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String homology, and closed geodesics on manifolds which are elliptic spaces

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Let M be a closed, simply connected, smooth manifold. Let $\mathbb{F}_p$ be the finite field with p elements, where $p > 0$ is a prime integer. Suppose that M is an $\mathbb{F}_p$–elliptic space in the sense of Félix, Halperin and Thomas (1991). We prove that if the cohomology algebra $H^*(M, \mathbb{F}_p)$ cannot be generated (as an algebra) by one element, then any Riemannian metric on M has an infinite number of geometrically distinct closed geodesics. The starting point is a classical theorem of Gromoll and Meyer (1969). The proof uses string homology, in particular the spectral sequence of Cohen, Jones and Yan (2004), the main theorem of McCleary (1987), and the structure theorem for elliptic Hopf algebras over $\mathbb{F}_p$ from Félix, Halperin and Thomas (1991).

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1 Introduction

We work over a ground field $\mathbb{F}$ and use $\mathbb{F}$ as the coefficients of homology and cohomology. Our main applications are in the case where this ground field is the finite field $\mathbb{F}_p$ with p elements, where $p > 0$ is a prime integer.

Let $\mathrm{HL}_*(M)$ denote the string homology algebra of a closed, simply connected manifold M. String homology is a graded commutative $\mathbb{F}$–algebra defined as follows. Let LM be the free loop space of M. In [4], Chas and Sullivan define the *string product*

$$H_p(LM) \otimes H_q(LM) \to H_{p+q-n}(LM),$$

where n is the dimension of M. This product is studied from the point of view of homotopy theory in Cohen and Jones [5]. The *string homology algebra* is defined by setting $\mathrm{HL}_s(M) = H_{s+n}(LM)$ and using the string product to define the product. It is proved that this string product makes $\mathrm{HL}_*(M)$ into a graded commutative $\mathbb{F}$–algebra in both [4] and [5].

Our main result about string homology is the following theorem. In the statement, ΩX refers to the based loop space of X.

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Theorem 1.1 *Let M be a simply connected, closed manifold. Suppose there is a constant C and an integer K such that*

$$\sum_{i \le n} \dim H_i(\Omega M; \mathbb{F}_p) \le C n^K.$$

Let K_0 be the minimal exponent which can occur in this bound. Then the string homology algebra $\mathrm{HL}_*(M; \mathbb{F}_p)$ contains a polynomial algebra P over $\mathbb{F}_p$ on K_0 generators and $\mathrm{HL}_*(M; \mathbb{F}_p)$ is a finitely generated free module over P.

If $H_*(\Omega M; \mathbb{F}_p)$ satisfies the growth hypotheses in the statement of this theorem, then we say that $H_*(\Omega M; \mathbb{F}_p)$ has *polynomial growth*. The main application of this theorem is the following result.

Theorem 1.2 *Let M be a simply connected, closed manifold. Suppose $H_*(\Omega M; \mathbb{F}_p)$ has polynomial growth and the algebra $H^*(M; \mathbb{F}_p)$ cannot be generated by one element. Then for any metric on M, there is an infinite number of geometrically distinct closed geodesics on M.*

To obtain this result from Theorem 1.1 we use the Gromoll–Meyer theorem relating closed geodesics and the topology of the free loop space. A metric on M defines a function, the *energy function*, on LM given by

$$\gamma \mapsto \int_{S^1} \langle \gamma'(t), \gamma'(t) \rangle \, dt.$$

If $\gamma : S^1 \to M$ is a closed geodesic parametrised by arc length then γ is a critical point of the energy function, as is the loop γ_n defined by $\gamma_n(z) = \gamma(z^n)$. Furthermore every critical point of the energy function is of the form γ_n, where γ is a closed geodesic parametrised by arc length; see Bott [3].

The circle S^1 acts on LM by rotating loops and the energy function is S^1–invariant. It follows that any closed geodesic γ parametrised by arc length generates an infinite number of critical S^1 orbits of the energy function. In general these orbits will not be isolated, but if there are only a finite number of geometrically distinct closed geodesics these orbits will be isolated.

