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On the periodic v_2–self-map of A_1

PRASIT BHATTACHARYA
PHILIP EGGER
MARK MAHOWALD

The spectrum $Y := M_2(1) \wedge C\eta$ admits eight v_1-self-maps of periodicity 1. These eight self-maps admit four different cofibers, which we denote by $A_1[ij]$ for $i, j \in \{0, 1\}$. We show that each of these four spectra admits a v_2-self-map of periodicity 32.

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This paper is dedicated to the memory of Mark Mahowald (1931–2013)

1 Introduction

Convention Throughout this paper, we work in the stable homotopy category of spectra localized at the prime 2.

Let $K(n)$ be the n^{th} Morava K–theory. Let $\mathcal{C}_0$ be the category of 2–local finite spectra, $\mathcal{C}_n \subset \mathcal{C}_0$ the full subcategory of $K(n-1)$–acyclics and $\mathcal{C}_\infty$ the full subcategory of contractible spectra. Hopkins and Smith [8] showed that the $\mathcal{C}_n$ are thick subcategories of $\mathcal{C}_0$ (in fact, they are the only thick subcategories of $\mathcal{C}_0$), and they fit into a sequence

$$\mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n \supset \cdots \supset \mathcal{C}_\infty.$$

We say a finite spectrum X is of type n if $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$.

A self-map $v\colon \Sigma^k X \to X$ of a finite spectrum X is called a *v_n–self-map* if

$$K(n)_*(v)\colon K(n)_*(X) \to K(n)_*(X)$$

is an isomorphism. For a finite spectrum X, a self-map $v\colon \Sigma^k X \to X$ can also be regarded as an element of $\pi_k(X \wedge DX)$, where DX is the Spanier–Whitehead dual of X.

For any ring spectrum E, let

$$\iota_{E*}\colon \pi_*(\_) \to E_*(\_)$$

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denote the E–Hurewicz natural transformation. Let $k(n)$ denote the connective cover of $K(n)$. If $v\colon S^k \to X \wedge DX$ is a v_n–self-map, then $\iota_{k(n)*}(v) \in k(n)_*(X \wedge DX)$ has to be the image of $v_n^m \in k(n)_* \cong \mathbb{F}_2[v_n]$, for some positive integer m, under the map

$$k(n)_*\iota_{X\wedge DX}\colon k(n)_* \to k(n)_*(X \wedge DX),$$

where $\iota_{X\wedge DX}\colon S^0 \to X \wedge DX$ is the unit map. The value m is called the *periodicity* of the v_n–self-map v. We call v a *minimal v_n–self-map* for X if v is a v_n–self-map with minimal periodicity. An easy consequence of [8, Theorem 9] is that the periodicity of a minimal v_n–self-map is always a power of 2.

Hopkins and Smith showed, among other things, that every type-n spectrum admits a v_n–self-map, and the cofiber of a v_n–self-map is of type $n+1$. However, not much is known about the minimal periodicity of such v_n–self-maps.

The sphere spectrum S^0 is a type-0 spectrum with a v_0–self-map $2\colon S^0 \to S^0$. The cofiber of this v_0–self-map is the type-1 spectrum $M(1)$. The spectrum $M(1)$ is known to admit a unique minimal v_1–self-map of periodicity 4. The cofiber of this v_1–self-map is denoted by $M(1,4)$. In 2008, Behrens, Hill, Hopkins and the third author [1] showed that the minimal v_2–self-map on $M(1,4)$ has periodicity 32.

Instead of S^0, we can start with the type-0 spectrum $C\eta$, the cofiber of $\eta\colon S^1 \to S^0$. The spectrum $C\eta$ admits a nonzero v_0–self-map $2 \wedge 1_{C\eta}\colon C\eta \to C\eta$, with cofiber $M(1) \wedge C\eta := Y$. The type-1 spectrum Y admits eight minimal v_1–self-maps of periodicity 1. These eight maps are constructed by Davis and the third author [3] using stunted projective spaces. The cofiber of any of the v_1–self-maps is referred to as A_1. Though there are eight different v_1–self-maps, there are only four different homotopy types of the cofibers A_1; see [3, Proposition 2.1].

Let $A(1)$ be the subalgebra of the Steenrod algebra A generated by Sq^1 and Sq^2. It turns out that the cohomology of any homotopy type of A_1 is a free $A(1)$–module on one generator. However, different homotopy types of A_1 have different A–module structures, which are distinguished by the action of Sq^4. We depict the cohomologies of the four different spectra A_1 in Figure 1 where the square brackets represent an action of Sq^4, the curved lines represent an action of Sq^2, and the straight lines represent an action of Sq^1. The subalgebra $A(1)$ has four different A–module structures, each of which corresponds to a homotopy type of A_1. Any A–module structure on $A(1)$ has a nontrivial Sq^4 action on the generator in degree 1 forced by the Adem relations. However, there are choices for Sq^4 actions to be trivial or nontrivial on generators in degree 0 and degree 2, thus giving us four different A–module structures. We denote the different homotopy types of A_1 using the notation $A_1[ij]$ where i and j are the indicator functions for the action of Sq^4 on the generators in degree 0 and degree 2, respectively.

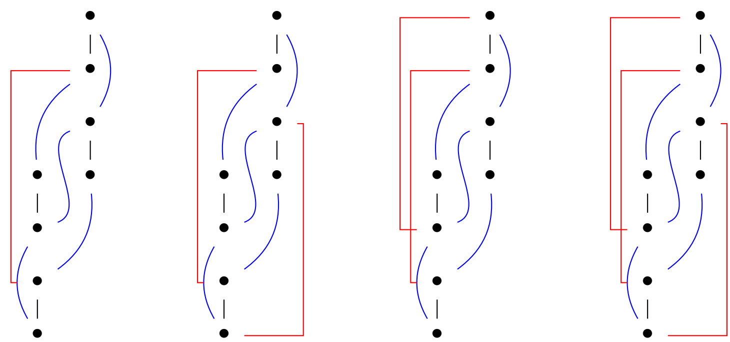

Figure 1: The A–module structures of $H^*(A_1[00])$, $H^*(A_1[10])$, $H^*(A_1[01])$ and $H^*(A_1[11])$

Remark 1.1 (determining A–module structure on Spanier–Whitehead duals) For every finite spectrum X, there is an isomorphism

$$H^* DX \cong DH^* X,$$

where we have Spanier–Whitehead duality on the left hand side and A–module duality on the right hand side. Thus, finding out the Spanier–Whitehead duality relations between the spectra $A_1[ij]$ boils down to finding the A–module duality relations between the A–modules depicted in Figure 1. The naïve guess is that dualizing these A–modules is equivalent to merely "flipping them upside down". However, this is not the case. For an A–module M and its dual DM, there is a pairing

$$\langle -, - \rangle \colon M \otimes DM \to \mathbb{F}_2$$

which is A–bilinear. Therefore, for elements $x, y \in M$ and $a \in A$, we have

$$\langle ax, y_* \rangle = \langle x, \chi(a) y_* \rangle,$$

where $\chi\colon A \to A$ is the antipode, and hence

$$(ax)_* = \sum_{\{g : ax = \chi(a) g\}} g_*.$$

Because $\chi(\mathrm{Sq}^1) = \mathrm{Sq}^1$ and $\chi(\mathrm{Sq}^2) = \mathrm{Sq}^2$, the naïve guess is correct when it comes to actions of Sq^1 and Sq^2. However, because we have $\chi(\mathrm{Sq}^4) = \mathrm{Sq}^4 + \mathrm{Sq}^3 Sq^1$, the naïve guess breaks down when considering the actions of Sq^4. Thus we find that $H^*(A_1[00])$ is dual to $H^*(A_1[11])$, while $H^*(A_1[10])$ and $H^*(A_1[01])$ are self-dual. It follows that the spectra $A_1[01]$ and $A_1[10]$ are Spanier–Whitehead self-dual, whereas $A_1[00]$ and $A_1[11]$ are Spanier–Whitehead dual to each other.

It is worth noting that A_1 is created in a way similar to $M(1,4)$, where $C\eta$ is analogous to S^0, and Y is analogous to $M(1)$. The minimal v_1–self-map of Y has periodicity 1, which is less than the periodicity of the minimal v_1–self-map on $M(1)$, which is 4. Hence, it is natural to ask if any of the four models of A_1 admit a v_2–self-map of periodicity less than that of $M(1,4)$.

In [3, Theorem 1.4(ii)], Davis and the third author claimed, incorrectly, that the periodicity of the minimal v_2–self-maps on $M(1,4)$ and the two self-dual models of A_1, namely $A_1[01]$ and $A_1[10]$, was 8. After successfully correcting the v_2–periodicity of $M(1,4)$ in [1], the v_2–periodicity of A_1 was called into question by the third author. He conjectured that the minimal v_2–self-map of A_1 should have periodicity 32, which is also the periodicity of the minimal v_2–self-map of $M(1,4)$.

The goal of this paper is to prove the following correction of [3, Theorem 1.4(ii)], as reported in Remark 1.4 of [1]:

Main Theorem *For all four models of A_1, the minimal v_2–self-map*

$$v\colon \Sigma^{192}A_1 \to A_1$$

has periodicity 32.

Notation 1.2 To lighten the notations, we use $\mathrm{Ext}_T^{s,t}(X)$ to denote $\mathrm{Ext}_T^{s,t}(H^*(X),\mathbb{F}_2)$, where T is a subalgebra of the Steenrod algebra A.

Notation 1.3 For any ring spectrum E, we denote the unit map by $\iota_E\colon S^0 \to E$. The unit map ι_E induces the Hurewicz natural transformation

$$\iota_{E*}\colon \pi_*(\_) \to E_*(\_)$$

as introduced earlier. When $E = A_1 \wedge DA_1$, we simply use $\iota\colon S^0 \to A_1 \wedge DA_1$ to denote the unit map. Let $i\colon S^0 \hookrightarrow A_1$ be the map that represents the inclusion of the bottom cell. Let $j\colon A_1 \wedge DA_1 \to A_1$ denote the map $1_{A_1} \wedge Di$. Given a map between two spectra $f\colon X \to Y$, the unit map ι_E induces a map in E–homology, which we denote by

$$E_*(f)\colon E_*X \to E_*Y,$$

and also a map of Adams spectral sequences, which we denote by

$$f_*^E\colon \mathrm{Ext}_A^{*,*}(E \wedge X) \to \mathrm{Ext}_A^{*,*}(E \wedge Y).$$

Outline

The proof of Main Theorem consists of two parts, namely

- the nonexistence part, where we eliminate the possibility of a v_2–self-map of A_1 of periodicity lower than 32,
- the existence part, where we show that there exists a v_2–self-map of A_1 of periodicity 32.

The proof makes use of several important differentials in the Adams spectral sequence that computes the homotopy groups of the spectrum *tmf*. As an A–module (see Hopkins and the third author [7]),

$$H^*(\mathit{tmf}) \cong A/\!\!/A(2),$$

where $A(2)$ is the subalgebra of A generated by Sq^1, Sq^2 and Sq^4. Therefore, by a change of rings formula, the E_2 page of that Adams spectral sequence simplifies to

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A(2)}(S^0) \Rightarrow \pi_{t-s}(\mathit{tmf}). \tag{1.4}$$

The E_2 page is periodic with the periodicity generator $b_{3,0}^4$, which lives in bidegree $(s,t) = (8, 8+48)$. The periodicity generator $b_{3,0}^4$ and its square $b_{3,0}^8$ are not present in the E_∞ page of the above spectral sequence. There exist differentials

$$d_2(b_{3,0}^4) = e_0 r \quad \text{and} \quad d_3(b_{3,0}^8) = wgr \tag{1.5}$$

in the Adams spectral sequence computing tmf_*. But in that spectral sequence, $b_{3,0}^{16}$ is a nonzero permanent cycle which detects the periodicity generator $\Delta^8 \in \pi_{192}(\mathit{tmf})$. All the details mentioned above are well documented by Henriques [6].

