Algebraic & Geometric Topology 16 (2016) 569–596

Random walk invariants of string links from R–matrices

THOMAS KERLER
YILONG WANG

We show that the exterior powers of the matrix valued random walk invariant of string links, introduced by Lin, Tian, and Wang, are isomorphic to the graded components of the tangle functor associated to the Alexander polynomial by Ohtsuki divided by the zero graded invariant of the functor. Several resulting properties of these representations of the string link monoids are discussed.

57M27; 57M25, 20F36, 57R56, 15A75, 17B37

1 Introduction, definitions, and results

1.1 The Burau representation and tangles

The aim of this article is to relate two generalizations of the Burau representation of the braid groups to certain types of tangles. Consider the standard presentation of the braid group in n strands:

$$B_n = \left\langle \sigma_i, i=1,\dots,n-1 \;\middle|\; \begin{array}{r} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, i=1,\dots,n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j|\geq 2 \end{array} \right\rangle. \tag{1}$$

The *unreduced* Burau representation B_n is defined on the free $\mathbb{Z}[t,t^{-1}]$–module of rank n, given by the homomorphism

$$\mathcal{B}_n\colon B_n \to \mathrm{End}(\mathbb{Z}[t,t^{-1}]^n), \quad \sigma_i \mapsto \mathcal{B}_n(\sigma_i) = \beta_i. \tag{2}$$

Here the Burau matrices β_i and their inverses, denoting $\bar{t} = t^{-1}$, are defined as

$$\beta_i = \mathbb{1}_{i-1} \oplus \begin{bmatrix} (1-t) & 1 \\ t & 0 \end{bmatrix} \oplus \mathbb{1}_{n-i-1} \quad \text{and} \quad \beta_i^{-1} = \mathbb{1}_{i-1} \oplus \begin{bmatrix} 0 & \bar{t} \\ 1 & (1-\bar{t}) \end{bmatrix} \oplus \mathbb{1}_{n-i-1}. \tag{3}$$

There are various generalizations of the Burau representation from braids to tangles. To explain these recall first the usual category of oriented tangles, denoted by $\mathcal{T}gl$, whose objects are tuples of signs, $\underline{\epsilon} = (\epsilon_1,\dots,\epsilon_n)$, writing $|\underline{\epsilon}| = n$. A morphism $T\colon \underline{\epsilon} \to \underline{\delta}$ is an equivalence class of oriented tangle diagrams in $\mathbb{R}\times[0,1]$ with endpoints $\{(j,0), j=1,\dots,|\underline{\epsilon}|\}$ at the bottom of the diagram and $\{(j,1), j=1,\dots,|\underline{\delta}|\}$ at the

Published: 23 February 2016 DOI: 10.2140/agt.2016.16.569

top, so that the orientation is upwards at the j^{th} position if $\epsilon_j = +$ and downwards if $\epsilon_j = -$. The equivalences are given by isotopies and the usual Reidemeister moves.

We denote by $\boldsymbol{\iota}^n = (+, \dots, +)$ the array of length n with all $+$ entries, which implies that the orientations of all strands at this object (either as source or target) must be pointing upwards.

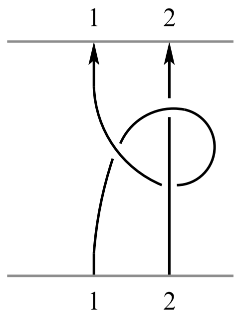

Figure 1: $S\colon \boldsymbol{\iota}^2 \to \boldsymbol{\iota}^2$.

A string link is a tangle class $T\colon \boldsymbol{\iota}^n \to \boldsymbol{\iota}^n$ with precisely n interval components, each of which has one endpoint $(i, 0)$ at the bottom and the other endpoint $(j, 1)$ at the top of the diagram. See Figure 1 for an example of a string link $S\colon \boldsymbol{\iota}^2 \to \boldsymbol{\iota}^2$ on two strands. String links form a monoid $\mathrm{Str}(n)$ with respect to the composition in $\mathcal{T}gl$, so we have the following inclusions of monoids:

$$B_n \subset \mathrm{Str}(n) \subset \mathrm{End}_{\mathcal{T}gl}(\boldsymbol{\iota}^n). \tag{4}$$

1.2 Random walk on string link diagrams

In [7], Lin, Tian, and Wang consider a generalization of $\mathcal{B}_n$ to $\mathrm{Str}(n)$ that is inspired by a remark of Vaughan Jones in his seminal paper [4], where he offers a probabilistic interpretation of the Burau representation. The description there is in terms of a bowling ball that runs along an arc in a braid diagram following its orientation. At positive crossings, the ball drops from an overcrossing strand to an undercrossing strand with probability $1 - t$ and remains on the overcrossing strand with probability t.

The situation is illustrated in Figure 2. An analogous rule is used for negative crossings in which t is replaced by $\bar{t} = t^{-1}$ (see Figure 3), so "probabilities" should rather be understood as weights whose values are allowed to be outside of the unit interval. Assuming the pictured strands are the i^{th} and $(i + 1)^{\text{st}}$ strands in a braid presentation on n strands, the Markov transition matrix from the array of probabilities just above the crossing to one just below is given by the unreduced Burau matrices in (3).

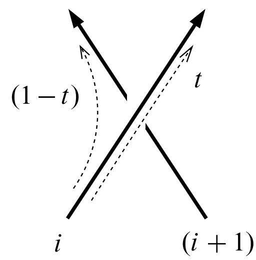

Figure 2

The construction in [7] extends the idea of weighted paths along an oriented string link diagram with the same rule as in Figure 2. As opposed to braid diagrams, it is possible to encounter loops, and thus infinite numbers of paths, between two endpoints. It is shown that, for t sufficiently close to 1, the resulting series of weights converge. For example, the diagram in Figure 1 contains a loop of combined weight $w = t(1-\bar{t})$ and associated geometric series $\sum_{k=0}^{\infty} w^k = (2-t)^{-1}$. The combined transition matrix in this example is given as

$$\mathcal{R}_2(S) = \frac{1}{2-t}\begin{bmatrix} 1 & \bar{t}-1 \\ 1-t & 3-t-\bar{t} \end{bmatrix}. \tag{5}$$

It is shown in [7] that the resulting transition matrix will always have rational functions in t as entries. The functorial nature of the construction (see Lemma 3 below) further implies a homomorphism of monoids,

$$\mathcal{R}_n\colon \mathrm{Str}(n) \longrightarrow \mathrm{End}(\mathbb{Q}(t,t^{-1})^n), \tag{6}$$

which restricts to $\mathcal{B}_n$ on the braid group B_n. The main result of this paper will also extend to the exterior powers of this representation

$$\begin{gathered} \textstyle\bigwedge^k \mathcal{R}_n\colon \mathrm{Str}(n) \to \mathrm{End}\big(\textstyle\bigwedge^k \mathbb{Q}(t,t^{-1})^n\big), \\ \textstyle\bigwedge^k \mathcal{R}_n(T)(x_1 \wedge \cdots \wedge x_k) = (\mathcal{R}_n(T)x_1) \wedge \cdots \wedge (\mathcal{R}_n(T)x_k). \end{gathered} \tag{7}$$

The construction of the random walk invariant in [7] has been applied to studies of the Jones polynomial and its relation to the Alexander polynomial in [8; 2] and has been generalized in [1; 6; 13].

1.3 Functorial invariants of tangles

A second generalization to the entire category of oriented tangles is given in Ohtsuki's book [10, Section 3.3] as operator invariants of tangles associated to the Alexander polynomial. The matrices assigned to generating tangles there have entries in the ring of Laurent polynomials in $t^{\pm 1/2}$ for a suitable basis. The construction can thus be summarized as a functor,

$$\mathcal{V}\colon \mathcal{T}gl \longrightarrow \mathbb{Z}[t^{1/2}, t^{-1/2}] - \mathrm{Fmod}, \tag{8}$$

from the tangle category to the category of free $\mathbb{Z}[t^{1/2}, t^{-1/2}]$–modules. The assignment on objects is given by $\mathcal{V}(\underline{\epsilon}) = V^{\epsilon_1} \otimes \cdots \otimes V^{\epsilon_n}$, where $V^{+} = V$ is the free $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ module of rank 2 generated by elements e_0 and e_1, and $V^{-} = V^{*}$ is the dual module with dual basis $\{e_0^*, e_1^*\}$. In fact, $\mathcal{V}$ is easily seen to be a tensor functor with respect to the tensor product on $\mathcal{T}gl$ defined by the usual juxtaposition.

As described in [10, Section 4.5], the tangle functor may be derived from the representation theory of the quantum group $U_{-1}(\mathfrak{sl}_2)$ following the methods originally introduced by Reshetikhin and Turaev in [12]. One implication of this observation, which can also be checked directly, is that for a tangle T the operator $\mathcal{V}(T)$ preserves a natural grading on the modules induced, for example, by $\deg(e_0) = \deg(e_0^*) = 0$, $\deg(e_1) = 1$, and $\deg(e_1^*) = -1$. For example, the module assigned to the object of length n with all positive orientations decomposes, as free $\mathbb{Z}[t^{1/2}, t^{-1/2}]$–modules, into its invariant graded components as follows:

$$\mathcal{V}(\iota^n) = V^{\otimes n} = \bigoplus_{j=0}^{n} W_{n,j} \quad \text{with } \dim(W_{n,j}) = \binom{n}{j}. \tag{9}$$

A consequence of this decomposition is that the representation of $\mathrm{Str}(n)$ on $\mathcal{V}(\iota^n)$ implied by the inclusion in (4) yields a series of representations as follows:

$$\mathcal{W}_{n,j}\colon \mathrm{Str}(n) \longrightarrow \mathrm{End}(W_{n,j})\colon T \mapsto \mathcal{V}(T)|_{W_{n,j}}. \tag{10}$$

We note that $W_{n,0}$ is of rank one so we may write $\mathcal{W}_{n,0}(T) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ as a well-defined polynomial-valued invariant. We will see in Lemma 9 that its specialization at $t = 1$ is one for all string links, so $\mathcal{W}_{n,0}(T) \neq 0$ also as an element in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$. We can therefore define representations of the string monoid as follows:

$$\mathcal{W}_{n,k/0}\colon \mathrm{Str}(n) \to \mathrm{End}(\mathbb{Q}(t^{1/2}, t^{-1/2})^n), \quad T \mapsto \frac{1}{\mathcal{W}_{n,0}(T)} \mathcal{W}_{n,k}(T). \tag{11}$$

It is shown in [10] that the restriction of the n–dimensional representation $\mathcal{W}_{n,1}\colon B_n \to \mathrm{GL}(W_{n,1})$ to the braid group is, up to a universal rescaling of generators, equivalent to the unreduced Burau representation $\mathcal{B}_n$ from (2).

