On intersecting subgroups of Brunnian link groups

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Let $G(L_n)$ be the link group of a Brunnian n–link L_n and R_i be the normal closure of the ith meridian in $G(L_n)$ for $1 \leq i \leq n$. In this article, we show that the intersecting subgroup $R_1 \cap R_2 \cap \cdots \cap R_m$ coincides with the iterated symmetric commutator subgroup $\prod_{\sigma \in \Sigma_m}[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(m)}]$ for $2 \leq m \leq n$ using the techniques of homotopy theory. Moreover, we give a presentation for the intersecting subgroup $R_1 \cap R_2 \cap \cdots \cap R_n$.

1 Introduction

An n–component link $L_n = \{l_1, l_2, \ldots, l_n\}$ is called a *Brunnian link*, named for Hermann Brunn [5], if it is a nontrivial link that becomes trivial when any component is removed. The simplest and best known example of Brunnian link are the Borromean rings. Milnor [25] classified Brunnian links up to link homotopy; he called them almost trivial links before Debrunner [9] named them Brunnian in 1961. The C_n–moves and the finite-type invariants have been studied by Habiro and Meilhan [13; 14; 15; 16] and Miyazawa and Yasuhara [26]. We refer to Mangum and Stanford [23], Meilhan and Yasuhara [24] and Ozawa [27] for more results. An important application of Brunnian links to mathematical chemistry was given by Chichak et al [6].

For an n–link $L_n = \{l_1, l_2, \ldots, l_n\}$, we define the link group $G(L_n)$ as the fundamental group of the link complement $S^3 \setminus L_n$. From the work of Mangum and Stanford [23], Brunnian links are determined by their link complements, so the link group $G(L_n)$ plays an important role in studying a Brunnian link L_n. Let $d_i L_n = \{l_1, l_2, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n\}$ be the $(n-1)$–link obtained by deleting the ith component l_i from the n–link L_n. There is a group homomorphism h_i from $G(L_n)$ to $G(d_i L_n)$, induced by the inclusion of the link complements $S^3 \setminus L_n \hookrightarrow S^3 \setminus d_i L_n$. The kernel of h_i is denoted by R_i. If we let $\langle \alpha_i \rangle$ be the normal closure of the meridian of l_i in $G(L_n)$, then $\langle \alpha_i \rangle = R_i$.

In this paper, we are interested in exploring the intersecting subgroup $R_1 \cap R_2 \cap \cdots \cap R_m$ of a Brunnian link group $G(L_n)$ for $2 \leq m \leq n$. Let β_i be the ith longitude of L_n. Since L_n is Brunnian, the homotopy class $[\beta_i]$ is in $R_1 \cap \cdots \cap R_{i-1} \cap R_{i+1} \cap \cdots \cap R_n$ for each i. For some Brunnian links L_n, for instance the Whitehead link, the homotopy classes of all of the longitudes lie in the full intersecting subgroup $R_1 \cap \cdots \cap R_n$.

Let N and H be subgroups of a group G. Recall that the commutator subgroup $[N, H]$ is defined as the subgroup of G generated by the commutators $[g, h] = g^{-1}h^{-1}gh$ for $g \in N$ and $h \in H$. If N and H are normal subgroups, then $[N, H]$ is contained in the intersecting subgroup $N \cap H$. Hence, for each permutation $\sigma \in \Sigma_m$, the iterated commutator subgroup

$$[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(m)}] \leq R_1 \cap R_2 \cap \cdots \cap R_m.$$

It follows that the *symmetric commutator subgroup*

$$[R_1, R_2, \ldots, R_m]_S = \prod_{\sigma \in \Sigma_m} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(m)}]$$

is a subgroup of $R_1 \cap R_2 \cap \cdots \cap R_m$. Using a recent result of Ellis and Mikhailov [10] together with the techniques of homotopy theory, we are able to determine the intersecting subgroup $R_1 \cap R_2 \cap \cdots \cap R_m$ for $2 \leq m \leq n$.

Theorem 1.1 *Suppose that $L_n = \{l_1, l_2, \ldots, l_n\}$ is an n–component Brunnian link and $G(L_n)$ is the link group of L_n. Let R_i be the normal closure of the ith meridian in $G(L_n)$. Then*

$$\bigcap_{i=1}^{m} R_i = [R_1, \ldots, R_m]_S$$

for $2 \leq m \leq n$.

Let $R_1, R_2, \ldots, R_m$ be subgroups of a group G. There is a Brown–Loday construction of non-commutative tensor product $T(R_1, R_2, \ldots, R_m)$ defined in terms of generator-relation system [4]. The construction of $T(R_1, R_2, \ldots, R_m)$ will be recalled in the next section.

Our second result gives a presentation of the intersecting subgroup $R_1 \cap \cdots \cap R_m$ in terms of the Brown–Loday construction.

Theorem 1.2 *Given the same assumptions of Theorem 1.1, we have:*

(1) *There is a short exact sequence of groups*

$$\bigoplus^{n-1} \mathbb{Z} \rightarrowtail T(R_1, R_2, \ldots, R_n) \twoheadrightarrow \bigcap_{i=1}^{n} R_i.$$

(2) *There is a short exact sequence of groups*

$$\bigoplus^{\infty} \mathbb{Z} \rightarrowtail T(R_1, R_2, \ldots, R_m) \twoheadrightarrow \bigcap_{i=1}^{m} R_i$$

for $2 \le m < n$.

