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Operad bimodules and composition products on André–Quillen filtrations of algebras

NICHOLAS J KUHN
LUÍS ALEXANDRE PEREIRA

If $\mathcal{O}$ is a reduced operad in a symmetric monoidal category of spectra (S–modules), an $\mathcal{O}$–algebra I can be viewed as analogous to the augmentation ideal of an augmented algebra. From the literature on topological André–Quillen homology, one can see that such an I admits a canonical (and homotopically meaningful) decreasing $\mathcal{O}$–algebra filtration $I \xleftarrow{\sim} I^1 \leftarrow I^2 \leftarrow I^3 \leftarrow \cdots$ satisfying various nice properties analogous to powers of an ideal in a ring.

We more fully develop such constructions in a manner allowing for more flexibility and revealing new structure. With R a commutative S–algebra, an $\mathcal{O}$–bimodule M defines an endofunctor of the category of $\mathcal{O}$–algebras in R–modules by sending such an $\mathcal{O}$–algebra I to $M \circ_{\mathcal{O}} I$. We explore the use of the bar construction as a derived version of this. Letting M run through a decreasing $\mathcal{O}$–bimodule filtration of $\mathcal{O}$ itself then yields the augmentation ideal filtration as above. The composition structure of the operad then induces pairings among these bimodules, which in turn induce natural transformations $(I^i)^j \to I^{ij}$, fitting nicely with previously studied structure.

As a formal consequence, an $\mathcal{O}$–algebra map $I \to J^d$ induces compatible maps $I^n \to J^{dn}$ for all n. This is an essential tool in the first author's study of Hurewicz maps for infinite loop spaces, and its utility is illustrated here with a lifting theorem.

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1 Introduction

Let S–mod be the category of symmetric spectra (see Hovey, Shipley and Smith [7]), one of the standard symmetric monoidal models for the category of spectra. Let S denote the sphere spectrum, and let $\mathcal{O}$ be a reduced operad in S–mod. If R is a commutative S–algebra, we let $\mathsf{Alg}_{\mathcal{O}}(R)$ denote the category of $\mathcal{O}$–algebras in R–modules.

The starting point of this paper is the observation that if M is a reduced $\mathcal{O}$–bimodule and $I \in \mathsf{Alg}_{\mathcal{O}}(R)$ then $M \circ_{\mathcal{O}} I$ is again in $\mathsf{Alg}_{\mathcal{O}}(R)$, and that many interesting constructions on $\mathcal{O}$–algebras are derived versions of functors of I of this form.

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Our first goal, presented in Section 2, is to develop the basic properties of a suitable derived version of $M \circ_{\mathcal{O}} I$, the bar construction $B(M, \mathcal{O}, I)$, noting particularly how structure on the category of $\mathcal{O}$–bimodules is reflected in the category of endofunctors of $\mathcal{O}$–algebras.

In Section 2.1 and Section 2.2, we begin by introducing the setting in which we wish to work. This includes the model structure on $\mathsf{Alg}_{\mathcal{O}}(R)$ developed by Harper [3], which piggybacks off of the "positive" model structure on S–mod first exploited by Shipley [15].

Theorem 2.11 lays out the basic properties of $B(M, \mathcal{O}, I)$ needed for homotopical analysis. For example, a levelwise homotopy fibration sequence in the bimodule variable M induces a homotopy fibration sequence in $\mathsf{Alg}_{\mathcal{O}}(R)$.

In Section 2.5 and Section 2.6, we describe $B(M, \mathcal{O}, I)$ in the case where M is concentrated in one level, in terms of the topological André–Quillen spectrum $\mathrm{TQ}(I) = B(\mathcal{O}(1), \mathcal{O}, I)$. This allows us to easily identify, in Section 2.7, the Goodwillie tower of $F_M(I) = B(M, \mathcal{O}, I)$, viewed as an endofunctor of $\mathsf{Alg}_{\mathcal{O}}(R)$. In particular, one learns that $\partial_* \mathrm{Id} = \mathcal{O}$, $\partial_* F_M = M$, and one gets the expected chain rule: $\partial_*(F_M \circ F_N) \simeq M \circ_{\mathcal{O}} N$. See Pereira [13] for more about Goodwillie calculus in this setting.

Remark 1.1 $\mathrm{TQ}(I)$ can be informally viewed as I/I^2: its study goes back to Basterra [1]. The results in Section 2, and their proofs, clearly have much in common with Harper and Hess [5, Section 4], and our definition of $\mathrm{TQ}(I)$ agrees with that in Harper [4]. However those authors use only one special family of bimodules in the M variable, whereas for applications in this paper, and in ongoing work, greater generality is essential. In particular, we try to make clear that, on the one hand, our constructions connect nicely to $\mathrm{TQ}(I)$ and, on the other, they are well suited to iteration using the monoidal properties of $\circ$.

Another new aspect of our work, also crucial to applications (see, for example, Kuhn [9]), is that throughout we also have change-of-rings statements allowing for passage from $\mathsf{Alg}_{\mathcal{O}}(R)$ to $\mathsf{Alg}_{\mathcal{O}}(R')$, given a map $R \to R'$ of commutative S–algebras.

An $\mathcal{O}$–algebra I can be viewed as similar to the augmentation ideal in an augmented ring. In Section 3, we apply our bar construction to a natural decreasing $\mathcal{O}$–bimodule filtration of $\mathcal{O}$ itself, defining, for $I \in \mathsf{Alg}_{\mathcal{O}}(R)$, a homotopically meaningful natural augmentation ideal filtration

$$I \xleftarrow{\sim} I^1 \leftarrow I^2 \leftarrow I^3 \leftarrow \cdots .$$

The results of the Section 2 show that I^n/I^{n+1} is determined by $\mathcal{O}(n)$ and $\mathrm{TQ}(I)$. Our model makes it easy to analyze connectivity properties: if R and $\mathcal{O}$ are connective and I is $(c-1)$–connected, then I^n will be $(nc-1)$–connected.