An closely related approach of constructing invariants of tangles associated to the Burau representation and Alexander polynomial makes use of the quantum groups $U_{\zeta}(\mathfrak{gl}(1|1))$; see for example [5; 9; 15].

1.4 Statement of main result

In view of the dominance of algebraically constructed functorial invariants it is natural to ask whether the representation $\mathcal{R}_n$ is really a special case of these. Among the obstacles in an identification is the peculiar analytic flavor of the construction involving geometric series. Particularly, the role of the denominators that are obtained by summing these series (as, for example, $(2-t)$ in (5)) are not obvious, and their algebraic or topological meaning is not at all immediate from the construction.

The main result of this article is to provide such algebraic interpretations in terms of an equivalence of string link representations as stated in the following theorem.

Theorem 1 *The representations $\bigwedge^k \mathcal{R}_n$ and $\mathcal{W}_{n,k/0}$ of the string link monoid* $\mathrm{Str}(n)$ *are isomorphic to each other.*

We note that the isomorphism will consist only of rescaling or reordering of basis vectors. In the case of $k=1$, we obtain a direct formula for the string link invariant,

$$\mathcal{I}_n^{-1}\mathcal{R}_n(T)\mathcal{I}_n = \frac{1}{\mathcal{W}_{n,0}(T)}\mathcal{W}_{n,1}(T), \tag{12}$$

where $\mathcal{I}_n$ is given as a bijection of canonical basis vectors up to multiplication with units of $\mathbb{Z}[t^{1/2}, t^{-1/2}]$. The denominator occurring in the random walk construction can thus also be interpreted as the zero-graded part of Ohtsuki's functor. Additional interpretations are suggested in Section 4.4.

Theorem 1 trivially holds true in the case $n=1$. Particularly, let $T\colon \iota^1 \to \iota^1$ be any 1–1–tangle and denote by L its closure. We note that [10, Theorem 3.12] implies $\mathcal{V}(T) = \Delta_L(t)\cdot \mathrm{id}_V$, where $\Delta_L(t)$ is the Alexander polynomial of the closed link L. In our notation, this means $\mathcal{W}_{1,0}(T) = \mathcal{W}_{1,1}(T) = \Delta_L(t)$, so we find $\mathcal{W}_{1,1/0}(T) = 1$ for the right-hand side of (12). Clearly, we also have $\mathcal{R}_1(T) = 1$ for any string link $T\colon \iota^1 \to \iota^1$, since a ball entering at the bottom of a diagram for T will have to emerge at the top of the diagram with probability one.

In the case of the basic string link $S\colon \iota^2 \to \iota^2$ introduced in Figure 1, we will also verify (12) and Theorem 1 in Section 4.3. There we explicitly compute the Ohtsuki Functor $\mathcal{V}(S)$ using standard functorial methods and decompositions. In (76), we obtain $\mathcal{W}_{2,0}(S) = (2-t)$ as well as

$$\mathcal{W}_{2,1}(S) = \begin{bmatrix} 3-t-\bar{t} & t^{-1/2}-t^{1/2} \\ t^{-1/2}-t^{1/2} & 1 \end{bmatrix} \tag{13}$$

in the basis $\{e_0 \otimes e_1, e_1 \otimes e_0\}$. Relation (12) is now easily verified using (5) as well as the basis transformation

$$\mathcal{I}_2 = \begin{bmatrix} 0 & t^{1/2} \\ t & 0 \end{bmatrix}. \tag{14}$$

An immediate implication of Theorem 1 is that $\mathcal{R}_n$ is dominated by finite type invariants since it is dominated by $\mathcal{V}$. This fact has been proved by more indirect means in [7]. Another consequence of Theorem 1 is the following.

Corollary 2 *Suppose $T \in \mathrm{Str}(n)$. Then $\mathcal{W}_{n,0}(T)^{k-1}$ divides all $k \times k$ minors of $\mathcal{W}_{n,1}(T)$ in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$.*

1.5 Overview of paper

The original random walk construction of $\mathcal{R}_n$ from [7] is reviewed and formalized in Section 2, where it is also applied to the situation of a string link T given as the closure of a braid b. Particularly, we obtain in Proposition 7 an expression for $\mathcal{R}_n(T)$ in terms of blocks of the Burau matrix for b. A consequence, stated in Corollary 8, is that $\mathcal{R}_n(T)$ admits an equilibrium state that is independent of T (and thus contains no information about T).

In Section 3, we review Ohtsuki's construction of the tangle functor $\mathcal{V}$ as well as the equivalence given in [10] of the implied braid representation with the exterior algebra extension of the unreduced Burau representation. The quantum trace, relevant to evaluating braid closures, is related to the natural supertrace on exterior algebras in Section 3.3. In addition, various grading and equivariance properties are discussed.

Section 4.3 contains the proofs for Theorem 1 and Corollary 2 after introducing several technical lemmas on relating traces, evaluations on top forms, and Schur complements of block matrices in Sections 4.1 and 4.2. Finally, we present additional points of view and possible further questions of study in Section 4.4.

Acknowledgment The first author thanks Craig Jackson for discussions and calculations on an early version of the conjecture that are documented in [3].

2 Random walk invariants of tangles

After a brief review of the random walk construction of [7], the main result of this section is a formula for the representation $\mathcal{R}_n$, in Proposition 7, in terms of block matrices of a Markov presentation of the string link.

2.1 Weighted path construction

We will review here the construction of [7], formalizing the intuition in terms of random walks given in the introduction. A string link $T \in \mathrm{Str}(n)$ is an oriented tangle $T\colon \iota^n \to \iota^n$ for which each component is an interval that starts at a bottom point $(i,0)$ and end at a top point $(\pi_T(i),1)$ with $\pi_T \in S_n$.

An admissible path P in a diagram of T is a path following the orientation of T at each piece of the diagram. At a crossing, the path must continue its direction on T if it is along the undercrossing piece of the crossing. If the path approaches a crossing along the overcrossing piece, it may either continue in the same direction or continue on the undercrossing piece in the respective direction. In addition, an admissible path needs to start at a bottom point $(i,0)$ and end at a top point $(j,1)$. Thus, locally, a path near a crossing may look like one of the dashed lines in Figures 2 or 3.

For braids, all admissible paths need to travel upwards so that they will pass through each crossing at most once. However, for a string link, such as the one in Figure 1, admissible paths may loop through a crossing arbitrarily often, so there are infinitely many admissible paths.

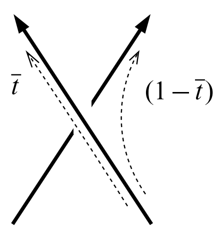

Figure 3

To an admissible path P that passes through M crossings (counting repetitions) we associate a weight $w(P) = w_1 \cdot \dots \cdot w_M$, where $w_k = 1$ if P approaches the k^{th} crossing along an undercrossing piece. If P approaches the k^{th} crossing along an overcrossing piece and the crossing is positive as in Figure 2, we set $w_k = t$ if P is continuing in the same direction and $w_k = 1-t$ if P continues on the undercrossing piece. For a negative crossing as in Figure 3, we assign $w_k = \bar{t} = t^{-1}$ if P is continuing in the same direction and $w_k = 1-\bar{t} = 1-t^{-1}$ otherwise.

We use these to assign to the diagram of a string link $T\colon \iota^n \to \iota^n$ an $n \times n$ matrix $\mathcal{R}_n(T)$ whose (i,j) entry is given by

$$\mathcal{R}_n(T)_{j,i} = \sum_{P \in \mathcal{P}_i^j} w(P) \in \mathbb{Q}(t,t^{-1}). \tag{15}$$

Here, the summation is over the set $\mathcal{P}_i^j$ of all admissible paths in a diagram of T from the point $(i,0)$ to $(j,1)$. The entry is 0 if there is no such path.

It is shown in [7] that, for t sufficiently close to 1, the summations (15) over all paths converge, indeed, to rational expressions in t, and that these expressions are not dependent on the diagram chosen to present a particular T.

Lemma 3 *The assignment in* (15) *has the following basic properties:*

(1) *The assignment $T \mapsto \mathcal{R}_n(T)$ obeys $\mathcal{R}_n(T)\mathcal{R}_n(S) = \mathcal{R}_n(T \circ S)$, where the composite $T \circ S$ in $\mathcal{T}gl$ is given by stacking T on top of S.*

(2) *When restricted to the braid group B_n, the assignment reduces to the Burau representation as defined in* (2).

(3) *Specializing $\mathcal{R}_n(T)$ to $t = 1$, we obtain the matrix of the permutation π_T associated to T, that is, $\mathcal{R}_n(T)_{j,i} = \delta_{j,\pi_T(i)}$.*

Proof For the first part, note that, at the boundary between T and S in $T \circ S$, all orientations are upwards, so no admissible path can return from T to S. Thus, any admissible path $Q\colon i \to k$ in $T \circ S$ is the composite of a path $P\colon i \to j$ in S and a path $R\colon j \to k$ in T for some j. Also, the weight is clearly multiplicative: $w(Q) = w(R)w(P)$. Summation over all R, P, and j thus yields the respective matrix element for $\mathcal{R}_n(T \circ S)$, which is thus the matrix product as desired.

As described in the introduction, the assignment coincides with the Burau representation on the generators, so, by the properties above, the two representations coincide on all elements on B_n.

For the third claim, notice that all paths that change direction will have weight zero and may thus be discarded. For given indices, the remaining paths that run strictly along a tangle component (preserving direction at every crossing) will have weight one. □

2.2 Braid closures and Burau matrix blocks

Given a braid $b \in B_{n+m}$ thought of as an isomorphism on ι^{n+m}, we can construct a tangle $T\colon \iota^n \to \iota^n$ by closing the last m strands by loops as indicated in Figure 4. The following lemma is a nearly straightforward generalization of Alexander's theorem for links with some additional attention given to orientations at the end points. A respective generalization of the Markov theorem also holds, but is not needed here, since we are only concerned with the comparison, rather than the construction, of invariants.

