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Detection of knots and a cabling formula for A–polynomials

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We say that a given knot $J \subset S^3$ is detected by its knot Floer homology and A–polynomial if whenever a knot $K \subset S^3$ has the same knot Floer homology and the same A–polynomial as J, then $K = J$. In this paper we show that every torus knot $T(p,q)$ is detected by its knot Floer homology and A–polynomial. We also give a one-parameter family of infinitely many hyperbolic knots in S^3 each of which is detected by its knot Floer homology and A–polynomial. In addition we give a cabling formula for the A–polynomials of cabled knots in S^3, which is of independent interest. In particular we give explicitly the A–polynomials of iterated torus knots.

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1 Introduction

One of the basic problems in knot theory is distinguishing knots in S^3 from each other using knot invariants. There are several knot invariants each powerful enough to determine if a given knot in S^3 is the unknot, such as knot Floer homology, the A–polynomial and Khovanov homology; see Ozsváth and Szabó [24], Boyer and Zhang [6] and Dunfield and Garoufalidis [11] and Kronheimer and Mrowka [21]. In other words, each of these invariants is an unknot-detector. It is also known that knot Floer homology can detect the trefoil knot and the figure 8 knot; see Ghiggini [14]. In this paper we first consider the problem of detecting the set of torus knots $T(p,q)$ in S^3 using knot invariants. To reach this goal neither knot Floer homology nor the A–polynomial alone is enough: The torus knot $T(4,3)$ has the same knot Floer homology as the $(2,3)$–cable over $T(3,2)$ (see Hedden [18]), while the torus knot $T(15,7)$ has the same A–polynomial as the torus knot $T(35,3)$. However when the two invariants are combined together, the job can be done. We have:

Theorem 1.1 *If a knot K in S^3 has the same knot Floer homology and the same A–polynomial as a torus knot $T(p,q)$, then $K = T(p,q)$.*

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We then go further to consider the detection problem for hyperbolic knots. For a one-parameter family of mutually distinct hyperbolic knots $k(l_*, -1, 0, 0)$ in S^3, where $l_* > 1$ is integer-valued, we succeed in showing that each knot in the family is detected by the combination of its A–polynomial and its knot Floer homology. A knot diagram for $k(l_*, -1, 0, 0)$ is illustrated in Figure 5. Note that $k(2, -1, 0, 0)$ is the $(-2, 3, 7)$–pretzel knot. Also note that the knots $k(l_*, -1, 0, 0)$ form a subfamily of the hyperbolic knots $k(l, m, n, p)$ (with some forbidden values on the integers l, m, n and p) given in Eudave-Muñoz [12], each of which admits a half-integral toroidal surgery (the slope formula is given in Eudave-Muñoz [13] and recalled in Section 4 of this paper), and by Gordon and Luecke [17], these hyperbolic knots $k(l, m, n, p)$ are the only hyperbolic knots in S^3 which admit nonintegral toroidal surgeries.

Theorem 1.2 *The knots in the family $\{k(l_*, -1, 0, 0) : l_* > 1, l_* \in \mathbb{Z}\}$ are mutually distinct hyperbolic knots in S^3. Let J_* be any fixed $k(l_*, -1, 0, 0)$ with $l_* > 1$. If a knot K in S^3 has the same knot Floer homology and the same A–polynomial as J_*, then $K = J_*$.*

To prove Theorem 1.1 (resp. Theorem 1.2), let $K \subset S^3$ be any knot with the same knot Floer homology and A–polynomial as the knot $T(p, q)$ (resp. J_*). Applying well-known results from knot Floer homology we get immediately the following three conditions on the knot K: K is fibered since $T(p, q)$ (resp. J_*) is, K has the same Alexander polynomial as $T(p, q)$ (resp. J_*) and K has the same Seifert genus as $T(p, q)$ (resp. J_*); see Ghiggini [14] and Ni [23], Ozsváth and Szabó [25] and Ozsváth and Szabó [24]. Combining these three derived conditions on K with the assumption on the A–polynomial of K will be sufficient for us in identifying K with $T(p, q)$ (resp. J_*). The proof of Theorem 1.1 will be given in Section 3 after we establish some general properties of A–polynomials in Section 2, where we also derive a cabling formula for A–polynomials of cabled knots in S^3 (Theorem 2.8) and in particular we give explicitly the A–polynomials of iterated torus knots (Corollary 2.12). The argument for Theorem 1.2 is more involved than that for Theorem 1.1, and so we need to make some more preparations for it (besides those made in Section 2) in the next three sections. In Section 4 we collect some topological properties about the family of knots $k(l, m, n, p)$; in particular, we give a complete genus formula for $k(l, m, n, p)$ and show that the knots $k(l_*, -1, 0, 0)$ form a class of small knots in S^3. In Section 5 we collect some information about the A–polynomials of the knots $k(l, m, n, p)$ without knowing the explicit formulas of the A–polynomials, and with such information we are able to show that if a hyperbolic knot K has the same A–polynomial as a given knot $J_* = k(l_*, -1, 0, 0)$, then K has the same half-integral toroidal surgery slope as J_* and K is one of the knots $k(l, m, 0, p)$ with l divisible by $2p-1$. We then in Section 6 identify each $J_* = k(l_*, -1, 0, 0)$ among the knots of the form $k(l, m, 0, p)$ where $(2p-1) \mid l$, using the genus formula and the half-integral toroidal slope formula for $k(l, m, 0, p)$. Results obtained in these three sections, together with some results from Section 2, are applied in Section 7 to complete the proof of Theorem 1.2.

Note that the A–polynomial $A_K(x, y)$ (for a knot K in S^3) used in this paper is a slightly modified version of the original A–polynomial given in Cooper, Culler, Gillet, Long and Shalen [8]. The only difference is that in the current version, the A–polynomial of the unknot is 1 and $y-1$ may possibly occur as a factor in $A_K(x, y)$ for certain knots K contributed by some component of the character variety of the knot exterior containing characters of irreducible representations, while in the original version $y-1$ is a factor of the A–polynomial for every knot contributed by the unique component of the character variety of the knot exterior consisting of characters of reducible representations (see Section 2 for details). The current version contains a bit more information than the original one.

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2 Some properties of A–polynomials

First we need to recall some background material on A–polynomials and set up some notations. For a finitely generated group Γ, the set of representations (ie group homomorphisms) from Γ to $SL_2(\mathbb{C})$ is denoted by $R(\Gamma)$. For each representation $\rho \in R(\Gamma)$, its character χ_ρ is the complex-valued function $\chi_\rho\colon \Gamma \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$ for $\gamma \in \Gamma$. Let $X(\Gamma)$ be the set of characters of representations in $R(\Gamma)$ and $t\colon R(\Gamma) \to X(\Gamma)$ the map sending ρ to χ_ρ. Then both $R(\Gamma)$ and $X(\Gamma)$ are complex affine algebraic sets such that t is a regular map (see [10] for details).

For an element $\gamma \in \Gamma$, the function $f_\gamma\colon X(\Gamma) \to \mathbb{C}$ is defined by $f_\gamma(\chi_\rho) = (\chi_\rho(\gamma))^2 - 4$ for each $\chi_\rho \in X(\Gamma)$. Each f_γ is a regular function on $X(\Gamma)$. Obviously $\chi_\rho \in X(\Gamma)$ is a zero point of f_γ if and only if either $\rho(\gamma) = \pm I$ or $\rho(\gamma)$ is a parabolic element. It is also evident that f_γ is invariant when γ is replaced by a conjugate of γ or the inverse of γ.

Note that if $\phi\colon \Gamma \to \Gamma'$ is a group homomorphism between two finitely generated groups, then it naturally induces a regular map $\tilde{\phi}\colon R(\Gamma') \to R(\Gamma)$ by $\tilde{\phi}(\rho') = \rho' \circ \phi$ and a regular map $\hat{\phi}\colon X(\Gamma') \to X(\Gamma)$ by $\hat{\phi}(\chi_{\rho'}) = \chi_{\tilde{\phi}(\rho')}$. Note that if X_0 is an irreducible subvariety of $X(\Gamma')$, then the Zariski closure of $\hat{\phi}(X_0)$ in $X(\Gamma)$ is also irreducible. If in addition the homomorphism ϕ is surjective, each of the regular maps $\widetilde{\phi}$ and $\widehat{\phi}$ is an embedding, in which case we may simply consider $R(\Gamma')$ and $X(\Gamma')$ as subsets of $R(\Gamma)$ and $X(\Gamma)$, respectively, and write $R(\Gamma') \subset R(\Gamma)$ and $X(\Gamma') \subset X(\Gamma)$.

For a compact manifold W, we denote $R(\pi_1(W))$ and $X(\pi_1(W))$, respectively, by $R(W)$ and $X(W)$.

The A–polynomial was introduced in [8]. We slightly modify its original definition for a knot K in S^3 as follows: Let M_K be the exterior of K in S^3 and let $\{\mu, \lambda\}$ be the standard meridian-longitude basis for $\pi_1(\partial M_K)$. Let $\widehat{i}_*\colon X(M_K) \to X(\partial M_K)$ be the regular map induced by the inclusion-induced homomorphism $i_*\colon \pi_1(\partial M_K) \to \pi_1(M_K)$, and let Λ be the set of diagonal representations of $\pi_1(\partial M_K)$, ie

$$\Lambda = \{\rho \in R(\partial M_K) : \rho(\mu) \text{ and } \rho(\lambda) \text{ are both diagonal matices}\}.$$

Then Λ is a subvariety of $R(\partial M_K)$ and $t|_\Lambda\colon \Lambda \to X(\partial M_K)$ is a degree-two, surjective, regular map. We may identify Λ with $\mathbb{C}^* \times \mathbb{C}^*$ through the eigenvalue map

$$E\colon \Lambda \to \mathbb{C}^* \times \mathbb{C}^*$$

given by

$$\rho \mapsto (x, y) \quad \text{if } \rho(\mu) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \text{ and } \rho(\lambda) = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}.$$

For every knot in S^3, there is a unique component in $X(M_K)$ consisting of characters of reducible representations, which we call the trivial component of $X(M_K)$. (The trivial component is of dimension one.) Now let $X^*(M_K)$ be the set of nontrivial components of $X(M_K)$ each of which has a one-dimensional image in $X(\partial M_K)$ under the map $\widehat{i}_*$. The set $X^*(M_K)$ is possibly empty, and in fact it is currently known that $X^*(M_K)$ is empty if and only if K is the unknot. So when K is a nontrivial knot, $(t|_\Lambda)^{-1}(\widehat{i}_*(X^*(M_K)))$ is one-dimensional in Λ and in turn $E\big((t|_\Lambda)^{-1}(\widehat{i}_*(X^*(M_K)))\big)$ is one-dimensional in $\mathbb{C}^* \times \mathbb{C}^* \subset \mathbb{C} \times \mathbb{C}$. Let D be the Zariski closure in $\mathbb{C}^2$ of $E\big((t|_\Lambda)^{-1}(\widehat{i}_*(X^*(M_K)))\big)$. Then D is a plane curve in $\mathbb{C}^2$ defined over $\mathbb{Q}$. Let $A_K(x, y)$ be a defining polynomial of D normalized so that $A_K(x, y) \in \mathbb{Z}[x, y]$ with no repeated factors and with 1 as the greatest common divisor of its coefficients. Then $A_K(x, y)$ is uniquely associated to K up to sign and is called the A–polynomial of K. For the unknot we define its A–polynomial to be 1. As remarked in the introduction, $y - 1$ might occur as a factor of $A_K(x, y)$ for certain knots. Also by [6] and [11], $A_K(x, y) = 1$ if and only if K is the unknot (in fact for every nontrivial knot K, the A–polynomial $A_K(x, y)$ contains a nontrivial factor which is not $y - 1$).

From the constructional definition of the A–polynomial, we see that each component X_0 of $X^*(M_K)$ contributes a factor $f_0(x, y)$ in $A_K(x, y)$, ie $f_0(x, y)$ is the defining polynomial of the plane curve D_0 which is the Zariski closure of $E((t|_\Lambda)^{-1}(\widehat{i}_*(X_0)))$ in $\mathbb{C}^2$, and moreover $f_0(x,y)$ is balanced, ie if (x,y) is a generic zero point of $f_0(x,y)$ then (x^{-1},y^{-1}) is also a zero point of $f_0(x,y)$. Also note that $f_0(x,y)$ is not necessarily irreducible over $\mathbb{C}$ but contains at most two irreducible factors over $\mathbb{C}$. We shall call such a $f_0(x,y)$ a balanced-irreducible factor of $A_K(x,y)$. Obviously $A_K(x,y)$ is a product of balanced-irreducible factors and the product decomposition is unique up to the ordering of the factors.

We now define a couple of functions which will be convenient to use in expressing the A–polynomials for torus knots and later on for cabled knots and iterated torus knots. Let (p,q) be a pair of relatively prime integers with $q \geq 2$. Define $F_{(p,q)}(x,y)$ to be the polynomial in $\mathbb{Z}[x,y]$ determined by the pair (p,q) by

$$F_{(p,q)}(x,y) = \begin{cases} 1+x^{2p}y & \text{if } q=2 \text{ and } p>0, \\ x^{-2p}+y & \text{if } q=2 \text{ and } p<0, \\ -1+x^{2pq}y^2 & \text{if } q>2 \text{ and } p>0, \\ -x^{-2pq}+y^2 & \text{if } q>2 \text{ and } p<0 \end{cases} \tag{2-1}$$

and define $G_{(p,q)}(x,y) \in \mathbb{Z}[x,y]$ to be the polynomial determined by the pair (p,q) by

$$G_{(p,q)}(x,y) = \begin{cases} -1+x^{pq}y & \text{if } p>0, \\ -x^{-pq}+y & \text{if } p<0. \end{cases} \tag{2-2}$$

Note that the ring $\mathbb{C}[x,y]$ is a unique factorization domain. The following lemma can be easily checked.

Lemma 2.1 *Among the polynomials in (2-1) and (2-2), the first two in (2-1) and the two in (2-2) are irreducible over $\mathbb{C}$, and the last two in (2-1) can be factored as the product of two irreducible polynomials over $\mathbb{C}$:*

$$\begin{aligned} -1+x^{2pq}y^2 &= (-1+x^{pq}y)(1+x^{pq}y) \quad \text{if } q>2 \text{ and } p>0, \\ -x^{-2pq}+y^2 &= (-x^{-pq}+y)(x^{-pq}+y) \quad \text{if } q>2 \text{ and } p<0. \end{aligned}$$

The set of nontrivial torus knots $T(p,q)$ is naturally indexed by pairs (p,q) satisfying $|p|>q\geq 2$, $(p,q)=1$. Note that $T(-p,q)$ is the mirror image of $T(p,q)$. The A–polynomial of a torus knot $T(p,q)$ is given by (eg [28, Example 4.1])

$$A_{T(p,q)}(x,y) = F_{(p,q)}(x,y) \tag{2-3}$$

In particular the A–polynomial distinguishes $T(p,q)$ from $T(-p,q)$.

For the exterior M_K of a nontrivial knot in S^3, we can consider the set of elements of $H_1(\partial M_K;\mathbb{Z}) \cong \pi_1(\partial M_K)$ as a subgroup of $\pi_1(M_K)$ which is well defined up to conjugation. In particular, the function f_α on $X(M_K)$ is well defined for each class $\alpha \in H_1(\partial M_K;\mathbb{Z})$. As f_α is also invariant under a change of the orientation of α, the function f_α is also well defined when α is a slope in ∂M_K. Later on for convenience we will often not make a distinction among a primitive class of $H_1(\partial M_K;\mathbb{Z})$, the corresponding element of $\pi_1(\partial M_K)$ and the corresponding slope in ∂M_K, so long as it is well defined.

It is known (eg [4]) that any irreducible curve X_0 in $X(M_K)$ belongs to one of the following three mutually exclusive types:

(a) For each slope α in ∂M_K, the function f_α is nonconstant on X_0.

(b) There is a unique slope α_0 in ∂M_K such that the function f_{α_0} is constant on X_0.

(c) For each slope α in ∂M_K, the function f_α is constant on X_0.

