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Solvable Lie flows of codimension 3

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In Appendix E of *Riemannian foliations* [Progress in Mathematics 73, Birkhäuser, Boston (1988)], É Ghys proved that any Lie $\mathfrak{g}$–flow is homogeneous if $\mathfrak{g}$ is a nilpotent Lie algebra. In the case where $\mathfrak{g}$ is solvable, we expect any Lie $\mathfrak{g}$–flow to be homogeneous. In this paper, we study this problem in the case where $\mathfrak{g}$ is a 3–dimensional solvable Lie algebra.

57R30; 53C12, 22E25

1 Introduction

Throughout this paper, we suppose that all manifolds are connected, smooth and orientable and all foliations are smooth and transversely orientable. In this paper, flows mean orientable 1–dimensional foliations.

Lie foliations were first defined by E Fedida [4]. A classical example of a Lie foliation is a homogeneous one. Through the results of several authors, it is recognized that the class of homogeneous Lie foliations is a large class in the class of Lie foliations, though of course these classes do not coincide. Therefore deciding which Lie foliations belong to the class of homogeneous Lie foliations is an important problem in Lie foliation theory.

P Caron and Y Carrière [2] proved that any Lie $\mathbb{R}^q$–flow without closed orbits is diffeomorphic to a linear flow on the $(q+1)$–dimensional torus, which is homogeneous. Carrière [3] proved that any Lie $\mathfrak{a}(2)$–flow is homogeneous. S Matsumoto and N Tsuchiya [13] proved that any Lie $\mathfrak{a}(2)$–foliation of a 4–dimensional manifold or its double covering is homogeneous.

In the case where $\mathfrak{g}$ is semisimple, M Llabrés and A Reventós constructed an example of Lie $\mathfrak{sl}_2(\mathbb{R})$–flow which is not homogeneous [12, Example 5.3].

In the case where $\mathfrak{g}$ is nilpotent, É Ghys [7] proved that any Lie $\mathfrak{g}$–flow is homogeneous. In the case where $\mathfrak{g}$ is solvable, we conjecture that any Lie $\mathfrak{g}$–flow is homogeneous.

In this paper, we study this problem in the case where $\mathfrak{g}$ is a 3–dimensional solvable Lie algebra.

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If a Lie $\mathfrak{g}$–flow $\mathcal{F}$ on M has a closed orbit, then any orbit is closed and M is an oriented S^1–bundle. In this case, the base space is diffeomorphic to a homogeneous space $\Gamma\backslash G$ and hence $\mathfrak{g}$ is unimodular. The total space M is, in general, not diffeomorphic to a homogeneous space. However, in the case where $\mathfrak{g}$ is of type (R) or $\mathfrak{g}$ is 3–dimensional, we can prove that the total space is a homogeneous space. More precisely, we obtain the following theorem.

Theorem A *Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathcal{F}$ be a Lie $\mathfrak{g}$–flow on a closed manifold M. Suppose that $\mathcal{F}$ has a closed orbit.*

(i) *If $\mathfrak{g}$ is of type (R) and unimodular, then $\mathcal{F}$ is diffeomorphic to the flow in Example 3.1.*

(ii) *If the dimension of $\mathfrak{g}$ is three and $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$, then $\mathcal{F}$ is diffeomorphic to the flow in Example 3.1.*

In particular, if $\mathfrak{g}$ is a 3–dimensional solvable Lie algebra and $\mathcal{F}$ has a closed orbit, then $\mathcal{F}$ is diffeomorphic to the flow in Example 3.1.

In the case where $\mathcal{F}$ has no closed orbits, we obtain the following theorem.

Theorem B *Let $\mathfrak{g}$ be a 3–dimensional solvable Lie algebra and $\mathcal{F}$ be a Lie $\mathfrak{g}$–flow on a closed manifold. Suppose that $\mathcal{F}$ has no closed orbits.*

(i) *If $\mathfrak{g}$ is isomorphic to either $\mathbb{R}^3$ or $\mathfrak{n}(3)$, then $\mathcal{F}$ is diffeomorphic to the flow in Example 3.1.*

(ii) *If $\mathfrak{g}$ is isomorphic to $\mathfrak{a}(3)$, then $\mathcal{F}$ is isomorphic to the flow in Example 3.3.*

(iii) *If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_2^k$, then $\mathcal{F}$ is isomorphic to the flow in Example 3.4.*

(iv) *If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^h$ and $h \neq 0$, then $\mathcal{F}$ is isomorphic to the flow in Example 3.5.*

(v) *If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$, then $\mathcal{F}$ is isomorphic to the flow in Example 3.6.*

Since (see Llabrés and Reventós [12]) there does not exist a Lie $\mathfrak{g}_1$–flow on a closed manifold, we have the following corollary.

Corollary 1.1 *For any 3–dimensional solvable Lie algebra $\mathfrak{g}$, any Lie $\mathfrak{g}$–flow on a closed manifold is homogeneous.*

The contents of this paper are the following: Section 2 is devoted to recalling some basic definitions and properties of Lie foliations and Lie algebras. In Section 2A, we recall some basic definitions of Lie algebras. In Section 2B, we recall the classification of 3–dimensional solvable Lie algebras. In Section 2C, we recall the definition and some properties of Lie foliations. In Section 3, we construct some important examples of Lie $\mathfrak{g}$–flows, which are models of codimension-3 solvable Lie $\mathfrak{g}$–flows. In Section 4, we prove Theorem A. In Section 5, we construct a diffeomorphism between Lie flows without closed orbits according to the construction of Ghys [7]. In Section 6, by using the diffeomorphism constructed in Section 5, we prove Theorem B.

2 Preliminaries

2A Solvable Lie groups and solvable Lie algebras

Let $\mathfrak{g}$ be a q–dimensional real Lie algebra. The descending central series of $\mathfrak{g}$ is defined inductively by

$$C^0\mathfrak{g} = \mathfrak{g} \quad \text{and} \quad C^k\mathfrak{g} = [\mathfrak{g}, C^{k-1}\mathfrak{g}].$$

Similarly the derived series of $\mathfrak{g}$ is defined inductively by

$$D^0\mathfrak{g} = \mathfrak{g} \quad \text{and} \quad D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}].$$

A Lie algebra $\mathfrak{g}$ is nilpotent if there exists an integer k such that $C^k\mathfrak{g} = \{0\}$, and a connected Lie group G is nilpotent if the Lie algebra of G is nilpotent. Similarly a Lie algebra $\mathfrak{g}$ is solvable if there exists an integer k such that $D^k\mathfrak{g} = \{0\}$, and a connected Lie group G is solvable if the Lie algebra of G is solvable.

Let H and G be Lie groups, and let Φ: $H \to \mathrm{Aut}(G)$ be a homomorphism. Then we can construct a new Lie group $H \ltimes_\Phi G$, which is called the semidirect product of H and G with respect to Φ, as follows. The semidirect product $H \ltimes_\Phi G$ is the direct product of the sets H and G endowed with the group structure via

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 \cdot h_2, g_1 \cdot \Phi(h_1)(g_2)).$$

The Lie group H is naturally a subgroup of $H \ltimes_\Phi G$, and G is naturally a normal subgroup of $H \ltimes_\Phi G$.

Let $\mathfrak{g}$ be a Lie algebra and ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$.

Definition 2.1 A solvable Lie algebra $\mathfrak{g}$ is said to be of type (R) if all the eigenvalues of $\mathrm{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ are real for any $X \in \mathfrak{g}$. A simply connected solvable Lie group is said to be of type (R) if the Lie algebra of G is of type (R).

It is well known that simply connected solvable Lie groups of type (R) have similar properties of simply connected nilpotent Lie groups; see [9].

2B Classification of 3–dimensional solvable Lie algebras

It is well known that 1–dimensional Lie algebras are isomorphic to $\mathbb{R}$ and that 2–dimensional Lie algebras are isomorphic to either $\mathbb{R}^2$ or $\mathfrak{a}(2)$, where

$$\mathfrak{a}(2) = \left\{ \begin{pmatrix} t & x \\ 0 & 0 \end{pmatrix} \;\middle|\; t, x \in \mathbb{R} \right\}$$

is the Lie algebra of $A(2)$, which is the affine transformation group of the real line.

Let $V = \langle T, X, Y \rangle_{\mathbb{R}}$ be a 3–dimensional vector space and consider the following Lie brackets on V:

- $\mathbb{R}^3$ (abelian): $[T, X] = [T, Y] = [X, Y] = 0$;
- $\mathfrak{n}(3)$ (Heisenberg): $[T, Y] = X$ and $[T, X] = [X, Y] = 0$;
- $\mathfrak{a}(3)$ (affine): $[T, X] = X$ and $[T, Y] = [X, Y] = 0$;
- $\mathfrak{g}_1$: $[T, X] = X + Y$, $[T, Y] = Y$ and $[X, Y] = 0$;
- $\mathfrak{g}_2^k$: $[T, X] = X$, $[T, Y] = kY$ and $[X, Y] = 0$ where $k \neq 0$;
- $\mathfrak{g}_3^h$: $[T, X] = Y$, $[T, Y] = -X + hY$ and $[X, Y] = 0$ where $h^2 < 4$.

