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On the infinite loop space structure of the cobordism category

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We show an equivalence of infinite loop spaces between the classifying space of the cobordism category with infinite loop space structure induced by taking disjoint union of manifolds and the infinite loop space associated to the Madsen–Tillmann spectrum.

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1 Introduction

In this article we show that there is an equivalence of infinite loop spaces between the classifying space of the d–dimensional cobordism category $B\,\mathrm{Cob}_\theta(d)$ and the 0^{th} space of the shifted Madsen–Tillmann spectrum $\mathrm{MT}\theta(d)[1]$. This extends a result by Galatius, Madsen, Tillmann and Weiss [5], who showed an equivalence of topological spaces

$$B\,\mathrm{Cob}_\theta(d) \simeq \mathrm{MT}\theta(d)[1]_0. \tag{1}$$

Note that both spaces in the equivalence above admit infinite loop space structures. The symmetric monoidal structure on the cobordism category, given by disjoint union of manifolds, induces an infinite loop space structure on $B\,\mathrm{Cob}_\theta(d)$, while the infinite loop space structure on $\mathrm{MT}\theta(d)[1]_0$ comes from it being the 0^{th} space of an Ω–spectrum. We will show that the equivalence (1) actually extends to an equivalence of infinite loop spaces with the above mentioned infinite loop space structures.

In more detail, our proof will rely on certain spaces of manifolds introduced by Galatius and Randal-Williams [4], which form an Ω–spectrum denoted here by ψ_θ. Using these spaces, they obtain a new proof of (1), which we record as the following theorem.

Theorem 1.1 *There are weak homotopy equivalences of spaces*

$$B\,\mathrm{Cob}_\theta(d) \simeq \psi_{\theta,0} \simeq \mathrm{MT}\theta(d)[1]_0.$$

In this article, we will show that the equivalences of the above theorem come from equivalences of spectra.

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Instead of directly constructing an equivalence of spectra, our strategy will be to construct Γ–spaces $\Gamma\,\mathrm{Cob}_\theta(d)$ and $\Gamma\psi_\theta$ with underlying spaces $B\,\mathrm{Cob}_\theta(d)$ and $\psi_{\theta,0}$ respectively, and we show that $\Gamma\psi_\theta$ is a model for the connective cover of the spectrum ψ_θ, denoted by $\psi_{\theta,\geq 0}$. This Γ–structure will be induced by taking disjoint union of manifolds. We then show that their associated spectra have the stable homotopy type of the connective cover of the shifted Madsen–Tillmann spectrum denoted by $\mathrm{MT}\theta(d)[1]_{\geq 0}$, by constructing a Γ–space model for $\mathrm{MT}\theta(d)[1]_{\geq 0}$ and exhibiting an equivalence of Γ–spaces. But more is true; we will see that the equivalences of Theorem 1.1 are the components of this equivalence of Γ–spaces and hence the main result of this article will be the following.

Main Theorem *There are stable equivalences of spectra*

$$\boldsymbol{B}\,\Gamma\,\mathrm{Cob}_\theta(d) \simeq \psi_{\theta,\geq 0} \simeq \mathrm{MT}\theta(d)[1]_{\geq 0}$$

such that the induced weak equivalences of spaces

$$\Omega^\infty \boldsymbol{B}\,\Gamma\,\mathrm{Cob}_\theta(d) \simeq \Omega^\infty\psi_\theta \simeq \Omega^\infty\mathrm{MT}\theta(d)[1]$$

are equivalent to the weak equivalences of Theorem 1.1.

Here, $\boldsymbol{B}\,\Gamma\,\mathrm{Cob}_\theta(d)$ is the spectrum associated to the symmetric monoidal category $\mathrm{Cob}_\theta(d)$. We would like to mention that a similar argument has been given by Madsen and Tillmann in [6] for the case $d = 1$.

This article is organized as follows. In the next section we recall some basic notions on spectra and Γ–spaces. This will also serve to fix notation and language. In Section 3 and Section 4 we review the proof of Theorem 1.1 of [4]. In Section 5 we will construct Γ–space models for the spectra ψ_θ and $\mathrm{MT}\theta(d)$, and in Section 6 we will show that these Γ–spaces are equivalent. Finally in Section 7, we will relate these Γ–spaces to the cobordism category with its infinite loop space structure induced by taking disjoint union of manifolds.

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2 Conventions on spectra and Γ–spaces

By a *space* we mean a compactly generated weak Hausdorff space. We denote by $\boldsymbol{S}$ the category of spaces and by $\boldsymbol{S}_*$ the category of based spaces. We fix a model for the circle by setting $S^1 := \mathbb{R} \cup \{\infty\}$.

We will work with the Bousfield–Friedlander model of sequential spectra; see Bousfield and Friedlander [2] or Mandell, May, Schwede and Shipley [7]. Recall that a *spectrum* E is a sequence of based spaces $E_n \in \boldsymbol{S}_*$, $n \in \mathbb{N}$ together with structure maps

$$s_n\colon S^1 \wedge E_n \to E_{n+1}.$$

A map of spectra $f\colon E \to F$ is a sequence of maps $f_n\colon E_n \to F_n$ commuting with the structure maps. We denote by **Spt** the category of spectra. A *stable equivalence* is a map of spectra inducing isomorphisms on stable homotopy groups. An Ω*–spectrum* is a spectrum E, where the adjoints of the structure maps $\Sigma E_n \to E_{n+1}$ are weak homotopy equivalences. There is a model structure on **Spt** with weak equivalences the stable equivalences and fibrant objects the Ω–spectra. Moreover, a stable equivalence between Ω–spectra is a levelwise weak homotopy equivalence. We obtain a Quillen adjunction

$$\Sigma^\infty\colon \boldsymbol{S}_* \longleftrightarrow \mathbf{Spt} :\Omega^\infty,$$

where Σ^∞ takes a based space to its suspension spectrum and Ω^∞ assigns to a spectrum its 0^{th} space.

A spectrum E is called *connective* if its negative homotopy groups vanish. The case that E is an Ω–spectrum is equivalent to E_n being $(n-1)$–connected for all $n \in \mathbb{N}$. Note that a map $f\colon E \to F$ between connective Ω–spectra is a stable equivalence if and only if $f_0\colon E_0 \to F_0$ is a weak homotopy equivalence. We denote by $\mathbf{Spt}_{\geq 0}$ the full subcategory of connective spectra. It is a reflective subcategory of **Spt** and we denote the left adjoint of the inclusion by

$$(-)_{\geq 0}\colon \mathbf{Spt} \to \mathbf{Spt}_{\geq 0}.$$

We will need two operations on spectra. The first one is the shift functor

$$(-)[1]\colon \mathbf{Spt} \to \mathbf{Spt}$$

defined on a spectrum E by setting $E[1]_n = E_{n+1}$ and obvious structure maps. The second operation is the loop functor

$$\Omega\colon \mathbf{Spt} \to \mathbf{Spt}$$

defined by $(\Omega E)_n = \Omega(E_n)$ and looping the structure maps.

