is a nontrivial extension when $n = p = 2$. For $n = 2$ and $p \geq 3$, $\mathbb{S}_2 \cong \mathbb{S}_2^1 \times \mathbb{Z}_p$.

One application of the algebraic duality resolution is given by the following theorem.

Theorem 1.2.4 *Let M be a profinite left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module. There is a first quadrant spectral sequence*

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathscr{C}_p, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M)$$

with differentials $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. Further, there are isomorphisms

$$E_1^{p,q} \cong \begin{cases} H^q(G_{24}, M) & \text{if } p = 0, \\ H^q(C_6, M) & \text{if } p = 1, 2, \\ H^q(G'_{24}, M) & \text{if } p = 3. \end{cases}$$

The spectral sequence of Theorem 1.2.4 is called the algebraic duality resolution spectral sequence. Its computational appeal is twofold. The E_1–term is composed of the cohomology of finite groups. Further, it collapses at the E_4–term. The d_1 differentials are induced by the maps of the exact sequence in Theorem 1.2.1. In order to compute the spectral sequence, it is necessary to have a detailed description of these maps, which we do in Theorem 1.2.6 below.

The following result introduces some important elements in $\mathbb{S}_2^1$.

Theorem 1.2.5 *There is an element α in K^1 such that $\mathbb{S}_2$ is topologically generated by the elements π, α, i and ω. The group $\mathbb{S}_2^1$ is topologically generated by the elements α, i and ω.*

To state the next result, for any element τ in G_{24}, let

$$\alpha_\tau = \tau\alpha\tau^{-1}\alpha^{-1}.$$

Let S_2^1 be the 2–Sylow subgroup of $\mathbb{S}_2^1$. The group S_2^1 admits a decreasing filtration, denoted $F_{n/2}S_2^1$ which will be defined in Section 2.2.

Theorem 1.2.6 *Let e be the canonical generator of $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ and e_p be the canonical generator of $\mathscr{C}_p$. For a subgroup G of S_2^1, let IG be the kernel of the augmentation $\varepsilon\colon \mathbb{Z}_2[\![G]\!] \to \mathbb{Z}_2$. The maps $\partial_p\colon \mathscr{C}_p \to \mathscr{C}_{p-1}$ of Theorem 1.2.1 can be chosen so that:*

(a) $\partial_1(e_1) = (e-\alpha)e_0$.

(b) $\partial_2(e_2) = \Theta e_1$ *for an element Θ in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]^{C_3}$ such that*

$$\Theta \equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k \mod \mathcal{J},$$

where $\mathcal{J}$ is the left ideal

$$\mathcal{J} = (IF_{4/2}K^1, (IF_{3/2}K^1)(IS_2^1), (IK^1)^7, 2(IK^1)^3, 4IK^1, 8).$$

In particular, $\Theta \equiv e+\alpha$ modulo $(2, (IS_2^1)^2)$.

(c) *There are isomorphisms of $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules $g_p\colon \mathscr{C}_p \to \mathscr{C}_p$ and differentials*

$$\partial'_{p+1}\colon \mathscr{C}_{p+1} \to \mathscr{C}_p$$

such that

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "g_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "g_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "g_1"] & \mathscr{C}_0 \arrow[r] \arrow[d, "g_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial'_3"] & \mathscr{C}_2 \arrow[r, "\partial'_2"] & \mathscr{C}_1 \arrow[r, "\partial'_1"] & \mathscr{C}_0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

is an isomorphism of complexes. The map $\partial'_3\colon \mathscr{C}_3 \to \mathscr{C}_2$ is given by

$$\partial'_3(e_3) = \pi(e+i+j+k)(e-\alpha^{-1})\pi^{-1}e_2.$$

Theorem 1.2.6 is the key to doing computations using the duality resolution spectral sequence. The most difficult part of Theorem 1.2.6 is giving a good estimate for $\partial_2\colon \mathscr{C}_2 \to \mathscr{C}_1$. A detailed description of the map ∂_2 is given in Section 3.4. Though such precision is not needed for our computations in [3], the hope is that it will be sufficient for most future computations using the duality resolution spectral sequence.

1.3 Organization of the paper

Section 2 is dedicated to the description of the Morava stabilizer group in the special case of $p = 2$ and $n = 2$. (A more general account of the structure of $\mathbb{S}_n$ can be found in Goerss, Henn, Mahowald and Rezk [14, Section 1].) We begin by recalling the standard filtration on $\mathbb{S}_2$ and defining the norm. This allows us to define the unit norm subgroup $\mathbb{S}_2^1$ and describe its finite subgroups. In particular, we give an explicit choice of maximal finite subgroup G_{24} in Lemma 2.4.3. In Section 2.5, we introduce a subgroup K such that $\mathbb{S}_2 \cong K \rtimes G_{24}$ and compute the cohomology of K and of its norm-one subgroup K^1. These results are due to Goerss and Henn but are not published. We finish the section with a proof of Theorem 1.2.5.

In Section 3, we introduce the finite resolution of the trivial $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module $\mathbb{Z}_2$. We construct the algebraic duality resolution spectral sequence. We describe the duality properties of the resolution and give a proof of Theorem 1.2.2. We end this section by giving a detailed description of the maps in the resolution, proving Theorem 1.2.6.

The appendix, contains the results on the cohomology of profinite p–adic analytic groups used in this paper.

1.4 Acknowledgements

I thank Mark Behrens, Irina Bobkova, Daniel G Davis, Matt Emerton, Niko Naumann, Doug Ravenel and Dylan Wilson for useful conversations. I thank the referee for a thorough reading and helpful comments. I thank Peter May for his support and for reading everything I have written multiple times. I thank Paul Goerss, Hans-Werner Henn and Charles Rezk for their generosity in sharing their ideas. In particular, I thank Paul and Hans-Werner for spending countless hours and hundreds of emails helping me understand their work and supporting mine. Finally, I will forever be indebted to Mark Mahowald for his incredible intuition and for the time he spent sharing it with me. I hope this paper and those to come will honor his tradition.

2 The structure of the Morava stabilizer group

2.1 A presentation of $\mathbb{S}_2$

Let F_2 be the Honda formal group law of height 2 at the prime 2. It is the 2–typical formal group law defined over $\mathbb{F}_2$ specified by the 2–series

$$[2]_{F_2}(x) = x^4.$$

The ring of endomorphisms of F_2 over $\mathbb{F}_4$ is isomorphic to the maximal order $\mathcal{O}_2$ in the central division algebra over $\mathbb{Q}_2$ of valuation $\frac{1}{2}$, which we denote by

$$\mathbb{D}_2 = D(\mathbb{Q}_2, \tfrac{1}{2}).$$

We begin by describing this isomorphism. More details can be found in Ravenel [26, A2.2; 27, Chapter 4].

Let $\mathbb{W} = W(\mathbb{F}_4)$ denote the ring of Witt vectors on $\mathbb{F}_4$. The ring $\mathbb{W}$ is isomorphic to the ring of integers of the unique unramified degree 2 extension of $\mathbb{Q}_2$. It is a complete local ring with residue field $\mathbb{F}_4$. The Teichmüller character defines a group homomorphism

$$\tau\colon \mathbb{F}_4^\times \to \mathbb{W}^\times.$$

Let ω be a choice of primitive third root of unity in $\mathbb{F}_4^\times$, and identify ω with its Teichmüller lift $\tau(\omega)$. Given such a choice, there is an isomorphism

$$\mathbb{W} \cong \mathbb{Z}_2[\omega]/(1+\omega+\omega^2).$$

The Galois group $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ is generated by the Frobenius σ. It is the $\mathbb{Z}_2$–linear automorphism of $\mathbb{W}$ determined by

$$\omega^\sigma = \omega^2.$$

The ring $\mathcal{O}_2$ is a noncommutative extension of $\mathbb{W}$. It is given by

$$\mathcal{O}_2 \cong \mathbb{W}\langle S\rangle/(S^2 = 2, aS = Sa^\sigma)$$

for a in $\mathbb{W}$. Note that any element of $\mathcal{O}_2$ can be expressed uniquely as a linear combination $a + bS$ for a and b in $\mathbb{W}$. The division algebra $\mathbb{D}_2$ is given by

$$\mathbb{D}_2 \cong \mathcal{O}_2 \otimes_{\mathbb{Z}_2} \mathbb{Q}_2.$$

The 2–adic valuation v on $\mathbb{Q}_2$ extends uniquely to a valuation v on $\mathbb{D}_2$ such that $v(S) = \frac{1}{2}$. Further, $\mathcal{O}_2 = \{x \in \mathbb{D} \mid v(x) \geq 0\}$ and $\mathcal{O}_2^\times = \{x \in \mathbb{D} \mid v(x) = 0\}$. Therefore, any finite subgroup $G \subseteq \mathbb{D}_2^\times$ is contained in $\mathcal{O}_2^\times$.

Next, we describe the ring of endomorphisms of F_2 and give an explicit isomorphism $\mathrm{End}(F_2) \cong \mathcal{O}_2$. A complete proof can be found in Ravenel [26, Section A2]. First, note that

$$\mathrm{End}(F_2) \subseteq \mathbb{F}_4[\![x]\!].$$

To avoid confusion with the elements $\mathbb{W} \subseteq \mathcal{O}_2$, let ζ be a choice of primitive third root of unity in the field of coefficients $\mathbb{F}_4[\![x]\!]$. Let $S(x)$ correspond to the endomorphism

$$S(x) = x^2$$

so that

$$[2]_{F_2}(x) = x^4 = S(S(x)) = S^2(x).$$

Define $\omega^i(x) = \zeta^i x$ and $0(x) = 0$. Given an element a in $\mathbb{W}$, one can write it uniquely as $a = \sum_{i=0}^\infty a_i 2^i$ where a_i in $\mathbb{W}$ satisfies the equation

$$x^4 - x = 0.$$

That is, a_i is in $\{0, 1, \omega, \omega^2\}$. Let $\gamma = a + bS$ be an element of $\mathcal{O}_2$. Let $a = \sum_{i\geq 0} a_{2i}2^i$ and $b = \sum_{i\geq 0} a_{2i+1}2^i$. Using the fact that $S^2 = 2$, the element γ can be expressed uniquely as a power series

$$\gamma = a_0 + 2a_2 + 4a_4 + \cdots + (a_1 + 2a_3 + 4a_6 + \cdots)S = \sum_{i\geq 0} a_i S^i. \tag{2.1.1}$$

One can show that

$$\gamma(x) = a_0(x) +_{F_2} a_1(x^2) +_{F_2} a_2(x^4) +_{F_2} \cdots +_{F_2} a_i(x^{2^i}) +_{F_2} \cdots$$

is a well-defined power series in $\mathbb{F}_4[\![x]\!]$. This determines a ring isomorphism from $\mathcal{O}_2$ to $\mathrm{End}(F_2)$.

The Morava stabilizer group $\mathbb{S}_2$ is the group of automorphisms of F_2. Thus,

$$\mathbb{S}_2 \cong \mathcal{O}_2^\times.$$

Any element of $\mathbb{S}_2$ can be expressed uniquely as a linear combination $a + bS$ for a in $\mathbb{W}^\times$ and b in $\mathbb{W}$. The center of $\mathbb{S}_2$ is given by the Galois invariant elements in $\mathbb{W}^\times$,

$$Z(\mathbb{S}_2) \cong \mathbb{Z}_2^\times.$$

Further, the element ω in $\mathbb{W}^\times$ generates a cyclic group of order 3 in $\mathbb{S}_2$, denoted C_3. The reduction of $\mathbb{W}$ modulo 2 induces an isomorphism $C_3 \cong \mathbb{F}_4^\times$.

The Galois group acts on $\mathbb{S}_2$ by

$$(a + bS)^\sigma = a^\sigma + b^\sigma S.$$

The extended Morava stabilizer group $\mathbb{G}_2$ is defined by

$$\mathbb{G}_2 := \mathbb{S}_2 \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

2.2 The filtration

In what follows, we use the presentation of $\mathbb{S}_2$ induced by the isomorphism $\mathbb{S}_2 \cong \mathcal{O}_2^\times$ that was described in Section 2.1. That is,

$$\mathbb{S}_2 \cong \big(\mathbb{W}\langle S\rangle/(S^2 = 2, aS = Sa^\sigma)\big)^\times$$

for a in $\mathbb{W}$. As described in Henn [17, Section 3], the group $\mathbb{S}_2$ admits the following filtration.

Recall from Section 2.1 that there is a valuation v on $\mathcal{O}_2$ such that

$$v(S) = \tfrac{1}{2}.$$

Regard $\mathbb{S}_2$ as the units in $\mathcal{O}_2$. For all $n \geq 0$, define

$$F_{n/2}\mathbb{S}_2 = \{x \in \mathbb{S}_2 \mid v(x-1) \geq n/2\}.$$

This filtration corresponds to the filtration of $\mathbb{S}_2$ by powers of S; that is, for $n \geq 1$,

$$F_{n/2}\mathbb{S}_2 = \{\gamma \in \mathbb{S}_2 \mid \gamma = 1 + a_n S^n + \cdots\}. \tag{2.2.1}$$

The motivation for indexing the filtration by half integers is that the induced filtration on $\mathbb{Z}_2^\times \subseteq \mathbb{S}_2$ is the usual 2–adic filtration by powers of 2.

Let

$$\mathrm{gr}_{n/2}\,\mathbb{S}_2 := F_{n/2}\mathbb{S}_2/F_{(n+1)/2}\mathbb{S}_2$$

and

$$\mathrm{gr}\,\mathbb{S}_2 = \bigoplus_{n\geq 0} \mathrm{gr}_{n/2}\,\mathbb{S}_2.$$

Define $S_2 := F_{1/2}\mathbb{S}_2$. The group S_2 is the 2–Sylow subgroup of $\mathbb{S}_2$. The map $\mathbb{S}_2 \to \mathbb{F}_4^\times$ which sends γ to a_0 has kernel S_2. It induces an isomorphism

$$\mathrm{gr}_{0/2}\,\mathbb{S}_2 \cong \mathbb{F}_4^\times.$$

Suppose that $n > 0$ and that γ is an element of $F_{n/2}\mathbb{S}_2$, so that

$$\gamma = 1 + a_n S^n + \cdots$$

for a_i as in (2.1.1). Let $\overline{\gamma}$ denote the image of γ in $\mathrm{gr}_{n/2}\,\mathbb{S}_2$. For $n \geq 1$, the map defined by $\overline{\gamma} \mapsto a_n$ gives a group isomorphism

$$\mathrm{gr}_{n/2}\,\mathbb{S}_2 \cong \mathbb{F}_4.$$

It follows from these observations that the subgroups $F_{n/2}\mathbb{S}_2$ form a basis of open neighborhoods for the identity, so that $\mathbb{S}_2$ is a profinite topological group.

Given any subgroup G of $\mathbb{S}_2$, the filtration on $\mathbb{S}_2$ induces a filtration on G, defined by $F_{n/2}G = F_{n/2}\mathbb{S}_2 \cap G$. Let

$$\mathrm{gr}\,G = \bigoplus_{n\geq 0} \mathrm{gr}_{n/2}\,G \tag{2.2.2}$$

be the associated graded for this filtration.

The associated graded $\mathrm{gr}\,S_2$ has the structure of a restricted Lie algebra. The bracket is induced by the commutator in S_2 and the restriction is induced by squaring. In [17, Lemma 3.1.4], Henn gives an explicit description of the structure of this Lie algebra. We record this result in the case when $p = 2$ and $n = 2$.

Lemma 2.2.1 (Henn) *For $n, m > 0$, let $a \in F_{n/2}S_2$ and $b \in F_{m/2}S_2$. Let $\overline{a}$ be the image of a in $\mathrm{gr}_{n/2} S_2$ and $\overline{b}$ be the image of b in $\mathrm{gr}_{m/2} S_2$. If $[a,b]$ denotes the commutator $aba^{-1}b^{-1}$, then*

$$\overline{[a,b]} \equiv \overline{a}\overline{b}^{2^n} + \overline{a}^{2^m}\overline{b} \in \mathrm{gr}_{(n+m)/2} S_2.$$

If $P(a) = a^2$, then

$$\overline{P(a)} \equiv \begin{cases} \overline{a}^3 \in \mathrm{gr}_{2/2} S_2 & \text{if } n = 1, \\ \overline{a} + \overline{a}^2 \in \mathrm{gr}_{4/2} S_2 & \text{if } n = 2, \\ \overline{a} \in \mathrm{gr}_{n/2+1} S_2 & \text{if } n > 2. \end{cases}$$

2.3 The norm and the subgroups $\mathbb{S}_2^1$ and $\mathbb{G}_2^1$

The group $\mathbb{S}_2 \cong \mathcal{O}_2^\times$ acts on $\mathcal{O}_2$ by right multiplication. This gives rise to a representation $\rho\colon \mathbb{S}_2 \to \mathrm{GL}_2(\mathbb{W})$:

$$\rho(a + bS) = \begin{pmatrix} a & 2b^\sigma \\ b & a^\sigma \end{pmatrix}. \tag{2.3.1}$$

The restriction of the determinant to $\mathbb{S}_2$ is given by

$$\det(a + bS) = aa^\sigma - 2bb^\sigma. \tag{2.3.2}$$

This defines a group homomorphism $\det\colon \mathbb{S}_2 \to \mathbb{Z}_2^\times$.

