

THH and base-change for Galois extensions of ring spectra

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We treat the question of base-change in THH for faithful Galois extensions of ring spectra in the sense of Rognes. Given a faithful Galois extension $A \rightarrow B$ of ring spectra, we consider whether the map $\mathrm{THH}(A) \otimes_A B \rightarrow \mathrm{THH}(B)$ is an equivalence. We reprove and extend positive results of Weibel and Geller, and McCarthy and Minasian, and offer new examples of Galois extensions for which base-change holds. We also provide a counterexample where base-change fails.

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1 Introduction

Let R be an \mathbb{E}_1 -ring spectrum. The *topological Hochschild homology* $\mathrm{THH}(R)$ of R is a spectrum constructed as the geometric realization of a certain cyclic object built from R , a homotopy-theoretic version of the Hochschild complex of an associative ring. Topological Hochschild homology has been studied in particular because of its connections with algebraic K -theory via the theory of trace maps. More generally, if R is an \mathbb{E}_1 -algebra in A -modules for an \mathbb{E}_∞ -ring A , then one can define a relative version $\mathrm{THH}^A(R)$.

Weibel and Geller [15] showed that Hochschild homology for commutative rings satisfies an étale base-change result. Equivalently, if k is a commutative ring and if $A \rightarrow B$ is an étale morphism of commutative k -algebras with A flat over k , then there is a natural equivalence

$$B \otimes_A \mathrm{THH}^k(A) \simeq \mathrm{THH}^k(B).$$

Weibel and Geller's result also applies in the nonflat case, although it cannot be stated in this manner.

One can hope to generalize the Weibel–Geller result to the setting of ring spectra. This leads to the following general question:

Question Let $A \rightarrow B$ be a morphism of \mathbb{E}_∞ -ring spectra. When is the map

$$(1) \quad \mathrm{THH}(A) \otimes_A B \rightarrow \mathrm{THH}(B)$$

an equivalence?

Following Lurie, we will use the following definition of étaleness:

Definition 1.1 A morphism $A \rightarrow B$ of \mathbb{E}_∞ -ring spectra is *étale* if $\pi_0(A) \rightarrow \pi_0(B)$ is étale and the natural map $\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_*(B)$ is an isomorphism.

McCarthy and Minasian [11] consider this question for an étale morphism¹ of connective \mathbb{E}_∞ -rings and prove the analog of the Weibel–Geller theorem, ie that (1) is an equivalence (see [11, Lemma 5.7]). In fact, they prove the result more generally for any THH-étale morphism of connective \mathbb{E}_∞ -rings.

In the setting of structured ring spectra, however, there are additional morphisms of nonconnective ring spectra that have formal properties similar to those of étale morphisms, though they are not étale on homotopy groups. The *faithful Galois extensions* of Rognes [14] are key examples here.

This note is primarily concerned with the following analog of the Weibel–Geller and McCarthy–Minasian question:

Question Let $A \rightarrow B$ be a G -Galois extension of \mathbb{E}_∞ -ring spectra, with G finite. When is the comparison map (1) an equivalence?

We make two main observations here. Our first observation uses the fact that THH, like algebraic K -theory, is an invariant not only of ring spectra but of stable ∞ -categories. We refer, for example, to Blumberg and Mandell [3] and Blumberg, Gepner and Tabuada [2] for a treatment of THH in this context. Using Galois descent, we observe that the map (1) is an equivalence if and only if the map $\mathrm{THH}(A) \rightarrow \mathrm{THH}(B)^{hG}$ is an equivalence. These maps are the comparison maps for the *Galois descent* problem in THH. Consequently, the results of Clausen, Mathew, Naumann and Noel [4] provide numerous examples in chromatic homotopy theory where (1) is an equivalence.

Our second observation is to reinterpret the base-change question for THH in terms of the formulation $\mathrm{THH}(R) \simeq S^1 \otimes R$ for \mathbb{E}_∞ -rings, due to McClure, Schwänzl and Vogt [12].