We recall Segal's infinite loop space machine [9], which provides many examples of connective spectra. We denote by Γ^{op} the skeleton of the category of finite pointed sets and pointed maps, ie its objects are the sets $m_+ := \{*, 1, \dots, m\}$. A Γ*–space* is a functor

$$\Gamma^{\mathrm{op}} \to \boldsymbol{S}_*$$

and we denote by $\Gamma\boldsymbol{S}_*$ the category of Γ–spaces and natural transformations.

There are distinguished maps $\rho_i\colon m_+ \to 1_+$ defined by $\rho_i(k) = *$ if $k \neq i$ and $\rho_i(i) = 1$. Let $A \in \Gamma\boldsymbol{S}_*$. The *Segal map* is the map

$$A(m_+) \xrightarrow{\prod_{i=1}^m \rho_i} \prod_m A(1_+).$$

A Γ–space is called *special* if the Segal map is a weak homotopy equivalence. If $A \in \Gamma\boldsymbol{S}_*$ is special, the set $\pi_0(A(1_+))$ is a monoid with multiplication induced by the span

$$A(1_+) \leftarrow A(2_+) \xrightarrow{\simeq} A(1_+) \times A(1_+),$$

where the left map is the map sending i to 1 for $i = 1, 2$, and the right map is the Segal map. A special Γ–space is called *very special* if this monoid is actually a group.

In [2], Bousfield and Friedlander construct a model structure on $\Gamma\boldsymbol{S}_*$ with fibrant objects the very special Γ–spaces and weak equivalences between fibrant objects levelwise weak equivalences.

There is a functor $\boldsymbol{B}\colon \Gamma\boldsymbol{S}_* \to \mathbf{Spt}$ defined as follows. Denote by $\mathbb{S}\colon \Gamma^{\mathrm{op}} \to \boldsymbol{S}_*$ the inclusion of finite pointed sets into pointed spaces. Given $A \in \Gamma\boldsymbol{S}_*$ we have an (enriched) left Kan extension along $\mathbb{S}$

$$\begin{tikzcd}
\Gamma^{\mathrm{op}} \arrow[r, "A"] \arrow[d, "\mathbb{S}"'] & \boldsymbol{S}_* \\
\boldsymbol{S}_* \arrow[ur, dashed] &
\end{tikzcd}$$

and we denote this left Kan extension by $L_{\mathbb{S}}A$. Now define $\boldsymbol{B}A_n := L_{\mathbb{S}}A(S^n)$. The structure maps are given by the image of the identity morphism $S^1 \wedge S^n \to S^1 \wedge S^n$ under the composite map

$$\begin{aligned}
\boldsymbol{S}_*(S^1 \wedge S^n, S^1 \wedge S^n) &\cong \boldsymbol{S}_*(S^1, \boldsymbol{S}_*(S^n, S^{n+1})) \\
&\to \boldsymbol{S}_*(S^1, \boldsymbol{S}_*(L_{\mathbb{S}}A(S^n), L_{\mathbb{S}}(A(S^{n+1})))) \\
&\cong \boldsymbol{S}_*(S^1 \wedge L_{\mathbb{S}}A(S^n), L_{\mathbb{S}}A(S^{n+1})).
\end{aligned}$$

By the Barratt–Priddy–Quillen theorem $L_{\mathbb{S}}\mathbb{S}$ is the sphere spectrum, hence the notation.

The functor $\boldsymbol{B}$ has a right adjoint $\boldsymbol{A}\colon \mathbf{Spt} \to \Gamma\boldsymbol{S}_*$ given by sending a spectrum $E \in \mathbf{Spt}$ to the Γ–space

$$n_+ \mapsto \mathbf{Spt}(\mathbb{S}^{\times n}, E)$$

using the topological enrichment of spectra. Moreover, the adjoint pair $\boldsymbol{B} \dashv \boldsymbol{A}$ is a Quillen pair which induces an equivalence of categories

$$\mathrm{Ho}(\Gamma\boldsymbol{S}_*) \simeq \mathrm{Ho}(\mathbf{Spt}_{\geq 0}).$$

In view of this equivalence we will say that a *Γ–space A is a model for a connective spectrum E* if there is a stable equivalence $\mathbb{L}\boldsymbol{B}A \simeq E$, where $\mathbb{L}\boldsymbol{B}$ is the left derived functor. The main theorem of Segal [9] states that $\boldsymbol{B}$ sends cofibrant-fibrant Γ–spaces to connective Ω–spectra.

Finally we make the following convention. We will refer to any zigzag of equivalences (of spaces, spectra or Γ–spaces) as simply an *equivalence*.

3 Recollection on spaces of manifolds

We recall the spaces $\Psi_\theta(\mathbb{R}^n)$ of embedded manifolds with tangential structure from Galatius and Randall-Williams [4]. Denote by $\mathrm{Gr}_d(\mathbb{R}^n)$ the Grassmannian manifold of d–dimensional planes in $\mathbb{R}^n$ and denote $BO(d) := \mathrm{colim}_{n\in\mathbb{N}} \mathrm{Gr}_d(\mathbb{R}^n)$ induced by the standard inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$. Let $\theta\colon X \to BO(d)$ be a Serre fibration and let $M \subset \mathbb{R}^n$ be a d–dimensional embedded smooth manifold. Then a *tangential θ–structure* on M is a lift

$$\begin{tikzcd} & X \arrow[d, "\theta"] \\ M \arrow[ur, dashed] \arrow[r, "\tau_M"'] & BO(d) \end{tikzcd}$$

where τ_M is the classifying map of the tangent bundle (determined by the embedding). The topological space $\Psi_\theta(\mathbb{R}^n)$ has as underlying set pairs (M, l), where M is a d–dimensional smooth manifold without boundary which is closed as a subset of $\mathbb{R}^n$, and $l\colon M \to X$ is a θ–structure. We refer to [4] and for a description of the topology. We will also in general suppress the tangential structure from the notation.

For $0 \leq k \leq n$, we have the subspaces $\psi_\theta(n,k) \subset \Psi_\theta(\mathbb{R}^n)$ of those manifolds $M \subset \mathbb{R}^n$, satisfying

$$M \subset \mathbb{R}^k \times (-1,1)^{n-k}.$$

In other words, $\psi_\theta(n,k)$ consists of manifolds with k possibly noncompact and $(n-k)$ compact directions. We denote

$$\Psi_\theta(\mathbb{R}^\infty) := \operatorname{colim}_{n\in\mathbb{N}} \Psi_\theta(\mathbb{R}^n)$$
$$\psi_\theta(\infty,k) := \operatorname{colim}_{n\in\mathbb{N}} \psi_\theta(n,k),$$

where the colimit is again induced by the standard inclusions. In [1] it is shown that the topological spaces $\Psi_\theta(\mathbb{R}^n)$ are metrizable and hence in particular compactly generated weak Hausdorff spaces.