Then any 3–dimensional solvable Lie algebra is isomorphic to one of the above Lie algebras.

It is well known that $\mathfrak{n}(3)$ is the Lie algebra of the 3–dimensional Heisenberg group

$$N(3) = \left\{ \begin{pmatrix} 1 & t & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

We will need an explicit description of simply connected Lie groups corresponding to the Lie algebras $\mathfrak{a}(3)$, $\mathfrak{g}_2^k$ and $\mathfrak{g}_3^h$. These Lie groups are given by

$$A(3) = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t, x, y \in \mathbb{R} \right\},$$

$$G_2^k = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{kt} & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t, x, y \in \mathbb{R} \right\},$$

$$G_3^{h=0} = \widetilde{\mathrm{SO}(2) \ltimes \mathbb{R}^2},$$

and

$$G_3^{h\neq 0} = \left\{ \begin{pmatrix} c(t)\cos(\phi+t) & -c(t)\sin t & x \\ c(t)\sin t & c(t)\cos(\phi-t) & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t,x,y\in\mathbb{R} \right\},$$

where $\widetilde{\mathrm{SO}(2)\ltimes\mathbb{R}^2}$ is the universal covering of the group of rigid motions $\mathrm{SO}(2)\ltimes\mathbb{R}^2$, $c(t)=(2/\alpha)e^{\beta t}$, $\beta=\tan\phi=h/\alpha$, and $\alpha=\sqrt{4-h^2}$; see [5].

Note that G_3^0 is isomorphic to the semidirect product $\mathbb{R}\ltimes_\rho\mathbb{R}^2$, where $\rho\colon \mathbb{R}\to\mathrm{Aut}(\mathbb{R}^2)$ is given by

$$\rho(t)=\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Note also that the Lie group $G_3^{h\neq 0}$ has another description

$$G_3^{h\neq 0} = \left\{ \begin{pmatrix} e^{\beta t}\cos t & -e^{\beta t}\sin t & x \\ e^{\beta t}\sin t & e^{\beta t}\cos t & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t,x,y\in\mathbb{R} \right\},$$

where $\beta=\tan\phi=h/\alpha$ and $\alpha=\sqrt{4-h^2}$. In this paper, we will use this description.

Lie algebras $\mathfrak{g}_2^k$ and $\mathfrak{g}_2^{k'}$ are isomorphic if and only if $k=k'$ or $k=1/k'$ and the Lie algebras $\mathfrak{g}_3^h$ and $\mathfrak{g}_3^{h'}$ are isomorphic if and only if $h=h'$ or $h=-h'$. The Lie algebra $\mathfrak{g}_2^k$ is unimodular if and only if $k=-1$. The Lie algebra $\mathfrak{g}_3^h$ is unimodular if and only if $h=-1$. The Lie algebra $\mathfrak{g}_3^h$ is not of type (R) for any h and the other 3–dimensional solvable Lie algebras are of type (R).

2C Lie foliations

Let $\mathcal{F}$ be a codimension-q foliation of a closed manifold M and $\mathfrak{g}$ be a q–dimensional real Lie algebra. A $\mathfrak{g}$–valued 1–form ω on M is said to be a Maurer–Cartan form if ω satisfies the equation $d\omega+\frac{1}{2}[\omega,\omega]=0$ and nonsingular if $\omega_x\colon T_xM\to\mathfrak{g}$ is surjective for each $x\in M$.

Definition 2.2 A codimension-q foliation $\mathcal{F}$ is a Lie $\mathfrak{g}$–foliation if there exists a nonsingular $\mathfrak{g}$–valued Maurer–Cartan form ω such that $\mathrm{Ker}(\omega)=T\mathcal{F}$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be foliations of M_1 and M_2, respectively. A smooth map $f\colon M_1\to M_2$ preserves foliations if $f(L)\in\mathcal{F}_2$ for every leaf $L\in\mathcal{F}_1$. We denote such a map by $f\colon (M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$. We call two foliations $\mathcal{F}_1$ of M_1 and $\mathcal{F}_2$ of M_2 diffeomorphic if there exists a foliation preserving map $f\colon (M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$ such that $f\colon M_1\to M_2$ is a diffeomorphism.

In this paper, we call two Lie $\mathfrak{g}$–foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ diffeomorphic if $\mathcal{F}_1$ is diffeomorphic to $\mathcal{F}_2$ as a foliation.

Fedida [4] proved that Lie $\mathfrak{g}$–foliations have a special property.

Theorem 2.3 [4] *Let $\mathcal{F}$ be a codimension-q Lie $\mathfrak{g}$–foliation of a closed manifold M and G be the simply connected Lie group of $\mathfrak{g}$. Let $p\colon \widetilde{M} \to M$ be the universal covering of M. Fix a Maurer–Cartan form $\omega \in A^1(M;\mathfrak{g})$ of $\mathcal{F}$. Then there exists a locally trivial fibration $D\colon \widetilde{M} \to G$ and a homomorphism $h\colon \pi_1(M) \to G$ such that*

(1) *$D(\alpha \cdot \tilde{x}) = h(\alpha) \cdot D(\tilde{x})$ for any $\alpha \in \pi_1(M)$ and any $\tilde{x} \in \widetilde{M}$, and*

(2) *the lifted foliation $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ coincides with the fibers of the fibration D.*

The fibration D is called the developing map, the homomorphism h is called the holonomy homomorphism and the image of h is called the holonomy group of the Lie $\mathfrak{g}$–foliation $\mathcal{F}$ with respect to the Maurer–Cartan form ω.

Conversely, if there exist D and h which satisfy condition (1) above, then the set of fibers of D defines a Lie $\mathfrak{g}$–foliation $\mathcal{F}$ of M such that the developing map is D and the holonomy homomorphism is h.

Example 2.4 Let G be a simply connected Lie group and $\widetilde{G}$ a simply connected Lie group with a uniform lattice Δ. Suppose that there exists a short exact sequence

$$0 \to K \to \widetilde{G} \xrightarrow{D_0} G \to 0.$$

Then the map D_0 defines a Lie $\mathfrak{g}$–foliation $\mathcal{F}_0$ of the homogeneous space $\Delta\backslash\widetilde{G}$.

We call Lie $\mathfrak{g}$–foliations constructed as in Example 2.4 homogeneous Lie $\mathfrak{g}$–foliations.

Definition 2.5 A Lie $\mathfrak{g}$–foliation $\mathcal{F}$ of a closed manifold M is homogeneous if $\mathcal{F}$ is diffeomorphic to a homogeneous Lie $\mathfrak{g}$–foliation.

Let $D\colon \widetilde{M} \to G$ be the developing map and $h\colon \pi_1(M) \to G$ be the holonomy homomorphism of a Lie $\mathfrak{g}$–foliation $\mathcal{F}$. Let $\Gamma = h(\pi_1(M))$ be the holonomy group of $\mathcal{F}$. Since the developing map D is h–equivariant, the map D induces a fibration

$$\overline{D}\colon M \to \overline{\Gamma}\backslash G,$$

where $\overline{\Gamma}$ is the closure of Γ. This fibration $\overline{D}$ is called the basic fibration, the homogeneous space $\overline{\Gamma}\backslash G$ the basic manifold, and the dimension of $\overline{\Gamma}\backslash G$ the basic dimension of $\mathcal{F}$.

Let $\overline{\mathcal{F}}$ be the foliation of M defined by the fibers of the fibration $\overline{D}$. By the definition of $\overline{D}$, we can see that any leaf F of $\overline{\mathcal{F}}$ is saturated by $\mathcal{F}$ and the foliation $\mathcal{F}|_F$ is a minimal foliation of F. Moreover the basic fibration $\overline{D}\colon M \to \overline{\Gamma}\backslash G$ induces a diffeomorphism from the leaf space $M/\overline{\mathcal{F}}$ to $\overline{\Gamma}\backslash G$.

3 Models of Lie flows

In this section, we construct some examples of homogeneous Lie flows which are important examples in this paper.

Example 3.1 Let

$$1 \to \mathbb{R} \to \widetilde{G} \xrightarrow{D_0} G \to 1$$

be a central exact sequence of Lie groups and Δ be a uniform lattice of $\widetilde{G}$. Then the surjective homomorphism $D_0\colon \widetilde{G} \to G$ defines a Lie $\mathfrak{g}$–flow $\mathcal{F}_0$ on $\Delta\backslash\widetilde{G}$.

The Lie $\mathfrak{g}$–flow construction in Example 3.1 is a special case of the construction of homogeneous Lie $\mathfrak{g}$–flows.

In the case in which the dimension of $\mathfrak{g}$ is three, by using the classification of 4–dimensional solvable Lie algebras (see [1]), we have more explicit descriptions of $\widetilde{G}$ and D_0.

