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E_n–cohomology with coefficients as functor cohomology

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Building on work of Livernet and Richter, we prove that E_n–homology and E_n–cohomology of a commutative algebra with coefficients in a symmetric bimodule can be interpreted as functor homology and cohomology. Furthermore, we show that the associated Yoneda algebra is trivial.

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1 Introduction

The little n–cubes operad was introduced to study n–fold loop spaces (see Boardman and Vogt [2] and May [13]). An E_n–operad is a Σ_*–cofibrant operad weakly equivalent to the operad formed by the singular chains on the little n–cubes operad, and algebras over such an operad are called E_n–algebras. Those are A_∞–algebras which are in addition commutative up to higher homotopies of a certain level depending on n. For a Σ_*–cofibrant operad one can define a suitable notion of homology and cohomology of algebras over this operad as a derived functor. For E_1–algebras this operadic notion of homology coincides with Hochschild homology. For E_∞–algebras one retrieves Γ–homology as defined by Robinson; see Robinson and Whitehouse [17]. In general, for a commutative algebra viewed as an E_n–algebra, E_n–homology can be seen to coincide with higher order Hochschild homology as defined in Pirashvili [14]; see Ginot, Tradler and Zeinalian [8] and Ziegenhagen [19].

Many notions of homology can be expressed as functor homology. The case of Hochschild homology and cyclic homology has been studied by Richter and Pirashvili in [16]. The same authors give a functor homology interpretation of Γ–homology in [15]. In [10], Hoffbeck and Vespa show that Leibniz homology of Lie algebras is functor homology. A more general approach to functor homology for algebras over an operad and their operadic homology is discussed in [6] by Fresse.

For the case of E_n–homology, functor homology interpretations of E_n–homology have been given by Livernet and Richter in [11] and Fresse in [4]. Both articles are exclusively concerned with the case of trivial coefficients. As proved in [5], E_n–homology with trivial coefficients coincides up to a suspension with the homology of a

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generalized iterated bar construction. Muriel Livernet and Birgit Richter use this in [11] to prove that E_n–homology of a commutative algebra with trivial coefficients can be interpreted as functor homology over a category of trees denoted by Epi_n. Fresse shows in [4] that this result can be extended to arbitrary E_n–algebras.

Recent work by Fresse and the author shows that E_n–homology and E_n–cohomology of a commutative algebra with coefficients in a symmetric bimodule can also be calculated via the iterated bar construction; see Fresse and Ziegenhagen [7]. We show in this article that the functor homology interpretation of Livernet and Richter can be extended to the case with coefficients and also holds for cohomology. More precisely, we introduce a category Epi_n^+ of trees extending the category Epi_n and a functor $b\colon {\mathrm{Epi}_n^+}^{\mathrm{op}} \to k$–mod, where k is any commutative unital ring. Then to a commutative nonunital k–algebra A and a symmetric A–bimodule M we associate Loday functors $\mathcal{L}(A;M)\colon \mathrm{Epi}_n^+ \to k$–mod and $\mathcal{L}^c(A;M)\colon {\mathrm{Epi}_n^+}^{\mathrm{op}} \to k$–mod and prove the following theorem:

Theorem 1.1 *We have an isomorphism*

$$H_*^{E_n}(A;M) \cong \mathrm{Tor}_*^{\mathrm{Epi}_n^+}(b, \mathcal{L}(A;M)),$$

and, if k is self-injective, an isomorphism

$$H^*_{E_n}(A;M) \cong \mathrm{Ext}^*_{{\mathrm{Epi}_n^+}^{\mathrm{op}}}(b, \mathcal{L}^c(A;M)).$$

This implies that there is an action on E_n–cohomology by the corresponding Yoneda algebra. We show that this algebra is trivial.

Outline We give an overview of the constructions of [11] in Section 2. In Section 3 we recall how to calculate E_n–homology and –cohomology of commutative algebras with coefficients in a symmetric bimodule via the iterated bar construction. To do this one introduces a twisting differential. In Section 4 we enlarge the category defined by Livernet and Richter to incorporate this twisting differential. We define E_n–homology and –cohomology for functors from this category to k–modules. Finally we show that there are Loday functors linking these notions to the usual notion of E_n–homology and –cohomology. We prove our main theorem in Section 5. In Section 6 we recall the definition of the Yoneda pairing and show that the Yoneda algebra is trivial.

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Conventions In the following we assume that $1 \leqslant n < \infty$. Let k be a commutative unital ring. We denote by A a commutative nonunital k–algebra and by M a symmetric A–bimodule. We often view A and M as differential graded k–modules concentrated in degree zero. Let $A_+ = A \oplus k$ be the unital augmented algebra obtained by adjoining a unit to A. We denote by $sc \in \Sigma C$ the element defined by $c \in C$ in the suspension of a graded k–module C. The k–module $k[X]$ is the free k–module generated by a set X. For $l \geqslant 0$ we denote by $[l]$ the set $[l] = \{0, \dots, l\}$.

2 The category Epi_n encoding the n–fold bar complex

In [5] Fresse proves that E_n–homology of E_n–algebras with trivial coefficients can be computed via the iterated bar complex. Livernet and Richter use this in [11] to give an interpretation of E_n–homology of commutative algebras with trivial coefficients as functor homology. They encode the information necessary to define an iterated bar complex in a category Epi_n of trees. We recall the construction of this category.

Definition 2.1 Let C be a differential graded nonunital algebra. The bar complex $B(C)$ is the differential graded k–module given by

$$B(C) = (\overline{T}^c(\Sigma C), \partial_B),$$

where $\overline{T}^c(\Sigma C)$ denotes the reduced tensor coalgebra on ΣC equipped with the differential induced by the differential of C. The twisting cochain ∂_B is defined by

$$\partial_B([c_1|\cdots|c_l]) = \sum_{i=1}^{l-1} (-1)^{i-1}[c_1|\cdots|c_i c_{i+1}|\cdots|c_l].$$

Here we use the classical bar notation and denote $sc_1 \otimes \cdots \otimes sc_l \in (\Sigma C)^{\otimes l}$ by $[c_1|\cdots|c_l]$. If C is commutative, the shuffle product

$$\mathrm{sh}\colon B(C) \otimes B(C) \to B(C)$$

is defined by

$$\mathrm{sh}([c_1|\cdots|c_j] \otimes [c_{j+1}|\cdots|c_{j+l}]) = \sum_{\sigma \in \mathrm{sh}(j,l)} \pm [c_{\sigma^{-1}(1)}|\cdots|c_{\sigma^{-1}(j+l)}],$$

with $\mathrm{sh}(j,l) \subset \Sigma_{j+l}$ the set of (j,l)–shuffles. For homogeneous elements $c_1, \dots, c_{j+l}$ the summand $[c_{\sigma^{-1}(1)}|\cdots|c_{\sigma^{-1}(j+l)}]$ is decorated by the graded signature $(-1)^{\epsilon}$, with

$$\epsilon = \prod_{\substack{i<l \\ \sigma(i)>\sigma(l)}} (|c_i|+1)(|c_l|+1).$$

The shuffle product makes $B(C)$ a commutative differential graded k–algebra.

We can iterate this construction and form the n–fold bar complex $B^n(A)$. The results in [5] for E_n–algebras imply that for any k–projective commutative nonunital k–algebra A we have

$$H_*^{E_n}(A;k) = H_*(\Sigma^{-n} B^n(A)).$$

Elements in the n–fold bar construction $B^n(A)$ correspond to sums of planar fully grown trees with leaves labelled by elements in A; see [3]. We fix some terminology concerning trees.

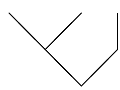
Definition 2.2 A planar fully grown n–level tree t is a sequence

$$t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$$

of order-preserving surjections. The element $i \in [r_j]$ is called the i^{th} vertex of the j^{th} level. The elements in $[r_n]$ are also called leaves. The degree of a tree t is given by the number of its edges, ie by

$$d(t) = \sum_{j=1}^{n} (r_j + 1).$$

For example, the 2–level tree



is given by the sequence $[2] \xrightarrow{f_2} [1]$ with $f_2(0) = f_2(1) = 0$, $f_2(2) = 1$.