Obviously a curve of type (a) or (b) has one-dimensional image in $X(\partial M_K)$ under the map $\widehat{i}_*$. Note that the trivial component of $X(M_K)$ is of type (b). Hence a curve of type (a) is contained in $X^*(M_K)$ and so is a curve of type (b) if it is not the trivial component of $X(M_K)$.

An irreducible curve of type (a) is named a *norm curve*. Indeed as the name indicates, a norm curve in $X(M_K)$ can be used to define a norm, known as the Culler–Shalen norm, on the real two-dimensional plane $H_1(\partial M_K;\mathbb{R})$ satisfying certain properties. Such a curve exists when M_K is hyperbolic: namely, any component of $X(M_K)$ which contains the character of a discrete faithful representation of $\pi_1(M_K)$ is a norm curve.

For an irreducible curve X_0 in $X(M_K)$, let $\widetilde{X}_0$ be the smooth projective completion of X_0 and let $\phi\colon \widetilde{X}_0 \to X_0$ be the birational map. The map ϕ is onto and is defined at all but finitely many points of $\widetilde{X}_0$. The points of $\widetilde{X}_0$ where ϕ is not defined are called ideal points and all other points of $\widetilde{X}_0$ are called regular points. The map ϕ induces an isomorphism from the function field of X_0 to that of $\widetilde{X}_0$. In particular every regular function f_γ on X_0 corresponds uniquely to its extension $\widetilde{f}_\gamma$ on $\widetilde{X}_0$ which is a rational function. If $\widetilde{f}_\gamma$ is not a constant function on $\widetilde{X}_0$, its degree, denoted $\deg(\widetilde{f}_\gamma)$, is equal to the number of zeros of $\widetilde{f}_\gamma$ in $\widetilde{X}_0$ counted with multiplicity, ie

$$\deg(\widetilde{f}_\gamma) = \sum_{v\in\widetilde{X}_0} Z_v(\widetilde{f}_\gamma),$$

where $Z_v(\widetilde{f}_\gamma)$ is the zero degree of $\widetilde{f}_\gamma$ at the point $v \in \widetilde{X}_0$.

We shall identify $H_1(\partial M_K;\mathbb{R})$ with the real xy–plane so that $H_1(\partial M_K;\mathbb{Z})$ consists of integer lattice points with $\mu = (1,0)$ being the meridian class and $\lambda = (0,1)$ the longitude class. So each slope m/n corresponds to the pair of primitive elements $\pm(m,n) \in H_1(\partial M_K;\mathbb{Z})$.

Theorem 2.2 *Let X_0 be a norm curve of $X(M_K)$. Then the associated Culler–Shalen norm $\|\cdot\|_0$ on $H_1(\partial M_K;\mathbb{R})$ has the following properties: Let*

$$s_0 = \min\{\|\alpha\|_0 : \alpha \neq 0, \alpha \in H_1(\partial M_K;\mathbb{Z})\}$$

and let B_0 be the disk in $H_1(\partial M_K;\mathbb{R})$ centered at the origin with radius s_0 with respect to the norm $\|\cdot\|_0$. Then:

(1) *For each nontrivial element $\alpha = (m,n) \in H_1(\partial M_K;\mathbb{Z})$, the norm $\|\cdot\|_0$ satisfies $\|\alpha\|_0 = \deg(\widetilde{f_\alpha}) \neq 0$ and thus $\|\alpha\|_0 = \|-\alpha\|_0$.*

(2) *The disk B_0 is a convex finite-sided polygon symmetric about the origin whose interior does not contain any nonzero element of $H_1(\partial M_K;\mathbb{Z})$ and whose boundary contains at least one but at most four nonzero classes of $H_1(\partial M_K;\mathbb{Z})$ up to sign.*

(3) *If (a,b) is a vertex of B_0, then there is a boundary slope m/n of ∂M_K such that $\pm(m,n)$ lie in the line passing through (a,b) and $(0,0)$. (That is, a/b is a boundary slope of ∂M_K for any vertex (a,b) of B_0.)*

(4) *If a primitive class $\alpha = (m,n) \in H_1(\partial M_K;\mathbb{Z})$ is not a boundary class and $M_K(\alpha)$ has no noncyclic representations, then $\alpha = (m,n)$ lies in ∂B (that is, $\|\alpha\|_0 = s_0$) and is not a vertex of B_0.*

(5) *If the meridian class $\mu = (1,0)$ is not a boundary class, then for any nonintegral class $\alpha = (m,n)$, if it is not a vertex of B_0 then it does not lie in ∂B and thus $\|\alpha\|_0 > \|\mu\|_0 = s_0$.*

Theorem 2.2 is originally from [9, Chapter 1], although it was assumed there that the curve X_0 contains the character of a discrete faithful representation of M_K. The version given here is contained in [4].

Recall that if $f_0(x,y) = \sum a_{i,j} x^i y^j \in \mathbb{C}[x,y]$ is a two-variable polynomial in x and y with complex coefficients, the Newton polygon N_0 of $f_0(x,y)$ is defined to be the convex hull in the real xy–plane of the set of points

$$\{(i,j) : a_{i,j} \neq 0\}.$$

The following theorem is proved in [5].

Theorem 2.3 *Let X_0 be a norm curve of $X(M_K)$ and let $f_0(x,y)$ be the balanced-irreducible factor of $A_K(x,y)$ contributed by X_0. Then the norm polygon B_0 determined by X_0 is dual to the Newton polygon N_0 of $f_0(x,y)$ in the following way: The set of slopes of vertices of B_0 is equal to the set of slopes of edges of N_0. In fact B_0 and N_0 mutually determine each other up a positive integer multiple.*

We remark that although in [5] there were some additional conditions imposed on X_0 and the version of the A–polynomial defined in [5] is mildly different from the one given here, the above theorem remains valid with identical reasoning as given in [5]. We only need to describe the exact relation between B_0 and N_0 as follows to see how they determine each other up to an integer multiple: Let Y_0 be the Zariski closure of the restriction $\widehat{i}_*(X_0)$ of X_0 in $X(\partial M_K)$ and let d_0 be the degree of the map $\widehat{i}_*\colon X_0 \to Y_0$. As explained in [5] (originally in [26]), the Newton polygon N_0 determines a width function w on the set of slopes given by

$$w(p/q) = k \in \mathbb{Z}$$

if $k+1$ is the number of lines in the xy–plane of slope q/p which contain points of both $\mathbb{Z}^2$ and N_0. The width function in turn defines a norm $\|\cdot\|_{N_0}$ on the xy–plane $H_1(\partial M_K;\mathbb{R})$ such that

$$\|(p,q)\|_{N_0} = w(p/q)$$

for each primitive class $(p,q) \in H_1(\partial M_K;\mathbb{Z})$. Finally

$$\|\cdot\|_0 = 2d_0\|\cdot\|_{N_0}.$$

Corollary 2.4 *If every balanced-irreducible factor of $A_K(x,y)$ over $\mathbb{C}$ has two monomials, then K is not a hyperbolic knot.*

Proof The condition of the corollary means that the Newton polygon of each balanced-irreducible factor of $A_K(x,y)$ consists of a single edge. On the other hand, for a hyperbolic knot, its character variety contains a norm curve component X_0 which contributes a balanced-irreducible factor $f_0(x,y)$ to the A–polynomial such that the Newton polygon of $f_0(x,y)$ has at least two edges of different slopes. □

An irreducible curve in $X^*(M_K)$ of type (b) (such a curve exists only for certain knots) is named a *seminorm curve* as suggested by the following theorem, which is contained in [4].

Theorem 2.5 *Suppose that $X_0 \subset X^*(M_K)$ is an irreducible curve of type (b) with α_0 being the unique slope such that f_{α_0} is constant on X_0. Then a seminorm $\|\cdot\|_0$ can be defined on $H_1(\partial M_K;\mathbb{R})$, with the following properties:*

(1) *For each slope $\alpha \neq \alpha_0$, the seminorm $\|\cdot\|_0$ satisfies $\|\alpha\|_0 = \deg(\widetilde{f_\alpha}) \neq 0$.*

(2) *$\|\alpha_0\|_0 = 0$ for the unique slope α_0 associated to X_0, and the slope α_0 is a boundary slope of M_K.*

(3) *If α is a primitive class and is not a boundary class and $M_K(\alpha)$ has no noncyclic representation, then $\Delta(\alpha,\alpha_0)=1$.*

(4) *Let $s_0=\min\{\|\alpha\|_0 : \alpha\neq\alpha_0$ is a slope$\}$. Then $\|\alpha\|_0=s_0\Delta(\alpha,\alpha_0)$, for any slope α.*

Note that for each torus knot $T(p,q)$, every nontrivial component in its character variety is a seminorm curve with pq as the associated slope.

Remark If K is a small knot, ie if its exterior M_K does not contain any closed essential surface, then every nontrivial component of $X(M_K)$ is a curve [8, Proposition 2.4] and is either a norm curve or a seminorm curve [4, Proposition 5.7].

We now proceed to get some properties of A–polynomials of satellite knots in S^3. Recall that a knot K in S^3 is a satellite knot if there is a pair of knots C and P in S^3, called a companion knot and a pattern knot respectively, associated to K, such that C is nontrivial, P is contained in a trivial solid torus V in S^3 but is not contained in a 3–ball of V and is not isotopic to the core circle of V, and there is a homeomorphism f from V to a regular neighborhood $N(C)$ of C in S^3 which maps a longitude of V (which bounds a disk in S^3) to a longitude of $N(C)$ (which bounds a Seifert surface for C) and maps a meridian of V to a meridian of $N(C)$, and finally $K=f(P)$. We sometimes write a satellite knot as $K=(P,C,V,f)$ to include the above defining information (K still depends on how P is embedded in V).

Lemma 2.6 *Let $K=(P,C,V,f)$ be a satellite knot in S^3. Then $A_P(x,y)\mid A_K(x,y)$ in $\mathbb{Z}[x,y]$.*

Proof The lemma is obviously true when P is the unknot in S^3. So we may assume that P is a nontrivial knot in S^3. Let M_K, M_C and M_P be the exteriors of K, C and P in S^3, respectively. There is a degree-one map $h\colon (M_K,\partial M_K)\to(M_P,\partial M_P)$ such that $h^{-1}(\partial M_P)=\partial M_K$ and $h|_{\partial M_K}\colon \partial M_K\to\partial M_P$ is a homeomorphism. The map h is given by a standard construction as follows: From the definition of the pattern knot given above, we see that if W is the exterior of P in the trivial solid torus V, then M_P is obtained by Dehn filling W along ∂V with a solid torus V' such that the meridian slope of V' is identified with the longitude slope of V. Also if we let $Y=f(W)$, then $M_K=M_C\cup Y$. Now the degree-one map $h\colon M_K\to M_P$ is defined as follows: on Y it is the homeomorphism $f^{-1}\colon Y\to W$ and on M_C it maps a regular neighborhood of a Seifert surface in M_C to a regular neighborhood of a meridian disk of V' in V' and maps the rest of M_C onto the rest of V' (which is a 3–ball).

The degree-one map h induces a surjective homomorphism $h_*\colon \pi_1(M_K) \to \pi_1(M_P)$ such that

$$h_*|_{\pi_1(\partial M_K)}\colon \pi_1(\partial M_K) \to \pi_1(\partial M_P)$$

is an isomorphism, mapping the meridian to the meridian and the longitude to the longitude. In turn h_* induces an embedding $\widehat{h}_*$ of $X(M_P)$ into $X(M_K)$ in such a way that the restriction of $\widehat{h}_*(X(M_P))$ on $X(\partial M_K)$ with respect to the standard meridian-longitude basis $\{\mu_K, \lambda_K\}$ of ∂M_K is the same as the restriction of $X(M_P)$ on $X(\partial M_P)$ with respect to the standard meridian-longitude basis $\{\mu_P, \lambda_P\}$ of ∂M_P, that is, for each $\chi_\rho \in X(M_P)$, we have $\chi_\rho(\mu_P) = \widehat{h}_*(\chi_\rho)(\mu_K)$, $\chi_\rho(\lambda_P) = \widehat{h}_*(\chi_\rho)(\lambda_K)$ and $\chi_\rho(\mu_P\lambda_P) = \widehat{h}_*(\chi_\rho)(\mu_K\lambda_K)$. The conclusion of the lemma now follows easily from the constructional definition of the A–polynomial. □

For polynomials $f(x,\bar{y}) \in \mathbb{C}[x,\bar{y}]$ and $g(y,\bar{y}) \in \mathbb{C}[y,\bar{y}]$ both with nonzero degree in $\bar{y}$, let

$$\mathrm{Res}_{\bar{y}}(f(x,\bar{y}), g(y,\bar{y}))$$

denote the resultant of $f(x,\bar{y})$ and $g(y,\bar{y})$ eliminating the variable $\bar{y}$. In general $\mathrm{Res}_{\bar{y}}(f(x,\bar{y}), g(y,\bar{y}))$ may have repeated factors even when both $f(x,\bar{y})$ and $g(y,\bar{y})$ are irreducible over $\mathbb{C}$. For a polynomial $f(x,y) \in \mathbb{C}[x,y]$, let

$$\mathrm{Red}[f(x,y)]$$

denote the polynomial obtained from $f(x,y)$ by deleting all its repeated factors.

Proposition 2.7 *Let $K = (P, C, V, f)$ be a satellite knot such that the winding number w of P in the solid torus V is nonzero. Then every balanced-irreducible factor $f_C(\bar{x},\bar{y})$ of the A–polynomial $A_C(\bar{x},\bar{y})$ of C extends to a balanced factor $f_K(x,y)$ of the A–polynomial $A_K(x,y)$ of K. More precisely:*

(1) *If the $\bar{y}$–degree of $f_C(\bar{x},\bar{y})$ is nonzero, then*

$$f_K(x,y) = \mathrm{Red}[\mathrm{Res}_{\bar{y}}(f_C(x^w,\bar{y}), \bar{y}^w - y)].$$

In particular, if $f_C(\bar{x},\bar{y}) = \bar{y} + \delta\bar{x}^n$ or $f_C(\bar{x},\bar{y}) = \bar{y}\bar{x}^n + \delta$ for n some nonnegative integer and $\delta \in \{1,-1\}$ (such a factor is irreducible and balanced), then $f_K(x,y) = y - (-\delta)^w x^{nw^2}$ or $f_K(x,y) = yx^{nw^2} - (-\delta)^w$, respectively.

(2) *If the $\bar{y}$–degree of $f_C(\bar{x},\bar{y})$ is zero, ie $f_C(\bar{x},\bar{y}) = f_C(\bar{x})$ is a function of $\bar{x}$ only, then $f_K = f_C(x^w)$.*

Proof Let M_K, M_C, Y, μ_C, λ_C, μ_K, λ_K be defined as in the proof of Lemma 2.6. We have $M_K = M_C \cup Y$. Note that $H_1(Y;\mathbb{Z}) = \mathbb{Z}[\mu_K] \oplus \mathbb{Z}[\lambda_C]$ and $[\lambda_K] = w[\lambda_C]$ and $[\mu_C] = w[\mu_K]$. Given a balanced-irreducible factor $f_C(\bar{x},\bar{y})$ of $A_C(\bar{x},\bar{y})$, let X_0 be a component of $X^*(M_C)$ which gives rise to the factor $f_C(\bar{x},\bar{y})$. For each element $\chi_\rho \in X_0$, the restriction of ρ on $\pi_1(\partial M_C)$ can be extended to an abelian representation of $\pi_1(Y)$, and thus ρ can be extended to a representation of $\pi_1(M_K)$, which we still denote by ρ, such that

$$\rho(\mu_C) = \rho(\mu_K^w) \quad \text{and} \quad \rho(\lambda_K) = \rho(\lambda_C^w).$$

It follows that X_0 extends to one or more components of $X^*(M_K)$ whose restrictions on ∂M_K are each one-dimensional and thus together give rise to a balanced factor $f_K(x,y)$ of $A_K(x,y)$. Moreover the variables x and y of $f_K(x,y)$ and the variables $\bar{x}$ and $\bar{y}$ of $f_C(\bar{x},\bar{y})$ are related by

$$\bar{x} = x^w \quad \text{and} \quad y = \bar{y}^w. \tag{2-4}$$

Therefore when the $\bar{y}$–degree of $f_C(\bar{x},\bar{y})$ is positive, $f_K(x,y)$ can be obtained by taking the resultant of $f_C(x^w,\bar{y})$ and $\bar{y}^w - y$, eliminating the variable $\bar{y}$, and then deleting possible repeated factors. In particular, if $f_C(\bar{x},\bar{y}) = \bar{y} + \delta\bar{x}^n$ or $\bar{y}\bar{x}^n + \delta$ for some nonnegative integer n and $\delta \in \{-1,1\}$, then the resultant of $\bar{y} + \delta x^{wn}$ or $\bar{y}x^{wn} + \delta$ with $\bar{y}^w - y$, eliminating the variable $\bar{y}$, is $y - (-\delta)^w x^{nw^2}$ or $yx^{nw^2} - (-\delta)^w$, respectively (which is irreducible over $\mathbb{C}$ and is balanced). Also if the degree of f_C in $\bar{y}$ is zero, then obviously $f_K = f_C(x^w)$. □

Remark In Proposition 2.7, the y–degree of $f_K(x,y)$ is at most equal to the $\bar{y}$–degree of $f_C(\bar{x},\bar{y})$ (and generically they are equal). This follows directly from the definition of the resultant; see [22, Chapter IV, Section 8].