Example 3.2 Let $\mathfrak{g}$ be a unimodular 3–dimensional solvable Lie algebra and G be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. Then any central extension

$$1 \to \mathbb{R} \to \widetilde{G} \xrightarrow{D_0} G \to 1$$

of G by $\mathbb{R}$ is given as follows:

(1) If $\mathfrak{g}$ is abelian, then $\widetilde{G}$ is isomorphic to either $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ or $\mathbb{R} \times N(3)$. If $\widetilde{G}$ is isomorphic to $\mathbb{R}^4$, then $D_0\colon \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by the natural projection

$$D_0\colon (t, x, y, z) \mapsto (x, y, z).$$

If $\widetilde{G}$ is isomorphic to $\mathbb{R} \times N(3)$, then $D_0\colon \mathbb{R} \times N(3) \to \mathbb{R}^3$ is given by

$$D_0\colon \left(s, \begin{pmatrix} 1 & t & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) \mapsto A \begin{pmatrix} s \\ t \\ y \end{pmatrix},$$

where $A \in \mathrm{GL}(3; \mathbb{R})$.

(2) If $\mathfrak{g}$ is isomorphic to $\mathfrak{n}(3)$, then $\widetilde{G}$ is isomorphic to $\mathbb{R} \times N(3)$ or to

$$N(4) = \left\{ \begin{pmatrix} 1 & t & \frac{1}{2}t^2 & x \\ 0 & 1 & t & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle|\, t, x, y, z \in \mathbb{R} \right\}.$$

If $\widetilde{G}$ is isomorphic to $\mathbb{R}\times N(3)$, then $D_0\colon \mathbb{R}\times N(3)\to N(3)$ is given by the natural projection. If $\widetilde{G}$ is isomorphic to $N(4)$, then $D_0\colon N(4)\to N(3)$ is given by

$$\begin{pmatrix}1 & t & \frac{1}{2}t^2 & x\\ 0 & 1 & t & y\\ 0 & 0 & 1 & z\\ 0 & 0 & 0 & 1\end{pmatrix}\mapsto\begin{pmatrix}1 & at+cz & \frac{1}{2}(abt^2+cdz^2)+(ad-1)tz+y\\ 0 & 1 & bt+dz\\ 0 & 0 & 1\end{pmatrix},$$

where either

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\qquad\text{or}\qquad\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 0\\ k & 1\end{pmatrix}\quad\text{for } k\in\mathbb{R}.$$

(3) If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_2^{-1}$, then $\widetilde{G}$ is isomorphic to $\mathbb{R}\times G_2^{-1}$ or the semidirect product $\mathbb{R}\ltimes_\Phi N(3)$ with respect to the homomorphism $\Phi\colon \mathbb{R}\to \mathrm{Aut}(N(3))$ defined by

$$\Phi(s)\colon \begin{pmatrix}1 & t & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\mapsto\begin{pmatrix}1 & e^s t & x\\ 0 & 1 & e^{-s}y\\ 0 & 0 & 1\end{pmatrix}.$$

If $\widetilde{G}$ is isomorphic to $\mathbb{R}\times G_2^{-1}$, then $D_0\colon \mathbb{R}\times G_2^{-1}\to G_2^{-1}$ is given by the natural projection. If $\widetilde{G}$ is isomorphic to $\mathbb{R}\ltimes_\Phi N(3)$, then the homomorphism

$$D_0\colon \mathbb{R}\ltimes_\phi N(3)\to G_2^{-1}$$

is given by

$$D_0\colon \left(s,\begin{pmatrix}1 & t & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\right)\mapsto\begin{pmatrix}e^s & 0 & t\\ 0 & e^{-s} & y\\ 0 & 0 & 1\end{pmatrix}.$$

(4) If $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$, then $\widetilde{G}$ is isomorphic to either $\mathbb{R}\times G_3^0$ or the semidirect product $\mathbb{R}\ltimes_\Psi N(3)$ with respect to the homomorphism $\Psi\colon \mathbb{R}\to\mathrm{Aut}(N(3))$ defined by

$$\begin{pmatrix}1 & t & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\mapsto\begin{pmatrix}1 & t\cos s-y\sin s & x-ty\sin^2 s+\frac{1}{4}(t^2-y^2)\sin 2s\\ 0 & 1 & t\sin s+y\cos s\\ 0 & 0 & 1\end{pmatrix}.$$

If $\widetilde{G}$ is isomorphic to $G_3^0\times\mathbb{R}$, then $D_0\colon \mathbb{R}\times G_3^0\to G_3^0$ is given by the natural projection. If $\widetilde{G}$ is isomorphic to $\mathbb{R}\ltimes_\Psi N(3)$, then the homomorphism

$$D_0\colon \mathbb{R}\ltimes_\Psi N(3)\to G_3^0=\mathbb{R}\ltimes_\rho\mathbb{R}^2$$

is given by

$$D_0\colon \left(s,\begin{pmatrix}1 & t & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\right)\mapsto\left(s,\begin{pmatrix}t\\ y\end{pmatrix}\right).$$

Define two classes of 4–dimensional solvable Lie groups $\widetilde{G}_2^k$ and $\widetilde{G}_3^h$ by

$$\widetilde{G}_2^k = \left\{ \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & e^{kt} & 0 & y \\ 0 & 0 & e^{-(1+k)t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, t, x, y, z \in \mathbb{R} \right\}$$

and

$$\widetilde{G}_3^h = \left\{ \begin{pmatrix} e^{\beta t}\cos t & -e^{\beta t}\sin t & 0 & x \\ e^{\beta t}\sin t & +e^{\beta t}\cos t & 0 & y \\ 0 & 0 & d^t & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, t, x, y, z \in \mathbb{R} \right\},$$

where $k \in \mathbb{R}$, $0 < h^2 < 4$, and $d = e^{-2\beta} \in \mathbb{R}$. We construct homogeneous Lie $\mathfrak{g}$–flows on the homogeneous spaces $\Delta\backslash\widetilde{G}_2^k$ and $\Delta\backslash\widetilde{G}_3^h$.

Example 3.3 Let Δ be a uniform lattice of $\widetilde{G}_2^0$. Define a homomorphism

$$D_0\colon \widetilde{G} \to A(3) \quad \text{by} \quad \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & e^{-t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then D_0 defines a homogeneous Lie $\mathfrak{a}(2)$–flow on $\Delta\backslash\widetilde{G}$.

Example 3.4 Assume that the Lie group $\widetilde{G}_2^k$ has a uniform lattice Δ. Define a homomorphism $D_0\colon \widetilde{G}_2^k \to G_2^k$ by

$$D_0\colon \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & e^{kt} & 0 & y \\ 0 & 0 & e^{-(1+k)t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^t & 0 & x \\ 0 & e^{kt} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then D_0 defines a homogeneous Lie $\mathfrak{g}_2^k$–flow on $\Delta\backslash\widetilde{G}$.

Example 3.5 We assume that $\widetilde{G}_3^h$ has a uniform lattice Δ. Define a homomorphism $D_0\colon \widetilde{G}_3^h \to G_3^h$ by

$$D_0\colon \begin{pmatrix} e^{\beta t}\cos t & -e^{\beta t}\sin t & 0 & x \\ e^{\beta t}\sin t & e^{\beta t}\cos t & 0 & y \\ 0 & 0 & d^t & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^{\beta t}\cos t & -e^{\beta t}\sin t & x \\ e^{\beta t}\sin t & e^{\beta t}\cos t & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then D_0 defines a homogeneous Lie $\mathfrak{g}_3^{h\neq 0}$–flow on $\Delta\backslash\widetilde{G}_3^h$.

Example 3.6 Let Δ be a uniform lattice of $G_3^0 \times \mathbb{R}$. Let $D_0\colon G_3^0 \times \mathbb{R} \to G_3^0$ be the natural homomorphism. Then D_0 defines a homogeneous Lie $\mathfrak{g}_3^0$–flow on $\Delta \backslash G_3^0$.

Remark We can extend the definition of $\widetilde{G}_3^h$ to the case when $h = 0$. Then $\widetilde{G}_3^0$ coincides with $\mathrm{SO}(2) \ltimes \mathbb{R}^2 \times \mathbb{R}$, which is not simply connected. The homomorphism $D_0\colon G_3^0 \times \mathbb{R} \to G_3^0$ defined in Example 3.6 coincides with the lifted homomorphism $D_0\colon \mathrm{SO}(2) \ltimes \mathbb{R}^2 \times \mathbb{R} \to \mathrm{SO}(2) \ltimes \mathbb{R}^2$ defined in Example 3.5.

4 Proof of Theorem A

Let $\mathcal{F}$ be a Lie $\mathfrak{g}$–flow on a closed manifold M. Assume that $\mathcal{F}$ has a closed orbit. Then any orbit of $\mathcal{F}$ is closed.

Let $D\colon \widetilde{M} \to G$ be the developing map and $h\colon \pi_1(M) \to G$ be the holonomy homomorphism. Since any orbit of $\mathcal{F}$ is closed, the holonomy group Γ is discrete in G and the basic fibration $\overline{D}\colon M \to \Gamma \backslash G$ is an oriented S^1–bundle over the homogeneous space $\Gamma \backslash G$.

Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$, which is naturally identified with the set of left-invariant 1–forms on G. Consider the inclusion map

$$\iota\colon \textstyle\bigwedge^* \mathfrak{g} \to A^*(\Gamma \backslash G)$$

and the induced map

$$\iota\colon H^*(\mathfrak{g}) \to H^*_{\mathrm{dR}}(\Gamma \backslash G),$$

where $H^*(\mathfrak{g})$ is the cohomology of the Lie algebra $\mathfrak{g}$, $A^*(\Gamma \backslash G)$ is the de Rham complex of $\Gamma \backslash G$, and $H^*_{\mathrm{dR}}(\Gamma \backslash G)$ is the de Rham cohomology of $\Gamma \backslash G$. We call a k–form $\omega \in A^k(\Gamma \backslash G)$ algebraic if ω is in $\iota(A^k(\mathfrak{g}))$.

Let $e(\overline{D}) \in H^2_{\mathrm{dR}}(\Gamma \backslash G)$ be the real Euler class of the S^1–bundle $\overline{D}$. We use the following lemma, which is a special case of [12, Theorem 5.1].

Lemma 4.1 *If $e(\overline{D})$ is represented by an algebraic 2–form, then $\mathcal{F}$ is homogeneous.*

Suppose the Euler class $e(\overline{D})$ is represented by an algebraic 2–form $\iota(\beta) \in A^2(\Gamma \backslash G)$. Then there exists a homogeneous Lie $\mathfrak{g}$–flow $(\Delta \backslash \widetilde{G}, \mathcal{F}_0)$ which is diffeomorphic to $(M, \mathcal{F})$. By the proof of [12, Theorem 5.1], the Lie algebra $\widetilde{\mathfrak{g}}$ of $\widetilde{G}$ coincides with the central extension

$$0 \to \mathbb{R} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of $\mathfrak{g}$ by $\mathbb{R}$ with the Euler class $-2[\beta] \in H^2(\mathfrak{g})$. Hence $\widetilde{G}$ is a central extension of G by $\mathbb{R}$. Therefore we have the following proposition.

Proposition 4.2 *If $e(\overline{D})$ is represented by an algebraic 2–form, then $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}$–flow in Example 3.1.*

Proof of Theorem A First, we assume that $\mathfrak{g}$ is a solvable Lie algebra of type (R). Let $e(\overline{D}) \in H^2_{\mathrm{dR}}(\Gamma\backslash G)$ be the real Euler class of the oriented S^1–bundle.

Since $\mathfrak{g}$ is of type (R), by Hattori [9, Theorem 4.1], the homomorphism

$$\iota\colon H^*(\mathfrak{g}) \to H^*_{\mathrm{dR}}(\Gamma\backslash G)$$

is an isomorphism. Therefore $e(\overline{D})$ is represented by an algebraic 2–form. Hence, by Proposition 4.2, $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}$–flow in Example 3.1.

Next, we assume that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$. By the classification of uniform lattices of G_3^0, the homogeneous space $\Gamma\backslash G_3^0$ is isomorphic to the mapping torus

$$T_A^3 = T^2 \times \mathbb{R}/(\boldsymbol{x}, t+1) \sim (A\boldsymbol{x}, t),$$

where $A \in \mathrm{SL}(2;\mathbb{Z})$ such that $A^p = I$ for some $p \in \mathbb{Z}$. If $A = I$, then the mapping torus T_A^3 is the three-dimensional torus T^3.

Define left-invariant 1–forms θ_T, θ_X and θ_Y on $G_3^0 = \mathbb{R} \ltimes_\rho \mathbb{R}^2$ by

$$\theta_T = dt, \quad \theta_X = \cos t \cdot dx + \sin t \cdot dy \quad \text{and} \quad \theta_Y = -\sin t \cdot dx + \cos t \cdot dy.$$

Then the second cohomology $H^2(\mathfrak{g}_3^0)$ of the Lie algebra $\mathfrak{g}_3^0$ is generated by the cohomology class $[\theta_X \wedge \theta_Y] = [dx \wedge dy]$.

On the other hand, we have

$$H^2_{\mathrm{dR}}(T_A^3) = \begin{cases} \mathbb{R}[dt \wedge dx] \oplus \mathbb{R}[dt \wedge dy] \oplus \mathbb{R}[dx \wedge dy] & \text{if } A = I, \\ \mathbb{R}[dx \wedge dy] & \text{if } A \neq I. \end{cases}$$

Therefore if $A \neq I$, then $H^2_{\mathrm{dR}}(T_A^3)$ is isomorphic to $H^2(\mathfrak{g}_3^0)$ and the Euler class $e(\overline{D})$ is represented by an algebraic 2–form. Hence $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}_3^0$–flow in Example 3.1.

In the case where $A = I$, by the following lemma, there exists a diffeomorphism $f\colon T^3 \to T^3$ such that the pullback $f^*e(\overline{D})$ is represented by an algebraic 2–form. Then the S^1–bundle M is diffeomorphic to the S^1–bundle f^*M, which is diffeomorphic to the Lie $\mathfrak{g}$–flow in Example 3.1. $\square$

Lemma 4.3 *For any $[\omega]=a[dt \wedge dx] + b[dt \wedge dy] + c[dx \wedge dy] \in H^2(T^3;\mathbb{Z})$, there exists an integer matrix $A \in \mathrm{SL}(3;\mathbb{Z}) \subset \mathrm{Diff}(T^3)$ such that $A^*[\omega] \in \mathbb{Z}[dx \wedge dy]$.*

Proof By the Smith normal form, we can show that there exist an integer d and an invertible 3×3 integer matrix B such that

$$B \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}.$$

Therefore there exists $A \in \mathrm{SL}(3;\mathbb{Z})$ and an integer n such that $A^*[\omega] = n[dx \wedge dy]$ for some $n \in \mathbb{Z}$. □

5 A construction of a diffeomorphism of flows

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Lie $\mathfrak{g}$–flows on closed manifolds M_1 and M_2 and let Γ_1 and Γ_2 be the holonomy groups of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ have no closed orbits and Γ_1 is conjugate to Γ_2 in G.

By replacing the developing map $D_1\colon \widetilde{M}_1 \to G$ and the holonomy homomorphism $h_1\colon \pi_1(M_1) \to G$ of $\mathcal{F}_1$ by

$$g \cdot D_1\colon \widetilde{M}_1 \to G \qquad \text{and} \qquad g^{-1} \cdot h_1 \cdot g\colon \pi_1(M_1) \to G$$

for some $g \in G$, we may assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same holonomy group Γ. The aim of this section is to construct a diffeomorphism between $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ according to Ghys's method; see [7; 6; 14].

By results of Haefliger [8, Section 3], a Lie $\mathfrak{g}$–foliation of a closed manifold M is a classifying space for (G, Γ) if every leaf of $\mathcal{F}$ is contractible. Thus $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are classifying spaces for (G, Γ). By the uniqueness of classifying spaces, there exists a homotopy equivalence $f\colon M_1 \to M_2$, which we may assume is smooth, such that $f^*\mathcal{F}_2 = \mathcal{F}_1$. In general, this map f is not a diffeomorphism. However, by using the averaging technique (see [7; 6]), we can modify f to a diffeomorphism from $(M_1, \mathcal{F}_1)$ to $(M_2, \mathcal{F}_2)$.

Parametrize $\mathcal{F}_1$ and $\mathcal{F}_2$ by ϕ_1^t and ϕ_2^t, respectively. Then we can define a smooth function

$$u\colon M_1 \times \mathbb{R} \to \mathbb{R}$$

by the equation

$$f(\phi_1^t(x)) = \phi_2^{u(x,t)}(f(x)).$$

The function u satisfies the cocycle condition

$$u(x, s+t) = u(x,t) + u(\phi_1^t(x), s).$$

By this equation and by the compactness of M, we obtain the following lemma.

Lemma 5.1 *There exists a constant $C > 0$ such that*

$$\left| \frac{\partial}{\partial t} u(x,t) \right| < C$$

for any $x \in M$ and $t \in \mathbb{R}$.

Let $\pi_1 = \overline{D}_1 \colon M_1 \to Q_1 = \overline{\Gamma} \backslash G$ be the basic fibration of $\mathcal{F}_1$. Fix a fiber F_1 of $\pi_1 \colon M_1 \to Q_1$. Let $\{(U_i, g_i)\}_{i=1}^k$ be a local trivialization of π_1, where $\{U_i\}_{i=1}^k$ is an open covering of Q_1 and $g_i \colon \pi_1^{-1}(U_i) \to U_i \times F_1$ is a diffeomorphism as fiber bundles. Let $\{V_1, \dots, V_k\}$ be a refinement of $\{U_1, \dots, U_k\}$ such that the closure $\overline{V}_i$ is contained in U_i for each $i = 1, \dots, k$.

Since f is transverse to the flow $\mathcal{F}_2$, by replacing $V_1, \dots, V_k$ with smaller ones if necessary, for each i there exists a codimension-one open ball $D_i \subset F_1$ such that

(1) $N_i := g_i^{-1}(V_i \times D_i)$ is transverse to $\mathcal{F}_1$,

(2) $f(N_i)$ is transverse to $\mathcal{F}_2$, and

(3) the restriction $f|_{N_i} \colon N_i \to f(N_i)$ is a diffeomorphism.