Definition 2.3 For a given vertex $i \in [r_j]$ the subtree $t_{j,i}$ is the $(n-j)$–level subtree of t given by

$$t_{j,i} = [|f_n^{-1} \cdots f_{j+1}^{-1}(i)|-1] \xrightarrow{g_n} [|f_{n-1}^{-1} \cdots f_{j+1}^{-1}(i)|-1] \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_{j+2}} [|f_{j+1}^{-1}(i)|-1],$$

with g_l the map making the diagram

$$\begin{tikzcd}
{[|f_l^{-1} \cdots f_{j+1}^{-1}(i)| - 1]} \arrow[r, "g_l"] \arrow[d, "\cong"] & {[|f_{l-1}^{-1} \cdots f_{j+1}^{-1}(i)| - 1]} \arrow[d, "\cong"] \\
f_l^{-1} \cdots f_{j+1}^{-1}(i) \arrow[r, "f_l"] & f_{l-1}^{-1} \cdots f_{j+1}^{-1}(i)
\end{tikzcd}$$

commute. Here the vertical maps are the unique order-preserving bijections.

Definition 2.4 [11, Definition 3.1] The category Epi_n has as objects planar fully grown trees with n levels. A morphism from

$$[r_n] \xrightarrow{f_n^r} \cdots \xrightarrow{f_2^r} [r_1] \quad \text{to} \quad [s_n] \xrightarrow{f_n^s} \cdots \xrightarrow{f_2^s} [s_1]$$

consists of surjections $h_i\colon [r_i] \to [s_i]$, $1 \leqslant i \leqslant n$, such that the diagram

$$\begin{tikzcd}
[r_n] \arrow[r, "f_n^r"] \arrow[d, "h_n"] & {[r_{n-1}]} \arrow[r, "f_{n-1}^r"] \arrow[d, "h_{n-1}"] & \cdots \arrow[r, "f_2^r"] & {[r_1]} \arrow[d, "h_1"] \\
{[s_n]} \arrow[r, "f_n^s"] & {[s_{n-1}]} \arrow[r, "f_{n-1}^s"] & \cdots \arrow[r, "f_2^s"] & {[s_1]}
\end{tikzcd}$$

commutes and such that h_i is order-preserving on the fibres $(f_i^r)^{-1}(l)$ of f_i^r for all $l \in [r_{i-1}]$. For $i = 1$ we require that the map h_1 is order-preserving on $[r_1]$. The composite of two morphisms $(g_n, \dots, g_1)\colon t^q \to t^r$ and $(h_n, \dots, h_1)\colon t^r \to t^s$ is given by $(h_n g_n, \dots, h_1 g_1)$.

Observe that since A is concentrated in degree zero, the degree of a labelled tree viewed as an element in $B^n(A)$ is given by the number of edges of the tree. Lemma 3.5 in [11] says that the maps in Epi_n decreasing the number of edges by one are exactly the summands of the differential of $B^n(A)$. This motivates the following definition.

Definition 2.5 [11, Definition 3.7] Let $F\colon \mathrm{Epi}_n \to k$–mod be a covariant functor. Let $\widetilde{C}^{E_n}(F)$ be the $(\mathbb{N} \cup \{0\})^n$–graded k–module with

$$\widetilde{C}^{E_n}_{(r_n,\dots,r_1)}(F) = \bigoplus F(t),$$

where the sum is indexed over all trees

$$t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1].$$

Let $d_i\colon [r_n] \to [r_n - 1]$ denote the order-preserving surjection which maps i and $i+1$ to i. For $1 \leqslant j \leqslant n$ let $\widetilde{\partial}_j\colon \widetilde{C}^{E_n} \to \widetilde{C}^{E_n}$ be the following map lowering the j^{th} degree by one:

- For $j = n$ define $\widetilde{\partial}_j$ restricted to $F(t)$ as
$$\sum_{\substack{0 \leqslant i < r_n \\ f_n(i) = f_n(i+1)}} (-1)^{s_{n,i}} F(d_i, \mathrm{id}_{[r_{n-1}]}, \dots, \mathrm{id}_{[r_1]}).$$
- Let $1 \leqslant j < n$, $0 \leqslant i < r_j$ and $\sigma \in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))$. Let $h = h_{i,\sigma}$ be the unique morphism of trees, exhibited in [11, Lemma 3.5], such that

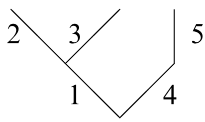
$h_j = d_i\colon [r_j] \to [r_j - 1]$, $h_l = \mathrm{id}$ for $l < j$, and h_{j+1} restricted to $f_{j+1}^{-1}(\{i, i+1\})$ acts like σ. Then $\widetilde{\partial}_j$ is the map whose restriction to $F(t)$ equals

$$\sum_{\substack{0 \leqslant i < r_j \\ f_j(i) = f_j(i+1)}} \sum_{\sigma \in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \epsilon(\sigma; t_{j,i}, t_{j,i+1})(-1)^{s_{j,i}} F(h_{i,\sigma}).$$

The signs arise from switching the degree -1 map d_i with suspensions, as well as from the graded signature of the permutation σ in the cases $j < n$. More precisely, we number the edges in the tree t from bottom to top and from left to right. For example, the 2–level tree

$$[2] \xrightarrow{f_2} [1] \quad \text{with } f_2(0) = f_2(1) = 0 \text{ and } f_2(2) = 1$$

is decorated as indicated in the following picture:



Then for $j < n$ we acquire a sign $(-1)^{s_{j,i}}$, where $s_{j,i}$ is the number of the rightmost top edge of the $(n-j)$–level subtree $t_{j,i}$ of t. For $j = n$ set $s_{n,i}$ to be the label of the edge whose leaf is the i^{th} leaf for $0 \leqslant i \leqslant n$.

For $j < n$ the map $F(h_{i,\sigma})$ is not only decorated by $(-1)^{s_{j,i}}$ but also by a sign associated to $\sigma \in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))$: Let $t_1, \dots, t_a$ be the $(n-j-1)$–level subtrees of t above the j–level vertex i, ie the $(n-j-1)$–level subtrees forming $t_{j,i}$. Similarly let $t_{a+1}, \dots, t_{a+b}$ denote the $(n-j-1)$–level subtrees above $i+1$. Then σ determines a shuffle of $\{t_1, \dots, t_a\}$ and $\{t_{a+1}, \dots, t_{a+b}\}$. The sign $\epsilon(\sigma; t_{j,i}, t_{j,i+1})$ picks up a factor $(-1)^{(d(t_x)+1)(d(t_y)+1)}$ whenever $x < y$ and $\sigma(x) > \sigma(y)$.

Lemma 2.6 *For any functor $F\colon \mathrm{Epi}_n \to k$–mod the $(\mathbb{N} \cup \{0\})^n$–graded module $\widetilde{C}^{E_n}(F)$ together with $\widetilde{\partial}_1, \dots, \widetilde{\partial}_n$ forms a multicomplex, which we again denote by $\widetilde{C}^{E_n}(F)$.*

Definition 2.7 [11, Definition 3.7] The homology

$$H_*^{E_n}(F) = H_*(\mathrm{Tot}(\widetilde{C}^{E_n}(F)))$$

of the total complex associated to $\widetilde{C}^{E_n}(F)$ is called the E_n–homology of $F\colon \mathrm{Epi}_n \to k$–mod.