The unit map $\iota_{k(2)}\colon S^0 \to k(2)$ factors through *tmf* (see [1, Remark 1.3]): ie we have

$$\iota_{k(2)}\colon S^0 \xrightarrow{\iota_{\mathit{tmf}}} \mathit{tmf} \xrightarrow{r} k(2). \tag{1.6}$$

The map induced by r in homotopy

$$r_*\colon \mathit{tmf}_* \to k(2)_*$$

maps $\Delta^{8n} \mapsto v_2^{32n}$, which is why *tmf* can detect periodic v_2–self-maps. This can be observed through a map of Adams spectral sequences. Since

$$H^*(k(2)) \cong A/\!\!/E(Q_2)$$

(due to Lellmann [9]), by a change of rings formula, we have

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E(Q_2)}(S^0) \Rightarrow \pi_{t-s}(k(2)).$$

The E_2 page is simply a polynomial algebra generated by v_2 in bidegree $(s,t) = (1, 1+6)$. The spectral sequence collapses due to sparseness, giving us the expected result $\pi_*(k(2)) = \mathbb{F}_2[v_2]$. The map $r\colon tmf \to k(2)$ induces a map of spectral sequences

$$\begin{tikzcd}
E_2 = \mathrm{Ext}^{s,t}_{A(2)}(S^0) \arrow[r, Rightarrow] \arrow[d, "r_*"'] & \pi_{t-s}(tmf) \arrow[d, "r_*"'] \\
E_2 = \mathrm{Ext}^{s,t}_{E(Q_2)}(S^0) \arrow[r, Rightarrow] & \pi_{t-s}(k(2))
\end{tikzcd}$$

which sends $b_{3,0}^{4n}$ to v_2^{8n} in the E_2 page, and therefore sends $b_{3,0}^{16n}$ to v_2^{32n} in the E_∞ page.

Next we study the commutative diagram of spectral sequences:

$$\begin{tikzcd}
& tmf_* \arrow[rr, "tmf_*\iota"] \arrow[dd] & & tmf_*(A_1 \wedge DA_1) \arrow[dd] \\
\mathrm{Ext}^{*,*}_{A(2)}(S^0) \arrow[ur, Rightarrow] \arrow[rr, "\iota_*^{tmf}"] \arrow[dd] & & \mathrm{Ext}^{*,*}_{A(2)}(A_1 \wedge DA_1) \arrow[ur, Rightarrow] \arrow[dd] & \\
& k(2)_* \arrow[rr, "k(2)_*\iota"] & & k(2)_*(A_1 \wedge DA_1) \\
\mathrm{Ext}^{*,*}_{E(Q_2)}(S^0) \arrow[ur, Rightarrow] \arrow[rr, "\iota_*^{k(2)}"'] & & \mathrm{Ext}^{*,*}_{E(Q_2)}(A_1 \wedge DA_1) \arrow[ur, Rightarrow] &
\end{tikzcd} \tag{1.7}$$

Since A_1 is a type-2 spectrum, Δ^8 has a nonzero image under the composite

$$tmf_* \xrightarrow{r_*} k(2)_* \xrightarrow{k(2)_*\iota} k(2)_*(A_1 \wedge DA_1).$$

Therefore, $tmf_*\iota(\Delta^{8n}) \in tmf_*(A_1 \wedge DA_1)$ is the lift of $k(2)_*\iota(v_2^{32n})$. Similarly, at the level of E_2 pages, we see that

$$\iota_*^{tmf}(b_{3,0}^{4n}) \in \mathrm{Ext}_{A(2)}(A_1 \wedge DA_1)$$

is the lift of $\iota_*^{k(2)}(v_2^{8n})$. In Section 3, we argue that the differentials in (1.5) induce a d_2 differential and a d_3 differential in the spectral sequence

$$\mathrm{Ext}^{s,t}_{A(2)}(A_1 \wedge DA_1) \Rightarrow tmf_*(A_1 \wedge DA_1),$$

supported by $\iota_*^{tmf}(b_{3,0}^4)$ and $\iota_*^{tmf}(b_{3,0}^8)$, respectively. This means that $k(2)_*\iota(v_2^8)$ and $k(2)_*\iota(v_2^{16})$ do not lift to $tmf_*(A_1 \wedge DA_1)$, thereby establishing the "nonexistence part" of Main Theorem.

The proof of the existence part of Main Theorem can roughly be divided into two parts:

- the lifting part, where we show that
$$\iota_*^{tmf}(b_{3,0}^{4n}) \in \mathrm{Ext}_{A(2)}^{8n,48n+8n}(A_1 \wedge DA_1)$$
lifts to an element $\widetilde{v_2^{8n}} \in \mathrm{Ext}_{A}^{8n,48n+8n}(A_1 \wedge DA_1)$ under the map
$$\iota_{tmf*}\colon \mathrm{Ext}_A^{*,*}(A_1 \wedge DA_1) \to \mathrm{Ext}_{A(2)}^{*,*}(A_1 \wedge DA_1),$$
- the survival part, where we show that $\widetilde{v_2^{32n}}$ is a nonzero permanent cycle in the Adams spectral sequence
$$E_2 = \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \Rightarrow \pi_{t-s}(A_1 \wedge DA_1)$$
for all $n > 0$.

To achieve the lifting part, we use a Bousfield–Kan spectral sequence
$$E_1^{s,t,n} := \mathrm{Ext}_{A(2)}^{s-n,t}(H^*(X) \otimes \overline{A/\!/A(2)}^{\otimes n}, \mathbb{F}_2) \Rightarrow \mathrm{Ext}_A^{s,t}(H^*(X), \mathbb{F}_2),$$
which is also otherwise known as the algebraic *tmf* spectral sequence.

For the survival part of the argument, we show that the d_2 and d_3 differentials of (1.5) lift along the zigzag of spectral sequences:

(1.8)
$$\begin{tikzcd}
 & & & \pi_{t-s}(A_1 \wedge DA_1) \arrow[dd, "\iota_{tmf*}"] \\
 & & \mathrm{Ext}_A(A_1 \wedge DA_1) \arrow[ur, Rightarrow] \arrow[dd, "\iota_{tmf*}"] & \\
 & \pi_{t-s}(tmf) \arrow[rr, "\iota_*^{tmf}"'] & & tmf_{t-s}(A_1 \wedge DA_1) \\
\mathrm{Ext}_{A(2)}^{s,t}(S^0) \arrow[ur, Rightarrow] \arrow[rr, "\iota_*^{tmf}"'] & & \mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Since $\widetilde{v_2^8}$ supports a d_2 differential and $\widetilde{v_2^{16}}$ supports a d_3 differential, $\widetilde{v_2^{32}}$ can only support a d_r differential for $r \geq 4$ by the Leibniz rule. There is another d_3 differential

(1.9) $$d_3(v_2^{20}h_1) = g^6$$

in the Adams spectral sequence for $\pi_*(\mathit{tmf})$ which lifts along (1.8). The lifts of the differentials in (1.5) and (1.9), along with the multiplicative structure, allow us to deduce that there is no nonzero element in the E_4 page of

$$\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \Rightarrow \pi_{t-s}(A_1 \wedge DA_1)$$

for $s \geq 36$ and $t - s = 191$. As a result, $\widetilde{v_2^{32}}$ is a nonzero permanent cycle, which detects a 32–periodic v_2–self-map of A_1.

Notation 1.10 Let T be any subalgebra of A, for example, $E(Q_2)$, $A(2)$ or A itself. Let X be any spectrum with a map $f\colon S^0 \to X$. Throughout the paper, we will denote any nonzero image of $a \in \mathrm{Ext}_T^{*,*}(S^0)$ under the map

$$f_*\colon \mathrm{Ext}_T^{*,*}(S^0) \to \mathrm{Ext}_T^{*,*}(X)$$

using the same notation.

Use of Bruner's Ext software

We will use Bruner's Ext software [2] for two purposes. Given any $A(2)$–module M which is finitely generated as an $\mathbb{F}_2$–vector space, the program can compute the groups $\mathrm{Ext}_{A(2)}^{s,t}(M, \mathbb{F}_2)$ to the extent of identifying generators in each bidegree within a finite range, determined by the user. Since we are interested in $\mathrm{Ext}_{A(2)}^{s,t}(X)$ for finite spectra X, such as $A_1 \wedge DA_1$, whose cohomology structures as $A(2)$–modules are known, this suits our task perfectly. The second purpose is the following: As any finite spectrum X is an S^0–module, $\mathrm{Ext}_{A(2)}^{*,*}(X)$ is a module over $\mathrm{Ext}_{A(2)}^{*,*}(S^0)$. Given an element $x \in \mathrm{Ext}_{A(2)}^{s,t}(X)$, the action of $\mathrm{Ext}_{A(2)}^{*,*}(S^0)$ can be computed using the `dolifts` functionality of the software.

One should also be aware that Main Theorem is by no means a consequence of the programming output. However, parts of the proof are reduced to pure algebraic computation, which can be performed using Bruner's program.

Organization of the paper

In Section 2, we use the May spectral sequence to compute $\mathrm{Ext}_{A(2)}^{*,*}(A_1)$. In particular, we establish a vanishing line of slope $\frac{1}{5}$, which will be useful for subsequent use of the algebraic *tmf* spectral sequence. In Section 3, we use the differentials in (1.5) to conclude that A_1 cannot admit a v_2–self-map of periodicity less than 32. We then use the algebraic *tmf* spectral sequence to lift the differentials in (1.5) along the zigzag (1.8), so that in the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \Rightarrow \pi_{t-s}(A_1 \wedge DA_1),$$

we have nonzero differentials $d_2(\widetilde{v_2^8})$ and $d_3(\widetilde{v_2^{16}})$. In Section 4, we use the algebraic *tmf* spectral sequence to lift the differential (1.9) along the zigzag (1.8). Finally, in Section 5, we complete the proof of Main Theorem.

In the Appendix, we provide a description of Bruner's Ext software to familiarize the readers with its usage. A summary of the output of the Bruner's program that is needed for some of the results in Section 5 is listed in the tables from the online supplement.

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2 Computation of $\mathrm{Ext}^{s,t}_{A(2)}(A_1)$ and its vanishing line

J P May in his thesis [10] introduced a filtration of the Steenrod algebra called the May filtration, which induces a filtration of the cobar complex $C(\mathbb{F}_2, A_*, \mathbb{F}_2)$. This filtration gives a trigraded spectral sequence

$$E_1^{s,t,u} = \mathbb{F}_2[h_{i,j} : i \geq 1, j \geq 0] \Rightarrow \mathrm{Ext}^{s,t}_A(S^0), \quad |h_{i,j}| = (1, 2^j(2^i-1), 2i-1),$$

with differentials d_r of tridegree $(1, 0, 1-2r)$, which converges to the E_2 page of the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_A(S^0) \Rightarrow \pi_{t-s}(S^0).$$

The element $h_{i,j}$ corresponds to the class $[\xi_i^{2^j}]$ in the cobar complex $C(\mathbb{F}_2, A_*, \mathbb{F}_2)$. We stick to the notation introduced by Tangora in his thesis [12]. For example, $h_{1,j}$ is abbreviated by h_j. Meanwhile, there are many elements $h_{i,j}$ that are not d_1–cycles in the May spectral sequence, however, even in these cases, the Leibniz rule means that $h_{i,j}^2$ will be d_1–cycles. To get around the awkwardness of talking about $h_{i,j}^2$ in later pages of the May spectral sequence, where $h_{i,j}$ may not even exist, Tangora uses $b_{i,j}$ to denote $h_{i,j}^2$ from the May E_2 page onwards.