For all $n \in \mathbb{N}$ and $1 \le k \le n-1$ we have a map

$$\begin{aligned} \mathbb{R} \times \psi_\theta(n,k) &\to \psi_\theta(n,k+1), \\ (t, M) &\mapsto M - t \cdot e_{k+1}, \end{aligned}$$

where e_{k+1} denotes the $(k+1)^{\mathrm{st}}$ standard basis vector. This descends to a map $S^1 \wedge \psi_\theta(n,k) \to \psi_\theta(n,k+1)$ when taking as basepoint the empty manifold.

Theorem 3.1 *The adjoint map*

$$\psi_\theta(n,k) \to \Omega\psi_\theta(n,k+1)$$

is a weak homotopy equivalence.

Proof See [4, Theorem 3.20]. □

Definition 3.2 Let ψ_θ be the spectrum with n^{th} space given by

$$(\psi_\theta)_n := \psi_\theta(\infty, n+1)$$

and structure maps given by the adjoints of the translations.

By the above theorem, the spectrum ψ_θ is an Ω–spectrum.

4 The weak homotopy type of $\psi_\theta(\infty, 1)$

This section contains a brief review of the main theorem of [5] as proven in [4]. Recall first the construction of the *Madsen–Tillmann spectrum* $\mathrm{MT}\theta(d)$ associated to a Serre fibration $\theta\colon X \to BO(d)$. Denote by $X(\mathbb{R}^n)$ the pullback

$$\begin{tikzcd} X(\mathbb{R}^n) \arrow[r] \arrow[d, "\theta_n"'] & X \arrow[d, "\theta"] \\ \mathrm{Gr}_d(\mathbb{R}^n) \arrow[r] & BO(d) \end{tikzcd}$$

and by $\gamma_{d,n}^{\perp}$ the orthogonal complement of the tautological bundle over $\mathrm{Gr}_d(\mathbb{R}^n)$. Then define the spectrum $T\theta(d)$ to have as n^{th} space the Thom space of the pullback bundle $T\theta(d)_n := \mathrm{Th}(\theta_n^* \gamma_{d,n}^{\perp})$. The structure maps are given by

$$S^1 \wedge \mathrm{Th}(\theta_n^* \gamma_{d,n}^{\perp}) \cong \mathrm{Th}(\theta_n^* \gamma_{d,n}^{\perp} \oplus \varepsilon) \to \mathrm{Th}(\theta_{n+1}^* \gamma_{d,n+1}^{\perp}),$$

where ε denotes the trivial bundle. Then define the Madsen–Tillmann spectrum $\mathrm{MT}\theta(d)$ to be a fibrant replacement of the spectrum $T\theta(d)$. Since the adjoints of the structure maps of $T\theta(d)$ are inclusions, we can give an explicit construction of $\mathrm{MT}\theta(d)$ as

$$\mathrm{MT}\theta(d)_n := \mathrm{colim}_k\, \Omega^k T\theta(d)_{n+k}.$$

Hence we have $\Omega^\infty \mathrm{MT}\theta(d) = \mathrm{colim}_k\, \Omega^k T\theta(d)_k$.

The passage from $\mathrm{MT}\theta(d)$ to our spaces of manifolds is as follows. We have a map

$$\mathrm{Th}(\theta_n^* \gamma_{d,n}^{\perp}) \to \Psi_\theta(\mathbb{R}^n)$$

given by sending an element (V, u, x), where $V \in \mathrm{Gr}_d(\mathbb{R}^n)$, $u \in V^\perp$ and $x \in X$, to the translated plane $V - u \in \Psi_\theta(\mathbb{R}^n)$ with constant θ–structure at x and sending the basepoint to the empty manifold.

Theorem 4.1 [4, Theorem 3.22] *The map* $\mathrm{Th}(\theta_n^* \gamma_{d,n}^{\perp}) \to \Psi_\theta(\mathbb{R}^n)$ *is a weak homotopy equivalence.*

On the other hand, by Theorem 3.1 we also have a weak homotopy equivalence

$$\psi_\theta(n,1) \to \Omega^{n-1}\Psi_\theta(\mathbb{R}^n).$$

Combining the two equivalences, we obtain

$$\Omega^{n-1}\,\mathrm{Th}(\theta_n^* \gamma_{d,n}^{\perp}) \xrightarrow{\simeq} \Omega^{n-1}\Psi_\theta(\mathbb{R}^n) \xleftarrow{\simeq} \psi_\theta(n,1).$$

Now we have a map

$$S^1 \wedge \Psi_\theta(\mathbb{R}^n) \to \Psi_\theta(\mathbb{R}^{n+1}),$$
$$(t, M) \mapsto M \times \{t\},$$

and we obtain the following commutative diagram:

$$\begin{tikzcd}
\Omega^{n-1}\,\mathrm{Th}(\theta_n^* \gamma_{d,n}^{\perp}) \arrow[r, "\simeq"] \arrow[d] & \Omega^{n-1}\Psi_\theta(\mathbb{R}^n) \arrow[d] & \psi_\theta(n,1) \arrow[l, "\simeq"'] \arrow[d] \\
\Omega^{n}\,\mathrm{Th}(\theta_{n+1}^* \gamma_{d,n+1}^{\perp}) \arrow[r, "\simeq"] & \Omega^{n}\Psi_\theta(\mathbb{R}^{n+1}) & \psi_\theta(n+1,1) \arrow[l, "\simeq"']
\end{tikzcd}$$

Finally, letting $n \to \infty$ we can determine the weak homotopy type of $\psi_\theta(\infty, 1)$.

Theorem 4.2 *There are weak equivalences of spaces*

$$\Omega^\infty \mathrm{MT}\theta(d)[1] \xrightarrow{\simeq} \operatorname{colim}_{n\in\mathbb{N}} \Omega^{n-1}\Psi_\theta(\mathbb{R}^n) \xleftarrow{\simeq} \psi_\theta(\infty,1).$$

5 Γ–space models for $\mathrm{MT}\theta(d)$ and ψ_θ

In this section we construct Γ–space models for the spectra $\mathrm{MT}\theta(d)$ and ψ_θ. The comparison of these Γ–spaces to the respective spectra relies heavily on results of May and Thomason [8].

We will encounter the following situation.

Definition 5.1 A functor $E\colon \Gamma^{\mathrm{op}} \to \mathbf{Spt}$ is called a Γ–*spectrum*. It is called a *special* Γ–spectrum if the Segal map

$$E(m_+) \to \prod_m E(1_+)$$

is a stable equivalence. Furthermore, we denote by $\Gamma^{(k)}E$ the Γ–space given by evaluating at the k^{th} space, that is,

$$\Gamma^{(k)}E(m_+) := E(m_+)_k.$$

The key proposition for showing that we have constructed the right Γ–spaces will be the following.

Proposition 5.2 *Let $E\colon \Gamma^{\mathrm{op}} \to \mathbf{Spt}$ be projectively fibrant and special. Then the Γ–space $\Gamma^{(k)}E$ is a model for the connective cover of $E(1_+)[k]$.*

Before we can prove the proposition, we will need some lemmas. The first one concerns the behavior of Segal's functor $\boldsymbol{B}$ with respect to the loop functor.