Livernet and Richter show that there is a Loday functor

$$\mathcal{L}(A;k)\colon \mathrm{Epi}_n \to k\text{–mod}$$

associated to every nonunital commutative algebra A such that

$$H_*^{E_n}(\mathcal{L}(A;k)) = H_*^{E_n}(A;k)$$

whenever A is k–projective. They then prove that E_n–homology of functors is indeed functor homology:

Theorem 2.8 [11, Theorem 4.1] *Let $\widetilde{b}\colon \mathrm{Epi}_n^{\mathrm{op}} \to k\text{–mod}$ be the functor given by*

$$\widetilde{b}(t) = \begin{cases} k & \text{if } t = [0] \to \cdots \to [0], \\ 0 & \text{otherwise.} \end{cases}$$

Then for $F\colon \mathrm{Epi}_n \to k\text{–mod}$ we have

$$H_*^{E_n}(F) = \mathrm{Tor}_*^{\mathrm{Epi}_n}(\widetilde{b}, F).$$

3 E_n–homology with coefficients via the iterated bar complex

Recent work by Fresse and the author (see [7]) shows that, at least for a commutative nonunital k–algebra A and a symmetric A–bimodule M, the iterated bar complex can also be used to calculate E_n–homology and –cohomology with coefficients. In order to incorporate the action of A on M one has to add a twisting cochain

$$\delta\colon A_+ \otimes B^n(A) \to A_+ \otimes B^n(A)$$

to the complex $A_+ \otimes B^n(A)$.

Definition 3.1 Given an n–level tree $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ and $a_0, \dots, a_{r_n} \in A$, let $t(a_0, \dots, a_{r_n})$ denote the element in $B^n(A)$ defined by t with leaves labelled by $a_0, \dots, a_{r_n}$. The twisting morphism $\delta\colon A_+ \otimes B^n(A) \to A_+ \otimes B^n(A)$ is given by

$$\delta(a \otimes t(a_0, \dots, a_{r_n})) = \sum_{\substack{0 \leqslant l \leqslant r_{n-1} \\ |f_n^{-1}(l)|>1 \\ x = \min f_n^{-1}(l)}} (-1)^{s_{n,x}-1} a a_x \otimes (t \setminus x)(a_0, \dots, \widehat{a}_x, \dots, a_{r_n}) + \sum_{\substack{0 \leqslant l \leqslant r_{n-1} \\ |f_n^{-1}(l)|>1 \\ y = \max f_n^{-1}(l)}} (-1)^{s_{n,y}} a_y a \otimes (t \setminus y)(a_0, \dots, \widehat{a}_y, \dots a_{r_n})$$

for $a \in A_+$. Here for $s \in [r_n]$ such that s is not the only element in the corresponding 1–fibre of t containing s, ie in the 1–fibre $f_n^{-1}(u)$ with $f_n(s) = u$, we let $t \setminus s$ be the tree obtained by deleting the leaf s. To be more precise,

$$t \setminus s = \ [r_n - 1] \xrightarrow{f_n'} [r_{n-1}] \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} [r_1]$$

with

$$f_n'(x) = \begin{cases} f_n(x), & x < s, \\ f_n(x+1), & x \geqslant s. \end{cases}$$

The sign $(-1)^{s_{n,i}}$ is as in Definition 2.5.

Remark 3.2 (a) Intuitively the map δ deletes leaves and acts with the corresponding label on the coefficient module A_+. The leaves which are deleted are either on the left or on the right of a 1–fibre of the tree. For $n = 1$ compare this to the complex calculating Hochschild homology $\mathrm{HH}(A; A_+)$: the standard differential maps $a \otimes a_0 \otimes \cdots \otimes a_l \in A_+ \otimes A^{\otimes l+1}$ to

$$aa_0 \otimes a_1 \otimes \cdots \otimes a_l + (-1)^{l+1} a_l a \otimes a_0 \otimes \cdots \otimes a_{l-1} \\ + \sum_{i=0}^{l-1} (-1)^{i+1} a \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_l.$$

The first two summands correspond to the twist δ, while the other summands correspond to ∂_B.

(b) In the definition of the map δ we only consider 1–fibres of cardinality at least two. If we wanted to take 1–fibres of cardinality one into account, we would add two summands for each such fibre: Both summands would replace t by a tree obtained by deleting the 1–fibre and then deleting further edges to obtain a fully grown tree again. One summand would multiply $a \in A_+$ from the right by the label a_x of the leaf x corresponding to the deleted fibre, the other summand would multiply by a_x from the left. Note that these summands are not of the appropriate degree, since we delete more than one edge. However, the two terms just described cancel each other out anyway, because for commutative A multiplying $a \in A_+$ with $a_x \in A$ from the left equals multiplying a with a_x from the right.

In Section 4 we will define E_n–homology and E_n–cohomology of functors defined on a category which extends the category Epi_n. The following theorem will allow us to argue in Remark 4.8 and Remark 4.10 that E_n–homology and E_n–cohomology of functors encompass E_n–homology and E_n–cohomology of commutative algebras with coefficients in a symmetric bimodule.

Theorem 3.3 [7] *For a commutative k–projective nonunital k–algebra A and a symmetric A–bimodule M we have*

$$H_*^{E_n}(A;M) = H_*(M \otimes_{A_+} (A_+ \otimes \Sigma^{-n} B^n(A), \delta))$$

and

$$H^*_{E_n}(A;M) = H^*(\mathrm{Hom}_{A_+}((A_+ \otimes \Sigma^{-n} B^n(A), \delta), M)).$$

4 The category Epi_n^+ encoding the n–fold bar complex with coefficients

We would like to establish a functor homology interpretation for E_n–homology of a commutative algebra A with coefficients in a symmetric A–bimodule M as well as for E_n–cohomology. To model E_n–homology with coefficients as functor homology we have to enlarge the category Epi_n to incorporate the summands of the twisting cochain δ.

Definition 4.1 The objects of the category Epi_n^+ are given by planar fully grown trees with n levels (see Definition 2.2). A morphism from

$$t^r = [r_n] \xrightarrow{f_n^r} \cdots \xrightarrow{f_2^r} [r_1] \quad \text{to} \quad t^s = [s_n] \xrightarrow{f_n^s} \cdots \xrightarrow{f_2^s} [s_1]$$

is represented by a sequence of maps $(h_n, \dots, h_1)$, where:

- For $i = 2, \dots, n-1$, the map $h_i\colon [r_i] \to [s_i]$ is a surjection which is order-preserving on the fibres $(f_i^r)^{-1}(l)$ for all $l \in [r_{i-1}]$. For $i = 1$ we require $h_1\colon [r_1] \to [s_1]$ to be order-preserving.
- The map
$$h_n\colon [r_n] \to [s_n]_+ := [s_n] \sqcup \{+\}$$
has $[s_n]$ in its image. We also require that the restriction of h_n to $h_n^{-1}([s_n])$ is order-preserving on the fibres of f_n^r. Furthermore, the intersection of $h_n^{-1}([s_n])$ with a fibre $(f_n^r)^{-1}(l)$ must be a (potentially empty) interval for all $l \in [r_{n-1}]$, ie of the form $\{a, a+1, \dots, a+b\}$ with $b \geqslant -1$.
- The diagram
$$\begin{tikzcd} h_n^{-1}([s_n]) \arrow[r, "f_n^r"] \arrow[d, "h_n"] & {[r_{n-1}]} \arrow[r] \arrow[d, "h_{n-1}"] & \cdots \arrow[r] & {[r_2]} \arrow[r, "f_1^r"] \arrow[d, "h_2"] & {[r_1]} \arrow[d, "h_1"] \\ {[s_n]} \arrow[r, "f_n^s"] & {[s_{n-1}]} \arrow[r] & \cdots \arrow[r] & {[s_2]} \arrow[r, "f_1^s"] & {[s_1]} \end{tikzcd}$$
commutes.

Finally, we identify certain morphisms by imposing the following equivalence relation on the set of morphisms from t^r to t^s: we identify morphisms h and h' if

- ${h_n}^{-1}(+) = {h'_n}^{-1}(+)$, and
- for all $1 \leqslant i \leqslant n$, the restrictions of h_i and h'_i to $f^r_{i+1} \dots f^r_n([r_n] \setminus h_n^{-1}(+))$ coincide.

The composition of two morphisms $(g_n, \dots, g_1)\colon t^q \to t^r$ and $(h_n, \dots, h_1)\colon t^r \to t^s$ is defined by composing componentwise and sending $+$ to $+$, ie

$$(h_n, \dots, h_1) \circ (g_n, \dots, g_1) := ((hg)_n, h_{n-1}g_{n-1}, \dots, h_1 g_1)$$

with

$$(hg)_n(x) = \begin{cases} + & \text{if } g_n(x) = +, \\ h_n g_n(x) & \text{otherwise.} \end{cases}$$

A straightforward calculation shows that composition in Epi_n^+ is well defined and associative.