One can use the same May filtration on the subalgebra $A(2)$ of A, to obtain a filtration on the cobar complex $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$. Thus we get a May spectral sequence with finitely many differentials

$$\mathbb{F}_2[h_0, h_1, h_2, h_{2,0}, h_{2,1}, h_{3,0}] \Rightarrow \mathrm{Ext}^{s,t}_{A(2)}(S^0),$$

all of which have been computed using techniques of [12]. The bigraded ring $\mathrm{Ext}^{s,t}_{A(2)}(S^0)$ is the Adams E_2 page for the homotopy groups of *tmf*.

We have obtained A_1 by a series of cofibrations

$$S^1 \xrightarrow{\eta} S^0 \to C\eta, \quad C\eta \xrightarrow{2} C\eta \to Y \quad \text{and} \quad \Sigma^2 Y \xrightarrow{v_1} Y \to A_1.$$

The maps 2, η and v_1 are detected by h_0, h_1 and $h_{2,0}$, respectively, in the May spectral sequence. Using the fact that cofiber sequences induce long exact sequences of E_1 pages of the May spectral sequence, we get that the E_1 page of the May spectral sequence converging to $\mathrm{Ext}^{s,t}_{A(2)}(A_1)$ is

$$\mathbb{F}_2[h_2, h_{2,1}, h_{3,0}] \Rightarrow \mathrm{Ext}^{s,t}_{A(2)}(A_1).$$

Alternatively, using a change of rings formula, we see that there is a quasi-isomorphism of cobar complexes

$$C(\mathbb{F}_2, A(2)_*, A(1)_*) \cong C(\mathbb{F}_2, (A(2)/\!\!/A(1))_*, \mathbb{F}_2).$$

Since, $C(\mathbb{F}_2, (A(2)/\!\!/A(1))_*, \mathbb{F}_2)$ is a quotient of $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$, the May filtration on $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$ induces a filtration on $C(\mathbb{F}_2, (A(2)/\!\!/A(1))_*, \mathbb{F}_2)$. As a result, we have a May spectral sequence

$$E_1^{s,t,u}(A_1) = \mathbb{F}_2[h_2, h_{2,0}, h_{3,0}] \Rightarrow \mathrm{Ext}^{s,t}_{A(2)}(A_1) \tag{2.1}$$

that is a module over the May spectral sequence for S^0,

$$E_1^{s,t,u}(S^0) = \mathbb{F}_2[h_0, h_1, h_2, h_{2,0}, h_{2,1}, h_{3,0}] \Rightarrow \mathrm{Ext}^{s,t}_{A(2)}(S^0). \tag{2.2}$$

The d_1 differentials in (2.2) come from the coproduct on $A(2)_*$. It is well known that $d_1(h_2) = 0$, $d_1(h_{2,1}) = h_1 h_2$ and $d_1(h_{3,0}) = h_0 h_{2,1} + h_2 h_{2,0}$. Under the quotient map

$$\mathbb{F}_2[h_0, h_1, h_2, h_{2,0}, h_{2,1}, h_{3,0}] \twoheadrightarrow \mathbb{F}_2[h_2, h_{2,1}, h_{3,0}],$$

all the images of the above differentials map to zero. Therefore, there are no d_1 differentials in (2.1).

One can use Nakamura's formula to compute higher May differentials. The operations Sq_i on the cobar complex of $C(\mathbb{F}_2, A_*, \mathbb{F}_2)$, defined by $\mathrm{Sq}_i(x) = x \cup_i x + \delta x \cup_{i+1} x$ (see [11]), satisfy

$$\mathrm{Sq}_0(h_{i,j}) = h_{i,j}^2, \quad \mathrm{Sq}_0(b_{i,j}) = b_{i,j}^2 \quad \text{and} \quad \mathrm{Sq}_1(h_{i,j}) = h_{i,j+1},$$

as well as Cartan's formulas (see [11, Propositions 4.4 and 4.5])

$$\mathrm{Sq}_0(xy) = \mathrm{Sq}_0(x)\mathrm{Sq}_0(y) \quad \text{and} \quad \mathrm{Sq}_1(xy) = \mathrm{Sq}_1(x)\mathrm{Sq}_0(y) + \mathrm{Sq}_0(x)\mathrm{Sq}_1(y),$$

whenever x and y are represented by elements in appropriate pages of the May spectral sequence. In particular, we have

$$\mathrm{Sq}_1(x^2) = 0$$

for every x. The differential δ in the cobar complex $C(\mathbb{F}_2, A_*, \mathbb{F}_2)$ satisfies the relation

$$\delta \mathrm{Sq}_i = \mathrm{Sq}_{i+1}\delta \tag{2.3}$$

for $i \geq 0$ (see [11, Lemma 4.1]), and is often called Nakamura's formula in the literature.

Since the May spectral sequence (2.2) is obtained by filtering the cobar complex, Nakamura's formula (2.3) helps to find differentials in (2.2). Furthermore, because the cobar complex $C(\mathbb{F}_2, (A(2)/\!\!/A(1))_*, \mathbb{F}_2)$ is a quotient of $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$, (2.3) can also help us to find differentials in (2.1).

Lemma 2.4 *In the May spectral sequence*

$$\mathbb{F}_2[h_2, h_{2,1}, h_{3,0}] \Rightarrow \mathrm{Ext}^{s,t}_{A(2)}(A_1),$$

we have the differentials

$$d_2(b_{2,1}) = h_2^3, \quad d_3(b_{3,0}) = h_2^2 h_{2,1} \quad \text{and} \quad d_4(b_{3,0}^2) = h_2 b_{2,1}^2,$$

and the spectral sequence collapses at E_5.

Proof In the May spectral sequence for S^0 (2.2), there is a differential

$$d_2(b_{2,1}) = h_2^3$$

which implies the corresponding d_2 differential in the May spectral sequence for A_1 (2.1). The element $b_{3,0}$ is represented by the element $[\xi_3|\xi_3]$ in the cobar complex $C(\mathbb{F}_2, A(2)_*, \mathbb{F}_2)$. Since $b_{3,0} = \mathrm{Sq}_0 h_{3,0}$, we apply Nakamura's formula (2.3) to obtain

$$\begin{aligned}\mathrm{Sq}_1(d_1(h_{3,0})) &= \mathrm{Sq}_1(h_0 h_{2,1} + h_2 h_{2,0}) \\ &= h_0^2 h_{2,2} + h_1 h_{2,1}^2 + h_2^2 h_{2,1} + h_3 h_{2,0}^2 \\ &= h_2^2 h_{2,1}\end{aligned}$$

in the May spectral sequence for A_1 (2.1). Therefore, it must be the case that, in the cobar complex $C(\mathbb{F}_2, (A(2)/\!\!/A(1))_*, \mathbb{F}_2)$,

$$\delta([\xi_3|\xi_3]) = [\xi_1^4|\xi_1^4|\xi_2^2] + \text{elements of higher May filtration.}$$

As a result, in (2.1), we have

$$d_3(b_{3,0}) = h_2^2 h_{2,1}.$$

Since $\mathrm{Sq}_0(b_{3,0}) = b_{3,0}^2$, we can apply Nakamura's formula (2.3) in a similar way to obtain

$$d_4(b_{3,0}^2) = h_2 b_{2,1}^2$$

in the May spectral sequence for S^0 (2.2) as well as A_1 (2.1).

For every r, we have that $E_r^{*,*,*}(A_1)$ is a differential graded module over $E_r^{*,*,*}(S^0)$. Since $b_{3,0}^4$ is a permanent cycle in (2.2), multiplication by $b_{3,0}^4$ commutes with differentials in (2.1). The elements of $E_5^{*,*,*}(A_1)$ that are not multiples of $b_{3,0}^4$ are permanent cycles by sparseness. Therefore, the elements of $E_5^{*,*,*}(A_1)$ that are multiples of $b_{3,0}^4$ are permanent cycles as well, and thus (2.1) collapses at the E_5 page. □

In Figure 2, the solid line of slope 1 represents multiplication by h_1, while the solid line of slope $\frac{1}{3}$ represents multiplication by h_2. The element $b_{3,0}^4$ is the periodicity generator of $\mathrm{Ext}_{A(2)}^{*,*}(A_1)$ and the solid lines in that part (right) are simply a repetition of the earlier pattern (left). This matches the output of Bruner's program [2] for $\mathrm{Ext}_{A(2)}^{s,t}(A_1)$, though different models of A_1 may have different hidden extensions some of which might not be detected in the May spectral sequence.

We have thus computed the E_∞ page of the May spectral sequence converging to $\mathrm{Ext}_{A(2)}^{s,t}(A_1)$. While Bruner's program [2] shows that different spectra have different hidden extensions, we are mainly interested in a vanishing line for $\mathrm{Ext}_{A(2)}^{s,t}(A_1)$, which will not be affected by these hidden extensions.

Lemma 2.5 *The group* $\mathrm{Ext}_{A(2)}^{s,t}(A_1)$ *is zero if*

$$s > \tfrac{1}{5}(t-s) + 1,$$

and for $t - s \geq 29$, *it is zero if*

$$s > \tfrac{1}{5}(t-s).$$

In other words, there is a vanishing line

$$y = \tfrac{1}{5}x + 1.$$

Proof Of the three generators of the E_1 page, h_2 has slope $\frac{1}{3}$, $h_{2,1}$ has slope $\frac{1}{5}$ and $h_{3,0}$ has slope $\frac{1}{6}$. However, while $\mathrm{Ext}_{A(2)}^{s,t}(A_1)$ contains infinitely large powers of $h_{2,1}$ and $h_{3,0}$, it only contains powers up to 2 of h_2. Hence, the vanishing line of $\mathrm{Ext}_{A(2)}^{s,t}(A_1)$ must have slope $\frac{1}{5}$, determined by $b_{2,1}^2$. Now, since $h_2 b_{2,1}^2 = 0$, the vanishing line for stems greater than 29 is $y = \frac{1}{5}x$ and a glance at Figure 2 gives us the y–intercept of the overall vanishing line. □

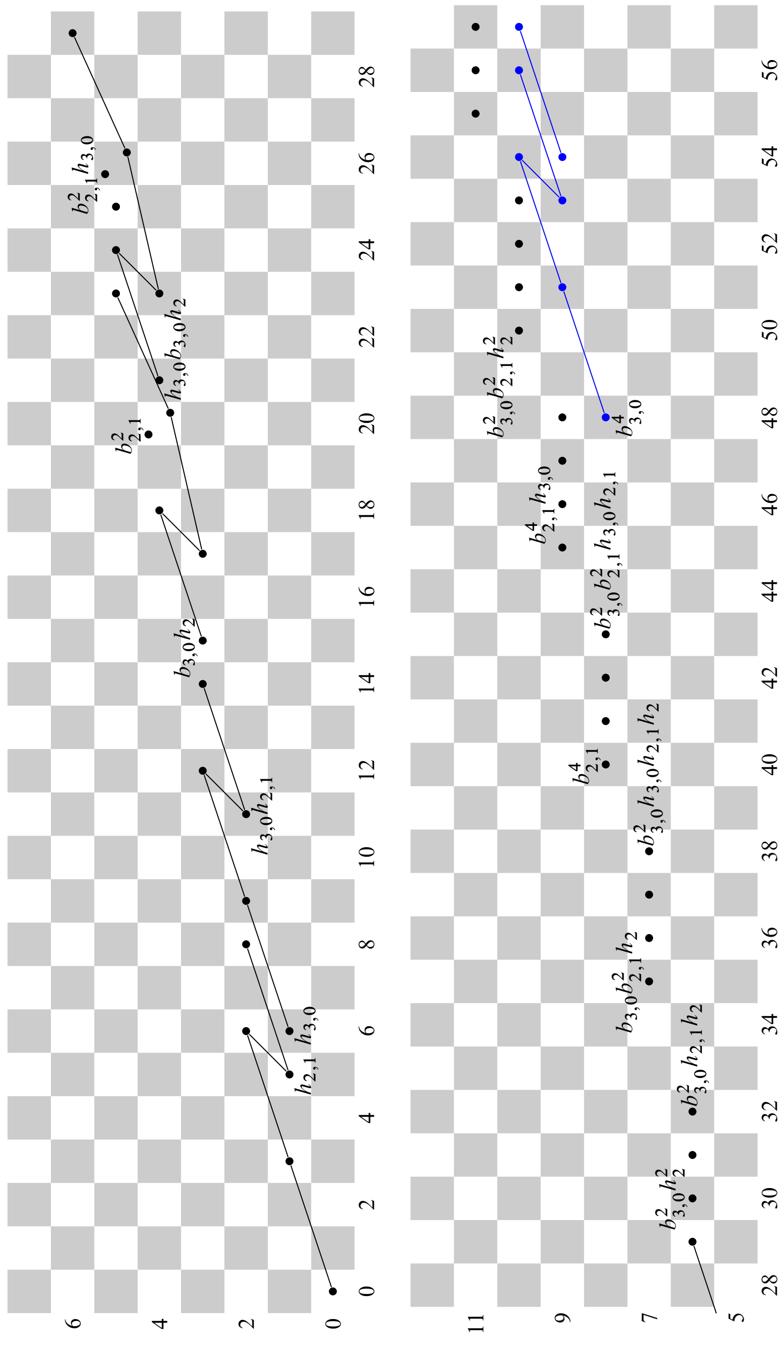


Figure 2: The E_∞ page of the May spectral sequence for $\mathrm{Ext}_{A(2)}(A_1)$