Remark The combinatorial problem of determining the intersecting subgroups is technically difficult in general, and related to the longstanding unsolved classical Whitehead asphericity question in low-dimensional topology; see Bogley [3]. The question on the determination of Brunnian braids answered in Bardakov, Mikhailov, Vershinin and Wu [1], Gurzo [12], Johnson [17], Levinson [18; 19] and Li and Wu [20] concerns the intersecting subgroups of braid groups with connections to homotopy theory; see Berrick, Cohen, Wong and Wu [2] and Cohen and Wu [7; 8]. As a development of Brown and Loday's program [4], recent progress made by Ellis and Mikhailov gives a possibility to determine certain intersecting subgroups in terms of commutator subgroups with obstructions given by homotopy groups. It was then discovered by Fang, Lei and Wu [11] and Li and Wu [20] that the quotient group $(R_1 \cap \cdots \cap R_n)/[R_1, \ldots, R_n]_S$ is isomorphic to the homotopy group $\pi_n(S^3)$ for a strongly nonsplittable n–link L, where R_i is the normal closure of the ith meridian and a link is called *strongly nonsplittable* if any of its nonempty sublink is nonsplittable. In other words, the determination of the intersecting subgroup of strongly nonsplittable links L is equally difficult to the determination of the homotopy group $\pi_n(S^3)$, even in the simplest case that L is a Hopf n–link. Since the Brunnian links are not strongly nonsplittable, we cannot apply the results of [11; 20] to answering the question of the intersecting subgroup of Brunnian links. However, we observe in Claim 2.2 that the connectivity hypothesis Ellis and Mikhailov [10] is satisfied for Brunnian link groups. This gives us the chance to determine the intersecting subgroup of Brunnian link groups in this article. As an application to Brunnian link groups, by Theorem 1.1 and Theorem 1.2(1) there is an exact sequence

$$1 \to \bigoplus^{n-1} \mathbb{Z} \to T(R_1, R_2, \ldots, R_n) \to G(L_n) \to F_{n-1}^{\times n},$$

where F_{n-1} is the free group of rank $n-1$. This gives a connection between Brown and Loday's theory on non-commutative tensor products and Brunnian links.

We should point out that by Theorem 1.1, the connectivity hypothesis of Ellis and Mikhailov [10] is also satisfied for *almost Brunnian link groups* along the lines of the proof of Lemma 2.1, where a link is called *almost Brunnian* if it becomes Brunnian after removing any of its components. Of course, the full determination of the intersecting subgroup of almost Brunnian link groups will technically depend on the determination

of the homotopy type of the homotopy colimit of a cubical diagram, as we will see in the case of Brunnian link groups in this article. The procedure can be continued for more general links until the intersecting subgroup differs from the symmetric commutator subgroup with obstruction given by the homotopy group of the homotopy colimit. This article highlights the homotopy theoretic methodology for determining the intersecting subgroup of link groups.

In the next section, we will lay out the necessary preliminaries for proving our theorems. In Section 3, we'll study the homotopy type of the space $X(L_n)^m$. The proofs of Theorems 1.1 and 1.2 will be given in Section 4.

2 Preliminaries

In this section we will review some results which we need to prove our main theorems, including symmetric commutator subgroups, colimits of classifying spaces and homotopy groups of homotopy colimits.

2A Symmetric commutator subgroups

Let G be a group with subgroups R_j for $1 \le j \le n$. The concept of the symmetric commutator subgroup of $R_1, R_2, \ldots, R_n$, denoted by $[R_1, R_2, \ldots, R_n]_S$, was introduced by Li and Wu [20] with the equation

$$(1) \qquad [R_1, R_2, \ldots, R_n]_S = \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}],$$

where Σ_n is the nth symmetric group and $[[R_{\sigma(1)}, R_{\sigma(2)}], \ldots, R_{\sigma(n)}]$ is the subgroup generated by the left iterated commutators $[[[g_1, g_2], g_3], \ldots, g_n]$ with $g_i \in R_{\sigma(i)}$.

In the same paper, the following theorem was also shown.

Theorem [20, Theorem 1.2] *Let R_j be any normal subgroup of a group G with $1 \le j \le n$. Then*

$$(2) \qquad [R_1, \ldots, R_n]_S = \prod_{\sigma \in \Sigma_{n-1}} [[R_1, R_{\sigma(2)}], \ldots, R_{\sigma(n)}],$$

where Σ_{n-1} acts on $\{2, 3, \ldots, n\}$.

A bracket arrangement of weight n in a group G is given by Magnus, Karrass and Solitar [22] as a map $\beta^n \colon G^n \to G$ defined inductively by

$$(3) \qquad \beta^1 = \mathrm{id}_G, \quad \beta^2(g_1, g_2) = [g_1, g_2],$$

where g_1 and g_2 are arbitrary elements in G and $[g_1, g_2]$ is the commutator of g_1 and g_2. Inductively, suppose that the bracket arrangements of weight k, β^k, are defined for $k = 1, 2, \ldots, n-1$. Then the bracket arrangements of weight n, β^n, is defined by the composition

$$(4) \qquad G^n = G^k \times G^{n-k} \xrightarrow{\beta^k \times \beta^{n-k}} G \times G \xrightarrow{\beta^2} G$$

for any β^k and β^{n-k}.

Closely related to the symmetric commutator subgroup is the fat commutator subgroup. For a group G with a sequence of subgroups $R_j, 1 \leq j \leq n$. The fat commutator subgroup $[[R_1, \ldots, R_n]]$ is defined to be the subgroup of G generated by all of the commutators

$$(5) \qquad \beta^t(g_{i_1}, \ldots, g_{i_t}),$$

where

(a) $1 \leq i_s \leq n$,

(b) $\{i_1, \ldots, i_t\} = \{1, \ldots, n\}$, ie each integer in $\{1, 2, \ldots, n\}$ appears as at least one of the integers i_s,

(c) $g_j \in R_j$,

(d) β^t runs over all of the bracket arrangements of weight $t, t \geq n$.

From the definition, it is clear that the symmetric commutator subgroup is a subgroup of the fat commutator subgroup. Moreover, the following result was also shown by Li and Wu [20], and plays an important role in our proof of Theorem 1.1.

Theorem [20, Theorem 1.1] *Let R_j be any normal subgroup of a group G with $1 \leq j \leq n$. Then*

$$(6) \qquad [[R_1, R_2, \ldots, R_n]] = [R_1, \ldots, R_n]_S.$$

This theorem tells us that the fat commutator subgroup is in fact the same as the symmetric commutator subgroup.