Lemma 2.3.1 *The determinant* $\det\colon \mathbb{S}_2 \to \mathbb{Z}_2^\times$ *is surjective.*

Before proving this lemma, we introduce elements of $\mathbb{S}_2$ that will play a key role in the remainder of this paper and in future computations. First, let

$$\pi := 1 + 2\omega. \tag{2.3.3}$$

By Hensel's lemma, $\mathbb{Z}_2$ contains two solutions of $f(x) = x^2 + 7$. One of them satisfies

$$\sqrt{-7} \equiv 1 + 4 \mod 8.$$

This allows us to define

$$\alpha := \frac{1 - 2\omega}{\sqrt{-7}}. \tag{2.3.4}$$

Note that the elements π and α are in $\mathbb{W}^\times \subseteq \mathbb{S}_2$.

Proof of Lemma 2.3.1. The group $\mathbb{Z}_2^\times$ is topologically generated by -1 and 3. It suffices to show that these values are in the image of the determinant. A direct computation shows that $\det(\pi) = 3$ and that $\det(\alpha) = -1$. □

Definition 2.3.2 The norm $N\colon \mathbb{S}_2 \to \mathbb{Z}_2^\times/\{\pm 1\}$ is the composite

$$\mathbb{S}_2 \xrightarrow{\det} \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\}.$$

For a prime p and an integer $i \geq 1$, let $U_i = \{x \in \mathbb{Z}_p^\times \mid x \equiv 1 \mod p^i\}$. For $p = 2$, there is a canonical identification

$$\mathbb{Z}_2^\times \cong \{\pm 1\} \times U_2. \tag{2.3.5}$$

Therefore, the image of the norm is canonically isomorphic to the group U_2. Further, the group U_2 is noncanonically isomorphic to the additive group $\mathbb{Z}_2$.

The subgroup $\mathbb{S}_2^1$ is defined by the short exact sequence

$$1 \to \mathbb{S}_2^1 \to \mathbb{S}_2 \xrightarrow{N} \mathbb{Z}_2^\times/\{\pm 1\} \to 1. \tag{2.3.6}$$

Any element γ such that $N(\gamma)$ is a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$ determines a splitting. The element π defined in (2.3.3) is an example. This gives a decomposition

$$\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2. \tag{2.3.7}$$

Note that the group $\mathbb{S}_2^1$ is closed in $\mathbb{S}_2$ as it is the intersection of the finite index subgroups which are the kernels of the norm followed by the projections $U_2 \to \mathbb{Z}/2^n$ for $n \geq 0$.

The norm N extends to a homomorphism

$$N\colon \mathbb{G}_2 \to \mathbb{Z}_2^\times/\{\pm 1\} \times \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2) \to \mathbb{Z}_2^\times/\{\pm 1\},$$

where the second map is the projection. The subgroup $\mathbb{G}_2^1$ is the kernel of the extended norm and

$$\mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2. \tag{2.3.8}$$

We note that there is no splitting which is equivariant with respect to the action of the Galois group.

The filtration on $\mathbb{S}_2$ induces a filtration on $\mathbb{S}_2^1$ and

$$S_2^1 := F_{1/2}\mathbb{S}_2^1 \tag{2.3.9}$$

is the 2–Sylow subgroup of $\mathbb{S}_2^1$.

Remark 2.3.3 Note that for odd primes p, there is a canonical isomorphism

$$\mathbb{Z}_p^\times \cong C_{p-1} \times U_1,$$

where C_{p-1} is a cyclic group of order $p-1$. The exact sequence analogous to (2.3.6) is given by

$$1 \to \mathbb{S}_2^1 \to \mathbb{S}_2 \to \mathbb{Z}_p^\times / C_{p-1} \to 1.$$

Further, it has a central splitting. Therefore, when p is odd, the Morava stabilizer group is a product

$$\mathbb{G}_2 \cong \mathbb{G}_2^1 \times \mathbb{Z}_p^\times / C_{p-1} \cong \mathbb{G}_2^1 \times \mathbb{Z}_p.$$

There is no central splitting at the prime $p=2$ and the extensions (2.3.7) and (2.3.8) are nontrivial.

We will need the following result in Section 2.5 to prove Theorem 2.5.7.

Lemma 2.3.4 *For $n \geq 1$,*

$$\mathrm{gr}_{n/2}\, S_2^1 = \begin{cases} \mathbb{F}_2 & \text{if } n \text{ is even,} \\ \mathbb{F}_4 & \text{if } n \text{ is odd.} \end{cases}$$

Proof Let $F_{0/2}\mathbb{Z}_2^\times = \mathbb{Z}_2^\times$ and, for $n \geq 2$ even,

$$F_{n/2}\mathbb{Z}_2^\times = F_{(n-1)/2}\mathbb{Z}_2^\times := U_{n/2} = \{\gamma \mid \gamma \equiv 1 \mod 2^{n/2}\}.$$

Let γ be in $\mathbb{S}_2$. Let $n \geq 2$ be even and suppose that γ has an expansion of the form

$$\gamma \equiv 1 + a_{n-1}S^{n-1} + a_n S^n \mod S^{n+1}.$$

By (2.3.2),

$$\det(\gamma) \equiv 1 + 2^{n/2}(a_n + a_n^\sigma) + a_{n-1}a_{n-1}^\sigma 2^{n-1} \mod 2^{n/2+1},$$

which is in $F_{n/2}\mathbb{Z}_2^\times$. Therefore, the determinant preserves this filtration. In fact, it induces short exact sequences of $\mathbb{F}_2$–vector spaces:

$$0 \to \mathrm{gr}_{n/2}\, \mathbb{S}_2^1 \to \mathrm{gr}_{n/2}\, \mathbb{S}_2 \to \mathrm{gr}_{n/2}\, \mathbb{Z}_2^\times \to 0.$$

The result then follows from the fact that

$$\mathrm{gr}_{n/2}\, \mathbb{Z}_2^\times = \begin{cases} \mathbb{F}_2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

and $\mathrm{gr}_{n/2}\, \mathbb{S}_2 \cong (\mathbb{F}_2)^2$ for $n \geq 1$. □

2.4 Finite subgroups of $\mathbb{S}_2$

In this section, we describe the finite subgroups of $\mathbb{S}_2$ that will be used in the construction of the resolution of Theorem 1.2.1. We also prove that there are two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$. This will be used in the proof of Theorem 1.2.1.

Proposition 2.4.1 is a special case of Hewett [19, Theorem 1.4].

Proposition 2.4.1 (Hewett) *Any maximal finite nonabelian subgroup of $\mathbb{S}_2$ is isomorphic to a binary tetrahedral group*

$$G_{24} \cong Q_8 \rtimes C_3.$$

Here, Q_8 is the quaternion group

$$Q_8 \cong \langle i, j \mid i^2 = j^2, iji = j \rangle,$$

and the action of C_3 permutes i, j and ij.

Our next goal is to prove that there are two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$. To do this, we will need some preliminary results. Note that the classification of conjugacy classes of maximal finite subgroups of $\mathbb{S}_2$ is addressed in Hewett [20] and in Bujard [8]. According to Bujard [8, Remark 1.36], Hewett's [20, Theorem 5.3] is incorrect. However, [8, Theorem 1.35] in the case $n = p = 2$ is also stated incorrectly. A correct statement can be found in [8, Theorem 4.30]. To avoid confusion, we restate the results we need.

Proposition 2.4.2 (Bujard) *There is a unique conjugacy class of groups isomorphic to Q_8, and one of groups isomorphic to G_{24}, in $\mathbb{S}_2$.*

Proof For Q_8, this is [8, Lemma 1.25]. For G_{24}, this is [8, Theorem 1.28]. □

It will be useful to have explicit choices of subgroups Q_8 and G_{24}. The proof of the following lemma is a direct computation.

Lemma 2.4.3 (Henn) *Let*

$$i := \frac{1}{1+2\omega}(1 - \alpha S).$$

Define $j = \omega i \omega^2$ and $k = \omega^2 i \omega = ij$. The elements i and j generate a quaternion subgroup of $\mathbb{S}_2$, denoted Q_8. The elements i and ω generate a subgroup isomorphic to G_{24}. Further, in $\mathbb{D}_2$,

$$\omega = -\frac{1 + i + j + k}{2}.$$

For H a subgroup of G, let $N_G(H)$ be the normalizer of H in G. Let $C_G(H)$ be the centralizer of H in G. Note that the element $1+i$ in $\mathbb{D}_2^\times$ is in $N_{\mathbb{D}_2^\times}(Q_8)$. Since the valuation $v(1+i)=\frac{1}{2}$, the restriction of the valuation to the normalizer is surjective. Therefore, there is an exact sequence

$$1 \to N_{\mathbb{S}_2}(Q_8) \to N_{\mathbb{D}_2^\times}(Q_8) \to \tfrac{1}{2}\mathbb{Z} \to 0.$$

Since $\mathbb{D}_2 \cong \mathbb{Q}_2(i,j)$, it follows by the Skolem–Noether theorem that $\mathrm{Aut}(Q_8)$ can be realized by inner conjugation in $\mathbb{D}_2^\times$. There is an exact sequence

$$1 \to C_{\mathbb{D}_2^\times}(Q_8) \to N_{\mathbb{D}_2^\times}(Q_8) \to \mathrm{Aut}(Q_8) \to 0.$$

The next proposition describes which of these automorphisms can be realized by conjugation in $\mathbb{S}_2$.

Proposition 2.4.4 (Henn) *The subgroup of $\mathrm{Aut}(Q_8)$ that can be realized by conjugation by an element of $\mathbb{S}_2$ is isomorphic to the alternating group A_4. It is generated by conjugation by the elements i, j and ω.*

Proof The group $\mathrm{Aut}(Q_8)$ is isomorphic to the symmetric group $\mathfrak{S}_4$. One verifies by a direct computation that conjugation by i, j and ω generates a subgroup of $\mathrm{Aut}(Q_8)$ isomorphic to A_4. Let $\mathrm{Out}_{\mathbb{S}_2}(Q_8)$ be the group of automorphisms of Q_8 that can be realized by conjugation in $\mathbb{S}_2$. Since $C_{\mathbb{D}_2^\times}(Q_8) \cong \mathbb{Q}_2^\times$ and $C_{\mathbb{S}_2}(Q_8) \cong \mathbb{Z}_2^\times$, there is a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_2^\times \arrow[r] \arrow[d] & \mathbb{Q}_2^\times \arrow[r, "v"] \arrow[d] & \mathbb{Z} \arrow[d] \\
N_{\mathbb{S}_2}(Q_8) \arrow[r] \arrow[d] & N_{\mathbb{D}_2^\times}(Q_8) \arrow[r, "v"] \arrow[d] & \frac{1}{2}\mathbb{Z} \arrow[d] \\
\mathrm{Out}_{\mathbb{S}_2}(Q_8) \arrow[r] & \mathfrak{S}_4 \arrow[r] & \mathbb{Z}/2
\end{tikzcd}$$

where the columns and rows are short exact. Therefore, $\mathrm{Out}_{\mathbb{S}_2}(Q_8) \cong A_4$. □

Lemma 2.4.5 *Let $G_{24} = Q_8 \rtimes C_3$. The normalizer of Q_8 in $\mathbb{S}_2$ is given by*

$$N_{\mathbb{S}_2}(Q_8) \cong U_2 \times G_{24}.$$

Proof By Proposition 2.4.4, there is a short exact sequence

$$1 \to C_{\mathbb{S}_2}(Q_8) \to N_{\mathbb{S}_2}(Q_8) \to A_4 \to 1.$$

The centralizer is the subgroup $\mathbb{Z}_2^\times \cong C_2 \times U_2$ of $\mathbb{S}_2$. Since G_{24} is defined by the extension

$$1 \to C_2 \to G_{24} \to A_4 \to 1,$$

and the elements of U_2 are in the centralizer of G_{24}, it follows that $N_{\mathbb{S}_2}(Q_8)$ is isomorphic to $U_2 \times G_{24}$. □

Since the image of the norm is torsion free, any finite subgroup G of $\mathbb{S}_2$ is contained in the kernel $\mathbb{S}_2^1$. Therefore, $\mathbb{S}_2^1$ has the same maximal finite subgroups as $\mathbb{S}_2$. However, there are more conjugacy classes in $\mathbb{S}_2^1$.

Proposition 2.4.6 *There are two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$. One is the conjugacy class of G_{24} defined in Lemma 2.4.3. The other is $\xi G_{24}\xi^{-1}$, where ξ is any element such that $N(\xi)$ is a topological generator of U_2.*

Proof Let $\mathbb{Z}_2^\times \subseteq \mathbb{S}_2$ be the center. Define

$$\mathbb{S}_2^0 := \mathbb{S}_2^1 \times U_2,$$

where U_2 is as in (2.3.5). The restriction of the determinant to U_2 surjects onto $(\mathbb{Z}_2^\times)^2$. Therefore, there is an exact sequence

$$1 \to \mathbb{S}_2^0 \to \mathbb{S}_2 \to \mathbb{Z}_2^\times/\big(\{\pm 1\}, (\mathbb{Z}_2^\times)^2\big) \to 1,$$

and $\mathbb{S}_2/\mathbb{S}_2^0 \cong \mathbb{Z}/2$. If $N(\xi)$ is a topological generator for $\mathbb{Z}_2^\times/\{\pm 1\}$, then ξ is a representative for the nontrivial coset in $\mathbb{S}_2/\mathbb{S}_2^0$.

By Proposition 2.4.2, there is a unique conjugacy class of subgroups isomorphic to G_{24} in $\mathbb{S}_2$. Since conjugation by any element of the center $\mathbb{Z}_2^\times$ is trivial, any two conjugacy classes in $\mathbb{S}_2^1$ differ by conjugation by an element of $\mathbb{S}_2/\mathbb{S}_2^0 \cong \mathbb{Z}/2$. Therefore, there are at most two conjugacy classes.

Next, we show that the conjugacy classes of G_{24} and $\xi G_{24}\xi^{-1}$ are distinct in $\mathbb{S}_2^1$. Conjugation acts on the 2–Sylow subgroups; hence, it suffices to prove the claim for the subgroup Q_8 of G_{24}. Suppose that there exists an element γ in $\mathbb{S}_2^1$ such that

$$\xi Q_8 \xi^{-1} = \gamma Q_8 \gamma^{-1}.$$

This would imply that $\gamma^{-1}\xi$ is in $N_{\mathbb{S}_2}(Q_8)$. By Lemma 2.4.5, $\gamma^{-1}\xi$ is a product $z\tau$ for z in U_2 and τ in G_{24}. This implies that $\xi = \gamma z\tau$. However, $\gamma z \tau$ is in $\mathbb{S}_2^0$. This is a contradiction, since the residue class of ξ in $\mathbb{S}_2/\mathbb{S}_2^0$ is nontrivial. Therefore, G_{24} and $\xi G_{24}\xi^{-1}$ represent distinct conjugacy classes in $\mathbb{S}_2^1$. □

A choice for ξ is the element π defined in (2.3.3). For the remainder of this paper, G'_{24} will denote

$$G'_{24} := \pi G_{24} \pi^{-1}, \tag{2.4.1}$$

so that G_{24} and G'_{24} are representatives for the two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$.

2.5 The Poincaré duality subgroups

In this section, we introduce the subgroups K and K^1 and we describe their continuous cohomology rings $H^*(K, \mathbb{F}_2)$ and $H^*(K^1, \mathbb{F}_2)$ as G_{24}–modules. The author learned the results of this section from Paul Goerss and Hans-Werner Henn. We refer the reader to the appendix for details on the cohomology of a profinite group.

Let K be the closure of the subgroup of $\mathbb{S}_2$ generated by α (as defined in (2.3.4)) and $F_{3/2}\mathbb{S}_2$. That is,

$$K = \overline{\langle \alpha, F_{3/2}\mathbb{S}_2 \rangle}.$$

Proposition 2.5.1 *The subgroup K is normal in $\mathbb{S}_2$. Further, $S_2 \cong K \rtimes Q_8$ and $\mathbb{S}_2 \cong K \rtimes G_{24}$.*

Proof There is an isomorphism $\mathbb{S}_2 \cong S_2 \rtimes C_3$ and α commutes with the group C_3. Further, for any element γ in S_2, it follows from Lemma 2.2.1 that the commutator $[\gamma, \alpha]$ is in $F_{3/2}\mathbb{S}_2$. Since $\mathbb{S}_2 \cong S_2 \rtimes C_3$, and $F_{3/2}\mathbb{S}_2$ is normal, K is also normal. The quotient S_2/K is a group of order 8 generated by the image of the elements i and j defined in Lemma 2.4.3. The inclusion of Q_8 followed by the projection to S_2/K is an isomorphism. This defines a splitting. Similarly, the group $\mathbb{S}_2/K$ is a group of order 24 generated by the image of ω and i, and this defines a splitting. □

Corollary 2.5.2 *If K^1 is the kernel of the norm restricted to K, then $\mathbb{S}_2^1 \cong K^1 \rtimes G_{24}$.*

Proof The elements α and π are in the group K since $\alpha^{-1}\pi$ is in $F_{3/2}\mathbb{S}_2$. Therefore, the norm restricted to K is surjective and $\mathbb{S}_2^1/K^1 \cong \mathbb{S}_2/K$. □

Our next goal is to compute the group cohomology of K and K^1. We will need a few preliminary results.