Remark 4.2 (a) It is clear that Epi_n is a subcategory of Epi_n^+ and that both categories share the same objects. Let $\delta_i\colon [r_n] \to [r_n - 1]_+$ be the map

$$\delta_i(x) = \begin{cases} x & \text{if } x < i, \\ + & \text{if } x = i, \\ x - 1 & \text{if } x > i. \end{cases}$$

Given a tree $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ such that i is the minimal or maximal element of a fibre $f_n^{-1}(l)$ containing at least two elements, let $\widehat{f_n}$ be given by

$$\widehat{f_n}(x) = \begin{cases} f_n(x) & \text{if } x < i, \\ f_n(x+1) & \text{if } x \geqslant i. \end{cases}$$

Let $t' = [r_n - 1] \xrightarrow{\widehat{f_n}} [r_{n-1}] \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} [r_1]$. Then, intuitively, the category Epi_n^+ is built from Epi_n by adding morphisms of the form $(\delta_i, \mathrm{id}, \dots \mathrm{id})\colon t \to t'$. The requirement that the elements of a fibre of f_n that are not mapped to $+$ form an interval reflects the fact that we have only added morphisms of the aforementioned kind.

(b) We only added morphisms $(\delta_i, \mathrm{id}, \dots, \mathrm{id})\colon t \to t'$ such that i is not the only element in the corresponding 1–fibre of t. Nevertheless, it is possible to map 1–fibres of cardinality one to $+$ by first applying maps which merge edges in lower levels. For example, the map

$$(h_2, h_1)\colon (\ [1] \xrightarrow{\mathrm{id}} [1]\ ) \to (\ [0] \xrightarrow{\mathrm{id}} [0]\ )$$

with $h_2(0)=+$, $h_2(1)=0$ and $h_1(0)=h_1(1)=0$ arises as the composite $(h_2'', h_1'')\circ(h_2', h_1')$ of the maps

$$(h_2', h_1')\colon (\ [1]\xrightarrow{\mathrm{id}}[1]\ )\to(\ [1]\xrightarrow{0,1\mapsto 0}[0]\ ),\quad h_2'=\mathrm{id},\quad h_1'(0)=h_1'(1)=0$$

and

$$(h_2'', h_1'')\colon (\ [1]\xrightarrow{0,1\mapsto 0}[0]\ )\to(\ [0]\xrightarrow{\mathrm{id}}[0]\ ),\quad h_2''=\delta_0,\quad h_1'=\mathrm{id}.$$

(c) The motivation for defining Epi_n^+ is to model the complex calculating E_n–homology of A with coefficients in M. Hence imposing the above equivalence relation on the set of morphisms is necessary: it should not matter what precisely happens to a subtree of a tree t if all its leaves get mapped to $+$, ie in which order and on what side of an element we act on with a family of elements of A.

After defining the category Epi_n^+ which also models the summands of the twisting cochain δ, we can proceed to define E_n–homology of a functor.

Definition 4.3 Let $F\colon \mathrm{Epi}_n^+\to k$–mod be a functor. As in Definition 2.5 set

$$C^{E_n}_{r_n,\dots,r_1}(F):=\bigoplus F(t),$$

where the sum is indexed over all trees

$$t=[r_n]\to\cdots\to[r_1].$$

Define maps $\partial_j\colon C^{E_n}_{r_n,\dots,r_j,\dots,r_1}(F)\to C^{E_n}_{r_n,\dots,r_j-1,\dots,r_1}(F)$ lowering the j^{th} degree by one by

$$\partial_j=\tilde{\partial}_j\quad\text{for } i<n\qquad\text{and}\qquad \partial_n=\tilde{\partial}_n+\delta_{\min}+\delta_{\max},$$

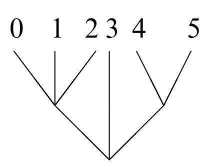
with

$$\delta_{\min}=\sum_{\substack{0\leqslant l\leqslant r_{n-1}\\ |f_n^{-1}(l)|>1}}(-1)^{s_{n,\min f_n^{-1}(l)}-1}F(\delta_{\min f_n^{-1}(l)},\mathrm{id},\dots,\mathrm{id}),$$

$$\delta_{\max}=\sum_{\substack{0\leqslant l\leqslant r_{n-1}\\ |f_n^{-1}(l)|>1}}(-1)^{s_{n,\max f_n^{-1}(l)}}F(\delta_{\max f_n^{-1}(l)},\mathrm{id},\dots,\mathrm{id}).$$

The integers $s_{n,i}$ are as in Definition 2.5.

Example 4.4 Let t be the 2–level tree



Then

$$\delta_{\min} = (-1)^1 F(\delta_0, \mathrm{id}) + (-1)^7 F(\delta_4, \mathrm{id}),$$
$$\delta_{\max} = (-1)^4 F(\delta_2, \mathrm{id}) + (-1)^9 F(\delta_5, \mathrm{id}).$$

We already know from [11, Lemma 3.8] that $(C^{E_n}, \widetilde{\partial}_1, \dots, \widetilde{\partial}_n)$ is a multicomplex. Hence it suffices to prove the following lemma, which can be done via a tedious but straightforward calculation; see [19, Lemma 4.14].

Lemma 4.5 *Let $F\colon \mathrm{Epi}_n^+ \to k$–mod. The maps defined above satisfy the identities*

$$(\delta_{\min} + \delta_{\max})\partial_j + \partial_j(\delta_{\min} + \delta_{\max}) = 0 \quad \textit{for all } j < n,$$
$$(\delta_{\min} + \delta_{\max})^2 + \widetilde{\partial}_n(\delta_{\min} + \delta_{\max}) + \widetilde{\partial}_n(\delta_{\min} + \delta_{\max}) = 0.$$

Hence $C^{E_n}(F)$ is a multicomplex.

Definition 4.6 Let $F\colon \mathrm{Epi}_n^+ \to k$–mod be a functor. The E_n–homology of F is

$$H_*^{E_n}(F) = H_*(\mathrm{Tot}(C^{E_n}(F))).$$

Remark 4.7 Given a functor $\widetilde{F}\colon \mathrm{Epi}_n \to k$–mod, we can extend $\widetilde{F}$ to a functor $F\colon \mathrm{Epi}_n^+ \to k$–mod by setting $F(h) = 0$ for every morphism $h\colon t^r \to t^s$ in Epi_n^+ such that $h([r_n]) \cap \{+\} \neq \varnothing$. With these definitions $H^{E_n}(F)$ coincides with the E_n–homology of $\widetilde{F}$ as defined in Definition 2.7. In this sense the definition of E_n–homology we just gave extends the definition given in [11, Definition 3.7].

We are specifically interested in calculating E_n–homology of commutative algebras, which is the E_n–homology of the following functors.

Remark 4.8 The Loday functor $\mathcal{L}(A; M)\colon \mathrm{Epi}_n^+ \to k$–mod is the following functor: For a given tree $t = [r_n] \to \dots \to [r_1]$ set

$$\mathcal{L}(A; M)(t) = M \otimes A^{\otimes r_n + 1}.$$

If $(h_n, \dots, h_1)\colon t^r \to t^s$ is a morphism, define

$$\mathcal{L}(A; M)(h_n, \dots, h_1)\colon M \otimes A^{\otimes r_n + 1} \to M \otimes A^{\otimes s_n + 1}$$

by

$$m \otimes a_0 \otimes \dots \otimes a_{r_n} \mapsto \Bigg(m \cdot \prod_{\substack{i \in [r_n] \\ h_n(i) = +}} a_i \Bigg) \otimes \Bigg(\prod_{\substack{i \in [r_n] \\ h_n(i) = 0}} a_i \Bigg) \otimes \dots \otimes \Bigg(\prod_{\substack{i \in [r_n] \\ h_n(i) = s_n}} a_i \Bigg).$$

Then

$$\mathrm{Tot}(C^{E_n}(\mathcal{L}(A;M))) = \Sigma^{-n}(M \otimes_{A_+} (A_+ \otimes B^n(A), \delta)).$$

In particular, by Theorem 3.3 we have

$$H_*^{E_n}(\mathcal{L}(A;M)) = H_*^{E_n}(A;M)$$

if A is k–projective. Note that $\mathcal{L}(A;k)$ agrees with the extension of the Loday functor defined by Livernet and Richter in [11, Definition 3.1] to Epi_n^+.