3 A d_2 and a d_3 differential

In this section, we first show that $b_{3,0}^4$ and $b_{3,0}^8$ in $\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1)$ support a d_2 and a d_3 differential, respectively. Then we show that these differentials lift to $\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1)$ under the map of spectral sequences:

$$\begin{tikzcd}
\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \arrow[r, Rightarrow] \arrow[d, "\iota_{tmf*}"] & \pi_{t-s}(A_1 \wedge DA_1) \arrow[d, "\iota_{tmf*}"] \\
\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \arrow[r, Rightarrow] & tmf_{t-s}(A_1 \wedge DA_1)
\end{tikzcd}$$

Some of the proofs in this section as well as in the subsequent sections use Bruner's program [2]. We provide the Appendix to help readers familiarize themselves with this software.

Lemma 3.1 *In the Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \Rightarrow tmf_{t-s}(A_1 \wedge DA_1),$$

we have $d_2(b_{3,0}^4) = e_0 r$ *and* $d_3(b_{3,0}^8) = wgr$.

Proof Recall the well known differentials (1.5) in the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{A(2)}^{s,t}(S^0) \Rightarrow tmf_{t-s}.$$

Using Bruner's program, we see that $e_0 r$ and wgr both have nonzero images in $\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1)$. Hence, in the map of Adams spectral sequences

$$\begin{tikzcd}
E_2^{s,t} = \mathrm{Ext}_{A(2)}^{s,t}(S^0) \arrow[r, Rightarrow] \arrow[d] & tmf_{t-s} \arrow[d] \\
E_2^{s,t} = \mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \arrow[r, Rightarrow] & tmf_{t-s}(A_1 \wedge DA_1)
\end{tikzcd}$$

we have established that in the (abusive) Notation 1.3, we have

$$\begin{aligned}
\mathrm{Ext}_{A(2)}^{s,t}(S^0) &\xrightarrow{\iota_*^{tmf}} \mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1), \\
b_{3,0}^4 &\mapsto b_{3,0}^4, \\
b_{3,0}^8 &\mapsto b_{3,0}^8, \\
e_0 r &\mapsto e_0 r, \\
wgr &\mapsto wgr.
\end{aligned}$$

Therefore, the d_2 differential of (1.5) forces a d_2 differential

$$d_2(b_{3,0}^4) = e_0 r$$

in the Adams spectral sequence for $tmf_*(A_1 \wedge DA_1)$. By the Leibniz rule, $d_2(b_{3,0}^8) = 0$ and hence $b_{3,0}^8$ is nonzero in the E_3 page. The d_3 differential in (1.5) will force a nonzero d_3 differential

$$d_3(b_{3,0}^8) = wgr$$

in the Adams spectral sequence for $tmf_*(A_1 \wedge DA_1)$ as claimed, provided the image of wgr is nonzero in the E_3 page. Thus we have to show that there does not exist a differential of the form $d_2(x) = wgr$.

Using Bruner's program [2], we check that $wgr \in \mathrm{Ext}_{A(2)}^{19,95+19}(S^0)$ maps nontrivially to $\mathrm{Ext}_{A(2)}^{19,95+19}(A_1)$. Therefore if we have $d_2(x) = wgr$ in

$$\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \Rightarrow tmf_{t-s}(A_1 \wedge DA_1),$$

then x must map to a nonzero element, say x', under the map

$$j_*\colon \mathrm{Ext}_{A(2)}^{17,96+17}(A_1 \wedge DA_1) \to \mathrm{Ext}_{A(2)}^{17,96+17}(A_1),$$

and we will have $d_2(x') = wgr$ in

$$\mathrm{Ext}_{A(2)}^{s,t}(A_1) \Rightarrow tmf_{t-s}(A_1).$$

There is exactly one generator of $\mathrm{Ext}_{A(2)}^{17,96+17}(A_1)$, and that generator is $b_{3,0}^4 \cdot y$ under the pairing

$$\mathrm{Ext}_{A(2)}^{8,48+8}(S^0) \otimes \mathrm{Ext}_{A(2)}^{9,48+9}(A_1) \to \mathrm{Ext}_{A(2)}^{17,96+17}(A_1).$$

It is clear that $d_2(y) = 0$ as $\mathrm{Ext}_{A(2)}^{11,47+11}(A_1) = 0$; see Figure 2. Thus using the Leibniz rule, we see that

$$d_2(b_{3,0}^4 y) = e_0 r \cdot y.$$

Using [2], we check that $e_0 r \cdot y = 0$. Therefore, wgr is nonzero in the E_3 page of the spectral sequence

$$\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \Rightarrow tmf_{t-s}(A_1 \wedge DA_1),$$

and therefore

$$d_3(b_{3,0}^8) = wgr$$

in this spectral sequence. □

The fact that $v_2^{16} \in k(2)_*(A_1 \wedge DA_1)$ does not lift to $tmf_*(A_1 \wedge DA_1)$ implies that $v_2^{2^k} \in k(2)_*(A_1 \wedge DA_1)$ for $1 \leq k \leq 4$ does not lift to $tmf_*(A_1 \wedge DA_1)$. Indeed, suppose that for $k = 0, 1, 2$ or 3 the element $v_2^{2^k} \in k(2)_*(A_1 \wedge DA_1)$ lifts to an element

$$x \in tmf_*(A_1 \wedge DA_1),$$

then $x^{2^{4-k}}$ would be a lift of v_2^{16} as $A_1 \wedge DA_1$ is a ring spectrum. This would contradict Lemma 3.1. Since the unit map for $k(2)$ factors through the unit map of *tmf* (1.6), Lemma 3.1 implies the following:

Theorem 3.2 *The spectrum A_1 cannot admit a v_2–self-map of periodicity 16 or less.*

Next we describe an algebraic resolution which will allow us to lift the d_2 differential and the d_3 differential of Lemma 3.1 to the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \Rightarrow \pi_{t-s}(A_1 \wedge DA_1).$$

We will briefly recall the resolution described in [1, Section 5], and how it is used to lift elements of Ext groups over $A(2)$ to Ext groups over A. Consider the A–module

$$A/\!\!/A(2) := A \otimes_{A(2)} \mathbb{F}_2,$$

and denote by $\overline{A/\!\!/A(2)}$ the kernel of the augmentation map

$$A/\!\!/A(2) \to \mathbb{F}_2.$$

When we consider the triangulated structure of the derived category of A–modules, we get maps

$$A/\!\!/A(2) \to \mathbb{F}_2 \to \overline{A/\!\!/A(2)}[1]$$

and a resulting diagram

$$\begin{tikzcd}
\mathbb{F}_2 \arrow[r] & \overline{A/\!\!/A(2)}[1] \arrow[r] & \overline{A/\!\!/A(2)}^{\otimes 2}[2] \arrow[r] & \cdots \\
A/\!\!/A(2) \arrow[u] & A/\!\!/A(2) \otimes \overline{A/\!\!/A(2)}[1] \arrow[u] & A/\!\!/A(2) \otimes \overline{A/\!\!/A(2)}^{\otimes 2}[2] \arrow[u] &
\end{tikzcd}$$

to which we shall apply the functor $\mathrm{Ext}_A^{s,t}(H^*(X) \otimes -, \mathbb{F}_2)$ to get a spectral sequence, which we shall refer to as the algebraic *tmf* spectral sequence to reflect the fact that $A/\!\!/A(2)$ is the cohomology of *tmf*. This spectral sequence will be trigraded, with E_1 page

$$\begin{aligned} E_1^{s,t,n} &= \mathrm{Ext}_A^{s,t}(H^*(X) \otimes A/\!\!/A(2) \otimes \overline{A/\!\!/A(2)}^{\otimes n}[n], \mathbb{F}_2) \\ &\cong \mathrm{Ext}_{A(2)}^{s-n,t}(H^*(X) \otimes \overline{A/\!\!/A(2)}^{\otimes n}, \mathbb{F}_2), \end{aligned}$$

which converges to

$$\mathrm{Ext}_A^{s,t}(H^*(X), \mathbb{F}_2).$$

For any element in the algebraic *tmf* spectral sequence in tridegree (s, t, n), we will refer to s as its Adams filtration, t as the internal degree and n as the algebraic *tmf* filtration. The differential d_r has tridegree $(1, 0, r)$. It is shown in [4] that

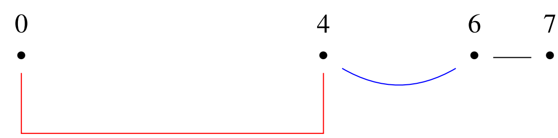
$$A/\!/A(2) \cong \bigoplus_{i \geq 0} H^*(\Sigma^{8i} bo_i),$$

where bo_i denotes the i^{th} bo–Brown–Gitler spectrum of [5]. As a result the E_1 page of the algebraic *tmf* spectral sequence simplifies to

$$E_1^{s,t,n} = \bigoplus_{i_1,\dots,i_n \geq 1} \mathrm{Ext}_{A(2)}^{s-n,t-8(i_1+\cdots+i_n)}(X \wedge bo_{i_1} \wedge \cdots \wedge bo_{i_n}) \Rightarrow \mathrm{Ext}_A^{s,t}(X).$$

We will attempt to exploit the relative sparseness of the E_1 page, especially its vanishing line properties, in the case when $X = A_1 \wedge DA_1$.

Remark 3.3 (the cellular structure of bo–Brown–Gitler spectra) The spectrum bo_0 is the sphere spectrum. The cohomology of the spectrum bo_1 as a module over the Steenrod algebra can be described through the following picture, with the generators labeled by cohomological degree:



where the straight line, curved line and square bracket describe the actions of Sq^1, Sq^2 and Sq^4, respectively. Note that the 4–skeleton of bo_1 is $C\nu$. Indeed, the bo_i fit together to form the following cofiber sequence

$$bo_{i-1} \to bo_i \to \Sigma^{4i} B(i),$$

where $B(i)$ is the i^{th} integral Brown–Gitler spectrum as described in [5]. Therefore for every $i \geq 1$, the 7–skeleton of bo_i is bo_1 and the 4–skeleton of bo_i is $C\nu$.

One can compute $\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1 \wedge bo_i)$ from $\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1)$ using the Atiyah–Hirzebruch spectral sequence or with Bruner's program [2].