The construction of such a filtration, or more precisely, the associated tower under I,

$$I^1/I^2 \leftarrow I/I^3 \leftarrow I/I^4 \leftarrow \cdots,$$

goes back to Minasian [12] and Kuhn [8] when $\mathcal{O} = \mathrm{Com}$. Harper and Hess [5] construct this tower in exactly the same way we do. However, we now add new structure by taking advantage of the observation that a pairing of bimodules

$$L \circ_{\mathcal{O}} M \to N$$

will induce a natural transformation of functors of $I \in \mathsf{Alg}_{\mathcal{O}}(R)$,

$$F_L(F_M(I)) \to F_N(I).$$

The multiplication $\mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ induces pairings among our $\mathcal{O}$–bimodule filtration of $\mathcal{O}$, and these in turn induce natural pairings

$$(I^n)^m \to I^{mn}$$

satisfying expected properties. As a formal consequence, an $\mathcal{O}$–algebra map $I \to J^d$ induces compatible maps $I^n \to J^{dn}$ for all n.

This seems to be fundamental structure which has not previously appeared in the literature. The following result is a consequence illustrating its utility.

Theorem 1.2 *Let $f\colon I \to J$ be a map in the homotopy category* $\mathrm{ho}\,\mathsf{Alg}_{\mathcal{O}}(R)$. *If f factors as $f = f_s \circ \cdots \circ f_1$ such that* $\mathrm{TQ}(f_i)$ *is null for all i, then there is a lifting in* $\mathrm{ho}\,\mathsf{Alg}_{\mathcal{O}}(R)$*:*

$$\begin{tikzcd} & J^{2^s} \arrow[d] \\ I \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & J \end{tikzcd}$$

We restate this, with slightly different notation, as Theorem 3.11.

Further applications in this spirit can be seen in the work of the first author on Hurewicz maps of infinite loop spaces [9], the project whose needs motivated this paper.

The deeper proofs from Section 2 are deferred to Section 4, which itself is supported by the Appendix. Much of the technical work consists of generalizing results and arguments from Pereira [14] from S–mod to R–mod for a general R.

2 General results about derived composition products

2.1 Our categories of modules and algebras

In this paper, the category of S–modules will mean the category of symmetric spectra as defined in [7]: $X \in S$–mod consists of a sequence $X_0, X_1, X_2, \dots$ of simplicial sets equipped with extra structure.

With the smash product as product and sphere spectrum S as unit, S–mod is a closed symmetric monoidal category. There is a notion of weak equivalence, and various model structures on S–mod compatible with these, such that the resulting quotient category models the standard stable homotopy category.

Recall that a symmetric sequence in S–mod then consists of a sequence

$$X(0), X(1), X(2), \dots,$$

where $X(n)$ is a symmetric spectrum equipped with an action of the n^{th} symmetric group Σ_n. The category of such symmetric sequences in S–mod, $\mathsf{Sym}(S)$, admits a composition product $\circ$ defined by

$$(1) \qquad (X \circ Y)(s) = \bigvee_r X(r) \wedge_{\Sigma_r} \Bigl(\bigvee_{\phi \colon \mathbf{s} \to \mathbf{r}} Y(s_1(\phi)) \wedge \cdots \wedge Y(s_r(\phi)) \Bigr),$$

where $\mathbf{s} = \{1, \dots, s\}$ and $s_k(\phi)$ is the cardinality of $\phi^{-1}(k)$. With this product, $(\mathsf{Sym}(S), \circ, S(1))$ is monoidal, where $S(1) = (*, S, *, *, \dots)$.

An operad $\mathcal{O}$ is then a monoid in this category, and one makes sense of left $\mathcal{O}$–modules, right $\mathcal{O}$–modules, and $\mathcal{O}$–bimodules in the usual way. Furthermore, if X is a right $\mathcal{O}$–module, and Y is a left $\mathcal{O}$–module, the symmetric sequence $X \circ_{\mathcal{O}} Y$ can be defined as the coequalizer in $\mathsf{Sym}(S)$ of the two evident maps

$$X \circ \mathcal{O} \circ Y \rightrightarrows X \circ Y.$$

Extra structure on X or Y can then induce evident extra structure on $X \circ_{\mathcal{O}} Y$.

For the purposes of this paper, it is natural to require that our operads $\mathcal{O}$ and bimodules M be reduced: $\mathcal{O}(0) = * = M(0)$. By contrast, an $\mathcal{O}$–algebra is a left $\mathcal{O}$–module I concentrated in level 0: $I(n) = *$ for all $n > 0$.

If R is a commutative S–algebra, these definitions and constructions extend to the category of R–modules. Furthermore, one can mix and match. For example, if X is a symmetric sequence in S–mod and Y is a symmetric sequence in R–mod, $X \circ Y$ will be the symmetric sequence in R–mod with

$$(X \circ Y)(s) = \bigvee_r X(r) \wedge_{\Sigma_r} \Bigl(\bigvee_{\phi \colon \mathbf{s} \to \mathbf{r}} Y(s_1(\phi)) \wedge_R \cdots \wedge_R Y(s_r(\phi)) \Bigr).$$

We denote by $\mathsf{Sym}(R)$ the category of symmetric sequences in R–mod, $\mathsf{Alg}_{\mathcal{O}}(R)$ the category of $\mathcal{O}$–algebras in R–mod and $\mathsf{Mod}^l_{\mathcal{O}}(R)$ the category of left $\mathcal{O}$–modules in $\mathsf{Sym}(R)$.

Remark 2.1 If $\mathcal{O}$ is an operad in S–mod, then the symmetric sequence $R \wedge \mathcal{O}$, defined as $(R \wedge \mathcal{O})(n) = R \wedge \mathcal{O}(n)$ is naturally an operad when viewed either in $\mathsf{Sym}(R)$ or $\mathsf{Sym}(S)$. It is easily checked that the category $\mathsf{Mod}^l_{\mathcal{O}}(R)$ is isomorphic to $\mathsf{Mod}^l_{R\wedge\mathcal{O}}(S)$, and that both of these are isomorphic to $\mathsf{Mod}^l_{R\wedge\mathcal{O}}$, the category of left $R \wedge \mathcal{O}$–modules in $\mathsf{Sym}(R)$. A similar remark holds for the three corresponding categories of algebras.