Proposition 2.5.3 *Any open subgroup of S_2 or of S_2^1 is a profinite 2–adic analytic group.*

Proof According to Dixon, Du Sautoy, Mann and Segal [11, Theorem 8.1], a topological group is 2–adic analytic if and only if it has an open subgroup which is a finitely generated powerful pro-2 group. (By [11, Definition 3.1], a pro-2 group H is powerful if the quotient $H/\overline{H^4}$ is abelian, where $H^4 = \langle h^4 \mid h \in H\rangle$.) By Lemma 2.2.1, if $n \geq 3$, then $F_{n/2}S_2$ is topologically generated by any finite set of elements that surjects onto $F_{n/2}S_2/F_{(n+2)/2}S_2$. Further, the image of P^2: $F_{n/2}S_2 \to F_{(n+4)/2}S_2$ is dense by Lemma 2.2.1. If $n \geq 4$, then

$$[F_{n/2}S_2, F_{n/2}S_2] \subseteq F_{(2n)/2}S_2 \subseteq F_{(n+4)/2}S_2.$$

This implies that $F_{n/2}S_2$ is powerful for $n \geq 4$. Since any open subgroup G of S_2 contains $F_{n/2}S_2$ for some large n, it is a profinite 2–adic analytic group.

The proof for open subgroups of S_2^1 is similar, using $F_{n/2}S_2^1$ instead of $F_{n/2}S_2$. $\square$

By Proposition 2.5.3, open subgroups of S_2 and S_2^1 are compact 2–adic analytic groups. This motivates our use of the following definition, which can be found in Symonds and Weigel [35, Section 4].

Definition 2.5.4 Let G be a compact p–adic analytic group. Then G is a Poincaré duality group of dimension n if G has cohomological dimension n and

$$H^s(G, \mathbb{Z}_p[\![G]\!]) \cong \begin{cases} \mathbb{Z}_p & \text{if } s = n, \\ 0 & \text{if } s \neq n \end{cases}$$

as abelian groups. The right $\mathbb{Z}_p[\![G]\!]$–module $H^n(G, \mathbb{Z}_p[\![G]\!])$ is denoted $D_p(G)$ and called the *compact dualizing module*. If the action of $\mathbb{Z}_p[\![G]\!]$ on $D_p(G)$ is trivial, the group G is called *orientable*.

Remark 2.5.5 For a Poincaré duality group G of dimension n, one can show that $H_n(G, D_p(G))$ is isomorphic to $\mathbb{Z}_p$; see Symonds and Weigel [35, Theorem 4.4.3]. Given a choice of generator $[G]$ for $H_n(G, D_p(G))$, the cap product induces a natural isomorphism

$$H^{n-*}(G, -) \xrightarrow{\cap[G]} H_*(G, D_p(G) \otimes_{\mathbb{Z}_p} -).$$

The following observations are useful to compute $D_p(G)$. Let $\phi_{D_p(G)}$: $G \to \mathbb{Z}_p^*$ be the representation associated to the action of G on $D_p(G)$. Let $L(G)$ be the $\mathbb{Q}_p$–Lie algebra associated to G, as defined in Lazard [24, Definition V.2.4.2.5]. The right conjugation action of G on itself induces a natural right action on $L(G)$, and thus a homomorphism Ad: $G \to \mathrm{Aut}(L(G))$. By [35, Corollary 5.2.5], if G is p–torsion free,

$$\phi_{D_p(G)}(g) = \det(\mathrm{Ad}(g)).$$

Proposition 2.5.6 *If an open subgroup U of $\mathbb{S}_n$ is a Poincaré duality group, then it is orientable.*

Proof This is the argument given by Strickland in the proof of [34, Proposition 5]. For any open subgroup U of $\mathbb{S}_n$, $L(U)$ is isomorphic to the central division algebra $\mathbb{D}_n$ over $\mathbb{Q}_p$ of valuation $1/n$. For g in U, the action $\mathrm{Ad}(g)$ is given by conjugation in $\mathbb{D}_n$, which has determinant one. □

The next result relies on Lazard's theory of groups which are *équi-p–valué*. We refer the reader who is unfamiliar with the theory of Lazard to Huber, Kings and Naumann [22, Section 2] for an overview of the terminology.

Theorem 2.5.7 *For $n \geq 3$, the group $F_{n/2}S_2$ is a Poincaré duality group of dimension 4. The continuous group cohomology $H^*(F_{n/2}S_2, \mathbb{F}_2)$ is the exterior algebra generated by*

$$H^1(F_{n/2}S_2, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathbb{F}_2}(\mathrm{gr}_{n/2} S_2 \oplus \mathrm{gr}_{(n+1)/2} S_2, \mathbb{F}_2) \cong \mathbb{F}_2^4.$$

Similarly, $F_{n/2}S_2^1$ is a Poincaré duality group of dimension 3 and $H^*(F_{n/2}S_2^1, \mathbb{F}_2)$ is the exterior algebra generated by

$$H^1(F_{n/2}S_2^1, \mathbb{F}_2) = \mathrm{Hom}_{\mathbb{F}_2}(\mathrm{gr}_{n/2} S_2^1 \oplus \mathrm{gr}_{(n+1)/2} S_2^1, \mathbb{F}_2) \cong \mathbb{F}_2^3.$$

Proof We define a filtration $w\colon F_{n/2}S_2 \to \mathbb{R}_+^* \cup \{\infty\}$ in the sense of Lazard [24, Definition II.1.1.1]. Let $w(1) = \infty$. For $k \geq 0$ and $x \in F_{(n+2k)/2}S_2 \backslash F_{(n+2k+2)/2}S_2$, let $w(x) = (n+2k)/2$. With this filtration, $F_{n/2}S_2$ is équi-p–valué of rank 4 in the sense of Lazard [24, V.2.2.7], with $\mathrm{gr}\, F_{n/2}S_2$ generated by

$$F_{n/2}S_2/F_{(n+2)/2}S_2 \cong \mathrm{gr}_{n/2} S_2 \oplus \mathrm{gr}_{(n+1)/2} S_2.$$

To verify that w is a filtration and that $F_{n/2}S_2$ is équi-p–valué with respect to w, one uses the formulas of Lemma 2.2.1, noting that the squaring map

$$P\colon F_{(n+2k)/2}S_2/F_{(n+2k+2)/2}S_2 \to F_{(n+2k+2)/2}S_2/F_{(n+2k+4)/2}S_2$$

is an isomorphism if and only if $n \geq 3$. The result then follows from [24, Proposition V.2.5.7.1], which states that $H^*(F_{n/2}S_2, \mathbb{F}_2)$ is an exterior algebra on the $\mathbb{F}_2$–linear dual of

$$F_{n/2}S_2/P(F_{n/2}S_2) \cong F_{n/2}S_2/F_{(n+2)/2}S_2 \cong \mathbb{F}_4^2 \cong \mathbb{F}_2^4.$$

According to Symonds and Weigel [35, Theorem 5.1.5], this also implies that $F_{n/2}S_2$ is a Poincaré duality group of dimension 4 (note that in [35], the authors imply in the third footnote that they use the terms *uniformly powerful pro-p* and *équi-p–valué* interchangeably.)

To prove the second claim, we use the same filtration, $F_{n/2}S_2^1$. By Lemma 2.3.4,

$$F_{n/2}S_2^1/F_{(n+2)/2}S_2^1 \cong \mathbb{F}_4 \oplus \mathbb{F}_2 \cong \mathbb{F}_2^3. \qquad \square$$

Recall that we use the convention that

$$\alpha_\tau = [\tau, \alpha] = \tau\alpha\tau^{-1}\alpha^{-1}.$$

The following congruences will be used in the computations of this section:

$$\begin{aligned} i &\equiv 1+S \mod S^2, & j &\equiv 1+\omega^2 S \mod S^2, \\ -1 &\equiv 1+S^2 \mod S^4, & \alpha &\equiv 1+\omega S^2 \mod S^4, \\ \alpha_i &\equiv 1+S^3 \mod S^4, & \alpha_j &\equiv 1+\omega^2 S^3 \mod S^4, \\ \alpha^2 &\equiv 1+S^4 \mod S^5, & \alpha\pi &\equiv 1+\omega S^4 \mod S^5. \end{aligned}$$

They are obtained by a direct computation using the definitions of π, α, i and j, which were given in (2.3.3), (2.3.4) and Lemma 2.4.3.

Definition 2.5.8 Let

$$\alpha_0 = \alpha, \quad \alpha_1 = \alpha_i, \quad \alpha_2 = \alpha_j, \quad \alpha_3 = \alpha^2, \quad \alpha_4 = \alpha\pi,$$

and let x_s in $\mathrm{Hom}_{\mathbb{F}_2}(\mathrm{gr}\,\mathbb{S}_2, \mathbb{F}_2)$ be the function dual to the image of α_s in $\mathrm{gr}\,\mathbb{S}_2$. The action of conjugation by an element τ on an element g is denoted by $\tau_*(g)$.

Remark 2.5.9 The action of conjugation by τ can be computed using Lemma 2.2.1 and the formula

$$[\tau, \gamma]\gamma = \tau_*(\gamma).$$

Corollary 2.5.10 *The continuous group cohomology $H^*(F_{3/2}S_2, \mathbb{F}_2)$ is the exterior algebra generated by*

$$H^1(F_{3/2}S_2, \mathbb{F}_2) \cong \mathbb{F}_2\{x_1, x_2, x_3, x_4\}$$

for x_s as in Definition 2.5.8. The action of α on $H^1(F_{3/2}S_2, \mathbb{F}_2)$ is trivial.

Similarly, $H^*(F_{3/2}S_2^1, \mathbb{F}_2)$ is the exterior algebra generated by

$$H^1(F_{3/2}S_2^1, \mathbb{F}_2) \cong \mathbb{F}_2\{x_1, x_2, x_3\}$$

with a trivial action by α.

Proof By Theorem 2.5.7, $H^*(F_{3/2}S_2, \mathbb{F}_2)$ is an exterior algebra generated by the $\mathbb{F}_2$–linear dual of $F_{3/2}S_2/F_{5/2}S_2$. This group is generated by the image of α_1, α_2, α_3 and α_4 of Definition 2.5.8. Therefore, $H^*(F_{3/2}S_2, \mathbb{F}_2)$ is the exterior algebra generated by $\mathbb{F}_2\{x_1, x_2, x_3, x_4\}$. Since α is in $F_{2/2}S_2$, if g is in $F_{3/2}S_2$, the commutator $[\alpha, g]$ is in $F_{5/2}S_2$. Using Remark 2.5.9, we conclude that the action of α on $H^1(F_{3/2}S_2, \mathbb{F}_2)$ is trivial.

The second claim follows in the same way from the fact that $F_{3/2}S_2^1/F_{5/2}S_2^1$ is generated by the image of α_1, α_2 and α_3. □

Lemma 2.5.11 *For α_i as defined in Definition 2.5.8, and $\overline{\alpha}_i$ its image in $H_1(K, \mathbb{Z}_2)$, there is an isomorphism*

$$H_1(K, \mathbb{Z}_2) \cong \mathbb{Z}/4\{\overline{\alpha}_0\} \oplus \mathbb{Z}/2\{\overline{\alpha}_1, \overline{\alpha}_2\} \oplus \mathbb{Z}_2\{\overline{\alpha}_4\},$$

where $2\overline{\alpha}_0$ is the image of $\alpha_3 = \alpha^2$. Similarly,

$$H_1(K^1, \mathbb{Z}_2) \cong \mathbb{Z}/4\{\overline{\alpha}_0\} \oplus \mathbb{Z}/2\{\overline{\alpha}_1, \overline{\alpha}_2\}.$$

The conjugation action of Q_8 on K factors through the quotient of Q_8 by the central subgroup C_2. The induced action on $H_1(K, \mathbb{Z}_2)$ is trivial on $\overline{\alpha}_4$. On the other generators, it is given by

$$\begin{aligned} i_*(\overline{\alpha}_0) &= \overline{\alpha}_0 + \overline{\alpha}_1, & j_*(\overline{\alpha}_0) &= \overline{\alpha}_0 + \overline{\alpha}_2, \\ i_*(\overline{\alpha}_1) &= \overline{\alpha}_1, & j_*(\overline{\alpha}_1) &= \overline{\alpha}_1 + 2\overline{\alpha}_0, \\ i_*(\overline{\alpha}_2) &= \overline{\alpha}_2 + 2\overline{\alpha}_0, & j_*(\overline{\alpha}_2) &= \overline{\alpha}_2. \end{aligned}$$

Hence, $H_1(K^1, \mathbb{Z}_2)$ is generated by the image of α as a G_{24}–module.

Proof First, we prove that the group $[K, K]$ is dense in $F_{5/2}S_2^1$. Note that K is contained in $F_{2/2}S_2$. Let a and b be in K. For $\overline{a}$ and $\overline{b}$ as in Lemma 2.2.1,

$$\overline{[a,b]} = \overline{a}\overline{b}^4 + \overline{a}^4\overline{b} \in \mathrm{gr}_{4/2}\, S_2.$$

Since $\overline{a}$ and $\overline{b}$ are in $\mathbb{F}_4$ and $x^4 = x$ for all x in $\mathbb{F}_4$, this implies that $\overline{[a,b]} = 0$ in $\mathrm{gr}_{4/2}\, S_2$. Therefore, $[a, b] \in F_{5/2}S_2$.

Since the norm is multiplicative, the elements of $[K, K]$ have norm one. Hence, $[K, K]$ is contained in $F_{5/2}S_2^1$. Further, the map from $[K, K]$ to $F_{5/2}S_2^1/F_{7/2}S_2^1$ induced by the inclusion is surjective. Indeed, $F_{5/2}S_2^1/F_{7/2}S_2^1$ is generated by the images of the elements $[\alpha, [i, \alpha]]$, $[\alpha, [j, \alpha]]$ and $[[i, \alpha], [j, \alpha]]$, all of which are in $[K^1, K^1]$. By Corollary 2.5.10, this implies that the composite

$$[K^1, K^1] \hookrightarrow [K, K] \to H_1(F_{5/2}S_2^1, \mathbb{F}_2)$$

is surjective. According to Behrens and Lawson [5, Theorem 2.1], it then follows from results of Koch and Serre that $[K^1, K^1]$ and $[K, K]$ are dense in $F_{5/2}S_2^1$. Therefore,

$$\overline{[K^1, K^1]} = \overline{[K, K]} = F_{5/2}S_2^1.$$

Hence, $H_1(K, \mathbb{Z}_2) \cong K/F_{5/2}S_2^1$ and $H_1(K^1, \mathbb{Z}_2) \cong K^1/F_{5/2}S_2^1$. Since α and π are in K, the norm $N\colon K \to \mathbb{Z}_2^\times/\{\pm 1\}$ is split surjective. The image of $\alpha_4 = \alpha\pi$ generates $\mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2$. Therefore,

$$H_1(K, \mathbb{Z}_2) \cong H_1(K^1, \mathbb{Z}_2) \oplus \mathbb{Z}_2\{\overline{\alpha}_4\}.$$

Finally, $H_1(K^1, \mathbb{Z}_2)$ is generated by the image of $\alpha_0 = \alpha$, $\alpha_1 = \alpha_i$ and $\alpha_2 = \alpha_j$. Since α_i and α_j are in $F_{3/2}S_2$, it follows from Lemma 2.2.1 that α_i^2 and α_j^2 are in

$$F_{5/2}S_2 = \overline{[K^1, K^1]}.$$

Therefore, the images of α_i and α_j have order 2 in $K^1/\overline{[K^1, K^1]}$. Finally, $\alpha^2 \equiv 1 + S^4$ modulo S^5, so that the image of α has order 4 in $K^1/\overline{[K^1, K^1]}$. We conclude that

$$H_1(K^1, \mathbb{Z}_2) \cong \mathbb{Z}/4\{\overline{\alpha}_0\} \oplus \mathbb{Z}/2\{\overline{\alpha}_1, \overline{\alpha}_2\}.$$

The action of Q_8 by conjugation factors through $C_2 = \{\pm 1\}$ since C_2 is in the center of $\mathbb{S}_2$. The action of the generators i and j is computed using Remark 2.5.9 and the following relations, which hold modulo S^5:

$$\begin{aligned}
&[i, \alpha] \equiv \alpha_i, \quad [i, \alpha_i] \equiv 1, \quad [i, \alpha_j] \equiv \alpha^2, \quad [i, \alpha\pi] \equiv 1, \\
&[j, \alpha] \equiv \alpha_j, \quad [j, \alpha_i] \equiv \alpha^2, \quad [j, \alpha_j] \equiv 1, \quad [j, \alpha\pi] \equiv 1.
\end{aligned}$$

These relations are obtained from Lemma 2.2.1. □

Corollary 2.5.12 *The group K is an orientable Poincaré duality group of dimension 4 and the group K^1 is an orientable Poincaré duality group of dimension 3.*

Proof It is a theorem of Serre [29, Section 1] that the cohomological dimension of a p–torsion free profinite group G is equal to the cohomological dimension of any of its open subgroups. The group K is a 2–group, and by Lemma 2.2.1, the squaring operation P on K has a trivial kernel. Therefore, K is torsion free. The group $F_{3/2}S_2$ is an open subgroup of K. Hence, the cohomological dimension of K is equal to the cohomological dimension of $F_{3/2}S_2$, so that K has cohomological dimension 4. Similarly, K^1 has cohomological dimension 3 since it contains $F_{3/2}S_2^1$ as an open subgroup. According to Symonds and Weigel [35, Proposition 4.4.1], a profinite group G of finite cohomological dimension is a Poincaré duality group if and only if it contains an open subgroup which is a Poincaré duality group. Therefore, both K and K^1 are Poincaré duality groups.