As a result, we obtain an example where (1) is not an equivalence.

Theorem 1.2 *There is a faithful G -Galois extension $A \rightarrow B$ of \mathbb{E}_∞ -ring spectra such that (1) is not an equivalence.*

¹We note that McCarthy and Minasian use the word “étale” differently in their paper.

Our counterexample is very simple; it is the map $C^*(S^1; \mathbb{F}_p) \rightarrow C^*(S^1; \mathbb{F}_p)$ induced by the degree- p cover $S^1 \rightarrow S^1$.

We in fact pinpoint exactly what goes wrong from a categorical perspective, and why this phenomenon cannot happen in the étale setting, thus proving a variant of the Weibel–Geller–McCarthy–Minasian theorem in the nonconnective setting:

Theorem 1.3 *Let R be an \mathbb{E}_∞ -ring, and let $A \rightarrow B$ be an étale morphism of \mathbb{E}_∞ - R -algebras (possibly nonconnective). Then the natural map $\mathrm{THH}^R(A) \otimes_A B \rightarrow \mathrm{THH}^R(B)$ is an equivalence.*

The use of categorical interpretation of THH in proving such base-change theorems is not new; McCarthy and Minasian [11] use this interpretation in a different manner.

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2 Categorical generalities

Let \mathcal{C} be a cocomplete ∞ -category, and let $x \in \mathcal{C}$. Given $x \in \mathcal{C}$, we can [5, Section 4.4.4] construct an object $S^1 \otimes x$.

Choose a basepoint $* \in S^1$. Then we have a diagram:

$$(2) \quad \begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ S^1 \otimes x & \longrightarrow & S^1 \otimes y \end{array}$$

As a result of this diagram, we have a natural map in \mathcal{C} ,

$$(3) \quad (S^1 \otimes x) \sqcup_x y \rightarrow S^1 \otimes y.$$

In order for (3) to be an equivalence, for any object $z \in \mathcal{C}$, the square of spaces

$$(4) \quad \begin{array}{ccc} \mathrm{Hom}(S^1, \mathrm{Hom}_{\mathcal{C}}(y, z)) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(y, z) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(S^1, \mathrm{Hom}_{\mathcal{C}}(x, z)) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(x, z) \end{array}$$

must be homotopy cartesian. This happens only in very special situations.

Proposition 2.1 *Let $f: X \rightarrow Y$ be a map of spaces. Then the diagram*

$$(5) \quad \begin{array}{ccc} \text{Hom}(S^1, X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Hom}(S^1, Y) & \longrightarrow & Y \end{array}$$

is homotopy cartesian if and only if, for every point $p \in X$, the map from the connected component of X containing p to that of Y containing $f(p)$ is a homotopy equivalence.

Proof Without loss of generality, we may assume that X and Y are connected spaces. In this case, choosing compatible basepoints in X and Y , we get equivalences

$$\Omega X \simeq \text{fib}(\text{Hom}(S^1, X) \rightarrow X), \quad \Omega Y \simeq \text{fib}(\text{Hom}(S^1, Y) \rightarrow Y),$$

and the fact that (5) is homotopy cartesian now implies that $\Omega X \rightarrow \Omega Y$ is a homotopy equivalence. Since X and Y are connected, this implies that $X \rightarrow Y$ is a homotopy equivalence. □

Definition 2.2 We will say that a map of spaces $X \rightarrow Y$ is a *split covering space* if the equivalent conditions of Proposition 2.1 are met. In particular, $X \rightarrow Y$ is a covering space that is trivial on each connected component of Y .

Observe that the base-change of a split covering space is still a split covering space.