2.2 Model structures

We specify model structures on the various categories just described.

We accept as given the S–model structure on symmetric spectra (called S–modules in this paper) as defined in [7, Definition 5.3.6] and [15, Theorem 2.4]. This is monoidal with respect to the smash product [7, Corollary 5.3.8].

We then give $\mathsf{Sym}(S)$ its associated *injective* model structure: weak equivalences and cofibrations are those morphisms which are levelwise weak equivalences and cofibrations in S–mod. That this structure exists was checked in [14, Theorem 3.8 and Section 5.3].[1]

As in [11, Section 15], [15], [5] and [14, Section 5.3], we need positive variants of these model structures. Weak equivalences will be as before, but there are fewer cofibrations: for $X \to Y$ in S–mod to be a positive cofibration, we now insist that $X_0 \to Y_0$ also be an isomorphism, and for $M \to N$ in $\mathsf{Sym}(S)$ to be a positive cofibration, we now insist that $M(0)_0 \to N(0)_0$ also be an isomorphism.[2] It is worth noting that if $M \in \mathsf{Sym}(S)$ is reduced, then it is positive cofibrant exactly when each $M(n)$ is cofibrant, when viewed in S–mod.

Given a commutative S–algebra R, the positive R–model structure on R–modules is then defined to be the projective structure induced from that on S–mod with its positive structure: weak equivalences and fibrations in R–mod are the maps which are weak equivalences and positive fibrations in S–mod. Similarly, we define the positive structure on $\mathsf{Sym}(R)$, the category of symmetric sequences in R–mod, to be the projective structure induced from that on $\mathsf{Sym}(S)$ with its positive structure: weak equivalences and fibrations in $\mathsf{Sym}(R)$ are the maps which are weak equivalences and positive fibrations in $\mathsf{Sym}(S)$.

[1] This structure is different from the associated projective structure used in [3; 4; 5].

[2] On $\mathsf{Sym}(S)$, this agrees with [14] but is different from [5], where it is required that $M(n)_0 \to N(n)_0$ be an isomorphism for all n.

Thanks to Remark 2.1, the following theorem is an immediate consequence of [14, Theorem 1.4]; see also [3]. Special cases go back to [15].

Theorem 2.2 *$\mathsf{Alg}_{\mathcal{O}}(R)$ has a projective model structure induced from the positive structure on R–mod: $f\colon I \to J$ is a weak equivalence if it is one in R–mod (and thus in S–mod), and a fibration if it is a positive fibration in R–mod (and thus in S–mod). Similarly, $\mathsf{Mod}^l_{\mathcal{O}}(R)$ has a projective model structure induced from the positive structure on $\mathsf{Sym}(R)$: $f\colon M \to N$ is a weak equivalence if it is one in $\mathsf{Sym}(R)$ (and thus in $\mathsf{Sym}(S)$), and a fibration if it is a positive fibration in $\mathsf{Sym}(R)$ (and thus in $\mathsf{Sym}(S)$).*

The next lemma says that the model structure on $\mathsf{Alg}_{\mathcal{O}}(R)$ is really the same as the model structure on $\mathsf{Mod}^l_{\mathcal{O}}(R)$, restricted to the subcategory of modules concentrated in degree 0.

Lemma 2.3 *An algebra map $I \to J$ is a cofibration in $\mathsf{Alg}_{\mathcal{O}}(R)$ if and only if it is a cofibration in $\mathsf{Mod}^l_{\mathcal{O}}(R)$, when I and J are regarded as objects in $\mathsf{Sym}(R)$ concentrated in level 0.*

Proof The inclusion $\mathsf{Alg}_{\mathcal{O}}(R) \hookrightarrow \mathsf{Mod}^l_{\mathcal{O}}(R)$ has right adjoint given by $M \mapsto M(0)$. This is a Quillen pair, as it is easily checked that this right adjoint preserves weak equivalences and fibrations. □

2.3 Cofibrancy assumption on $\mathcal{O}$ and first consequences

Unless stated otherwise, we make the following standing cofibrancy assumption about our operad $\mathcal{O}$.

Assumption 2.4 *The map $S(1) \to \mathcal{O}$ is assumed to be a positive cofibration in $\mathsf{Sym}(S)$.*

As $\mathcal{O}(0) = *$ has been assumed earlier, equivalently this means that, in S–mod, $S \to \mathcal{O}(1)$ is a cofibration, and $\mathcal{O}(n)$ is cofibrant for all n.

Notation 2.5 Let $\mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$ be the full subcategory of $\mathsf{Alg}_{\mathcal{O}}(R)$ consisting of $\mathcal{O}$–algebras in R–mod which are cofibrant when just viewed as R–modules.

A key advantage of our particular model structure on $\mathsf{Alg}_{\mathcal{O}}(R)$ is that the following property holds.

Proposition 2.6 *The forgetful functor $\mathsf{Alg}_{\mathcal{O}}(R) \to R$–mod preserves cofibrations between cofibrant objects. In particular, if I is cofibrant in $\mathsf{Alg}_{\mathcal{O}}(R)$, then $I \in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$.*

When $R = S$, this is [14, Theorem 1.5]. We defer the proof of the general case to Section 4.

It follows that a functorial cofibrant replacement functor on $\mathsf{Alg}_{\mathcal{O}}(R)$ takes values in $\mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$. More elementary but also useful is that $\mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$ is well behaved under change of rings.

Lemma 2.7 *Let $R \to R'$ be a map of commutative S–algebras. Then*

$$R' \wedge_R \_\_ \colon \mathsf{Alg}_{\mathcal{O}}(R) \to \mathsf{Alg}_{\mathcal{O}}(R')$$

restricts to a functor

$$R' \wedge_R \_\_ \colon \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}} \to \mathsf{Alg}_{\mathcal{O}}(R')^{\mathrm{c}}$$

which preserves weak equivalences.