Next we are going to consider cabled knots. Let (p,q) be a pair of relatively prime integers with $|q| \geq 2$, and K be the (p,q)–cabled knot over a nontrivial knot C. That is, K is a satellite knot with C as a companion knot and with $T(p,q)$ as a pattern knot which lies in the defining solid torus V as a standard (p,q)–cable with winding number $|q|$. As the $(-p,-q)$–cable over a knot is equal to the (p,q)–cable over the same knot, we may always assume $q \geq 2$. The following theorem gives a cabling formula for the A–polynomial of a cabled knot K over a nontrivial knot C, in terms of the A–polynomial $A_C(\bar{x},\bar{y})$ of C.

Theorem 2.8 *Let K be the (p,q)–cabled knot over a nontrivial knot C, with $q \geq 2$. Then*

$$A_K(x,y) = \mathrm{Red}\big[F_{(p,q)}(x,y)\,\mathrm{Res}_{\bar{y}}\big(A_C(x^q,\bar{y}),\bar{y}^q - y\big)\big]$$

if the $\bar{y}$–degree of $A_C(\bar{x},\bar{y})$ is nonzero and

$$A_K(x,y) = F_{(p,q)}(x,y)A_C(x^q)$$

if the $\bar{y}$–degree of $A_C(\bar{x},\bar{y})$ is zero.

Proof For a polynomial $f(\bar{x},\bar{y})\in\mathbb{C}[\bar{x},\bar{y}]$, define

$$\operatorname{Ext}^q[f(\bar{x},\bar{y})] = \begin{cases}\operatorname{Red}\big[\operatorname{Res}_{\bar{y}}\big(f(x^q,\bar{y}),\bar{y}^q-y\big)\big] & \text{if the degree of } f(\bar{x},\bar{y}) \text{ in } \bar{y} \text{ is nonzero,}\\ f(x^q) & \text{if the degree of } f(\bar{x},\bar{y}) \text{ in } \bar{y} \text{ is zero.}\end{cases}$$

Then Proposition 2.7 says that

$$f_K(x,y)=\operatorname{Ext}^w[f_C(\bar{x},\bar{y})],$$

and Theorem 2.8 says that

$$A_K(x,y)=\operatorname{Red}\big[F_{(p,q)}(x,y)\operatorname{Ext}^q[A_C(\bar{x},\bar{y})]\big].$$

Note that $\operatorname{Ext}^q[A_C(\bar{x},\bar{y})]=\operatorname{Red}\big[\prod\operatorname{Ext}^q[f_C(\bar{x},\bar{y})]\big]$ where the product runs over all balanced-irreducible factors $f_C(\bar{x},\bar{y})$ of $A_C(\bar{x},\bar{y})$ and thus by Proposition 2.7, $\operatorname{Ext}^q[A_C(\bar{x},\bar{y})]$ is a balanced factor of $A_K(x,y)$. So we only need to show:

Claim 2.9 *$F_{(p,q)}(x,y)$ is a balanced factor of $A_K(x,y)$. (Note that each irreducible factor of $F_{(p,q)}(x,y)$ is balanced.)*

Claim 2.10 *Besides $F_{(p,q)}(x,y)$ and $\operatorname{Ext}^q[A_C(\bar{x},\bar{y})]$, the A–polynomial $A_K(x,y)$ has no other balanced factors.*

To prove the above two claims, let M_K, M_C, Y, μ_C, λ_C, μ_K, λ_K be defined as in Lemma 2.6 with respect to $K=(P,C,f,V)$ where $P=T(p,q)$ is embedded in V as a standard (p,q)–cable. We have $M_K=M_C\cup Y$ and $\partial Y=\partial M_C\cup\partial M_K$. For convenience in the present argument, we give a direct description of Y as follows: We may consider $N=N(C)$ as $D\times C$, where D is a disk of radius 2, such that $\{x\}\times C$ has slope zero for each point $x\in\partial D$. Let D_* be the concentric subdisk in D with radius 1. Then $N_*=D_*\times C$ is a solid torus in $N(C)$ sharing the same core circle C. We may assume that the knot K is embedded in the boundary of N_* as a standard (p,q)–curve, where ∂N_* has the meridian-longitude coordinates consistent with that of $\partial M_C=\partial N$ (ie $\{x\}\times C$ is a longitude of ∂N_* for any point $x\in\partial D_*$). Then Y is the exterior of K in N.

Note that Y is a Seifert fibered space whose base orbifold is an annulus with a single cone point of order q, a Seifert fiber of Y in ∂M_C has slope p/q, and a Seifert fiber of Y in ∂M_K has slope pq. Let γ_C be a Seifert fiber of Y lying in ∂M_C and γ_K be a Seifert fiber of Y lying in ∂M_K. Up to conjugation, we may consider γ_C and γ_K as elements of $\pi_1(Y)$. Also note that γ_C is conjugate to γ_K in $\pi_1(Y)$. It is well known that each of γ_C and γ_K lies in the center of $\pi_1(Y)$, which is independent of conjugation. It follows that if $\rho\in R(Y)$ is an irreducible representation then $\rho(\gamma_C)=\rho(\gamma_K)=\epsilon I$, for some fixed $\epsilon \in \{1, -1\}$, where I is the identity matrix. Hence if X_0 is an irreducible subvariety of $X(Y)$ which contains the character of an irreducible representation, then for every $\chi_\rho \in X_0$, we have $\rho(\gamma_C) = \rho(\gamma_K) = \epsilon I$, which is due to the fact that the characters of irreducible representations are dense in X_0.

Now we are ready to prove Claim 2.9. If $|p| > 1$, then by Lemma 2.6 and (2-3), $F_{(p,q)}(x, y)$ is a factor of $A_K(x, y)$. So we may assume that $|p| = 1$. Under this assumption, one can see that the fundamental group of Y has the presentation

$$\pi_1(Y) = \langle \alpha, \beta : \alpha^q \beta = \beta \alpha^q \rangle \tag{2-5}$$

such that

$$\mu_K = \alpha\beta,$$

where α is a based simple loop free homotopic in Y to the center circle of N and β is a based simple loop free homotopic in Y to λ_C in ∂M_C. To see these assertions, note that Y contains the essential annulus $A_2 = \partial N_* \cap Y$ with $\partial A_2 \subset \partial M_K$ of the slope pq (the cabling annulus, consisting of Seifert fibers of Y) and A_2 decomposes Y into two pieces U_1 and U_2, such that U_1 is a solid torus (which is $N_* \cap Y$) and U_2 is topologically ∂M_C times an interval. The above presentation for $\pi_1(Y)$ is obtained by applying the van Kampen theorem associated to the splitting of $Y = U_1 \cup_{A_2} U_2$ along A_2. We should note that as $|p| = 1$, a longitude in ∂N_* intersects K geometrically exactly once. It is this curve pushed into U_1 which yields the element α and pushed into U_2 which yields β. Also because $|p| = 1$, we have that $\{\gamma_C, \beta\}$ forms a basis for $\pi_1(U_2)$. Therefore by van Kampen, $\pi_1(Y)$ is generated by α, β and γ_C with relations $\alpha^q = \gamma_C$ and $\gamma_C \beta = \beta \gamma_C$, which yields presentation (2-5) after canceling the element γ_C. With a suitable choice of orientation for β, ie replacing β by its inverse if necessary, we also have $\mu_K = \alpha\beta$. See Figure 1 for an illustration when $q = 3$.

By [15, Lemma 7.2], $N_K(pq)$, which denotes the manifold obtained by Dehn surgery on K in the solid torus N with the slope pq, is homeomorphic to $L(q, p)\#(D^2 \times S^1)$, and $M_K(pq)$, which denotes the manifold obtained by Dehn filling of M_K with the slope pq, is homeomorphic to $L(q, p) \# M_C(p/q)$ (where the meaning of the notation $M_C(p/q)$ should be obvious). $M_C(p/q)$ is a homology sphere since $|p| = 1$. By [20], $R(M_C(p/q)) \subset R(M_C)$ contains at least one irreducible representation ρ_C. Note that the restriction of ρ_C on $\pi_1(\partial M_C)$ is not contained in $\{I, -I\}$. That is, we have $\rho_C(\gamma_C) = I$ and $\rho_C(\beta) \neq \pm I$ (we may consider β as the longitude λ_C of $\pi_1(\partial M_C)$). If $q > 2$, we can extend ρ_C to a curve of representations of $\pi_1(M_K)$ with one-dimensional characters. In fact for every $A \in \mathrm{SL}_2(\mathbb{C})$, we may define $\rho_A \in R(M_K)$ as follows: On $\pi_1(M_C)$, let $\rho_A = \rho_C$, so that in particular $\rho_A(\beta) = \rho_C(\beta)$, and define $\rho_A(\alpha) = ABA^{-1}$ where B is a fixed order q matrix in $\mathrm{SL}_2(\mathbb{C})$. It's routine to

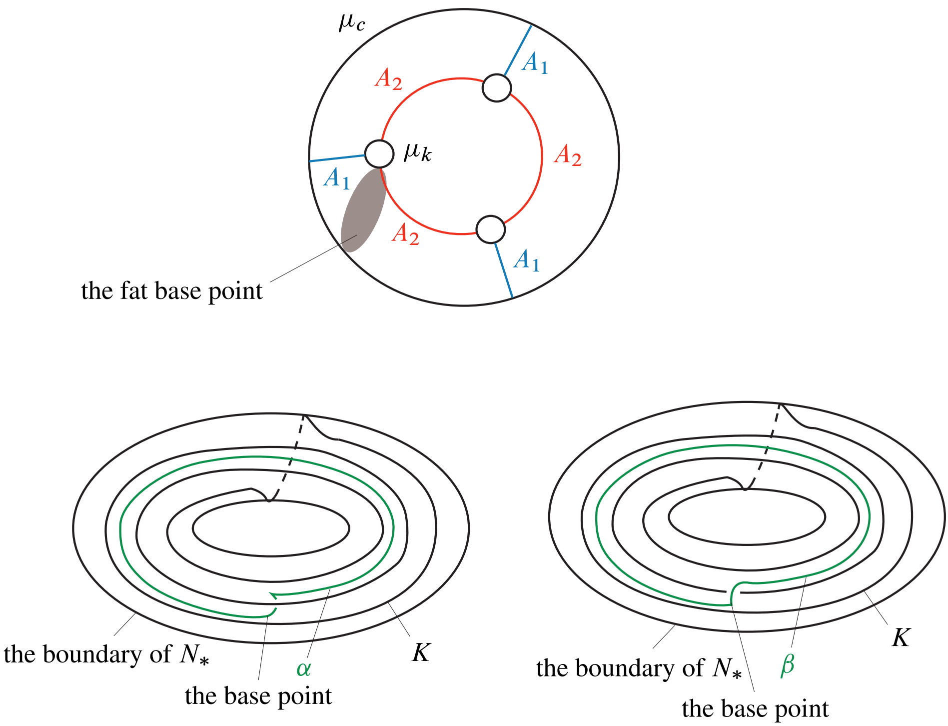

Figure 1: The cross section $D \cap Y$ of Y and the elements α and β of $\pi_1(Y)$ when $q=3$

check that ρ_A is well defined, the trace of $\rho_A(\mu_K) = ABA^{-1}\rho_C(\beta)$ varies as A runs over $\mathrm{SL}_2(\mathbb{C})$, and $\rho_A(\gamma_K) = I$. Hence we get a curve in $X(M_K)$ whose restriction on ∂M_K is one-dimensional, and moreover this curve generates $-1 + yx^q$ as a factor of $A_K(x,y)$ if $p=1$ or generates $-x^q + y$ as a factor of $A_K(x,y)$ if $p=-1$.

We now show that $1 + yx^q$ is a factor of $A_K(x,y)$ if $p=1$ or $x^q + y$ is a factor of $A_K(x,y)$ if $p=-1$, for all $q \geq 2$. For the representation ρ_C given in the last paragraph, define $\rho_C^\epsilon \in R(M_C)$ by

$$\rho_C^\epsilon(\gamma) = \epsilon(\gamma)\rho_C(\gamma) \quad \text{for } \gamma \in \pi_1(M_C),$$

where ϵ is the onto homomorphism $\epsilon\colon \pi_1(M_C) \to \{I, -I\}$. As μ_C is a generator of $H_1(M_C;\mathbb{Z})$, we have that $\rho_C^\epsilon(\mu_C) = -\rho_C(\mu_C)$. Similarly, as λ_C is trivial in $H_1(M_C;\mathbb{Z})$, we also have that $\rho_C^\epsilon(\lambda_C) = \rho_C(\lambda_C)$. Hence we have that $\rho_C^\epsilon(\gamma_C) = \rho_C^\epsilon(\lambda_C^q \mu_C^p) = -\rho_C(\gamma_C) = -I$. We can now extend ρ_C^ϵ to ρ_A^ϵ over M_K similarly as for ρ_C to ρ_A, only this time we choose B as a fixed order $2q$ matrix in $\mathrm{SL}_2(\mathbb{C})$, so that the trace of $\rho_A^\epsilon(\mu_K) = ABA^{-1}\rho_C(\beta)$ varies as A runs over $\mathrm{SL}_2(\mathbb{C})$, and $\rho_A^\epsilon(\gamma_K) = -I$. The existence of the factor of $A_K(x,y)$ that we set to prove now follows. This completes the proof of Claim 2.9.

Last we prove Claim 2.10. Given a balanced-irreducible factor $f_0(x,y)$ of $A_K(x,y)$, let X_0 be an irreducible component of $X^*(M_K)$ over $\mathbb{C}$ which produces $f_0(x,y)$. Let X_0^Y and X_0^C be the Zariski closure of the restriction of X_0 on Y and M_C, respectively. Note that each of X_0^Y and X_0^C is irreducible. Also, X_0^Y is at least one-dimensional since its restriction on ∂M_K is one-dimensional. If X_0^Y does not contain irreducible characters, then the restriction of X_0^Y on ∂M_C is also one-dimensional. So the restriction of X_0^C on ∂M_C is one-dimensional and X_0 is an abelian extension of X_0^C (since the character of a reducible representation is also the character of an abelian representation). It follows that X_0^C cannot be the trivial component in $X(M_C)$ for otherwise X_0 would be the trivial component of $X(M_K)$. Hence $X_0^C \in X^*(M_C)$ and X_0 is its abelian extension, which means that $f_0(x,y)$ is a factor of $\mathrm{Ext}^q[A_C(\bar{x},\bar{y})]$.

Hence we may assume that X_0^Y contains irreducible characters. Then, as noted before, we have $\rho(\gamma_K) = \epsilon I$ for each ρ with $\chi_\rho \in X_0^Y$. If $\epsilon = -1$, then $1 + x^{pq} y$ or $x^{pq} + y$ is the factor contributed by X_0 to $A_K(x,y)$ corresponding to whether p is positive or negative, respectively. That is, $f_0(x,y)$ is a factor of $F_{(p,q)}(x,y)$. Hence we may assume that $\epsilon = 1$. It follows that X_0^Y is a positive-dimensional component of $N_K(pq)$. Therefore $q > 2$ (because $N_K(pq) = L(q,p) \,\#\, (D^2 \times S^1)$) and X_0 contributes the factor $-1 + x^{pq} y$ or $-x^{pq} + y$ to $A_K(x,y)$ corresponding to whether p is positive or negative respectively. That is, f_0 is a factor of $F_{(p,q)}(x,y)$. This proves Claim 2.10 and also completes the proof of the theorem.