We use the following terminology for graded vector spaces V. If each V_i is finite-dimensional we say V has *finite type*. If V has finite type then we say it has *finite dimension* if $\dim V_i$ is zero for all but a finite number of i, *infinite dimension* if $\dim V_i$ is non-zero for an infinite number of i, and *doubly infinite dimension* if the sequence of numbers $\dim V_i$ is unbounded. Note that polynomial growth with exponent at least 2 is the same as doubly infinite dimension. Using Morse–Bott theory, Gromoll and Meyer showed in [12] that the relation between critical points of the energy function and closed geodesics leads to the following theorem.

Theorem 1.3 *Let M be a simply connected closed manifold. If $H_*(LM;\mathbb{F})$ has doubly infinite dimension for some field $\mathbb{F}$, then for any metric on M there is an infinite number of geometrically distinct closed geodesics on M.*

If $\pi_1(M)$ is finite, then we can apply this theorem to the universal cover $\widetilde{M}$ of M. If $\pi_1(M)$ is infinite and $\pi_1(M)$ has an infinite number of conjugacy classes, then LM has an infinite number of components. Given a metric on M we can choose a minimiser of the energy function in each component of LM and it follows that this metric has an infinite number of geodesics; see Ballmann, Thorbergsson and Ziller [1]. This leaves the case where $\pi_1(M)$ is infinite but only has a finite number of conjugacy classes. Very little is known about this case; see Bangert and Hingston [2].

In [20] Sullivan and Vigué-Poirrier took up the case where $\mathbb{F}=\mathbb{Q}$ and, as an application of the theory of minimal models in rational homotopy, proved the following theorem.

Theorem 1.4 *Suppose M is a closed, simply connected manifold and the algebra $H^*(M,\mathbb{Q})$ is not generated by one element. Then $H_*(LM,\mathbb{Q})$ is doubly infinite.*

There are other interesting applications of the Gromoll–Meyer theorem in Halperin and Vigué-Poirrier [13] and Ndombol and Thomas [18]. Both these papers assume connectivity hypotheses of the following type: if M is a simply connected closed manifold of dimension n, then there are explicitly given constants $a\neq 0$ and b for which $H_i(M;\mathbb{F})=0$ for $2\leq i\leq r$, where $r\geq an+b$.

A very important ingredient in the proof of Theorem 1.2 is the following theorem from McCleary [14].

Theorem 1.5 *Let X be a simply connected space such that the algebra $H^*(X;\mathbb{F}_p)$ cannot be generated by one element. Then $H_*(\Omega X;\mathbb{F}_p)$ is doubly infinite.*

Indeed the main idea which led to this paper is to use string homology with coefficients in $\mathbb{F}_p$ to convert this theorem into a result about string homology. The first step in this process is to use the spectral sequence of Cohen, Jones and Yan [6] to relate string homology and the homology of the based loop space. The second is to use the structure theorems for elliptic Hopf algebras over $\mathbb{F}_p$ from Félix, Halperin and Thomas [9] to obtain the information about the E_2–term of this spectral sequence required for the proof.

This paper is set out as follows. In Section 2 we deal with those aspects of string homology our main results require. The primary objective in Section 2 is to prove Theorem 2.1. In Section 3 we give applications of Theorem 2.1. For example we explain how this theorem applies to the main examples of McCleary and Ziller [15; 16]. In Section 4 we summarise the results from [9] we need and complete the proofs of the main theorems. Finally in Section 5 we give applications of the main theorem to homogeneous spaces.

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2 String homology

In [6, Theorem 1], it is shown that there is a multiplicative second quadrant spectral sequence $(E_r^{s,t}, d_r^{s,t})$ with

$$d_r^{s,t}\colon E_r^{s,t} \to E_r^{s-r,t+r-1}, \quad E_2^{s,t} = H^{-s}(M) \otimes H_t(\Omega M),$$

and converging to $\mathrm{HL}_*(M)$. We will refer to it as the *CJY spectral sequence*.