We now consider E_n–cohomology. The definition of E_n–cohomology is dual to the definition of E_n–homology.

Definition 4.9 Let $G\colon \mathrm{Epi}_n^{+\mathrm{op}} \to k$–mod be a functor. The E_n–cohomology of G is defined as

$$H^*_{E_n}(G) = H_*(\mathrm{Tot}(C_{E_n}(G))),$$

with the multicomplex $C_{E_n}(G)$ defined as follows. We set

$$C_{E_n}^{r_n,\dots,r_1}(G) = \bigoplus G(t),$$

where the sum is indexed over trees

$$t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1].$$

The differentials

$$\partial_j\colon C_{E_n}^{r_n,\dots,r_j,\dots,r_1}(G) \to C_{E_n}^{r_n,\dots r_j+1,\dots,r_1}(G)$$

raise the j^{th} degree by one. For $j=n$ define ∂_n restricted to $G(t)$ as

$$\sum_{\substack{0\leqslant i<r_n \\ f_n(i)=f_n(i+1)}} (-1)^{s_{n,i}} G(d_i, \mathrm{id},\dots,\mathrm{id}) + \sum_{\substack{0\leqslant l\leqslant r_{n-1} \\ |f_n^{-1}(l)|>1}} (-1)^{s_{n,\min f_n^{-1}(l)}-1} G(\delta_{\min f_n^{-1}(l)}, \mathrm{id},\dots,\mathrm{id}) + \sum_{\substack{0\leqslant l\leqslant r_{n-1} \\ |f_n^{-1}(l)|>1}} (-1)^{s_{n,\max f_n^{-1}(l)}} G(\delta_{\max f_n^{-1}(l)}, \mathrm{id},\dots,\mathrm{id}).$$

For $1\leqslant j<n$ the map ∂_j restricted to $G(t)$ is given by

$$\sum_{\substack{0\leqslant i<r_j \\ f_j(i)=f_j(i+1)}} \sum_{\sigma\in \mathrm{sh}(f_{j+1}^{-1}(i), f_{j+1}^{-1}(i+1))} \epsilon(\sigma; t_{j,i}, t_{j,i+1})(-1)^{s_{j,i}} G(h_{i,\sigma}).$$

Here $h=h_{i,\sigma}$ again denotes the unique morphism of trees, exhibited in [11, Lemma 3.5], such that $h_j = d_i\colon [r_j] \to [r_j - 1]$, $h_l = \mathrm{id}$ for $l < j$, and h_{j+1} restricted to $f_{j+1}^{-1}(\{i, i+1\})$ acts like σ.

As was the case for E_n–homology, this definition generalizes E_n–cohomology of commutative algebras with coefficients in a symmetric bimodule:

Remark 4.10 Define $\mathcal{L}^c(A;M)\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k\text{–mod}$ on $t = [r_n] \to \cdots \to [r_1]$ by

$$\mathcal{L}^c(A;M)(t) = \mathrm{Hom}_k(A^{\otimes r_n+1}, M).$$

If $(h_n,\dots,h_1)$ is a morphism from t^r to t^s, define

$$\mathcal{L}^c(A;M)(h_n,\dots,h_1)\colon \mathrm{Hom}_k(A^{\otimes s_n+1}, M) \to \mathrm{Hom}_k(A^{\otimes r_n+1}, M)$$

by

$$(\mathcal{L}^c(A;M)(h_n,\dots,h_1)(f))(a_0 \otimes \cdots \otimes a_{r_n}) = \Bigg(\prod_{\substack{i\in[r_n]\\ h_n(i)=+}} a_i\Bigg) \cdot f\Bigg(\Bigg(\prod_{\substack{i\in[r_n]\\ h_n(i)=0}} a_i\Bigg) \otimes \cdots \otimes \Bigg(\prod_{\substack{i\in[r_n]\\ h_n(i)=s_n}} a_i\Bigg)\Bigg).$$

Then $\mathrm{Tot}(C_{E_n}(\mathcal{L}^c(A;M)))$ coincides with the complex computing E_n–cohomology of A with coefficients in M. Theorem 3.3 hence yields that

$$H^*_{E_n}(\mathcal{L}^c(A;M)) = H^*_{E_n}(A;M)$$

if A is k–projective.

5 E_n–cohomology as functor cohomology

In [11, Theorem 4.1] Livernet and Richter show that E_n–homology with trivial coefficients can be interpreted as functor homology. We now extend this result to E_n–homology and E_n–cohomology with arbitrary coefficients. As in [11], we prove that E_n–homology coincides with functor homology by using the axiomatic characterizations of Tor and Ext. For a background on functor homology we refer the reader to [16]. We first show that certain projective functors are acyclic. Recall that for a small category $\mathcal{C}$ a functor $F\colon \mathcal{C} \to k$–mod is called projective if it has the usual lifting property with respect to objectwise surjective natural transformations. For $t \in \mathrm{Epi}_n^+$ define projective functors P_t and P^t by

$$P_t = k[\mathrm{Epi}_n^+(t,-)]\colon \mathrm{Epi}_n^+ \to k\text{–mod} \quad \text{and} \quad P^t = k[\mathrm{Epi}_n^+(-,t)]\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k\text{–mod}.$$

In the proof of the following lemma, we will consider trees obtained by restricting a given tree to certain leaves.

Definition 5.1 Let $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ be a tree. For fixed $I \subset [r_n]$ set $r_i^I = |f_{i+1} \cdots f_n(I)| - 1$. Define a tree t^I as the upper row in

$$\begin{tikzcd}
{[r_n^I]} \arrow[r, "f_n^I"] \arrow[d] & {[r_{n-1}^I]} \arrow[r, "f_{n-1}^I"] \arrow[d] & \cdots \arrow[r, "f_2^I"] & {[r_1^I]} \arrow[d] \\
I \arrow[r, "f_n"] & f_n(I) \arrow[r, "f_{n-1}"] & \cdots \arrow[r, "f_2"] & f_2 \cdots f_n(I)
\end{tikzcd}$$

Here the vertical morphisms are determined by requiring that they are bijective and order-preserving, while the maps f_n^I are defined by requiring that all squares commute. Intuitively t^I is the subtree of t given by restricting t to edges connecting leaves labelled by I with the root (the bottom vertex of the tree t).

Lemma 5.2 *Let $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ be a tree. Let $I \subset [r_n]$ be a set such that $I \cap f_n^{-1}(i)$ is a (possibly empty) interval for all $i \in [r_{n-1}]$. Then we can define a morphism*

$$h^I = (h_n^I, \dots, h_1^I) \colon t \to t^I$$

in Epi_n^+ *as follows: The map h_n^I maps all $x \in [r_n] \setminus I$ to $+$ and is an order-preserving bijection restricted to I. For $i < n$ we require that h_i^I restricted to $f_{i+1} \cdots f_n(I)$ is the order-preserving bijection to $[r_i^I]$ and that h_i^I be order-preserving on the whole set $[r_i]$.*

Proof Recall that a morphism in Epi_n^+ is an equivalence class with respect to the equivalence relation introduced in Definition 4.1. Since $I = [r_n] \setminus (h_n^I)^{-1}(+)$ the above requirements uniquely determine h^I up to equivalence. The maps h_i^I assemble to a morphism in Epi_n^+ since they are chosen to be order-preserving and the squares

$$\begin{tikzcd}
f_{i+1} \cdots f_n(I) \arrow[r, "f_i"] \arrow[d, "h_i^I"] & f_i \cdots f_n(I) \arrow[d, "h_{i-1}^I"] \\
{[r_i^I]} \arrow[r, "f_i^I"] & {[r_{i-1}^I]}
\end{tikzcd}$$

commute by definition of f_i^I. Furthermore $(h_n^I)^{-1}(+) \cap f_n^{-1}(i) = I \cap f_n^{-1}(i)$ is an interval. $\square$

Now we are in the position to compute the E_n–homology of the representable projectives.