Lemma 4 *Every oriented tangle $T\colon \iota^n \to \iota^n$ is given as the partial closure of a braid $b \in B_{n+m}$ for some m (as in Figure 4).*

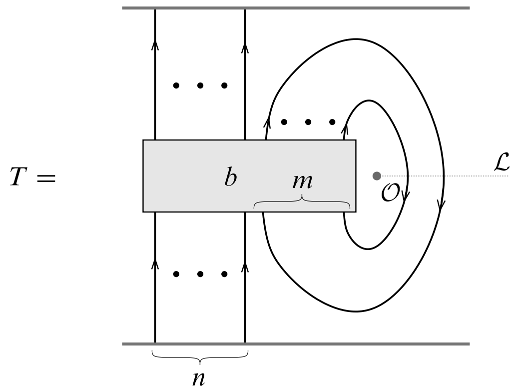

Figure 4: Markov presentation.

Proof Most of the proof is nearly verbatim the same as, for example, the one in [11, Section 6.5]. The diagram is assumed to be polygonal with vertical endpoints and a rotation point $\mathcal{O}$ to the right of the diagram at midheight. We may assume that no line segments are in radial direction from $\mathcal{O}$ and, using subdivision, that each segment has at most one crossing point. Thus each segment is either in clockwise or counterclockwise direction around $\mathcal{O}$. The *Alexander trick* is applied to every line segment in counterclockwise direction as depicted in [11, Figure 6.4], depending on the orientation of a possible crossing. As a result, all segments will be in clockwise direction.

Observe that the line segments at the top and bottom end points are already in clockwise direction, so they are not affected by the algorithm. In the resulting tangle diagram, we will have m segments intersecting the horizontal line $\mathcal{L}$ as indicated in Figure 4. Each of these segments can be arranged to be vertical and, thus, oriented downwards. To these we apply the cut and stretch process described in [11, Figure 6.3]. As a result, segments at all crossings are oriented upwards and to the left of $\mathcal{O}$, so we obtain the desired braid closure presentation. □

For a given tangle $T\colon \iota^n \to \iota^n$ that is the closure of a braid $b \in B_{n+m}$ in the above fashion, consider the block form of the $(n+m)\times(n+m)$ Burau matrix associated to b and its specialization at $t=1$:

$$\mathcal{B}_n(b) = \begin{bmatrix} X & Y \\ Z & Q \end{bmatrix} \quad \text{and} \quad \pi_b = \mathcal{B}_n(b)|_{t=1} = \begin{bmatrix} \overline{X} & \overline{Y} \\ \overline{Z} & \overline{Q} \end{bmatrix}. \tag{16}$$

Here, X is an $n\times n$ matrix and Q is an $m\times m$ matrix, each with coefficients in $\mathbb{Z}[t,t^{-1}]$. The dimensions of the matrices $\overline{X}$, $\overline{Y}$, $\overline{Z}$, and $\overline{Q}$ are the same, but all matrix entries are either 0 or 1.

Lemma 5 *Suppose $T\colon \iota^n \to \iota^n$ is a tangle that is the closure of a braid $b \in B_{n+m}$ and that has associated matrices as in* (16)*. Then T is a string link if and only if $\bar{Q}$ is nilpotent.*

Proof We note first that, since $\bar{Q}$ is the block of a permutation matrix π_b, it has at most one 1 in each row and each column and 0 for all other entries. That is, it is the incidence matrix of an oriented graph I_Q with each vertex having at most one incoming and one outgoing edge. Clearly, the components of such a graph are either oriented intervals or oriented circles. Thus, by reordering the basis, the matrix can be brought into a block diagonal form with two types of blocks. The first, corresponding to interval components of I_Q, are nilpotent Jordan blocks given by square matrices N with $N_{i,j} = \delta_{j,i+1}$. The second type, for the circle components, are cyclic $k \times k$ matrices C with $C_{i,j} = 1$ if $j \equiv i + 1 \mod k$ and $C_{i,j} = 0$ otherwise. Hence, $\bar{Q}$ is nilpotent if and only if there are no cyclic blocks.

Consider first a matrix block of the first nilpotent type N. Denoting the canonical basis $\{e_j : j = 1, \ldots, n+m\}$, we have, for some p and q with $n < p < q \le n+m$, that $\pi_b e_s = N e_s = e_{s-1}$ for $s = p+1, \ldots, q$. Since π_b encodes which points at the bottom of b are connected by strands to which points at the top of b, we find that the arcs at positions p through q are consecutively connected to each other by intervals in b. They thus form one interval J in T that is starting at the q^{th} position at the top of b, and ending at the p^{th} position at the bottom of b, and intersecting $\mathcal{L}$ in $q-p+1$ arcs. Since N, and thus $\bar{Q}$, has only 0 entries in the p^{th} column, we must have a 1 entry in the p^{th} column for $\bar{Y}$ in some k^{th} row (with $k \le n$), which means J is connected to the k^{th} top point of the diagram for T. Similarly, since N, and thus $\bar{Q}$, has only 0 entries in the q^{th} row, $\bar{Z}$ must have a 1 entry in the q^{th} row in some l^{th} column, so J must be connected to the l^{th} start point at the bottom of the tangle diagram.

Thus, if all matrix blocks of $\bar{Q}$ are of nilpotent type, the components of all closing arcs are connected to top and bottom points of the tangle diagram. Components of T that are disjoint from closing arcs are always oriented upwards and thus also must connect to top and bottom points of the diagram. Thus, if $\bar{Q}$ is nilpotent, T is indeed a string link.

Conversely, suppose $\bar{Q}$ contains a cyclic block with $\pi_b e_s = C e_s = e_{s-1}$ for $s = p+1, \ldots, q$ and $C e_p = e_q$. Then the arcs connected at positions p through q are again consecutively connected by intervals of b into one component, but now the ends of the component are also connected by an interval in b forming a closed component of T. However, closed components are not allowed for a string link. $\square$

2.3 Block matrix formula for string links

The following consequence of nilpotency will be useful to control geometric series occurring in the random walk picture.

Lemma 6 *Suppose Q is an $m \times m$ matrix that depends continuously on a real (or complex) parameter t in a vicinity of $t = 1$. Assume also that the specialization $\bar{Q}$ at $t = 1$ is nilpotent. Then, for any $d > 0$, there exist an $\varepsilon > 0$ and a $C > 0$ such that, for all t with $|t-1| < \varepsilon$ and all integers $N > 0$, we have*

$$\|Q^N\| \le C d^N. \tag{17}$$

Proof Assume that Q is continuous for t with $|t-1| \le \varepsilon_1$, and let $M > d$ be an upper bound of $\|Q\|$ for these t. Now since $\bar{Q}^m = 0$ by assumption, we have that $\|Q^m\|$ is a continuous function vanishing at 0, so there is an $\varepsilon \in (0, \varepsilon_1)$ such that $\|Q^m\| < d^m$ whenever $|t-1| < \varepsilon$. Writing $N = cm + r$ with $r = 0, \dots, m-1$, we thus have $\|Q^N\| = \|Q^r Q^{mc}\| \le \|Q\|^r \|Q^m\|^c \le M^r d^{mc} = (M/d)^r d^N \le C d^N$, where $C = (M/d)^{m-1}$. □

In the proposition below, we establish a relation between the blocks of the Burau representation of a braid and the random walk invariant of the string link obtained by closing the same braid.

Proposition 7 *Suppose $T \in \mathrm{Str}(n)$ is a string link obtained as the closure of a braid $b \in B_{n+m}$, and let X, Y, Z, and Q be the matrix blocks of $\mathcal{B}_n(b)$ as in (16). Then $(\mathbb{1} - Q)$ is invertible over $\mathbb{Q}(t, t^{-1})$, and we have*

$$\mathcal{R}_n(T) = X + Y(\mathbb{1} - Q)^{-1} Z. \tag{18}$$

Proof We first note that, for $D(t) = \det(\mathbb{1} - Q)$, we have $D(1) = 1$ since $\bar{Q}$ is nilpotent, so $D(t) \ne 0$, and $(\mathbb{1} - Q)$ is indeed invertible over $\mathbb{Q}(t, t^{-1})$. Thus, the expression on the right side of (18) is always well defined.

In order to evaluate $\mathcal{R}_n(T)_{ji}$, we partition the set of admissible paths $\mathcal{P}_i^j$ further. Note that every path $P \in \mathcal{P}_i^j$ is characterized by how often and in which order it will pass through the m closing arcs attached to positions $n+1$ through $n+m$ of b as in Figure 4. Denote by $\mathcal{P}_i^j(i_1 \dots i_k)$ the set of admissible paths that pass through arcs in positions $i_1, \dots, i_k \in \{n+1, \dots, n+m\}$ in this order (and with repetitions allowed). Denote also by $\mathcal{M}_s^t$ the set of admissible paths in b that start at the s^{th} position at the bottom of b and end at the t^{th} position at the top of b. Then it is clear that each path in the former is put together in a unique fashion by pieces from the latter sets, yielding a natural bijection as follows:

$$\mathcal{P}_i^j(i_1\dots i_k)\cong \mathcal{M}_{i_k}^j\times\mathcal{M}_{i_{k-1}}^{i_k}\times\cdots\times\mathcal{M}_{i_1}^{i_2}\times\mathcal{M}_i^{i_1}. \tag{19}$$

Special cases are $\mathcal{P}_i^j(i_1)=\mathcal{M}_{i_1}^j\times\mathcal{M}_i^{i_1}$ for $k=1$ and $\mathcal{P}_i^j(\varnothing)=\mathcal{M}_i^j$ for $k=0$. For $k\geq 1$, we compute, using multiplicative property of weights of composed strands,

(20)
$$\begin{aligned}
\sum_{P\in\mathcal{P}_i^j(i_1\dots i_k)} w(P) &= \sum_{P_k\in\mathcal{M}_{i_k}^j,\dots,P_1\in\mathcal{M}_{i_1}^{i_2},P_0\in\mathcal{M}_i^{i_1}} w(P_k)\cdots w(P_1)w(P_0)\\
&=\Big(\sum_{P_k\in\mathcal{M}_{i_k}^j} w(P_k)\Big)\cdots\Big(\sum_{P_1\in\mathcal{M}_{i_1}^{i_2}} w(P_1)\Big)\Big(\sum_{P_0\in\mathcal{M}_i^{i_1}} w(P_0)\Big)\\
\text{(by Lemma 3 (2))}\quad &= \mathcal{B}_n(b)_{ji_k}\cdots\mathcal{B}_n(b)_{i_2i_1}\mathcal{B}_n(b)_{i_1i}\\
\text{(using block form)}\quad &= Y_{ji_k}\cdots Q_{i_2i_1}Z_{i_1i}.
\end{aligned}$$