Since K is an open subgroup of $\mathbb{S}_2$, it follows from Proposition 2.5.6 that it is orientable. It remains to prove that K^1 is orientable. Let $\mathbb{Z}_2^\times$ be the center of $\mathbb{S}_2$. Let $U_2 \subseteq \mathbb{Z}_2^\times$ be as in (2.3.5). The group $H = K^1 \times U_2$ is an open subgroup of $\mathbb{S}_2$, and hence H is orientable. Further, $L(H) \cong L(K^1) \oplus L(U_2)$, and the action of H preserves the summands. Recall from Remark 2.5.5 that the representation $\phi_{D_2(H)}\colon H \to \mathbb{Z}_2^*$ is given by the determinant of the adjoint action of H on $L(H)$. For g in H,

$$\det(\mathrm{Ad}(g)) = \det(\mathrm{Ad}(g)|_{L(K^1)}) \det(\mathrm{Ad}(g)|_{L(U_2)}).$$

Since U_2 is abelian, $\det(\mathrm{Ad}(g)|_{L(U_2)}) = 1$. It follows from the orientability of H that $\det(\mathrm{Ad}(g)|_{L(K^1)}) = 1$. In particular, this holds for any g in K^1, and the representation $\phi_{D_2(K^1)}$ is trivial. □

Theorem 2.5.13 (Goerss and Henn, unpublished) *As an $\mathbb{F}_2$–algebra,*

$$H^*(K, \mathbb{F}_2) \cong \mathbb{F}_2[x_0, x_1, x_2, x_4]/(x_0^2, x_1^2 + x_0x_1, x_2^2 + x_0x_2, x_4^2),$$

where x_s has degree one and is as in Definition 2.5.8. Further,

$$H^*(K^1, \mathbb{F}_2) \cong \mathbb{F}_2[x_0, x_1, x_2]/(x_0^2, x_1^2 + x_0x_1, x_2^2 + x_0x_2).$$

The conjugation action of Q_8 factors through $Q_8/C_2 \cong C_2 \times C_2$. It is trivial on x_0 and x_4. On x_1 and x_2, it is described by

$$\begin{aligned} i_*(x_1) &= x_0 + x_1, & j_*(x_1) &= x_1, \\ i_*(x_2) &= x_2, & j_*(x_2) &= x_0 + x_2, \end{aligned}$$

so that the induced representation on $H^1(K^1, \mathbb{F}_2)$ is isomorphic to the augmentation ideal $I(Q_8/C_2)$, and $H^2(K^1, \mathbb{F}_2)$ is isomorphic to the coaugmentation ideal $I(Q_8/C_2)^*$.

Proof The spectral sequence for the group extension

$$1 \to F_{3/2}\mathbb{S}_2 \to K \to \mathbb{Z}/2\{\overline{\alpha}_0\} \to 1$$

has E_2–term given by

$$\mathbb{F}_2[x_0] \otimes E(x_1, x_2, x_3, x_4).$$

It follows from Lemma 2.5.11 and Lemma A.1.5 of the appendix that $x_0^2 = 0$. Since x_3 is the function dual to the image of α^2 in $\operatorname{gr}\mathbb{S}_2$, we have that $d_2(x_3) = x_0^2$. Using the isomorphism $H^1(K, \mathbb{F}_2) \cong \operatorname{Hom}(K, \mathbb{F}_2)$ and Lemma 2.5.11, one computes that

$$H^1(K, \mathbb{F}_2) \cong \mathbb{F}_2\{x_0, x_1, x_2, x_4\}.$$

Hence, $d_r(x_i) = 0$ for $i \neq 3$. All other differentials are determined by these differentials, and

$$E_3 \cong E_\infty \cong E(x_0, x_1, x_2, x_4).$$

Similarly, the E_2–term for the extension

$$1 \to F_{3/2}\mathbb{S}_2^1 \to K^1 \to \mathbb{Z}/2\{\overline{\alpha}_0\} \to 1$$

is given by $\mathbb{F}_2[x_0] \otimes E(x_1, x_2, x_3)$, and $E_3 \cong E_\infty \cong E(x_0, x_1, x_2)$.

Now we determine the multiplicative extensions. First, note that it follows from Lemma A.1.5 that $x_4^2 = 0$ since x_4 is dual to a class that lifts to the free class $\overline{\alpha}_4$ in $H_1(K, \mathbb{Z}_2)$. Similarly, x_1^2 and x_2^2 are nonzero since they lift to 2–torsion classes $\overline{\alpha}_1$ and $\overline{\alpha}_2$ in $H_1(K, \mathbb{Z}_2)$. Therefore, x_1^2 and x_2^2 are linear combinations of x_0x_1 and x_0x_2. We will show that $x_1^2 = x_0x_1$. The proof that $x_2^2 = x_0x_2$ is similar.

Let N be the closure of the normal subgroup of K^1 generated by $F_{6/2}K^1$, α^2 and α_j. That is,

$$N = \overline{\langle F_{6/2}K^1, \alpha^2, \alpha_j \rangle}.$$

Since $[K^1, F_{3/2}K^1] \subseteq F_{6/2}K^1$, and $[\alpha, \alpha_j] = \alpha_j^2$, the group K^1/N is a group of order 8 generated by the image a of α and the image b of α_i. The order of a is 2 and the order of b is 4. Further, since $[\alpha, \alpha_i] = \alpha_i^2$, the group K^1/N is isomorphic to the dihedral group D_8. Now, note that

$$H_1(D_8, \mathbb{F}_2) \cong \mathbb{F}_2\{a, b\}.$$

It is proved in Adem and Milgram [1, Chapter IV, Theorem 2.7] that

$$H^*(D_8, \mathbb{F}_2) \cong \mathbb{F}_2[x, y, w]/(xy),$$

where x is the function dual to a and y is the function dual to $a + b$. Changing the basis of $H_1(D_8, \mathbb{F}_2)$ from $\langle a, a+b \rangle$ to $\langle a, b \rangle$ sends the basis $\langle x, y \rangle$ of $H^1(D_8, \mathbb{F}_2)$ to the basis $\langle y_0, y_1 \rangle = \langle x + y, y \rangle$. We obtain the following presentation

$$H^*(K^1/N, \mathbb{F}_2) \cong \mathbb{F}_2[y_0, y_1, w]/(y_1^2 + y_0y_1).$$

The projection induces a map

$$f\colon H^*(K^1/N, \mathbb{F}_2) \to H^*(K^1, \mathbb{F}_2)$$

with $f(y_0) = x_0$ and $f(y_1) = x_1$. Therefore, $x_1^2 + x_0x_1 = 0$ in $H^2(K^1/N, \mathbb{F}_2)$.

The action of Q_8 follows from Lemma 2.5.11. The isomorphism between the representation $H^1(K^1, \mathbb{F}_2)$ and the representation $I(Q_8/C_2)$ defined by

$$0 \to I(Q_8/C_2) \to \mathbb{F}_2[Q_8/C_2] \xrightarrow{\varepsilon} \mathbb{F}_2 \to 0$$

is given by sending x_0 to the invariant $e+i+j+ij$, x_1 to $e+j$ and x_2 to $e+i$. $\square$

The following description of the integral homology of K^1 will be used heavily in the proof of Theorem 1.2.1.

Corollary 2.5.14 (Goerss and Henn, unpublished) *The integral homology of K^1 is given by*

$$H_n(K^1,\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2 & \text{if } n=0,3,\\ \mathbb{Z}/4\oplus(\mathbb{Z}/2)^2 & \text{if } n=1,\\ 0 & \text{if } n=2.\end{cases}$$

Proof The result for $n=1$ is Lemma 2.5.11. The homology $H_*(K^1,\mathbb{F}_2)$ is dual to $H^*(K^1,\mathbb{F}_2)$, computed in Theorem 2.5.13. The groups $H_n(K^1,\mathbb{Z}_2)$ for $n=2,3$ are computed from the long exact sequence associated to

$$0\to\mathbb{Z}_2\xrightarrow{2}\mathbb{Z}_2\to\mathbb{F}_2\to 0,$$

using the fact that $H_n(K^1,\mathbb{F}_2)$ and $H_1(K^1,\mathbb{Z}_2)$ are known. $\square$

We finish this section by proving Theorem 1.2.5.

Proof of Theorem 1.2.5 Since $\mathbb{S}_2\cong S_2\rtimes C_3$ and ω generates C_3, it suffices to show that S_2 is generated by π, α, i and $j=\omega i\omega^{-1}$. Further, according to Behrens and Lawson [5, Theorem 2.1], it suffices to prove that the inclusion $\langle\pi,\alpha,i,j\rangle\to S_2$ induces a surjective map

$$H_1(S_2,\mathbb{F}_2)\cong S_2/\overline{S_2^*},$$

where S_2^* is the group $S_2^2[S_2,S_2]$. The claim then follows from the isomorphism $S_2\cong K\rtimes Q_8$, the surjectivity of the map

$$\langle\pi,\alpha,\alpha_i,\alpha_j\rangle\to K/\overline{K^*}$$

and the fact that i and j generate Q_8. The argument for $\mathbb{S}_2^1$ is similar. $\square$

3 The algebraic duality resolution

This section is devoted to the construction of the algebraic duality resolution and the description of its properties. We refer the reader to the appendix for background on the cohomology of profinite groups.

3.1 The resolution

From now on, we fix $p = 2$. The goal of this section is to prove Theorem 1.2.1, which was stated in Section 1.2, and is restated as Theorem 3.1.7 below. The proof is broken into a series of results given in Lemma 3.1.1, Lemma 3.1.2, Lemma 3.1.3 and Theorem 3.1.6. All results in this section are due to Goerss, Henn, Mahowald and Rezk.

Let G_{24} be the maximal finite subgroup of $\mathbb{S}_2$ defined in Lemma 2.4.3. Recall that $G'_{24} = \pi G_{24} \pi^{-1}$ for $\pi = 1 + 2\omega$ in $\mathbb{S}_2$. It was shown in Proposition 2.4.6 that there are two conjugacy classes of maximal finite subgroups in $\mathbb{S}_2^1$, and that G_{24} and G'_{24} are representatives. Recall that $C_2 = \{\pm 1\}$ is the subgroup generated by $[-1](x)$ and $C_6 = C_2 \times C_3$. The group K^1 is the Poincaré duality subgroup of $\mathbb{S}_2^1$ which was defined in Section 2.5.

Lemma 3.1.1 *Let $\mathscr{C}_0 = \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}]\!]$ with canonical generator e_0. Let $\varepsilon\colon \mathscr{C}_0 \to \mathbb{Z}_2$ be the augmentation*

$$\varepsilon\colon \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}]\!] \to \mathbb{Z}_2 \otimes_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]} \mathbb{Z}_2[\![\mathbb{S}_2^1]\!] \otimes_{\mathbb{Z}_2[G_{24}]} \mathbb{Z}_2 \cong \mathbb{Z}_2.$$

Let N_0 be defined by the short exact sequence

$$0 \to N_0 \to \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0. \tag{3.1.1}$$

Then N_0 is the left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–submodule of $\mathscr{C}_0$ generated by $(e-\alpha)e_0$, for e the unit in $\mathbb{S}_2^1$ and α as defined in (2.3.4).

Proof Since $\mathbb{S}_2^1 \cong K^1 \rtimes G_{24}$, $\mathscr{C}_0 \cong \mathbb{Z}_2[\![K^1]\!]$ as a $\mathbb{Z}_2[\![K^1]\!]$–module. Therefore, N_0 is isomorphic to the augmentation ideal IK^1. Lemma A.1.4 of the appendix implies that $H_1(K^1, \mathbb{Z}_2) \cong H_0(K^1, N_0)$, where an isomorphism sends the image of g in $K^1/\overline{[K^1, K^1]}$ to the image of $e - g$ in $IK^1/\overline{(IK^1)^2}$. It was shown in Lemma 2.5.11 that $K^1/\overline{[K^1, K^1]}$ is generated by α as a G_{24}–module. This implies that, as a G_{24}–module, $H_0(K^1, N_0)$ is generated by the image of $(e-\alpha)e_0$. Therefore, the map $F\colon \mathbb{Z}_2[\![\mathbb{S}_2^1]\!] \to N_0$ defined by $F(\gamma) = \gamma(e-\alpha)e_0$ induces a surjective map

$$\mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![K^1]\!]} F\colon \mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![K^1]\!]} \mathbb{Z}_2[\![\mathbb{S}_2^1]\!] \to \mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![K^1]\!]} N_0.$$

By Lemma A.1.3 of the appendix, F itself is surjective, and $(e-\alpha)e_0$ generates N_0 as a $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module. □

Lemma 3.1.2 *Let N_0 be as in Lemma 3.1.1. Let $\mathscr{C}_1 = \mathbb{Z}_2[\![\mathbb{S}_2^1/C_6]\!]$ with canonical generator e_1. There is a map $\partial_1\colon \mathscr{C}_1 \to N_0$ defined by*

$$\partial_1(\gamma e_1) = \gamma(e-\alpha)e_0 \tag{3.1.2}$$

for γ in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$. Further, let N_1 be defined by the short exact sequence

$$0 \to N_1 \to \mathscr{C}_1 \xrightarrow{\partial_1} N_0 \to 0, \tag{3.1.3}$$

and let Θ_0 in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ be any element such that $\Theta_0 e_1$ is in the kernel of ∂_1 and

$$\Theta_0 e_1 \equiv (3+i+j+k)e_1 \mod (4, IK^1).$$

Then $\Theta_0 e_1$ generates N_1 over $\mathbb{S}_2^1$.

Proof The element α satisfies $\tau\alpha = \alpha\tau$ for $\tau \in C_6$. Therefore, the map ∂_1 given by (3.1.2) is well defined.

Let N_1 be the kernel of ∂_1. Note that $\mathbb{Z}_2[\![\mathbb{S}_2^1/C_6]\!] \cong \mathbb{Z}_2[\![K^1]\!]^4$ as $\mathbb{Z}_2[\![K^1]\!]$–modules, generated by e_1, ie_1, je_1 and ke_1. Therefore, there is an isomorphism of G_{24}–modules

$$H_0(K^1, \mathscr{C}_1) \cong \mathbb{Z}_2[G_{24}/C_6].$$

As $\mathbb{Z}_2[\![K^1]\!]$–modules, $H_0(K^1, \mathscr{C}_1) \cong \mathbb{Z}_2^4$ generated by the image of the classes e_1, ie_1, je_1 and ke_1. Since $N_0 \cong IK^1$, Lemma A.1.4 of the appendix and Corollary 2.5.14 imply that

$$H_1(K^1, N_0) \cong H_2(K^1, \mathbb{Z}_2) = 0.$$

Therefore, the long exact sequence on cohomology gives rise to a short exact sequence

$$0 \to H_0(K^1, N_1) \to H_0(K^1, \mathscr{C}_1) \to H_0(K^1, N_0) \to 0.$$

By Lemma A.1.4 of the appendix and Lemma 2.5.11,

$$H_0(K^1, N_0) \cong H_1(K^1, \mathbb{Z}_2) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^2,$$

which is all torsion. Thus, we can identify $H_0(K^1, N_1)$ with a free submodule of $H_0(K^1, \mathscr{C}_1)$. Further, it must have rank 4 over $\mathbb{Z}_2$. This can be made explicit as follows.