Here second quadrant means that $E_r^{s,t}$ is zero if $s > 0$ or $t < 0$. Multiplicative means that each term $E_r^{*,*}$ is a bigraded algebra, d_r is a bigraded derivation of the product, and the E_∞ term of the spectral sequence is the bigraded algebra associated to a filtration of $\mathrm{HL}_*(M)$. The edge homomorphism $h\colon \mathrm{HL}_*(M) \to E_\infty^{0,*} \subseteq H_*(\Omega M)$ is the natural algebra homomorphism $h\colon \mathrm{HL}_*(M) \to H_*(\Omega M)$. This gives us a method of relating the algebras $H_*(\Omega M)$ and $\mathrm{HL}_*(M)$.

The simplest way to construct this spectral sequence is to use the string topology spectrum $\mathcal{S}(M) = LM^{-TM}$ introduced in [6]. The skeletal filtration of M induces a filtration of LM using the evaluation map $LM \to M$, and this in turn induces a filtration of $\mathcal{S}(M)$. The spectral sequence is the spectral sequence obtained from this filtration of $\mathcal{S}(M)$.

Our main application of this spectral sequence is the following theorem.

Theorem 2.1 *Let M be a closed oriented manifold. Then $\mathrm{HL}_*(M;\mathbb{F}_p)$ contains a polynomial algebra over $\mathbb{F}_p$ on k generators if and only if the centre of $H_*(\Omega M;\mathbb{F}_p)$ contains a polynomial algebra over $\mathbb{F}_p$ on k generators.*

The first step is to prove the following lemma.

Lemma 2.2 *Let M be a closed manifold. The kernel of the ring homomorphism $h\colon \mathrm{HL}_*(M) \to H_*(\Omega M)$ is a nilpotent ideal.*

Proof Let

$$0 = F^{-n-1} \subseteq F^{-n} \subseteq \cdots \subseteq F^0 = \mathrm{HL}_*(M)$$

be the (negatively indexed) filtration of $\mathrm{HL}_*(M)$ coming from the CJY spectral sequence. Here n is the dimension of the manifold M. Then

$$F^{-i} F^{-j} \subseteq F^{-i-j}$$

and so $(F^{-1})^{n+1} = 0$. The proposition follows since F^{-1} is exactly the kernel of the edge homomorphism of this spectral sequence. □

Next we give a simple but very useful lemma.

Lemma 2.3 *Suppose M is a closed, simply connected manifold of dimension n. Let C be the centre of the algebra $H_*(\Omega M;\mathbb{F}_p)$. Then for any $x \in C$,*

$$x^{p^{n-2}} \in \mathrm{im}\big(h\colon \mathrm{HL}_*(M;\mathbb{F}_p) \to H_*(\Omega M;\mathbb{F}_p)\big).$$

Proof Since h is the edge homomorphism in the CJY spectral sequence we know that an element $y \in H_*(\Omega M;\mathbb{F}_p) = E_2^{0,*}$ is in the image of h if and only if it is an infinite cycle in this spectral sequence. Let $x \in H_*(\Omega M;\mathbb{F}_p) = E_2^{0,*}$ be a central element. Now x may or may not be a cycle for d_2 in the CJY spectral sequence. But d_2 is a derivation and x is central so it follows that

$$d_2 x^p = p x^{p-1} d_2 x.$$

Since the ground field is $\mathbb{F}_p$ it follows that $d_2 x^p = 0$. It may or may not be the case that x^p is a cycle for d_3 but the same argument shows that $x^{p^2} = (x^p)^p$ is a cycle for d_3. Because M has dimension n, it follows that $d_r = 0$ for $r \geq n+1$. Since M is simply connected $H^1(M;\mathbb{F}_p) = H^{n-1}(M;\mathbb{F}_p) = 0$. It follows that there are at most $n-2$ differentials on $E_2^{0,*}$ which could be non-zero, starting with d_2. Repeating this argument at most $n-2$ times shows that $x^{p^{n-2}} \in E_2^{0,*}$ is an infinite cycle and it follows that $x^{p^{n-2}}$ is in the image of h. □

We will also need the following result of [11].