Perhaps we should note that in the above proof a consistent choice of basepoints can be made for all the relevant manifolds, such as M_K, M_C, Y, A_2, U_1, U_2, ∂M_C, ∂M_K, $N_K(pq)$, $M_C(p/q)$, $M_K(pq)$, so that their fundamental groups are all well defined as relevant subgroups or quotient groups. In fact we can choose a simply connected region in $D \cap Y$ (as shown in Figure 1) so that its intersection with each relevant manifold listed above is a simply connected region which is served as the "fat basepoint" of that manifold. □

Example 2.11 Let C be the figure 8 knot. Its A–polynomial is [8, Appendix]

$$A_C(x,y) = x^4 + (-1 + x^2 + 2x^4 + x^6 - x^8)y + x^4 y^2.$$

If K is the $(p,2)$–cable over C, where $p > 0$, then

$$\begin{aligned} A_K(x,y) &= \mathrm{Red}\big[F_{(p,2)}(x,y)\,\mathrm{Res}_{\bar{y}}\big(A_C(x^2,\bar{y}), \bar{y}^2 - y\big)\big] \\ &= (1 + x^{2p} y)\big[x^{16} + \big(-1 + 2x^4 + 3x^8 - 2x^{12} - 6x^{16} \\ &\qquad - 2x^{20} + 3x^{24} + 2x^{28} - x^{32}\big)y + x^{16} y^2\big]. \end{aligned}$$

If K is the $(p,3)$–cable over C, where $p>0$, then

$$\begin{aligned}A_K(x,y) &= \mathrm{Red}\big[F_{(p,3)}(x,y)\,\mathrm{Res}_{\bar y}\big(A_C(x^3,\bar y),\bar y^3-y\big)\big]\\ &= (-1+x^{6p}y^2)\big[x^{36}+\big(-1+3x^6+3x^{12}-8x^{18}-12x^{24}+6x^{30}+20x^{36}\\ &\qquad +6x^{42}-12x^{48}-8x^{54}+3x^{60}\\ &\qquad +3x^{66}-x^{72}\big)y+x^{36}y^2\big].\end{aligned}$$

Finally we would like to give an explicit formula for the A–polynomials of iterated torus knots. Let

$$K=[(p_1,q_2),(p_2,q_2),\dots,(p_n,q_n)]$$

be an n^{th} iterated torus knot, ie $T(p_n,q_n)$ is a nontrivial torus knot and when $n>1$, for each i satisfying $n>i\geq 1$, the knot $[(p_i,q_i),(p_{i+1},q_{i+1}),\dots,(p_n,q_n)]$ is a satellite knot with $[(p_{i+1},q_{i+1}),\dots,(p_n,q_n)]$ as a companion knot and with $T(p_i,q_i)$, $q_i>1$, as a pattern knot lying in the trivial solid torus V as a (p_i,q_i)–cable with winding number q_i, where $|p_i|$ may be less than q_i and $|p_i|=1$ is also allowed.

Corollary 2.12 *Let $K=[(p_1,q_1),(p_2,q_2),\dots,(p_n,q_n)]$ be an iterated torus knot. If q_i is odd for each $1\leq i<n$, then*

$$\begin{aligned}&A_K(x,y)\\ &\quad = F_{(p_1,q_1)}(x,y)F_{(p_2,q_2)}(x^{q_1^2},y)F_{(p_3,q_3)}(x^{q_1^2q_2^2},y)\cdots F_{(p_n,q_n)}(x^{q_1^2q_2^2\cdots q_{n-1}^2},y),\end{aligned}$$

and if q_i is even for some $1\leq i<n$ and we let m be the smallest such integer, then

$$\begin{aligned}&A_K(x,y)\\ &\quad = F_{(p_1,q_1)}(x,y)F_{(p_2,q_2)}(x^{q_1^2},y)F_{(p_3,q_3)}(x^{q_1^2q_2^2},y)\cdots F_{(p_m,q_m)}(x^{q_1^2q_2^2\cdots q_{m-1}^2},y)\\ &\qquad \cdot G_{(p_{m+1},q_{m+1})}(x^{q_1^2q_2^2\cdots q_m^2},y)G_{(p_{m+2},q_{m+2})}(x^{q_1^2q_2^2\cdots q_{m+1}^2},y)\\ &\qquad \cdots G_{(p_n,q_n)}(x^{q_1^2q_2^2\cdots q_{n-1}^2},y)\end{aligned}$$

Remark (1) In the A–polynomial given in the corollary, each

$$F_{(p_i,q_i)}(x^{q_1^2q_2^2\cdots q_{i-1}^2},y)\quad\text{or}\quad G_{(p_i,q_i)}(x^{q_1^2q_2^2\cdots q_{i-1}^2},y)$$

is a nontrivial polynomial even when $|p_i|=1$.

(2) The polynomial expression for $A_K(x,y)$ given in the corollary has no repeated factors.

(3) The boundary slopes detected by $A_K(x,y)$ are precisely the following n integer slopes:

$$p_1q_1,\quad p_2q_2q_1^2,\quad p_3q_3q_1^2q_2^2,\quad \dots,\quad p_nq_nq_1^2q_2^2\cdots q_{n-1}^2.$$

Proof The proof goes by induction on n, applying Theorem 2.8. When $n = 1$, the proposition holds obviously. Suppose for $n-1 \geq 1$ the proposition holds. Note that K has $C = [(p_2, q_2), \dots, (p_n, q_n)]$ as a companion knot and $P = T(p_1, q_1)$ as the corresponding pattern knot (which maybe a trivial knot in S^3, which occurs exactly when $|p_1| = 1$). By induction, the A–polynomial $A_C(x, y)$ of C is of the corresponding form as described by the corollary. Now applying Theorem 2.8 one more time to the pair (C, P) we see that the corollary holds. We omit the routine details. □

For instance the A–polynomial of the (r, s)–cable over the (p, q)–torus knot is

$$A(x,y) = \begin{cases} F_{(r,s)}(x,y)F_{(p,q)}(x^{s^2}, y) & \text{if } s \text{ is odd,} \\ F_{(r,s)}(x,y)G_{(p,q)}(x^{s^2}, y) & \text{if } s \text{ is even.} \end{cases}$$

3 Proof of Theorem 1.1

Suppose that K is a knot in S^3 with the same knot Floer homology and the same A–polynomial as a given torus knot $T(p, q)$. Our goal is to show that $K = T(p, q)$.

By (2-3) and Corollary 2.4, K is not a hyperbolic knot. So K is either a torus knot or a satellite knot.

Lemma 3.1 *Suppose that $T(r, s)$ is a torus knot whose A–polynomial divides that of the torus knot $T(p, q)$ and whose Alexander polynomial divides that of $T(p, q)$. Then $T(r, s) = T(p, q)$.*

Proof Since $A_{T(r,s)}(x, y) \mid A_{T(p,q)}(x, y)$, from (2-3), we have $rs = pq$. From the condition $\Delta_{T(r,s)}(t) \mid \Delta_{T(p,q)}(t)$, ie

$$\frac{(t^{rs}-1)(t-1)}{(t^r-1)(t^s-1)} \;\Big|\; \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)},$$

we have

$$(t^p-1)(t^q-1) \mid (t^r-1)(t^s-1). \tag{3-1}$$

Now if $T(r, s) \neq T(p, q)$, then either $q > s$ or $|p| > |r|$. If $q > s$, then from (3-1) we must have $q \mid r$. By our convention for parametrizing torus knots, $|p| > q \geq 2$. Thus we must also have $p \mid r$. But since p and q are relatively prime, we have $pq \mid r$, which contradicts the earlier conclusion that $rs = pq$ since $s > 1$. If $|p| > |r|$, again by our convention $|r| > s \geq 2$, and we see that $(t^p - 1)$ does not divide $(t^r - 1)(t^s - 1)$ and so (3-1) cannot hold. This contradiction completes the proof of the lemma. □

By Lemma 3.1, we see that if K is a torus knot, then $K = T(p,q)$.

We are going to show that it is impossible for K to be a satellite knot, which will complete the proof of Theorem 1.1. Suppose that K is a satellite knot. We need to derive a contradiction from this assumption. Let (C,P) be a pair of associated companion knot and pattern knot to K, and let w be the winding number of P in its defining solid torus V (recall the definition of a satellite knot given in Section 2). As $T(p,q)$ is a fibered knot, K is also fibered (because by assumption they have the same knot Floer homology). According to [7, Corollary 4.15 and Proposition 8.23], each of C and P is a fibered knot in S^3, the winding number w is at least 1, and the Alexander polynomials of these knots satisfy the equality

$$\Delta_K(t) = \Delta_C(t^w)\Delta_P(t). \tag{3-2}$$

We may choose C such that C is itself not a satellite knot, and thus is either a hyperbolic knot or a torus knot.

Lemma 3.2 *The companion knot C cannot be a hyperbolic knot.*

Proof Suppose that C is hyperbolic. Then $A_C(\bar{x},\bar{y})$ contains a balanced-irreducible factor $f_C(\bar{x},\bar{y})$ whose Newton polygon detects at least two distinct boundary slopes of C. As $w \geq 1$, by Proposition 2.7, $f_C(\bar{x},\bar{y})$ extends to a balanced factor $f_K(x,y)$ of $A_K(x,y) = A_{T(p,q)}(x,y)$. Moreover from (2-4) we see that the Newton polygon of $f_K(x,y)$ detects at least two distinct boundary slopes of K. But clearly the Newton polygon of $A_K(x,y) = A_{T(p,q)}(x,y)$ only detects one boundary slope. We arrive at a contradiction. □

So $C = T(r,s)$ is a torus knot.

Lemma 3.3 *The pattern knot P of K cannot be the unknot.*

Proof Suppose otherwise that P is the unknot. Then as noted in [19] the winding number w of P in its defining solid torus V is larger than 1. Equation (3-2) becomes $\Delta_K(t) = \Delta_C(t^w)$ for some integer $w > 1$. On the other hand it is easy to check that the degrees of the leading term and the second term of $\Delta_{T(p,q)}(t)$ differ by 1 and thus $\Delta_K(t) = \Delta_{T(p,q)}(t)$ cannot be of the form $\Delta_C(t^w)$ with $w > 1$. This contradiction completes the proof. □

If $w = 1$, then by Proposition 2.7, the A–polynomial $A_C(x,y) = A_{T(r,s)}(x,y)$ divides the A–polynomial $A_K(x,y) = A_{T(p,q)}(x,y)$, and by (3-2), $\Delta_C(t) = \Delta_{T(r,s)}(t)$ divides $\Delta_K(t) = \Delta_{T(p,q)}(t)$. Hence by Lemma 3.1 we have $C = T(r,s) = T(p,q)$.

By Lemma 3.3, P is a nontrivial knot. Hence from (3-2) we see that the genus of $C = T(p,q)$ is less than that of K. But the genus of K is equal to that of $T(p,q)$ (because by assumption they have the same knot Floer homology). We derive a contradiction.

Hence $w > 1$. By Lemma 2.6, $A_P(x,y)$ divides $A_K(x,y)$. Now if P is itself a satellite knot with its own companion knot C_1 and pattern knot P_1, then again each of C_1 and P_1 is a fibered knot and the winding number w_1 of P_1 with respect to C_1 is larger than zero. Arguing as above, we see that C_1 may be assumed to be a torus knot and that $w_1 > 1$. Also we have

$$\Delta_K(t) = \Delta_C(t^w)\Delta_{C_1}(t^{w_1})\Delta_{P_1}(t),$$

from which we see that P_1 cannot be the trivial knot just as in the proof of Lemma 3.3.

So after finitely many such steps (the process must terminate by [27]), we end up with a pattern knot P_m for P_{m-1} such that P_m is nontrivial but is no longer a satellite knot. Thus P_m is either a hyperbolic knot or a torus knot. By Corollary 2.4 and Lemma 2.6, P_m cannot be hyperbolic. Hence P_m is a torus knot. Again because $A_{P_m}(x,y)$ divides $A_K(x,y) = A_{T(p,q)}(x,y)$ and $\Delta_{P_m}(t)$ divides $\Delta_K(t) = \Delta_{T(p,q)}(t)$, we have $P_m = T(p,q)$ by Lemma 3.1. But once again we would have $g(T(p,q)) < g(T(p,q))$. This gives a final contradiction.

4 The knots $k(l,m,n,p)$

In [12] a family of hyperbolic knots $k(l,m,n,p)$ in S^3 (where at least one of p and n has to be zero) was constructed such that each knot in the family admits one (and only one) half-integral toroidal surgery. To be hyperbolic, the following restrictions on the values for l, m, n and p are imposed:

(4-1)
$$\begin{aligned}&\text{If } p=0, \text{ then } l\neq 0,\pm 1,\ m\neq 0,\ (l,m)\neq(2,1),(-2,-1),\\ &\qquad\qquad \text{and } (m,n)\neq(1,0),(-1,1);\\ &\text{if } n=0, \text{ then } l\neq 0,\pm 1,\ m\neq 0,1, \text{ and } (l,m,p)\neq(-2,-1,0),(2,2,1).\end{aligned}$$

From now on we assume that any given $k(l,m,n,p)$ is hyperbolic, ie l, m, n and p satisfy the above restrictions.

The half-integral toroidal slope $r = r(l,m,n,p)$ of $k(l,m,n,p)$ was explicitly computed in [13, Proposition 5.3] as

(4-2)
$$r = \begin{cases} l(2m-1)(1-lm) + n(2lm-1)^2 - \frac{1}{2} & \text{when } p=0,\\ l(2m-1)(1-lm) + p(2lm-l-1)^2 - \frac{1}{2} & \text{when } n=0.\end{cases}$$

It turns out that the knots $k(l,m,n,p)$ are the only hyperbolic knots in S^3 which admit nonintegral toroidal surgeries.

Theorem 4.1 [17] *If a hyperbolic knot K in S^3 admits a nonintegral toroidal surgery, then K is one of the knots $k(l,m,n,p)$.*

Proposition 4.2 *The knots $k(l,m,n,p)$ have the following properties:*

(a) $k(l,m,n,0)$ *is the mirror image of* $k(-l,-m,1-n,0)$.

(b) $k(l,m,0,p)$ *is the mirror image of* $k(-l,1-m,0,1-p)$.

(c) $k(l,\pm1,n,0)=k(-l\pm1,\pm1,n,0)$.

(d) $k(2,-1,n,0)=k(-3,-1,n,0)=k(2,2,0,n)$.

Proof This follows from [12, Proposition 1.4]. □

Explicit closed braid presentations for the knots $k(l,m,n,p)$ are given in [13]. Figure 2 shows the braids whose closures are the knots $k(l,m,n,p)$. The left two pictures are a reproduction of [13, Figure 12], but the right two pictures are different from [13, Figure 13]. Here an arc with label s means s parallel strands, and a box with label t means t positive full-twists when $t>0$ and $|t|$ negative full-twists when $t<0$. We only give the picture for the case $l>0$, since the case $l<0$ can be treated by applying Proposition 4.2.