We'll also need another result of Wu.

Theorem [28, Corollary 3.5] *Let $G = F(S)$ be a free group generated by S and $F(T_i)$ be subgroups of G freely generated by $T_i \subset S$ for $1 \leq i \leq k$. Let R_i be the kernel of the projection homomorphism $\pi_i \colon G \to F(T_i)$. Then*

$$\bigcap_{i=1}^{k} R_i = [[R_1, R_2, \dots, R_k]].$$

Commutator subgroups have many important applications, especially to homotopy groups and geometric groups. In [20], Li and Wu used the commutator subgroups to give a connection between links and homotopy groups. In the present paper, we apply the commutator subgroups to study the homotopy groups of the homotopy colimits of classifying spaces for Brunnian link groups.

2B Homotopy colimits of classifying spaces

In this section, we review some facts about the colimits of classifying spaces. Our terminology here is coincident with that of Ellis and Mikhailov [10].

Roughly speaking, a classifying space BG of a topological group G is a topological space that is the quotient of a weakly contractible space (ie a topological space whose homotopy groups are all trivial) by a free action of G. The existence of a classifying space for discrete groups is a classical result. For a discrete group G, the classifying space BG is a path-connected topological space whose fundamental group is isomorphic to G and all higher homotopy groups are trivial, ie BG is a $K(G, 1)$–space. Of course, for a group G, the classifying space is not unique. It was shown that if X is a classifying space for G and Y is a classifying space for K, then $X \times Y$ is a classifying space for $G \times K$. Some of the simple examples of classifying spaces are:

(1) The circle S^1 is a classifying space for the infinite cyclic group $\mathbb{Z}$.

(2) The n–torus T^n is a classifying space for the free abelian group of rank n, $\mathbb{Z}^n$.

(3) The wedge of n circles $\bigvee^n S^1$ is a classifying space for the free group of rank n. Suppose that L_n is an n–Brunnian link and R_i is the normal closure of the nth meridian in $G(L_n)$. For each proper subset $I \subset \{1, 2, \dots, n\}$, $G(L_n)/\prod_I R_i$ is a free group of rank $n - |I|$, so $B(G(L_n)/\prod_I R_i) = \bigvee^{n-|I|} S^1$.

(4) It has been shown [11, Theorem 2.1] that a link in S^3 is nonsplittable if and only if the link complement is a $K(\pi, 1)$–space. So for an n–Brunnian link L_n with link group $G(L_n)$, $BG(L_n) = S^3 \setminus L_n$ is a classifying space.

We are interested in the homotopy colimits of classifying spaces. In order to have an intuitive image of homotopy colimits, we describe the homotopy push-out (colimit) of a diagram of spaces

$$Z \xleftarrow{\ g\ } X \xrightarrow{\ f\ } Y.$$

We want the homotopy push-out to preserve the homotopy type of X. So, we will first thicken X to get $X \times [0, 1]$ and then identify $X \simeq X \times \{0\}$ with $f(X) \subset Y$ and $X \simeq X \times \{1\}$ with $g(X) \subset Z$. Therefore, the homotopy push-out is just the quotient space

$$Q = Y \amalg (X \times [0, 1]) \amalg Z / \sim,$$

where $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$.

In a more general sense than homotopy push-out, the homotopy colimit will be a functor from the category of diagrams of spaces into the category of topological spaces $\mathcal{T}$. Suppose that $\mathcal{D}$ is a small category, ie a category where both the class of objects and class of morphisms are actually just sets. Denote by $\mathcal{T}^{\mathcal{D}}$ the category of functors $F \colon \mathcal{D} \to \mathcal{T}$ and natural transformations. That is, $\mathcal{T}^{\mathcal{D}}$ is the category of diagrams of spaces with the shape of $\mathcal{D}$. Our homotopy colimit is a functor

$$\operatorname{hocolim}_{\mathcal{D}} \colon \mathcal{T}^{\mathcal{D}} \to \mathcal{T}$$

which satisfies the following properties:

(1) (Homotopy invariance property) Suppose that $F, F' \colon \mathcal{D} \to \mathcal{T}$ are two functors and there is a natural transformation $\tau \colon F \to F'$ such that $\tau_d \colon F(d) \to F'(d)$ is a homotopy equivalence for all $d \in \mathcal{D}$, then

$$\operatorname{hocolim}(F) \simeq \operatorname{hocolim}(F').$$

(2) If all the maps between spaces are cofibrations, then the homotopy colimit is coincident with the usual colimit, ie if $F(\alpha) \colon F(d) \to F(d')$ is a cofibration for every $\alpha \colon d \to d'$ in $\mathcal{D}$, then

$$\operatorname{hocolim}(F) \simeq \operatorname{colim}(F).$$

What inspired us was the work on homotopy colimits given by cubical diagrams by Ellis and Mikhailov [10]. In the next section, we will review some of their results and settle down some important lemmas we need later for the proofs of our theorems.

2C Homotopy groups of homotopy colimits

Let G be a group. We recall the definition of connectedness for m–tuples of normal subgroups $(R_1, R_2, \ldots, R_m)$ from [10].

Definition The m–tuple $(R_1, R_2, \ldots, R_m)$ is *connected* if either $m \leq 2$ or $m \geq 3$ and for all subsets $I, J \subset \{1, 2, \ldots, m\}$ with $|I| \geq 2, |J| \geq 1$ we have

$$(7) \qquad \left(\bigcap_{i \in I} R_i \right) \prod_{j \in J} R_j = \bigcap_{i \in I} \left(R_i \prod_{j \in J} R_j \right).$$

In this paper, we study the homotopy colimit X of the spaces $B(G/\prod_{i \in I} R_i)$, where I ranges over all strict subsets $I \subsetneq \{1, 2, \ldots, m\}$. The following theorem of Ellis and Mikhailov is important for our proofs.