Theorem 2.4 *The image of $h\colon \mathrm{HL}_*(M;\mathbb{F}_p) \to H_*(\Omega M;\mathbb{F}_p)$ is contained in the centre of $H_*(\Omega M,\mathbb{F}_p)$.*

To prove Theorem 2.1 we simply combine the previous three results.

Proof of Theorem 2.1 The kernel of $h\colon \mathrm{HL}_*(M;\mathbb{F}_p) \to H_*(\Omega M;\mathbb{F}_p)$ is a nilpotent ideal, and the image of h is contained in the centre of $H_*(\Omega M;\mathbb{F}_p)$. So if $\mathrm{HL}_*(M;\mathbb{F}_p)$ contains a polynomial algebra on k generators, then so does the centre of $H_*(\Omega M;\mathbb{F}_p)$. On the other hand, if the centre of $H_*(\Omega M;\mathbb{F}_p)$ contains the polynomial algebra $\mathbb{F}_p[x_1,\dots,x_k]$, then Lemma 2.3 shows that every element of the subalgebra of the E_2–term of the CJY spectral sequence

$$\mathbb{F}_p[(x_1)^{p^{n-2}},\dots,(x_k)^{p^{n-2}}] \subset H_*(\Omega M;\mathbb{F}_p) = E_2^{0,*}$$

is an infinite cycle. It follows that $\mathrm{HL}_*(M;\mathbb{F}_p)$ contains a polynomial algebra on k generators. □

3 Applications of Theorem 2.1

3.1 Sphere bundles over spheres

Let M be a k–sphere bundle over S^l. If l is odd then M has the same cohomology ring as the product of spheres $S^k \times S^l$ and the theorem of Sullivan and Vigué-Poirrier, Theorem 1.4, shows that any metric on M has an infinite number of closed geodesics. If l is even and $k \neq l-1$ the same argument applies. We are left with the case of a $2n-1$ sphere bundle over S^{2n}. So let $Q = Q_{2n,e}$ denote the sphere bundle

$$S^{2n-1} \to Q \to S^{2n}$$

with Euler class $e \in \mathbb{Z}$. We choose an orientation of S^{2n} to identify the Euler class with an integer. If $e \neq 0$ then the rational cohomology ring of $Q_{2n,e}$ is generated by one element and so we will not be able to use the theorem of Sullivan and Vigué-Poirrier.

There are three special cases to deal with, $2n = 2, 4, 8$. In these dimensions there is a $2n-1$ sphere bundle over S^{2n} with Euler class ± 1 but the non-existence of elements with Hopf invariant one shows that these are the only dimensions in which this can happen. In these special cases $Q_{2n,\pm 1}$ is a homotopy sphere and we cannot use the Gromoll–Meyer theorem for any coefficient field $\mathbb{F}$. The remaining cases are dealt with by the following theorem.

Proposition 3.1 *If $e \neq 0, \pm 1$, for any metric on $Q = Q_{2n,e}$, there is an infinite number of closed geodesics on Q.*

Proof Choose a prime p such that p divides e. Standard basic calculations in algebraic topology show that

$$H^*(Q;\mathbb{F}_p) = E[a_{2n-1}, b_{2n}] \quad \text{and} \quad H_*(\Omega Q;\mathbb{F}_p) = P[u_{2n-2}, v_{2n-1}].$$

Here E denotes the exterior algebra over $\mathbb{F}_p$ and P denotes the polynomial algebra over $\mathbb{F}_p$. The subscripts are the degrees of the elements. If $p = 2$, then the algebra $P[u_{2n-2}, v_{2n-1}]$ is not graded commutative since $v_{2n-1}^2 \neq 0$. However the centre of $H_*(\Omega Q; \mathbb{F}_p)$ is precisely $P[u_{2n-2}, v_{2n-1}^2]$. Theorem 2.1 shows that $\mathrm{HL}_*(Q)$ contains a polynomial algebra on two generators and so $H_*(LQ; \mathbb{F}_p)$ has doubly infinite dimension. The Gromoll–Meyer theorem shows that for any metric on Q, there is an infinite number of distinct closed geodesics. □