Lemma 5.2 *There exists $T_0 > 0$ such that, for any $i = 1, \dots, k$ and any $x \in \pi_1^{-1}(\overline{V}_i)$, the orbit $O_{\phi_1}(x; (0, T_0)) := \{\phi_1^t(x) \mid 0 < t < T_0\}$ intersects N_i.*

Proof Define a function $r_i \colon \pi_1^{-1}(U_i) \to \mathbb{R}$ by

$$r_i(x) = \inf\{t > 0 \mid \phi_1^t(x) \in N_i\}.$$

Since $\mathcal{F}_1$ is minimal on each fiber, the function r_i is well-defined and upper semi-continuous. Since $\pi_1^{-1}(\overline{V}_i)$ is compact, for each $i \in \{1, \dots, k\}$, there exists an upper bound T_i. Then we should take $T_0 = \max\{T_1, \dots, T_k\} + 1$. □

For any $x \in N_i$, we define $s_i(x) \in \mathbb{Z}$ to be the number of times that the orbit $O_{\phi_2}\big(f(x); (-2CT_0 - \delta, 2CT_0 + \delta)\big)$ intersects $f(N_i)$, where δ is a small positive number and C and T_0 are the constants in Lemmas 5.1 and 5.2, respectively. By the choice of N_i, the function $s \colon N_i \to \mathbb{Z}$ is bounded.

For any $x \in N_i$, we consider the set

$$T_i(x) = \big\{t \in \mathbb{R} \;\big|\; \phi_1^t(x) \in N_i \text{ and } |u(x,t)| < 2CT + \delta\big\}.$$

If t and t' satisfy $u(x,t) = u(x,t')$ and $\phi_1^t(x)$ and $\phi_1^{t'}(x)$ are in N_i, then we have $f(\phi_1^t(x)) = f(\phi_1^{t'}(x))$. Since $f|_{N_i}$ is a diffeomorphism, we have $\phi_1^t(x) = \phi_1^{t'}(x)$. Since $\mathcal{F}_1$ has no closed orbits, this implies that $t = t'$. Hence, if t and t' are distinct points of $T_i(x)$, then $u(x,t) \neq u(x,t')$. Therefore the number of elements of $T_i(x)$ is less than $s_i(x)$ and hence bounded.

For an arbitrary point $x \in N_i$, we can take a sufficiently small connected neighborhood A_x of x in N_i and a sufficiently large number $t_x > 0$ such that t_x is an upper bound of $T_i(y)$ for any $y \in A_x$. Then we have

$$|u(y,t)| \geq 2CT_0 + \delta > 2CT_0$$

if $t > t_x$, $y \in A_x$ and $\phi_1^t(y) \in N_i$. Since the basic manifold $Q_1 = \overline{\Gamma}\backslash G$ is compact, we can choose connected open subsets $A_{i_1}, \dots, A_{i_{k_i}}$ of N_i and large numbers t_{i_j} such that

(1) $\{\pi_1(A_{i_j}) \mid i = 1, \dots, k,\ j = 1, \dots, k_i\}$ is a refinement of $\{V_1, \dots, V_k\}$, and
(2) t_{i_j} is an upper bound of $T_i(y)$ for any $y \in A_{i_j}$.

Let $t_0 > \max\{t_{i_j} \mid i = 1, \dots, k,\ j = 1, \dots, k_i\}$ be an arbitrary number. Then, for any $x \in A_{i_j}$, we have

$$|u(x,t)| > 2CT_0$$

if $t \geq t_0$ and $\phi_1^t(x) \in N_i$.

Lemma 5.3 *For any i_j, one of the following holds:*

(a) $u(x,t) > CT_0$ *for any* $x \in A_{i_j}$ *and any* $t \geq t_0$.
(b) $u(x,t) < -CT_0$ *for any* $x \in A_{i_j}$ *and any* $t \geq t_0$.

Proof Let $x \in A_{i_j}$ be an arbitrary point. Let $t_0 = s_0 < s_1 < s_2 < \cdots$ be the maximal sequence such that $\phi_1^{s_l}(x) \in N_i$ for $l \geq 1$. By Lemma 5.2, we obtain $s_{l+1} - s_l < T_0$ for any $l \geq 0$. On the other hand, we have $|u(x,s_l)| > 2CT_0$ for any $l \geq 1$.

Lemma 5.1 implies that

$$|u(x,s_{l+1}) - u(x,s_l)| < C(s_{l+1} - s_l) < CT_0.$$

Hence we have either $u(x,s_l) > 2CT_0$ for any $l \geq 1$ or $u(x,s_l) < -2CT_0$ for any $l \geq 1$.

For any $t \geq t_0$, there exists $l \geq 0$ such that $s_l \leq t \leq s_{l+1}$. By Lemma 5.1, we have

$$|u(x,s_{l+1}) - u(x,t)| \leq C(s_{l+1} - t) < C(s_{l+1} - s_l) < CT_0.$$

Therefore we have either $u(x,t) > CT_0$ for any $t \geq t_0$ or $u(x,t) < -CT_0$ for any $t \geq t_0$. By the continuity of u, we have either $u(x,t) > CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$ or $u(x,t) < -CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$. □

Let $W_{i_j} = \pi_1(A_{i_j})$ and $E_{i_j} = \pi_1^{-1}(W_{i_j})$.

Lemma 5.4 *There exists τ_0 such that, for each E_{i_j}, one of the following holds:*

(a) $u(x,t) > 0$ *for any* $x \in E_{i_j}$ *and any* $t \geq \tau_0$.

(b) $u(x,t) < 0$ *for any* $x \in E_{i_j}$ *and any* $t \geq \tau_0$.

Proof By Lemma 5.2, we have

$$S_{i_j} = \sup_{x \in E_{i_j}} \inf\{t > 0 \mid \phi_1^t(x) \in A_{i_j}\} < \infty$$

for any i_j. Let

$$\begin{aligned} S &= \max\{S_{i_j} \mid 1 \leq i \leq k,\ 1 \leq j \leq k_i\}, \\ \alpha_+ &= \max\{u(x,t) \mid x \in M,\ 0 \leq t \leq S\}, \\ \alpha_- &= \max\{-u(x,t) \mid x \in M,\ 0 \leq t \leq S\}, \end{aligned}$$

and

$$\alpha = \max\{\alpha_+, \alpha_-\}.$$

Take an integer n and a constant τ_0 satisfying

$$nCT_0 > \alpha \quad \text{and} \quad \tau_0 > n(t_0 + S).$$

Fix i_j. By Lemma 5.3, we have either $u(y,t) > CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$ or $u(x,t) < -CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$.

First, we suppose that $u(x,t) > CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$. Fix an arbitrary point $x \in E_{i_j}$ and any $t \geq \tau_0$. Define a sequence $0 \leq v_1 < v_2 < \cdots < v_n$ inductively as follows: Let v_1 be the first arrival time of x to A_{i_j}. Thus we have $\phi_1^{v_1}(x) \in A_{i_j}$ and $0 \leq v_1 \leq S$. For $l \geq 1$, let v_{l+1} be the first arrival time to A_{i_j} of x after the time $v_l + t_0$. Thus we have $\phi_1^{v_{l+1}} \in A_{i_j}$ and $v_l + t_0 \leq v_{l+1} \leq v_l + t_0 + S$.

Since $v_1 \leq S$ and $v_{l+1} - v_l \leq t_0 + S$, we have

$$v_n \leq S + (n-1)(t_0 + S) < \tau_0 - t_0 \leq t - t_0.$$

Since $v_{l+1} - v_l \geq t_0$ and $t - v_n > t_0$, we have

$$\begin{aligned} u(x,t) &= u\left(x, v_1 + \sum_{l=1}^{n-1}(v_{l+1} - v_l) + t - v_n\right) \\ &= u(x, v_1) + \sum_{l=1}^{n-1} u(\phi_1^{v_l}(x), v_{l+1} - v_l) + u(\phi_1^{v_n}(x), t - v_n) \\ &> -\alpha_- + (n-1)CT_0 + CT_0 \\ &> -\alpha + nCT_0 > 0. \end{aligned}$$

In the case where $u(x,t) < -CT_0$ for any $x \in A_{i_j}$ and any $t \geq t_0$, by the same argument, we have

$$\begin{aligned}
u(x,t) &= u\left(x, v_1 + \sum_{l=1}^{n-1}(v_{l+1} - v_l) + t - v_n\right) \\
&= u(x,v_1) + \sum_{l=1}^{n-1} u(\phi_1^{v_l}(x), v_{l+1} - v_l) + u(\phi_1^{v_n}(x), t - v_n) \\
&< \alpha_+ - (n-1)CT_0 - CT_0 \\
&< \alpha - nCT_0 < 0. \qquad \square
\end{aligned}$$

Finally, we prove the following lemma.