Theorem [10, Theorem 1] *If the $(m-1)$–tuple $(R_1, \ldots, \hat{R}_i, \ldots, R_m)$ is connected for each $1 \leq i \leq m$, then the homotopy colimit X satisfies*

$$(8) \qquad \pi_m(X) \cong \frac{R_1 \cap \ldots \cap R_m}{\prod_{I \cup J = \{1,2,\ldots,m\}, I \cap J = \varnothing} [\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j]},$$

$$(9) \qquad \pi_{m+1}(X) \cong \ker(\partial \colon T(R_1, \ldots, R_m) \to G).$$

The connectivity condition in this theorem is crucial. The following lemma ensures the connectivity property of the subgroups in our results.

Lemma 2.1 *For a Brunnian link $L_n = \{l_1, l_2, \ldots, l_n\}$ as described above, the k–tuple $(R_{i_1}, \ldots, R_{i_k})$ is connected for each $1 \leq k \leq n-1$ and $i_j \in \{1, \ldots, n\}$.*

Before we prove the lemma, we make the following claim for simplicity.

Claim 2.2 *Let $\pi \colon G \to G/\prod_{j \in J} R_j$; $x \mapsto \bar{x}$ be the quotient map. Equation (7) in the definition of connectivity is equivalent to the equality*

$$(10) \qquad \overline{\bigcap_{i \in I} R_i} = \bigcap_{i \in I} \bar{R}_i.$$

Proof First, suppose that (10) is true, ie

$$\overline{\bigcap_{i \in I} R_i} = \left(\bigcap_{i \in I} R_i \right) \Big/ \prod_{j \in J} R_j = \bigcap_{i \in I} \left(R_i / \prod_{j \in J} R_j \right) = \bigcap_{i \in I} \bar{R}_i.$$

Notice that $(\bigcap_{i\in I} R_i)(\prod_{j\in J} R_j) \subseteq R_i(\prod_{j\in J} R_j)$. Therefore $(\bigcap_{i\in I} R_i)(\prod_{j\in J} R_j) \subseteq \bigcap_{i\in I}(R_i(\prod_{j\in J} R_j))$.

On the other hand, we pick any $x \in \bigcap_{i\in I}(R_i(\prod_{j\in J} R_j))$, ie $x \in R_i(\prod_{j\in J} R_j)$ for any i. The image of x, $\pi(x)$, belongs to $\overline{R_i}$ for all i. So $\pi(x) \in \bigcap_{i\in I} \overline{R_i} = \overline{\bigcap_{i\in I} R_i}$, where the last equality follows from (10). There exists $y \in \bigcap_{i\in I} R_i$ such that $\pi(y) = \pi(x)$. So $\pi(y^{-1}x) = e$, ie $y^{-1}x \in \ker \pi = \prod_{j\in J} R_j$. Then $x = y \cdot y^{-1}x \in (\bigcap_{i\in I} R_i)(\prod_{j\in J} R_j)$. Equation (7) holds.

For the other direction, we suppose (7) is true. $\overline{\bigcap_{i\in I} R_i} \subseteq \overline{R_i}$ for any i, so $\overline{\bigcap_{i\in I} R_i} \subseteq \bigcap_{i\in I} \overline{R_i}$. Now for any $\bar{x} \in \bigcap_{i\in I} \overline{R_i}$, $\bar{x} \in \overline{R_i}$ for any i. Then $x \in \pi^{-1}(\bar{x}) \subset \bigcap_{i\in I}(R_i(\prod_{j\in J} R_j)) = (\bigcap_{i\in I} R_i)(\prod_{j\in J} R_j)$, where the last equality follows from (7). So $\pi(x) \in \overline{\bigcap_{i\in I} R_i}$. Equation (10) holds. $\square$

Proof of Lemma 2.1 For the cases that $n \leq 3$, the proofs are trivial. So without loss of generality, we assume $n > 3$, $I = \{1, \ldots, n-2\}$ and $J = \{n-1\}$. For other cases, the proofs are similar.

Suppose that $\overline{G} = G(L_n)/\prod_{j\in J} R_j$. We notice that $\overline{G}$ is a free group generated by $\{\overline{\alpha_i} = \alpha_i \mid i \notin J\}$ since J is nonempty. So $\overline{G} = F(\overline{\alpha_1}, \ldots, \overline{\alpha_{n-2}}, \overline{\alpha_n}) = F(\alpha_1, \ldots, \alpha_{n-2}, \alpha_n)$. There is a group homomorphism

$$F(\overline{\alpha_1}, \ldots, \overline{\alpha_{n-2}}) \to F(\overline{\alpha_1}, \ldots, \overline{\alpha_{i-1}}, \overline{\alpha_{i+1}}, \ldots \overline{\alpha_{n-2}}),$$

so that $\overline{R_i} = \langle \overline{\alpha_i} \rangle$ is the kernel. By [1, Lemma 5.1], we have $\overline{R_1} \cap \cdots \cap \overline{R_{n-2}} = [\overline{R_1}, \ldots, \overline{R_{n-2}}]_S$. We also notice that $[R_1, \ldots, R_{n-2}]_S \leq \bigcap_{k=1}^{n-2} R_k$. Now, if we think of the quotient map

$$G(L_n) \xrightarrow{q} \overline{G},$$

we have

$$(11) \qquad \overline{\bigcap_{k\in I} R_k} = q\left(\bigcap_{k\in I} R_k\right) \geq q([R_1, R_2, \ldots, R_{n-2}]_S)$$
$$= [\overline{R_1}, \overline{R_2}, \ldots, \overline{R_{n-2}}]_S = \bigcap_{k\in I} \overline{R_k}.$$