Lemma 3.4 *The group*

$$\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \cdots \wedge bo_{i_n})$$

is zero if $s > \frac{1}{5}((t-s)+6)$.

Proof We showed in Lemma 2.5 that $\mathrm{Ext}^{s,t}_{A(2)}(\mathrm{A}_1)$ has a vanishing line $s = \frac{1}{5}(t-s)$ for $t-s \geq 30$ and a vanishing line of $s = \frac{1}{5}(t-s)+1$ overall. The only generator of $\mathrm{Ext}^{s,t}_{A(2)}(A_1)$ with a slope greater than $\frac{1}{5}$ is h_2, so if we kill off h_2 by considering $\mathrm{Ext}^{s,t}_{A(2)}(A_1 \wedge C\nu)$ then the vanishing line is precisely $s = \frac{1}{5}(t-s)$.

As we mentioned in Remark 3.3, the 4–skeleton of any bo_i is $C\nu$ and the next cell is in dimension 6. So we can build bo_i by attaching finitely many cells of dimension at least 6 to $C\nu$. Hence by using the Atiyah–Hirzebruch spectral sequence and the fact that $\frac{1}{5}(x-6)+1 < \frac{1}{5}x$, one can see that the vanishing line of $A_1 \wedge bo_i$ is $s = \frac{1}{5}(t-s)$. One can build $A_1 \wedge bo_{i_1} \wedge \cdots \wedge bo_{i_n}$ from $A_1 \wedge bo_{i_1}$, iteratively using cofiber sequences, which depend on the cell structure of $bo_{i_2} \wedge \cdots \wedge bo_{i_n}$. Since we have already established that $\mathrm{Ext}^{s,t}_{A(2)}(A_1 \wedge bo_{i_1})$ has vanishing line $s = \frac{1}{5}(t-s)$ and that $bo_{i_2} \wedge \cdots \wedge bo_{i_n}$ is a connected spectrum, we conclude, using the Atiyah–Hirzebruch spectral sequence, that the vanishing line for $\mathrm{Ext}^{s,t}_{A(2)}(A_1 \wedge bo_{i_1} \wedge \cdots \wedge bo_{i_n})$ is $s = \frac{1}{5}(t-s)$.

However, DA_1 has cells in negative dimension, in fact the bottom cell is in dimension -6. Again by using the Atiyah–Hirzebruch spectral sequence, one concludes that the vanishing line for $\mathrm{Ext}^{s,t}_{A(2)}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \cdots \wedge bo_{i_n})$ is

$$s = \tfrac{1}{5}(t-s+6)$$

for any $i_k \geq 1$, completing the proof. □

Corollary 3.5 *The group* $\mathrm{Ext}^{s,t}_{A}(A_1 \wedge DA_1)$ *is zero if*

$$s > \tfrac{1}{5}(t-s) + \tfrac{11}{5},$$

and for $t-s \geq 23$, *it is zero if*

$$s > \tfrac{1}{5}(t-s) + \tfrac{6}{5}.$$

The result is a straightforward consequence of Lemma 2.5, Lemma 3.4 and the algebraic *tmf* spectral sequence.

Lemma 3.6 *The element*

$$b_{3,0}^4 \in \mathrm{Ext}^{8,48+8}_{A(2)}(A_1 \wedge DA_1)$$

lifts to an element $\widetilde{v_2^8}$ *under the map*

$$\iota_{tmf*}\colon \mathrm{Ext}^{8,48+8}_{A}(A_1 \wedge DA_1) \to \mathrm{Ext}^{8,48+8}_{A(2)}(A_1 \wedge DA_1).$$

Proof Consider the algebraic *tmf* spectral sequence:

$$\begin{array}{c} E_1^{s,t,n} = \displaystyle\bigoplus_{i_1 \geq 1, \dots, i_n \geq 1} \mathrm{Ext}_{A(2)}^{s-n, t-8(i_1+\cdots+i_n)}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \dots bo_{i_n}) \\ \Downarrow \\ \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \end{array}$$

The element $b_{3,0}^4$ has tridegree $(s,t,n) = (8, 48+8, 0) = (8, 56, 0)$ in the above spectral sequence. The element $d_n(b_{3,0}^4)$ has tridegree $(9, 56, n)$ and hence belongs to

$$\mathrm{Ext}_{A(2)}^{9-n, 56-8(i_1+\cdots+i_n)}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \cdots \wedge bo_{i_n})$$

for some $(i_1, \dots, i_n)$ where $i_k \geq 1$. We will show that the above group is zero for all $n \geq 1$ and for all tuples $(i_1, \dots, i_n)$ where $i_k \geq 1$.

By Lemma 3.4 the above group is zero if

$$\tfrac{1}{5}(56 - 8(i_1 + \cdots + i_n) - 9 + n + 6) < 9 - n, \tag{3.7}$$

which is trivially satisfied for $n > 4$.

For $n = 1$, (3.7) becomes

$$\tfrac{1}{5}(54 - 8i_1) < 8,$$

thus $i_1 > 1$, so it suffices to verify that

$$\mathrm{Ext}_{A(2)}^{8,48}(A_1 \wedge DA_1 \wedge bo_1) = 0.$$

For $n = 2$, (3.7) becomes

$$\tfrac{1}{5}(55 - 8(i_1 + i_2)) < 7,$$

thus $i_1 + i_2 > 2$, so it suffices to verify that

$$\mathrm{Ext}_{A(2)}^{7,40}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1) = 0.$$

For $n = 3$, (3.7) becomes

$$\tfrac{1}{5}(56 - 8(i_1 + i_2 + i_3)) < 6,$$

thus $i_1 + i_2 + i_3 > 3$, so it suffices to verify that

$$\mathrm{Ext}_{A(2)}^{6,32}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1 \wedge bo_1) = 0.$$

For $n = 4$, (3.7) becomes

$$\tfrac{1}{5}(57 - 8(i_1 + i_2 + i_3 + i_4)) < 5,$$

thus $i_1+i_2+i_3+i_4>4$, so it suffices to verify that

$$\mathrm{Ext}^{5,24}_{A(2)}(A_1\wedge DA_1\wedge bo_1\wedge bo_1\wedge bo_1\wedge bo_1)=0.$$

For all four models of A_1, Bruner's program [2] shows that all the groups we expected to be zero are in fact zero. □

Corollary 3.8 *For all $n\in\mathbb{N}$, the elements $b^{4n}_{3,0}\in\mathrm{Ext}^{8n,48n+8n}_{A(2)}(A_1\wedge DA_1)$ lift to an element $\widetilde{v_2^{8n}}\in\mathrm{Ext}^{8n,48n+8n}_{A}(A_1\wedge DA_1)$ under the map ι_{tmf*}.*

Proof Since $A_1\wedge DA_1$ is a ring spectrum, it follows that the map

$$\iota_{tmf*}\colon \mathrm{Ext}^{s,t}_A(A_1\wedge DA_1)\to\mathrm{Ext}^{s,t}_{A(2)}(A_1\wedge DA_1)$$

is a map of algebras. By Lemma 3.6, $b^4_{3,0}$ lifts and thus $b^{4n}_{3,0}$ lifts for every $n\in\mathbb{N}$. □

Remark 3.9 The lift of $\widetilde{v_2^{8n}}$ in Corollary 3.8 may not be unique. The indeterminacy in the choice of $\widetilde{v_2^{8n}}$ consists of elements of higher algebraic *tmf* filtration.

Lemma 3.10 *In the Adams spectral sequence*

$$E_2^{s,t}=\mathrm{Ext}^{s,t}_A(A_1\wedge DA_1)\Rightarrow\pi_{t-s}(A_1\wedge DA_1),$$

there is a d_2–differential

$$d_2(\widetilde{v_2^8})=e_0r+R$$

and a d_3–differential

$$d_3(\widetilde{v_2^{16}})=wgr+S$$

for some R and S in algebraic tmf filtration greater than zero.

Proof Recall that e_0r and wgr are elements in $\mathrm{Ext}^{*,*}_A(S^0)$ (see [12]), which maps nontrivially (see Lemma 3.1) under the composite

$$\mathrm{Ext}^{*,*}_A(S^0)\to\mathrm{Ext}^{*,*}_{A(2)}(S^0)\to\mathrm{Ext}^{*,*}_{A(2)}(A_1\wedge DA_1).$$

Therefore, by inspecting the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^{*,*}_A(S^0) \arrow[r, "\iota_*"] \arrow[d, "\iota_{tmf*}"'] & \mathrm{Ext}^{*,*}_A(A_1\wedge DA_1) \arrow[d, "\iota_{tmf*}"] \\
\mathrm{Ext}^{*,*}_{A(2)}(S^0) \arrow[r, "\iota_*^{tmf}"] & \mathrm{Ext}^{*,*}_{A(2)}(A_1\wedge DA_1)
\end{tikzcd} \tag{3.11}$$

we see that e_0r and wgr are nonzero image in $\mathrm{Ext}^{*,*}_A(A_1\wedge DA_1)$. Since $\widetilde{v_2^8}$ and $\widetilde{v_2^{16}}$ are lifts of $b^4_{3,0}$ and $b^8_{3,0}$, respectively, the differentials of Lemma 3.1 force the differentials as claimed. □

4 Another d_3 differential

The goal of this section is to lift the d_3 differential (1.9) in the spectral sequence for tmf_* to a d_3 differential

$$d_3(\widetilde{v_2^{20}h_1}) = g^6$$

in the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \Rightarrow \pi_*(A_1 \wedge DA_1)$$

along the zigzag (1.8).

The element $g \in \mathrm{Ext}_A^{4,20+4}(S^0)$ is Tangora's name [12] for the element detected by $b_{2,1}^2$ in the May spectral sequence

$$\mathbb{F}_2[h_{i,j} : i > 0, j \geq 0] \Rightarrow \mathrm{Ext}_A^{s,t}(S^0).$$

In the literature, the same name is adopted for its image in $\mathrm{Ext}_{A(2)}^{4,20+4}(S^0)$.

Lemma 4.1 *In the Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \Rightarrow tmf_{t-s}(A_1 \wedge DA_1),$$

the element g^6 is hit by a d_3 differential

$$d_3(v_2^{20}h_1) = g^6.$$

Proof From the calculation in Lemma 2.4, it is clear that $g^6 = b_{2,1}^{12}$ has a nonzero image in $\mathrm{Ext}_{A(2)}^{24,120+24}(A_1)$. Since we have a factorization of maps

$$\mathrm{Ext}_{A(2)}^{24,120+24}(S^0) \to \mathrm{Ext}_{A(2)}^{24,120+24}(A_1 \wedge DA_1) \to \mathrm{Ext}_{A(2)}^{24,120+24}(A_1),$$

we have that g^6 must also be nonzero in the Adams E_2 page for $tmf_*(A_1 \wedge DA_1)$.