Corollary 2.3 *Suppose $x \rightarrow y$ is a morphism in \mathcal{C} as above. Then the natural map $(S^1 \otimes x) \sqcup_x y \rightarrow S^1 \otimes y$ is an equivalence if and only if, for every object $z \in \mathcal{C}$, the induced map of spaces $\text{Hom}_{\mathcal{C}}(y, z) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$ is a split cover.*

Proof Our map is an equivalence if and only if (4) is homotopy cartesian for each $z \in \mathcal{C}$. By Proposition 2.1, we get the desired claim. □

We now give this class of morphisms a name.

Definition 2.4 A morphism $x \rightarrow y$ in an ∞ -category \mathcal{C} is said to be *strongly 0-cotruncated* if, for every $z \in \mathcal{C}$, the map $\text{Hom}_{\mathcal{C}}(y, z) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$ is a split covering space.

Corollary 2.3 states that $x \rightarrow y$ has the property that $(S^1 \otimes x) \sqcup_x y \rightarrow S^1 \otimes y$ is an equivalence if and only if the map is strongly 0-cotruncated.

For passage to a relative setting, we will find the following useful:

Proposition 2.5 *Let \mathcal{C} be a cocomplete ∞ -category, let $a \in \mathcal{C}$, and let $x \rightarrow y$ be a morphism in $\mathcal{C}_{a/}$. If $x \rightarrow y$ is strongly 0-cotruncated when regarded as a morphism in \mathcal{C} , then it is strongly 0-cotruncated when regarded as a morphism in $\mathcal{C}_{a/}$.*

Proof Suppose $a \rightarrow z$ is an object of $\mathcal{C}_{a/}$. Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_{a/}}(y, z) &= \mathrm{fib}(\mathrm{Hom}_{\mathcal{C}}(y, z) \rightarrow \mathrm{Hom}_{\mathcal{C}}(a, z)), \\ \mathrm{Hom}_{\mathcal{C}_{a/}}(x, z) &= \mathrm{fib}(\mathrm{Hom}_{\mathcal{C}}(x, z) \rightarrow \mathrm{Hom}_{\mathcal{C}}(a, z)). \end{aligned}$$

Since $\mathrm{Hom}_{\mathcal{C}}(y, z) \rightarrow \mathrm{Hom}_{\mathcal{C}}(x, z)$ is a split cover, it follows easily that the same holds after taking homotopy fibers over the basepoint in $\mathrm{Hom}_{\mathcal{C}}(a, z)$. In fact, we can assume without loss of generality that $\mathrm{Hom}_{\mathcal{C}}(x, z)$ is connected, in which case $\mathrm{Hom}_{\mathcal{C}}(y, z)$ is a disjoint union $\bigsqcup_S \mathrm{Hom}_{\mathcal{C}}(x, y)$. Taking fibers over the map to $\mathrm{Hom}_{\mathcal{C}}(a, z)$ preserves the disjoint union as desired, so the map on fibers is a split cover. \square

3 \mathbb{E}_{∞} -ring spectra

We let CAlg denote the ∞ -category of \mathbb{E}_{∞} -ring spectra. The construction THH in this case can be interpreted (by [12]) as tensoring with S^1 ; that is, we have

$$\mathrm{THH}(A) \simeq S^1 \otimes A, \quad A \in \mathrm{CAlg}.$$

If one works in a relative setting, under an \mathbb{E}_{∞} -ring R , then $\mathrm{THH}^R(A) \simeq S^1 \otimes A$, where the tensor product is computed in $\mathrm{CAlg}_{R/}$.

Given a morphism in $\mathrm{CAlg}_{R/}$, $A \rightarrow B$, we can use the setup of the previous section and obtain a morphism

$$\mathrm{THH}^R(A) \otimes_A B \rightarrow \mathrm{THH}^R(B),$$

which is a special case of (3). The base-change problem for THH asks when this is an equivalence.

By Corollary 2.3, this is equivalent to the condition that the morphism $A \rightarrow B$ in $\mathrm{CAlg}_{R/}$ should be strongly 0-cotruncated. We can now prove Theorem 1.3 from the introduction, which we restate for convenience.