Lemma 5.3 *For $A \in \Gamma\boldsymbol{S}_*$ there is a natural map of spectra*

$$\boldsymbol{B}\Omega A \to \Omega\boldsymbol{B}A$$

which is the identity on 0^{th} spaces.

Proof Since $\mathbb{S}\colon \Gamma^{\mathrm{op}} \to \boldsymbol{S}_*$ is fully faithful, we have a strictly commutative diagram of functors:

$$\begin{tikzcd}
\Gamma^{\mathrm{op}} \arrow[r, "\Omega A"] \arrow[d, "\mathbb{S}"'] & \boldsymbol{S}_* \\
\boldsymbol{S}_* \arrow[ur, "L_{\mathbb{S}}\Omega A"'] &
\end{tikzcd}$$

The composition of the loop functor with the left Kan extension $\Omega L_{\mathbb{S}} A$ also gives a strictly commutative diagram:

$$\begin{tikzcd} \Gamma^{\mathrm{op}} \arrow[r, "\Omega A"] \arrow[d, "\mathbb{S}"'] & \boldsymbol{S}_* \\ \boldsymbol{S}_* \arrow[ur, "\Omega L_{\mathbb{S}} A"'] & \end{tikzcd}$$

Hence by the universal property of the left Kan extension we get a natural transformation $\gamma\colon L_{\mathbb{S}}\Omega A \Rightarrow \Omega L_{\mathbb{S}} A$. Now the components at the spheres assemble into a map of spectra $\boldsymbol{B}\Omega A \to \Omega \boldsymbol{B} A$, since by naturality we have a commutative diagram:

$$\begin{tikzcd} S^1 \wedge L_{\mathbb{S}}\Omega A(S^n) \arrow[r] \arrow[d, "\mathrm{id}\wedge\gamma"'] & L_{\mathbb{S}}\Omega A(S^{n+1}) \arrow[d, "\gamma"] \\ S^1 \wedge \Omega L_{\mathbb{S}} A(S^n) \arrow[r] & \Omega L_{\mathbb{S}} A(S^{n+1}) \end{tikzcd}$$

Finally, since $S^0 = 1_+ \in \Gamma^{\mathrm{op}}$ the map of spectra is the identity on 0^{th} spaces. □

In general for any $A \in \Gamma\boldsymbol{S}_*$ the spectrum $\boldsymbol{B}A$ might not have the right stable homotopy type, as the functor $\boldsymbol{B}$ only preserves weak equivalences between cofibrant objects. However for very special Γ–spaces, there is a more convenient replacement, which gives the right homotopy type. As a second lemma we record the following fact from [8], which generalizes a construction of [9].

Lemma 5.4 *There is a functor $W\colon \Gamma\boldsymbol{S}_* \to \Gamma\boldsymbol{S}_*$ such that the following hold for all very special $X \in \Gamma\boldsymbol{S}_*$:*

- *The spectrum $\boldsymbol{B}WX$ is a connective Ω–spectrum.*
- *The Γ–space WX is very special and there is a weak equivalence $WX \to X$.*
- *If X, Y are very special and there is a weak equivalence $X \simeq Y$, then $\boldsymbol{B}WX \simeq \boldsymbol{B}WY$.*
- *There is a weak equivalence $W\Omega X \to \Omega WX$.*

Proof See [8, Appendix B]. □

The important thing for us will be that if $X \in \Gamma\boldsymbol{S}_*$ is very special, then $\boldsymbol{B}WX$ has the right stable homotopy type.

Lemma 5.5 (up and across lemma [8; 3]) *Let E^i, $i \in \mathbb{N}$ be a sequence of connective Ω–spectra together with stable equivalences $f^i\colon E^i \to \Omega E^{i+1}$. Let E_0 be the spectrum with $(E_0)_n := E_0^n$ and structure maps given by $f_0^n\colon E_0^n \to \Omega E_0^{n+1}$. Then there is a natural stable equivalence $E^0 \simeq E_0$.*

Note that in particular E_0 is connective. We are now ready to prove our key proposition.

Proof of Proposition 5.2 We prove the proposition for $k = 0$. The argument for higher k is completely analogous.

We first show that the Γ–space $\Gamma^{(0)}E$ is very special. Note that the Γ–spaces $\Gamma^{(k)}E$ are special, since E is projectively fibrant and thus the Segal map is a levelwise equivalence. It remains to show that $\pi_0(\Gamma^{(0)}E(1_+))$ is a group. To this end we compose with the functor $\boldsymbol{A}\colon \mathbf{Spt} \to \Gamma \boldsymbol{S}_*$ to obtain a functor

$$\Gamma^{\mathrm{op}} \xrightarrow{E} \mathbf{Spt} \xrightarrow{\boldsymbol{A}} \Gamma \boldsymbol{S}_*$$

which is equivalently a functor

$$\widehat{A} := \Gamma^{\mathrm{op}} \times \Gamma^{\mathrm{op}} \to \boldsymbol{S}_*.$$

Fixing the first variable gives a Γ–space

$$\widehat{A}(k_+)(-)\colon \Gamma^{\mathrm{op}} \to \boldsymbol{S}_*$$

which is obtained by first evaluating the Γ–spectrum E at k_+ and then applying the functor $\boldsymbol{A}$ to the spectrum $E(k_+)$. In particular, we have

$$\widehat{A}(1_+)(-) = \boldsymbol{A}(E(1_+))\colon \Gamma^{\mathrm{op}} \to \boldsymbol{S}_*,$$

which is very special by construction.

Fixing the second variable gives a Γ–space

$$\widehat{A}(-)(k_+)\colon \Gamma^{\mathrm{op}} \to \boldsymbol{S}_*$$

which is obtained as the composition

$$\Gamma^{\mathrm{op}} \xrightarrow{E} \mathbf{Spt} \xrightarrow{\boldsymbol{A}} \Gamma \boldsymbol{S}_* \xrightarrow{ev_{k_+}} \boldsymbol{S}_*,$$

where the last functor is given by evaluating a Γ–space at the object k_+. In particular, we have

$$\widehat{A}(-)(1_+) = \boldsymbol{A}(E(-))(1_+) = \Gamma^{(0)}E\colon \Gamma^{\mathrm{op}} \to \boldsymbol{S}_*,$$

which is special since $\Gamma^{(0)}E$ is special.

Now we have the following diagram, where the middle square commutes by functoriality:

$$\begin{tikzcd}
 & & \widehat{A}(1_+)(1_+)\times\widehat{A}(1_+)(1_+) \\
 & \widehat{A}(2_+)(2_+) \arrow[r] \arrow[d] & \widehat{A}(2_+)(1_+) \arrow[u, "\simeq"'] \arrow[d] \\
\widehat{A}(1_+)(1_+)\times\widehat{A}(1_+)(1_+) & \widehat{A}(1_+)(2_+) \arrow[l, "\simeq"] \arrow[r] & \widehat{A}(1_+)(1_+)
\end{tikzcd}$$

By the above identification of the Γ–spaces $\widehat{A}(-)(1_+)$ and $\widehat{A}(1_+)(-)$ we see that the right vertical span represents the monoid structure of $\Gamma^{(0)}E$ and the lower horizontal span represents the monoid structure of $\boldsymbol{A}E(1_+)$. In other words, the maps into the products in the lower left and upper right corner are given by the Segal maps while the maps into the lower right corner are the respective multiplications induced by the nontrivial map $2_+ \to 1_+$, as are the remaining maps.