The set of admissible paths $\mathcal{P}_i^j[k]$ from $(i,0)$ to $(j,1)$ that intersect $\mathcal{L}$ exactly k times is the union of all $\mathcal{P}_i^j(i_1\dots i_k)$ for fixed i, j, and k. Hence, for $k\geq 1$, from (20) and by summation over all intermediate indices $i_1,\dots,i_k$, we obtain

$$\sum_{P\in\mathcal{P}_i^j[k]} w(P)=(YQ^{k-1}Z)_{ji}. \tag{21}$$

In the case $k=0$, we have $\mathcal{P}_i^j[0]=\mathcal{M}_i^j$, so

$$\sum_{P\in\mathcal{P}_i^j[0]} w(P)=X_{ji}. \tag{22}$$

Summing terms in (21) and (22), we thus obtain

$$\sum_{P\in\mathcal{P}_i^j,|P\cap\mathcal{L}|\leq N} w(P)=\Big(X+Y\Big(\sum_{r=0}^{N-1}Q^r\Big)Z\Big)_{ji}. \tag{23}$$

Given that T is a string link, we know by Lemma 5 that Q specializes to a nilpotent matrix at $t=1$, so we can apply Lemma 6. Thus, if we choose any $d<1$ in the latter lemma, the geometric series in (23) will converge, as $N\to\infty$ for t in some ε–vicinity of 1, to the right hand side expression of (18). This has to coincide with the rational function limit asserted in [7, Theorem A]. By uniqueness of meromorphic continuations, we thus have the desired equality as rational functions for all t. $\square$

We conclude with an observation related to the initial interpretation of $\mathcal{R}_n(T)$ as a stochastic matrix (at least for positive string links), namely that there are right and left eigenvectors independent of T. To this end, we denote the n–dimensional row vector $\boldsymbol{e}_n = (1,\dots,1)$ and let $\boldsymbol{v}_n$ be the n–dimensional column vector with $\boldsymbol{v}_n^T = (1, t, \dots, t^{n-1})$.

Corollary 8 *For any string link $T \in \mathrm{Str}(n)$, we have*

$$\boldsymbol{e}_n \mathcal{R}_n(T) = \boldsymbol{e}_n \quad \text{and} \quad \mathcal{R}_n(T)\boldsymbol{v}_n = \boldsymbol{v}_n. \tag{24}$$

Proof We first note that (24) holds for braids since $\boldsymbol{e}_n$ and $\boldsymbol{v}_n$ are easily verified to be eigenvectors for the braid generators in (1). In particular, for a braid $b \in B_{n+m}$ whose m–closure is T, we have $\boldsymbol{e}_{n+m}\mathcal{B}_n(b) = \boldsymbol{e}_{n+m}$ and $\mathcal{B}_n(b)\boldsymbol{v}_{n+m} = \boldsymbol{v}_{n+m}$. Since $\boldsymbol{e}_{n+m} = (\boldsymbol{e}_n, \boldsymbol{e}_m)$, the former implies $\boldsymbol{e}_n X + \boldsymbol{e}_m Z = \boldsymbol{e}_n$ and $\boldsymbol{e}_n Y + \boldsymbol{e}_m Q = \boldsymbol{e}_m$, which can also be written as $\boldsymbol{e}_n Y = \boldsymbol{e}_m(\mathbb{1} - Q)$. Using (18), we find

$$\begin{aligned}\boldsymbol{e}_n \mathcal{R}_n(T) &= \boldsymbol{e}_n X + \boldsymbol{e}_n Y(\mathbb{1} - Q)^{-1} Z \\ &= \boldsymbol{e}_n X + \boldsymbol{e}_m(\mathbb{1} - Q)(\mathbb{1} - Q)^{-1} Z = \boldsymbol{e}_n X + \boldsymbol{e}_m Z = \boldsymbol{e}_n.\end{aligned}$$

A similar calculation, again using Proposition 7, shows that $\boldsymbol{v}_n$ is a right eigenvector. □

The fact that $\boldsymbol{e}_n$ is a left eigenvector means that all column sums of $\mathcal{R}_n(T)$ are one, supplementing a formal proof to the intuition for this fact provided in [7]. Assuming that T is a positive string link and $t \in [0, 1]$ so that all entries in $\mathcal{R}_n(T)$ are nonnegative, we thus have that $\mathcal{R}_n(T)$ is indeed a stochastic matrix. After suitable renormalization, the positive eigenvector $\boldsymbol{v}_n$ thus represents an equilibrium state,

$$\boldsymbol{p}_n = \frac{1}{\langle \boldsymbol{e}_n, \boldsymbol{v}_n \rangle} \boldsymbol{v}_n = (p_1, \dots, p_n)^T,$$

with probability of finding a ball in j^{th} position given by $p_j = t^j(1-t)/(1-t^n)$. If T is, in addition, nonseparable and $t \in (0, 1)$, this is the unique equilibrium; see [7].

Since the stationary state $\boldsymbol{p}$ is independent of T, it clearly contains no information about T, answering [7, Remark (4)] in the negative. An interpretation of these right and left eigenvectors in terms of the representation theory of $U_{-1}(\mathfrak{sl}_2)$ is given in Section 4.4.

3 Tangle functors and exterior algebras

3.1 Tangle functor associated to the Alexander polynomial

In this section we review, with slight variations, Ohtsuki's description in [10] of the tangle functor in (9), which is associated to the Alexander polynomial and generalizes the Burau representation. The construction is based in Turaev's set of relations for R–matrices identified in [14], which imply the extension to a functor on oriented tangles, as stated in [10, Theorem 3.7]. The R–matrix associated in [10] to the Alexander polynomial is given as an endomorphism on $V \otimes V$, where V is the free $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ module with generators e_0 and e_1. In the basis $\{e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1\}$, it has the form

$$R = \begin{bmatrix} t^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & t^{-1/2} - t^{1/2} & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{bmatrix}. \tag{25}$$

This R–matrix implements a representation of $\psi_n \colon B_n \to \mathrm{End}(V^{\otimes n})$ in the usual manner (see [10, (2.1)]), so $\psi_n(b)$ coincides with $\mathcal{V}(b)$ for a braid $b \in B_n$.

We note that our convention for orientations is the opposite of that in [10], where downwards arrows are considered positive orientations. However, diagrams in [10] there are easily translated to our convention by simply reversing all arrows.

As already indicated in Section 1.3 of the introduction, the tangle functor preserves a natural grading on the associated vector spaces which can be expressed more formally as follows. Specifically, define an endomorphism $\boldsymbol{\theta}(\underline{\epsilon})$ on the module $\mathcal{V}(\underline{\epsilon}) = V^{\epsilon_1} \otimes \cdots \otimes V^{\epsilon_n}$ acting diagonally in the natural basis by

$$\boldsymbol{\theta}(\underline{\epsilon})(e_{i_1}^{\epsilon_1} \otimes \cdots \otimes e_{i_n}^{\epsilon_n}) = \left(\sum_{s=1}^{n} \epsilon_s i_s \right) e_{i_1}^{\epsilon_1} \otimes \cdots \otimes e_{i_n}^{\epsilon_n}. \tag{26}$$

Here we use the convention $e_j^+ = e_j$ and $e_j^- = e_j^*$ for basis vectors of $V^+ = V$ and $V^- = V^*$, respectively. The eigenspaces of $\boldsymbol{\theta}(\underline{\epsilon})$ are thus the graded components of $\mathcal{V}(\underline{\epsilon})$. It is readily checked that the morphism in (26) gives, in fact, rise to a natural transformation $\boldsymbol{\theta} \colon \mathcal{V} \xrightarrow{\bullet} \mathcal{V}$.

In the case when all signs are positive, we denote further $\theta_n^{\otimes} := \boldsymbol{\theta}(\iota^n) \in \mathrm{End}(V^{\otimes n})$, which has eigenvalue $k = |\{s : i_s = 1\}|$ on $e_{i_1} \otimes \cdots \otimes e_{i_n}$. The eigenspace

$$W_{n,k} = \ker(\theta_n^{\otimes} - k\mathbb{1}) \tag{27}$$

is thus the k–graded component of rank $\binom{n}{k}$ in $V^{\otimes n}$ (as noted in (9)) and is invariant under the braid group action.

Observe also that the R–matrix in (25) specializes for $t = 1$ to a signed permutation given by $R(e_i \otimes e_j) = (-1)^{ij} e_j \otimes e_i$, so, in particular, $R = R^{-1}$. The latter implies that, in this case, crossings of a tangle T can be changed arbitrarily without changing $\mathcal{V}(T)$, so we may assume T to be a braid. Hence, $\mathcal{V}(T)$ is a composition of signed permutations. Moreover, on $W_{n,0} = \langle e_0^{\otimes n} \rangle$, all R act as the identity at $t = 1$, which implies the following statement.

Lemma 9 *The endomorphism $\mathcal{W}_{n,k}(T)$ reduces to signed permutations on the canonical basis in the specialization $t = 1$. In particular, $\mathcal{W}_{n,0}(T) = 1$ at $t = 1$.*

Another useful property of the tangle functor is its equivariance with respect to the $U_{-1}(\mathfrak{sl}_2)$ action given in [10]. In the particular case of a tangle $T\colon \iota^n \to \iota^n$, this property implies that $\mathcal{V}(T) \in \mathrm{End}(V^{\otimes n})$ commutes with operators

$$h^{\otimes n} = t^{n/2}(-1)^{\theta_n^{\otimes}}, \quad \text{where } h = t^{1/2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{28}$$

as well as

$$\widetilde{E}_n = \sum_{i=1}^{n} \mathbb{1}^{\otimes i-1} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes (h^{-1})^{\otimes n-i}, \quad \widetilde{F}_n = \sum_{i=1}^{n} h^{\otimes i-1} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \mathbb{1}^{\otimes n-i}. \tag{29}$$

The operators in (28) and (29) describe the actions of n–fold coproducts of rescaled generators of $U_{-1}(\mathfrak{sl}_2)$, and they fulfill basic relations. For example, $h^{\otimes n}$ anticommutes with both $\widetilde{E}_n$ and $\widetilde{F}_n$, and $[\widetilde{E}_n, \widetilde{F}_n] = (t^{1/2} - t^{-1/2})^{-1}(h^{\otimes n} - (h^{-1})^{\otimes n})$.