To show that it is also nonzero in the Adams E_3 page, we must exclude the possibility that $g^6 \in \mathrm{Ext}_{A(2)}^{24,120+24}(A_1 \wedge DA_1)$ might be hit by a d_2 differential

$$d_2(\hat{x}) = g^6$$

for some elements $\hat{x} \in \mathrm{Ext}_{A(2)}^{22,121+22}(A_1 \wedge DA_1)$. In such a case, $\hat{x}$ would have to map to a nonzero element $x \in \mathrm{Ext}_{A(2)}^{22,121+22}(A_1)$ and there would exist a differential

$$d_2(x) = g^6 \tag{4.2}$$

in the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A(2)}(A_1) \Rightarrow \mathit{tmf}_{t-s}(A_1)$$

as $g^6 \neq 0 \in \mathrm{Ext}^{24,120+24}_{A(2)}(A_1)$. From the calculations of Lemma 2.4, there is exactly one possible nonzero $x \in \mathrm{Ext}^{22,121+22}_{A(2)}(A_1)$. Using Bruner's program [2] (see (A.2)) we see that this x is a multiple of $gb^4_{3,0}$ under the pairing

$$\mathrm{Ext}^{12,68+12}_{A(2)}(S^0) \otimes \mathrm{Ext}^{10,53+10}_{A(2)}(A_1) \to \mathrm{Ext}^{22,121+22}_{A(2)}(A_1), \qquad gb^4_{3,0} \otimes \bar{x} \mapsto x.$$

Clearly $d_2(\bar{x}) = 0$ as $\mathrm{Ext}^{12,52+12}_{A(2)}(A_1) = 0$, and hence by the Leibniz rule, we get

$$d_2(x) = ge_0 r \cdot \bar{x}.$$

However, $ge_0 r = 0$ in $\mathrm{Ext}^{14,67+14}_{A(2)}(S^0)$, therefore $d_2(x) = 0$. It follows that the d_2 differential in (4.2) cannot exist and g^6 is a nonzero element in the E_3 page of the spectral sequence

$$\mathrm{Ext}^{s,t}_{A(2)}(A_1 \wedge DA_1) \Rightarrow \mathit{tmf}_{t-s}(A_1 \wedge DA_1).$$

Thus the d_3 differential of (1.9) in Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{A(2)}(S^0) \Rightarrow \mathit{tmf}_{t-s}$$

forces the d_3 differential

$$d_3(v_2^{20} h_1) = g^6$$

in the Adams spectral sequence for $\mathit{tmf}_*(A_1 \wedge DA_1)$ as claimed. □

Our next goal is to lift this d_3 differential to the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{A}(A_1 \wedge DA_1) \Rightarrow \pi_{t-s}(A_1 \wedge DA_1).$$

The main tool at our disposal is the algebraic *tmf* spectral sequence, described in Section 3.

Lemma 4.3 *The elements g^6 and $v_2^{20} h_1$ lift to $\mathrm{Ext}^{s,t}_{A}(A_1 \wedge DA_1)$ under the map*

$$\iota_{\mathit{tmf}*}\colon \mathrm{Ext}^{s,t}_{A}(A_1 \wedge DA_1) \to \mathrm{Ext}^{s,t}_{A(2)}(A_1 \wedge DA_1).$$

Proof In the proof of Lemma 4.1, we showed that g^6 is a nonzero element if $\mathrm{Ext}^{24,120+24}_{A(2)}(A_1 \wedge DA_1)$. Since g^6 is an element of $\mathrm{Ext}^{24,120+24}_{A}(S^0)$, from the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}_A^{*,*}(S^0) \arrow[r, "\iota_*"] \arrow[d, "\iota_{tmf*}"'] & \mathrm{Ext}_A^{*,*}(A_1 \wedge DA_1) \arrow[d, "\iota_{tmf*}"] \\
\mathrm{Ext}_{A(2)}^{*,*}(S^0) \arrow[r, "\iota_*^{tmf}"] & \mathrm{Ext}_{A(2)}^{*,*}(A_1 \wedge DA_1)
\end{tikzcd}$$

it easily follows that g^6 lifts to $\mathrm{Ext}_A^{24,120+24}(A_1 \wedge DA_1)$ under the map ι_{tmf*}.

It is known that $v_2^{20}h_1 = b_{3,0}^8 \cdot v_2^4 h_1$ under the pairing

$$\mathrm{Ext}_{A(2)}^{16,96+16}(S^0) \otimes \mathrm{Ext}_{A(2)}^{5,25+5}(S^0) \to \mathrm{Ext}_{A(2)}^{21,121+21}(S^0), \qquad b_{3,0}^8 \otimes v_2^4 h_1 \mapsto v_2^{20} h_1.$$

Therefore the same relation $v_2^{20}h_1 = b_{3,0}^8 \cdot v_2^4 h_1$ is true in $\mathrm{Ext}_{A(2)}^{21,121+21}(A_1 \wedge DA_1)$ as

$$\iota_*^{tmf}\colon \mathrm{Ext}_{A(2)}^{s,t}(S^0) \to \mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1)$$

is a map of algebras. From Corollary 3.8, we already know that $b_{3,0}^8$ lifts to

$$\widetilde{v_2^{16}} \in \mathrm{Ext}_A^{16,96+16}(A_1 \wedge DA_1).$$

Using the algebraic *tmf* spectral sequence

$$E_1^{s,t,n} = \bigoplus_{i_1 \geq 1, \dots, i_n \geq 1} \mathrm{Ext}_{A(2)}^{s-n, t-8(i_1+\cdots+i_n)}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \cdots \wedge bo_{i_n}) \Longrightarrow \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1)$$

and the vanishing lines established in Lemma 3.4, we see $v_2^4 h_1 \in \mathrm{Ext}_{A(2)}^{5,25+5}(A_1 \wedge DA_1)$ also has a lift

$$\widetilde{v_2^4 h_1} \in \mathrm{Ext}_A^{5,25+5}(A_1 \wedge DA_1).$$

Therefore,

$$\widetilde{v_2^{16}} \cdot \widetilde{v_2^4 h_1} \in \mathrm{Ext}_A^{21,121+21}(A_1 \wedge DA_1)$$

is a lift of $v_2^{20}h_1$, as

$$\iota_{tmf*}\colon \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \to \mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1)$$

is a map of algebras. □

We will denote any lift of $v_2^{20}h_1$ by $\widetilde{v_2^{20}h_1} \in \mathrm{Ext}_A^{21,121+21}(A_1 \wedge DA_1)$. One should be aware that the choice of $\widetilde{v_2^{20}h_1}$ is not unique. The indeterminacy in the choice of $\widetilde{v_2^{20}h_1}$ consists of elements of higher algebraic *tmf* filtration. This does not cause problems later in the paper because of the following technical lemma.

Lemma 4.4 *Suppose that we have a nontrivial differential $d_r(x) = y$ in the Adams spectral sequence for a spectrum X,*

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(X) \Rightarrow \pi_{t-s}(X).$$

If x has algebraic tmf filtration greater than zero, then so does y.

Proof If the algebraic *tmf* filtration of x is greater than zero then the map of spectral sequences

$$\begin{tikzcd}
\mathrm{Ext}_A^{s,t}(X) \arrow[r, Rightarrow] \arrow[d, "\iota_{tmf*}"'] & \pi_{t-s}(X) \arrow[d, "\iota_{tmf*}"] \\
\mathrm{Ext}_{A(2)}^{s,t}(X) \arrow[r, Rightarrow] & tmf_{t-s}(X)
\end{tikzcd}$$

sends x to 0. Therefore,

$$\begin{aligned}
\iota_{tmf*}(y) &= \iota_{tmf*}(d_r(x)) \\
&= d_r(\iota_{tmf*}(x)) \\
&= 0,
\end{aligned}$$

which means that the algebraic *tmf* filtration of y is greater than zero. □

Lemma 4.5 *In the Adams spectral sequence*

$$\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \Rightarrow \pi_{t-s}(A_1 \wedge DA_1),$$

there exists a d_3 differential

$$d_3(\widetilde{v_2^{20}h_1}) = g^6.$$

Proof It is easy to check that Lemma 4.1, along with the map of Adams spectral sequences

$$\begin{tikzcd}
E_2^{s,t} = \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \arrow[r, Rightarrow] \arrow[d] & \pi_{t-s}(A_1 \wedge DA_1) \arrow[d] \\
E_2^{s,t} = \mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \arrow[r, Rightarrow] & tmf_{t-s}(A_1 \wedge DA_1)
\end{tikzcd}$$

induced by ι_{tmf}, forces a d_3 differential (also see Remark 4.7)

$$d_3(\widetilde{v_2^{20}h_1}) = g^6 + R, \tag{4.6}$$

where R is an element of algebraic *tmf* filtration greater than zero. Studying the algebraic *tmf* spectral sequence for $A_1 \wedge DA_1$, using the vanishing lines of Lemma 3.4 and using the fact that (checked using Bruner's program)

$$\mathrm{Ext}_{A(2)}^{23,113+23}(A_1 \wedge DA_1 \wedge bo_1) = 0 \quad \text{and} \quad \mathrm{Ext}_{A(2)}^{22,106+22}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1) = 0,$$

we conclude that R is in fact zero. □

Remark 4.7 Lemma 4.4 in particular eliminates the possibility of a differential of the form

$$d_r(S) = g^6,$$

where S is in the higher algebraic *tmf* filtration. This is needed for the conclusion of (4.6).

5 Proof of Main Theorem

Recall from Corollary 3.8 that there are candidates in the E_2 page of the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \Rightarrow \pi_{t-s}(A_1 \wedge DA_1), \tag{5.1}$$

denoted by $\widetilde{v_2^{8n}}$, that can detect an $8n$–periodic v_2–self-map. Since $\widetilde{v_2^8}$ supports a d_2 differential and $\widetilde{v_2^{16}}$ supports a d_3 differential (see Lemma 3.10), by the Leibniz formula $\widetilde{v_2^{32}}$ is a nonzero d_3–cycle. The only way $\widetilde{v_2^{32}}$ can fail to detect a 32–periodic v_2–self-map is by supporting a nonzero d_r differential for $r \geq 4$ in the Adams spectral sequence (5.1). So we look for candidates in the E_2 page of (5.1) that can potentially be the target of a nonzero d_r differential supported by $\widetilde{v_2^{32}}$ for $r \geq 4$. Such elements will live in $\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1)$ with $t - s = 191$ and Adams filtration $s \geq 36$. We use the algebraic *tmf* spectral sequence to detect such candidates. The goal of this section is to argue that any such candidate is either zero or not present in the E_4 page of the spectral sequence (5.1).

From Section 3, we recall the algebraic *tmf* spectral sequence:

$$\begin{array}{c} E_1^{s,t,n} = \bigoplus_{i_1,\ldots,i_n \geq 1} \mathrm{Ext}_{A(2)}^{s-n,t-8(i_1+\cdots+i_n)}(bo_{i_1} \wedge \cdots \wedge bo_{i_n} \wedge A_1 \wedge DA_1) \\ \Downarrow \\ \mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1) \end{array}$$

An easy consequence of the vanishing line established in Lemma 3.4 is the following.

Lemma 5.2 *The only potential contributors to* $\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1)$ *for* $t-s=191$ *and* $s \geq 36$ *come from the following summands of the algebraic tmf* E_1 *page:*

$$\begin{aligned}\mathrm{Ext}_{A(2)}^{s,t}(A_1 \wedge DA_1) \oplus & \bigoplus_{1\leq i\leq 3} \mathrm{Ext}_{A(2)}^{s-1,t-8i}(A_1 \wedge DA_1 \wedge bo_i)\\ \oplus & \bigoplus_{1\leq i\leq 2} \mathrm{Ext}_{A(2)}^{s-2,t-8-8i}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_i)\\ \oplus & \,\mathrm{Ext}_{A(2)}^{s-3,t-24}(A_1 \wedge DA_1 \wedge bo_1 \wedge bo_1 \wedge bo_1).\end{aligned}$$

While the result holds for all models of A_1, the computations will be slightly different for different models, and so we will treat these models separately. Since $A_1[00]$ and $A_1[11]$ are Spanier–Whitehead dual to each other, we can treat the cases of $A_1[00]$ and $A_1[11]$ as one case. We will then have to treat the cases of the self-dual spectra $A_1[01]$ and $A_1[10]$ separately. The completeness of the tables in this section will be justified by the more detailed tables in the online supplement.