Theorem *Let R be an \mathbb{E}_{∞} -ring and let $A \rightarrow B$ be an étale morphism (as in Definition 1.1) in $\mathrm{CAlg}_{R/}$. Then the natural morphism $\mathrm{THH}^R(A) \otimes_A B \rightarrow \mathrm{THH}^R(B)$ is an equivalence.*

This is closely related to [15, Theorem 0.1] and includes it in the case of a flat extension $R \rightarrow A$ of discrete \mathbb{E}_∞ -rings. For connective \mathbb{E}_∞ -rings, this result is [11, Lemma 5.7] (who treat more generally the case of a THH-étale morphism).

Proof Given an étale morphism $A \rightarrow B$ in $\text{CAlg}_{R/}$, we need to argue that it is strongly 0-cotruncated. By Proposition 2.5, we may reduce to the case where $R = S^0$. Given $C \in \text{CAlg}$, we have a homotopy cartesian square

$$\begin{CD} \text{Hom}_{\text{CAlg}}(B, C) @>>> \text{Hom}_{\text{Ring}}(\pi_0 B, \pi_0 C) \\ @VVV @VVV \\ \text{Hom}_{\text{CAlg}}(A, C) @>>> \text{Hom}_{\text{Ring}}(\pi_0 A, \pi_0 C) \end{CD}$$

by eg [8, Section 7.5]. Here Ring is the category of rings. Since the right vertical map is a map of discrete spaces and therefore a split covering, it follows that $\text{Hom}_{\text{CAlg}}(B, C) \rightarrow \text{Hom}_{\text{CAlg}}(A, C)$ is a split covering, as desired. \square

We also note in passing that the étale descent theorem has a partial converse in the setting of connective \mathbb{E}_∞ -rings. We note that this rules out nonalgebraic Galois extensions.

Corollary 3.1 *Let $A \rightarrow B$ be a morphism of connective \mathbb{E}_∞ -rings which is almost of finite presentation [8, Section 7.2.4]. Suppose the map $\text{THH}(A) \otimes_A B \rightarrow \text{THH}(B)$ is an equivalence. Then $A \rightarrow B$ is étale.*

Proof Indeed, B defines a 0-cotruncated object (Definition 5.1) of $\text{CAlg}_{A/}$ and it is well known that this, combined with the fact that B is almost of finite presentation, implies that B is étale. We reproduce the argument for the convenience of the reader.

In fact, since B is 0-cotruncated, one finds that for any B -module M , the space of maps² $\text{Hom}_{\text{CAlg}_{A//B}}(B, B \oplus M)$ is homotopy discrete, where the \mathbb{E}_∞ -ring $B \oplus M$ is given the square-zero multiplication. Replacing M by ΣM , it follows that

$$\text{Hom}_{\text{CAlg}_{A//B}}(B, B \oplus M) \simeq \Omega \text{Hom}_{\text{CAlg}_{A//B}}(B, B \oplus \Sigma M)$$

is actually contractible. Thus the cotangent complex $L_{B/A}$ vanishes, which implies that B is étale over A by [6, Lemma 8.9]. The connectivity is used in this last step. \square

The above argument also appears in [14, Section 9.4], where it is shown that a map $A \rightarrow B$ which is 0-cotruncated as in Definition 5.1 below (which Rognes calls

²For an ∞ -category \mathcal{C} and a morphism $x \rightarrow y$, we let $\mathcal{C}_{x//y}$ denote $(\mathcal{C}_x)_{/y}$, where $y \in \mathcal{C}_x$ via the given morphism.

formally symmetrically étale, and which has been called THH–étale in [11]) has to have vanishing cotangent complex (which is called TAQ–étale); see [14, Lemma 9.4.4]. The key point is that in the connective setting, TAQ–étaleness plus a weak finiteness condition is enough to imply étaleness. This entirely breaks down when one works with nonconnective \mathbb{E}_∞ –ring spectra.