Proof This is immediate since $R' \wedge_R \_\_$ is left adjoint to a forgetful functor that is easily seen to be right Quillen. □

2.4 General properties of the bar construction

We will make much use of the bar construction. Given an $\mathcal{O}$–bimodule M and $I \in \mathsf{Alg}_{\mathcal{O}}(R)$, $B(M,\mathcal{O},I) \in \mathsf{Alg}_{\mathcal{O}}(R)$ is defined as the geometric realization of the simplicial object $B_\bullet(M,\mathcal{O},I)$ in R–mod defined by

$$B_n(M,\mathcal{O},I) = M \circ \overbrace{\mathcal{O} \circ \cdots \circ \mathcal{O}}^{n} \circ I.$$

Similarly if M and N are $\mathcal{O}$–bimodules, then $B(M,\mathcal{O},N)$ is again an $\mathcal{O}$–bimodule.

The theme of the next set of results is that this construction is well behaved when the $\mathcal{O}$–bimodules are positive cofibrant in $\mathsf{Sym}(S)$, and $I \in \mathsf{Alg}_{\mathcal{O}}(R)$ is cofibrant in R–mod. (We recall that a reduced $M \in \mathsf{Sym}(S)$ is positively cofibrant exactly when it is levelwise cofibrant.)

Proposition 2.8 *Let M, N be levelwise cofibrant $\mathcal{O}$–bimodules. Then $B(M,\mathcal{O},N)$ is again levelwise cofibrant. Similarly, if M is levelwise cofibrant and I is in $\mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$, then $B(M,\mathcal{O},I) \in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$.*

The first statement is immediately implied by [14, Theorem 1.6] which says that $B_\bullet(M,\mathcal{O},N)$ is Reedy cofibrant in the category of simplicial objects in $\mathsf{Sym}(S)$. We defer the proof of the second statement for general R to Section 4.

We also record the following, which shows that the bar construction can be usefully used as a derived circle product. This will also be proved in Section 4.

Proposition 2.9 *Let M be a levelwise cofibrant right $\mathcal{O}$–module. If I is cofibrant in $\mathsf{Alg}_{\mathcal{O}}(R)$, the natural map $B(M,\mathcal{O},I) \to M \circ_{\mathcal{O}} I$ is a weak equivalence. Similarly, if N is cofibrant in $\mathsf{Mod}^{l}_{\mathcal{O}}(S)$, then $B(M,\mathcal{O},N) \to M \circ_{\mathcal{O}} N$ is a weak equivalence.*

To emphasize the functors defined by levelwise cofibrant bimodules, we change notation.

Definition 2.10 If M is a levelwise cofibrant $\mathcal{O}$–bimodule, define

$$F^R_M\colon \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}} \to \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$$

by the formula $F^R_M(I) = B(M, \mathcal{O}, I)$.

Theorem 2.11 *The F^R_M construction satisfies the following properties:*

(a) *The functor sending (M, I) to $F^R_M(I)$ takes weak equivalences in either the M or I variable to weak equivalences in $\mathsf{Alg}_{\mathcal{O}}(R)$.*

(b) *A levelwise homotopy fibration[3] sequence of levelwise cofibrant $\mathcal{O}$–bimodules*

$$L \to M \to N$$

induces a homotopy fibration sequence in $\mathsf{Alg}_{\mathcal{O}}(R)$

$$F^R_L(I) \to F^R_M(I) \to F^R_N(I).$$

(c) *There is a natural isomorphism of functors*

$$F^R_M \circ F^R_N \simeq F^R_{B(M,\mathcal{O},N)}.$$

(d) *Let $R \to R'$ be a map of commutative S–algebras. There is a natural isomorphism in $\mathsf{Alg}_{\mathcal{O}}(R')$*

$$F^{R'}_M(R' \wedge_R I) \simeq R' \wedge_R F^R_M(I).$$

Parts (a) and (b) will be proved in Section 4. By contrast, parts (c) and (d) are straightforward. Part (c) follows from the natural isomorphism

$$B(M, \mathcal{O}, B(N, \mathcal{O}, I)) \simeq B(B(M, \mathcal{O}, N), \mathcal{O}, I),$$

while part (d) follows from the natural isomorphism

$$R' \wedge_R B(M, \mathcal{O}, I) \simeq B(M, \mathcal{O}, R' \wedge_R I).$$

Remark 2.12 As there is a natural map $B(M, \mathcal{O}, N) \to M \circ_{\mathcal{O}} N$, it follows that a bimodule pairing

$$\mu\colon M \circ_{\mathcal{O}} N \to L$$

[3] Equivalently, we could say cofibration, as levelwise homotopy fibration sequences agree with levelwise homotopy cofibration sequences.

induces a natural transformation

$$\mu\colon F^R_M \circ F^R_N \to F^R_L$$

defined as the composite

$$F^R_M \circ F^R_N \simeq F^R_{B(M,\mathcal{O},N)} \to F^R_{M\circ_{\mathcal{O}} N} \to F^R_L.$$

See Section 3 for examples of this.

2.5 Topological André–Quillen homology

In the next two subsections, we consider our constructions when M is concentrated in just one level, ie there exists an n such that $M(m) = *$ for all $m \neq n$. We show that then $F^R_M(I)$ is determined by $M(n)$ together with the topological André–Quillen homology of I.

We first need to define this last term in our context. The S–module $\mathcal{O}(1)$ will be an associative S–algebra, and can be viewed as an operad concentrated in level 1. From this point of view, the evident maps $\mathcal{O}(1) \to \mathcal{O}$ and $\mathcal{O} \to \mathcal{O}(1)$ are both maps of operads, and the second of these gives $\mathcal{O}(1)$ the structure of an $\mathcal{O}$–bimodule concentrated in level 1.

Let $R\mathcal{O}(1)$–mod be the category of $R \wedge \mathcal{O}(1)$–modules. It is illuminating to note that this category is also $\mathsf{Alg}_{\mathcal{O}(1)}(R)$, when one views $\mathcal{O}(1)$ as an operad. The map $\mathcal{O} \to \mathcal{O}(1)$ induces a functor

$$z\colon R\mathcal{O}(1)\text{–mod} \to \mathsf{Alg}_{\mathcal{O}}(R)$$

with left adjoint

$$Q = \mathcal{O}(1) \circ_{\mathcal{O}} \_\colon \mathsf{Alg}_{\mathcal{O}}(R) \to R\mathcal{O}(1)\text{–mod}$$

making the pair of functors into a Quillen pair.