Notation 5.3 The elements of $E_1^{s,t,n}$, the E_1 page of the algebraic *tmf* spectral sequence for $A_1 \wedge DA_1$, which are nonzero permanent cycles, will detect nonzero elements of $\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1)$. Therefore we place an element $x \in E_1^{s,t,n}$ in bidegree $(t-s-n, s+n)$. Thus the elements that may contribute to the same bidegree of $\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1)$ are placed together. With this arrangement any differential in the algebraic *tmf* spectral sequence will look like Adams d_1 differential. The generators of

$$E_1^{s,t,n} = \bigoplus_{i_1,\dots,i_n\geq 1} \mathrm{Ext}_{A(2)}^{s-n,t-8(i_1+\cdots+i_n)}(A_1 \wedge DA_1 \wedge bo_{i_1} \wedge \cdots \wedge bo_{i_n})$$

will be denoted by dots in the following manner (recall that $bo_0 = S^0$):

- elements with $n=0$ are denoted by a $\bullet$,
- elements with $n=1, i_1=1$ are denoted by a $\circ^1$,
- elements with $n=1, i_1=2$ are denoted by a $\circ^2$,
- elements with $n=2, i_1=1, i_2=1$ are denoted by a $\odot$,
- and N/A stands for "not applicable," ie coordinates of the table which are irrelevant to our arguments.

5.1 The case $A_1 = A_1[00]$ or $A_1 = A_1[11]$

We begin by laying out, in Table 1, the elements of the E_1 page of the algebraic *tmf* spectral sequence, in Notation 5.3. The table makes it clear that all elements

$s \setminus t-s$	189	190	191
40	0	0	0
39	0	$\langle \bullet\bullet \rangle := Y^0_{39}$	$\langle \bullet\bullet\bullet \rangle := X^0_{39}$
38	N/A	$\langle \bullet\bullet\bullet\bullet\bullet \rangle := Y^0_{38}$	$\langle \bullet\bullet\bullet \rangle := X^0_{38}$
37	N/A	$\langle \bullet\bullet\bullet\bullet\bullet \rangle$ $\langle \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1 \circ^1 \rangle$	$\langle \bullet\bullet\bullet\bullet\bullet \rangle := X^0_{37}$ $\langle \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1 \circ^1 \rangle := X^1_{37}$
36	N/A	N/A	$\langle \bullet\bullet\bullet \rangle := X^0_{36}$ $\langle \circ^1 \circ^1 \rangle := X^1_{36}$ $\langle \odot\, \odot\, \odot\, \odot\, \odot \odot \rangle := X^{1,1}_{36}$

Table 1: E_1 page of the algebraic *tmf* spectral sequence for $\mathrm{Ext}^{s,t}_A(A_1 \wedge DA_1)$, where $A_1 = A_1[00]$ or $A_1 = A_1[11]$, stem 189–191.

$s \setminus t-s$	70	71
15	$\langle \bullet\bullet \rangle = g^{-6} Y^0_{39}$	$\langle \bullet\bullet\bullet \rangle = g^{-6} X^0_{39}$
14	$\langle \bullet\bullet\bullet\bullet\bullet \rangle = g^{-6} Y^0_{38}$	$\langle \bullet\bullet\bullet\bullet \rangle = g^{-6} X^0_{38}$
13	$\langle \bullet\bullet\bullet\bullet\bullet \rangle$ $\langle \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1 \circ^1 \rangle$	$\langle \bullet\bullet\bullet\bullet\bullet\bullet\bullet \rangle = g^{-6} X^0_{37}$ $\langle \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1 \circ^1 \rangle = g^{-6} X^1_{37}$
12	N/A	$\langle \circ^1\, \circ^1\, \circ^1\, \circ^1\, \circ^1 \circ^1 \rangle = g^{-6} X^1_{36}$ $\langle \odot\, \odot\, \odot\, \odot\, \odot \odot \rangle = g^{-6} X^{1,1}_{36}$

Table 2: E_1 page of the algebraic *tmf* spectral sequence for $\mathrm{Ext}^{s,t}_A(A_1 \wedge DA_1)$, where $A_1 = A_1[00]$ or $A_1 = A_1[11]$, stem 70–71.

with $t-s = 191$, with the possible exception of those in X^0_{36}, are permanent cycles in the algebraic *tmf* spectral sequence. Our goal is to show that every linear combination of elements in $X^{i_1,\dots,i_n}_s$ is either absent or zero in the E_4 page of the Adams spectral sequence. Using Bruner's program (for details see Tables 1–4 from the online supplement), we observe that a lot of these elements are multiples of g^6 in the E_1 page of the algebraic *tmf* spectral sequence, which we record in Table 2.

Lemma 5.4 *Every element of*

$$X^0_{39} \oplus X^0_{38} \oplus X^0_{37} \oplus X^1_{37} \oplus X^1_{36} \oplus X^{1,1}_{36}$$

is present in the Adams E_2 page, but is either zero or absent in the Adams E_4 page.

Proof Tables 1–4 of the online supplement make clear that multiplication by g^6 surjects onto $X^0_{39} \oplus X^0_{38} \oplus X^0_{37} \oplus X^1_{37} \oplus X^1_{36} \oplus X^{1,1}_{36}$. Notice that for any

$$x = g^6 \cdot y \in X^0_{39} \oplus X^0_{38} \oplus X^0_{37} \oplus X^1_{37} \oplus X^1_{36} \oplus X^{1,1}_{36},$$

both x and y are nonzero permanent cycles in the algebraic *tmf* spectral sequence. Indeed, the target of any differential supported by y, must have algebraic *tmf* filtration greater than y and from Table 2 it is clear no such element is present in the appropriate bidegree. Hence y is present in the Adams E_2 page. The same argument holds for x.

Case 1 When $x = g^6 \cdot y \in X^0_{39} \oplus X^0_{38} \oplus X^1_{37} \oplus X^1_{36} \oplus X^{1,1}_{36}$, then both x and y are permanent cycles in the algebraic *tmf* spectral sequence as the differentials must increase algebraic *tmf* filtration. In fact these elements are permanent cycles in the Adams spectral sequence for either degree reasons or by Lemma 4.4. If y is a target of a differential in the algebraic *tmf* spectral sequence or an Adams d_2 differential, then y is zero in the E_3 page. Consequently, $x = g^6 \cdot y$ is zero in the E_3 page as well. If y is not a target of such differentials, then we have

$$d_3(\widetilde{v_2^{20}h_1} \cdot y) = \widetilde{v_2^{20}h_1} \cdot d_3(y) + d_3(\widetilde{v_2^{20}h_1}) \cdot y = g^6 \cdot y = x.$$

In either case, x is zero in the E_4 page.

Case 2 When $x = g^6 \cdot y \in X^0_{37}$ and y is a permanent cycle, then we can argue $x = g^6 \cdot y$ is zero in the E_4 page as we did in the previous cases. If

$$d_2(y) = y',$$

then y' must belong to $g^{-6}Y^0_{39}$. Since multiplication by g^6 is a bijection between $g^{-6}Y^0_{39}$ and Y^0_{39}, we get

$$d_2(x) = d_2(g^6 \cdot y) = g^6 \cdot d_2(y) + d_2(g^6) \cdot y = g^6 \cdot y' \neq 0.$$

Therefore, x is absent in the E_4 page. □

Thus we are left with the case when $x \in X^0_{36}$.

Lemma 5.5 *Every element of X^0_{36} is either zero or absent in the Adams E_4 page.*

Proof X^0_{36} is spanned by three generators $\{s_1, t_1, t_2\}$. Using Bruner's program, we explore the following relations in the E_1 page of the algebraic *tmf* spectral sequence:

$$\begin{aligned} s_1 &= b^4_{3,0} \cdot x_1, & Y^0_{38} &\ni e_0 r \cdot x_1 \neq 0, & & \\ t_1 &= b^4_{3,0} \cdot y_1 = b^8_{3,0} \cdot z_1, & e_0 r \cdot y_1 &= 0, & Y^0_{39} &\ni wgr \cdot z_1 \neq 0, \\ t_2 &= b^4_{3,0} \cdot y_2 = b^8_{3,0} \cdot z_2, & e_0 r \cdot y_2 &= 0, & Y^0_{39} &\ni wgr \cdot z_2 \neq 0, \end{aligned}$$

$s\downarrow\ t{-}s \rightarrow$	94	95	$s\downarrow\ t{-}s \rightarrow$	142	143
23	0	0	30	0	0
22	0	0	29	$\langle \bullet\bullet\bullet\bullet\bullet \rangle$	$\langle \bullet\bullet\bullet\bullet\bullet \rangle$
21	0	0	28	N/A	$\langle \bullet{=}x_1, \bullet{=}y_1, \bullet{=}y_2 \rangle := Z_{28}$ $\langle \circ^1 \circ^1 \rangle$
20	N/A	$\langle \bullet{=}z_1, \bullet{=}z_2 \rangle := Z_{20}$			

Table 3: E_1 page of the algebraic *tmf* spectral sequence for $\mathrm{Ext}_A^{s,t}(A_1 \wedge DA_1)$, where $A_1 = A_1[00]$ or $A_1 = A_1[11]$.

and $wgr \cdot z_1$ and $wgr \cdot z_2$ are linearly independent. In Bruner's notation, $s_1 = 36_{64}$, $t_1 = 36_{65}$, $t_2 = 36_{66}$, $x_1 = 28_{32}$, $e_0 r \cdot x_1 = 38_{25}$, $y_1 = 28_{33}$, $y_2 = 28_{34}$, $z_1 = 20_1$, $wgr \cdot z_1 = 39_1$, $z_2 = 20_2$ and $wgr \cdot z_2 = 39_2$; see Table 5 from the online supplement.

From Table 3, it is clear that any element in Z_{20} and Z_{28} are permanent cycles.

Case 1 If $x = \epsilon_1 s_1 + \delta_1 t_1 + \delta_2 t_2 \neq 0$ in the Adams E_2 page with $\epsilon_1 \neq 0$, then

$$d_2(x) = \epsilon_1 d_2(\widetilde{v_2^8} \cdot x_1) = \epsilon_1 (e_0 r \cdot x_1) \neq 0.$$

Thus x is not present in the E_4 page.

Case 2 If $x = \delta_1 t_1 + \delta_2 t_2 \neq 0$, then

$$d_2(x) = 0.$$

If $x \neq 0$ in the Adams E_3 page, then

$$d_3(x) = \delta_1 d_3(\widetilde{v_2^{16}} \cdot z_1) + \delta_2 d_3(\widetilde{v_2^{16}} \cdot z_2) = wgr \cdot (\delta_1 z_1 + \delta_2 z_2) \neq 0.$$

Thus x is not present in the E_4 page. □

This proves Main Theorem in the cases $A_1 = A_1[00]$ or $A_1 = A_1[11]$.

5.2 The case $A_1 = A_1[01]$ or $A_1 = A_1[10]$

A priori, $A_1[01]$ and $A_1[10]$ are two different spectra and we must therefore give two different proofs of Main Theorem. However, it turns out that Tables 4 and 5 are identical for $A_1[01]$ and $A_1[10]$, and therefore the exact same arguments will apply to both spectra. For $A_1[01]$, refer to Tables 6–9 of the online supplement, and for $A_1[10]$, refer to Tables 10–13 of the online supplement, to observe that most of the elements in Table 4 are multiples by g^6 of elements in Table 5.

$s\backslash t-s$	190	191
39	0	$\langle\bullet\rangle := X^0_{39}$
38	$\langle\bullet\bullet\bullet\bullet\rangle := Y^0_{38}$	$\langle\bullet\rangle := X^0_{38}$
37	$\langle\bullet\bullet\bullet\bullet\rangle$ $\langle\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\circ^1\rangle$	$\langle\bullet\bullet\bullet\bullet\bullet\rangle := X^0_{37}$ $\langle\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\circ^1\rangle := X^1_{37}$
36	N/A	$\langle\odot\odot\rangle := X^{1,1}_{36}$

Table 4: E_1 page of the algebraic *tmf* spectral sequence for $\mathrm{Ext}^{s,t}_A(A_1 \wedge DA_1)$, where $A_1 = A_1[01]$, stem 190–191.