Functoriality also implies that the operator invariant of a tangle $T\colon \iota^n \to \iota^n$ is given by the partial quantum trace over the invariant for a braid $b \in B_{n+m}$ if T is given as the closure of b in the sense of Figure 4. More precisely, for an endomorphism $f \in \mathrm{End}(V^{\otimes n})$, define its quantum trace in terms of the canonical trace as follows:

$$\mathrm{Tr}_n(f) = \mathrm{trace}_{V^{\otimes n}}(h^{\otimes n} f). \tag{30}$$

It follows from the evaluations and coevaluations associated to extrema in [10, Section 3.3] (again with reversed orientation convention) that closing off a right most strand of a tangle diagram with an arc corresponds to applying Tr_1 to contract the respective indices of the associated operator. Iterating the process, we obtain, for the closure T of a braid b as above,

$$\mathcal{V}(T) = \mathrm{id}^{\otimes n} \otimes \mathrm{Tr}_m(\mathcal{V}(b)), \tag{31}$$

where we suppress notation for the natural isomorphism

$$\mathrm{End}(V^{\otimes n+m}) \cong \mathrm{End}(V^{\otimes n}) \otimes \mathrm{End}(V^{\otimes n}).$$

3.2 Isomorphisms with exterior algebra representations

In this section we outline, again with some variations in conventions and normalizations, the equivalence of the braid representation ψ_n obtained from $\mathcal{V}$ and the exterior algebra extension of the Burau representation given in [10, Appendix C].

We denote by M_n the free $\mathbb{Z}[t^{1/2}, t^{-1/2}]$–module with basis $\{v_1, \dots, v_n\}$ and by $\mathcal{B}_n\colon B_n \to \mathrm{End}(M_n)$ the braid group representation as given in (2) and (3). The extension to the exterior algebra is thus

$$\bigwedge^* \mathcal{B}_n\colon B_n \to \mathrm{End}\big(\bigwedge^* M_n\big), \quad b \mapsto \bigwedge^* \mathcal{B}_n(b). \tag{32}$$

The action is clearly also graded with invariant submodules $\bigwedge^k M_n$. As before, it is useful to encode the grading as an operator on $\bigwedge^* M_n$ defined by

$$\theta_n^{\wedge}(\omega) = k\omega \quad \text{for } \omega \in \bigwedge^k M_n. \tag{33}$$

This allows us to define, similar to the quantum trace above, a supertrace str_n on morphisms $f \in \mathrm{End}\big(\bigwedge^* M_n\big)$ by

$$\mathrm{str}_n(f) = \mathrm{trace}_{\bigwedge^* M_n}((-1)^{\theta_n^{\wedge}} f). \tag{34}$$

Analogous to [10, (C.3)], we next define, for each n, an isomorphism $\mathcal{I}_n\colon V^{\otimes n} \to \bigwedge^* M_n$ by induction as

$$\begin{aligned} \mathcal{I}_n\colon V^{\otimes n} \xrightarrow{\mathcal{I}_{n-1} \otimes \mathrm{id}_V} \bigwedge^* M_{n-1} \otimes V &\longrightarrow \bigwedge^* M_n, \\ \alpha_0 \otimes e_0 + \alpha_1 \otimes e_1 &\longmapsto \alpha_0 + t^{n/2} \alpha_1 \wedge v_n. \end{aligned} \tag{35}$$

Some immediate properties of these isomorphisms implied by (35) include that they preserve the respective gradings, that is,

$$\mathcal{I}_n \theta_n^{\otimes} = \theta_n^{\wedge} \mathcal{I}_n, \tag{36}$$

and that they factor, up to scaling, with respect to products of spaces in the sense of the following commutative diagram.

$$\begin{tikzcd} \bigwedge^* M_{n+m} \arrow[r, equal] & \bigwedge^*(M_n \oplus M_m) \arrow[r, equal] & \bigwedge^* M_n \otimes \bigwedge^* M_m \\ V^{\otimes n+m} \arrow[u, "\mathcal{I}_{n+m}"] \arrow[r, equal] & V^{\otimes n} \otimes V^{\otimes m} \arrow[ur, "\mathcal{I}_n \otimes t^{(n/2)\theta_m^{\wedge}} \mathcal{I}_m"'] & \end{tikzcd} \tag{37}$$

Here, double lines indicate obvious canonical isomorphisms, and the factor $t^{n/2\theta_m^{\wedge}}$ stems from the shift in the basis labeling for $M_{n+m} \cong M_n \oplus M_m$.

In order to state the equivalence of braid group representations, we also consider the representation $\widehat{\psi}_n\colon B_n \to \mathrm{End}(V^{\otimes n})$ obtained from the rescaled R–matrix $\widehat{R} = t^{1/2}R$ with R as in (25). It is related to the original representation by

$$\widehat{\psi}_n(b) = t^{(1/2)\varpi(b)}\psi_n(b) = t^{(1/2)\varpi(b)}\mathcal{V}(b), \tag{38}$$

where $\varpi(b)$ is the writhe of b given by the number of positive crossing minus the number of negative crossings of b.

The following lemma is essentially identical to [10, Lemma C.1] and is verified by direct computation using (35) and (37).

Lemma 10 [10] *The isomorphisms defined in* (35) *provide an equivalence between the representations ψ_n and $\bigwedge^*\mathcal{B}_n$ of B_n. That is, for any $b \in B_n$, we have the following commutative diagram.*

$$\begin{array}{ccc} V^{\otimes n} & \xrightarrow{\ \mathcal{I}_n\ } & \bigwedge^* M_n \\ {\scriptstyle \widehat{\psi}_n(b)}\big\downarrow & & \big\downarrow{\scriptstyle \bigwedge^*\mathcal{B}_n(b)} \\ V^{\otimes n} & \xrightarrow[\ \mathcal{I}_n\ ]{} & \bigwedge^* M_n \end{array} \tag{39}$$

Moreover, all isomorphisms preserve gradings, so (39) *also holds when restricted to subrepresentations $W_{n,k}$ and $\bigwedge^k M_n$ instead of $V^{\otimes n}$ and $\bigwedge^* M_n$.*

3.3 Relations between traces

The aim of this section is to replace the quantum trace formula for braid closures (31) by traces over exterior algebras and thus reduce the proof of Theorem 1 to the exterior algebra of the Burau representation. We begin with notation for the partial supertrace given by the following composite of natural isomorphisms and str_n as in (34):

(40)

$$\begin{aligned}\mathrm{Str}_n^{n+m}\colon \mathrm{End}\big(\textstyle\bigwedge^* M_{n+m}\big) &= \mathrm{End}\big(\textstyle\bigwedge^* M_n \otimes \bigwedge^* M_m\big)\\ &= \mathrm{End}\big(\textstyle\bigwedge^* M_n\big) \otimes \mathrm{End}\big(\textstyle\bigwedge^* M_m\big) \xrightarrow{\mathrm{id}\otimes \mathrm{str}_m} \mathrm{End}\big(\textstyle\bigwedge^* M_n\big).\end{aligned}$$

Moreover, for an endomorphism $f \in \mathrm{End}\big(\bigwedge^* M_n\big)$, we denote the conjugate

$$f^{\mathcal{I}} = \mathcal{I}_n^{-1} f \mathcal{I}_n \in \mathrm{End}(V^{\otimes n}). \tag{41}$$

The explicit relation between supertrace and quantum trace is stated in this terminology in the next lemma.

Lemma 11 *Suppose* $f \in \mathrm{End}\big(\bigwedge^* M_{n+m}\big)$. *Then*

$$\mathrm{id}\otimes \mathrm{Tr}_m(f^{\mathcal{I}}) = t^{m/2}\mathrm{Str}_n^{m+n}(f)^{\mathcal{I}}. \tag{42}$$

Proof By linearity, we may assume that $f = a \otimes b$ with $a \in \mathrm{End}\big(\bigwedge^* M_n\big)$ and $b \in \mathrm{End}\big(\bigwedge^* M_m\big)$ modulo the isomorphism indicated in (40). Using (37), we thus find

$$f^{\mathcal{I}} = \mathcal{I}_{n+m}^{-1} f \mathcal{I}_{n+m} = \mathcal{I}_n^{-1} a \mathcal{I}_n \otimes \mathcal{I}_m^{-1} t^{-(n/2)\theta_m^{\wedge}} b t^{(n/2)\theta_m^{\wedge}} \mathcal{I}_m,$$

so we find for the partial quantum trace

$$\begin{aligned} \mathrm{id}\otimes \mathrm{Tr}_m(f^{\mathcal{I}}) &= \mathcal{I}_n^{-1} a \mathcal{I}_n \cdot \mathrm{trace}_{V^{\otimes m}}(h^{\otimes m}\mathcal{I}_m^{-1} t^{-(n/2)\theta_m^{\wedge}} b t^{(n/2)\theta_m^{\wedge}} \mathcal{I}_m) \\ &= a^{\mathcal{I}} \cdot \mathrm{trace}_{\bigwedge^* M_m}(Pb). \end{aligned} \tag{43}$$

In the last step, we use cylicity of the canonical trace to combine the isomorphisms appearing in the isomorphism P, which is given and evaluated as follows:

$$\begin{aligned} P &= t^{(n/2)\theta_m^{\wedge}} \mathcal{I}_m h^{\otimes m} \mathcal{I}_m^{-1} t^{-(n/2)\theta_m^{\wedge}} \\ \text{(by (28))} \quad &= t^{m/2} t^{(n/2)\theta_m^{\wedge}} \mathcal{I}_m (-1)^{\theta_m^{\otimes}} \mathcal{I}_m^{-1} t^{-(n/2)\theta_m^{\wedge}} \\ \text{(by (36))} \quad &= t^{m/2} t^{(n/2)\theta_m^{\wedge}} (-1)^{\theta_m^{\wedge}} t^{-(n/2)\theta_m^{\wedge}} \\ &= t^{m/2}(-1)^{\theta_m^{\wedge}}. \end{aligned} \tag{44}$$

On the other hand, we have $\mathrm{Str}_n^{n+m}(f) = a\cdot \mathrm{str}_m(b) = a\cdot \mathrm{trace}_{\bigwedge^* M_m}((-1)^{\theta_m^{\wedge}} b)$ by definitions in (34) and (40), from which (42) readily follows. □

From the traces' equivalence, we can now compute the tangle functor $\mathcal{V}$ on string links from the Burau representation.