The other direction is obvious. $\square$

The group $T(R_1, R_2, \ldots, R_m)$ in [10, Theorem 1] is generated by symbols

$$a \otimes_{A,B} b,$$

where $A \sqcup B = \{1, 2, \ldots, m\}, a \in R_A, b \in R_B$ (here for any $I \subset \{1, 2, \ldots, m\}, R_I = \bigcap_{i \in I} R_i$), and it satisfies the relations

$$a \otimes_{A,B} b = (b \otimes_{B,A} a)^{-1},$$

$$aa' \otimes_{A,B} b = (^a a' \otimes_{A,B} {}^a b)(a \otimes_{A,B} b),$$

$$(^u[u^{-1}, v] \otimes_{U \cup V, W} {}^u w)(^w[w^{-1}, u] \otimes_{W \cup U, V} {}^w v)(^v[v^{-1}, w] \otimes_{V \cup W, U} {}^v u) = 1,$$

$$(a \otimes_{A,B} b)(a' \otimes_{A',B'} b')(a \otimes_{A,B} b)^{-1} = {}^{[a,b]}a' \otimes_{A',B'} {}^{[a,b]}b'$$

for $A \sqcup B = A' \sqcup B' = \{1, 2, \ldots, m\}, a \in N_A, a' \in N_{A'}, b \in N_B, b' \in N_{B'}, U \sqcup V \sqcup W = \{1, 2, \ldots, m\}, u \in N_U, v \in N_V, w \in N_W$. Here we define ${}^x y = xyx^{-1}$.

For an m–tuple $\Delta \in \{0, 1\}^m$, there is an m–cube of spaces $B = B(G, R_1, \ldots, R_m)$: $\{0, 1\}^m \to$ (spaces) for which

$$B_\Delta = B\left(\frac{G}{\prod_{i \in \Delta} R_i}\right)$$

is the classifying space of the quotient group $G / \prod_{i \in \Delta} R_i$. Here $i \in \Delta$ means the ith coordinate of Δ equals 1. This idea can be easily extended to an m–tuple $\Delta \in \{-1, 0, 1\}^m$. It was shown [4; 21] that each m–cube of spaces corresponds to an m–cube of fibrations $F: \{-1, 0, 1\}^m \to$ (spaces). The homomorphism

$$T(R_1, \ldots, R_m) \to \pi_1(F_{-1,-1,\ldots,-1,0}) \leq G, \quad x \otimes y \mapsto [x, y]$$

has a crossed module structure $\partial: T(R_1, \ldots, R_m) \to G, x \otimes y \mapsto [x, y]$ such that for any $g \in G, {}^g(x \otimes y) = (^g x \otimes {}^g y)$.

3 Homotopy type of $X(L_n)^m$

In this section we will study the homotopy type of the classifying space $X(L_n)^m$. When $n = 2$, it's immediate to see that $X(L_2)^2 \simeq S^3$, because S^3 is the push-out of $K(\pi, 1)$–spaces $S^3 \setminus d_2 L_2 \leftarrow S^3 \setminus L_2 \to S^3 \setminus d_1 L_2$. We choose the case $n = 4$ as an experimental example; the case $n = 3$ is similar. The more general cases will be studied inductively.

3A The experimental example $n = 4$

To get $X(L_4)^2$ we study the cubical diagram

$$
\begin{array}{ccc}
BG(L_4) = S^3 \setminus L_4 & \xrightarrow{h_1^*} & B(G(L_4)/R_1) = \bigvee^3 S^1 \\
{\scriptstyle h_2^*} \swarrow & & \swarrow \\
B(G(L_4)/R_2) = \bigvee^3 S^1 \longrightarrow & X(L_4)^2 &
\end{array}
$$

where h_1^* and h_2^* are induced from the group homomorphisms h_1 and h_2.

In order to get the homotopy type of the push-out $X(L_4)^2$ we need to look into the structure of $B(G(L_4)/R_1)$ and $B(G(L_4)/R_2)$. Let $d_1 L_4 = \{l_2, l_3, l_4\}$ be the 3–link obtained by deleting the first component of L_4. The link complement $S^3 \setminus d_1 L_4$ has same the homotopy type as $(\bigvee^3 S^1) \vee (\bigvee^2 S^2)$ since $d_1 L_4$ is splittable; we refer to [11, Proposition 2.2] for more details. The 2–sphere $S_i^2, i = 2, 3$, is a separating sphere which separates l_i from the other components. Then the classifying space $B(G(L_4)/R_1)$ is obtained by filling in the 2–spheres from $(\bigvee^3 S^1) \vee (\bigvee^2 S^2)$ with 3–balls, ie $B(G(L_4)/R_1) = (S^3 \setminus d_1 L_4) \cup (\bigcup^2 D^3) = ((\bigvee^3 S^1) \vee (\bigvee^2 S^2)) \cup (\bigcup^2 D^3) = \bigvee^3 S^1$. The left side of Figure 1 will show us the idea more clearly.
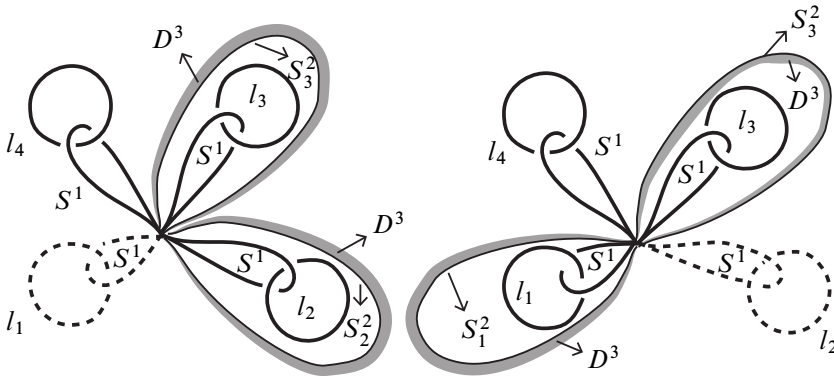

Figure 1: The classifying space $B(G(L_4)/R_1)$ is obtained by filling in the 2–spheres from $(\bigvee^3 S^1) \vee (\bigvee^2 S^2)$ with 3–balls (left); $B(G(L_4)/R_2)$ is obtained similarly (right).
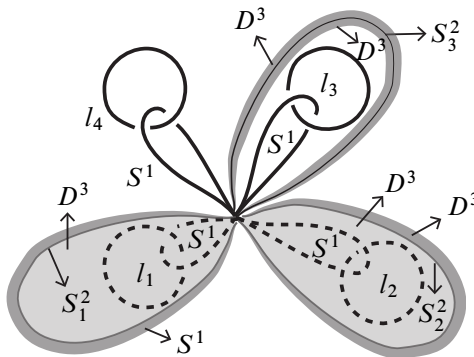

Figure 2: $X(L_4)^2$ is homotopic to $(\bigvee^2 S^1) \vee (\bigvee^3 S^3)$.