3.2 The Grassmannian of oriented two planes in $\mathbb{R}^{2n+1}$

Let $G_2^+(\mathbb{R}^{2n+1})$ denote the Grassmannian of oriented 2–planes in $\mathbb{R}^{2n+1}$. Recall the following two calculations from the theory of characteristic classes.

Suppose 2 is a unit in the coefficient field $\mathbb{F}$. Then

(1) $H^*(G_2^+(\mathbb{R}^{2n+1}); \mathbb{F}) = P[x_2]/(x_2^{2n})$,

(2) $H^*(G_2^+(\mathbb{R}^{2n+1}); \mathbb{F}_2) = P[x_2]/(x_2^n) \otimes E(y_{2n})$.

So the algebra $H^*(G_2^+(\mathbb{R}^{2n+1}); \mathbb{F}_p)$ can be generated by a single generator for $p \neq 2$, but in the case $p = 2$ it requires at least two generators. Another standard calculation in algebraic topology shows that

$$H_*(\Omega G_2^+(\mathbb{R}^{2n+1}); \mathbb{F}_2) = E(u_1) \otimes P[v_{2n-2}] \otimes P[w_{2n-1}] \cong H_*(\Omega(\mathbb{CP}^n \times S^{2n}); \mathbb{F}_2).$$

Evidently this contains a central polynomial algebra generated by two elements. The following theorem follows from the Gromoll–Meyer theorem in the case of $\mathbb{F}_2$ coefficients.

Theorem 3.2 *Any metric on $G_2^+(\mathbb{R}^{2n+1})$ has an infinite number of closed geodesics.*

3.3 The list of examples from McCleary–Ziller

There is a list in McCleary and Ziller [15], based on the work of [19], consisting of one representative from each diffeomorphism class of homogeneous spaces G/K, where G is a compact connected Lie group and K is a connected closed subgroup, with two properties:

- G/K is not diffeomorphic to a sphere, a real, complex, or quaternionic projective space, nor is it diffeomorphic to the Cayley projective plane.
- The algebra $H^*(G/K; \mathbb{Q})$ is generated by one element.

In other words it is the list of examples of homogeneous spaces to which we would like to apply the theorem of Gromoll–Meyer, but cannot do so over the ground field $\mathbb{Q}$. This list contains two infinite families:

- The Stiefel manifold $V_2(\mathbb{R}^{2n+1})$ of two frames in $\mathbb{R}^{2n+1}$. This is a $2n-1$ sphere bundle over S^{2n} with Euler class 2, and Proposition 3.1 shows that any metric on $V_2(\mathbb{R}^{2n+1})$ has an infinite number of geometrically distinct closed geodesics.
- The Grassmannian of oriented 2–planes in $\mathbb{R}^{2n+1}$. Theorem 3.2 shows that any metric on this manifold has an infinite number of geometrically distinct closed geodesics.

There are another seven homogeneous spaces on this list. The first two are $\mathrm{SU}(2)/\mathrm{SO}(3)$ and $\mathrm{Sp}(2)/\mathrm{SU}(2)$, and the other five are homogeneous spaces for G_2. It is possible to go through these seven examples by direct calculations with loop spaces. However, we will deal with them in Section 5 as examples of our main theorem.

4 The proof of Theorem 1.1 and Theorem 1.2

We next need results contained in a series of interrelated papers by Félix, Halperin, Lemaire and Thomas on the homology of based loop spaces. We give a brief summary of the results we need.