Lemma 5.3 *Fix a tree $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$. Then*

$$H_*^{E_n}(P_t) = \begin{cases} 0 & \text{if } * > 0, \\ \bigoplus_{i \in [r_n]} k & \text{if } * = 0. \end{cases}$$

Proof Set $C := \mathrm{Tot}(C^{E_n}(P_t))$. We define an ascending filtration by subcomplexes of C by

$$F^p C_{s_n,\dots,s_1} := \bigoplus k\big[\{(h_n,\dots,h_1) \in P_t(t^s) : |h_n^{-1}([s_n])| \leqslant p+1\}\big],$$

where the sum is indexed over trees

$$t^s = [s_n] \xrightarrow{f_n^s} \cdots \xrightarrow{f_2^s} [s_1].$$

Hence $F^p C$ is generated by morphisms that map at least $r_n - p$ leaves to $+$. This yields a first quadrant spectral sequence

$$E^1_{p,q} = H_{p+q}(F^p C / F^{p-1} C) \implies H_{p+q}(C).$$

The quotient $F^p C / F^{p-1} C$ can be identified with the free k–module generated by morphisms $(h_n,\dots,h_1) \in k[\mathrm{Epi}_n^+(t,t^s)]$ with $|h_n^{-1}([s_n])| = p+1$. The differentials $\delta_{\min}$ and $\delta_{\max}$ vanish on this quotient. The remaining summands of ∂_n and the differentials $\partial_{n-1},\dots,\partial_1$ do not change the number of leaves that get mapped to $+$. We conclude that $F^p C / F^{p-1} C$ is isomorphic to D as a complex, where

$$D_{s_n,\dots,s_1} = \bigoplus k\big[\{(h_n,\dots,h_1) \in P_t(t^s) : |h_n^{-1}([s_n])| = p+1\}\big]$$

with differentials $\partial_1,\dots,\partial_{n-1}$ and $\widehat{\partial}_n = \partial_n - \delta_{\min} - \delta_{\max}$, and where the sum is indexed over trees t^s as above. The complex D can be decomposed further: The remaining differentials do not only respect the number of deleted leaves but also the set of deleted leaves itself. Hence D is the direct sum of subcomplexes D^I with

$$D^I_{s_n,\dots,s_1} = \bigoplus k\big[\{(h_n,\dots,h_1) \in P_t(t^s) : h_n^{-1}([s_n]) = I\}\big]$$

such that I is a subset of $[r_n]$ of cardinality $p+1$, and the sum is over trees t^s as above. Notice that the differentials of D and D^I look like the differentials used in Definition 2.5 to define E_n–homology of functors from Epi_n to k–mod. We will show that D^I in fact can be identified with the complex associated to such a functor. More precisely, D^I is the complex computing E_n–homology of the representable functor $k[\mathrm{Epi}_n(t^I,-)]\colon \mathrm{Epi}_n \to k$–mod: Denote by $h^I\colon t \to t^I$ the morphism defined in Lemma 5.2. We define

$$\Psi\colon \widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-)) \to D^I$$

by mapping $j \in \mathrm{Epi}_n(t^I,t^s)$ to $\Psi(j) = j \circ h^I$. Since j does not delete any leaves this yields an element of D^I. We define an inverse Φ to Ψ by mapping $h \in D^I$ to the composite of the columns in

$$\begin{tikzcd}
{[r_n^I]} \arrow[r, "f_n^I"] \arrow[d] & {[r_{n-1}^I]} \arrow[r, "f_{n-1}^I"] \arrow[d] & \cdots \arrow[r, "f_2^I"] & {[r_1^I]} \arrow[d] \\
I \arrow[r, "f_n"] \arrow[d, "h_n"] & f_n(I) \arrow[r, "f_{n-1}"] \arrow[d, "h_{n-1}"] & \cdots \arrow[r, "f_2"] & f_2 \cdots f_n(I) \arrow[d, "h_1"] \\
{[s_n]} \arrow[r, "g_n"] & {[s_{n-1}]} \arrow[r, "g_{n-1}"] & \cdots \arrow[r, "g_2"] & {[s_1]}
\end{tikzcd}$$

Here the upper vertical maps are order-preserving bijections. We see that $\Phi(h)_i$ only depends on $h_{i|f_{i+1}\cdots f_n(I)}$, ie Φ is well defined on equivalence classes. It is obvious that each $\Phi(h)_i$ is surjective and that the usual requirements on commutativity are satisfied. Consider a fibre $(f_i^I)^{-1}(l)$: The map $\Phi(h)_i$ first sends it order-preservingly and surjectively to $f_{i+1}\cdots f_n(I) \cap f_i^{-1}(l') \subset [r_i]$, where l' denotes the image of l under the map $[r_{i-1}^I] \to f_i \cdots f_n(I)$. Since h_i preserves the order on fibres of f_i we see that $\Phi(h)_i$ is order-preserving on the fibres of f_i^I. Hence Φ is indeed a map from D^I to $\widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-))$.

Finally, we note that obviously $\Phi \circ \Psi$ is the identity. To show that Ψ is a left inverse for Φ one writes down $(\Psi\circ\Phi)(h)$ for a given h and uses that $((\Psi\circ\Phi)(h))_i$ only needs to coincide with h_i on $f_{i+1}\cdots f_n(I)$. The maps Φ and Ψ commute with composition, hence also with applying the differentials. Since the signs in the differentials applied to a morphism h are determined by the target tree t^s of h, there is no trouble with signs either. Hence we have constructed an isomorphism

$$D^I \cong \widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-))$$

of complexes.

We know from [11, Section 4] that $H_*(\mathrm{Tot}(\widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-)))) = 0$ for $* > 0$ and that

$$H_0(\mathrm{Tot}(\widetilde{C}^{E_n}(\mathrm{Epi}_n(t^I,-)))) = \begin{cases} k & \text{if } t^I = [0] \to [0] \to \cdots \to [0], \\ 0 & \text{otherwise.} \end{cases}$$

Since $t^I = [0] \to [0] \to \cdots \to [0]$ implies $p+1 = |I| = 1$, we see that the E^1–term of our spectral sequence is

$$E^1_{p,q} = H_{p+q}(F^pC/F^{p-1}C) = \begin{cases} \bigoplus_{i\in[r_n]} k & \text{if } p=q=0, \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence collapses and the claim follows. □

Having proved that $H_*^{E_n}(P_t)$ is acyclic we can use the axiomatic description of Tor (see eg [9, Chapter 2]).

Theorem 5.4 *Denote by* $b\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k\text{–mod}$ *the functor given by the cokernel of*

$$(\delta_0, \mathrm{id}, \dots, \mathrm{id})_* - (d_0, \mathrm{id}, \dots, \mathrm{id})_* + (\delta_1, \mathrm{id}, \dots, \mathrm{id})_*\colon P^{[1]\to[0]\to\cdots\to[0]} \to P^{[0]\to\cdots\to[0]}.$$

Then for any $F\colon \mathrm{Epi}_n^+ \to k\text{–mod}$ *we have*

$$H_*^{E_n}(F) \cong \mathrm{Tor}_*^{\mathrm{Epi}_n^+}(b, F),$$

and this isomorphism is natural in F.