By the same argument, we can determine the homotopy type and structure of the space $B(G(L_4)/R_2)$, which is illustrated on the right in Figure 1.

By definition, the homotopy push-out $X(L_4)^2$ is homotopic to $(\bigvee^2 S^1) \vee (\bigvee^3 S^3)$ as indicated in Figure 2.

Actually, the space $X(L_4)^2$ is obtained from the disjoint union of $B(G(L_4)/R_1) = ((\bigvee^3 S^1) \vee (\bigvee^2 S^2)) \cup (\bigcup^2 D^3)$, $B(G(L_4)/R_2) = ((\bigvee^3 S^1) \vee (\bigvee^2 S^2)) \cup (\bigcup^2 D^3)$ and $(S^3 \setminus L_4) \times [0,1]$ by identifying $(S^3 \setminus L_4) \times \{0\}$ (respectively $(S^3 \setminus L_4) \times \{1\}$) with $h_1^*(S^3 \setminus L_4) \subset B(G(L_4)/R_1)$ (respectively $h_2^*(S^3 \setminus L_4) \subset B(G(L_4)/R_2)$). Notice that now the component l_1 (respectively l_2) is filled in, so one side of the 2–sphere S_1^2 (respectively S_2^2) is $(D^3 \setminus l_1) \cup l_1 = D^3$ (respectively $(D^3 \setminus l_2) \cup l_2 = D^3$) while the other side is also a 3–ball, ie there is a 3–sphere.
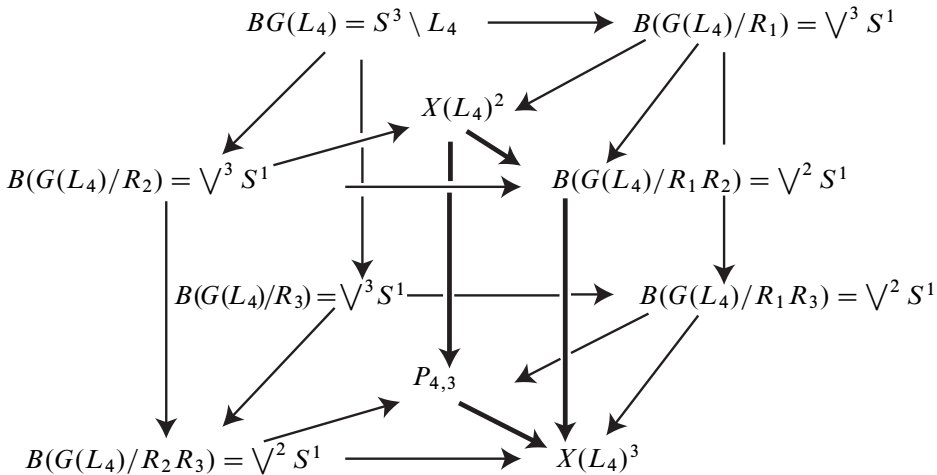
For $m = 3$, we refer to the cubical diagram in Figure 3.

Figure 3

There is a natural inclusion from $X(L_4)^2 = (\bigvee^2 S^1) \vee (\bigvee^3 S^3)$ to $B(G(L_4)/R_1 R_2) = \bigvee^2 S^1$. Actually the space $B(G(L_4)/R_1 R_2)$ can be thought of as obtained from $X(L_4)^2$ by filling in the three S^3 components by 4–balls, ie $B(G(L_4)/R_1 R_2) = X(L_4)^2 \cup (\bigcup^3 D^4) = ((\bigvee^2 S^1) \vee (\bigvee^3 S^3)) \cup (\bigcup^3 D^4) = \bigvee^2 S^1$.

Now, let's study the bottom level of the cubical diagram. We have $B(G(L_4)/R_3) = (S^3 \setminus d_3 L_4) \cup (\bigcup^2 D^3) = \bigvee^3 S^1$. $B(G(L_4)/R_1 R_3)$ (respectively $B(G(L_4)/R_2 R_3)$) can be thought of as obtained from the space $B(G(L_4)/R_3)$ by filling in the l_1 (respectively l_2) component and then capping off the resulting S^3 boundary by a D^4, ie $B(G(L_4)/R_1 R_3) = (S^3 \setminus d_3 L_4) \cup (\bigcup^2 D^3) \cup l_1 \cup D^4 = \bigvee^2 S^1$ (respectively $B(G(L_4)/R_2 R_3) = (S^3 \setminus d_3 L_4) \cup (\bigcup^2 D^3) \cup l_2 \cup D^4 = \bigvee^2 S^1$), as shown in Figure 4.

By the definition of homotopy push-out, the space $P_{4,3}$ is determined by the diagram

$$B(G(L_4)/R_3) = \bigvee^3 S^1 \longrightarrow B(G(L_4)/R_1 R_3) = \bigvee^2 S^1$$

$$B(G(L_4)/R_2 R_3) = \bigvee^2 S^1 \longrightarrow P_{4,3},$$

and $P_{4,3}$ is isomorphic to S^1; see Figure 5. On the other hand, $P_{4,3}$ can also be thought of as obtained from $X(L_4)^2$ by removing the l_3 component and capping off all three S^3 components by 4–balls; see Figure 6.