The map $H_0(K^1, \partial_1)$ sends the residue class of τe_1 to that of $\tau(e-\alpha)e_0$. For τ in G_{24}, $\tau^{-1}e_0 = e_0$, hence $\tau(e-\alpha)e_0 = (e - \tau_*(\alpha))e_0$, where $\tau_*(\alpha) = \tau\alpha\tau^{-1}$. Again, using the boundary isomorphism $H_1(K^1, \mathbb{Z}_2) \cong H_0(K^1, N_0)$ of Lemma A.1.4, the formulas of Lemma 2.5.11 together with the fact that $k = ij$ can be used to compute

$$\partial_1(e_1) \equiv \overline{\alpha}, \quad \partial_1(ie_1) \equiv \overline{\alpha} + \overline{\alpha}_i, \quad \partial_1(je_1) \equiv \overline{\alpha} + \overline{\alpha}_j, \quad \partial_1(ke_1) \equiv 3\overline{\alpha} + \overline{\alpha}_i + \overline{\alpha}_j.$$

Here, $\overline{a}$ is the image of a in $H_1(K^1, \mathbb{Z}_2)$. As α generates a group isomorphic to $\mathbb{Z}/4$, and α_i and α_j both generate groups isomorphic to $\mathbb{Z}/2$, a set of $\mathbb{Z}_2$ generators for the kernel of $H_0(K^1,\partial_1)$ is given by the elements

$$f_1=-4e_1,\quad f_2=2(i-e)e_1,\quad f_3=2(j-e)e_1,\quad f_4=(k-i-j-e)e_1.$$

Let

$$f=(3e+i+j+k)e_1\in H_0(K^1,N_1).$$

Then f generates $H_0(K^1,N_1)\cong\mathbb{Z}_2[G_{24}/C_6]$ as a G_{24}–module. Indeed, using the fact that $G_{24}/C_6\cong Q_8/C_2$, one computes

$$f_1=1/3(i+j+k-5)f,\quad f_2=if-f,\quad f_3=jf-f,\quad f_4=-k(f+f_1).$$

(Note that $-\tau$ denotes $(-1)\cdot\tau$ for the coefficient -1 in $\mathbb{Z}_2$, as opposed to the generator of the central C_2 in Q_8.)

Next, we show that if

$$f'\equiv f\mod (4,IK^1),$$

then f' also generates $H_0(K^1,N_1)$ as a G_{24}–module. To do this, note that $\mathbb{Z}_2[Q_8/C_2]$ is a complete local ring with maximal ideal $\mathfrak{m}=(2,IQ_8/C_2)$. Hence, any element congruent to 1 modulo $\mathfrak{m}$ is invertible. Therefore, if $f'=f+\epsilon f$ for ϵ in $\mathfrak{m}$, then f' is also a generator. However, for a in $H_0(K_1,\mathscr{C}_1)$,

$$4ae_1=a\tfrac{1}{3}((e-i)+(e-j)+(e-k)+2e)f.$$

Hence, $a\frac{1}{3}((e-i)+(e-j)+(e-k)+2e)$ is in $\mathfrak{m}$. Therefore, $4H_0(K_1,\mathscr{C}_1)$ is contained in $\mathfrak{m}f$.

Let Θ_0 in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ be such that

$$\Theta_0e_1\equiv(3+i+j+k)e_1\mod (4,IK^1).$$

Let $F\colon \mathbb{Z}_2[\![\mathbb{S}_2^1]\!]\to N_1$ be the map defined by $F(\gamma)=\gamma\Theta_0e_1$. It induces a surjective map

$$\mathbb{F}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]}F\colon \mathbb{F}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]}\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]\to\mathbb{F}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]}N_1.$$

By Lemma A.1.3 of the appendix, F itself is surjective, and Θ_0e_1 generates N_1 as a $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module. □

Define $\mathrm{tr}_{C_3}\colon \mathbb{Z}_2[\![\mathbb{S}_2^1]\!]\to\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ to be the $\mathbb{Z}_2$–linear map induced by

$$\mathrm{tr}_{C_3}(g)=g+\omega g\omega^{-1}+\omega^{-1}g\omega \tag{3.1.4}$$

for g in $\mathbb{S}_2^1$ and ω our chosen generator of C_3.

Lemma 3.1.3 *Let $\mathscr{C}_2=\mathbb{Z}_2[\![\mathbb{S}_2^1/C_6]\!]$ with canonical generator e_2. Let Θ in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ satisfy:*

(1) $\tau\Theta = \Theta\tau$ *for* τ *in* C_6,

(2) Θe_1 *is in the kernel of* $\partial_1\colon \mathscr{C}_1 \to \mathscr{C}_0$,

(3) $\Theta e_1 \equiv (3+i+j+k)e_1$ *modulo* $(4, IK^1)$.

Then the map of $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$*–modules* $\partial_2\colon \mathscr{C}_2 \to \mathscr{C}_1$ *defined by*

$$\partial_2(\gamma e_2) = \gamma\Theta e_2 \tag{3.1.5}$$

surjects onto $N_1 = \ker(\partial_1)$. *Further, if* N_2 *is defined by the exact sequence*

$$0 \to N_2 \to \mathscr{C}_2 \xrightarrow{\partial_2} N_1 \to 0, \tag{3.1.6}$$

then $N_2 \cong \mathbb{Z}_2[\![K^1]\!]$ *as* $\mathbb{Z}_2[\![K^1]\!]$*–modules.*

Proof Choose an element Θ_0 which generates N_1 as in Lemma 3.1.2. Recall that $C_6 \cong C_2 \times C_3$ and that C_2 is in the center of $\mathbb{S}_2$. Therefore, for tr_{C_3} as defined by (3.1.4),

$$\Theta = \tfrac{1}{3}\,\mathrm{tr}_{C_3}(\Theta_0)$$

satisfies properties (1), (2) and (3). The map ∂_2 given by (3.1.5) is well defined and surjects onto N_1 by Lemma A.1.3.

Let $N_2 \subseteq \mathscr{C}_2$ be the kernel of ∂_2 as in the statement of the result. The map ∂_2 induces an isomorphism $H_0(K^1, \mathscr{C}_2) \cong H_0(K^1, N_1)$. Hence, for all n,

$$H_n(K^1, N_2) \cong H_{n+1}(K^1, N_1) \cong H_{n+2}(K^1, N_0) \cong H_{n+3}(K^1, \mathbb{Z}_2).$$

This implies:

$$H_n(K^1, N_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Choose an element e' in N_2 such that e' reduces to a generator of $\mathbb{Z}_2$ in $H_0(K^1, N_2)$. Define $\phi\colon \mathbb{Z}_2[\![K^1]\!] \to N_2$ by $\phi(k) = ke'$. Then

$$\mathrm{Tor}_0^{\mathbb{Z}_2[\![K^1]\!]}(\mathbb{F}_2, \phi)$$

is an isomorphism, and

$$\mathrm{Tor}_1^{\mathbb{Z}_2[\![K^1]\!]}(\mathbb{F}_2, \phi)$$

is surjective. By Lemma A.1.3 of the appendix, ϕ is an isomorphism of $\mathbb{Z}_2[\![K^1]\!]$–modules. □

Splicing the exact sequences (3.1.1), (3.1.3) and (3.1.6) gives an exact sequence

$$0 \to N_2 \to \mathscr{C}_2 \to \mathscr{C}_1 \to \mathscr{C}_0 \to \mathbb{Z}_2 \to 0, \tag{3.1.7}$$

which is a free resolution of $\mathbb{Z}_2$ as a trivial $\mathbb{Z}_2[\![K^1]\!]$–module. The next goal is to find an isomorphism $N_2 \cong \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}']\!]$, where $G_{24}' = \pi G_{24}\pi^{-1}$ represents the other conjugacy class of maximal finite subgroups in $\mathbb{S}_2^1$. To prove this, we will need a few results. Before stating these, we introduce some notation.

Let G be a subgroup of $\mathbb{S}_2$ which contains the central subgroup C_2. We define

$$PG := G/C_2.$$

We let

$$A_4 := PG_{24}, \tag{3.1.8}$$

$$A_4' := PG_{24}'. \tag{3.1.9}$$

The choice of notation is justified by the fact that both of these groups are isomorphic to the alternating group on four letters. Note also that, since C_2 is central, $PC_6 \cong C_3$ and $P\mathbb{S}_2^1 \cong K^1 \rtimes A_4$. Therefore, for any G which contains C_2,

$$\mathbb{Z}_2[\![\mathbb{S}_2^1/G]\!] \cong \mathbb{Z}_2[\![P\mathbb{S}_2^1/PG]\!]$$

as $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules. To prove that $N_2 \cong \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}']\!]$, it will thus be sufficient to prove that

$$N_2 \cong \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]$$

as $\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]$–modules.

We showed in Corollary 2.5.12 that K^1 is a Poincaré duality group (see Definition 2.5.4). Further, there is an isomorphism of $\mathbb{Z}_2[\![K^1]\!]$–modules

$$\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!] \cong \mathbb{Z}_2[\![K^1]\!].$$

Hence,

$$H^n(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \cong \begin{cases} \mathbb{Z}_2 & \text{if } n=3, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.10}$$

Lemma 3.1.4 *The inclusion $\iota\colon K^1 \to P\mathbb{S}_2^1$ induces an isomorphism*

$$\iota^*\colon H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \to H^3(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]).$$

Proof The action of A_4 on $H^3(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$ is trivial. This follows from the fact that there are no nontrivial one-dimensional representations of A_4. Indeed,

$$\mathrm{Hom}(A_4, Gl_1(\mathbb{Z}_2)) = H^1(A_4, \mathbb{Z}_2^\times).$$

and $H^1(A_4, \mathbb{Z}_2^\times) = 0$. Since $P\mathbb{S}_2^1 \cong K^1 \rtimes A_4$, there is a spectral sequence

$$H^p(A_4, H^q(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])) \Longrightarrow H^{p+q}(P\mathbb{S}_2^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]).$$

Because the action of A_4 on $H^3(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$ is trivial, (3.1.10) implies that the edge homomorphism

$$H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \to H^0(A_4, H^3(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]))$$

induced by the inclusion $\iota\colon K^1 \to P\mathbb{S}_2^1$ is an isomorphism. □

Lemma 3.1.5 *There are surjections*

$$\eta\colon \mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]}(N_2, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \to H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]),$$
$$\eta'\colon \mathrm{Hom}_{\mathbb{Z}_2[\![K^1]\!]}(N_2, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \to H^3(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$$

making the following diagram commute

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]}(N_2,\ \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \arrow[r, "\eta"] \arrow[d, "\iota^*"'] & H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \arrow[d, "\iota^*"] \\ \mathrm{Hom}_{\mathbb{Z}_2[\![K^1]\!]}(N_2, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \arrow[r, "\eta'"] & H^3(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \end{tikzcd} \tag{3.1.11}$$

where ι^* is the map induced by the inclusion $\iota\colon K^1 \to P\mathbb{S}_2^1$.

Proof Let $\mathscr{B}_p = \mathscr{C}_p$ for $0 \le p < 3$ and $\mathscr{B}_3 = N_2$. Resolving $\mathscr{B}_p$ by projective $\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]$–modules gives rise to spectral sequences

$$E_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]}(\mathscr{B}_p, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \Longrightarrow H^{p+q}(P\mathbb{S}_2^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$$

and

$$F_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[\![K^1]\!]}(\mathscr{B}_p, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \Longrightarrow H^{p+q}(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]).$$

These are first quadrant cohomology spectral sequences, with differentials

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

and

$$d_r\colon F_r^{p,q} \to F_r^{p+r,q-r+1}.$$

Further, $\iota\colon K^1 \to P\mathbb{S}_2^1$ induces a map of spectral sequences

$$\iota^*\colon E_r^{p,q} \to F_r^{p,q}.$$

Let η be the edge homomorphism

$$\eta\colon E_1^{3,0} \to H^3(P\mathbb{S}_2^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]),$$

and let η' be the edge homomorphism

$$\eta'\colon F_1^{3,0} \to H^3(K^1, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]).$$

First, note that since the modules $\mathscr{B}_p$ are projective $\mathbb{Z}_2[\![K^1]\!]$–modules, $F_r^{p,q}$ collapses with $F_\infty^{p,q}=0$ for $q>0$ so that

$$F_\infty^{3,0} \to H^3(K^1; \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$$

is surjective. Hence, η' is surjective.

In order to show that η is surjective, it is sufficient to show that $E_1^{3-q,q}=0$ for $q>0$. For $q=1$ and $q=2$, this follows from the fact that $\mathbb{Z}_2[\![P\mathbb{S}_2^1/C_3]\!]$ is a projective $\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]$–module. Hence, if $q>0$, then

$$\mathrm{Ext}^q_{\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]}(\mathbb{Z}_2[\![P\mathbb{S}_2^1/C_3]\!], \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) = 0.$$

It remains to show that

$$E_1^{0,3} = \mathrm{Ext}^3_{\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]}(\mathscr{B}_0, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$$

is zero, where $\mathscr{B}_0 = \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4]\!]$.

Let $V \cong C_2 \times C_2$ be the 2–Sylow subgroup of A_4. Then

$$\begin{aligned} E_1^{0,3} = \mathrm{Ext}^3_{\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]}(\mathscr{B}_0, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) &\cong H^3(A_4, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \\ &\cong H^3(V, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])^{C_3}. \end{aligned}$$

Let $G_n = PF_{n/2}S_2^1 \rtimes A_4'$ and $X_n = P\mathbb{S}_2^1/G_n$. The profinite A_4–set $P\mathbb{S}_2^1/A_4'$ is isomorphic to the inverse limit of the finite A_4–sets X_n. There is an exact sequence

$$0 \to \lim{}^1 H^2(V, \mathbb{Z}_2[X_n]) \to H^3(V, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \to \lim_n H^3(V, \mathbb{Z}_2[X_n]) \to 0.$$

Since the groups $H^2(V, \mathbb{Z}_2[X_n])$ are finite, the Mittag–Leffler condition is satisfied and $\lim{}^1 H^2(V, \mathbb{Z}_2[X_n]) = 0$. Hence,

$$H^3(V, \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \cong \lim_n H^3(V, \mathbb{Z}_2[X_n]).$$

We will show that there is an integer N such that $H^3(V,\mathbb{Z}_2[X_n])=0$ for all $n\geq N$. This implies that $H^3(V,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$ is zero, so that $E_1^{0,3}=0$.

Note that

$$H^3(V,\mathbb{Z}_2[X_n])\cong\bigoplus_{x\in V\backslash X_n/G_n}H^3(V,\mathbb{Z}_2[V/V_x])\cong\bigoplus_{x\in V\backslash X_n/G_n}H^3(V_x,\mathbb{Z}_2)$$

for $V_x=\{g\in V\mid gxG_n=xG_n\}$. If the inclusion $V_x\subseteq V$ is an equality, then $x^{-1}Vx$ is a subgroup of G_n. We show that there exists an integer N such that, for all $n\geq N$, there is no element x in $P\mathbb{S}_2^1$ such that $x^{-1}Vx\subseteq G_n$. This implies that, for $n\geq N$, for all choices of coset representatives $x\in V\backslash X_n/G_n$, the group V_x is either trivial or it has order 2. In both cases, $H^3(V_x,\mathbb{Z}_2)=0$. Hence, for $n\geq N$, $H^3(V,\mathbb{Z}_2[X_n])=0$.

Suppose that there is a sequence of integers n_m and elements x_{n_m} such that $x_{n_m}^{-1}Vx_{n_m}\subseteq G_{n_m}$. Since $P\mathbb{S}_2^1$ is compact, we can choose the sequence (x_{n_m}) to converge to some element y. The groups G_n are closed and nested, so the continuity of the group multiplication implies that $y^{-1}Vy\subseteq G_n$ for all $n\in\mathbb{N}$. Therefore,

$$y^{-1}Vy\subseteq\bigcap_n G_n=A_4',$$

and hence $y^{-1}Vy=V'$, where V' is the 2–Sylow subgroup of A_4'. However, it follows from Proposition 2.4.6 that V and V' are not conjugate in $P\mathbb{S}_2^1$. Therefore, such a sequence cannot exist, and there must be some integer N such that, for all $n\geq N$, there is no x in $P\mathbb{S}_2^1$ such that $x^{-1}Vx\subseteq G_n$. □

Theorem 3.1.6 *There is an isomorphism of left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules*

$$\phi\colon \mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}']\!]\to N_2,$$

where $G_{24}'=\pi G_{24}\pi^{-1}$.

Proof It suffices to construct an isomorphism $\varphi\colon N_2\to\mathbb{Z}_2[\![P\mathbb{S}_2^1/PG_{24}']\!]$ of left $\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]$–modules. The result then follows by letting $\phi=\varphi^{-1}$, considered as a map of $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules.