Definition 2.13 Define $\mathrm{TQ}\colon \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}} \to R\mathcal{O}(1)$–mod by the formula $\mathrm{TQ}(I) = B(\mathcal{O}(1), \mathcal{O}, I)$.

The next proposition is a special case of Proposition 2.9.

Proposition 2.14 *If I is cofibrant in $\mathsf{Alg}_{\mathcal{O}}(R)$, the natural map $\mathrm{TQ}(I) \to Q(I)$ is an equivalence.*

As TQ is thus equivalent to the left derived functor of the left Quillen functor Q, one has the next two consequences. To state the first, we let $[I, J]_{\mathsf{Alg}}$ denote morphisms between I and J in the homotopy category of $\mathsf{Alg}_{\mathcal{O}}(R)$, and we similarly let $[M, N]_{\mathsf{Mod}}$ denote morphisms between M and N in the homotopy category of $R\mathcal{O}(1)$–mod.

Corollary 2.15 *There is an adjunction in the associated homotopy categories*

$$[\mathrm{TQ}(I), N]_{\mathsf{Mod}} \simeq [I, z(N)]_{\mathsf{Alg}}.$$

Corollary 2.16 *If $I \to J \to K$ is a homotopy cofibration sequence in $\mathsf{Alg}_{\mathcal{O}}(R)$, then*

$$\mathrm{TQ}(I) \to \mathrm{TQ}(J) \to \mathrm{TQ}(K)$$

is a homotopy cofibration sequence in $R\mathcal{O}(1)$–mod.

The next result is a particular instance of Theorem 2.11(d).

Proposition 2.17 *Let $R \to R'$ be a map of commutative S–algebras. There is a natural isomorphism*

$$\mathrm{TQ}(R' \wedge_R I) \simeq R' \wedge_R \mathrm{TQ}(I).$$

The first TQ here is with respect to the S–algebra R'.

2.6 $\mathcal{O}$–bimodules with one term

As before, we can view $\mathcal{O}(1)$ as either a commutative S–algebra or an operad concentrated in level 1.

Suppose $M \in \mathsf{Sym}(S)$ is a right $\mathcal{O}(1)$–module with the operad interpretation, ie one has $M \circ \mathcal{O}(1) \to M$ making appropriate diagrams commute. Unraveling the definitions, one sees that this structure map amounts to Σ_n–equivariant maps

$$M(n) \wedge \mathcal{O}(1)^{\wedge n} \to M(n)$$

exhibiting $M(n)$ as an $\mathcal{O}(1)^{\wedge n}$–module. Equivalently, each $M(n)$ will be a right $\Sigma_n \wr \mathcal{O}(1)$–module, where $\Sigma_n \wr \mathcal{O}(1)$ is the associative algebra with underlying S–module $\bigvee_{\sigma \in \Sigma_n} \mathcal{O}(1)^{\wedge n}$, and evident "twisted" multiplication.

From this, it is easy to see that if $J \in \mathsf{Alg}_{\mathcal{O}(1)}(R) = R\mathcal{O}(1)$–mod, then

$$M \circ_{\mathcal{O}(1)} J = \bigvee_n M(n) \wedge_{\Sigma_n \wr \mathcal{O}(1)} J^{\wedge_R n}.$$

Now suppose, given $M(n)$, an $(\mathcal{O}(1), \Sigma_n \wr \mathcal{O}(1))$–bimodule. Abusing notation, we will also write $M(n)$ for the symmetric sequence concentrated at level n:

$$M(n) = (*, \dots, *, M(n), *, \dots).$$

From this point of view, $M(n)$ is precisely an $\mathcal{O}(1)$–bimodule, where $\mathcal{O}(1)$ is viewed as an operad. Furthermore, an $\mathcal{O}$–bimodule structure on $M(n)$ will necessarily be an $\mathcal{O}(1)$–bimodule structure pulled back via the map of operads $\mathcal{O} \to \mathcal{O}(1)$.

Theorem 2.18 *Suppose $M(n)$ is also a cofibrant S–module. Then, for $I \in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$, there is a natural isomorphism*

$$F^R_{M(n)}(I) = z(M(n) \wedge_{\Sigma_n \wr \mathcal{O}(1)} \mathrm{TQ}(I)^{\wedge_R n}),$$

and a natural equivalence

$$z(B(M(n), \mathcal{O}(1), \mathrm{TQ}(I))) \xrightarrow{\sim} F^R_{M(n)}(I).$$

Proof We suppress some applications of z, the pullback along $\mathcal{O} \to \mathcal{O}(1)$. Firstly, one has natural isomorphisms

$$\begin{aligned} M(n) \wedge_{\Sigma_n \wr \mathcal{O}(1)} \mathrm{TQ}(I)^{\wedge_R n} &= M(n) \wedge_{\Sigma_n \wr \mathcal{O}(1)} B(\mathcal{O}(1), \mathcal{O}, I)^{\wedge_R n} \\ &= M(n) \circ_{\mathcal{O}(1)} B(\mathcal{O}(1), \mathcal{O}, I) \\ &= B(M(n), \mathcal{O}, I) \\ &= F^R_{M(n)}(I). \end{aligned}$$

Secondly, the equivalence $B(M(n), \mathcal{O}(1), \mathcal{O}(1)) \xrightarrow{\sim} M(n)$ induces the equivalence

$$\begin{aligned} B(M(n), \mathcal{O}(1), \mathrm{TQ}(I)) &= B(M(n), \mathcal{O}(1), B(\mathcal{O}(1), \mathcal{O}, I)) \\ &= B(B(M(n), \mathcal{O}(1), \mathcal{O}(1)), \mathcal{O}, I) \\ &\xrightarrow{\sim} B(M(n), \mathcal{O}, I) \\ &= F^R_{M(n)}(I). \end{aligned}$$ □

Corollary 2.19 *Let $f\colon I \to J$ be a morphism in $\mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$. With $M(n)$ as in the theorem, if $\mathrm{TQ}(f)$ is a weak equivalence, so is $F^R_{M(n)}(f)$.*

2.7 The Goodwillie tower of F^R_M

The second author has studied Goodwillie calculus on the category $\mathsf{Alg}_{\mathcal{O}}(R)$ [13]. Here we sketch how our results above lead to an understanding of the Goodwillie tower of the functor F^R_M.