4 Connection with descent

In this section, we will show that the question of base-change in THH is equivalent to a descent-theoretic question. We will then use some of the descent results of [4] to obtain examples where base-change for THH holds. Let $A \rightarrow B$ be a faithful G –Galois extension of \mathbb{E}_∞ –rings for G a finite group.

To begin with, we will need to recall a fact about Galois descent.

Proposition 4.1 (see [13, Chapter 6], [1, Theorem 2.8] or [10, Theorem 9.4], for example) *If $A \rightarrow B$ is a faithful G –Galois extension, then we have an equivalence of symmetric monoidal ∞ –categories*

$$\text{Mod}(A) \simeq \text{Mod}(B)^{hG},$$

where the left adjoint is extension of scalars along $A \rightarrow B$ and the right adjoint is given by taking homotopy fixed points.

We can restate the above equivalence in the following manner:

Corollary 4.2 *Let $\text{Fun}(BG, \text{Sp})$ be the symmetric monoidal ∞ –category of G –spectra equipped with a G –action. Then we have a natural equivalence*

$$\text{Mod}_{\text{Fun}(BG, \text{Sp})}(B) \simeq \text{Mod}_{\text{Sp}}(A)$$

given by taking homotopy fixed points.

Proof This follows from Proposition 4.1 using the fact that the construction of forming modules in a symmetric monoidal ∞ –category is compatible with homotopy limits of symmetric monoidal ∞ –categories. \square

Let $\mathcal{C} = \text{Fun}(BG, \text{CAlg})$ be the ∞ –category of \mathbb{E}_∞ –algebras equipped with a G –action, so that B defines an object of \mathcal{C} . We have therefore have natural equivalences of ∞ –categories

$$(6) \quad \mathcal{C}_B \simeq \text{CAlg}(\text{Fun}(BG, \text{Sp}))_B \simeq \text{CAlg}(\text{Mod}_{\text{Fun}(BG, \text{Sp})}(B)) \simeq \text{CAlg}(\text{Mod}(A)),$$

where the last equivalence is given by taking homotopy fixed points. We now obtain:

Proposition 4.3 *For a faithful G -Galois extension $A \rightarrow B$, the following two statements are equivalent:*

- $\mathrm{THH}(A) \otimes_A B \rightarrow \mathrm{THH}(B)$ is an equivalence.
- $\mathrm{THH}(A) \rightarrow \mathrm{THH}(B)$ is a faithful G -Galois extension.
- The map $\mathrm{THH}(A) \simeq (\mathrm{THH}(A) \otimes_A B)^{hG} \rightarrow \mathrm{THH}(B)^{hG}$ is an equivalence.

Proof In this case, the maps $B \rightarrow \mathrm{THH}(A) \otimes_A B \rightarrow \mathrm{THH}(B)$ that we obtain are G -equivariant, as they are natural in the \mathbb{E}_∞ - A -algebra B . Therefore, the map $\mathrm{THH}(A) \otimes_A B \rightarrow \mathrm{THH}(B)$ is naturally a morphism in $\mathrm{CAlg}(\mathrm{Fun}(BG, \mathrm{Sp}))_{B/}$. By (6), the map is an equivalence if and only if it induces an equivalence on homotopy fixed points.

Finally, if the map $\mathrm{THH}(A) \otimes_A B \rightarrow \mathrm{THH}(B)$ is an equivalence, then the morphism $\mathrm{THH}(A) \rightarrow \mathrm{THH}(B)$ is a base-change of the faithful G -Galois extension $A \rightarrow B$ and is thus a faithful G -Galois extension itself. Conversely, if $\mathrm{THH}(A) \rightarrow \mathrm{THH}(B)$ is a faithful G -Galois extension, then the descent map $\mathrm{THH}(A) \rightarrow \mathrm{THH}(B)^{hG}$ is an equivalence. \square

In particular, the map $A \rightarrow B$ is strongly 0-cotruncated if and only if one has *Galois descent* for THH along the map $A \rightarrow B$. In [4], we give a general criterion for proving descent in telescopically localized THH .