Hence we obtain two monoid structures on $\pi_0(\widehat{A}(1_+)(1_+))$ induced by $\boldsymbol{A}E(1_+)$ and $\Gamma^{(0)}E$. The commutativity of the middle square is now precisely the statement that they are compatible, or in other words that one is a homomorphism for the other, thus by the Eckmann–Hilton argument they agree. We now observe that the monoid $\boldsymbol{A}E(1_+)$ is actually a group, since $\pi_0(\boldsymbol{A}E(1_+)(1_+))$ is the 0^{th} stable homotopy group of $E(1_+)$. It follows that $\Gamma^{(0)}E$ is very special.

As a next step, we compose with taking connective covers to obtain a special Γ–spectrum in connective Ω–spectra

$$E_{\geq 0}\colon \Gamma^{\mathrm{op}} \to \mathbf{Spt}_{\geq 0}.$$

Note that $\Gamma^{(0)}E \simeq \Gamma^{(0)}E_{\geq 0}$ and hence $\Gamma^{(0)}E_{\geq 0}$ is very special. For $k \geq 1$, the Γ–spaces $\Gamma^{(k)}E_{\geq 0}$ will automatically be very special since $E_{\geq 0}(1_+)$ is connective and hence $\pi_0(\Gamma^{(k)}E(1_+)) \cong \pi_0(E_{\geq 0}(1_+)_k) = 0$.

We now consider the spectra associated to the very special Γ–spaces $\Gamma^{(k)}E_{\geq 0}$, ie we apply May and Thomason's replacement followed by Segal's functor to obtain a sequence of connective Ω–spectra

$$\boldsymbol{B}W\Gamma^{(k)}E_{\geq 0} \quad \text{for } k \in \mathbb{N}.$$

Now by Lemma 5.4 we have the following equivalence:

$$\boldsymbol{B}W\Gamma^{(k)}E_{\geq 0} \xrightarrow{\simeq} \boldsymbol{B}W\Omega\Gamma^{(k+1)}E_{\geq 0} \xrightarrow{\simeq} \boldsymbol{B}\Omega W\Gamma^{(k+1)}E_{\geq 0}.$$

By Lemma 5.3 we have a map $\boldsymbol{B}\Omega W\Gamma^{(k)}E \to \Omega\boldsymbol{B}W\Gamma^{(k)}E$ which is the identity on 0^{th} spaces. In particular, since both spectra are Ω–spectra, this map is an equivalence on connective covers. We now observe that since $E(1_+)_{\geq 0}$ is a connective Ω–spectrum, we have

$$\pi_0(\boldsymbol{B}\,W\Gamma^{(k)}E) = \pi_0(E(1^+)_{\geq 0,k}) = 0$$

for $k \geq 1$ and hence $\Omega\boldsymbol{B}\,W\Gamma^{(k)}E$ is connective. Thus we obtain a stable equivalence

$$\boldsymbol{B}\Omega W\Gamma^{(k)}E \to \Omega\boldsymbol{B}\,W\Gamma^{(k)}E$$

for $k \geq 1$. Putting all these maps together we obtain a sequence of connective Ω–spectra $\boldsymbol{B}\,W\Gamma^{(k)}E_{\geq 0}$ together with stable equivalences

$$\boldsymbol{B}\,W\Gamma^{(k)}E_{\geq 0} \xrightarrow{\simeq} \boldsymbol{B}\,W\Omega\Gamma^{(k+1)}E_{\geq 0} \xrightarrow{\simeq} \boldsymbol{B}\Omega W\Gamma^{(k+1)}E_{\geq 0} \xrightarrow{\simeq} \Omega\boldsymbol{B}\,W\Gamma^{(k+1)}E_{\geq 0},$$

that is, we have $\boldsymbol{B}\,W\Gamma^{(k)}E_{\geq 0} \xrightarrow{\simeq} \Omega\boldsymbol{B}\,W\Gamma^{(k+1)}E_{\geq 0}$. Thus we are in the situation of Lemma 5.5 and conclude that

$$\boldsymbol{B}\,W\Gamma^{(0)}E_{\geq 0} = \boldsymbol{B}\,W\Gamma^{(0)}E \simeq E(1_+)_{\geq 0}. \qquad \square$$

In light of obtaining the right stable homotopy type, we will from now on assume that we replace a Γ–space A by WA before applying the functor $\boldsymbol{B}$, ie in what follows, $\boldsymbol{B}A$ will mean $\boldsymbol{B}\,WA$.

We start with constructing a Γ–space model for the (connective cover of the) spectrum ψ_θ. Recall that ψ_θ has as n^{th} space the space $\psi_\theta(\infty, n+1)$ and structure maps given by translation of manifolds in the $(n+1)^{\text{st}}$ coordinate. The idea is that the spaces $\psi_\theta(\infty, n)$ come with a preferred monoid structure, namely taking disjoint union of manifolds. To make this precise, we introduce the following notation.

Definition 5.6 Let $\theta\colon X \to BO(d)$ be a Serre fibration. We obtain for each $m \in \mathbb{N}$ the Serre fibration

$$\coprod_m \theta\colon \coprod_m X \to BO(d).$$

We denote this Serre fibration by $\theta(m_+)$.

We can now associate to each $m_+ \in \Gamma^{\mathrm{op}}$ the space $\Psi_{\theta(m_+)}(\mathbb{R}^n)$. We think of elements of $\Psi_{\theta(m_+)}(\mathbb{R}^n)$ as manifolds with components labeled by nonbasepoint elements of m_+ together with θ–structures on those labeled components.

Lemma 5.7 *For all $n \in \mathbb{N}$, the spaces $\Psi_{\theta(m_+)}(\mathbb{R}^n)$ assemble into a Γ–space.*

Proof We have to define the induced maps. Let $\sigma\colon m_+ \to k_+$ be a map of based sets. We obtain a map

$$\coprod_{\sigma^{-1}(k_+\backslash\{*\})} X \to \coprod_{k_+\backslash\{*\}} X.$$

Now define the induced map $\Psi_{\theta(m_+)}(\mathbb{R}^n) \to \Psi_{\theta(k_+)}(\mathbb{R}^n)$ as follows. The image of a pair (M, l) is given by the manifold

$$M' := l^{-1}\Big(\coprod_{\sigma^{-1}(k_+ \setminus \{*\})} X \Big)$$

together with $\theta(k_+)$–structure given by the composition

$$M' \xrightarrow{l|_{M'}} \coprod_{\sigma^{-1}(k_+ \setminus \{*\})} X \to \coprod_{k_+ \setminus \{*\}} X.$$

In other words, we relabel the components of M and forget about those components, which get labeled by the basepoint. Taking the empty manifold as basepoint, it is easy to see that this is functorial in Γ^{op}. □

Note that $\Psi_{\theta(0_+)} \cong *$ since it consists of only the empty manifold and we have $\Psi_{\theta(1_+)}(\mathbb{R}^n) = \Psi_\theta(\mathbb{R}^n)$. Also note that we obtain by restriction for any $k \geq 1$ the Γ–spaces

$$m_+ \mapsto \psi_{\theta(m_+)}(\infty, k).$$

As mentioned above, the Γ–structure can be thought of as taking disjoint union of manifolds. Below we will see that, when stabilizing to $\mathbb{R}^\infty$, taking disjoint union gives a homotopy coherent multiplication on our spaces of manifolds.