Given a levelwise cofibrant $\mathcal{O}$–bimodule M, let $M^{\le n}$ denote the $\mathcal{O}$–bimodule with

$$M^{\le n}(k) = \begin{cases} M(k) & \text{if } k \le n, \\ * & \text{if } k > n. \end{cases}$$

Definition 2.20 Let $P_n F^R_M = F^R_{M^{\le n}}\colon \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}} \to \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$.

Theorem 2.21 *The Goodwillie tower of the functor F^R_M identifies with*

$$P_1 F^R_M \leftarrow P_2 F^R_M \leftarrow P_3 F^R_M \leftarrow \cdots,$$

and its n^{th} derivative $\partial_n F^R_M$ identifies with $M(n)$.

Sketch of proof The sequence of $\mathcal{O}$–bimodules

$$M(n) \to M^{\le n} \to M^{\le (n-1)}$$

satisfies the hypothesis of Theorem 2.11(b). Thus the homotopy fiber of the map

$$P_n F^R_M(I) \to P_{n-1} F^R_M(I)$$

identifies as $F^R_{M(n)}(I)$, which Theorem 2.18 tells us is

$$z(M(n) \wedge_{\Sigma_n \wr \mathcal{O}(1)} \mathrm{TQ}(I)^{\wedge_R n}).$$

This is a homogeneous n–excisive functor; note that Corollary 2.16 first tells us that TQ is a homogeneous linear functor. See [13, Theorem 3.2] for more detail.

It follows that $P_n F^R_M$ is n–excisive. With a bit more care, one can now check that the natural transformation $F^R_M \to P_n F^R_M$ identifies with the map from F^R_M to its n–excisive quotient: the proof of [13, Theorem 4.3] generalizes immediately to our setting. □

Under connectivity hypotheses, one gets very concrete convergence estimates. Say that $X \in \mathsf{Sym}(S)$ is connective if each $X(n) \in S$–mod is connective, ie -1–connected.

Proposition 2.22 *If R, M, and $\mathcal{O}$ are connective, and I is $(c-1)$–connected, then the map $F^R_M(I) \to P_n F^R_M(I)$ is $(n+1)c$–connected.*

Proof We need to show that the homotopy fiber is $((n+1)c-1)$–connected. By Theorem 2.11(b), this homotopy fiber identifies with $B(M^{>n}, \mathcal{O}, I)$, where

$$M^{>n}(k) = \begin{cases} M(k) & \text{if } k > n, \\ * & \text{if } k \le n. \end{cases}$$

This fiber then is the homotopy colimit (in R–modules) of a diagram of R–modules of the form

$$M(r) \wedge \mathcal{O}(s_1) \wedge \cdots \wedge \mathcal{O}(s_k) \wedge I^{\wedge_R t},$$

with $t \geq r > n$. In particular, it is a homotopy colimit of a diagram of $((n+1)c-1)$–connected R–modules, and so is itself $((n+1)c-1)$–connected. □

These results also show the following, when combined with Corollary 2.19.

Theorem 2.23 *Let $f\colon I \to J$ be a morphism in $\mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$. If $\mathrm{TQ}(f)$ is a weak equivalence, so is $P_n F_M^R(f)$ for any n and any levelwise cofibrant $\mathcal{O}$–bimodule M. Furthermore, if R, M and $\mathcal{O}$ are connective, and I and J are 0–connected, then $F_M^R(f)$ is itself a weak equivalence.*

Special cases of this theorem appear in [8] and [5].

3 Application to the augmentation ideal filtration

In our constructions, when the $\mathcal{O}$–bimodule is $\mathcal{O}$ itself, the resulting functor, sending an $\mathcal{O}$–algebra I to $F_{\mathcal{O}}^R(I) = B(\mathcal{O},\mathcal{O},I)$, is naturally weakly equivalent to the identity. In this section we study structure on the augmentation ideal filtration of I arising from using the levelwise bimodule filtration of $\mathcal{O}$ in conjunction with the operad structure $\mathcal{O} \circ \mathcal{O} \to \mathcal{O}$.

3.1 Construction and basic properties of the filtration

Definitions 3.1 Let $1 \leq i < m \leq \infty$.

(a) Let $\mathcal{O}_i^m$ denote the $\mathcal{O}$–bimodule with $\mathcal{O}_i^m(k) = \begin{cases} \mathcal{O}(k) & \text{if } i \leq k < m, \\ * & \text{otherwise.} \end{cases}$

(b) For $I \in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$, let $I_m^i = F_{\mathcal{O}_i^m}^R(I) = B(\mathcal{O}_i^m, \mathcal{O}, I)$.

Note that there is a natural weak equivalence $I_\infty^1 \to I$. We sometimes write I^i for I_∞^i, and readers are encouraged to view I_m^i as I^i/I^m; see Theorem 3.4(b) below.

For $j \leq i$ and $n \leq m$, it is not hard to see that the evident map

$$\mathcal{O}_i^m \to \mathcal{O}_j^n$$

is a map of $\mathcal{O}$–bimodules, and thus induces natural maps

$$I_m^i \to I_n^j$$

for all $I \in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$.

Special cases of these are illustrated in the next examples.

Example 3.2 $I \in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$ has a natural augmentation ideal filtration

$$I \xleftarrow{\sim} I^1 \leftarrow I^2 \leftarrow I^3 \leftarrow \cdots .$$

Example 3.3 $I^1_n = P_{n-1} F^R_{\mathcal{O}}(I)$ in the notation of the last section, so the tower

$$I^1_2 \leftarrow I^1_3 \leftarrow I^1_4 \leftarrow \cdots$$

identifies with the Goodwillie tower of the identity functor on $\mathsf{Alg}_{\mathcal{O}}(R)$. This tower, defined as we do here, is the subject of study in [5].