$s\backslash t-s$	70	71
15	0	$\langle\bullet\rangle = g^{-6}X^0_{39}$
14	$\langle\bullet\bullet\bullet\bullet\rangle = g^{-6}Y^0_{38}$	$\langle\bullet\bullet\rangle = g^{-6}X^0_{38}$
13	$\langle\bullet\bullet\bullet\bullet\bullet\rangle$ $\langle\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\circ^1\rangle$	$\langle\bullet\bullet\bullet\bullet\bullet\bullet\bullet\rangle = g^{-6}X^0_{37}$ $\langle\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\circ^1\rangle = g^{-6}X^1_{37}$
12	N/A	$\langle\circ^1\,\circ^1\,\circ^1\,\circ^1\,\circ^1\circ^1\rangle$ $\langle\odot\odot\rangle = g^{-6}X^{1,1}_{36}$

Table 5: E_1 page of the algebraic *tmf* spectral sequence for $\mathrm{Ext}^{s,t}_A(A_1 \wedge DA_1)$, where $A_1 = A_1[01]$, stem 70–71.

Lemma 5.6 *All elements of*

$$X^0_{39} \oplus X^0_{38} \oplus X^0_{37} \oplus X^1_{37} \oplus X^{1,1}_{36} \tag{5.7}$$

are present in the Adams E_2 page, but are zero in the Adams E_4 page.

Proof Differentials in the algebraic *tmf* spectral sequence increase algebraic *tmf* filtration. Therefore, as Tables 4 and 5 make clear, all elements of (5.7) are permanent cycles in the algebraic *tmf* spectral sequence and are therefore present in the Adams E_2 page. Furthermore, all these elements are permanent cycles in the Adams spectral sequence, either for degree reasons or by Lemma 4.4.

Tables 6–13 of the online supplement make clear that multiplication by g^6 is surjective onto (5.7). Therefore, any element $x = g^6 \cdot y$ in (5.7) which is not zero in the Adams E_3 page is a target of a d_3 differential

$$d_3(\widetilde{v_2^{20}h_1} \cdot y) = d_3(\widetilde{v_2^{20}h_1}) \cdot y + \widetilde{v_2^{20}h_1} \cdot d_3(y) = g^6 \cdot y = x,$$

hence zero in the E_4 page. □

Appendix: General remarks on the use of Bruner's program

Since many of our proofs relied on the output of Bruner's program, we append some facts about the program to justify our claims.

The program takes as input a graded module M over A (or $A(2)$) that is a finite dimensional $\mathbb{F}_2$–vector space and computes $\mathrm{Ext}_A^{s,t}(M, \mathbb{F}_2)$ (or $\mathrm{Ext}_{A(2)}^{s,t}(M, \mathbb{F}_2)$) for t in a user-defined range, and $0 \le s \le \mathtt{MAXFILT}$, where one has $\mathtt{MAXFILT} = 40$ by default. The structure of M as an A–module is encoded in a text file named `M`, placed in the directory `A/samples` in the way we will now describe.

The first line of the text file `M` consists of a positive integer n, the dimension of M as an $\mathbb{F}_2$–vector space, whose basis elements we will call $g_0, \dots, g_{n-1}$. The second line consists of an ordered list of integers $d_0, \dots, d_{n-1}$, which are the respective degrees of the g_i. Every subsequent line in the text file describes a nontrivial action of some Sq^k on some generator g_i. For instance, if we have

$$\mathrm{Sq}^k(g_i) = g_{j1} + \cdots + g_{jl},$$

we would encode this fact by writing the line

```
i k l j1 ...jl
```

followed by a new line. Every action not encoded by such a line is assumed to be trivial. To ensure that such a text file in fact represents an honest A–module, we must run the `newconsistency` script, which will alert us if:

- the text file contains a line

  ```
  i k l j1 ...jl
  ```

  and it turns out that one of the d_j is not equal to $d_i + k$, or
- the module taken as a whole fails to satisfy a particular Adem relation.

Example A.1 Consider the A–module given by Figure 3, where generators are depicted by dots and actions of Sq^1, Sq^2 and Sq^4 are depicted by straight lines, curved lines and square brackets, respectively.

Based on this picture, we get the text file in Figure 4, which we call `A1-00_def`. We go to the directory `A2` and run:

```
./newmodule A1-00 ../A/samples/A1-00_def
cd A1-00
```

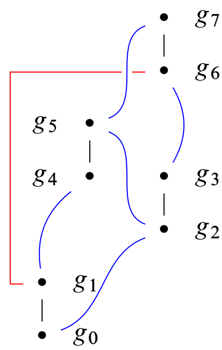


Figure 3: $H^*A_1[00]$ as an A–module

Now we are ready to compute. Running the script

```
./dims 0 250
```

will compute $\mathrm{Ext}^{s,t}_{A(2)}(A_1[00])$ for $0 \le s \le \mathtt{MAXFILT} = 40$ and $0 \le t \le 250$. To see the Ext group, one runs

```
./report summary
./vsumm A1-00 > A1-00.tex
pdflatex A1-00.tex
```

to produce a pdf document `A1-00.pdf` as in the online supplement.

As this file makes apparent, the generators of the Ext group (as an $\mathbb{F}_2$ vector space) are stored in the computer with names such as s_g, where s is the Adams filtration of the generator, and g is some way of ordering all generators of filtration s. It should be emphasized that g is not the stem of the generator. In `A1-00.pdf` from the online supplement, for instance, the generator 1_2 is the second generator of filtration 1, but it is in stem 6. This file also tells us the action of the Hopf elements h_0 through h_3, so that in our example, h_2 multiplied by the generator 1_2 equals the generator 2_2.

By running

```
./display 0 A1-00_
```

to produce single-page pdf documents `A1-00_1.pdf`, `A1-00_2.pdf`, …, it is also possible to see the Ext group in the visually more appealing form of a chart, as shown in `A1-00_1.pdf` from the online supplement.

The program is also capable of computing dual modules via the `dualizeDef` script, and tensor products via the `tensorDef` script. Both executables are conveniently located in

```
8

0 1 2 3 3 4 5 6

0 1 1 1
0 2 1 2
0 3 1 3
0 6 1 7
1 2 1 4
1 3 1 5
1 4 1 6
1 5 1 7
2 1 1 3
2 2 1 5
3 2 1 6
3 3 1 7
4 1 1 5
5 2 1 7
6 1 1 7
```

Figure 4: The text file `A/samples/A1-00_def`

the `A/samples` directory where we put our module definition text files. Thus, running

```
./dualizeDef A1-00_def DA1-00_def
./tensorDef A1-00_def DA1-00_def ADA1-00_def
```

produces the text file `ADA1-00_def`, with which we proceed in the same way as earlier with `A1-00_def`.

While `ADA1-00.pdf` only shows the action of the Hopf elements h_0 through h_3, the scripts `cocycle` and `dolifts` enable the user to input a specific generator and find the action of much of $\mathrm{Ext}^{s,t}_{A(2)}(S^0)$ on that specific generator. Let us do this with the generator $0_6 \in \mathrm{Ext}^{0,0}_{A(2)}(A_1[00] \wedge DA_1[00])$ by going to directory `A2` and running

```
./cocycle ADA1-00 0 6
```

which will create a subdirectory `A2/ADA1-00/0_6`. To find the action of all elements of $\mathrm{Ext}^{s,t}_{A(2)}(S^0)$ with $0 \leq s \leq 20$ on 0_6, we go back to directory `A2/ADA1-00` and run:

```
./dolifts 0 20 maps
```

Now `ADA1-00/0_6` will contain several text files, among them `brackets.sym` (which contains information about Massey products) and `Map.aug` (which contains information about the action of $\mathrm{Ext}^{s,t}_{A(2)}(S^0)$ on 0_6).

The generators of $\mathrm{Ext}^{s,t}_{A(2)}(S^0)$ are stored in the computer in the format s_g. Here we include a list of important elements of $\mathrm{Ext}^{s,t}_{A(2)}(S^0)$ and their s_g representations:

$$\begin{aligned} g &= 4_8 \in \mathrm{Ext}^{4,20+4}_{A(2)}(S^0) \\ b^4_{3,0} &= 8_{19} \in \mathrm{Ext}^{8,48+8}_{A(2)}(S^0) \\ e_0 r &= 10_{18} \in \mathrm{Ext}^{10,47+10}_{A(2)}(S^0) \\ b^8_{3,0} &= 16_{54} \in \mathrm{Ext}^{16,96+16}_{A(2)}(S^0) \\ wgr &= 19_{56} \in \mathrm{Ext}^{19,95+19}_{A(2)}(S^0) \\ v_2^{20} h_1 &= 21_{85} \in \mathrm{Ext}^{21,121+21}_{A(2)}(S^0) \\ g^6 &= 24_{90} \in \mathrm{Ext}^{24,120+24}_{A(2)}(S^0) \end{aligned}$$

We'd like to know what $s_g(0_6) \in \mathrm{Ext}_{A(2)}(A_1[00] \wedge DA_1[00])$ is in the notation of `ADA1-00.pdf`. Of course, $s_g(0_6)$ is in filtration s, so we only need to specify which of the generators in filtration s make up $s_g(0_6)$. If, for instance, we have

$$s_g(0_6) = s_{g1} + \cdots + s_{gn},$$

then `ADA1-00/0_6/Map.aug` will contain the lines:

```
s g1 g
s g2 g
  ⋮
s gn g
```

Now, in the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{A(2)}(S^0) \Rightarrow tmf_{t-s},$$

we have

$$d_2(b^4_{3,0}) = e_0 r = 10_{18} \in \mathrm{Ext}^{10,47+10}_{A(2)}(S^0) \quad \text{and} \quad d_3(b^8_{3,0}) = 19_{56} \in \mathrm{Ext}^{19,95+19}_{A(2)}(S^0).$$

It follows that if

$$10_{18}(0_6) = 10_x \in \mathrm{Ext}^{8,8+47}_{A(2)}(A_1 \wedge DA_1)$$

and

$$19_{56}(0_6) = 19_y \in \mathrm{Ext}^{19,19+95}_{A(2)}(A_1 \wedge DA_1),$$

then $b_{3,0}^4 \in \mathrm{Ext}_{A(2)}^{8,48+8}(A_1 \wedge DA_1)$ and $b_{3,0}^8 \in \mathrm{Ext}_{A(2)}^{16,96+16}(A_1 \wedge DA_1)$ support a d_2 differential and a d_3 differential, respectively. By doing the above steps for all four versions of A_1, and checking the respective `Map.aug` files, each contain lines

```
10 x 18
19 y 56
```

justifying the claim in Lemma 3.1.

Using the tools we have so far described, it is easy to verify the claim from the proof of Lemma 4.1, that for all four models of A_1 we have

$$gb_{3,0}^4 \cdot 10_3 = 22_7. \tag{A.2}$$

It is similarly easy to verify that if $A_1 = A_1[00]$ or $A_1 = A_1[11]$, we have

$$ge_0 r \cdot 10_3 = 0,$$

while if $A_1 = A_1[01]$ or $A_1 = A_1[10]$, we have

$$ge_0 r \cdot 10_3 = 24_0 = g^6.$$

Finally, in order to run the algebraic *tmf* spectral sequence, we will also need do computations involving the *bo*–Brown–Gitler spectra. We give the $A(2)$–module definitions for the cohomologies of bo_1 and bo_2 in `bo1_def` and `bo2_def` from the online supplement.

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Department of Mathematics, University of Notre Dame
106 Hayes-Healy Hall, Notre Dame, IN 46556, United States

Department of Mathematics, Pennsylvania State University
235 McAllister, University Park, PA 16802, United States

Department of Mathematics, Northwestern University
Evanston, IL 60208, United States

pbhattac@nd.edu, pee3@psu.edu, markmah@me.com

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