Lemma 5.8 *The spectra $\psi_{\theta(m_+)}$ assemble into a projectively fibrant Γ–spectrum.*

Proof By the above lemma we have for each $n \in \mathbb{N}$ and each map of finite pointed sets $\sigma\colon m_+ \to k_+$ a map

$$\sigma_*^n\colon \psi_{\theta(m_+)}(\infty, n+1) \to \psi_{\theta(k_+)}(\infty, n+1)$$

which is functorial in Γ^{op} for fixed n. Thus, we have to show that these maps commute with the structure maps, that is we need to show that the diagram

$$\begin{tikzcd} S^1 \wedge \psi_{\theta(m_+)}(\infty, n+1) \arrow[r] \arrow[d, "\mathrm{id}\wedge\sigma_*^n"'] & \psi_{\theta(m_+)}(\infty, n+2) \arrow[d, "\sigma_*^{n+1}"] \\ S^1 \wedge \psi_{\theta(k_+)}(\infty, n+1) \arrow[r] & \psi_{\theta(k_+)}(\infty, n+2) \end{tikzcd}$$

commutes. But this is clear since the structure maps just translate the manifolds in the $(n+1)^{\mathrm{st}}$ coordinate, while the map σ_*^n relabels the components. □

Definition 5.9 We denote by $\Gamma\psi_\theta$ the Γ–spectrum

$$m_+ \mapsto \psi_{\theta(m_+)}.$$

To avoid awkward notation, we will denote the induced Γ–spaces $\Gamma^{(k)}(\Gamma\psi_\theta)$ simply by $\Gamma^{(k)}\psi_\theta$.

Proposition 5.10 *The Γ–space $\Gamma^{(0)}\psi_\theta$ is a model for the connective cover of ψ_θ, ie there is a stable equivalence*

$$\boldsymbol{B}\Gamma^{(0)}\psi_\theta \simeq \psi_{\theta,\geq 0}.$$

Proof We show that $\Gamma\psi_\theta$ is a special Γ–spectrum. The assertion then follows from Proposition 5.2. Since $\psi_{\theta(m_+)}$ is an Ω–spectrum for all $m_+ \in \Gamma^{\mathrm{op}}$, it suffices to show that $\Gamma^{(k)}\psi_\theta$ is a special Γ–space for every k.

We observe that the Segal map for $\Gamma^{(k)}\psi_\theta$

$$\Gamma^{(k)}\psi_\theta(m_+) \to \prod_m \Gamma^{(k)}\psi_\theta(m_+)$$

is an embedding, and we identify its image with a subspace of the product space. This subspace can be characterized as follows. A tuple $(M_1, \dots, M_m)$ lies in this subspace if and only if $M_i \cap M_j = \varnothing \subset \mathbb{R}^\infty$ for all $i \neq j$. We show that this subspace is a weak deformation retract of the product space

$$\prod_m \Gamma^{(k)}\psi_\theta(m_+) = \prod_m \psi_\theta(\infty, k+1).$$

To this end, we need a map that makes manifolds (or more generally any subsets) disjoint inside $\mathbb{R}^\infty$. Consider the maps

$$F\colon \mathbb{R}^\infty \to \mathbb{R}^\infty,$$
$$(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots),$$

as well as for any $a \in \mathbb{R}$ the map

$$G_a\colon \mathbb{R}^\infty \to \mathbb{R}^\infty,$$
$$(x_1, x_2, \dots) \mapsto (a + x_1, x_2, \dots).$$

These maps are clearly homotopic to the identity via a straight line homotopy. Choosing $a \in (-1, 1)$, the composition $G_a \circ F\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ induces a self-map

$$\psi_\theta(\infty, k+1) \to \psi_\theta(\infty, k+1)$$

which is homotopic to the identity. Using for each factor of the product space $\prod_m \psi_\theta(\infty, k+1)$ a different (fixed) real number gives a map

$$\prod_m \psi_\theta(\infty, k+1) \to \prod_m \psi_\theta(\infty, k+1),$$

which is our desired deformation retract; this is also illustrated in Figure 1. □

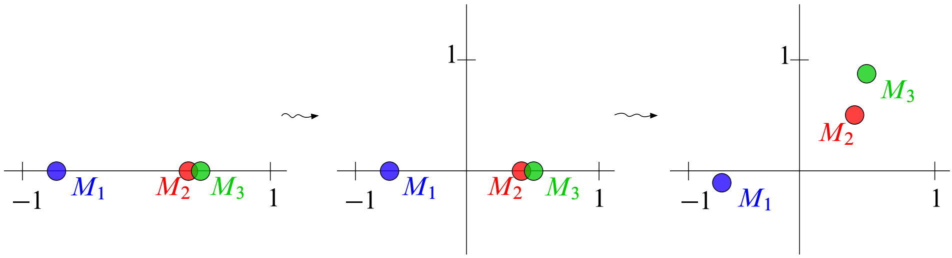

Figure 1: Making manifolds disjoint

Recall from Lemma 5.7 that the association

$$m_+ \mapsto \Psi_{\theta(m_+)}(\mathbb{R}^n)$$

defines a Γ–space for all $n \in \mathbb{N}$.

Definition 5.11 Denote by $\Gamma\Psi_\theta$ the (levelwise) colimit of Γ–spaces

$$\Gamma\Psi_\theta(m_+) := \operatorname{colim}_{n\in\mathbb{N}} \Omega^{n-1}\Psi_{\theta(m_+)}(\mathbb{R}^n).$$

From Theorem 4.2 we obtain for each $m_+ \in \Gamma^{\mathrm{op}}$ equivalences

$$\Gamma^{(0)}\psi_\theta(m_+) = \psi_{\theta(m_+)}(\infty, 1) \xrightarrow{\simeq} \operatorname{colim}_{n\in\mathbb{N}} \Omega^{n-1}\Psi_{\theta(m_+)}(\mathbb{R}^n) = \Gamma\Psi_\theta(m_+)$$

which are clearly functorial in Γ^{op}. Hence we obtain a levelwise equivalence of Γ–spaces $\Gamma^{(0)}\psi_\theta \xrightarrow{\simeq} \Gamma\Psi_\theta$.