Then the space $X(L_4)^3$ is determined by the diagram

$$X(L_4)^2 \longrightarrow B(G(L_4)/R_1 R_2) = \bigvee^2 S^1$$

$$P_{4,3} = S^1 \longrightarrow X(L_4)^3,$$

where the space $B(G(L_4)/R_1 R_2)$ is obtained from $X(L_4)^2$ by filling in the three S^3 components by 4–balls, as shown in Figure 7. We conclude that $X(L_4)^3 = S^1 \vee (\bigvee^3 S^4)$.
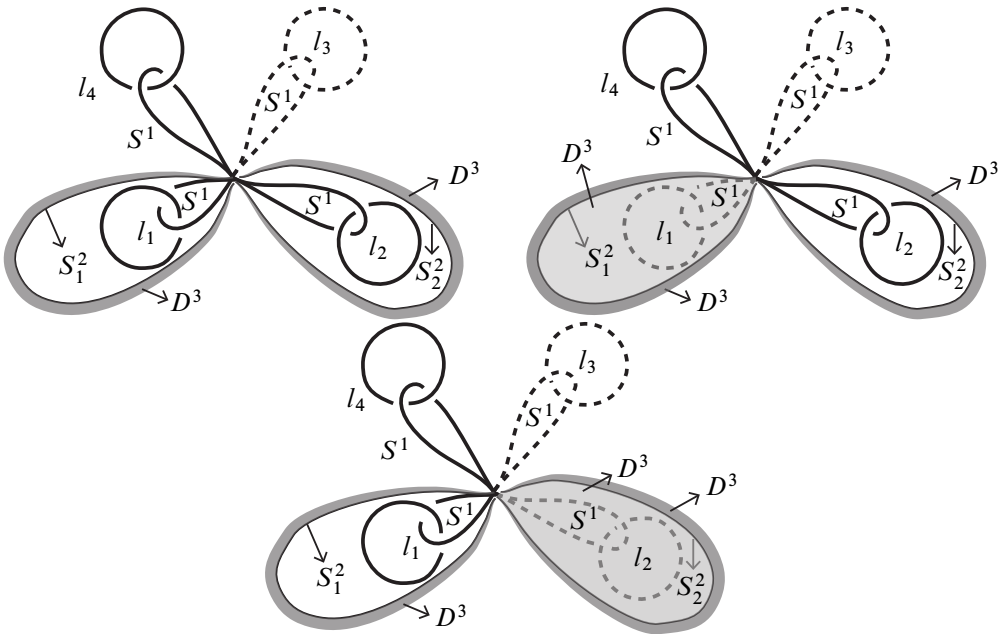

Figure 4: The classifying spaces $B(G(L_4)/R_3)$, $B(G(L_4)/R_1 R_3)$, and $B(G(L_4)/R_2 R_3)$, respectively

The last case left is $X(L_4)^4$, which is determined by the following diagram:

$$X(L_4)^3 = S^1 \vee (\bigvee^3 S^4) \longrightarrow B(G(L_4)/R_1 R_2 R_3) = X(L_4)^3 \cup (\bigcup^3 D^5) = S^1$$

$$P_{4,4} \longrightarrow X(L_4)^4,$$

where the space $B(G(L_4)/R_1 R_2 R_3)$ is obtained from $X(L_4)^3$ by capping off the S^4 components by 5–balls, and the space $P_{4,4}$ is obtained by removing the l_4 component
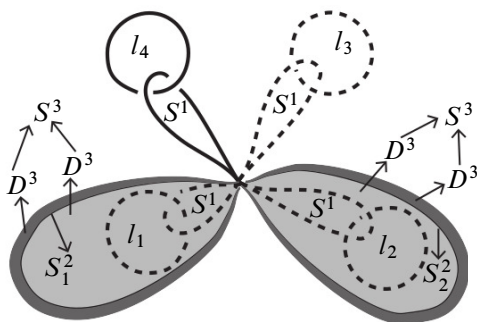

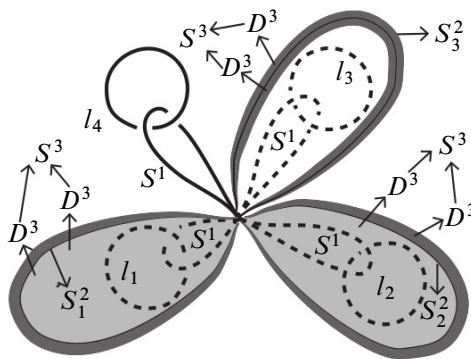Figure 5: The pushout $P_{4,3}$ is isomorphic to S^1.

Figure 6: On the other hand, $P_{4,3}$ can be thought of as obtained from $X(L_4)^2$ by removing the l_3 component and capping off all the three S^3 components by D^4's.
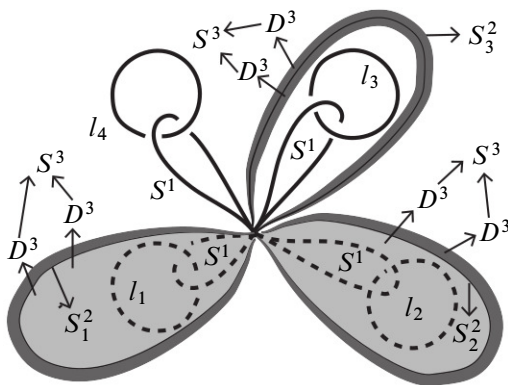

Figure 7: The space $B(G(L_4)/R_1 R_2)$ is obtained from $X(L_4)^2$ by filling in the three S^3 components by D^4's.

from $X(L_4)^3$ and capping off the S^4 components by 5–balls, ie $P_{4,4}$ is isomorphic to a point. So $X(L_4)^4 = \bigvee^3 S^5$.

3B   The general cases

Based on the experimental example we studied in detail in the previous section, we'll sketch the procedure of finding the homotopy types of the homotopy colimits very roughly in this section.