4.1 Elliptic Hopf algebras

Let Γ be a graded Hopf algebra over the ground field $\mathbb{F}$. The *lower central series* of Γ is the sequence

$$\Gamma = \Gamma^{(0)} \supset \Gamma^{(1)} \supset \Gamma^{(2)} \supset \cdots \supset \Gamma^{(n)} \supset \cdots,$$

where $\Gamma^{(i+1)} = [\Gamma, \Gamma^{(i)}]$. By definition Γ is *nilpotent* if $\Gamma^{(s)} = \mathbb{F}$ for some s. Although the definition of the $\Gamma^{(i)}$ depends only on the algebra structure of Γ, it is straightforward to check that the $\Gamma^{(i)}$ are normal Hopf subalgebras of Γ.

We say that Γ is *connected* if $\Gamma_i = 0$ when $i < 0$ and $\Gamma_0 = \mathbb{F}$, and that Γ is *finitely generated* if it is finitely generated as an algebra. From [9] we have the following definition.

Definition 4.1 Fix a ground field $\mathbb{F}$. A Hopf algebra Γ over $\mathbb{F}$ is *elliptic* if it is connected, co-commutative, finitely generated, and nilpotent.

Note that the only part of the definition of an elliptic Hopf algebra which refers to the coproduct is the condition that it is co-commutative.

Here are some examples. In these examples we assume that the Hopf algebras in question are connected and co-commutative over a fixed ground field $\mathbb{F}$.

(1) If Γ is a finite-dimensional Hopf algebra, then Γ is elliptic. To prove this first note that since Γ is connected $\Gamma^{(i)}$ is $(i+1)$–connected. Since Γ is finite-dimensional it follows that $\Gamma^{(i)} = \mathbb{F}$ for sufficiently large i. So Γ is nilpotent. Since Γ is finite, it is finitely generated.

(2) If Γ is commutative, then Γ is elliptic if and only if Γ is finitely generated.

(3) Let L be a Lie algebra. Let $U(L)$ be the universal enveloping algebra of L. This becomes a Hopf algebra by defining the coproduct to be the unique coproduct which makes the elements of L primitive. Then $U(L)$ is an elliptic Hopf algebra if and only if L is a finitely generated nilpotent Lie algebra.

The structure theorem for elliptic Hopf algebras proved in [9] tells us that essentially these examples generate the class of all elliptic Hopf algebras by taking extensions.

Theorem 4.2 *Let $\mathbb{F}$ be a field and let Γ be a connected, finitely generated, co-commutative Hopf algebra over $\mathbb{F}$.*

- *If $\mathbb{F}$ has characteristic zero, then Γ is elliptic if and only if $\Gamma = U(L)$, where L is a finitely generated, nilpotent Lie algebra over $\mathbb{F}$.*
- *If $\mathbb{F}$ has characteristic $p \neq 0$, then Γ is elliptic if and only if it contains a finitely generated, central Hopf subalgebra C, such that $\Gamma /\!/ C$ is finite.*

The statement of the second clause of the theorem is not quite the same as the statement of [9, Theorem B(ii)] but it is easily seen to be equivalent to it. From [17] we know that Γ is isomorphic to $C \otimes \Gamma /\!/ C$ as a C algebra. Since C is finitely generated and commutative it follows from a theorem of Borel [17] that as an algebra C is isomorphic to $P \otimes A$, where P is a polynomial algebra over $\mathbb{F}$ in a finite number of variables and A is a finite-dimensional algebra. It follows that Γ is isomorphic to $P \otimes A \otimes \Gamma /\!/ C$ as a P module. Since both A and $\Gamma /\!/ C$ are finite-dimensional it follows that Γ is a finitely generated free module over P. This is the condition given in [9].