Theorem 4.4 [4] *Suppose $A \rightarrow B$ is a G -Galois extension such that the map $K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$ induced by restriction of scalars is surjective. Fix an implicit prime p and a height n . Fix a weakly additive (see [4, Definition 3.11]) invariant E of κ -compact small idempotent-complete A -linear ∞ -categories taking values in a presentable stable ∞ -category. Then the natural morphisms*

$$L_n^f E(\mathrm{Perf}(A)) \rightarrow L_n^f E(\mathrm{Perf}(B))^{hG} \rightarrow (L_n^f E(\mathrm{Perf}(B)))^{hG}$$

are equivalences, where L_n^f denotes finitary L_n -localization. In particular, one can take $E = K$, THH or TC .

As a result, we can prove that the base-change map is an equivalence in a large class of “chromatic” examples of Galois extensions.

Theorem 4.5 *Suppose $A \rightarrow B$ is a faithful G -Galois extension of \mathbb{E}_∞ -rings. Assume that for every prime p , the localization $A_{(p)}$ is L_n^f -local for some $n = n(p)$. Suppose the map $K_0(B) \otimes \mathbb{Q} \rightarrow K_0(A) \otimes \mathbb{Q}$ is surjective (or equivalently has image containing the unit). Then the base-change map $\mathrm{THH}(A) \otimes_A B \rightarrow \mathrm{THH}(B)$ is an equivalence.*

Proof To check that the map $\mathrm{THH}(A) \otimes_A B \rightarrow \mathrm{THH}(B)$ is an equivalence, it suffices to localize at p , so we may assume A and B are p -local, and therefore L_n^f -local. Since L_n^f is a smashing localization, it follows that all THH terms in sight are automatically L_n^f -localized. In this case, the result follows by combining [Proposition 4.3](#) and [Theorem 4.4](#). \square

Example 4.6 Most classes of examples of faithful Galois extensions in chromatic homotopy theory satisfy the conditions of [Theorem 4.4](#). We refer to [[4](#), Section 5] for a detailed treatment. For example:

- (1) The C_2 -Galois extension $KO \rightarrow KU$ or the C_{p-1} -Galois extension $L \rightarrow \widehat{KU}_p$.
- (2) The G -Galois extension $E_n^{hG} \rightarrow E_n$ if G is a finite subgroup of the extended Morava stabilizer group (see [[4](#), Appendix B] by Meier, Naumann and Noel).
- (3) Any Galois extension of $\mathrm{TMF}[1/n]$, $\mathrm{Tmf}_0(n)$ or related spectra.

It follows that the comparison map in THH is an equivalence for these Galois extensions.

5 A counterexample

In this section, we will give an example over \mathbb{F}_p where the comparison (or equivalently descent) map for THH is not an equivalence. We begin with a useful weakening of [Definition 2.4](#).

Definition 5.1 A morphism $x \rightarrow y$ in an ∞ -category \mathcal{C} is said to be *0-cotruncated* if, for every $z \in \mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(y, z) \rightarrow \mathrm{Hom}_{\mathcal{C}}(x, z)$ is a covering space (ie has discrete homotopy fibers over any basepoint). An object $x \in \mathcal{C}$ is said to be *0-cotruncated* if $\mathrm{Hom}_{\mathcal{C}}(x, z)$ is discrete for any $z \in \mathcal{C}$.

The condition that $x \rightarrow y$ should be cotruncated is equivalent to the statement that $y \in \mathcal{C}_x$ should define a 0-cotruncated object. Note that an object $x \in \mathcal{C}$ is 0-cotruncated if and only if the natural map $x \rightarrow S^1 \otimes x$ is an equivalence.