Recall from Corollary 2.5.12 that K^1 is an orientable Poincaré duality group of dimension 3, as in Definition 2.5.4. That is, the compact dualizing module $D_2(K^1)$ is isomorphic to the trivial $\mathbb{Z}_2[\![K^1]\!]$–module $\mathbb{Z}_2$ and $H_3(K^1,\mathbb{Z}_2)\cong\mathbb{Z}_2$. Choose a generator $[K^1]$ of $H_3(K^1,\mathbb{Z}_2)$. As in Remark 2.5.5, there is a natural isomorphism

$$H^{3-*}(K^1,-)\xrightarrow{\cap[K^1]}H_*(K^1,-).$$

Let

$$\nu\colon H_3(K^1,\mathbb{Z}_2)\to \mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]} N_2$$

be the edge homomorphism for the homology spectral sequence obtained from (3.1.7). Define

$$\mathrm{ev}\colon \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]} N_2, \mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]} \mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])\to H_0(K^1,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$$

by

$$\mathrm{ev}(f)=f(\nu([K^1])).$$

Let $\iota\colon K^1\to P\mathbb{S}_2^1$ be the inclusion. Let η and η' be the edge homomorphisms of Lemma 3.1.5. We obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]}(N_2,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \arrow[r, "\eta"] \arrow[d, "\iota^*"'] & H^3(P\mathbb{S}_2^1,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \arrow[d, "\iota^*"] \\
\mathrm{Hom}_{\mathbb{Z}_2[\![K^1]\!]}(N_2,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \arrow[r, "\eta'"] \arrow[d, "\mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]}-"'] & H^3(K^1,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \arrow[d, "\cap[K^1]"] \\
\mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]}N_2,\mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]}\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]) \arrow[r, "\mathrm{ev}"] & H_0(K^1,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])
\end{tikzcd}$$

Since $\cap[K^1]\circ\eta'$ is surjective, so is the map ev. Both N_2 and $\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]$ are free of rank one over $\mathbb{Z}_2[\![K^1]\!]$. Hence, $\mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]} N_2$ and $\mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]}\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!]$ are abstractly isomorphic to $\mathbb{Z}_2$. Since ev is a surjective group homomorphism from $\mathbb{Z}_2$ to itself, it is an isomorphism. It follows from Lemma A.1.3 that any element of $\mathrm{Hom}_{\mathbb{Z}_2[\![K^1]\!]}(N_2,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$ that becomes a unit after applying $\mathbb{Z}_2\otimes_{\mathbb{Z}_2[\![K^1]\!]}-$ is an isomorphism. By Lemma 3.1.5, the composite $\cap[K^1]\circ\iota^*\circ\eta$ is surjective. Therefore, we can choose φ in $\mathrm{Hom}_{\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]}(N_2,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$ such that $\cap[K^1]\circ\iota^*\circ\eta(\varphi)$ is a generator of $H_0(K^1,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$. Then $\iota^*(\varphi)$ in $\mathrm{Hom}_{\mathbb{Z}_2[\![K^1]\!]}(N_2,\mathbb{Z}_2[\![P\mathbb{S}_2^1/A_4']\!])$ is an isomorphism, and hence φ must be an isomorphism. □

Combining the previous results, we can finally prove Theorem 1.2.1. We restate it here for convenience.

Theorem 3.1.7 *Let $\mathbb{Z}_2$ be the trivial $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module. There is an exact sequence of complete $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules*

$$0\to\mathscr{C}_3\xrightarrow{\partial_3}\mathscr{C}_2\xrightarrow{\partial_2}\mathscr{C}_1\xrightarrow{\partial_1}\mathscr{C}_0\xrightarrow{\varepsilon}\mathbb{Z}_2\to 0,$$

where $\mathscr{C}_0\cong\mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}]\!]$ and $\mathscr{C}_1\cong\mathscr{C}_2\cong\mathbb{Z}_2[\![\mathbb{S}_2^1/C_6]\!]$ and $\mathscr{C}_3=\mathbb{Z}_2[\![\mathbb{S}_2^1/G_{24}']\!]$. Further, this is a free resolution of the trivial $\mathbb{Z}_2[\![K^1]\!]$–module $\mathbb{Z}_2$.

Proof Let
$$\mathscr{C}_3 := \mathbb{Z}_2[\![\mathbb{S}_2^1/G'_{24}]\!].$$
Let $\phi\colon \mathscr{C}_3 \to N_2$ be the isomorphism of Theorem 3.1.6. Let $\partial_3\colon \mathscr{C}_3 \to \mathscr{C}_2$ be the isomorphism ϕ followed by the inclusion of N_2 in $\mathscr{C}_2$. This gives an exact sequence

$$0 \to \mathscr{C}_3 \to \mathscr{C}_2 \xrightarrow{\partial_2} N_1 \to 0. \tag{3.1.12}$$

Splicing the exact sequences of (3.1.1), (3.1.3) and (3.1.12) finishes the proof. □

The exact sequence of Theorem 3.1.7 is called the *algebraic duality resolution*. The duality properties it satisfies will be described in Section 3.3.

3.2 The algebraic duality resolution spectral sequence

The algebraic duality resolution gives rise to a spectral sequence called the *algebraic duality resolution spectral sequence*, which we describe here. The following result is a refinement of Theorem 1.2.4, which was stated in Section 1.2. We define
$$Q'_8 := \pi Q_8 \pi^{-1}.$$
We also let V be the 2–Sylow subgroup of A_4 and V' be the 2–Sylow subgroup of A'_4, where $A_4 \cong PG_{24}$ and $A'_4 = PG'_{24}$ as defined in (3.1.8) and (3.1.9).

Theorem 3.2.1 *Let M be a profinite $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module. There is a first quadrant spectral sequence*
$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathscr{C}_p, M) \Longrightarrow H^{p+q}(\mathbb{S}_2^1, M)$$
with differentials $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$. Further,
$$E_1^{p,q} \cong \begin{cases} H^q(G_{24}, M) & \text{if } p = 0, \\ H^q(C_6, M) & \text{if } p = 1, 2, \\ H^q(G'_{24}, M) & \text{if } p = 3. \end{cases}$$
Similarly, there are first quadrant spectral sequences
$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![G]\!]}(\mathscr{C}_p, M) \Longrightarrow H^{p+q}(G, M),$$
where G is S_2^1, $P\mathbb{S}_2^1$ or PS_2^1. The E_1–term is
$$E_1^{p,q} \cong \begin{cases} H^p(Q_8; M) & \text{if } q = 0, \\ H^p(C_2; M) & \text{if } q = 1, 2, \\ H^p(Q'_8; M) & \text{if } q = 3 \end{cases}$$

when G *is* S_2^1,

$$E_1^{p,q} \cong \begin{cases} H^p(A_4;M) & \text{if } q=0, \\ H^p(C_3;M) & \text{if } q=1,2, \\ H^p(A_4';M) & \text{if } q=3 \end{cases}$$

when G *is* $P\mathbb{S}_2^1$ *and*

$$E_1^{p,q} \cong \begin{cases} H^p(V;M) & \text{if } q=0, \\ H^p(\{e\};M) & \text{if } q=1,2, \\ H^p(V';M) & \text{if } q=3 \end{cases}$$

when G *is* PS_2^1.

Proof There are two equivalent constructions. First, recall that the algebraic duality resolution is spliced from the exact sequences

$$0 \to N_i \to \mathscr{C}_i \to N_{i-1} \to 0, \tag{3.2.1}$$

with $\mathscr{C}_3 = N_2$ and $N_{-1} = \mathbb{Z}_2$. The exact couple

$$\begin{tikzcd} \mathrm{Ext}(N_*,M) \arrow[rr, "\delta_*"] & & \mathrm{Ext}(N_{*-1},M) \arrow[dl, "r_*"] \\ & \mathrm{Ext}(\mathscr{C}_*,M) \arrow[ul, "i^*"] & \end{tikzcd}$$

gives rise to the algebraic duality resolution spectral sequence.

Alternatively, one can resolve each $\mathscr{C}_\bullet \to \mathbb{Z}_2$ into a double complex of projective finitely generated $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules. The total complex $\mathrm{Tot}(P_{p,q})$ for $p \geq 0$ is a projective resolution of $\mathbb{Z}_2$ as a $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module. The homology of the double complex $\mathrm{Hom}_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathrm{Tot}(P_{p,q}),M)$ is

$$\mathrm{Ext}^{p+q}_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathbb{Z}_2,M) \cong H^{p+q}(\mathbb{S}_2^1,M).$$

The identification of the E_1–term follows from Shapiro's Lemma A.1.2 of the appendix. Indeed, any finite subgroup H of $\mathbb{S}_2^1$ is closed. Further, since $\mathbb{S}_2^1 \cong S_2^1 \rtimes C_3$,

$$\begin{aligned} \mathrm{Ext}^q_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathbb{Z}_2[\![\mathbb{S}_2^1]\!] \otimes_{\mathbb{Z}_2[H]} \mathbb{Z}_2, M) &\cong \left(\mathrm{Ext}^q_{\mathbb{Z}_2[\![S_2^1]\!]}(\mathbb{Z}_2[\![S_2^1]\!] \otimes_{\mathbb{Z}_2[\mathrm{Syl}_2(H)]} \mathbb{Z}_2, M)\right)^{C_3} \\ &\cong \left(\mathrm{Ext}^q_{\mathbb{Z}_2[\mathrm{Syl}_2(H)]}(\mathbb{Z}_2,M)\right)^{C_3} \cong H^q(H,M). \end{aligned}$$

For the groups S_2^1, $P\mathbb{S}_2$ and PS_2^1, one applies the same construction, keeping the following isomorphisms in mind. Let $H \subseteq \mathbb{S}_2^1$ be a finite subgroup which contains C_6

and let $PH = H/C_2$. There are isomorphisms

$$\mathbb{Z}_2[\![\mathbb{S}_2^1/H]\!] \cong \mathbb{Z}_2[\![P\mathbb{S}_2^1]\!] \otimes_{\mathbb{Z}_2[PH]} \mathbb{Z}_2 \cong \mathbb{Z}_2[\![P\mathbb{S}_2^1/PH]\!]$$

and

$$\mathbb{Z}_2[\![\mathbb{S}_2^1/H]\!] \cong \mathbb{Z}_2[\![PS_2^1]\!] \otimes_{\mathbb{Z}_2[\mathrm{Syl}_2(PH)]} \mathbb{Z}_2 \cong \mathbb{Z}_2[\![P\mathbb{S}_2^1/\operatorname{Syl}_2(PH)]\!]$$

as $\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]$ and $\mathbb{Z}_2[\![PS_2^1]\!]$–modules, respectively. □

3.3 The duality

The algebraic duality resolution of Theorem 3.1.7 owes its name to the fact that it satisfies a certain twisted duality. This duality is crucial for computations as it allows us to understand the map $\partial_3\colon \mathscr{C}_3 \to \mathscr{C}_2$.

Let $\mathrm{Mod}(\mathbb{S}_2^1)$ denote the category of finitely generated left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules. Let $\pi = 1 + 2\omega$ in $\mathbb{S}_2$ be as defined in (2.3.3). For M in $\mathrm{Mod}(\mathbb{S}_2^1)$, let $c_\pi(M)$ denote the left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module whose underlying $\mathbb{Z}_2$–module is M, but for which the action of γ in $\mathbb{S}_2^1$ on an element m in $c_\pi(M)$ is given by

$$\gamma \cdot m = \pi\gamma\pi^{-1} m.$$

If $\phi\colon M \to N$ is a morphism of left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules, let $c_\pi(\phi)\colon c_\pi(M) \to c_\pi(N)$ be given by

$$c_\pi(\phi)(m) = \phi(m).$$

Then $c_\pi\colon \mathrm{Mod}(\mathbb{S}_2^1) \to \mathrm{Mod}(\mathbb{S}_2^1)$ is a functor. In fact, c_π is an involution, since $\pi^2 = -3$ is in the center of $\mathbb{S}_2$. We can now prove Theorem 1.2.2, which is restated here for convenience.

Theorem 3.3.1 (Henn, Karamanov and Mahowald, unpublished) *There exists an isomorphism of complexes of left $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules:*

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & c_\pi(\mathscr{C}_0^*) \arrow[r, "c_\pi(\partial_1^*)"] & c_\pi(\mathscr{C}_1^*) \arrow[r, "c_\pi(\partial_2^*)"] & c_\pi(\mathscr{C}_2^*) \arrow[r, "c_\pi(\partial_3^*)"] & c_\pi(\mathscr{C}_3^*) \arrow[r, "\overline{\varepsilon}"] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

Proof The proof is similar to the proof of Henn, Karamanov, Mahowald [18, Proposition 3.8]. Let $\mathscr{C}_p^* = \mathrm{Hom}_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathscr{C}_p, \mathbb{Z}_2[\![\mathbb{S}_2^1]\!])$ and $\partial_p^* = \mathrm{Hom}_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\partial_p, \mathbb{Z}_2[\![\mathbb{S}_2^1]\!])$ be the $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–duals of $\mathscr{C}_p$ and ∂_p in the sense of Equation (A.1.1). The resolution of Theorem 3.1.7 gives rise to a complex

$$0 \to \mathscr{C}_0^* \xrightarrow{\partial_1^*} \mathscr{C}_1^* \xrightarrow{\partial_2^*} \mathscr{C}_2^* \xrightarrow{\partial_3^*} \mathscr{C}_3^* \to 0. \tag{3.3.1}$$

Because K^1 has finite index in $\mathbb{S}_2^1$, the induced and coinduced modules of $\mathbb{Z}_2[\![K^1]\!]$ are isomorphic; see Symonds and Weigel [35, Section 3.3]. Therefore

$$\mathrm{Hom}_{\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]}(\mathscr{C}_p, \mathbb{Z}_2[\![\mathbb{S}_2^1]\!]) \cong \mathrm{Hom}_{\mathbb{Z}_2[\![K^1]\!]}(\mathscr{C}_p, \mathbb{Z}_2[\![K^1]\!])$$

and the homology of the complex (3.3.1) is $H^n(K^1, \mathbb{Z}_2[\![K^1]\!])$. By Corollary 2.5.12, $H^n(K^1, \mathbb{Z}_2[\![K^1]\!])$ is 0 for $n \neq 3$ and $\mathbb{Z}_2$ for $n=3$. Further, the action of G_{24} on $H^3(K^1, \mathbb{Z}_2[\![K^1]\!]) \cong \mathbb{Z}_2$ is trivial, as there are no nontrivial one dimensional 2–adic representations of G_{24}. Hence, (3.3.1) is a resolution of $\mathbb{Z}_2$ as a trivial $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–module.

The module $\mathscr{C}_p^*$ is of the form $\mathbb{Z}_2[\![\mathbb{S}_2^1/H]\!]$ via the isomorphism t defined in (A.1.2). Let $\overline{\varepsilon}$ be the augmentation

$$\overline{\varepsilon}\colon \mathscr{C}_3^* \to \mathbb{Z}_2.$$

Because the augmentation $\varepsilon\colon \mathbb{Z}_2[\![K^1]\!] \to \mathbb{Z}_2$ induces an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_2[\![K^1]\!]}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathrm{Hom}_{\mathbb{Z}_2[\![K^1]\!]}(\mathbb{Z}_2[\![K^1]\!], \mathbb{Z}_2),$$

one can choose an isomorphism $H^3(K, \mathbb{Z}_2[\![K^1]\!]) \to \mathbb{Z}_2$ making the following diagram commute:

$$\begin{tikzcd} \mathscr{C}_3^* \arrow[r] \arrow[d, "\overline{\varepsilon}"] & H^3(K^1, \mathbb{Z}_2[\![K^1]\!]) \arrow[dl, dotted] \\ \mathbb{Z}_2 & \end{tikzcd}$$

Therefore, the dual resolution is given by

$$0 \to \mathscr{C}_0^* \xrightarrow{\partial_1^*} \mathscr{C}_1^* \xrightarrow{\partial_2^*} \mathscr{C}_2^* \xrightarrow{\partial_3^*} \mathscr{C}_3^* \xrightarrow{\overline{\varepsilon}} \mathbb{Z}_2 \to 0.$$

Take the image of this resolution in $\mathrm{Mod}(\mathbb{S}_2^1)$ under the involution c_π. Let e_3^π be the canonical generator of $c_\pi(\mathscr{C}_3^*)$. The map $f_0\colon \mathscr{C}_0 \to c_\pi(\mathscr{C}_3^*)$ defined by

$$f_0(e_0) = e_3^\pi$$

is an isomorphism of $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules, and the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\
c_\pi(\mathscr{C}_3^*) \arrow[r, "\overline{\varepsilon}"] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

Therefore, f_0 induces an isomorphism $\ker\varepsilon \cong \ker\overline{\varepsilon}$. As both

$$\mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \ker\varepsilon$$

and

$$c_\pi(\mathscr{C}_2^*) \xrightarrow{c_\pi(\partial_2^*)} \mathscr{C}_3^* \xrightarrow{c_\pi(\partial_3^*)} \ker\overline{\varepsilon}$$

are the beginning of projective resolutions of $\ker\varepsilon$ and $\ker\overline{\varepsilon}$ as $\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]$–modules, f_0 lifts to a chain map:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \ker\varepsilon \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & c_\pi(\mathscr{C}_0^*) \arrow[r, "c_\pi(\partial_1^*)"] & c_\pi(\mathscr{C}_1^*) \arrow[r, "c_\pi(\partial_2^*)"] & c_\pi(\mathscr{C}_2^*) \arrow[r, "c_\pi(\partial_3^*)"] & \ker\overline{\varepsilon} \arrow[r] & 0
\end{tikzcd}$$

Let $PS_2^1 = S_2^1/C_2$, where S_2^1 denotes the 2–Sylow subgroup of $\mathbb{S}_2^1$. By construction, f_0 is an isomorphism, which implies that $\mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![PS_2^1]\!]} f_1$ and $\mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![PS_2^1]\!]} f_2$ are isomorphisms. As $\mathscr{C}_p$ and $c_\pi(\mathscr{C}_p^*)$ are projective $\mathbb{Z}_2[\![P\mathbb{S}_2^1]\!]$–modules for $p = 1, 2$, Lemma A.1.3 of the appendix implies that f_1 and f_2 are isomorphisms. Finally, f_3 must be an isomorphism by the five lemma. □

3.4 A description of the maps

This section is dedicated to proving the statements in Theorem 1.2.6. The first statement of Theorem 1.2.6 is that

$$\partial_1(e_1) = (e-\alpha)e_0.$$

This was shown in Theorem 3.1.7. In this section, we prove the remaining statements of that theorem.