4.2 Depth and the Gorenstein condition

Let A be a graded augmented algebra over the ground field $\mathbb{F}$. We will assume that A is connected. We can form the vector spaces

$$\mathrm{Ext}_A^{i,j}(\mathbb{F}, A).$$

The *depth* of A, depth A, is defined as follows:

$$\operatorname{depth} A = \inf\{s \mid \mathrm{Ext}_A^{s,*}(\mathbb{F}, A) \neq 0\}.$$

If $n = \operatorname{depth} A$, then $\operatorname{Ext}_A^{s,t}(\mathbb{F}, A) = 0$ for $s < n$ and there is an integer t such that $\operatorname{Ext}_A^{n,t}(\mathbb{F}, A) \neq 0$. In particular the depth of A could be infinite, that is, $\operatorname{Ext}_A^{s,t}(\mathbb{F}, A) = 0$ for all (s,t).

The graded algebra A is *Gorenstein* if there is a pair of integers (n,m) such that

- $\operatorname{Ext}_A^{s,t}(\mathbb{F}, A) = 0$ if $(s,t) \neq (n,m)$,
- $\operatorname{Ext}_A^{n,m}(\mathbb{F}, A) = \mathbb{F}$.

The definitions of the Gorenstein condition and depth first appeared in classical commutative ring theory. Gorenstein rings generalise complete intersection rings.

It is straightforward to check that

- $\operatorname{depth} A \otimes B = \operatorname{depth} A + \operatorname{depth} B$,
- $A \otimes B$ is Gorenstein if and only if both A and B are Gorenstein.

In the case of a polynomial algebra $\mathbb{F}[x]$ with one generator of degree k,

$$\operatorname{Ext}^{1,k}_{\mathbb{F}[x]}(\mathbb{F}, \mathbb{F}[x]) = \mathbb{F}, \quad \operatorname{Ext}^{s,t}_{\mathbb{F}[x]}(\mathbb{F}, \mathbb{F}[x]) = 0 \quad (s,t) \neq (1,k).$$

In the case where $A = \mathbb{F}[x]/(x^n)$ is a truncated polynomial with generator of degree k,

$$\operatorname{Ext}_A^{0,-k(n-1)}(\mathbb{F}, A) = \mathbb{F}, \quad \operatorname{Ext}_A^{s,t}(\mathbb{F}, A) = 0 \quad (s,t) \neq (0, -k(n-1)).$$

The most elementary method for doing these calculations is to use the minimal resolution of $\mathbb{F}$ over $\mathbb{F}[x]$ and the minimal resolution of $\mathbb{F}$ over $\mathbb{F}[x]/(x^n)$. It follows that both the algebras $\mathbb{F}[x]$ and $\mathbb{F}[x]/(x^n)$ are Gorenstein, and

$$\operatorname{depth} \mathbb{F}[x] = 1, \quad \operatorname{depth} \mathbb{F}[x]/(x^n) = 0.$$

The following lemma is [8, Proposition 1.7].

Lemma 4.3 *Suppose A is an infinite tensor product of algebras. Then the depth of A is infinite.*

Suppose Γ is a connected Hopf algebra that is commutative as an algebra. By a theorem of Borel [17, Theorem 7.11] it follows that Γ is isomorphic as an algebra to a tensor product of polynomial algebras and truncated polynomial algebras. If Γ is not finitely generated then Lemma 4.3 shows that Γ has infinite depth. If Γ is finitely generated, then it has finite depth and it is isomorphic to $P \otimes A$, where P is a polynomial algebra with $m = \operatorname{depth} \Gamma$ variables and A is a finite tensor product of truncated polynomial algebras. This proves Theorem 4.2 in the case where Γ is commutative. One way to think of the proof of Theorem 4.2 is that it works by reducing the general case to the commutative case by using the condition that Γ is nilpotent.

The results of [7] and [8] show the relevance of the Gorenstein condition to topology. We summarise these results as the following theorem.

Theorem 4.4 *Let X be a simply connected finite complex.*

(1) *The Hopf algebra $H_*(\Omega X;\mathbb{F})$ has finite depth. In fact,* depth $X \le$ LSCat X, *where* LSCat X *denotes the Lyusternik–Schnirelman category of X.*

(2) *If the Hopf algebra $H_*(\Omega X;\mathbb{F})$ is Gorenstein, then X is a Poincaré duality space.*

In [8] Félix, Halperin and Thomas extend the Gorenstein condition to differential graded algebras and show that a finite complex X is a Poincaré duality space if and only if the cochain algebra $S^*(X;\mathbb{F})$ is a Gorenstein differential graded algebra. While it is true that if $H^*(X;\mathbb{F})$ is Gorenstein then so is $S^*(X;\mathbb{F})$, the reverse implication is not true; see [8, Examples 3.3].