Corollary 12 *Suppose* $T\colon \iota^n \to \iota^n$ *is a string link presented as the closure of a braid* $b \in B_{n+m}$. *Then*

$$\mathcal{V}(T) = t^{(1/2)(m+\varpi(b))}\cdot \mathrm{Str}_n^{n+m}\big(\textstyle\bigwedge^* \mathcal{B}_n(b)\big)^{\mathcal{I}}. \tag{45}$$

Proof The proof is a direct computation combining previous results:

$$\begin{aligned} \text{Right-hand side of (45)} &= t^{(1/2)\varpi(b)} t^{m/2}\cdot \mathrm{Str}_n^{n+m}\big(\textstyle\bigwedge^* \mathcal{B}_n(b)\big)^{\mathcal{I}} \\ \text{(by Lemma 11)} \quad &= t^{(1/2)\varpi(b)}\, \mathrm{id}\otimes \mathrm{Tr}_m\big(\textstyle\bigwedge^* \mathcal{B}_n(b)^{\mathcal{I}}\big) \\ \text{(by Lemma 10)} \quad &= t^{(1/2)\varpi(b)}\, \mathrm{id}\otimes \mathrm{Tr}_m(\hat{\psi}_{n+m}(b)) \\ \text{(by (38))} \quad &= \mathrm{id}\otimes \mathrm{Tr}_m(\psi_{n+m}(b)) = \mathrm{id}\otimes \mathrm{Tr}_m(\mathcal{V}(b)) \\ \text{(by (31))} \quad &= \mathcal{V}(T). \end{aligned} \tag{46}$$

□

Note that Corollary 12 not only implies that the maps $\mathrm{Str}_n^{n+m}\big(\bigwedge^* \mathcal{B}_n(b)\big)$ preserve the natural grading, but also that these operators commute with the actions of the conjugates $\breve{E}_n = \mathcal{I}_n \widetilde{E}_n \mathcal{I}_n^{-1}$ and $\breve{F}_n = \mathcal{I}_n \widetilde{F}_n \mathcal{I}_n^{-1}$ of the operators in (29). In fact, similar to [10, Lemma C.3], we have

$$\breve{F}_n \alpha = t^{-1/2} \boldsymbol{v}_n \wedge \alpha \tag{47}$$

for any $\alpha \in \bigwedge^* M_n$ and $\boldsymbol{v}_n$ as in Corollary 8. In order to find a respective expression for $\breve{E}_n$, we define $\boldsymbol{\chi}_j \colon \bigwedge^k M_n \to \bigwedge^{k-1} M_n$ by $\boldsymbol{\chi}_j(\alpha) = 0$ and $\boldsymbol{\chi}_j(\alpha \wedge v_j) = \alpha$, where $\alpha = v_{i_1} \wedge \cdots \wedge v_{i_s}$ with $j \notin \{i_1, \ldots, i_s\}$. A basic calculation with forms then yields

$$\breve{E}_n = t^{-n/2} \sum_{j=1}^{n} \boldsymbol{\chi}_j . \tag{48}$$

For a string link T given as the closure of a braid b, we denote the restriction of the operator to degree k forms as

$$\mathcal{Y}_{n,k}(T,b) = \mathrm{Str}_n^{n+m}\big(\bigwedge^* \mathcal{B}_n(b)\big)\big|_{\bigwedge^k M_n}. \tag{49}$$

This corresponds, via conjugation by $\mathcal{I}_n$, to the restrictions defined in (10), so we have from (45) that $\mathcal{W}_{n,k}(T) = t^{(1/2)(m+\varpi(b))} \cdot \mathcal{Y}_{n,k}(T,b)^{\mathcal{I}}$. Let us also define

$$\mathcal{Y}_{n,k/0}(T,b) = \frac{1}{\mathcal{Y}_{n,0}(T,b)} \mathcal{Y}_{n,k}(T,b). \tag{50}$$

Canceling the factors, we thus find that

$$\mathcal{W}_{n,k/0}(T) = \mathcal{Y}_{n,k/0}(T,b)^{\mathcal{I}}, \tag{51}$$

where the morphism on the left is as defined in (11). Consequently, we have reduced the proof of Theorem 1 to showing that

$$\bigwedge^k \mathcal{R}_n(T) = \mathcal{Y}_{n,k/0}(T,b) \quad \text{for all } T, \tag{52}$$

which will be the objective of the next section.

4 Proof of main results and conclusions

Before proving Theorem 1 in Section 4.3, we provide several technical lemmas on exterior algebras that relate partial traces, actions on top forms and their dual contractions, as well as Schur complements. At the end of this section, we comment on various implications of our result and possible generalizations.

4.1 Supertrace from top forms

The aim of this section is to generalize the well known relation $\mathrm{str}_m\big(\bigwedge^* f\big) = \det(\mathbb{1} - f)$, where $f \in \mathrm{End}(M_m)$ is any endomorphism on a free module M_m of rank m, to partial traces with respect to endomorphisms on $U \oplus M_m$, where U is another module of rank n.

Let M_m be generated by a basis $\{w_1, \dots, w_m\}$, and denote by $\mathcal{P}_m$ the set of subsets of $\{1, \dots, m\}$. For any $S = \{i_1, \dots, i_k\} \in \mathcal{P}_m$ with $i_1 < i_2 < \cdots < i_k$, denote $\boldsymbol{\tau}^S = w_{i_1} \wedge \cdots \wedge w_{i_k}$, so $\{\boldsymbol{\tau}^S : S \in \mathcal{P}_m\}$ is a basis of $\bigwedge^* M_m$. Denote also the top form $\boldsymbol{\tau}^m = \boldsymbol{\tau}^{\{1,\dots,m\}} = w_1 \wedge \cdots \wedge w_m$. We define contractions with respective dual forms on the combined exterior algebra as

$$\boldsymbol{\chi}_S \colon \textstyle\bigwedge^*(U \oplus M_m) \to \bigwedge^* U, \quad \text{with } \boldsymbol{\chi}_S(\alpha \wedge \boldsymbol{\tau}^T) = \delta_{S,T}\alpha, \tag{53}$$

for all $\alpha \in \bigwedge^* U$ and $S, T \in \mathcal{P}_m$. For future application, we also record here the following elementary property, which is immediate from (53):

$$\boldsymbol{\chi}_S(\gamma \wedge \delta) = \gamma \wedge \boldsymbol{\chi}_S(\delta) \quad \text{for any } \gamma \in \textstyle\bigwedge^* U \text{ and } \delta \in \bigwedge^*(U \oplus M_m). \tag{54}$$

We again use abbreviated notation $\boldsymbol{\chi}_m = \boldsymbol{\chi}_{\{1,\dots,m\}}$ for contractions with the respective top form. Using contractions, the partial supertrace for an endomorphism $f \in \mathrm{End}\big(\bigwedge^*(U \oplus M_m)\big)$ acting on an element $\alpha \in \bigwedge^* U$ may thus be reexpressed by the following formula:

$$\mathrm{Str}_n^{n+m}(f)\alpha = \sum_{S \in \mathcal{P}_m} (-1)^{|S|} \boldsymbol{\chi}_S(f(\alpha \wedge \boldsymbol{\tau}^S)). \tag{55}$$

Writing S^c for the complement of S we have the relation

$$\boldsymbol{\tau}^m = \sigma_S \boldsymbol{\tau}^S \wedge \boldsymbol{\tau}^{S^c}, \tag{56}$$

where $\sigma_S \in \{\pm 1\}$ is the signature of the respective shuffle permutation. More generally, for $\beta \in \bigwedge^* U$, we have that $\beta \wedge \boldsymbol{\tau}^T \wedge \boldsymbol{\tau}^{S^c}$ is a nonzero multiple of $\beta \wedge \boldsymbol{\tau}^m$ only if $T = S$. This observation and (56) thus imply the relation

$$\boldsymbol{\chi}_S(\eta) = \sigma_S \boldsymbol{\chi}_m(\eta \wedge \boldsymbol{\tau}^{S^c}). \tag{57}$$

Lemma 13 *Let $A \in \mathrm{End}(U \oplus M_m)$ and $\alpha \in \bigwedge^* U$. Then*

$$\mathrm{Str}_n^{n+m}\big(\textstyle\bigwedge^* A\big)\alpha = \boldsymbol{\chi}_m\big(\big(\bigwedge^* A\alpha\big) \wedge \big(\bigwedge^*(\mathbb{1} - A)\boldsymbol{\tau}^m\big)\big). \tag{58}$$

Proof We first compute the action of $(\mathbb{1}-A)$ on the top form:

$$\begin{aligned}\textstyle\bigwedge^*(\mathbb{1}-A)\boldsymbol{\tau}^m &= (\mathbb{1}-A)w_1\wedge\cdots\wedge(\mathbb{1}-A)w_m \\ &= \sum_{\epsilon_1,\dots,\epsilon_n\in\{0,1\}} (-1)^{\sum_i \epsilon_i} A^{\epsilon_1}w_1\wedge\cdots\wedge A^{\epsilon_n}w_n \\ &= \sum_{S\in\mathcal{P}_m}(-1)^{|S|}A_S\tau_m = \sum_{S\in\mathcal{P}_m}\sigma_S(-1)^{|S|}A_S(\boldsymbol{\tau}^S\wedge\boldsymbol{\tau}^{S^c}) \\ &= \sum_{S\in\mathcal{P}_m}\sigma_S(-1)^{|S|}\big(\textstyle\bigwedge^{|S|}A\boldsymbol{\tau}^S\big)\wedge\boldsymbol{\tau}^{S^c}.\end{aligned}\tag{59}$$