Lemma 5.5 *One of the following holds:*

(a) $u(x,t) > 0$ *for any* $x \in M_1$ *and any* $t \geq \tau_0$*.*

(b) $u(x,t) < 0$ *for any* $x \in M_1$ *and any* $t \geq \tau_0$*.*

Proof By the continuity of u and the connectedness of M, if there exists i_j such that $u(x,t) > 0$ for any $x \in E_{i_j}$ and any $t \geq \tau_0$, then $u(x,t) > 0$ for any $x \in M$ and any $t \geq \tau_0$. Similarly, if there exists i_j such that $u(x,t) < 0$ for any $x \in E_{i_j}$ and any $t \geq \tau_0$, then $u(x,t) < 0$ for any $x \in M$ and any $t \geq \tau_0$. $\square$

We construct a diffeomorphism from $(M_1, \mathcal{F}_1)$ to $(M_2, \mathcal{F}_2)$. Let $T \in \mathbb{R}$ be a positive constant such that $T \geq \tau_0$. Define $\nu_T\colon M_1 \to \mathbb{R}$ and $f_T\colon M_1 \to M_2$ by

$$\nu_T(x) = \frac{1}{T}\int_0^T u(x,\tau)\,d\tau \quad \text{and} \quad f_T(x) = \phi_2^{\nu(x)}(f(x)).$$

By the equation

$$\begin{aligned}
\nu_T(\phi_1^t(x)) &= \frac{1}{T}\int_0^T u(\phi_1^t(x),\tau)\,d\tau \\
&= \frac{1}{T}\int_0^T \{u(x,t+\tau) - u(x,t)\}\,d\tau,
\end{aligned}$$

we have

$$f_T(\phi_1^t(x)) = \phi_2^{\frac{1}{T}\int_0^T u(x,t+\tau)\,d\tau}(f(x)).$$

Therefore, for any $x \in M_1$, we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{T}\int_0^T u(x,t+\tau)\,d\tau\right) &= \frac{1}{T}(u(x,t+T) - u(x,t)) \\
&= \frac{1}{T}u(\phi_1^t(x),T).
\end{aligned}$$

By Lemma 5.5, $u(x,T) \neq 0$ for any $x \in M_1$. Therefore $f_T\colon M_1 \to M_2$ is a local diffeomorphism. Since M_1 is closed, f_T is a covering map. Since f_T is homotopic to f via

$$F\colon M_1 \times [0,1] \to M_2 \quad \text{defined by } F(x,t) = \phi_2^{t\nu(x)}(f(x))$$

and f is a homotopy equivalence, the map f_T is a diffeomorphism.

Therefore we obtain the following theorem.

Theorem 5.6 [7] *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be Lie $\mathfrak{g}$–flows on closed manifolds M_1 and M_2, respectively. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ have no closed orbits and the holonomy group of $\mathcal{F}_1$ is conjugate to the holonomy group of $\mathcal{F}_2$ in G. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are diffeomorphic.*

6 Proof of Theorem B

Let $\mathfrak{g}$ be a 3–dimensional solvable Lie algebra and let $\mathcal{F}$ be a Lie $\mathfrak{g}$–flow on a closed manifold M which has no closed orbits. Let Γ be the holonomy group of $\mathcal{F}$. Since $\mathcal{F}$ has no closed orbits, the holonomy homomorphism $h\colon \pi_1(M) \to G$ is injective. Hence the fundamental group $\pi_1(M)$ is isomorphic to the holonomy group Γ.

If the Lie algebra $\mathfrak{g}$ is nilpotent, then the Lie algebra $\mathfrak{g}$ is isomorphic to either $\mathbb{R}^3$ or $\mathfrak{n}(3)$. By the theorem of Ghys [7, Section 2], $(M,\mathcal{F})$ is diffeomorphic to a homogeneous Lie $\mathfrak{g}$–flow $(\Delta\backslash\widetilde{G}, \mathcal{F}_0)$, where $\widetilde{G}$ is a simply connected nilpotent Lie group. Since any 1–dimensional ideal of a nilpotent Lie algebra is contained in its center, the kernel of the induced homomorphism

$$dD_0\colon \tilde{\mathfrak{g}} \to \mathfrak{g}$$

is contained in the center of $\tilde{\mathfrak{g}}$. Hence $\widetilde{G}$ is a central extension of G by $\mathbb{R}$ and $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}$–flow in Example 3.1.

We suppose that $\mathfrak{g}$ is not nilpotent. First, we consider the case where $\mathfrak{g}$ is isomorphic to $\mathfrak{a}(3)$.

6A $\mathfrak{a}(3)$ case

Let $\mathcal{F}$ be a Lie $\mathfrak{a}(3)$–flow on a closed manifold M without closed orbits, and fix a nonsingular $\mathfrak{a}(3)$–valued Maurer–Cartan form ω of $\mathcal{F}$. The Lie algebra $\mathfrak{a}(3)$ has the explicit description

$$\mathfrak{a}(3) = \left\{ \begin{pmatrix} t & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \; t,x,y \in \mathbb{R} \right\}.$$

Then there exist nonsingular 1–forms ω_T, ω_X and ω_Y on M such that

$$\omega = \begin{pmatrix} \omega_T & 0 & \omega_X \\ 0 & 0 & \omega_Y \\ 0 & 0 & 0 \end{pmatrix}.$$

Since ω is a Maurer–Cartan form, we have the following equations:

$$d\omega_T = 0 \quad \text{and} \quad d\omega_X = \tfrac{1}{2}\omega_T \wedge \omega_X \quad \text{and} \quad d\omega_Y = 0.$$

Therefore ω_T and ω_Y are nonsingular closed 1–forms and

$$\omega' = \begin{pmatrix} \omega_T & \omega_X \\ 0 & 0 \end{pmatrix}$$

is a nonsingular $\mathfrak{a}(2)$–valued Maurer–Cartan form of M. The nonsingular closed 1–form ω_T and the nonsingular Maurer–Cartan form ω' define two foliations $\mathcal{G}$ and $\mathcal{F}'$ of M whose codimensions are one and two, respectively. Since ω' is a $\mathfrak{a}(2)$–valued Maurer–Cartan form, the foliation $\mathcal{F}'$ is a Lie $\mathfrak{a}(2)$–foliation. By an observation of Matsumoto and Tsuchiya [13, Section 7], we can see that the closed 1–form ω_T is a rational form. Therefore each leaf of $\mathcal{G}$ is compact and the leaf space $M/\mathcal{G}$ is diffeomorphic to S^1.

Let $\pi\colon M \to S^1 = M/\mathcal{G}$ be the natural projection and fix a fiber N of π. Since the tangent bundle $T\mathcal{F}$ coincides with $\mathrm{Ker}(\omega)$ and $\mathrm{Ker}(\omega_T)$ includes $\mathrm{Ker}(\omega)$, each orbit of the Lie $\mathfrak{a}(3)$–flow $\mathcal{F}$ is tangent to the fibers of π.

Let $\mathcal{F}|_N$ be the foliation defined by the restriction of $\mathcal{F}$ to the fiber N.

Lemma 6.1 *The fiber N is diffeomorphic to the 3–dimensional torus and the flow $\mathcal{F}|_N$ is diffeomorphic to a linear flow.*

Proof The tangent bundle TN coincides with $\mathrm{Ker}(\omega_T|_N)$. By the equation

$$d\omega_X = \tfrac{1}{2}\omega_T \wedge \omega_X,$$

the 1–form $\omega_X|_N$ on N is a nonsingular closed 1–form. Since $T\mathcal{F}$ coincides with

$$\mathrm{Ker}(\omega) = \mathrm{Ker}(\omega_T) \cap \mathrm{Ker}(\omega_X) \cap \mathrm{Ker}(\omega_Y),$$

$T\mathcal{F}|_N$ coincides with $\mathrm{Ker}(\omega_X|_N) \cap \mathrm{Ker}(\omega_Y|_N)$. Since the 1–forms $\omega_X|_N$ and $\omega_Y|_N$ are closed, the nonsingular $\mathbb{R}^2$–valued 1–form

$$\eta_N = \begin{pmatrix} \omega_X|_N \\ \omega_Y|_N \end{pmatrix}$$

is a Maurer–Cartan form. Hence the flow $\mathcal{F}|N$ is a Lie $\mathbb{R}^2$–flow on N. By the theorem of Caron and Carrière [2, Theorem 1], the manifold N is diffeomorphic to T^3 and the flow $\mathcal{F}|_N$ is diffeomorphic to a linear flow. $\square$

By Lemma 6.1, the manifold M is a T^3–bundle over S^1. Let $F \in \mathrm{Diff}_+(T^3)$ be the monodromy map of the T^3–bundle $\pi\colon M \to S^1$. Fix generators α_1, α_2 and α_3 of $\pi_1(T^3) \simeq \mathbb{Z}^3$ and an element β of $\pi_1(M)$ such that $\pi_*(\beta) \in \pi_1(S^1)$ is a generator. Then the induced map $F_*\colon \pi_1(T^3) \to \pi_1(T^3)$ defines an integer matrix $A \in \mathrm{SL}(3;\mathbb{Z})$ and the fundamental group $\pi_1(M)$ is isomorphic to $\mathbb{Z} \ltimes_A \mathbb{Z}^3$.