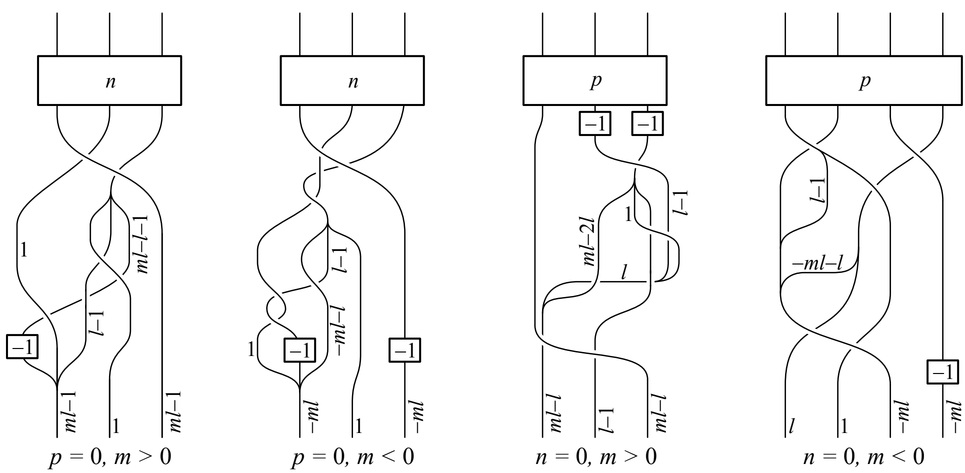

Figure 2: The braid whose closure is $k(l,m,n,p)$, where $l>0$

Proposition 4.3 *Suppose that $l > 0$. Let*

$$N = \begin{cases} 2ml - 1 & \text{if } p = 0, n \neq 0, m > 0, \\ -2ml + 1 & \text{if } p = 0, n \neq 0, m < 0, \\ 2ml - l - 1 & \text{if } n = 0, m > 0, \\ -2ml + l + 1 & \text{if } n = 0, m < 0. \end{cases}$$

Then the genus of $k(l,m,n,0)$ is

$$g = \tfrac{1}{2}|n|N(N-1) + \begin{cases} m^2l^2 - \frac{1}{2}ml(l+5) + l + 1 & \text{if } m > 0, n \leq 0, \\ -m^2l^2 + \frac{1}{2}ml(l+1) - l + 1 & \text{if } m > 0, n > 0, \\ m^2l^2 - \frac{1}{2}ml(l-1) & \text{if } m < 0, n \leq 0, \\ -m^2l^2 + \frac{1}{2}ml(l+3) & \text{if } m < 0, n > 0, \end{cases}$$

and the genus of $k(l,m,0,p)$ is

$$g = \tfrac{1}{2}|p|N(N-1) + \begin{cases} m^2l^2 - \frac{1}{2}ml(l+5) + l + 1 & \text{if } m > 0,\ p \leq 0, \\ -m^2l^2 + \frac{1}{2}ml(l+1) + 1 & \text{if } m > 0,\ p > 0, \\ m^2l^2 - \frac{1}{2}ml(l-1) & \text{if } m < 0,\ p \leq 0, \\ -m^2l^2 + \frac{1}{2}ml(l+3) - l & \text{if } m < 0,\ p > 0. \end{cases}$$

Proof In [13], it is noted that $k(l,m,n,p)$ is the closure of a positive or negative braid. Hence it is fibered, and the genus can be computed by the formula

$$g(k) = \tfrac{1}{2}(C - N + 1),$$

where C is the crossing number in the positive or negative braid, and N is the braid index.

When $p = 0$, $m > 0$ and $n \leq 0$, the braid is a negative braid, $N = 2ml - 1$, and

$$\begin{aligned} C = {} & -n(2ml-1)(2ml-2) + ml(ml-1) \\ & + ml - 2 + l(ml-l-1) + (ml-l-1)(ml-l-2), \end{aligned}$$

and so

$$g = -n(2ml-1)(ml-1) + m^2l^2 - \tfrac{1}{2}ml(l+5) + l + 1.$$

When $p = 0$, $m > 0$ and $n > 0$, we can cancel all the negative crossings in the braid to get a positive braid of index $N = 2ml - 1$. We have

$$\begin{aligned} C = {} & n(2ml-1)(2ml-2) \\ & - \big(ml(ml-1) + ml - 2 + l(ml-l-1) + (ml-l-1)(ml-l-2)\big), \end{aligned}$$

and so

$$g = n(2ml-1)(ml-1) - m^2l^2 + \tfrac{1}{2}ml(l+1) - l + 1.$$

The computations for other cases are similar. □

Using Proposition 4.3 and parts (a) and (b) of Proposition 4.2, we can compute the genus of $k(l,m,n,p)$ when $l<0$. For example, when $p\le 0$, the genus of $k(l,m,0,p)$ is

$$g=\begin{cases} -\frac{1}{2}p(2ml-l-1)(2ml-l-2) \\ \qquad\qquad +m^2l^2-\frac{1}{2}ml(l+5)+l+1 & \text{if } l>0,\, m>0, \\ -\frac{1}{2}p(-2ml+l+1)(-2ml+l)+m^2l^2-\frac{1}{2}ml(l-1) & \text{if } l>0,\, m<0, \\ -\frac{1}{2}p(-2ml+l+1)(-2ml+l)+m^2l^2-\frac{1}{2}ml(l-1) & \text{if } l<0,\, m>0, \\ -\frac{1}{2}p(2ml-l-1)(2ml-l-2) \\ \qquad\qquad +m^2l^2-\frac{1}{2}ml(l+5)+l+2 & \text{if } l<0,\, m<0. \end{cases} \tag{4-3}$$

The r–surgery on the knot $J=k(l,m,n,p)$ which was explicitly given in [12] is the double branched cover of S^3 with the branched set in S^3 being a link shown in Figure 3. From the tangle decomposition of the branched link one can see that $M_J(r)$ is a graph manifold obtained by gluing two Seifert fibered spaces, each over a disk with two cone points, together along their torus boundaries. For our purpose, we need to give a more detailed description of the graph manifold $M_J(r)$ as follows:

Let (B,t) denote a two string tangle, ie B is a 3–ball and t is a pair of disjoint properly embedded arcs in B. Here we may assume that B is the unit 3–ball in the xyz–space $\mathbb{R}^3$ (with the xy–plane horizontal) and that the four endpoints of t lie in the lines $z=y$, $x=0$ and $z=-y$, $x=0$. Let D be the unit disk in B which is the intersection of B with the yz–plane. Then the four endpoints of t divides ∂D into four arcs, naturally named the east, west, north and south arcs. The *denominator closure* of (B,t) is the link in S^3 obtained by capping off t with the east and west arcs, and the *numerator closure* of (B,t) is the link in S^3 obtained by capping off t with the north and south arcs.

Let (B_i,t_i), for $i=1,2$, be the two tangles shown in Figure 3 and let X_i be the double branched cover of (B_i,t_i). The denominator closure of (B_1,t_1) is the twisted knot of type $(2,p)$ (which is the trivial knot when $p=0$ or 1 and the trefoil knot when $p=-1$) and therefore the double branched cover of S^3 over the link is the lens space of order $|2p-1|$. The numerator closure of (B_1,t_1) gives a composite link in S^3 and in fact the composition of two nontrivial rational links corresponding to the rational numbers $-l$ and $\big(-lm(2p-1)+pl+2p-1\big)/\big(-m(2p-1)+p\big)$.

The double branched cover of the east arc (and also the west arc) is a simple closed essential curve in ∂X_1, which we denote by μ_1, with which Dehn filling of X_1 is a lens space of order $|2p-1|$. The double branched cover of the north arc (and also the south arc)

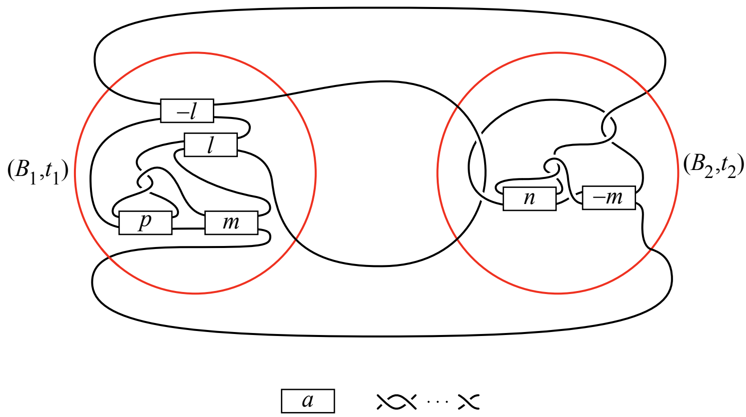

Figure 3: The tangle decomposition of the branched link in S^3 for the half-integral toroidal surgery

is a Seifert fiber of X_1, which we denote by σ_1, with which Dehn filling of X_1 is a connected sum of two nontrivial lens spaces of orders $|-l|$ and $\big|-lm(2p-1)+pl+2p-1\big|$.

Similarly the numerator closure of (B_2,t_2) is the twisted knot of type $(2,n)$ and the denominator closure of (B_2,t_2) is a composite link of two nontrivial rational links that correspond to the rational numbers -2 and $\big(2(2n-1)(m-1)+4n-1\big)/\big((2n-1)(m-1)+n\big)$. So the double branched cover of the north arc of (B_2,t_2) is a simple closed essential curve in ∂X_2, denoted μ_2, with which Dehn filling of X_2 is a lens space of order $|2n-1|$, and the double branched cover of the west arc is a Seifert fiber of X_2, denoted by σ_2, with which Dehn filling is a connected sum of two nontrivial lens spaces of orders 2 and $\big|2(2n-1)(m-1)+4n-1\big|$.

Finally $M_J(r)$ is obtained by gluing X_1 and X_2 along their boundary tori such that μ_1 is identified with σ_2 and σ_1 with μ_2.

Note that X_1 is the exterior of a torus knot in S^3 when $p=0$ or 1, and X_2 is the exterior of a $(2,a)$–torus knot in S^3 when $n=0$ or 1.

Lemma 4.4 *Let $J=k(l,m,n,p)$, let r be the unique half-integral toroidal slope of J, and let M_J be the exterior of J. Up to isotopy, there is a unique closed orientable incompressible surface in $M_J(r)$, which is a torus.*

Proof As discussed above, $M_J(r)$ is a graph manifold with the torus decomposition $M_J(r)=X_1\cup X_2$. We just need to show that any connected closed orientable incompressible surface S in $M_J(r)$ is isotopic to ∂X_1. Suppose otherwise that S is not isotopic to ∂X_1. As each X_i does not contain closed essential surfaces, S must intersect ∂X_1. We may assume that $F_i = S \cap X_i$ is incompressible and boundary incompressible, for $i = 1, 2$. As X_i is Seifert fibered, F_i is either horizontal (ie consisting of Seifert fibers) or vertical (transverse to Seifert fibers) in X_i, up to isotopy. So the boundary slope of F_i is either σ_i (when F_i is horizontal) or is the rational longitude of X_i (when F_i is vertical). But σ_i is identified with μ_j, for $i = 1, 2$, where $\{i, j\} = \{1, 2\}$. So the boundary slope of F_i must be the rational longitude of X_i for each i. It follows that $H_1(M_J(r); \mathbb{Z})$ is infinite, yielding a contradiction. □

Lemma 4.5 *For each $J = k(l, m, n, p)$, its meridian slope is not a boundary slope.*

Proof If the meridian slope of J is a boundary slope, then the knot exterior M_J contains a connected closed orientable incompressible surface S of genus larger than one such that S remains incompressible in any nonintegral surgery, by [9, Theorem 2.0.3]. But by Lemma 4.4, $M_J(r)$ does not contain any closed orientable incompressible surface of genus larger than one. This contradiction completes the proof. □

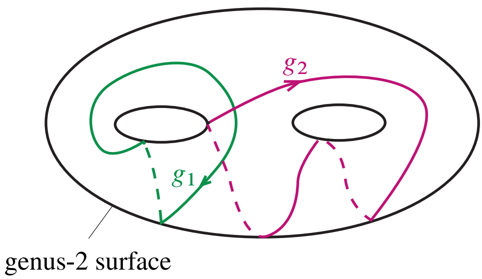

Figure 4: The elements g_1 and g_2 in ∂H

Last in this section we are going to show that the knots $k(l_*, -1, 0, 0)$ form a class of small knots.

Let H be a standard genus two handlebody in S^3 and let g_1 and g_2 be oriented loops in the surface ∂H as shown in Figure 4. As observed in [2, Figure 8], a regular neighborhood of $g_1 \cup g_2$ in ∂H is a genus one Seifert surface of a trefoil knot. Under the basis $\{[g_1], [g_2]\}$, the monodromy of the trefoil is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

From [13, Figure 11], the knot $k(l_*, -1, 0, 0)$ has a knot diagram shown in Figure 5 (left), which in turn can be embedded, as shown in Figure 5 (right), in the Seifert surface with the homology class $l_*[g_1] + (l_* + 1)[g_2]$. For any knot J lying in the Seifert surface, an algorithm to classify all closed essential surfaces in the knot exterior of J is given in [1, Theorem 10.1(3)]. For simplicity, we only state the part that we need:

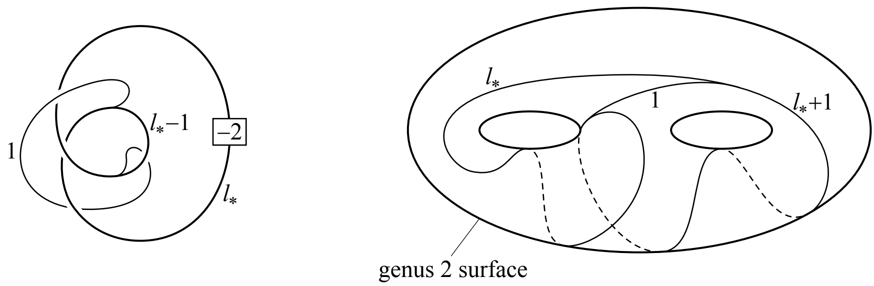

Figure 5: The knot $k(l_*,-1,0,0)$ (left) and corresponding embedding (right)

Theorem 4.6 *Suppose that L is a simple closed curve on the genus one Seifert surface of a trefoil knot in the homology class $a_1[g_1]+a_2[g_2]$. The slope a_1/a_2 has continued fraction expansion*

$$\frac{a_1}{a_2}=[b_1,\dots,b_k]=b_1-\cfrac{1}{b_2-\cfrac{1}{\cdots-\cfrac{1}{b_k}}},$$

where the coefficients alternate signs, $b_i\neq 0$ when $i\geq 2$, and $|b_k|\geq 2$. If $b_1=0$ and $b_2=-1$, then every closed essential surface in the complement of L corresponds to a solution of the following equation:

$$0=\sum_{i\in I}-b_i+\sum_{j\in J}b_j+\begin{cases}0 & \text{if } 3\in J,\\ -1 & \text{otherwise,}\end{cases} \tag{4-4}$$

where I and J are subsets of $\{3,\dots,k\}$ each not containing consecutive integers and $3\notin I\cap J$.

Lemma 4.7 *Each knot $k(l_*,-1,0,0)$ is small. Namely, the exterior of $k(l_*,-1,0,0)$ contains no closed essential surfaces.*

Proof In this case $a_1=l_*$, $a_2=l_*+1$ and $a_1/a_2=[0,-1,l_*]$. Clearly, (4-4) has no solution. So $k(l_*,-1,0,0)$ is small. □

5 Information on A–polynomials of the knots $k(l,m,n,p)$

To obtain explicit expressions for the A–polynomials of the knots $k(l,m,n,p)$ could be a very tough task. But we can obtain some useful information about the A–polynomials of these knots without computing them explicitly.

Lemma 5.1 *Suppose a graph manifold W is obtained by gluing together two torus knot exteriors X_1, X_2, such that the meridian of X_i is glued to the Seifert fiber of X_{i+1} for $i = 1, 2$, where $X_3 = X_1$. Then $\pi_1(W)$ has no noncyclic $\mathrm{SL}_2(\mathbb{C})$ representations.*

To prove this lemma, we will use the following well-known fact whose proof is elementary.

Lemma 5.2 *Suppose that $A, B \in \mathrm{SL}_2(\mathbb{C})$ are two commuting matrices with $A \neq \pm I$.*

(i) *If there exists $P \in \mathrm{SL}_2(\mathbb{C})$ such that*

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{for some } \lambda \in \mathbb{C} \setminus \{0, \pm 1\}, \tag{5-1}$$

then

$$PBP^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \quad \text{for some } \mu \in \mathbb{C} \setminus \{0\}.$$

(ii) *If there exists $P \in \mathrm{SL}_2(\mathbb{C})$ such that*

$$PAP^{-1} = \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{for some } a \in \mathbb{C} \setminus \{0\}, \tag{5-2}$$

then

$$PBP^{-1} = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{for some } b \in \mathbb{C}.$$

Proof of Lemma 5.1 As W has cyclic homology group, it is equivalent to show that every $\mathrm{SL}_2(\mathbb{C})$ representation of $\pi_1(W)$ is abelian.