Corollary 5.12 *The Γ–space $\Gamma\Psi_\theta$ is a model for the connective cover of the spectrum ψ_θ.*

We now construct a Γ–space model for the Madsen–Tillmann spectrum $\mathrm{MT}\theta(d)$ and we will show in the next section that this Γ–space is equivalent to $\Gamma\Psi_\theta$. As before, we will use the Serre fibrations $\theta(m_+)$. First note that the construction of the Madsen–Tillmann spectrum commutes with coproducts over $BO(d)$, that is we have $\mathrm{MT}\theta(m_+)(d) \cong \bigvee_m \mathrm{MT}\theta(d)$.

Definition 5.13 Define the Γ–spectrum $\Gamma\mathrm{MT}\theta(d)\colon \Gamma^{\mathrm{op}} \to \mathbf{Spt}$ by setting

$$\Gamma\mathrm{MT}\theta(d)(m_+) := \mathrm{MT}\theta(m_+)(d).$$

For any based map $\sigma\colon m_+ \to k_+$, define the induced map to be the fold map

$$\Gamma\mathrm{MT}\theta(d)(m_+) \cong \bigvee_m \mathrm{MT}\theta(d) \to \bigvee_k \mathrm{MT}\theta(d) \cong \Gamma\mathrm{MT}\theta(d)(k_+).$$

As before, we will denote the induced Γ–spaces by $\Gamma^{(k)}\mathrm{MT}\theta(d)$ for all $k \in \mathbb{N}$.

Proposition 5.14 *The Γ–space $\Gamma^{(1)}\mathrm{MT}\theta(d)$ is a model for the connective cover of the spectrum $\mathrm{MT}\theta(d)[1]$.*

Proof Again it suffices to show that $\Gamma\mathrm{MT}\theta(d)$ is special. But this follows easily since in **Spt** we have a stable equivalence

$$\mathrm{MT}\theta(m_+)(d) \cong \bigvee_m \mathrm{MT}\theta(d) \simeq \prod_m \mathrm{MT}\theta(d).$$

Thus by Proposition 5.2 we obtain a stable equivalence

$$\boldsymbol{B}\,\Gamma^{(1)}\mathrm{MT}\theta(d) \simeq \mathrm{MT}\theta(d)[1]_{\geq 0}. \qquad \square$$

6 Equivalence of Γ–space models

In the previous section we have constructed the Γ–space models $\Gamma\Psi_\theta$ for ψ_θ and $\Gamma^{(1)}\mathrm{MT}\theta(d)$ for $\mathrm{MT}\theta(d)[1]_{\geq 0}$. But more is true; by Theorem 4.2 we have for each $m_+ \in \Gamma^{\mathrm{op}}$ a weak equivalence of spaces

$$\begin{aligned}\Gamma^{(1)}\mathrm{MT}\theta(d)(m_+) &= \Omega^\infty \mathrm{MT}\theta(m_+)(d)[1] \\ &\xrightarrow{\simeq} \operatorname{colim}_{n\in\mathbb{N}} \Omega^{n-1}\Psi_{\theta(m_+)}(\mathbb{R}^n) = \Gamma\Psi_\theta(m_+).\end{aligned}$$

The following lemma shows that these equivalences define a levelwise equivalence of Γ–spaces.

Lemma 6.1 *The weak equivalences of Theorem 4.1*

$$\mathrm{Th}\big(\theta_n^*\gamma_{d,n}^\perp\big) \xrightarrow{\simeq} \Psi_\theta(\mathbb{R}^n)$$

assemble into a map of Γ–spaces. In particular, we obtain a levelwise equivalence

$$\Gamma^{(1)}\mathrm{MT}\theta(d) \xrightarrow{\simeq} \Gamma\Psi_\theta.$$

Proof We need to show that for any map of based sets $\sigma\colon m_+ \to k_+$ the diagram

$$\begin{tikzcd} \mathrm{Th}\big(\theta_n(m_+)^*\gamma_{d,n}^\perp\big) \arrow[r] \arrow[d, "\sigma_*"'] & \Psi_{\theta(m_+)}(\mathbb{R}^n) \arrow[d, "\sigma_*"] \\ \mathrm{Th}\big(\theta_n(k_+)^*\gamma_{d,n}^\perp\big) \arrow[r] & \Psi_{\theta(k_+)}(\mathbb{R}^n) \end{tikzcd}$$

commutes. But this follows easily since the left-hand vertical map is just the fold map. In particular one can view this map as relabeling components of the wedge and mapping components labeled by $*$ to the basepoint. On the other hand this is precisely the description of the right-hand vertical map. $\square$

We can now prove the first part of our main theorem.

Theorem 6.2 *There is an equivalence of spectra*

$$\mathrm{MT}\theta(d)[1]_{\geq 0} \simeq \psi_{\theta,\geq 0}.$$

Proof By Lemma 6.1, we have an equivalence of Γ–spaces

$$\Gamma^{(1)}\mathrm{MT}\theta(d) \xrightarrow{\simeq} \Gamma\Psi_\theta.$$

By Proposition 5.14, $\Gamma^{(1)}\mathrm{MT}\theta(d)$ is a model for the spectrum $\mathrm{MT}\theta(d)[1]_{\geq 0}$, while by Proposition 5.10 and its corollary, the Γ–space $\Gamma\Psi_\theta$ is a model for the connective cover of ψ_θ. Hence we obtain equivalences

$$\mathrm{MT}\theta(d)[1]_{\geq 0} \simeq \boldsymbol{B}\Gamma^{(1)}\mathrm{MT}\theta(d) \simeq \boldsymbol{B}\Gamma\Psi_\theta \simeq \psi_{\theta,\geq 0}. \qquad \square$$

7 The cobordism category

In the previous section we have exhibited an equivalence between the connective covers of the spectra $\mathrm{MT}\theta(d)[1]$ and ψ_θ. It remains to relate these spectra to the (classifying space of the) topological cobordism category.

Classically, the d–dimensional cobordism category has as objects closed $(d-1)$–dimensional manifolds and morphisms given by diffeomorphism classes of cobordisms. It is a symmetric monoidal category with monoidal product given by taking disjoint union of manifolds. We will see that this is also true for the topological variant in a sense we will make precise below. In particular, having a symmetric monoidal structure endows the classifying space of the cobordism category with the structure of an infinite loop space, and we will see that it is equivalent as such to the infinite loop space associated to $\mathrm{MT}\theta(d)[1]$.

Recall that a *topological category* $\mathcal{C}$ has a space of objects $\mathcal{C}_0$ and a space of morphisms $\mathcal{C}_1$ together with source and target maps $s,t\colon \mathcal{C}_1 \to \mathcal{C}_0$, a composition map $c\colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$, and a unit map $e\colon \mathcal{C}_0 \to \mathcal{C}_1$, which satisfy the usual associativity and unit laws. There have appeared several definitions of the cobordism category as a topological category, which all have equivalent classifying spaces. The relevant model for us will be the topological poset model of [4]. We recall its definition. Define D_θ to be the subspace

$$D_\theta \subset \mathbb{R} \times \psi_\theta(\infty, 1)$$

consisting of pairs (t, M) where $t \in \mathbb{R}$ is a regular value of the projection onto the first coordinate $M \subset \mathbb{R} \times (-1,1)^\infty \to \mathbb{R}$. Order its elements by $(t,M) \leq (t',M')$ if and only if $t \leq t'$ with the usual order on $\mathbb{R}$ and $M = M'$.