The following result provides a description of the maps $\partial_3\colon \mathscr{C}_3 \to \mathscr{C}_2$ and proves the last part of Theorem 1.2.6. It is a consequence of Theorem 3.3.1.

Theorem 3.4.1 *There are isomorphisms of $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$–modules $g_p\colon \mathscr{C}_p \to \mathscr{C}_p$ and differentials*

$$\partial'_{p+1}\colon \mathscr{C}_{p+1} \to \mathscr{C}_p$$

such that

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "g_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "g_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "g_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "g_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial'_3"] & \mathscr{C}_2 \arrow[r, "\partial'_2"] & \mathscr{C}_1 \arrow[r, "\partial'_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] & \mathbb{Z}_2 \arrow[r] & 0 \end{tikzcd} \tag{3.4.1}$$

is an isomorphism of complexes. The map $\partial'_3\colon \mathscr{C}_3 \to \mathscr{C}_2$ is given by

$$\partial'_3(e_3) = \pi(e+i+j+k)(e-\alpha^{-1})\pi^{-1}e_2. \tag{3.4.2}$$

Proof We will construct a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \mathscr{C}_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \mathscr{C}_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & c_\pi(\mathscr{C}_0^*) \arrow[r, "c_\pi(\partial_1^*)"] \arrow[d, "q_3"] & c_\pi(\mathscr{C}_1^*) \arrow[r, "c_\pi(\partial_2^*)"] \arrow[d, "q_2"] & c_\pi(\mathscr{C}_2^*) \arrow[r, "c_\pi(\partial_3^*)"] \arrow[d, "q_1"] & c_\pi(\mathscr{C}_3^*) \arrow[r, "\overline{\varepsilon}"] \arrow[d, "q_0"] & \mathbb{Z}_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{C}_3 \arrow[r, "\partial'_3"] & \mathscr{C}_2 \arrow[r, "\partial'_2"] & \mathscr{C}_1 \arrow[r, "\partial'_1"] & \mathscr{C}_0 \arrow[r, "\varepsilon"] & \mathbb{Z}_2 \arrow[r] & 0 \end{tikzcd}$$

The maps g_p will be the composites of the vertical maps. First, let $e_p^\pi \in c_\pi(\mathscr{C}_p^*)$ be the canonical generator. Define isomorphisms $q_p\colon c_\pi(M_{3-p}^*) \to M_p$ by

$$q_p(e_{3-p}^\pi) = e_p.$$

Define $g_p\colon \mathscr{C}_p \to \mathscr{C}_p$ by

$$g_p = q_p f_p$$

and $\partial'_{p+1}\colon \mathscr{C}_{p+1} \to \mathscr{C}_p$ by

$$\partial'_{p+1} = q_p c_\pi(\partial_{3-p}^*) q_{p+1}^{-1}.$$

By construction, (3.4.1) is commutative.

In order to compute ∂'_3, it is necessary to understand ∂_1^*. By definition,

$$\partial_1^*(e_0^*)(e_1) = e_0^*((e-\alpha)e_1) = (e-\alpha)\sum_{h \in G_{24}} h.$$

However,

$$(e-\alpha)\sum_{h\in G_{24}} h = (e-\alpha)\sum_{h\in C_6} h(e+i^{-1}+j^{-1}+k^{-1})$$
$$= \sum_{h\in C_6} h(e-\alpha)(e+i^{-1}+j^{-1}+k^{-1})$$
$$= ((e+i+j+k)(e-\alpha^{-1})e_1^*)(e_1).$$

Hence,

$$\partial_1^*(e_0^*) = (e+i+j+k)(e-\alpha^{-1})e_1^*.$$

A diagram chase shows that ∂_3' is given by (3.4.2). □

The maps $\partial_1\colon \mathscr{C}_1 \to \mathscr{C}_0$ and $\partial_3\colon \mathscr{C}_3 \to \mathscr{C}_2$ now have explicit descriptions up to isomorphisms. The map $\partial_2\colon \mathscr{C}_2 \to \mathscr{C}_1$ is harder to describe. Theorem 3.4.5 and Corollary 3.4.6 below give an estimate for this map. These are technical results which will be used in our computations in [3]. Note that Theorem 3.1.7, Theorem 3.4.1 and Corollary 3.4.6 below prove Theorem 1.2.6, which was stated in Section 1.2.

Recall that

$$\alpha_\tau = [\tau,\alpha] = \tau\alpha\tau^{-1}\alpha^{-1}.$$

We will need the following result to describe the element Θ of Lemma 3.1.3.

Lemma 3.4.2 *Let $n \geq 2$ and x be in $IF_{n/2}K^1$. There exist h_0, h_1 and h_2 in $\mathbb{Z}_2[\![F_{n/2}K^1]\!]$ such that*

$$x = \begin{cases} h_0(e-\alpha^{2^{m-1}}) + h_1(e-\alpha_i^{2^{m-1}}) + h_2(e-\alpha_j^{2^{m-1}}) & \text{if } n = 2m, \\ h_0(e-\alpha^{2^{m}}) + h_1(e-\alpha_i^{2^{m-1}}) + h_2(e-\alpha_j^{2^{m-1}}) & \text{if } n = 2m+1. \end{cases} \tag{3.4.3}$$

Proof Define a map of $\mathbb{Z}_2[\![F_{n/2}K^1]\!]$–modules

$$p\colon \bigoplus_{i=0}^{2} \mathbb{Z}_2[\![F_{n/2}K^1]\!]_i \to IF_{n/2}K^1$$

by sending (h_0,h_1,h_2) to the element given by (3.4.3). It is sufficient to show that the map induced by p surjects onto

$$H_1(F_{n/2}K^1,\mathbb{F}_2) \cong \mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![F_{n/2}K^1]\!]} IF_{n/2}K^1.$$

By Lemma 2.2.1, $H_1(F_{n/2}K^1, \mathbb{F}_2)$ is generated by the classes

$$\begin{aligned} &\alpha^{2^{m-1}},\ \alpha_i^{2^{m-1}},\ \alpha_j^{2^{m-1}} \quad \text{if } n=2m, \\ &\alpha^{2^{m}},\ \alpha_i^{2^{m-1}},\ \alpha_j^{2^{m-1}} \quad \text{if } n=2m+1. \end{aligned}$$

Therefore, $\mathbb{F}_2 \otimes_{\mathbb{Z}_2[\![F_{n/2}K^1]\!]} p$ is surjective, and hence so is p. □

The ideal

$$\mathcal{I} = ((IK^1)^7, 2(IK^1)^3, 4(IK^1), 8)$$

will play a crucial role in the following estimates.

Corollary 3.4.3 *Let e_0 be the canonical generator of $\mathscr{C}_0$ and g be in $F_{8/2}K^1$. There exists h in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ such that $(e-g)e_0 = h(e-\alpha)e_0$ with $h \equiv 0 \mod \mathcal{I}$.*

Proof By Lemma 3.4.2, there exist h_0, h_1 and h_2 in $\mathbb{Z}_2[\![F_{8/2}K^1]\!]$ such that

$$e-g = h_0(e-\alpha^8) + h_1(e-\alpha_i^8) + h_2(e-\alpha_j^8).$$

Since

$$(e-x^8) = \Big(\sum_{s=0}^{7} x^s\Big)(e-x),$$

this implies that

$$e-g = h_0\Big(\sum_{s=0}^{7}\alpha^s\Big)(e-\alpha) + h_1\Big(\sum_{s=0}^{7}\alpha_i^s\Big)(e-\alpha_i) + h_2\Big(\sum_{s=0}^{7}\alpha_j^s\Big)(e-\alpha_j).$$

Let

$$h = h_0\Big(\sum_{s=0}^{7}\alpha^s\Big) + h_1\Big(\sum_{s=0}^{7}\alpha_i^s\Big)(i-\alpha_i) + h_2\Big(\sum_{s=0}^{7}\alpha_j^s\Big)(j-\alpha_j).$$

If $\tau \in G_{24}$, then $\tau e_0 = e_0$. Hence,

$$(\tau-\alpha_\tau)(e-\alpha)e_0 = (e-\alpha_\tau)e_0.$$

Using this fact, one verifies that $(e-g)e_0 = h(e-\alpha)e_0$. Further,

$$\sum_{s=0}^{7} x^s \equiv (1-x)^7 + 2x^4(x-1)^3 + 4x^2(x-1) \mod (8).$$

Since α, α_i and α_j are in K^1 and K^1 is a normal subgroup, this implies that

$$h \equiv 0 \mod ((IK^1)^7, 2(IK^1)^3, 4(IK^1), 8).$$ □

We will use the following result.

Lemma 3.4.4 *The element $\alpha_i\alpha_j\alpha_k$ is in $F_{4/2}K^1$. The element $\alpha_i\alpha_j\alpha_k\alpha^2$ is in $F_{8/2}K^1$.*

Proof Let $T=\alpha S$ in $\mathcal{O}_2\cong \mathrm{End}(F_2)$. Then $T^2=-2$, and $aT=Ta^{\sigma}$ for a in $\mathbb{W}$. As defined in (2.3.4) and Lemma 2.4.3, we have

$$\alpha=\frac{1}{\sqrt{-7}}(1-2\omega),\qquad i=-\tfrac{1}{3}(1+2\omega)(1-T),$$
$$j=-\tfrac{1}{3}(1+2\omega)(1-\omega^2T),\quad k=-\tfrac{1}{3}(1+2\omega)(1-\omega T).$$

Further,

$$\alpha^{-1}=-\frac{1}{\sqrt{-7}}(1-2\omega^2).$$

We use the fact that $\frac{1}{3}$ and $\frac{1}{\sqrt{-7}}$ are in $Z(\mathbb{S}_2)$. We also use the fact $\tau^{-1}=-\tau$ for $\tau=i$, j and k and the fact that $S^4=4$ and $S^8=16$.

First, note that

$$\begin{aligned}i\alpha&=-\frac{1}{3\sqrt{-7}}(1+2\omega)(1-T)(1-2\omega)\\&=-\frac{1}{3\sqrt{-7}}(1+2\omega)((1-2\omega)-(1-2\omega^2)T)\\&=-\frac{1}{3\sqrt{-7}}((5+4\omega)+(1-4\omega)T).\end{aligned}$$

Further,

$$\begin{aligned}i^{-1}\alpha^{-1}&=-\frac{1}{3\sqrt{-7}}(1+2\omega)(1-T)(1-2\omega^2)\\&=-\frac{1}{3\sqrt{-7}}(1+2\omega)((1-2\omega^2)-(1-2\omega)T)\\&=-\frac{1}{3\sqrt{-7}}((-1+4\omega)-(5+4\omega)T).\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_i=i\alpha i^{-1}\alpha^{-1}&=-\tfrac{1}{63}((5+4\omega)+(1-4\omega)T)((-1+4\omega)-(5+4\omega)T)\\&\equiv 13+(2+8\omega)T \mod S^8.\end{aligned}$$

Using the fact that $\alpha_j=\omega\alpha_i\omega^2$ and $\alpha_k=\omega^2\alpha_i\omega$, this implies that

$$\alpha_j\equiv 13+\omega^2(2+8\omega)T \mod S^8,\quad \alpha_k\equiv 13+\omega(2+8\omega)T \mod S^8.$$

Hence,

$$\begin{aligned}\alpha_i\alpha_j &\equiv (13+(2+8\omega)T)(13+\omega^2(2+8\omega)T)\\ &\equiv \big(9+\omega^2(10+8\omega)T+(10+8\omega)T+(2+8\omega)(\omega(2+8\omega^2))T^2\big)\\ &\equiv 9+8\omega+(8+14\omega)T \mod S^8,\end{aligned}$$

so that

$$\begin{aligned}\alpha_i\alpha_j\alpha_k &\equiv (9+8\omega+(8{+}14\omega)T)(13+\omega(2{+}8\omega)T)\\ &\equiv (5+8\omega+(8{+}6\omega)T+(9{+}8\omega)\omega(2{+}8\omega)T+(8{+}14\omega)\omega^2(2{+}8\omega^2)T^2)\\ &\equiv 13+8\omega \mod S^8.\end{aligned}$$

This shows that $\alpha_i\alpha_j\alpha_k \equiv 1$ modulo S^4. Finally, note that

$$\begin{aligned}\alpha_i\alpha_j\alpha_k\alpha^2 \equiv (13+8\omega)\Big(\frac{1}{\sqrt{-7}}(1-2\omega)\Big)^2 \equiv -\tfrac{1}{7}(13+8\omega)(1-2\omega)^2 \equiv -\tfrac{9}{7}\\ \equiv 1 \mod S^8,\end{aligned}$$

which shows that $\alpha_i\alpha_j\alpha_k\alpha^2$ is in $F_{8/2}K^1$. □

Theorem 3.4.5 *There exists Θ in $\mathbb{Z}_2[\![\mathbb{S}_2^1]\!]$ satisfying the conditions of Lemma 3.1.3 such that*

$$\begin{aligned}\Theta \equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k\\ -\tfrac{1}{3}\operatorname{tr}_{C_3}((e-\alpha_i)(j-\alpha_j)+(e-\alpha_i\alpha_j)(k-\alpha_k)+(e-\alpha_i\alpha_j\alpha_k)(e+\alpha))\end{aligned}$$

modulo $\mathcal{I} = ((IK^1)^7, 2(IK^1)^3, 4(IK^1), 8)$, where tr_{C_3} is defined by (3.1.4).

Proof We will use the following facts. First, note that

$$\tau e_q = e_q$$

for $\tau \in G_{24}$ and $q = 0$, or for $\tau \in C_6$ and $q = 1$. This implies that

$$\tau(e-\alpha)e_0 = (e-\alpha_\tau\alpha)e_0.$$

Since $j = \omega i\omega^{-1}$ and $k = \omega^{-1}i\omega$, it also implies that

$$\omega i e_q = j e_q, \quad \omega^2 i e_q = k e_q.$$

The element $\alpha \in \mathbb{W}^\times \subseteq \mathbb{S}_2$ commutes with ω. This implies that

$$\omega\alpha_i e_q = \alpha_j e_q.$$

We will use the fact that for $\tau \in G_{24}$,

$$(\tau - \alpha_\tau)(e-\alpha)e_0 = (e-\alpha_\tau)e_0.$$

We will also use the identity

$$e - gh = (e-g) + (e-h) - (e-g)(e-h).$$

Let $\Theta_0 = e + i$. Then $\mathrm{tr}_{C_3}(\Theta_0)e_1 = (3+i+j+k)e_1$ and

$$\begin{aligned}
\partial_1(\Theta_0 e_1) &= (e+i)(e-\alpha)e_0 \\
&= (e-\alpha)e_0 + (e-\alpha_i\alpha)e_0 \\
&= 2(e-\alpha)e_0 + (e-\alpha_i)e_0 - (e-\alpha_i)(e-\alpha)e_0 \\
&= (e-\alpha^2)e_0 + (e-\alpha)^2 e_0 + (e-\alpha_i)e_0 - (e-\alpha_i)(e-\alpha)e_0.
\end{aligned}$$

Let $\Theta_1 = e + i - (e-\alpha) + (e-\alpha_i)$. Then,

$$\partial_1(\Theta_1 e_1) = (e-\alpha^2)e_0 + (e-\alpha_i)e_0.$$

Therefore,

$$\begin{aligned}
&\partial_1(\mathrm{tr}_{C_3}(\Theta_1)e_1) \\
&\quad = 3(e-\alpha^2)e_0 + (e-\alpha_i)e_0 + (e-\alpha_j)e_0 + (e-\alpha_k)e_0 \\
&\quad = 3(e-\alpha^2)e_0 + (e-\alpha_i)(e-\alpha_j)e_0 + (e-\alpha_i\alpha_j)e_0 + (e-\alpha_k)e_0 \\
&\quad = 3(e-\alpha^2)e_0 + (e-\alpha_i)(e-\alpha_j)e_0 + (e-\alpha_i\alpha_j)(e-\alpha_k)e_0 + (e-\alpha_i\alpha_j\alpha_k)e_0 \\
&\quad = 2(e-\alpha^2)e_0 + (e-\alpha_i)(e-\alpha_j)e_0 + (e-\alpha_i\alpha_j)(e-\alpha_k)e_0 \\
&\qquad\qquad + (e-\alpha_i\alpha_j\alpha_k)(e-\alpha^2)e_0 + (e-\alpha_i\alpha_j\alpha_k\alpha^2)e_0.
\end{aligned}$$