Here A_S acts on $\boldsymbol{\tau}^m$ by the formula in the previous line with $\epsilon_j=1$ if $j\in S$ and $\epsilon_j=0$ otherwise. Moreover, we are making use of (56) in the third line. By further evaluation,

$$\begin{aligned}\text{Right hand side of (58)} &= \sum_{S\in\mathcal{P}_m}\sigma_S(-1)^{|S|}\boldsymbol{\chi}_m\big(\big(\textstyle\bigwedge^*A\alpha\big)\wedge\big(\textstyle\bigwedge^{|S|}A\boldsymbol{\tau}^S\big)\wedge\boldsymbol{\tau}^{S^c}\big) \\ &= \sum_{S\in\mathcal{P}_m}\sigma_S(-1)^{|S|}\boldsymbol{\chi}_m\big(\big(\textstyle\bigwedge^*A(\alpha\wedge\boldsymbol{\tau}^S)\big)\wedge\boldsymbol{\tau}^{S^c}\big) \\ \text{(by (57))}\quad &= \sum_{S\in\mathcal{P}_m}(-1)^{|S|}\boldsymbol{\chi}_S\big(\textstyle\bigwedge^*A(\alpha\wedge\boldsymbol{\tau}^S)\big) \\ \text{(by (55))}\quad &= \mathrm{Str}_n^{n+m}\big(\textstyle\bigwedge^*A\big)\alpha,\end{aligned}\tag{60}$$

which is the desired form and thus completes the proof. □

4.2 Partial traces from Schur complements

Consider an endomorphism $B\in\mathrm{End}(U\oplus M_m)$ and assume, for the respective block form

$$B=\begin{bmatrix}H & J\\ K & L\end{bmatrix},\tag{61}$$

that $L\in\mathrm{End}(M_m)$ is invertible. Then B has a block-UL factorization

$$B=B_uB_l,\quad\text{with}\quad B_u=\begin{bmatrix}\mathbb{1} & G\\ 0 & \mathbb{1}\end{bmatrix}\quad\text{and}\quad B_l=\begin{bmatrix}D & 0\\ K & L\end{bmatrix},\tag{62}$$

where

$$D=H-JL^{-1}K\quad\text{and}\quad G=JL^{-1}.\tag{63}$$

The endomorphism D is also called the *Schur complement* of L in B. A basic relation between these endomorphisms is $\det(B)=\det(L)\det(D)$ which we generalize in the next lemma for our purposes.

Lemma 14 *Suppose B has a block form as in* (61), *with L invertible and $D \in \mathrm{End}(U)$ its Schur complement as in* (63). *Then, for $\alpha \in \bigwedge^* U$, we have*

$$\chi_m\big(\textstyle\bigwedge^* B(\alpha \wedge \boldsymbol{\tau}^m)\big) = \det(L) \textstyle\bigwedge^* D\alpha. \tag{64}$$

Proof We first note that, since B_l maps M_m to itself, and since its restriction to M_m is L, we have $\bigwedge^* B_l \boldsymbol{\tau}^m = \bigwedge^* L\boldsymbol{\tau}^m = \det(L)\boldsymbol{\tau}^m$. Furthermore, if $\alpha = x_1 \wedge \cdots \wedge x_k$ for $x_j \in U$, we have

$$\textstyle\bigwedge^* B_l\alpha = (Dx_1 + Kx_1) \wedge \cdots \wedge (Dx_k + Kx_k) = \bigwedge^* D\alpha + \rho,$$

where $\rho = \sum_i \gamma_i \wedge w_i$, since each $Kx_i \in M_m$ and is hence a combination of the w_i. This form then implies $\big(\bigwedge^* B_l\alpha\big) \wedge \boldsymbol{\tau}^m = \big(\bigwedge^* D\alpha\big) \wedge \boldsymbol{\tau}^m$. Combining these formulas for $\bigwedge^* B_l$, we thus obtain

$$\begin{aligned} \textstyle\bigwedge^* B_l(\alpha \wedge \boldsymbol{\tau}^m) = \big(\bigwedge^* B_l\alpha\big) \wedge \big(\bigwedge^* B_l\boldsymbol{\tau}^m\big) &= \det(L)\big(\textstyle\bigwedge^* B_l\alpha\big) \wedge \boldsymbol{\tau}^m \\ &= \det(L)\big(\textstyle\bigwedge^* D\alpha\big) \wedge \boldsymbol{\tau}^m. \end{aligned} \tag{65}$$

Continuing with the action of B_u on forms, we note that $\bigwedge^* B_u\eta = \eta$ for any $\eta \in \bigwedge^* U$ since B_u is the identity on U. Furthermore, we have that

$$\textstyle\bigwedge^* B_u\boldsymbol{\tau}^m = (w_1 + Gw_1) \wedge \cdots \wedge (w_n + Gw_n) = \boldsymbol{\tau}^m + \displaystyle\sum_{S \in \mathcal{P}_m : |S| < m} \psi_S \wedge \boldsymbol{\tau}^S,$$

where $\psi_S \in \bigwedge^* U$, since all $Gw_i \in U$. As a result, we have

$$\textstyle\bigwedge^* B_u(\eta \wedge \boldsymbol{\tau}^m) = \eta \wedge \bigwedge^* B_u\boldsymbol{\tau}^m = \eta \wedge \bigwedge^* B_u\boldsymbol{\tau}^m = \eta \wedge \boldsymbol{\tau}^m + \rho',$$

where ρ' is the summation of terms $\eta \wedge \psi_S \wedge \boldsymbol{\tau}^S$ with $|S| < m$ and $\eta \wedge \psi_S \in \bigwedge^* U$. Since all of these terms are, by (53), in the kernel of χ_m, we find

$$\chi_m\big(\textstyle\bigwedge^* B_u(\eta \wedge \boldsymbol{\tau}^m)\big) = \eta. \tag{66}$$

Combining the actions in (65) and (66) for $\eta = \det(L)\bigwedge^* D\alpha$, we thus find

$$\begin{aligned} \chi_m\big(\textstyle\bigwedge^* B_u B_l(\alpha \wedge \boldsymbol{\tau}^m)\big) &= \chi_m\big(\textstyle\bigwedge^* B_u\big(\bigwedge^* B_l(\alpha \wedge \boldsymbol{\tau}^m)\big)\big) \\ &= \chi_m\big(\textstyle\bigwedge^* B_u\big(\det(L)\big(\bigwedge^* D\alpha\big)\big) \wedge \boldsymbol{\tau}^m\big) \\ &= \det(L)\big(\textstyle\bigwedge^* D\alpha\big), \end{aligned} \tag{67}$$

which is the desired form and thus completes the proof. □

The next lemma extends this result to expressions as those in Lemma 13.

Lemma 15 *Let B, L, D, and α be as in Lemma 14 above. Then*

$$\chi_m\big(\big(\textstyle\bigwedge^*(\mathbb{1} - B)\alpha\big) \wedge \big(\bigwedge^* B\boldsymbol{\tau}^m\big)\big) = \det(L)\textstyle\bigwedge^*(\mathbb{1} - D)\alpha. \tag{68}$$

Proof The calculations for this proof are very similar to those of Lemma 13. We may assume $\alpha = x_1 \wedge \cdots \wedge x_k$ for independent generators $x_i \in U$. As before, denote $\alpha_S = x_{i_1} \wedge \cdots \wedge x_{i_p}$ for $S = \{i_1, \ldots, i_p\} \in \mathcal{P}_k$ with $i_1 < \cdots < i_p$, so

$$\alpha = \sigma'_S \alpha_{S^c} \wedge \alpha_S, \tag{69}$$

where σ'_S is the signature of the respective shuffle permutation. We thus obtain, by a calculation analogous to that in (59), that

$$\textstyle\bigwedge^*(\mathbb{1} - B)\alpha = \sum_{S \in \mathcal{P}_k} \sigma'_S (-1)^{|S|} \alpha_{S^c} \wedge \big(\bigwedge^* B \alpha_S\big). \tag{70}$$

Forming the wedge product of this expression with $\bigwedge^* B \boldsymbol{\tau}^m$, we thus obtain

$$\big(\textstyle\bigwedge^*(\mathbb{1} - B)\alpha\big) \wedge \big(\bigwedge^* B \boldsymbol{\tau}^m\big) = \sum_{S \in \mathcal{P}_k} \sigma'_S (-1)^{|S|} \alpha_{S^c} \wedge \big(\bigwedge^* B(\alpha_S \wedge \boldsymbol{\tau}^m)\big).$$

The remainder of the proof is a computation:

$$\begin{aligned}
\text{Left-hand side of (68)} &= \sum_{S \in \mathcal{P}_k} \sigma'_S (-1)^{|S|} \boldsymbol{\chi}_m\big(\alpha_{S^c} \wedge \big(\textstyle\bigwedge^* B(\alpha_S \wedge \boldsymbol{\tau}^m)\big)\big) \\
\text{(by (54))} \quad &= \sum_{S \in \mathcal{P}_k} \sigma'_S (-1)^{|S|} \alpha_{S^c} \wedge \boldsymbol{\chi}_m\big(\textstyle\bigwedge^* B(\alpha_S \wedge \boldsymbol{\tau}^m)\big) \\
\text{(by Lemma 14)} \quad &= \det(L) \sum_{S \in \mathcal{P}_k} \sigma'_S (-1)^{|S|} \alpha_{S^c} \wedge \textstyle\bigwedge^* D \alpha_S \\
&= \det(L) \textstyle\bigwedge^*(\mathbb{1} - D)\alpha,
\end{aligned} \tag{71}$$

where the last step is analogous to (70). □

Substituting $B = \mathbb{1} - A$, we combine this with Lemma 13 to obtain the following.

Corollary 16 *Suppose an endomorphism $A \in \operatorname{End}(U \oplus M_m)$ is of the form*

$$A = \mathbb{1}_{n+m} - \begin{bmatrix} \mathbb{1}_n & G \\ 0 & \mathbb{1}_m \end{bmatrix} \begin{bmatrix} D & 0 \\ K & L \end{bmatrix}, \tag{72}$$

with L invertible. Then we have

$$\operatorname{Str}_n^{n+m}\big(\textstyle\bigwedge^* A\big) = \det(L) \bigwedge^*(\mathbb{1} - D). \tag{73}$$

4.3 Proof of main results and example

Proof of Theorem 1 As noted in Section 3.3 it suffices to prove the relation in (52). For a string link $T\colon \iota^n \to \iota^n$ presented by the closure braid $b \in B_{n+m}$, consider the Burau matrix $\mathcal{B}_n(b)$ with block form as in (16).

In order to evaluate $\mathrm{Str}_n^{n+m}\big(\bigwedge^* \mathcal{B}_n(b)\big)$ used in the definition of $\mathcal{Y}_{n,k}(T,b)$, we apply Corollary 16 using $A = \mathcal{B}_n(b)$ and $U = M_n$. The condition in (72) then translates to the set of block conditions $Z = -K$, $Q = \mathbb{1} - L$, $Y = -GL$, and $X = \mathbb{1} - (D + GK) = \mathbb{1} - H$.