Proof A short exact sequence $0 \to F \to G \to H \to 0$ of functors yields a short exact sequence of chain complexes

$$0 \to \mathrm{Tot}(C^{E_n}(F)) \to \mathrm{Tot}(C^{E_n}(G)) \to \mathrm{Tot}(C^{E_n}(H)) \to 0.$$

This in turn gives rise to a long exact sequence on homology. We already showed that $H_*^{E_n}(P_t)$ is zero in positive degrees. Every projective functor from Epi_n^+ to k–mod receives a surjection from a sum of functors of the form of P_t. It hence is a direct summand of this sum. Therefore $H_*^{E_n}(P)$ vanishes in positive degrees for all projective functors P. Finally, the zeroth E_n–homology of a functor F is given by the cokernel of

$$(-1)^{n-1}F(\delta_0, \mathrm{id}, \dots, \mathrm{id}) + (-1)^n F(d_0, \mathrm{id}, \dots, \mathrm{id}) + (-1)^{n+1} F(\delta_1, \mathrm{id}, \dots, \mathrm{id}).$$

Using the natural isomorphism $P^t \otimes_{\mathrm{Epi}_n^+} F \cong F(t)$ of k–modules and that tensor products are right exact, one sees that this coincides with $b \otimes_{\mathrm{Epi}_n^+} F$. □

Every functor $F\colon \mathrm{Epi}_n^+ \to k\text{–mod}$ gives rise to a functor $F^*\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k\text{–mod}$, its dual, by setting $F^*(t) = \mathrm{Hom}_k(F(t), k)$. Since we just proved that E_n–homology of projective functors vanishes, we can relate E_n–homology with E_n–cohomology via the following spectral sequence.

Proposition 5.5 (see eg [18, Theorem 10.49]) *If $F(t)$ is k–free for every $t \in \mathrm{Epi}_n^+$, there is a first quadrant spectral sequence*

$$E^2_{p,q} = \mathrm{Ext}^q_k(H_p^{E_n}(F), k) \implies H^{p+q}_{E_n}(F^*).$$

In particular, whenever k is injective as a k–module, E_n–homology of F and E_n–cohomology of its dual are dual to each other.

Examples of commutative self-injective rings include fields, group algebras of finite commutative groups over a self-injective ring, quotients R/I of a principal ideal domain R with $I \neq 0$, and commutative Frobenius rings [1, Chapter 5, Section 18]. The product of self-injective rings is again self-injective.

Theorem 5.6 *Suppose that k is injective as a k–module and let $G\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k$–mod be a functor. Then there is an isomorphism*

$$H^*_{E_n}(G) \cong \mathrm{Ext}^*_{\mathrm{Epi}_n^{+\,\mathrm{op}}}(b, G).$$

This isomorphism is natural in G.

Proof That $H^*_{E_n}$ maps short exact sequences to long exact sequences follows as in the homological case. Since the projective functor P_t is finitely generated and k–free, the functor P_t^* is injective. The universal coefficient spectral sequence (Proposition 5.5) yields that these modules are acyclic. But then all other injective modules are acyclic too, since they are direct summands of products of these. Finally, let $G\colon \mathrm{Epi}_n^{+\,\mathrm{op}} \to k$–mod be an arbitrary functor. Then the zeroth E_n–cohomology of G is by definition the kernel of

$$(-1)^{n-1}G(\delta_0, \mathrm{id}, \dots, \mathrm{id}) + (-1)^n G(d_0, \mathrm{id}, \dots, \mathrm{id}) + (-1)^{n+1}G(\delta_1, \mathrm{id}, \dots, \mathrm{id}).$$

The Yoneda lemma and the left exactness of $\mathrm{Nat}_{\mathrm{Epi}_n^{+\,\mathrm{op}}}(-, G)$ yield that this kernel results from applying $\mathrm{Nat}_{\mathrm{Epi}_n^{+\,\mathrm{op}}}(-, G)$ to b. □

6 Functor cohomology and cohomology operations

We recall the definition of the Yoneda pairing on Ext. The Yoneda pairing is usually defined in the context of modules over a ring (see eg [12, Chapter III, Sections 5–6]). But it is well known to be easily generalized to suitable abelian categories with enough projectives and injectives. We assume that k is self-injective in this section.

Definition 6.1 Let F, G and H be functors from $\mathrm{Epi}_n^{+\mathrm{op}}$ to k–mod. Let P_F denote a projective resolution of F and I_H an injective resolution of H. There is a pairing

$$\mu\colon \mathrm{Ext}^*_{\mathrm{Epi}_n^{+\mathrm{op}}}(G,H)\otimes \mathrm{Ext}^*_{\mathrm{Epi}_n^{+\mathrm{op}}}(F,G)\to \mathrm{Ext}^*_{\mathrm{Epi}_n^{+\mathrm{op}}}(F,H),$$

defined as the composite

$$\begin{array}{c}
\mathrm{Ext}^m_{\mathrm{Epi}_n^{+\mathrm{op}}}(G,H)\otimes \mathrm{Ext}^n_{\mathrm{Epi}_n^{+\mathrm{op}}}(F,G)\\
\Big\Vert\\
H_m(\mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(G,I_H))\otimes H_n(\mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(P_F,G))\\
\Big\downarrow\\
H_{n+m}\big(\mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(G,I_H)\otimes \mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(P_F,G)\big)\\
\Big\downarrow\\
H_{n+m}(\mathrm{Nat}_{\mathrm{Epi}_n^{+\mathrm{op}}}(P_F,I_H)) = \mathrm{Ext}^{n+m}_{\mathrm{Epi}_n^{+\mathrm{op}}}(F,H).
\end{array}$$

Here the second map is induced by composing natural transformations. This associative pairing is called the Yoneda pairing.

In particular, there is a natural action of

$$\mathrm{Ext}^*_{\mathrm{Epi}_n^{+\mathrm{op}}}(b,b) = H^*_{E_n}(b)$$

on E_n–cohomology. One could hope to find cohomology operations via this action. For example, if the characteristic of k is a prime p, Hochschild cohomology $\mathrm{HH}^*(A;A_+)$ is a p–restricted Gerstenhaber algebra, ie the Lie algebra structure on $\Sigma^{-1}\,\mathrm{HH}^*(A;A_+)$ comes with a restriction. We will determine $H^*_{E_n}(b)$ to see whether we can find new or old cohomology operations using the Yoneda pairing. For the remainder of this section we will denote $b\colon \mathrm{Epi}_n^{+\mathrm{op}}\to k$–mod by b_n since we will have to consider trees of varying levels. Since we are going to work homologically we make b_n^*, the dual of b_n, explicit. Intuitively, b_n^* is the functor assigning to a tree its set of leaves.

Proposition 6.2 *The functor b_n^* assigns $k\langle[r_n]\rangle = k[\{0,\dots,r_n\}]$ to a given tree $t = [r_n]\to\cdots\to[r_1]$. Denoting the generators of $k\langle[r_n]\rangle$ by $\alpha_0,\dots,\alpha_{r_n}$, it induces the maps*

$$b_n^*(\tau_n,\dots,\tau_{j+1},d_i,\mathrm{id},\dots,\mathrm{id})\colon k\langle[r_n]\rangle\to k\langle[r_n]\rangle,\quad \alpha_m\mapsto \alpha_{\tau_n^{-1}(m)}$$

for suitable $\tau_{j+1} \in \Sigma_{[r_{j+1}]}, \dots, \tau_n \in \Sigma_{[r_n]}$ *as in* [11, Lemma 3.5],

$$b_n^*(d_i, \mathrm{id}, \dots, \mathrm{id})\colon k\langle[r_n+1]\rangle \to k\langle[r_n]\rangle, \quad \alpha_m \mapsto \begin{cases} \alpha_m & \text{if } m \leqslant i, \\ \alpha_{m-1} & \text{if } m > i, \end{cases}$$

$$b_n^*(\delta_i, \mathrm{id}, \dots, \mathrm{id})\colon k\langle[r_n+1]\rangle \to k\langle[r_n]\rangle, \quad \alpha_m \mapsto \begin{cases} \alpha_m & \text{if } m < i, \\ 0 & \text{if } m = i, \\ \alpha_{m-1} & \text{if } m > i. \end{cases}$$

We will show that b_n^* is indeed acyclic with respect to E_n–homology. The case $n = 1$ can be easily calculated:

Proposition 6.3 *For* $n = 1$ *we have*

$$H^r_{E_1}(b_1) \cong H^{E_1}_r(b_1^*) = 0$$

for $r > 0$ *and*

$$H^0_{E_1}(b_1) \cong H^{E_1}_0(b_1^*) = k.$$

For $n > 1$ we derive the acyclicity of b_n^* from the case $n = 1$. For this we need the following lemma. Recall that the differential ∂_n is induced by morphisms which act on the top level of a given tree. Intuitively, the following lemma states that ∂_n can be split into parts that correspond to morphisms acting on the different fibres.