We choose a basepoint of W on the common boundary torus T of X_1 and X_2. Then $\pi_1(T)$, $\pi_1(X_1)$ and $\pi_1(X_2)$ are naturally subgroups of $\pi_1(W)$. Suppose that $\rho\colon \pi_1(W) \to \mathrm{SL}_2(\mathbb{C})$ is a representation.

Let $f_i \in \pi_1(T)$ represent the Seifert fiber of X_i for $i = 1, 2$. We first consider the case that one of $\rho(f_1)$ and $\rho(f_2)$, say $\rho(f_1)$, is in $\{\pm I\}$, which is the center of $\mathrm{SL}_2(\mathbb{C})$. Since f_1 represents the meridian of X_2, the Seifert fiber f_1 normally generates $\pi_1(X_2)$, hence $\rho(\pi_1(X_2))$ is contained in $\{\pm I\}$. Since $f_2 \in \pi_1(X_2)$, we see that $\rho(f_2)$ is also in $\{\pm I\}$, and thus for the same reason as just given $\rho(\pi_1(X_1))$ is in $\{\pm I\}$. Hence $\rho(\pi(W))$ is contained in $\{\pm I\}$.

Now suppose that neither $\rho(f_1)$ nor $\rho(f_2)$ is in $\{\pm I\}$. Then there exists $P \in \mathrm{SL}_2(\mathbb{C})$ such that $P\rho(f_1)P^{-1}$ is in the form of either (5-1) or (5-2). Since f_1 is in the center of $\pi_1(X_1)$, by Lemma 5.2, $P\rho(\pi_1(X_1))P^{-1}$ (including $P\rho(f_2)P^{-1}$) is contained in an abelian subgroup A of $\mathrm{SL}_2(\mathbb{C})$ (A is either the set of diagonal matrices or the set of upper triangular trace ± 2 matrices). Since f_2 is in the center of $\pi_1(X_2)$, again by Lemma 5.2, $P\rho(\pi_1(X_2))P^{-1}$ is also contained in A. Hence $\rho(\pi_1(W))$, being generated by $\rho(\pi_1(X_1))$ and $\rho(\pi_1(X_2))$, is an abelian group. □

Corollary 5.3 *The fundamental group of the half-integral toroidal surgery on the knot $J = k(l, m, 0, 0)$ has no noncyclic $\mathrm{SL}_2(\mathbb{C})$ representations and has no noncyclic $\mathrm{PSL}_2(\mathbb{C})$ representations.*

Proof By the discussion preceding Lemma 4.4, the half-integral toroidal surgery, $M_J(r)$, is a graph manifold satisfying the conditions of Lemma 5.1. Hence $M_J(r)$ has no noncyclic $\mathrm{SL}_2(\mathbb{C})$ representations. The manifold cannot have noncyclic $\mathrm{PSL}_2(\mathbb{C})$ representations either since every $\mathrm{PSL}_2(\mathbb{C})$ representation of $M_J(r)$ lifts to a $\mathrm{SL}_2(\mathbb{C})$ representation because the manifold has odd cyclic first homology. □

Lemma 5.4 *Let M_J be the knot exterior of a given hyperbolic knot $J = k(l, m, 0, 0)$. Let X_0 be a norm curve in $X(M_J)$ and B_0 the norm polygon determined by X_0. Then the half-integral toroidal slope $r = d/2$ of J is associated to a vertex of B_0 as described in Theorem 2.2(3), ie $2/d$ is the slope of a vertex of B_0 in the xy–plane $H_1(\partial M_J; \mathbb{R})$.*

Proof Suppose otherwise that r is not associated to a vertex of B_0. As the meridian slope of M_J is not a boundary slope by Lemma 4.5, it follows from parts (4) and (5) of Theorem 2.2 that μ is contained in ∂B_0 but r is not, which means that we have $Z_v(\tilde{f}_r) > Z_v(\tilde{f}_\mu)$ for some point $v \in \tilde{X}_0$. As $M_J(r)$ has no noncyclic representations by Corollary 5.3, the point v cannot be a regular point of $\tilde{X}_0$ (by [9, Proposition 1.5.2] or [4, Proposition 4.8]). So v is an ideal point of $\tilde{X}_0$. As r is not a slope associated to any vertex of B_0, we have that $\tilde{f}_\alpha(v)$ is finite for every class α in $H_1(\partial M_J; \mathbb{Z})$. Now we may apply [4, Proposition 4.12] to see that M_J contains a closed essential surface S such that if S compresses in $M_J(r)$ and $M_J(\alpha)$, then $\Delta(r, \alpha) \le 1$. By Lemma 4.4, S must compress in $M_J(r)$ and of course S compresses in $M(\mu)$. But $\Delta(r, \mu) = 2$. We arrive at a contradiction. □

Lemma 5.5 *Suppose that $A_J(x, y)$ is the A–polynomial of a given hyperbolic knot $J = k(l, m, 0, 0)$. Let $r = d/2$ be the half-integral toroidal slope of J. If (x_0, y_0) is a solution of the system*

$$\begin{cases} A_J(x, y) = 0 \\ x^d y^2 - 1 = 0 \end{cases}$$

then $x_0 \in \{0, 1, -1\}$.

Proof Suppose otherwise that $x_0 \notin \{0, 1, -1\}$. Then by the constructional definition of the A–polynomial, there is a component X_1 in $X^*(M_J)$ which contributes a factor $f_0(x, y)$ in $A_J(x, y)$ such that (x_0, y_0) is a solution of

$$\begin{cases} f_0(x, y) = 0 \\ x^d y^2 - 1 = 0. \end{cases}$$

Let Y_0 be the Zariski closure of $\hat{i}_*(X_1)$ in $X(\partial M_J)$. Then Y_0 is an irreducible curve. We may find an irreducible curve X_0 in X_1 such that Y_0 is also the Zariski closure of $\hat{i}_*(X_0)$. Now it also follows from the constructional definition of the A–polynomial that there is a convergent sequence of regular points $\{v_i\} \subset \widetilde{X}_0$ such that $\widetilde{f}_r(v_i) \to 0$ and $\widetilde{f}_\mu(v_i) \to (x_0 + x_0^{-1})^2 - 4$, ie if v is the limit point of v_i in $\widetilde{X}_0$, then

$$\widetilde{f}_r(v) = 0 \quad \text{and} \quad \widetilde{f}_\mu(v) = (x_0 + x_0^{-1})^2 - 4 \neq 0.$$

Note that $\widetilde{f}_r$ is not constant on X_0. For otherwise X_0 would be a seminorm curve with r as the associated slope, which leads to a contradiction with Theorem 2.5(3) when replacing α_0 and α there by r and μ here respectively (note that by Lemma 4.5 μ is not a boundary slope).

So we have $Z_v(\widetilde{f}_r) > Z_v(\widetilde{f}_\mu) = 0$. Again v cannot be a regular point of $\widetilde{X}_0$ by the same reason given in the proof of Lemma 5.4. So v is an ideal point of $\widetilde{X}_0$ such that $\widetilde{f}_r(v) = 0$ and $\widetilde{f}_\mu(v) = (x_0 + x_0^{-1})^2 - 4$ is finite since $x_0 \neq 0$. Hence $\widetilde{f}_\alpha(v)$ is finite for every class α in $H_1(\partial M_J; \mathbb{Z})$. We can now get a contradiction with [4, Proposition 4.12] as in the proof of Lemma 5.4. □

Lemma 5.6 *Let K be a hyperbolic knot in S^3. For a given slope p/q, if every solution (x_0, y_0) of the system of equations*

$$\begin{cases} A_K(x, y) = 0 \\ x^p y^q - 1 = 0 \end{cases}$$

has $x_0 \in \{1, -1, 0\}$, then $M_K(p/q)$ is not a hyperbolic manifold.

Proof Some of the ideas for the proof come from [3]. Suppose otherwise that $M_K(p/q)$ is a hyperbolic 3–manifold. Then $\pi_1(M_K(p/q))$ has a discrete faithful representation $\bar{\rho}_0$ into $\mathrm{PSL}_2(\mathbb{C})$. By Thurston [10, Proposition 3.1.1], this representation can be lifted to an $\mathrm{SL}_2(\mathbb{C})$ representation ρ_0. It follows from Mostow rigidity that the character χ_{ρ_0} of ρ_0 is an isolated point in $X(M_K(p/q))$. Note that ρ_0 can be considered as an element in $R(M_K)$ and χ_{ρ_0} can be considered as an element in $X(M_K)$ since $R(M_K(p/q))$ embeds in $R(M_K)$ and $X(M_K(p/q))$ embeds in $X(M_K)$. Of course we have $\rho_0(\mu^p\lambda^q) = I$ but $\rho_0(\mu) \neq \pm I$. Let X_0 be a component of $X(M_K)$ which contains χ_{ρ_0}. By Thurston [10, Proposition 3.2.1], X_0 is positive-dimensional.

Claim 5.7 *The function $f_{\mu^p\lambda^q}$ is nonconstant on X_0.*

Suppose otherwise. Then $f_{\mu^p\lambda^q}$ is constantly zero on X_0 since $f_{\mu^p\lambda^q}(\chi_{\rho_0}) = 0$. So for every $\chi_\rho \in X_0$, the image $\rho(\mu^p\lambda^q)$ is either I or $-I$ or is a parabolic element. Let X_1 be an irreducible curve in X_0 which contains the point χ_{ρ_0}. For a generic point $\chi_\rho \in X_1$, the image $\rho(\mu^p\lambda^q)$ cannot be a parabolic element since otherwise $\rho(\mu)$ is either I or $-I$ or is parabolic for all $\chi_\rho \in X_1$, and this happens in particular at the point χ_{ρ_0}, yielding a contradiction (since ρ_0 is a discrete faithful representation of a closed hyperbolic 3–manifold, its image does not contain any parabolic elements). For a generic point $\chi_\rho \in X_1$, the image $\rho(\mu^p\lambda^q)$ cannot be $-I$ either for otherwise, by continuity, $\rho(\mu^p\lambda^q) = -I$ for every point $\chi_\rho \in X_1$, and this happens in particular at the point χ_{ρ_0}, yielding another contradiction. So for a generic point $\chi_\rho \in X_1$, $\rho(\mu^p\lambda^q) = I$ and again by continuity, $\rho(\mu^p\lambda^q) = I$ for every point $\chi_\rho \in X_1$. So X_1 factors though the p/q–surgery on K and becomes a subvariety of $X(M_K(p/q))$. But this contradicts the fact that χ_{ρ_0} is an isolated point of $X(M_K(p/q))$. The claim is thus proved.

It also follows from the proof of Claim 5.7 that X_0 is one-dimensional. For otherwise there would be a curve X_1 in X_0 such that $\chi_{\rho_0} \in X_1$ and $f_{\mu^p\lambda^q}$ is constantly zero on X_1, which is impossible by the proof of Claim 5.7.

It follows from Claim 5.7 that the restriction of X_0 in $X(\partial M_K)$ is one-dimensional and thus $X_0 \in X^*(M_K)$ and contributes a factor $f_0(x,y)$ to $A_K(x,y)$.

We may assume, up to conjugation of ρ_0, that

$$\rho_0(\mu) = \begin{pmatrix} x_0 & a \\ 0 & x_0^{-1} \end{pmatrix} \quad \text{and} \quad \rho_0(\lambda) = \begin{pmatrix} y_0 & b \\ 0 & y_0^{-1} \end{pmatrix}.$$

Note that $x_0 \neq \pm 1$ (as $\rho_0(\mu) \neq \pm I$ and cannot be a parabolic element of $\mathrm{SL}_2(\mathbb{C})$). By the construction of $A_K(x,y)$, the pair (x_0, y_0) is a solution of the system

$$\begin{cases} f_0(x,y) = 0 \\ x^p y^q - 1 = 0 \end{cases}$$

and thus is a solution of

$$\begin{cases} A_K(x,y) = 0 \\ x^p y^q - 1 = 0. \end{cases}$$

We get a contradiction with the assumption of the lemma. □

Proposition 5.8 *If $K \subset S^3$ is a hyperbolic knot whose A–polynomial divides the A–polynomial of $J = k(l, m, 0, 0)$, then K has the same half-integral toroidal slope as J and thus K is one of the knots $k(l, m, n, p)$.*

Proof Since K is hyperbolic, $X(M_K)$ contains a norm curve component X_0' which contributes a balanced-irreducible factor $f_0(x, y)$ to $A_K(x, y)$ such that the Newton polygon of $f_0(x, y)$ is dual to the norm polygon determined by X_0' by Theorem 2.3. By the assumption that $A_K(x, y)$ divides $A_J(x, y)$, the factor $f_0(x, y)$ is also a factor of $A_J(x, y)$. Thus there is a curve X_0 in a component of $X^*(M_J)$ which contributes $f_0(x, y)$ and X_0 must be a norm curve whose norm polygon B_0 is dual to the Newton polygon of $f_0(x, y)$. By Lemma 5.4, the half-integral toroidal slope $r = d/2$ of J is associated to a vertex of B_0 and thus $r = d/2$ is also associated to an edge of the Newton polygon of $f_0(x, y)$. Hence r is also a boundary slope of K.

Again by the assumption that $A_K(x, y)$ divides $A_J(x, y)$, together with Lemma 5.5, we see that if (x_0, y_0) is a solution of the system

$$\begin{cases} A_K(x, y) = 0 \\ x^d y^2 - 1 = 0 \end{cases}$$

then $x_0 \in \{0, 1, -1\}$. Now applying Lemma 5.6, we see that $M_K(r)$ is not a hyperbolic manifold. Applying [9, Theorem 2.0.3] and [16] we see that $M_K(r)$ must be a Haken manifold and thus must be a toroidal manifold (as it has finite first homology). Finally K is one of the knots $k(l, m, n, p)$ by Theorem 4.1. □

Lemma 5.9 *The half-integral toroidal r–surgery on $J = k(l, m, n, 0)$ with $n \neq 0, 1$ is a manifold with an irreducible $\mathrm{SL}_2(\mathbb{C})$ representation ρ_0 whose image contains no parabolic elements.*

Proof We know from Section 4 that $M_J(r) = X_1 \cup X_2$, where X_1 is an (a, b)–torus knot exterior and X_2 is Seifert fibered with base orbifold $D^2(2, c)$ for some odd integer $c > 1$, such that the meridian slope μ_1 of X_1 is identified with the Seifert fiber slope σ_2 of X_2 and the Seifert fiber slope σ_1 of X_1 is identified with a lens space filling slope μ_2 of X_2 (and $|2n - 1|$ is the order of the lens space).

Perhaps it is easier to construct a $\mathrm{PSL}_2(\mathbb{C})$ representation $\bar\rho_0$ of $\pi_1(M_J(r))$ with the required properties. As $M_J(r)$ has zero $\mathbb{Z}_2$–homology, every $\mathrm{PSL}_2(\mathbb{C})$ representation of $\pi_1(M_J(r))$ lifts to an $\mathrm{SL}_2(\mathbb{C})$ representation.

The representation $\bar\rho_0$ will send $\pi_1(X_2)$ to a cyclic group of order $|2n - 1|$, which is possible by factoring through $X_2(\mu_2)$. So $\bar\rho_0(\mu_2) = \mathrm{id}$. We claim $\bar\rho_0(\sigma_2)$ is not the identity element. For otherwise $\bar{\rho}_0$ factors through the group

$$\langle x, y : x^2 = y^c = 1, xy = 1\rangle,$$

which is the trivial group as c is odd.

Hence the order of $\bar{\rho}_0(\sigma_2)$ is an odd number $q > 1$ which is a factor of $2n-1$.