We note that, by Proposition 7, the block $L = \mathbb{1} - Q$ is indeed invertible. Moreover, $\mathbb{1} - D = X + GK = X - GZ = X + YL^{-1}Z = X + Y(\mathbb{1} - Q)^{-1}Z = \mathcal{R}_n(T)$, also by Proposition 7. This implies, by (73), that

$$\mathrm{Str}_n^{n+m}\big(\textstyle\bigwedge^* \mathcal{B}_n(b)\big) = \det(\mathbb{1} - Q)\textstyle\bigwedge^* \mathcal{R}_n(T). \tag{74}$$

From (49) we thus find, by restriction to k–forms, that

$$\mathcal{Y}_{n,k}(T,b) = \det(\mathbb{1} - Q)\textstyle\bigwedge^k \mathcal{R}_n(T). \tag{75}$$

In particular, $\mathcal{Y}_{n,0}(T,b) = \det(\mathbb{1} - Q)$. Relation (52) now follows immediately, completing the proof of Theorem 1. □

Proof of Corollary 2 The $k \times k$ minors of $\mathcal{W}_{n,1}$ are just the matrix elements of $\bigwedge^k \mathcal{W}_{n,1}(T)$, which is, by (12), up to relabeling and rescaling of basis, the same as $\bigwedge^k \big(\mathcal{W}_{n,0}(T)\mathcal{R}_n(T)\big) = \mathcal{W}_{n,0}(T)^k \bigwedge^k \mathcal{R}_n(T)$. At the same time, $\mathcal{W}_{n,0}(T)\bigwedge^k \mathcal{R}_n(T)$ is, again up to relabeling and rescaling of basis, the same as $\mathcal{W}_{n,k}(T)$ by Theorem 1. Hence, up to permutations and multiplications by units in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$, the matrix elements of $\bigwedge^k \mathcal{W}_{n,1}(T)$ are the same as those of $\mathcal{W}_{n,0}(T)^{k-1}\mathcal{W}_{n,k}(T)$. Since $\mathcal{W}_{n,k}(T)$ is a matrix over $\mathbb{Z}[t^{1/2}, t^{-1/2}]$, this implies the assertion. □

As an example, consider again the string link $S\colon \iota^2 \to \iota^2$ from Figure 1. The Ohtsuki's tangle functor can be readily computed, for example, by using skein relations $R - R^{-1} = (t^{-1/2} - t^{1/2})\mathbb{1}$ and the fact that isolated components render a diagram zero. Organized by graded components and using the basis $\{e_0 \otimes e_1, e_1 \otimes e_0\}$ for $W_{2,1}$, we obtain

$$\begin{aligned} \mathcal{V}(S) &= \mathcal{W}_{2,0}(S) \oplus \mathcal{W}_{2,1}(S) \oplus \mathcal{W}_{2,2}(S) \\ &= (2 - t) \oplus \begin{bmatrix} 3 - t - \bar{t} & t^{-1/2} - t^{1/2} \\ t^{-1/2} - t^{1/2} & 1 \end{bmatrix} \oplus (2 - \bar{t}). \end{aligned} \tag{76}$$

We find from (35) that $\mathcal{I}_2\colon W_{2,1} \to \bigwedge^1 M_2 = M_2$ is given by $\mathcal{I}_2(e_0 \otimes e_1) = t v_2$ and $\mathcal{I}_2(e_1 \otimes e_0) = t^{1/2} v_1$, so, in the basis $\{v_1, v_2\}$ for M_2, we obtain

$$\mathcal{I}_2 \mathcal{V}(S) \mathcal{I}_2^{-1} = (2-t) \oplus \begin{bmatrix} 1 & \bar{t}-1 \\ 1-t & 3-t-\bar{t} \end{bmatrix} \oplus (2-\bar{t}). \tag{77}$$

Given that $\mathcal{W}_{2,0}(S) = (2-t)$, we immediately have that $\mathcal{W}_{2,1/0}(S)$ is, up to basis change, the same as $\bigwedge^1 \mathcal{R}_2(S) = \mathcal{R}_2(S)$ as in (5). Moreover, we readily compute

$$\textstyle\bigwedge^2 \mathcal{R}_2(S) = \det(\mathcal{R}_2(S)) = \dfrac{2-\bar{t}}{2-t} = \mathcal{W}_{2,2/0}(S), \tag{78}$$

thus verifying the statement of Theorem 1 for all k in this example.

4.4 Concluding comments and outlook

We begin with remarks on the probabilistic motivation initially given in [4] and expanded upon in [7]. As noted earlier, to have a true stochastic matrix, we need to confine ourselves to diagrams T with only positive crossings and $t \in [0,1]$ in order to have probabilities in $[0,1]$ (or all negative crossings and $t^{-1} \in (0,1]$).

Recall from the comments following Corollary 8 that the unique equilibrium state $\boldsymbol{p}_n = c\boldsymbol{v}_n$ of $\mathcal{R}_n(T)$ for nonseparable T is fixed and thus contains no topological information about T. The size of the space of equilibrium states for general T, however, may be used to provide a measure of separability for a given string link and thus may be of interest for further study.

The existence of the fixed equilibrium state and the interpretation of $\mathcal{R}_n(T)$ as a stochastic matrix rely on the existence of left and right eigenvectors with eigenvalue 1 as in Corollary 8. From the point of view of the tangle functor, their existence is actually a basic consequence of the underlying representation theory of $U_{-1}(\mathfrak{sl}_2)$ as outlined in the following.

In particular, we have, by equivariance and accounting for degrees, for the operators defined in (29), that $\widetilde{E}_n \mathcal{W}_{n,k+1}(T) = \mathcal{W}_{n,k}(T)\widetilde{E}_n$ and $\widetilde{F}_n \mathcal{W}_{n,k}(T) = \mathcal{W}_{n,k+1}(T)\widetilde{F}_n$. Now, let $\mathbf{1} = e_0^{\otimes n}$ be the generating vector of $W_{n,0}$ and $\mathbf{1}^*$ the respective dual vector in $W_{n,0}^*$. The intertwining relations imply, for $k=0$, that $\mathcal{W}_{n,1}(T)(\widetilde{F}_n\mathbf{1}) = \widetilde{F}_n \mathcal{W}_{n,0}(T)\mathbf{1} = \mathcal{W}_{n,0}(T)(\widetilde{F}_n\mathbf{1})$, so $\mathcal{W}_{n,1/0}(T)(\widetilde{F}_n\mathbf{1}) = (\widetilde{F}_n\mathbf{1})$. From Theorem 1, we thus have that $\mathcal{I}_n\widetilde{F}_n\mathbf{1}$ is also an eigenvector, with eigenvalue one, for $\mathcal{R}_n(T)$.

Using (47), we find that this is, by $\mathcal{I}_n\widetilde{F}_n\mathbf{1} = \breve{F}_n\mathcal{I}_n\mathbf{1} = \breve{F}_n 1 = t^{-1/2}\boldsymbol{v}_n$, indeed proportional to the eigenvector found directly in Corollary 8. A similar equivariance argument shows that a left eigenvector for $\mathcal{R}_n(T)$ is given by $\mathbf{1}^*\widetilde{E}_n\mathcal{I}_n^{-1} = \mathbf{1}^*\mathcal{I}_n^{-1}\breve{E}_n = \breve{E}_n\big|_{M_n} = t^{-n/2}\boldsymbol{e}_n$ by (48). We remark also that the *reduced* Burau representation can be viewed in this context as the further restriction of M_n to the "highest" weight space, $\ker(\widetilde{E}_n)$.

Another question not treated here, but of possible further interest, is that of duality relations between degree k and $(n-k)$ representations. The existence of a duality principle is strongly suggested by the obvious symmetry in the example in (76), but also basic algebraic properties of the R–matrix used for $\mathcal{V}$ and its relations to exterior algebras. From a topological point of view, such a principle may be expected via Poincaré duality in local coefficient cohomologies of underlying configuration spaces.

Observe also that the multiplicative function $\mathcal{W}_{n,0}\colon \mathrm{Str}(n) \to \mathbb{Z}[t^{1/2}, t^{-1/2}]$ allows us to define the assignment

$$\mathcal{X}\colon \mathrm{Str}(n) \to \mathbb{Z}^{0,+}, \quad T \mapsto \mathrm{span}(\mathcal{W}_{n,0}(T)), \tag{79}$$

where the span of a Laurent polynomial is given by the difference of its highest and lowest nonvanishing powers in $t^{1/2}$. Clearly, this is an additive function with $\mathcal{X}(T \circ S) = \mathcal{X}(T) + \mathcal{X}(S)$, and it vanishes on braids, that is, the invertible elements of $\mathrm{Str}(n)$.

Further objects of study are thus the size of the kernel of $\mathcal{X}$ beyond the braid group, as well as relations of $\mathcal{X}(T)$ with the number of simple loops in the random walk picture, or other topological properties of string links.

We finally point out the generalization of Ohtsuki's functor indicated by the two-variable R–matrix provided at the end of [10, Section 4.5]. This yields, in the same manner as before, a tangle functor from which we obtain, analogous to (10), a representation

$$p\mathcal{W}_{n,j}\colon \mathrm{pStr}(n) \to \mathrm{End}\big(\mathbb{Z}[t_1^{\pm 1/4}, \dots, t_n^{\pm 1/4}]^{\binom{n}{j}}\big) \tag{80}$$

on the monoid $\mathrm{pStr}(n)$ of *pure* string links. In the tangle functor picture, the variables t_j label representations of $U_{-1}(\mathfrak{sl}_2)$ and thus fit into the framework of TQFTs, where the t_j are interpreted as "colors" or "charges". Analogous to (11), we can also define quotient representations $p\mathcal{W}_{n,j/0}$.

Extending the construction of the classical Gassner representation of the pure braid groups, Kirk, Livingston, and Wang obtain, in [6], a representation

$$\gamma_n^{KLW}\colon \mathrm{pStr}(n) \to \mathrm{End}(\mathbb{Q}(t_1, \dots, t_n)^n).$$

We thus conclude this article with a conjecture that naturally extends Theorem 1 and for which we expect the proof to follow a similar strategy.

Conjecture 17 *The representation γ_n^{KLW} is equivalent to $p\mathcal{W}_{n,1/0}$.*

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Department of Mathematics, Ohio State University
Columbus, OH 43210, USA

kerler@math.ohio-state.edu, wang.3003@math.ohio-state.edu

Received: 23 February 2015 Revised: 22 May 2015