Set $A = (a_{ij})$. Then we have

$$\beta\alpha_j\beta^{-1} = \alpha_1^{a_{1j}}\alpha_2^{a_{2j}}\alpha_3^{a_{3j}} \quad \text{for } j = 1, 2, 3.$$

Since $\mathcal{F}$ has no closed orbits, the holonomy homomorphism $h\colon \pi_1(M) \to \Gamma$ is an isomorphism. Let Γ' be the abelian subgroup of Γ generated by $h(\alpha_1)$, $h(\alpha_2)$ and $h(\alpha_3)$. Since $\pi_1(N)$ is a normal subgroup of $\pi_1(M)$, Γ' is a normal subgroup of Γ.

Lemma 6.2 *Let H be an abelian subgroup of $A(3)$. Then H is contained in either*

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle|\, x, y \in \mathbb{R} \right\} \quad \text{or} \quad H_{(t_0,x_0)} = \left\{ \begin{pmatrix} e^t & 0 & \frac{1-e^t}{1-e^{t_0}}x_0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle|\, t, y \in \mathbb{R} \right\}$$

for some $x_0 \in \mathbb{R}$ and $t_0 \neq 0$.

Proof Suppose that H is not contained in $\mathbb{R}^2$. Then there exists

$$g_0 = \begin{pmatrix} e^{t_0} & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \in H$$

such that $t_0 \neq 0$. Let

$$g = \begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in H$$

be an arbitrary element. Since H is abelian, we have $g_0 g = g g_0$. Then we obtain the equation

$$x = \frac{1-e^t}{1-e^{t_0}}x_0. \qquad \square$$

Lemma 6.3 *The abelian subgroup Γ' of Γ is contained in $\mathbb{R}^2$.*

Proof Suppose that Γ' is not contained in $\mathbb{R}^2$. By Lemma 6.2, Γ' is contained in $H_{(t_0,x_0)}$ for some $x_0 \in \mathbb{R}$ and $t_0 \neq 0$. Since $\Gamma' \not\subset \mathbb{R}^2$, there exists

$$g = \begin{pmatrix} e^t & 0 & \frac{1-e^t}{1-e^{t_0}} x_0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma'$$

such that $t \neq 0$.

Set

$$h(\beta) = \begin{pmatrix} e^{t'} & 0 & x' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \in A(3).$$

Since Γ' is normal in Γ, we have

$$h(\beta) g h(\beta)^{-1} \in \Gamma' \subset H_{(t_0,x_0)}.$$

Then we obtain the equation

$$(1-e^t)\bigl\{(1-e^{t_0})x' - (1-e^{t'})x_0\bigr\} = 0.$$

Since $t \neq 0$, this equation implies that $h(\beta) \in H_{(t_0,x_0)}$. Thus Γ is contained in $H_{(t_0,x_0)}$. However, this contradicts the fact that the holonomy group Γ is uniform in $A(3)$. Therefore Γ' is contained in $\mathbb{R}^2$. □

Set

$$h(\alpha_i) = \begin{pmatrix} 1 & 0 & x_i \\ 0 & 1 & y_i \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } i = 1,2,3 \qquad \text{and} \qquad h(\beta) = \begin{pmatrix} e^{t_0} & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ is uniform in $A(3)$, we have $t_0 \neq 0$. Moreover, by conjugating in $A(3)$, we may assume that $x_0 = 0$.

Lemma 6.4 *A is conjugate to the matrix*

$$\begin{pmatrix} e^{t_0} & 0 & 0 \\ 0 & e^{-t_0} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof By the equation

$$\beta \alpha_j \beta^{-1} = \alpha_1^{a_{1j}} \alpha_2^{a_{2j}} \alpha_3^{a_{3j}},$$

we have

$$\begin{pmatrix} 1 & 0 & e^{t_0}x_j \\ 0 & 1 & y_j \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_{1j}x_1 + a_{2j}x_2 + a_{3j}x_3 \\ 0 & 1 & a_{1j}y_1 + a_{2j}y_2 + a_{3j}y_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus we have the equations

$${}^tA \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = e^{t_0} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad {}^tA \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Since Γ is uniform in $A(3)$, we can show that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \mathbf{0} \quad \text{and} \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \neq \mathbf{0}.$$

Therefore e^{t_0}, 1 and e^{-t_0} are the eigenvalues of A. □

Define elements $\widehat{\alpha}_i$ and $\widehat{\beta}$ of $\widetilde{G}_2^{-1}$ by

$$\widehat{\alpha}_i = \begin{pmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & z_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \widehat{\beta} = \begin{pmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & e^{-t_0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue e^{t_0}. Let Δ be the subgroup of $\widetilde{G}_2^{-1}$ generated by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ and $\widehat{\beta}$. Since

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

are eigenvectors of $A \in \mathrm{SL}(3;\mathbb{Z})$ corresponding to the eigenvalues e^{t_0}, 1 and e^{-t_0}, respectively, the subgroup Δ is discrete in $\widetilde{G}_2^{-1}$. Therefore Δ is a uniform lattice of $\widetilde{G}_2^{-1}$.

Define an submersion homomorphism $D_0\colon \widetilde{G}_2^{-1} \to A(3)$ by

$$D_0\colon \begin{pmatrix} e^t & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & e^{-t} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} e^t & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then D_0 defines a Lie $\mathfrak{a}(3)$–flow $\mathcal{F}_0$ on $\Delta\backslash\widetilde{G}_2^{-1}$ whose holonomy group coincides with Γ. Therefore, by Theorem 5.6, the Lie $\mathfrak{a}(3)$–flow $\mathcal{F}$ is diffeomorphic to $\mathcal{F}_0$. Hence $\mathcal{F}$ is diffeomorphic to the flow in Example 3.3.

6B $\mathfrak{g}_2^k$ case

We consider the case where $\mathfrak{g}$ is isomorphic $\mathfrak{g}_2^k$. In this case, the basic dimension of $\mathcal{F}$ is one; see [5; 11]. Hence the manifold M is diffeomorphic to a T^3–bundle over S^1.

Let α_1, α_2, α_3 and β be the same as defined in Section 6A. Then there exists an integer matrix $A = (a_{ij}) \in \mathrm{SL}(3;\mathbb{Z})$ such that the fundamental group $\pi_1(M)$ is isomorphic to $\mathbb{Z} \ltimes_A \mathbb{Z}^3$.

Let Γ' be the normal abelian subgroup of Γ generated by $h(\alpha_1)$, $h(\alpha_2)$ and $h(\alpha_3)$.

Lemma 6.5 *Let H be an abelian subgroup of G_2^k. Then H is contained in either*

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{or} \quad H_{(t_0,x_0,y_0)} = \left\{ \begin{pmatrix} e^t & 0 & \frac{1-e^t}{1-e^{t_0}}x_0 \\ 0 & e^{kt} & \frac{1-e^{kt}}{1-e^{kt_0}}y_0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

for some $x_0, y_0 \in \mathbb{R}$ and $t_0 \neq 0$.

Proof Suppose that H is not contained in $\mathbb{R}^2$. Then there exists

$$g_0 = \begin{pmatrix} e^{t_0} & 0 & x_0 \\ 0 & e^{kt_0} & y_0 \\ 0 & 0 & 1 \end{pmatrix} \in H$$

such that $t_0 \neq 0$. Let

$$g = \begin{pmatrix} e^t & 0 & x \\ 0 & e^{kt} & y \\ 0 & 0 & 1 \end{pmatrix} \in H$$

be an arbitrary element. Then $gg_0 = g_0g$ implies the equations

$$(1-e^{t_0})x = (1-e^t)x_0 \quad \text{and} \quad (1-e^{t_0})x = (1-e^t)y_0.$$

Since $t_0 \neq 0$, these equations imply that $g \in H_{(t_0,x_0,y_0)}$. □

Lemma 6.6 *The normal abelian subgroup* Γ' *of* Γ *is contained in* $\mathbb{R}^2$*.*

Proof Suppose that Γ' is not contained in $\mathbb{R}^2$. By Lemma 6.5, there exist $x_0, y_0 \in \mathbb{R}$ and $t_0 \neq 0$ such that Γ' is contained in $H_{(t_0,x_0,y_0)}$. Since $\Gamma' \not\subset \mathbb{R}^2$, there exists

$$g = \begin{pmatrix} e^t & 0 & x \\ 0 & e^{kt} & y \\ 0 & 0 & 1 \end{pmatrix} \in \Gamma'$$

such that $t \neq 0$.