These examples are related: the filtration of the first example appears as the homotopy fibers of the maps from I to the tower in the second example. More precisely, there are homotopy fiber sequences

$$I^n \to I^1 \to I^1_n.$$

This is a special case of property (b) in the next theorem.

Theorem 3.4 *The functors sending I to I^i_n satisfy the following properties:*

(a) *They preserve weak equivalences in the variable $I \in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$.*

(b) *For $1 \le i < m < l \le \infty$, the sequence $I^m_l \to I^i_l \to I^i_m$ is a homotopy fiber sequence. In particular, $I^m \to I^i \to I^i_m$ is a homotopy fiber sequence.*

(c) *There are natural isomorphisms $I^1_2 = z(\mathrm{TQ}(I))$, and more generally, $I^k_{k+1} = z(\mathcal{O}(k) \wedge_{\Sigma_k \wr \mathcal{O}(1)} \mathrm{TQ}(I)^{\wedge_R k})$.*

(d) *Let $R \to R'$ be a map of commutative S–algebras. There is a natural isomorphism $R' \wedge_R I^i_n \simeq (R' \wedge_R I)^i_n$.*

All of these properties follow immediately from the more general results of Section 2. For example, part (b) follows from Theorem 2.11(b) applied to the sequence of $\mathcal{O}$–bimodules

$$\mathcal{O}^l_m \to \mathcal{O}^l_i \to \mathcal{O}^m_i.$$

Our connectivity estimates of Section 2.7 give the following.

Proposition 3.5 *Suppose R and $\mathcal{O}$ are connective. If I is $(c-1)$–connected, then I^n is $(nc-1)$–connected.*

3.2 Composition properties of the filtration

Now we look at composition structure. As will be shown in the proof of the next proposition, it is not hard to see that the operad composition

$$\mu\colon \mathcal{O}\circ\mathcal{O}\to\mathcal{O}$$

induces maps of $\mathcal{O}$–bimodules

$$\mu\colon \mathcal{O}_i^\infty \circ_{\mathcal{O}} \mathcal{O}_j^\infty \to \mathcal{O}_{ij}^\infty,$$

and these pairings, in turn, define natural maps

$$\mu\colon (I^j)^i \to I^{ij}$$

for all $I\in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$.

With a little more care, one can check the following.

Proposition 3.6 *Given $i<m$, $j<n$, and $ij<N\le \min(ij+(n-j), mj)$, the operad structure map*

$$\mu\colon \mathcal{O}\circ\mathcal{O}\to\mathcal{O}$$

induces maps of $\mathcal{O}$–bimodules

$$\mu\colon \mathcal{O}_i^m \circ_{\mathcal{O}} \mathcal{O}_j^n \to \mathcal{O}_{ij}^N$$

making the following diagram commute:

$$\begin{tikzcd}
\mathcal{O}\circ_{\mathcal{O}}\mathcal{O} \arrow[d, "\mu", "\wr"'] & \mathcal{O}_i^\infty\circ_{\mathcal{O}}\mathcal{O}_j^\infty \arrow[l, hook'] \arrow[r, two heads] \arrow[d, "\mu"] & \mathcal{O}_i^m\circ_{\mathcal{O}}\mathcal{O}_j^n \arrow[d, "\mu"] \\
\mathcal{O} & \mathcal{O}_{ij}^\infty \arrow[l, hook'] \arrow[r, two heads] & \mathcal{O}_{ij}^N
\end{tikzcd}$$

These thus induce natural maps $\mu\colon (I_n^j)_m^i \to I_N^{ij}$ making the following diagram commute:

$$\begin{tikzcd}
I \arrow[d, equal] & (I^j)^i \arrow[l] \arrow[r] \arrow[d, "\mu"] & (I_n^j)_m^i \arrow[d, "\mu"] \\
I & I^{ij} \arrow[l] \arrow[r] & I_N^{ij}
\end{tikzcd}$$

Proof We begin by observing that $(\mathcal{O}_i^\infty\circ\mathcal{O}_j^\infty)(s)$ equals a wedge of S–modules of the form

$$\mathcal{O}(r)\wedge\mathcal{O}(s_1)\wedge\cdots\wedge\mathcal{O}(s_r)$$

such that $s=s_1+\cdots+s_r$, $r\ge i$, and $s_k\ge j$ for all k.

These conditions force $s \geq ij$, and thus the dotted arrow exists in the following diagram:

$$\begin{tikzcd} \mathcal{O} \circ \mathcal{O} \arrow[r, "\mu"] & \mathcal{O} \\ \mathcal{O}_i^\infty \circ \mathcal{O}_j^\infty \arrow[u, hook] \arrow[r, dotted, "\mu"] & \mathcal{O}_{ij}^\infty \arrow[u, hook] \end{tikzcd}$$

Similarly, if $ij < N \leq \min(ij + (n - j), mj)$, then the dotted arrow exists in this diagram:

$$\begin{tikzcd} \mathcal{O}_i^\infty \circ \mathcal{O}_j^\infty \arrow[r, "\mu"] \arrow[d, two heads] & \mathcal{O}_{ij}^\infty \arrow[d, two heads] \\ \mathcal{O}_i^m \circ \mathcal{O}_j^n \arrow[r, dotted, "\mu"] & \mathcal{O}_{ij}^N \end{tikzcd}$$

To see this, note that a wedge summand as above maps to $*$ under the quotient $\mathcal{O}_i^\infty \circ \mathcal{O}_j^\infty \twoheadrightarrow \mathcal{O}_i^m \circ \mathcal{O}_j^n$ (the left map of the diagram) exactly when either $r \geq m$ or $s_k \geq n$ for at least one k. In the first case, it follows that $s \geq mj$. In the second case, it follows that $s \geq (r-1)j + n \geq (i-1)j + n = ij + (n-j)$. We conclude that if $N \leq \min(ij + (n-j), mj)$, then $s \geq N$, so this summand also maps to $*$ under the composite $\mathcal{O}_i^\infty \circ \mathcal{O}_j^\infty \xrightarrow{\mu} \mathcal{O}_{ij}^\infty \twoheadrightarrow \mathcal{O}_{ij}^N$. Thus the dotted arrow in the diagram exists.