On the X_1 side, we need to have $\bar{\rho}_0(\sigma_1) = \mathrm{id}$ and $\bar{\rho}_0(\mu_1) = \bar{\rho}_0(\sigma_2)$ of order q. So $\bar{\rho}_0$ factors through the triangle group

$$\langle x, y : x^a = y^b = (xy)^q = 1\rangle.$$

Such a representation exists and can be required to be irreducible. Also, as at least one of a and b is odd and q is odd, we may require the image of $\bar{\rho}_0$ to contain no parabolic elements. In fact the triangle group is either a spherical or a hyperbolic triangle group and so we may simply choose $\bar{\rho}_0$ to be a discrete faithful representation of the triangle group into $\mathrm{SO}(3) \subset \mathrm{PSL}_2(\mathbb{C})$ (when the triangle group is spherical) or into $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$ (when the triangle group is hyperbolic), and thus the image of $\bar{\rho}_0$ has no parabolic elements. □

Lemma 5.10 *For any given $J = k(l, m, 0, p)$ with $p \neq 0, 1$ and l not divisible by $|2p-1|$, the half-integral toroidal surgery on J is a manifold with an irreducible $\mathrm{SL}_2(\mathbb{C})$ representation ρ_0 whose image contains no parabolic elements.*

Proof The proof is similar to that of Lemma 5.9.

We know that $M_J(r) = X_1 \cup X_2$, where X_2 is a $(2, a)$–torus knot exterior and X_1 is Seifert fibered with base orbifold $D^2\big(|l|, |-lm(2p-1) + pl + 2p - 1|\big)$, such that the meridian slope μ_2 of X_2 is identified with the Seifert fiber slope σ_1 of X_1 and the Seifert fiber slope σ_2 of X_2 is identified with a lens space filling slope μ_1 of X_1 (and $|2p-1|$ is the order of the lens space).

As in Lemma 5.9, we just need to construct a $\mathrm{PSL}_2(\mathbb{C})$ representation $\bar{\rho}_0$ of $\pi_1(M_J(r))$ which is irreducible and whose image contains no parabolic elements.

The representation $\bar{\rho}_0$ will send $\pi_1(X_1)$ to a cyclic group of order $|2p-1|$, which is possible by factoring through $X_1(\mu_1)$. So $\bar{\rho}_0(\mu_1) = \mathrm{id}$. We claim $\bar{\rho}_0(\sigma_1)$ is not the identity element. For otherwise $\bar{\rho}_0$ factors through the group

$$\langle x, y : x^l = y^{-lm(2p-1)+pl+2p-1} = 1, xy = 1\rangle,$$

which is a cyclic group of order less than $|2p-1|$ since l is not divisible by $2p-1$ by our assumption.

Hence the order of $\bar{\rho}_0(\sigma_2)$ is an odd number $q > 1$ which is a factor of $2p-1$.

On the X_2 side, we need to have $\bar{\rho}_0(\sigma_2) = \mathrm{id}$ and $\bar{\rho}_0(\mu_2) = \bar{\rho}_0(\sigma_1)$ of order q. So $\bar{\rho}_0$ factors through the triangle group

$$\langle x, y : x^2 = y^a = (xy)^q = 1\rangle.$$

Such a representation exists and can be required to be irreducible. Also, as both a and q are odd, we may require the image of $\bar{\rho}_0$ to contain no parabolic elements. □

Lemma 5.11 *The A–polynomial of any $J = k(l,m,n,0)$ with $n \neq 0, 1$ does not divide the A–polynomial of any $J' = k(l', m', 0, 0)$.*

Proof Suppose otherwise that $A_J(x, y) \mid A_{J'}(x, y)$. Then, by Proposition 5.8, the knots J and J' have the same half-integral toroidal slope $r = d/2$ with d odd.

Let ρ_0 be an irreducible representation of $M_J(r)$ provided by Lemma 5.9. Then $\rho_0(r) = I$ but $\rho_0(\mu) \neq \pm I$. We know that χ_{ρ_0} is contained in a positive-dimensional component X_1 of $X(M_J)$. Let X_0 be an irreducible curve in X_1 containing χ_{ρ_0}.

Claim *f_r is not constant on X_0.*

Otherwise f_r is constantly equal to 0 on X_0. If f_μ is not a constant on X_0, then X_0 would be a seminorm curve with r as its associated boundary slope, which is impossible by Theorem 2.5 as μ is not a boundary slope and $\Delta(\mu, r) = 2$. So f_μ is a constant not equal to 0 on X_0 since $\rho_0(\mu) \neq \pm I$ and is not parabolic. So for any point $\chi_\rho \in X_0$, $\rho(r)$ cannot be a parabolic element (for otherwise $\rho(\mu)$ is also parabolic and thus $f_\mu(\chi_\rho) = 0$). So $\rho(r) = I$ for any $\chi_\rho \in X_0$. We now get a contradiction with [4, Proposition 4.10], which proves the claim.

So f_r is not constant on X_0 which means that the component $X_1 \supset X_0$ belongs to $X^*(M_J)$ and thus contributes a factor in $A_J(x, y)$. Moreover the point χ_{ρ_0} contributes a root (x_0, y_0) to the system

$$\begin{cases} A_J(x, y) = 0 \\ x^d y^2 = 1 \end{cases}$$

such that $x_0 \neq \pm 1, 0$. As $A_J(x, y) \mid A_{J'}(x, y)$, the pair (x_0, y_0) is also a solution of the system

$$\begin{cases} A_{J'}(x, y) = 0 \\ x^d y^2 = 1, \end{cases}$$

which contradicts Lemma 5.5. □

With a similar proof replacing Lemma 5.9 by Lemma 5.10, we have:

Lemma 5.12 *The A–polynomial of any $k(l,m,0,p)$ with $p\neq 0,1$ and l not divisible by $2p-1$ does not divide the A–polynomial of any $k(l',m',0,0)$.*

Lemma 5.13 *Let K be a hyperbolic knot in S^3. For a given slope p/q with p odd, if every solution (x_0,y_0) of the system of equations*

$$\begin{cases} A_K(x,y)=0 \\ x^p y^q - 1 = 0 \end{cases}$$

has $x_0\in\{1,-1,0\}$, then $M_K(p/q)$ cannot be a Seifert fibered space whose base orbifold is a 2–sphere with exactly three cone points.

Proof The proof is similar to that of Lemma 5.6. Suppose otherwise that $M_K(p/q)$ is a Seifert fibered space whose base orbifold is a 2–sphere with exactly three cone points. As p is odd, the base orbifold is either spherical or hyperbolic. Hence $\pi_1(M_K(p/q))$ has an irreducible $\mathrm{PSL}_2(\mathbb{C})$ representation $\bar\rho_0$ which factors through the orbifold fundamental group of the base orbifold such that the image of $\bar\rho_0$ contains no parabolic elements. As p is odd, the $\mathrm{PSL}_2(\mathbb{C})$ representation $\bar\rho_0$ lifts to an $\mathrm{SL}_2(\mathbb{C})$ representation ρ_0. It is well known that the character χ_{ρ_0} of ρ_0 is an isolated point in $X(M_K(p/q))$. Considered as a point in $X(M_K)$, the character χ_{ρ_0} is contained in a positive-dimensional component X_0 of $X(M_K)$. Arguing exactly as in Claim 5.7 we have $f_{\mu^p\lambda^q}$ is nonconstant on X_0. Hence X_0 is contained $X^*(M_K)$ and contributes a factor $f_0(x,y)$ to $A_K(x,y)$. Exactly as in the proof of Lemma 5.6, the point χ_{ρ_0} provides a solution (x_0,y_0) to the system

$$\begin{cases} A_K(x,y)=0 \\ x^p y^q - 1 = 0 \end{cases}$$

such that $x_0\notin\{1,-1,0\}$, giving a contradiction with the assumption of the lemma. □

6 Distinguishing $k(l_*,-1,0,0)$ from $k(l,m,0,p)$

The goal of this section is to prove the following proposition:

Proposition 6.1 *Suppose that two knots $k(l_*,-1,0,0)$ and $k(l,m,0,p)$ have the same genus g and the same half-integral toroidal slope r, where $l_*>1$ and $(1-2p)\mid l$. Then $k(l_*,-1,0,0)=k(l,m,0,p)$.*

Let

$$s=r+\tfrac{1}{2} \quad\text{and}\quad d=-s-2g.$$

For $k(l,m,0,p)$, by (4-2),

$$-s=-p(2ml-l-1)^2+(2ml-l)(ml-1). \tag{6-1}$$

When $p\le 0$, using (4-3) , we get

$$d=\begin{cases}-p(2ml-l-1)+3ml-l-2\alpha & \text{if } lm>0,\\ -p(-2ml+l+1)-3ml+l & \text{if } lm<0,\end{cases} \tag{6-2}$$

where

$$\alpha=\begin{cases}1 & \text{if } l>0,\ m>0,\\ 2 & \text{if } l<0,\ m<0.\end{cases}$$

Consider the family of knots $k(l_*,-1,0,0)$, where $l_*>1$. In this case, d and s are given by

$$\begin{cases} d=4l_*,\\ s=-3l_*(l_*+1),\end{cases} \tag{6-3}$$

so

$$-\tfrac{4}{3}s+1=\left(\tfrac{1}{2}d+1\right)^2. \tag{6-4}$$

Lemma 6.2 *Suppose that $p<0$ and $(1-2p)\,|\,l$. Then the knot $k(l,m,0,p)$ has different (g,r) from the knot $k(l_*,-1,0,0)$, where $l_*>1$.*

Proof Otherwise, assume $k(l,m,0,p)$ has the same (g,r) as $k(l_*,-1,0,0)$. Using (6-2), we see that

$$d+2>-p|2ml-l-1|+|2ml-l-1|.$$

Using (6-1) and (4-1), we have

$$-s+\tfrac{3}{4}<-p(2ml-l-1)^2+(2ml-l)(ml-1)+1<(1-p)(2ml-l-1)^2.$$

Using (6-4) and the previous two inequalities, we get

$$\begin{aligned}\tfrac{1}{4}(1-p)^2(2ml-l-1)^2&<\left(\tfrac{1}{2}d+1\right)^2\\ &=-\tfrac{4}{3}s+1\\ &<\tfrac{4}{3}(1-p)(2ml-l-1)^2.\end{aligned}$$

So $1-p<\frac{16}{3}$, and hence $-p\le 4$.

If $p=-1$ or -4, then $3\,|\,l$. By (6-1), $3\nmid -s$, which contradicts (6-3).

If $p=-3$, then $7\,|\,l$. By (6-1), $-s\equiv 3 \pmod 7$. It follows from (6-4) that 5 is a quadratic residue modulo 7, which is not true.

If $p=-2$, then $5\,|\,l$ and $-s\equiv 2 \pmod 5$. It follows from (6-4) that 2 is a quadratic residue modulo 5, which is not true. □

For the family of knots $k(l,m,0,0)$, the following proposition expresses l and m in terms of s and d.

Proposition 6.3 *For the family of knots $k(l,m,0,0)$, the pair (l,m) is determined by s and d by the following formulas:*

If $lm>0$, then

$$l=\frac{d+2\alpha+3-3\sqrt{(d+2\alpha-1)^2+4s}}{2},$$

$$m=\frac{1}{3}\left(\frac{2d+4\alpha}{d+2\alpha+3-3\sqrt{(d+2\alpha-1)^2+4s}}+1\right).$$

If $lm<0$, then

$$l=\frac{-d+3+3\sqrt{(d+1)^2+4s}}{2},$$

$$m=\frac{1}{3}\left(1-\frac{2d}{-d+3+3\sqrt{(d+1)^2+4s}}\right).$$

Proof Using (6-2), we can express ml as a linear function of d and l. Substituting such an expression of ml into (6-1), we get

(6-5) $$-s=\begin{cases}\frac{1}{9}(-l+2d+4\alpha)(l+d+2\alpha-3) & \text{if } lm>0,\\ \frac{1}{9}(-l-2d)(l-d-3) & \text{if } lm<0.\end{cases}$$

If $lm>0$, by (6-5), l is a root of the quadratic polynomial

(6-6) $$(x-2d-4\alpha)(x+d+2\alpha-3)-9s,$$

whose two roots are

$$\frac{d+2\alpha+3\pm 3\sqrt{(d+2\alpha-1)^2+4s}}{2}.$$

By (6-2), $d>0$ and $|2l|\le d+2\alpha$, so

$$l=\frac{d+2\alpha+3-3\sqrt{(d+2\alpha-1)^2+4s}}{2}.$$

Using (6-2) again, we can compute m as in the statement.

If $lm<0$, then l is a root of the quadratic polynomial

(6-7) $$(x+2d)(x-d-3)-9s,$$

whose two roots are

$$\frac{-d+3\pm 3\sqrt{(d+1)^2+4s}}{2}.$$

By (6-2), $d>0$ and $|2l|\le d$, so

$$l=\frac{-d+3+3\sqrt{(d+1)^2+4s}}{2}.$$

Using (6-2) again, we can compute m as in the statement. □

Lemma 6.4 *Suppose that two knots $k(l,m,0,0)$ and $k(l_*,m_*,0,0)$ have the same g and r. Suppose further that*

$$lm>0,\quad l_*m_*>0,\quad l>0>l_*.$$

Then the quadruple (l,m,l_*,m_*) is either $(2,2,-3,-1)$ or $(6,m,-2,-3m+1)$ for some $m\ge 2$.

Proof Using Proposition 6.3 and (4-1), we get

$$l=\frac{d+5-3\sqrt{(d+1)^2+4s}}{2}\ge 2 \tag{6-8}$$

and

$$l_*=\frac{d+7-3\sqrt{(d+3)^2+4s}}{2}\le -2. \tag{6-9}$$

Hence

$$\tfrac{1}{9}(d+11)^2-(d+3)^2\le 4s\le -\tfrac{8}{9}(d+1)^2, \tag{6-10}$$

which implies

$$\tfrac{1}{9}(d-7)^2\le (d+1)^2+4s\le \tfrac{1}{9}(d+1)^2.$$

From (6-2) and (4-1), we can conclude that

$$d\ge 8. \tag{6-11}$$

By (6-8), $(d+1)^2+4s$ is a perfect square which has the same parity as $d+1$, hence

$$(d+1)^2+4s=\tfrac{1}{9}(d+1-2c)^2 \tag{6-12}$$

for some $c\in\{0,1,2,3,4\}$. Then

$$l=c+2\quad\text{and}\quad m=\frac{d+c+4}{3(c+2)}.$$

Using (6-10), we also get

$$\tfrac{1}{9}(d+11)^2 \le (d+3)^2+4s \le \tfrac{1}{9}(d^2+38d+73) < \tfrac{1}{9}(d+19)^2.$$

By (6-9), $(d+3)^2+4s$ is a perfect square with the same parity as $d+3$, so

$$(d+3)^2+4s = \tfrac{1}{9}(d+11+2c_*)^2 \tag{6-13}$$

for some $c_* \in \{0,1,2,3\}$. Then

$$l_* = -c_* - 2 \quad \text{and} \quad m_* = -\frac{d-c_*+2}{3(c_*+2)}.$$

Comparing (6-12) and (6-13), we get

$$(4-c_*-c)d = (5+c_*+c)(6+c_*-c)-18. \tag{6-14}$$

Moreover, since both $\frac{1}{3}(d+1-2c)$ and $\frac{1}{3}(d+11+2c_*)$ are integers, $\frac{1}{3}(10+2c_*+2c)$ is an integer, so

$$c+c_* \in \{1,4,7\}.$$

If $c+c_*=7$, then $c=4$ and $c_*=3$. By (6-14), $d=-14$, a contradiction to (6-11).

If $c+c_*=4$, using (6-14) we get $c=4$ and $c_*=0$. Hence $l=6$, $m=\frac{1}{18}(d+8)$, $l_*=-2$ and $m_*=-\frac{1}{6}(d+2)=-3m+1$.

If $c+c_*=1$, using (6-14) we get $d=6+2(c_*-c)$. Using (6-11), the only possible case is $c=0$, $c_*=1$ and $(l,m,l_*,m_*)=(2,2,-3,-1)$. □

Proof of Proposition 6.1 When $p<0$, this result follows from Lemma 6.2.

When $p>0$, from (6-1) it is easy to see $s>0$ for $k(l,m,0,p)$, but $s=-3l_*(l_*+1)<0$ for $k(l_*,-1,0,0)$.

Now we consider the case $p=0$. By Proposition 6.3, for any given (g,r), there are at most three knots, $k(l_i,m_i,0,0)$ for $i=1,2,3$, having this (g,r). There is at most one pair (l_i,m_i) in each of the three cases

- $l>0,\ m>0$,
- $l<0,\ m<0$,
- $lm<0$.