In the setting of \mathbb{E}_{∞} -ring spectra, étale morphisms are far from the only examples of 0-cotruncated morphisms. For example, any faithful G -Galois extension in the sense of Rognes [[14](#)] is 0-cotruncated. This is essentially [[14](#), Lemma 9.2.6]. However, we show that faithful Galois extensions need not be *strongly* 0-cotruncated. Equivalently, base-change for THH can fail for them.

Proof of Theorem 1.2 Consider the degree- p map $S^1 \rightarrow S^1$, which is a \mathbb{Z}/p -torsor. Let k be a separably closed field of characteristic p . For a space X , we let $C^*(X; k) = F(X_+; k)$ denote the \mathbb{E}_∞ -rings of k -valued cochains on X . The induced map of \mathbb{E}_∞ -rings $\phi: C^*(S^1; k) \rightarrow C^*(S^1; k)$ is a faithful \mathbb{Z}/p -Galois extension of \mathbb{E}_∞ -ring spectra. This follows from [14, Proposition 5.6.3(a)] together with the criterion for the faithfulness via vanishing of the Tate construction [14, Proposition 6.3.3]. See also [10, Theorem 7.13].

We will show, nonetheless, that ϕ does not satisfy base-change for THH, or equivalently that it is not strongly 0-cotruncated. It suffices to show this in $\text{CAlg}_{k/}$ in view of Proposition 2.5.

By p -adic homotopy theory [9] (see also [7], which does not assume $k = \overline{\mathbb{F}}_p$), the natural map

$$S^1 \rightarrow \text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k)$$

exhibits $\text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k)$ as the p -adic completion of S^1 . In particular, $\text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k) \simeq K(\mathbb{Z}_p, 1)$ and the map given by precomposition with ϕ

$$\text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k) \xrightarrow{\phi^*} \text{Hom}_{\text{CAlg}_{k/}}(C^*(S^1; k), k),$$

is identified with the multiplication by p map $K(\mathbb{Z}_p, 1) \rightarrow K(\mathbb{Z}_p, 1)$. In particular, while this is a covering map, it is *not* a split covering map, so that ϕ is not strongly 0-cotruncated. □

The use of cochain algebras in providing such counterexamples goes back to an idea of Mandell [11, Example 3.5], who gives an example of a morphism of \mathbb{E}_∞ -ring spectra with trivial cotangent complex (ie is TAQ-étale) which is not THH-étale. Namely, Mandell shows that if $n > 1$ then the map $C^*(K(\mathbb{Z}/p, n); \mathbb{F}_p) \rightarrow \mathbb{F}_p$ has trivial cotangent complex.

We close by observing that it is the fundamental group that it is at the root of these problems.

Proposition 5.2 *Let X be a simply connected, pointed space and let $A \rightarrow B$ be a faithful G -Galois extension of \mathbb{E}_∞ -rings. In this case, the map of \mathbb{E}_∞ -rings*

$$(X \otimes A) \otimes_A B \rightarrow X \otimes B$$

is an equivalence.

In particular, one does have base-change for higher topological Hochschild homology (ie where $X = S^n$ with $n > 1$).

Proof Following the earlier reasoning, it suffices to show that whenever $C \in \text{CAlg}$, the square

$$\begin{array}{ccc} \text{Hom}(X, \text{Hom}_{\text{CAlg}}(B, C)) & \longrightarrow & \text{Hom}_{\text{CAlg}}(B, C) \\ \downarrow & & \downarrow \\ \text{Hom}(X, \text{Hom}_{\text{CAlg}}(A, C)) & \longrightarrow & \text{Hom}_{\text{CAlg}}(A, C) \end{array}$$

is homotopy cartesian. However, this follows since $\text{Hom}_{\text{CAlg}}(B, C) \rightarrow \text{Hom}_{\text{CAlg}}(A, C)$ is a covering space, and X is simply connected. \square

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