Set

$$h(\beta) = \begin{pmatrix} e^{t'} & 0 & x' \\ 0 & e^{kt'} & y' \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ' is a normal subgroup of Γ, we have

$$h(\beta) g h(\beta)^{-1} \in \Gamma' \subset H_{(t_0,x_0,y_0)}.$$

Thus we obtain the equations

$$(1-e^t)\{(1-e_0^t)x_0-(1-e^{t'})x_0\}=0 \quad \text{and} \quad (1-e^{kt})\{(1-e^{kt})x'-(1-e^{kt'})x_0\}=0.$$

Since $t \neq 0$, these equations imply that $h(\beta) \in H_{(t_0,x_0,y_0)}$. This contradicts the fact that Γ is uniform in G_2^k. Therefore we have that Γ' is contained in $\mathbb{R}^2$. □

Set

$$h(\alpha_j) = \begin{pmatrix} 1 & 0 & x_j \\ 0 & 1 & y_j \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h(\beta) = \begin{pmatrix} e^{t_0} & 0 & x_0 \\ 0 & e^{kt_0} & y_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ is uniform in G_2^k, we have $t_0 \neq 0$. Moreover, by conjugating in G_2^k, we may assume that $x_0 = 0$ and $y_0 = 0$.

By the same argument as the proof of Lemma 6.4, we can prove the following lemma.

Lemma 6.7 $A \in \mathrm{SL}(3;\mathbb{Z})$ *is conjugate to the matrix*

$$\begin{pmatrix} e^{t_0} & 0 & 0 \\ 0 & e^{kt_0} & 0 \\ 0 & 0 & e^{(-1-k)t_0} \end{pmatrix}.$$

Define elements $\widehat{\alpha}_i$ and $\widehat{\beta}$ of $\widetilde{G}_2^k$ by

$$\widehat{\alpha}_i = \begin{pmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & z_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \widehat{\beta} = \begin{pmatrix} e^{t_0} & 0 & 0 & 0 \\ 0 & e^{kt_0} & 0 & 0 \\ 0 & 0 & e^{-(1+k)t_0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue e^{t_0}. Let Δ be the subgroup of $\widetilde{G}_2^k$ generated by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ and $\widehat{\beta}$. Then Δ is a uniform lattice of $\widetilde{G}_2^k$. Since Δ coincides with Γ via the homomorphism $D\colon \widetilde{G}_2^k \to G_2^k$ defined in Example 3.4, the Lie $\mathfrak{g}_2^k$–flow $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}_2^k$–flow in Example 3.4.

6C $\mathfrak{g}_3^{h\neq 0}$ case

In the case in which $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^{h\neq 0}$, the basic dimension of $\mathcal{F}$ is one; see [5; 11]. Hence the manifold M is diffeomorphic to a T^3–bundle over S^1 and $\pi_1(M) = \mathbb{Z} \ltimes_A \mathbb{Z}^3$ for some $A \in \mathrm{SL}(3;\mathbb{Z})$.

By the same argument as in Section 6B, we can prove the following lemma.

Lemma 6.8 *The normal subgroup Γ' of Γ is contained in $\mathbb{R}^2$.*

By Lemma 6.8, we have

$$h(\alpha_i) = \begin{pmatrix} 1 & 0 & x_i \\ 0 & 1 & y_i \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{\beta t_0}\cos t_0 & -e^{\beta t_0}\sin t_0 & x_0 \\ e^{\beta t_0}\sin t_0 & e^{\beta t_0}\cos t_0 & y_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since Γ is uniform in G_3^h, we have $t_0 \neq 0$. By conjugating in G_3^h, we may assume that $x_0 = 0$ and $y_0 = 0$.

By the equation

$$\beta\alpha_j\beta^{-1} = \alpha_1^{a_{1j}}\alpha_2^{a_{2j}}\alpha_3^{a_{3j}},$$

we have

$$A\boldsymbol{x} = e^{\beta t_0}(\cos t_0\boldsymbol{x} - \sin t_0\boldsymbol{y}) \quad \text{and} \quad A\boldsymbol{y} = e^{\beta t_0}(\sin t_0\boldsymbol{x} + \cos t_0\boldsymbol{y}),$$

where

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

By an easy calculation, we can show that $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$. Hence $\boldsymbol{x} + i\,\boldsymbol{y}$ and $\boldsymbol{x} - i\,\boldsymbol{y}$ are eigenvectors corresponding to the eigenvalues $e^{(\beta+i)t_0}$ and $e^{(\beta-i)t_0}$, respectively.

Let $d^{t_0} = e^{-2\beta t_0}$ be the other eigenvalue of A, and let

$$\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

be an eigenvector of A corresponding to the eigenvalue d^{t_0}. Let

$$\widehat{\alpha}_i = \begin{pmatrix} 1 & 0 & 0 & x_i \\ 0 & 1 & 0 & y_i \\ 0 & 0 & 1 & z_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \widehat{\beta} = \begin{pmatrix} e^{\beta t_0}\cos t_0 & -e^{\beta t_0}\sin t_0 & 0 & 0 \\ e^{\beta t_0}\sin t_0 & e^{\beta t_0}\cos t_0 & 0 & 0 \\ 0 & 0 & d^{t_0} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be elements of $\widetilde{G}_3^h$ and Δ be the subgroup of $\widetilde{G}_3^h$ generated by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ and $\widehat{\beta}$. Then the subgroup Δ is a uniform lattice of $\widetilde{G}_3^h$

Since Δ coincides with Γ via the homomorphism $D\colon \widetilde{G}_3^h \to G_3^h$ defined in Example 3.5, the Lie $\mathfrak{g}_3^h$–flow $\mathcal{F}$ is diffeomorphic to the Lie $\mathfrak{g}_3^h$–flow in Example 3.5.

6D $\mathfrak{g}_3^0$ case

Suppose that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_3^0$. By [10, Corollaries 2.4 and 2.7] and the theorem of Caron and Carrière [2, Theorem 1], the manifold M is diffeomorphic to the 4–dimensional torus T^4.

Fix generators α_1, α_2, α_3 and α_4 of $\pi_1(M) \simeq \mathbb{Z}^4$ and set

$$h(\alpha_i) = \left(t_i, \begin{pmatrix} x_i \\ y_i \end{pmatrix} \right) \in G_3^0 = \mathbb{R} \ltimes_\rho \mathbb{R}^2.$$

Since Γ is uniform in G_3^0, Γ is not contained in $\{0\} \times \mathbb{R}^2$. Hence we may assume that $t_1 \neq 0$.

Lemma 6.9 *$t_i \in 2\pi\mathbb{Z}$, for each i.*

Proof Suppose that there exists i such that $t_i \notin 2\pi\mathbb{Z}$. We may assume that $t_1 \notin 2\pi\mathbb{Z}$.

Since Γ is abelian, we can show that Γ is contained in H, where

$$H = \left\{ \left(t, \begin{pmatrix} 1-\cos t & \sin t \\ -\sin t & 1-\cos t \end{pmatrix} \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} \right) \,\middle|\, t \in \mathbb{R} \right\}$$

is a simply connected 1–dimensional closed subgroup of G_3^0 and

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} 1-\cos t_1 & \sin t_1 \\ -\sin t_1 & 1-\cos t_1 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Since Γ is uniform in G_3^0, the homogeneous space $H\backslash G_3^0$ is compact.

On the other hand, $H\backslash G_3^0$ is homeomorphic to $\mathbb{R}^2$, since H is a simply connected 1–dimensional closed subgroup of G_3^0. This is a contradiction. $\square$

By Lemma 6.9, we have $t_i = 2\pi n_i$. Define a diffeomorphism $F\colon G_3^0 \to \mathbb{R}^3$ by

$$F\colon \left(t, \begin{pmatrix} x \\ y \end{pmatrix} \right) \mapsto \begin{pmatrix} t \\ x \\ y \end{pmatrix}.$$

Then $F|_\Gamma\colon \Gamma \to \mathbb{R}^3$ is a homomorphism and F is $F|_\Gamma$–equivariant, that is,

$$F(\gamma \cdot g) = F|_\Gamma(\gamma) \cdot F(g)$$

for any $\gamma \in \Gamma$ and any $g \in G_3^0$. Therefore the rank of the matrix

$$\begin{pmatrix} 2\pi n_1 & 2\pi n_2 & 2\pi n_3 & 2\pi n_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix}$$

is three.

We may assume that

$$\begin{pmatrix} 2\pi n_1 \\ x_1 \\ y_1 \end{pmatrix}, \quad \begin{pmatrix} 2\pi n_2 \\ x_2 \\ y_2 \end{pmatrix}, \quad \begin{pmatrix} 2\pi n_3 \\ x_3 \\ y_3 \end{pmatrix}$$

are linearly independent. Consider the subgroup Δ of $G_3^0 \times \mathbb{R} = \mathbb{R} \ltimes_\rho \mathbb{R}^2 \times \mathbb{R}$ generated by $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ and $\widehat{\alpha}_4$, where

$$\widehat{\alpha}_i = \left(2\pi n_i, \begin{pmatrix} x_i \\ y_i \end{pmatrix}, 0 \right) \quad \text{for } i = 1,2,3 \quad \text{and} \quad \widehat{\alpha}_4 = \left(2\pi n_4, \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}, 1 \right).$$

Then Δ is a uniform lattice of $G_3^0 \times \mathbb{R}$ and Δ coincides with Γ via the homomorphism $D\colon G_3^0 \times \mathbb{R} \to G_3^0$ in Example 3.6. Therefore the Lie $\mathfrak{g}_3^0$–flow is diffeomorphic to the flow in Example 3.6.

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