Definition 7.1 The *d–dimensional cobordism category* $\mathrm{Cob}_\theta(d)$ is the topological category associated to the topological poset D_θ. That is, its space of objects is given by $\mathrm{ob}(\mathrm{Cob}_\theta(d)) = D_\theta$, and its space of morphisms is given by the subspace $\mathrm{mor}(\mathrm{Cob}_\theta(d)) \subset \mathbb{R}^2 \times \psi_\theta(\infty, 1)$ consisting of triples (t_0, t_1, M), where $t_0 \le t_1$. The source and target maps are simply given by forgetting regular values.

Given a topological category $\mathcal{C}$ we can take its internal nerve yielding a simplicial space

$$N_\bullet \mathcal{C}\colon \Delta^{\mathrm{op}} \to \boldsymbol{S}$$

as follows. The space of 0–simplices and 1–simplices is given by $\mathcal{C}_0$ and $\mathcal{C}_1$ respectively. For $n \ge 2$ the space of n–simplices is given by the n–fold fiber product

$$N_n\mathcal{C} := \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1.$$

The face and the degeneracy maps are obtained from the structure maps of the topological category. The associativity and unit laws ensure that we indeed obtain a simplicial space. Applying this construction to the cobordism category now yields a simplicial space

$$N_\bullet \mathrm{Cob}_\theta(d)\colon \Delta^{\mathrm{op}} \to \boldsymbol{S}.$$

We will also write $\mathrm{Cob}_\theta(d)$ for the simplicial space obtained from taking the nerve and write $\mathrm{Cob}_\theta(d)_k$ for the space of k–simplices.

Considering $\psi_\theta(\infty, 1)$ as a constant simplicial space, we have a forgetful map of simplicial spaces $\mathrm{Cob}_\theta(d) \to \psi_\theta(\infty, 1)$ defined on k–simplices by

$$\mathrm{Cob}_\theta(d)_k \to \psi_\theta(\infty, 1),$$
$$(\underline{t}, M) \mapsto M.$$

Theorem 7.2 *The forgetful map induces a weak equivalence*

$$B\,\mathrm{Cob}_\theta(d) \xrightarrow{\simeq} \psi_\theta(\infty, 1),$$

where $B\,\mathrm{Cob}_\theta(d)$ is the realization of the simplicial space $\mathrm{Cob}_\theta(d)$.

Proof See [4, Theorem 3.10]. □

We now encode the symmetric monoidal structure of $\mathrm{Cob}_\theta(d)$ in terms of a Γ–structure.

Lemma 7.3 *The simplicial spaces $\mathrm{Cob}_{\theta(m_+)}(d)$ assemble into a Γ–object in simplicial spaces*

$$\mathrm{Cob}_{\theta(-)}(d)\colon \Gamma^{\mathrm{op}} \to \boldsymbol{S}^{\Delta^{\mathrm{op}}}.$$

Proof For $m_+ \in \Gamma^{\mathrm{op}}$ the k–simplices are given as subspaces

$$\mathrm{Cob}_{\theta(m_+)}(d)_k \subset \mathbb{R}^{k+1} \times \psi_{\theta(m_+)}(\infty, 1) = \mathbb{R}^{k+1} \times \Gamma^{(0)}\psi_\theta(m_+).$$

Thus for a map $\sigma\colon m_+ \to n_+$, we define the map

$$\mathrm{Cob}_{\theta(m_+)}(d) \to \mathrm{Cob}_{\theta(n_+)}(d)$$

on k–simplices to be induced by the map

$$\mathrm{id} \times \sigma_*\colon \mathbb{R}^{k+1} \times \Gamma^{(0)}\psi_\theta(m_+) \to \mathbb{R}^{k+1} \times \Gamma^{(0)}\psi_\theta(n_+),$$

where σ_* comes from the functoriality in Γ^{op} of the Γ–space $\Gamma^{(0)}\psi_\theta$. From this description it is clear that the maps just defined are functorial in Δ^{op} and hence define a map of simplicial spaces. □

Definition 7.4 Denote by $\Gamma\,\mathrm{Cob}_\theta(d)$ the Γ–object in simplicial spaces

$$\begin{aligned} \Gamma\,\mathrm{Cob}_{\theta(m_+)}(d) &\to \boldsymbol{S}^{\Delta^{\mathrm{op}}}, \\ m_+ &\mapsto \mathrm{Cob}_{\theta(m_+)}(d). \end{aligned}$$

Composing with the realization of simplicial spaces we get a functor

$$B\Gamma\,\mathrm{Cob}_\theta(d)\colon \Gamma^{\mathrm{op}} \to \boldsymbol{S}.$$

We obtain a Γ–space by choosing as basepoints the elements $(\underline{0}, \varnothing) \in \mathrm{Cob}_\theta(d)_k$ for all $k \in \mathbb{N}$.

Lemma 7.5 *The forgetful map induces a levelwise equivalence of Γ–spaces*

$$B\Gamma\,\mathrm{Cob}_\theta(d) \xrightarrow{\simeq} \Gamma^{(0)}\psi_\theta.$$

Proof By construction it is clear that the forgetful maps are functorial in Γ^{op} so that they indeed define a map of Γ–spaces. By Theorem 7.2, these maps are weak equivalences and hence we obtain a levelwise equivalence of Γ–spaces. □

In particular, the Γ–space $B\Gamma\,\mathrm{Cob}_\theta(d)$ is very special, and applying Segal's functor we obtain a connective Ω–spectrum, which we denote by $\boldsymbol{B}\Gamma\,\mathrm{Cob}_\theta(d)$ to avoid awkward notation. In conclusion, we obtain an equivalence of spectra

$$\boldsymbol{B}\Gamma\,\mathrm{Cob}_\theta(d) \xrightarrow{\simeq} \boldsymbol{B}\Gamma^{(0)}\psi_\theta.$$

Combining with Theorem 6.2, we obtain our main theorem.

Main Theorem *There are stable equivalences of spectra*

$$\boldsymbol{B}\Gamma\,\mathrm{Cob}_\theta(d) \simeq \boldsymbol{B}\Gamma^{(0)}\psi_\theta \simeq \mathrm{MT}\theta(d)[1]_{\geq 0},$$

such that the induced equivalences

$$\Omega^\infty \boldsymbol{B}\Gamma\,\mathrm{Cob}_\theta(d) \simeq \Omega^\infty\psi_\theta \simeq \Omega^\infty \mathrm{MT}\theta(d)[1]$$

are equivalent to the equivalences of Theorem 4.2 and Theorem 7.2.

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