Let

$$\begin{aligned}
\Theta_2 = \mathrm{tr}_{C_3}\big(e+i-(e-\alpha)+(e-\alpha_i)\big) - 2(e+\alpha) \\
-(e-\alpha_i)(j-\alpha_j) - (e-\alpha_i\alpha_j)(k-\alpha_k) - (e-\alpha_i\alpha_j\alpha_k)(e+\alpha).
\end{aligned}$$

Then $\Theta_2 \equiv 3+i+j+k \mod (4, IK^1)$. Further,

$$\partial_1(\Theta_2 e_1) = (e-\alpha_i\alpha_j\alpha_k\alpha^2)e_0.$$

By Lemma 3.4.4, $\alpha_i\alpha_j\alpha_k\alpha^2 \in F_{8/2}K^1$. By Corollary 3.4.3, there exists h such that

$$(e-\alpha_i\alpha_j\alpha_k\alpha^2)e_0 = h(e-\alpha)e_0$$

and $h \equiv 0$ modulo $\mathcal{I}$, where $\mathcal{I} = ((IK^1)^7, 2(IK^1)^3, 4IK^1, 8)$. Therefore,

$$\partial_1((\Theta_2 - h)e_1) = (e-\alpha_i\alpha_j\alpha_k\alpha^2)e_0 - h(e-\alpha)e_0 = 0.$$

Define

$$\Theta = \tfrac{1}{3}\,\mathrm{tr}_{C_3}(\Theta_2 - h).$$

Then Θ satisfies the conditions of Lemma 3.1.3. Further,

$$\Theta \equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k \\ -\tfrac{1}{3}\,\mathrm{tr}_{C_3}((e-\alpha_i)(j-\alpha_j)+(e-\alpha_i\alpha_j)(k-\alpha_k)+(e-\alpha_i\alpha_j\alpha_k)(e+\alpha))$$

modulo $\mathcal{I}$. □

Corollary 3.4.6 *Let $\mathcal{J} = (IF_{4/2}K^1, (IF_{3/2}K^1)(IS_2^1), \mathcal{I})$. The element Θ from Theorem 3.4.5 satisfies*

$$\Theta \equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k \quad \mathrm{mod}\ \mathcal{J},$$

and $\Theta \equiv e+\alpha$ modulo $(2,(IS_2^1)^2)$.

Proof First, note that $\alpha_\tau \in F_{3/2}K^1$ for $\tau \in G_{24}$. Further, by Lemma 3.4.4, $\alpha_i\alpha_j\alpha_k$ is in $F_{4/2}K^1$. Hence, it follows from Theorem 3.4.5 that

$$\Theta \equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k \quad \mathrm{mod}\ \mathcal{J}.$$

For the second claim, we first prove that $\mathcal{J} \subseteq (2,(IS_2^1)^2)$. It is clear that

$$((IF_{3/2}K^1)(IS_2^1), \mathcal{I}) \subseteq (2,(IS_2^1)^2).$$

Further, it follows from Lemma 3.4.2 and the fact that $(e-x^{2^k}) \equiv (e-x)^{2^k}$ modulo (2) that

$$IF_{4/2}K^1 \subseteq (2,(IS_2^1)^2).$$

Therefore, $\mathcal{J} \subseteq (2,(IS_2^1)^2)$. Hence,

$$\Theta \equiv e+\alpha+i+j+k-\alpha_i-\alpha_j-\alpha_k \quad \mathrm{mod}\ (2,(IS_2^1)^2).$$

Further, $(e-i)(e-j) \equiv e+i+j+k$ modulo (2) and

$$e-\alpha_i = i\alpha((e-\alpha^{-1})(e-i^{-1})-(e-i^{-1})(e-\alpha^{-1})).$$

Therefore, $e+i+j+k$ and $e-\alpha_\tau$ are in $(2,(IS_2^1)^2)$. We conclude that

$$\Theta \equiv e+\alpha \quad \mathrm{mod}\ (2,(IS_2^1)^2).$$ □

Appendix: Background on profinite groups

We use the terminology of Ribes and Zalesskii [28, Section 5]. Let G be a profinite p–adic analytic group and $\{U_k\}$ be a system of open normal subgroups of G such that $\bigcap_k U_k = \{e\}$. The *completed group ring of* G is

$$\mathbb{Z}_p[\![G]\!] := \lim_{n,k} \mathbb{Z}/(p^n)[G/U_k].$$

The *augmentation* is the continuous homomorphism of $\mathbb{Z}_p$–modules $\varepsilon\colon \mathbb{Z}_p[\![G]\!] \to \mathbb{Z}_p$ defined by $\varepsilon(g) = 1$ for $g \in G$. The *augmentation ideal* IG is the kernel of ε.

A *left* $\mathbb{Z}_p[\![G]\!]$*–module* is a $\mathbb{Z}_p[\![G]\!]$–module M which is a Hausdorff topological abelian group with a continuous structure map $\mathbb{Z}_p[\![G]\!] \times M \to M$. The module M is *finitely generated* if it is the closure of the $\mathbb{Z}_p[\![G]\!]$–module generated by a finite subset of M. It is *discrete* if it is the union of its finite $\mathbb{Z}_p[\![G]\!]$–submodules and *profinite* if it is the inverse limit of its finite $\mathbb{Z}_p[\![G]\!]$–submodule quotients; see [28, Lemma 5.1.1]. The module M is complete with respect to the IG–adic topology if

$$M \cong \lim_{n,k} \mathbb{Z}_p/(p^n)[G/U_k] \otimes_{\mathbb{Z}_p[\![G]\!]} M.$$

It is a theorem of Lazard that $\mathbb{Z}_p[\![G]\!]$ is Noetherian; see Symonds and Weigel [35, Theorem 5.1.2]. Finitely generated $\mathbb{Z}_p[\![G]\!]$–modules are thus both profinite and complete with respect to the IG–adic topology.

Let $M = \lim_i M_i$ be a profinite $\mathbb{Z}_p[\![G]\!]$–bimodule and $N = \lim_j N_j$ a profinite left $\mathbb{Z}_p[\![G]\!]$–module. Then

$$M \otimes_{\mathbb{Z}_p[\![G]\!]} N = \lim_{i,j} M_i \otimes_{\mathbb{Z}_p[\![G]\!]} N_j$$

denotes the completed tensor product, which is itself a profinite left $\mathbb{Z}_p[\![G]\!]$–module [28, Section 5.5]. The abelian group of continuous $\mathbb{Z}_p[\![G]\!]$–homomorphisms is denoted by

$$\mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, N).$$

This is a topological space with the compact open topology. If M is finitely generated, then it is compact; see [35, Section 3.7].

Lazard also proves in [24, V.3.2.7] that the trivial $\mathbb{Z}_p[\![G]\!]$–module $\mathbb{Z}_p$ admits a resolution by finitely generated $\mathbb{Z}_p[\![G]\!]$–modules. A $\mathbb{Z}_p[\![G]\!]$–module M which admits a projective resolution $P_\bullet \to M$ by finitely generated $\mathbb{Z}_p[\![G]\!]$–modules is said to be of type $\mathbf{FP}^\infty$; see [35, Section 3.7]. For such M, we let

$$\mathrm{Ext}^n_{\mathbb{Z}_p[\![G]\!]}(M, N) = H^n(\mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(P_\bullet, N))$$

and

$$\mathrm{Tor}_n^{\mathbb{Z}_p[\![G]\!]}(M, N) = H_n(P_\bullet \otimes_{\mathbb{Z}_p[\![G]\!]} N).$$

These functors are studied by Symonds and Weigel in [35, Section 3.7]. There are isomorphisms

$$H^n(G, N) \cong \mathrm{Ext}^n_{\mathbb{Z}_p[\![G]\!]}(M, N),$$

where $H^n(G, N)$ is the cohomology computed with continuous cochains and

$$H_n(G, N) \cong \mathrm{Tor}_n^{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p, N);$$

see Neukirch, Schmidt and Wingberg [25, Propositions 5.2.6, 5.2.14] or the discussion in Kohlhaase [23, Section 3]. Therefore, these functors satisfy the usual properties of group cohomology; see Ribes and Zalesskii [28, Section 6]. In particular, for $[G, G]$ the commutator subgroup, $G^* = G^p[G, G]$, and $\mathbb{Z}_p$ and $\mathbb{F}_p$ the trivial modules, we have

$$H_1(G, \mathbb{Z}_p) \cong G/\overline{[G, G]}, \qquad H_1(G, \mathbb{F}_p) \cong G/\overline{G^*},$$
$$H^1(G, \mathbb{Z}_p) \cong \mathrm{Hom}(G, \mathbb{Z}_p), \quad H^1(G, \mathbb{F}_p) \cong \mathrm{Hom}(G, \mathbb{F}_p).$$

Examples A.1.1 We give examples, which we use in this paper, in [3] and in [2].

(a) The modules

$$\mathbb{Z}_p[\![G/H]\!] := \mathbb{Z}_p[\![G]\!] \otimes_{\mathbb{Z}_p[H]} \mathbb{Z}_p$$

for H a finite subgroup of G and $\mathbb{Z}_p$ the trivial $\mathbb{Z}_p[H]$–module are finitely generated, and thus profinite and complete.

(b) The $\mathbb{Z}_p[\![G]\!]$–dual of a finitely generated $\mathbb{Z}_p[\![G]\!]$–module M is defined as

$$M^* := \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(M, \mathbb{Z}_p[\![G]\!]), \tag{A.1.1}$$

with the action of $g \in G$ on $\phi \in M^*$ defined by

$$(g\phi)(m) = \phi(m)g^{-1}.$$

This gives M^* the structure of a finitely generated left $\mathbb{Z}_p[\![G]\!]$–module; see Symonds and Weigel [35, 3.7.1] and Henn, Karamanov and Mahowald [18, Section 3.4]. For example, if $H \subseteq G$ is a finite subgroup and $[g]$ denotes the coset gH, there is a canonical isomorphism

$$t\colon \mathbb{Z}_p[\![G/H]\!] \to \mathbb{Z}_p[\![G/H]\!]^* \tag{A.1.2}$$

which sends $[g]$ to the map $[g]^*\colon \mathbb{Z}_p[\![G/H]\!] \to \mathbb{Z}_p[\![G]\!]$ defined by

$$[g]^*([x]) = x \sum_{h \in H} hg^{-1}.$$

We refer the reader to [18, Section 3.4] for a detailed discussion of $\mathbb{Z}_p[\![G]\!]$–duals.

(c) In the case when $G = S_n$ is the p–Sylow subgroup of $\mathbb{S}_n$, an important example is the continuous $\mathbb{Z}_p[\![S_n]\!]$–module $(E_n)_*X = \pi_* L_{K(n)}(E_n \wedge X)$ for a spectrum X; see Goerss, Henn, Mahowald and Rezk [14, Section 2]. In the case when X is a finite spectrum, $(E_n)_*X$ is profinite, although it is not known if, in general, it is finitely generated over $\mathbb{Z}_p[\![S_n]\!]$. For a more extensive discussion, see the work of Kohlhaase in [23].

Lemma A.1.2 (Shapiro's Lemma) *Let G be a profinite p–analytic group and let H be a closed subgroup. Let M be a $\mathbb{Z}_p[\![H]\!]$–module of type $\mathbf{FP}^\infty$ and let $N = \lim_i N_i$ be a profinite $\mathbb{Z}_p[\![G]\!]$–module, which is also a $\mathbb{Z}_p[\![H]\!]$–module via restriction. Then*

$$\mathrm{Ext}^*_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p[\![G]\!] \otimes_{\mathbb{Z}_p[\![H]\!]} M, N) \cong \mathrm{Ext}^*_{\mathbb{Z}_p[\![H]\!]}(M, N).$$

Proof Let $P_\bullet \to M$ be a projective resolution of M by finitely generated $\mathbb{Z}_p[\![H]\!]$–modules. According to Brumer [7, Lemma 4.5], $\mathbb{Z}_p[\![G]\!]$ is a projective $\mathbb{Z}_p[\![H]\!]$–module. Hence, the functor $\mathbb{Z}_p[\![G]\!] \otimes_{\mathbb{Z}_p[\![H]\!]} (-)$ is exact, and $\mathbb{Z}_p[\![G]\!] \otimes_{\mathbb{Z}_p[\![H]\!]} P_\bullet$ is a projective resolution of $\mathbb{Z}_p[\![G]\!] \otimes_{\mathbb{Z}_p[\![H]\!]} M$ by finitely generated $\mathbb{Z}_p[\![G]\!]$–modules. Finally, note that

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p[\![G]\!] \otimes_{\mathbb{Z}_p[\![H]\!]} P_\bullet, N) &\cong \lim_i \mathrm{Hom}_{\mathbb{Z}_p[\![G]\!]}(\mathbb{Z}_p[\![G]\!] \otimes_{\mathbb{Z}_p[\![H]\!]} P_\bullet, N_i) \\
&\cong \lim_i \mathrm{Hom}_{\mathbb{Z}_p[\![H]\!]}(P_\bullet, N_i) \\
&\cong \mathrm{Hom}_{\mathbb{Z}_p[\![H]\!]}(P_\bullet, N),
\end{aligned}$$

where the first isomorphism is proved by Symonds and Weigel [35, (3.7.1)] and the second follows from Ribes and Zalesskii [28, Proposition 5.5.4(c)]. □

The following result is Lemma 4.3 of Goerss, Henn, Mahowald and Rezk [14]. It is a version of Nakayama's lemma in this setting.

Lemma A.1.3 *Let G be a finitely generated profinite p–group. Let M and N be finitely generated complete $\mathbb{Z}_p[\![G]\!]$–modules and $f\colon M \to N$ be a map of complete $\mathbb{Z}_p[\![G]\!]$–modules. If the induced map*

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p[\![G]\!]} f\colon \mathbb{F}_p \otimes_{\mathbb{Z}_p[\![G]\!]} M \to \mathbb{F}_p \otimes_{\mathbb{Z}_p[\![G]\!]} N$$

is surjective, then so is f. If the map

$$\mathrm{Tor}_q^{\mathbb{Z}_p[\![G]\!]}(\mathbb{F}_p, f)\colon \mathrm{Tor}_q^{\mathbb{Z}_p[\![G]\!]}(\mathbb{F}_p, M) \to \mathrm{Tor}_q^{\mathbb{Z}_p[\![G]\!]}(\mathbb{F}_p, N)$$

is an isomorphism for $q = 0$ and surjective for $q = 1$, then f is an isomorphism.

The following is a restatement of some of the results which can be found in Ribes and Zalesskii [28, Lemma 6.8.6].

Lemma A.1.4 *Let G be a profinite group and let IG be the augmentation ideal. For a profinite $\mathbb{Z}_p[\![G]\!]$–module M, the boundary map for the short exact sequence*

$$0 \to IG \to \mathbb{Z}_p[\![G]\!] \xrightarrow{\varepsilon} \mathbb{Z}_p \to 0$$

induces an isomorphism

$$H_{n+1}(G, M) \cong \mathrm{Tor}_n^{\mathbb{Z}_p[\![G]\!]}(IG, M).$$

For the trivial module $M = \mathbb{Z}_p$, this isomorphism sends g in $G/\overline{[G,G]}$ to the residue class of $e - g$ in $H_0(G, IG) \cong IG/\overline{IG^2}$. Let G^* be the subgroup generated by $[G,G]$ and G^p. For $M = \mathbb{F}_p$, it sends g in $G/\overline{G^*}$ to the residue class of $e - g$ in $\mathbb{F}_p \otimes_{\mathbb{Z}_p} IG/\overline{IG^2}$.

Finally, we note the following classical result.

Lemma A.1.5 *Let G be a profinite 2–analytic group and suppose that $H_1(G, \mathbb{Z}_2) \cong G/\overline{[G,G]}$ is a finitely generated 2–group. Suppose that the residue class of an element g in $G/\overline{[G,G]}$ generates a summand isomorphic to $\mathbb{Z}/2^k$. Let x in $H^1(G, \mathbb{Z}/2) \cong \mathrm{Hom}(G, \mathbb{Z}/2)$ be the homomorphism dual to g. Then x^2 is nonzero in $H^2(G, \mathbb{Z}/2)$ if and only if $k = 1$.*

Proof This follows from the fact that x in $H^1(G, \mathbb{Z}/2)$ has a nonzero Bockstein in the long exact sequence associated to the extension of trivial modules

$$1 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1$$

if and only if g generates a $\mathbb{Z}/2$ summand. □

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Received: 19 December 2014 Revised: 30 March 2015