If X is a finite complex, then we know that $H_*(\Omega X;\mathbb{F})$ has finite type and finite depth. The following theorem gives some useful practical ways to deduce, in addition, that $H_*(\Omega X;\mathbb{F})$ is elliptic. For the proof see [9, Theorem C].

Theorem 4.5 *Suppose Γ is a connected, co-commutative Hopf algebra over $\mathbb{F}$ of finite type and that Γ has finite depth. Then the following are equivalent:*

(1) *Γ is elliptic.*

(2) *Γ is nilpotent.*

(3) *Γ has polynomial growth.*

(4) *Γ is Gorenstein.*

4.3 The proof of Theorem 1.1

If M is a closed, connected, oriented manifold of finite dimension, then $H_*(\Omega M;\mathbb{F}_p)$ is connected and co-commutative, and it has finite type and finite depth. We are assuming it has polynomial growth. It follows from Theorem 4.5 that $H_*(\Omega M;\mathbb{F}_p)$ is elliptic. Therefore, from Theorem 4.2, it is a finitely generated free module over a central subalgebra P that is a polynomial algebra on a finite number, say l, of variables. It follows that $H_*(\Omega M;\mathbb{F}_p)$ has polynomial growth with exponent l and indeed l is the minimal exponent which can occur in the inequality for polynomial growth. In the notation of Theorem 1.1, $l = K_0$. This proves Theorem 1.1.

4.4 The proof of Theorem 1.2

It follows from Theorem 4.2 that if Γ is an elliptic Hopf algebra over $\mathbb{F}_p$, then Γ is doubly infinite if and only if the centre of Γ contains a polynomial algebra on two generators. Now let M be a simply connected closed manifold satisfying the hypotheses of Theorem 1.2. Then, as in the proof of Theorem 1.1, it follows that $H_*(\Omega M;\mathbb{F}_p)$ is an elliptic Hopf algebra. Suppose in addition that the algebra $H^*(M;\mathbb{F}_p)$ cannot be generated by one element. From Theorem 1.5, it follows that $H_*(\Omega M;\mathbb{F}_p)$ is doubly infinite and so the centre of $H_*(\Omega M;\mathbb{F}_p)$ contains a polynomial algebra on two generators. By Theorem 2.1 it follows that $\mathrm{HL}_*(M;\mathbb{F}_p)$ contains a polynomial algebra on two generators and therefore $H_*(LM;\mathbb{F}_p)$ is doubly infinite. The Gromoll–Meyer theorem, Theorem 1.3, completes the proof.

5 Application to homogeneous spaces

The following theorem is [10, Example 3.2].

Theorem 5.1 *Let G be a simply connected, compact Lie group and K a connected, closed subgroup of G. Then the homogeneous space G/K is $\mathbb{F}_p$ elliptic for any prime p.*

The proof uses the fibration

$$\Omega G \to \Omega(G/K) \to K$$

for which the fundamental group $\pi_1(K)$ acts trivially on the groups $H_*(\Omega G; F_p)$. Then a Leray–Serre spectral sequence argument may be applied because K and ΩG are both elliptic and hence have polynomial growth.

Now return to the list from [15]. The seven examples of homogeneous spaces in this list not covered by Proposition 3.1 and Theorem 3.2 are $\mathbb{F}_p$ elliptic spaces for any prime p by Theorem 5.1. Furthermore, in each case, there is a prime p such that the cohomology algebra of the homogeneous space cannot be generated by a single element. Therefore by Theorem 1.2 any metric has an infinite number of geometrically distinct closed geodesics.

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