Lemma 6.4 *Let* $F\colon \mathrm{Epi}_n^+ \to k$–mod *be a functor and* $r_1, \dots, r_{n-1} \geqslant 0$. *Consider the* $r_{n-1}+1$*–fold multicomplex*

$$M_{x_0,\dots,x_{r_{n-1}}}(F) = \bigoplus_{\substack{t=[x_0+\dots+x_{r_{n-1}}] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1] \\ |f_n^{-1}(0)|=x_0+1,\, |f_n^{-1}(i)|=x_i \text{ for all } 1 \leqslant i \leqslant r_{n-1}}} F(t)$$

where the i^{th} *differential* d^i *of the multicomplex is the part of* ∂_n *induced by morphisms operating on the fibre* $f_n^{-1}(i)$. *Then*

$$\mathrm{Tot}(M) \cong \Sigma^{-r_1-\dots-r_{n-1}}(C^{E_n}_{(*,r_{n-1},\dots,r_1)}(F), \partial_n).$$

Furthermore, we can split M *into submulticomplexes corresponding to the underlying* $(n-1)$*–level tree* T: *Let* $t^{x_0+1,x_1,\dots,x_{r_{n-1}}}$ *be the tree extending* T *with top-level fibres of cardinality* $x_0+1, x_1, \dots, x_{r_{n-1}}$. *Let*

$$M^T_{x_1,\dots,x_{r_{n-1}}} = F(t^{x_0+1,x_1,\dots,x_{r_{n-1}}}).$$

Then

$$M_{*,\dots,*}(F) = \bigoplus_{T=[r_{n-1}]\to\dots\to[r_1]} (M^T_{*,\dots,*}, d^0, \dots, d^{r_{n-1}}).$$

Proof The differential ∂_n is the sum of the maps d^i for $0 \leqslant i \leqslant r_{n-1}$, each of them leaving all 1–fibres except for $f_n^{-1}(i)$ unchanged. Two such differentials d^i and d^j commute except for their signs: Since d^i deletes or merges edges left of $f_n^{-1}(j)$ for $i < j$, we find that $d^i d^j = -d^j d^i$. Hence it is clear that up to a shift we can interpret $C^{E_n}_{(*,r_{n-1},\dots,r_1)}(F)$ as a total complex as above. All the differentials d^i leave the lower levels of a tree t as they were. Hence the splitting above holds, allowing us to consider one $(n-1)$–tree shape at a time. □

Theorem 6.5 *For all $n \geqslant 0$ we have*

$$H_s^{E_n}(b_n^*) = \begin{cases} k & \text{if } s = 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Proof We will prove that

$$H_*(C^{E_n}_{(*,r_{n-1},\dots,r_1)}(b_n^*), \partial_n) = 0$$

except when $r_{n-1} = 0$. Note that if $r_{n-1} = 0$ this forces $r_{n-2}, \dots, r_1 = 0$, and

$$(C^{E_n}_{(*,0,\dots,0)}(b_n^*), \partial_n) \cong C_*^{E_1}(b_1^*).$$

By Proposition 6.3 and Lemma 6.4 this gives rise to a copy of k in $H_0^{E_n}(b_n^*)$.

Now fix $r_{n-1} \geqslant 1$, $r_{n-2}, \dots, r_1 \geqslant 0$. Let $T = [r_{n-1}] \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} [r_1]$ be an $(n-1)$–level tree. Consider the corresponding summand M^T of the multicomplex $M(b_n^*)$ discussed in Lemma 6.4. According to the lemma it suffices to show that the homology of the total complex associated to M^T is trivial for all trees T as above. Let us start by calculating the homology of M^T in the zeroth direction, ie for each given $x_1, \dots, x_{r_{n-1}} \geqslant 1$ we consider the complex

$$(M^T_{*,x_1,\dots,x_{r_{n-1}}}, d^0) = \Bigg(\bigoplus_{\substack{t=[*+x_1+\cdots+x_{r_{n-1}}] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1] \\ |f_n^{-1}(0)|=*+1,\, |f_n^{-1}(i)|=x_i}} b_n^*(t), d^0 \Bigg).$$

Since we fixed T, for each p there is exactly one tree $t = [r_n] \xrightarrow{f_n} \cdots \xrightarrow{f_2} [r_1]$ with $|f_n^{-1}(0)| = p+1$ and $|f_n^{-1}(i)| = x_i$ for $1 \leqslant i \leqslant r_{n-1}$. Let $q = r_n - p$. The differential d^0 maps $\alpha_j \in b_n^*(t) = k\langle \alpha_0, \dots, \alpha_{p+q} \rangle$ to

$$(-1)^{n-1} b_n^*(\delta_0, \mathrm{id}, \dots, \mathrm{id})(\alpha_j) + \sum_{i=0}^{p-1} (-1)^{n+i} b_n^*(d_i, \mathrm{id}, \dots, \mathrm{id})(\alpha_j) + (-1)^{n+p} b_n^*(\delta_p, \mathrm{id}, \dots, \mathrm{id})(\alpha_j).$$

Thus for $j \leqslant p$ the element $d^0(\alpha_j)$ coincides up to a sign $(-1)^{n-1}$ with the image of $\alpha_j \in b_1^*([p])$ under the differential d_{E_1} of $C_*^{E_1}(b_1^*)$. If $j > p$ all the induced morphisms are the identity. Hence $(M^T_{*,x_1,\dots,x_{r_{n-1}}}, d^0)$ is isomorphic to

$$\cdots \xrightarrow{d_{E_1}\oplus 0} b_1^*([3]) \oplus k^q \xrightarrow{d_{E_1}\oplus \mathrm{id}} b_1^*([2]) \oplus k^q \xrightarrow{d_{E_1}\oplus 0} b_1^*([1]) \oplus k^q \xrightarrow{d_{E_1}\oplus \mathrm{id}} b_1^*([0]) \oplus k^q$$

and $H_p(M^T_{*,x_1,\dots,x_{r_{n-1}}}, d^0)$ is concentrated in degree $p=0$, where it is k. We showed in Proposition 6.3 that $H_0^{E_1}(b_1^*) = b_1^*([0])$. Hence a cycle in $H_0(M^T_{*,x_1,\dots,x_{r_{n-1}}}, d^0)$ is given by $\alpha_0 \in b_n^*(t^{1,x_1,\dots,x_{r_{n-1}}})$, where $t^{1,x_1,\dots,x_{r_{n-1}}}$ is the tree which extends T with top-level fibres of cardinality $1, x_1, \dots, x_{r_{n-1}}$.

We now determine how d^1 acts on these cycles. The differential d^1 is induced by morphisms acting on leaves in the second-to-left top-level fibre. All of these morphisms leave the leftmost leaf invariant and therefore each of the induced maps sends α_0 to α_0. Hence for fixed $x_2, \dots, x_{r_{n-1}} \geqslant 1$ the chain complex $(H_0(M^T_{*,*,x_2\dots,x_{r_{n-1}}}, d^0), d^1)$ is one-dimensional on the generator α_0 in each degree r with differential

$$d^1(\alpha_0) = (-1)^{2n-1} \sum_{i=0}^{r+1} (-1)^i \alpha_0.$$

We see that the homology of $(H_0(M^T_{*,*,x_2\dots,x_{r_{n-1}}}, d^0), d^1)$ vanishes completely and the homology of the total complex of M^T is zero. Hence $(C^{E_n}_{(*,r_{n-1},\dots,r_1)}(b_n^*), \partial_n)$ has trivial homology as well, whenever $r_{n-1} \geqslant 1$. □

Corollary 6.6 *No nontrivial cohomology operations arise on E_n–cohomology via the Yoneda pairing defined in Definition 6.1.*

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