For any $n > 4$ and $2 \leq m \leq n$, we want to show that

$$X(L_n)^m = \left(\bigvee^{n-m} S^1\right) \vee \left(\bigvee^{n-1} S^{m+1}\right),$$

noticing that for $n = 2, 3, 4$ this equality holds. Now, for any $n > 4$, we assume that the spaces $X(L_n)^k, 2 \leq k \leq m-1$ satisfy this equation. By induction, we notice that $X(L_n)^m$ is determined by the diagram

$$
\begin{array}{ccc}
\begin{array}{c} X(L_n)^{m-1} = (\bigvee^{n-m+1} S^1) \\ \vee (\bigvee^{n-1} S^m) \end{array} & \xrightarrow{\hspace{1cm}} & \begin{array}{c} B(G(L_n)/R_1 \cdots R_{m-1}) \\ = X(L_n)^{m-1} \cup (\bigcup^{n-1} D^{m+1}) \\ = \bigvee^{n-m+1} S^1 \end{array} \\
\downarrow & & \downarrow \\
P_{n,m} & \xrightarrow{\hspace{3cm}} & X(L_n)^m,
\end{array}
$$

where $B(G(L_n)/R_1 \cdots R_{m-1})$ can be thought of as obtained from $X(L_n)^{m-1}$ by filling in the S^m components, and $P_{n,m}$ is obtained from $X(L_n)^{m-1}$ by removing the l_m component and capping off all the sphere components with solid balls. Then $P_{n,m} = \bigvee^{n-m} S^1$, and the colimit $X(L_n)^m = (\bigvee^{n-m} S^1) \vee (\bigvee^{n-1} S^{m+1})$.

4   Proofs of theorems

In this section, we first study the nth and $(n+1)^{\text{st}}$ homotopy groups of the homotopy colimits, then the proofs of our theorems are just corollaries.

4A   The homotopy groups of the homotopy colimits

We have already shown that $X(L_n)^m = (\bigvee^{n-m} S^1) \vee (\bigvee^{n-1} S^{m+1})$.

For the case $m = n$, the homotopy groups of $X(L_n)^n$ can be obtained by applying the Hurewicz theorem. In fact, the homology groups of $X(L_n)^n$ satisfy the equalities

$$(12) \qquad H_n(X(L_n)^n) = H_n\left(\bigvee^{n-1} S^{n+1}\right) = \bigoplus^{n-1} H_n(S^{n+1}) = 0,$$

$$(13) \qquad H_{n+1}(X(L_n)^n) = H_{n+1}\left(\bigvee^{n-1} S^{n+1}\right) = \bigoplus^{n-1} H_{n+1}(S^{n+1}) = \bigoplus^{n-1} \mathbb{Z}.$$

So by the Hurewicz theorem, $\pi_n(X(L_n)^n) = 0$ and $\pi_{n+1}(X(L_n)^n) = \bigoplus^{n-1} \mathbb{Z}$.

For the case $m < n$, $X(L_n)^m = (\bigvee^{n-m} S^1) \vee (\bigvee^{n-1} S^{m+1})$. We consider the universal covering space $\widetilde{X}$ of $X(L_n)^m$, which is an infinite tree with every vertex of valency $2(n-m)$, and with $n-1$ $(m+1)$–spheres attached to each vertex. So we obtain

$$(14) \qquad \pi_m(X(L_n)^m) = \pi_m(\widetilde{X}) = 0,$$

$$(15) \qquad \pi_{m+1}(X(L_n)^m) = \pi_{m+1}(\widetilde{X}) = \bigoplus^{\infty} \mathbb{Z}.$$

4B Proof of Theorem 1.1

According to (8),

$$0 = \pi_m(X(L_n)^m) \cong \frac{R_1 \cap \cdots \cap R_m}{\prod_{I \cup J = \{1,2,\ldots,m\}, I \cap J = \varnothing}[\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j]}$$

for all m, ie $R_1 \cap \cdots \cap R_m = \prod_{I \cup J = \{1,2,\ldots,m\}, I \cap J = \varnothing}[\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j]$. Obviously, $[R_1, \ldots, R_m]_S \leq R_1 \cap \cdots \cap R_m$. The following lemma will conclude the proof.

Lemma 4.1 *Based on the assumptions of Theorem 1.1,*

$$[R_1, \ldots, R_m]_S \geq \prod_{I \cup J = \{1,2,\ldots,m\}, I \cap J = \varnothing} \left[\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j\right] = R_1 \cap \cdots \cap R_m.$$

Proof According to [28, Corollary 3.5], $\bigcap_{i \in I} R_i = [[R_{i_1}, \ldots, R_{i_{|I|}}]]$ and $\bigcap_{j \in J} R_j = [[R_{j_1}, \ldots, R_{j_{|J|}}]]$, so

$$\prod_{I \cup J = \{1,2,\ldots,m\}, I \cap J = \varnothing} \left[\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j\right] \leq [[R_1, \ldots, R_m]] = [R_1, \ldots, R_m]_S. \qquad \square$$

4C   Proof of Theorem 1.2

According to (9),

$$\ker(\partial\colon T(R_1,\ldots,R_m)\to G(L_n))=\pi_{m+1}(X(L_n)^m)=\begin{cases}\bigoplus^{n-1}\mathbb{Z} & \text{when } m=n\\ \bigoplus^{\infty}\mathbb{Z} & \text{when } m<n.\end{cases}$$

For all disjoint subsets $A, B \subset \{1, 2, \ldots, m\}$ and any $x \in R_A, y \in R_B$, we have $\partial(x \otimes y) = [x, y] \in [\bigcap_{i \in A} R_i, \bigcap_{j \in B} R_j]$. By the arbitrariness of A and B, we claim that $\mathrm{Im}(\partial) = \prod_{I \cup J = \{1,2,\ldots,m\}, I \cap J = \varnothing}[\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j]$. Further, we notice that there is an exact sequence

$$\ker(\partial) = \pi_{m+1}(X(L_n)^m) \rightarrowtail T(R_1, R_2, \ldots, R_m)$$

$$\twoheadrightarrow \mathrm{Im}(\partial) = \prod_{I \cup J = \{1,2,\ldots,m\}, I \cap J = \varnothing}\left[\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j\right]$$

$$\rightarrowtail \bigcap_{i=1}^{m} R_i \to \mathrm{Coker}(\partial) = \pi_m(X(L_n)^m) = 0.$$

This concludes the proof. □

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