The pair $(l_*, -1)$ is in the third case. By Proposition 4.2(c), $k(-l_*-1, -1, 0, 0)$ is equal to $k(l_*, -1, 0, 0)$, and the pair $(-l_*-1, -1)$ is in the second case. Suppose that there is also a pair (l, m) in the first case with the same (g, r), then Lemma 6.4 implies that $(l, m) = (2, 2)$ and $-l_*-1 = -3$. By Proposition 4.2(d), the three knots $k(2, 2, 0, 0)$, $k(-3, -1, 0, 0)$ and $k(2, -1, 0, 0)$ are the same. □

7 Proof of Theorem 1.2

Recall that the unique half-integral toroidal slope of $k(l_*, -1, 0, 0)$ is $r = \frac{1}{2}(2s-1)$, where $s = -3l_*(l_*+1)$. Hence the knots in the family $\{k(l_*, -1, 0, 0) : l_* > 1\}$ are mutually distinct.

Suppose that $K \subset S^3$ is a knot which has the same A–polynomial and the same knot Floer homology as a given knot $J_* = k(l_*, -1, 0, 0)$ with $l_* > 1$. Our goal is to show that $K = J_*$.

As J_* is hyperbolic, K cannot be a torus knot by Theorem 1.1.

Suppose K is hyperbolic. Then by Proposition 5.8, K has the same half-integral toroidal slope r as J_* and K is one of the knots $k(l, m, n, p)$. Applying Lemma 5.11, we have $n = 0$ or 1. Since the half-integral toroidal slope of $k(l, m, 1, 0)$ is positive while the half-integral toroidal slope of $J_* = k(l_*, -1, 0, 0)$ is negative, we see that n must be zero. Similarly applying Lemma 5.12, we have p is nonpositive, and $2p-1$ divides l. That is, we have $K = k(l, m, 0, p)$ with $(2p-1) \mid l$. Now by Proposition 6.1, we have $K = J_*$.

It remains to show that K cannot be a satellite knot. Suppose otherwise that K is a satellite knot. We are going to derive a contradiction from this assumption.

Lemma 7.1 *The A–polynomial $A_{J_*}(x, y)$ of $J_* = k(l_*, -1, 0, 0)$ does not contain any factor of the form $x^j y + \delta$ or $y + \delta x^{-j}$ for $j \in \mathbb{Z}$ and $\delta \in \{-1, 1\}$.*

Proof Suppose otherwise that $A_{J_*}(x, y)$ contains a factor of the form $x^j y + \delta$ or $y + \delta x^{-j}$. As this factor is irreducible and balanced, it is contributed by a curve component X_0 in $X^*(M_{J_*})$. Moreover X_0 is a seminorm curve with $\mu^j \lambda$ as the unique associated boundary slope.

We claim that $\mu^j \lambda$ is either $\mu^s \lambda$ or $\mu^{s-1}\lambda$. From Theorem 2.5(3) and Lemma 4.5, we see that the meridian slope μ has the minimal seminorm s_0. To prove the claim, we just need to show, by Theorem 2.5(4), that for the half-integral toroidal slope r of J_*, we have $\|r\|_0 = \|\mu\|_0$, which is equivalent to showing that

$$Z_v(\widetilde{f}_r) \le Z_v(\widetilde{f}_\mu)$$

for every $v \in \widetilde{X}_0$. As $M_{J_*}(r)$ has no noncyclic representation by Corollary 5.3, at every regular point $v \in \widetilde{X}_0$ we have $Z_v(\widetilde{f}_r) \le Z_v(\widetilde{f}_\mu)$. If at an ideal point v of $\widetilde{X}_0$ we have $Z_v(\widetilde{f}_r) > Z_v(\widetilde{f}_\mu)$, then $\widetilde{f}_\alpha(v)$ is finite for every class α in $H_1(\partial M_{J_*};\mathbb{Z})$ (as both $\widetilde{f}_r$ and $\widetilde{f}_{\mu^j\lambda}$ are finite at v). We can now derive a contradiction with [4, Proposition 4.12] just as in the proof of Lemma 5.4, and the claim is thus proved.

On the other hand it is shown in [13, Proposition 5.4] that $M_{J_*}(s)$ and $M_{J_*}(s-1)$ each are small Seifert fibered spaces. As J_* is a small knot, each of $\mu^s\lambda$ and $\mu^{s-1}\lambda$ cannot be a boundary slope by [9, Theorem 2.0.3] and thus cannot be the slope $\mu^j\lambda$. We arrive at a contradiction. □

Since J_* is a fibered knot by [13], K is also fibered. Hence if (C, P) is any pair of companion knot and pattern knot associated to K, then each of C and P is fibered and the winding number w of P with respect to C is larger than zero.

Lemma 7.2 *The satellite knot K has a companion knot C which is hyperbolic.*

Proof As is true for any satellite knot, K has a companion knot C which is either a torus knot or a hyperbolic knot. So we just need to rule out the possibility that C is a torus knot. If C is a (p,q)–torus knot, then by (2-3), $A_C(x,y)$ contains a factor of the form $yx^k + \delta$ or $y + \delta x^{-k}$ for some integer k and $\delta \in \{1,-1\}$. As the winding number w of the pattern knot P with respect to C is nonzero, by Proposition 2.7, $A_K(x,y)$ contains a factor of one of the forms $yx^{w^2k} - (-\delta)^w$ or $y - (-\delta)^w x^{-w^2k}$. But this contradicts Lemma 7.1. □

We now fix a hyperbolic companion knot C for K which exists by Lemma 7.2 and let P be the corresponding pattern knot.

Lemma 7.3 *For any hyperbolic knot C in S^3, any surgery with a slope j/k, where j and k are relatively prime, $k > 2$ and j is odd, will produce either a hyperbolic manifold or a Seifert fibered space whose base orbifold is S^2 with exactly three cone points.*

Proof $M_C(j/k)$ is irreducible [16] and atoroidal [17], as $k > 2$. Thus $M_C(j/k)$ is either a hyperbolic manifold or an atoroidal Seifert fibered space. In latter case, the Seifert fibered space has noncyclic fundamental group [9]. As j is odd, the base orbifold of the Seifert fibered space cannot be nonorientable. Thus the base orbifold is a 2–sphere with exactly three cone points. □

Lemma 7.4 *The integer $d = 2s - 1$ is divisible by w^2.*

Proof Suppose otherwise. Let d_1/q_1 be the rational number $d/(2w^2)$ in its reduced form, ie $d_1 = d/\gcd(d, w^2)$ and $q_1 = 2w^2/\gcd(d, w^2)$. Then $q_1 > 2$ and d_1 is odd. So by Lemma 7.3, the surgery on C with the slope d_1/q_1 will yield either a hyperbolic manifold or a Seifert fibered space whose base orbifold is a 2–sphere with exactly three cone points. Applying either Lemma 5.6 or Lemma 5.13, we see that the A–polynomial $A_C(\bar{x}, \bar{y})$ of C has a zero point $(\bar{x}_0, \bar{y}_0)$ such that $\bar{x}_0^{d_1}\bar{y}_0^{q_1} = 1$ and $\bar{x}_0 \notin \{0, 1, -1\}$. Now from Proposition 2.7 and its proof, we see that $A_C(\bar{x}, \bar{y})$ can be extended to a factor $f(x, y)$ of $A_K(x, y)$ with the variables of $A_C(\bar{x}, \bar{y})$ and $f(x, y)$ satisfying the relations $\bar{x} = x^w$ and $\bar{y}^w = y$. In particular, for some (x_0, y_0) we have $\bar{x}_0 = x_0^w$ and $\bar{y}_0^w = y_0$, and (x_0, y_0) is a zero point of $f(x, y)$. Obviously $x_0 \notin \{0, 1, -1\}$. From $(\bar{x}_0^{d_1}\bar{y}_0^{q_1})^w = 1$, we have $x_0^{w^2 d_1} y_0^{q_1} = 1$, ie $x_0^d y_0^2 = 1$.

As $f(x, y)$ is a factor in $A_K(x, y) = A_{J_*}(x, y)$, we see that the system

$$\begin{cases} A_{J_*}(x, y) = 0 \\ x^d y^2 - 1 = 0 \end{cases}$$

has a solution (x_0, y_0) with $x_0 \notin \{0, 1, -1\}$. We get a contradiction with Lemma 5.5. □

Note that $s - 1$ is a cyclic surgery slope of J_* (provided by [13, Proposition 5.4]).

Lemma 7.5 *If (x_0, y_0) is a solution of the system*

$$\begin{cases} A_{J_*}(x, y) = 0 \\ x^{s-1} y - 1 = 0 \end{cases}$$

then x_0 is either 1 or -1.

Proof If x_0 is neither 1 nor -1, it follows that there is a curve component X_0 in $X^*(M)$ such that $\widetilde{X}_0$ has a point at which $\widetilde{f}_{\mu^{s-1}\lambda} = 0$ but $\widetilde{f}_\mu \neq 0$. This is impossible as $\widetilde{f}_{\mu^{s-1}\lambda}$ has the minimal zero degree at every point of $\widetilde{X}_0$ (because J_* is a small knot, the cyclic surgery slope $s - 1$ cannot be a boundary slope by [9, Theorem 2.0.3]). □

Lemma 7.6 *The integer $s - 1$ is divisible by w^2.*

Proof The proof is similar to that of Lemma 7.4, only replacing Lemma 5.5 by Lemma 7.5. First note that $s - 1$ is an odd number (as $s = -3l_*(l_* + 1)$ is even). So if $s - 1$ is not divisible by w^2, then the reduced form d_1/q_1 of the rational number $(s-1)/w^2$ has denominator $q_1 > 2$. Now arguing as in the proof of Lemma 7.4 starting from the d_1/q_1–surgery on C, we see that the system

$$\begin{cases} A_{J_*}(x,y) = 0 \\ x^{s-1}y - 1 = 0 \end{cases}$$

has a solution (x_0, y_0) with $x_0 \neq 1, -1$. This gives a contradiction with Lemma 7.5. □

Corollary 7.7 *The winding number w equals 1.*

Proof This follows immediately from Lemmas 7.4 and 7.6 □

Lemma 7.8 *The companion knot C has the same half-integral toroidal slope as J_* and C is one of the knots $k(l,m,0,p)$ with p nonpositive and with $(2p-1) \mid l$.*

Proof It follows from Corollary 7.7 and Proposition 2.7 that $A_C(x,y)$ is a factor of $A_K(x,y) = A_{J_*}(x,y)$. So Proposition 5.8 says that r is also a toroidal slope of C, and C is one of the knots $k(l,m,n,p)$. Now arguing as in the case when K is hyperbolic, we see that C is one of the knots $k(l,m,0,p)$ with p nonpositive and with $(2p-1) \mid l$. □

Lemma 7.9 *For $C = k(l,m,0,p)$ given by Lemma 7.8, we have $m = -1$ unless $C = k(2,2,0,0)$ or $C = k(-2,m,0,0)$.*

Proof If $m \neq -1$, then by [12, Theorem 2.1(d)], the $(s-1)$–surgery on the knot $C = k(l,m,0,p)$ is a Seifert fibered manifold whose base orbifold is a 2–sphere with exactly three cone points, except when C is one of the knots $k(-2,m,0,p)$, $k(2,2,0,0)$, $k(2,3,0,1)$, $k(3,2,0,1)$ or $k(2,2,0,2)$. As we know that p is nonpositive and $2p-1$ divides l, these exceptional cases can be excluded except for $k(2,2,0,0)$ or $k(-2,m,0,0)$. So we just need to deal with the case when the $(s-1)$–surgery on C is a Seifert fibered manifold whose base orbifold is a 2–sphere with exactly three cone points. Note that $s-1$ is odd. Hence by Lemma 5.13, the system

$$\begin{cases} A_C(x,y) = 0 \\ x^{s-1}y - 1 = 0 \end{cases}$$

has a solution (x_0, y_0) with $x_0 \neq 1, -1$. As $A_C(x,y)$ is a factor of $A_{J_*}(x,y)$, the point (x_0, y_0) is also a solution of the system

$$\begin{cases} A_{J_*}(x,y)=0 \\ x^{s-1}y-1=0, \end{cases}$$

which yields a contradiction with Lemma 7.5. □

Lemma 7.10 $$|s-2| \le 4g(C).$$

Proof Recall that C is one of the knots $k(l,-1,0,p)$ with p nonpositive, $k(2,2,0,0)$, or $k(-2,m,0,0)$ and has the same r slope as J_* and thus has the same s slope as J_*. From (4-2) and (4-3) we have that for $k(l,-1,0,p)$ with p nonpositive, its s slope and genus g are given by

$$s=-3l(l+1)+p(-3l-1)^2$$

and

$$g=\begin{cases} -\frac{1}{2}p(3l+1)3l+l^2+\frac{1}{2}l(l-1) & \text{if } l>0, \\ -\frac{1}{2}p(-3l-1)(-3l-2)+l^2+\frac{1}{2}l(l+5)+l+2 & \text{if } l<0; \end{cases}$$

for $k(2,2,0,0)$, its s and g values are

$$s=-18 \quad \text{and} \quad g=5;$$

and for $k(-2,m,0,0)$,

$$s=-2(2m-1)(2m+1) \quad \text{and} \quad g=\begin{cases} 4m^2-3m & \text{if } m>0, \\ 4m^2+3m & \text{if } m<0. \end{cases}$$

In each case, one can check directly that $|s-2| \le 4g$ holds, keeping in mind some forbidden values on l, m and p given by (4-1). □

As noted in the proof of Theorem 1.1, when the winding number w equals 1, the pattern knot P is a nontrivial knot. We also have, by Lemma 2.6, that $A_P(x,y)$ is a factor of $A_K(x,y)=A_{J_*}(x,y)$. Combining this fact with (2-3) and Lemma 7.1 we know that P cannot be a torus knot. So P is either a hyperbolic knot or a satellite knot.

If P is a hyperbolic knot, then arguing as in the proofs of Lemmas 7.8, 7.9 and 7.10, we have that P has the same half-integral toroidal slope as J_*, that P is one of the knots $k(l',-1,0,p')$ with p' nonpositive or $k(2,2,0,0)$ or $k(-2,m',0,0)$, and that $|s-2| \le 4g(P)$.

Now from

$$\Delta_{J_*}(t)=\Delta_K(t)=\Delta_C(t)\Delta_P(t)$$

we have that the genus of the given satellite knot K (which is equal to that of J_*) is equal to the sum of the genus of C and the genus of P. So the genus of one of C and P, say C (the argument is the same for P), is less than or equal to the half of the genus of J_*, ie

$$g(C) \leq \tfrac{1}{2} g(J_*).$$

So

$$|s-2| \leq 2g(J_*). \tag{7-1}$$

But $s = -3l_*(l_*+1)$ and $g(J_*) = l_*^2 + \frac{1}{2} l_*(l_*-1)$, which do not satisfy (7-1). This contradiction shows that P cannot be hyperbolic.

So P is a satellite knot. Let (C_1, P_1) be a pair companion knot and pattern knot for P. Once again, as P is fibered, each of C_1 and P_1 is fibered, and the winding number w_1 of P_1 with respect to C_1 is larger than zero. Making use of the fact that $A_P(x,y) \mid A_{J_*}(x,y)$, one can show, similarly as for the pair (C, P), that C_1 can be assumed to be hyperbolic, that $w_1 = 1$, that C_1 has the same r and s values as J_*, that C_1 is $k(l'', -1, 0, p'')$ for some nonpositive p'' or $k(2,2,0,0)$ or $k(-2, m'', 0, 0)$, and that $|s-2| \leq 4g(C_1)$. Now from the equality

$$\Delta_{J_*}(t) = \Delta_K(t) = \Delta_C(t)\Delta_{C_1}(t)\Delta_{P_1}(t)$$

we see that one of $g(C)$ and $g(C_1)$ is less than or equal to $\frac{1}{2} g(J_*)$. This leads to a contradiction just as in the preceding paragraph. So P cannot be a satellite knot, and this final contradiction completes the proof of Theorem 1.2.

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