Thus, the bimodule map $\mathcal{O}_i^m \circ \mathcal{O}_j^n \to \mathcal{O}_{ij}^{\min(ij+(n-j),mj)}$ induces an $\mathcal{O}$–bimodule map $\mathcal{O}_i^m \circ_{\mathcal{O}} \mathcal{O}_j^n \to \mathcal{O}_{ij}^{\min(ij+(n-j),mj)}$. This follows formally from the fact that each of the maps $\mathcal{O} \hookrightarrow \mathcal{O}_i^\infty \twoheadrightarrow \mathcal{O}_i^m$ are maps of $\mathcal{O}$–bimodules, combined with the evident fact that the operad pairing $\mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ induces a map $\mathcal{O} \circ_{\mathcal{O}} \mathcal{O} \to \mathcal{O}$. □

Addendum 3.7 *The construction shows a bit more compatibility than listed above: given $i \leq i'$, $m \leq m'$, $j \leq j'$, $n \leq n'$, $N \leq N'$, with $i < m$, $i' < m'$, $j < n$, $j' < m'$, $ij < N \leq \min(ij + (n-j), mj)$, and $i'j' < N' \leq \min(i'j' + (n'-j'), m'j')$, there is a commutative diagram of $\mathcal{O}$–bimodules*

$$\begin{tikzcd} \mathcal{O}_{i'}^{m'} \circ_{\mathcal{O}} \mathcal{O}_{j'}^{n'} \arrow[r, two heads] \arrow[d, "\mu"] & \mathcal{O}_i^m \circ_{\mathcal{O}} \mathcal{O}_j^n \arrow[d, "\mu"] \\ \mathcal{O}_{i'j'}^{N'} \arrow[r, two heads] & \mathcal{O}_{ij}^N \end{tikzcd}$$

and thus a commutative diagram

$$\begin{tikzcd} (I_{n'}^{j'})_{m'}^{i'} \arrow[r] \arrow[d, "\mu"] & (I_n^j)_m^i \arrow[d, "\mu"] \\ I_{N'}^{i'j'} \arrow[r] & I_N^{ij} \end{tikzcd}$$

Example 3.8 For simplicity, let $D_i(M) = \mathcal{O}(i) \wedge_{\Sigma_i \wr \mathcal{O}(1)} M^{\wedge_R i}$, for $M \in R\mathcal{O}(1)$–mod, and let $T = \mathrm{TQ}$. With this notation, Theorem 2.18 tells us that there is an isomorphism $I^i_{i+1} \simeq zD_iT(I)$. Then there is a commutative diagram

$$\begin{tikzcd}
(I^j_{j+1})^i_{i+1} \arrow[rr, "\mu"] \arrow[d, equal] & & I^{ij}_{ij+1} \arrow[d, equal] \\
zD_iT(zD_jT(I)) \arrow[r] & zD_iD_jT(I) \arrow[r] & zD_{ij}T(I)
\end{tikzcd}$$

where the lower left map is induced by the counit $TzM \to M$ (which one can check is a projection onto a wedge summand), and the lower right map is induced by the operad structure map $\mathcal{O}(i) \wedge \mathcal{O}(j)^{\wedge i} \to \mathcal{O}(ij)$. Diagrams like this suggest that our composition structure should be useful in doing calculations in spectral sequences associated to the augmentation ideal filtration. There are hints of how this might go in [10, Theorem 1.6].

3.3 Application to lifting filtrations

Theorem 3.9 *Let $I, J \in \mathsf{Alg}_{\mathcal{O}}(R)^{\mathrm{c}}$, and let $f\colon I \to J^d$ be a morphism in $\mathsf{Alg}_{\mathcal{O}}(R)$. Then f induces $\mathcal{O}$–algebra maps $f_n\colon I^n \to J^{dn}$ for all n, and the assignment sending f to f_n is both functorial and preserves weak equivalences. Furthermore, the maps f_n are compatible as n varies: for $m < n$, the following diagram commutes:*

$$\begin{tikzcd}
I_n \arrow[r, "f_n"] \arrow[d] & J^{dn} \arrow[d] \\
I^m \arrow[r, "f_m"] & J^{dm}
\end{tikzcd}$$

Proof Let f_n be the composite $I^n \xrightarrow{f^n} (J^d)^n \xrightarrow{\mu} J^{dn}$. □

Definition 3.10 Say that a map $f \in [I, J]_{\mathsf{Alg}}$ has AQ–filtration[4] at least s if f factors in $\mathrm{ho}(\mathsf{Alg}_{\mathcal{O}}(R))$ as the composition of s maps

$$I = I(0) \xrightarrow{f(1)} I(1) \xrightarrow{f(2)} I(2) \to \cdots \to I(s-1) \xrightarrow{f(s)} I(s) = J$$

such that $\mathrm{TQ}(f(i))$ is null for each i.

[4] The reader can decide if AQ stands for André–Quillen or Adams–Quillen.

Theorem 3.11 *Let $f \in [I, J]_{\mathsf{Alg}}$ have* AQ*–filtration at least s. Then there exists $\tilde{f} \in [I, J^{2^s}]_{\mathsf{Alg}}$ such that*

$$\begin{tikzcd} & J^{2^s} \arrow[d] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & J \end{tikzcd}$$

commutes in $\mathrm{ho}(\mathsf{Alg}_{\mathcal{O}}(R))$.

Proof We work in $\mathrm{ho}(\mathsf{Alg}_{\mathcal{O}}(R))$.

Let $f = f(s) \circ \cdots \circ f(1)$ as in Definition 3.10.

For each i between 1 and s, there is an exact sequence of pointed sets

$$[I(i-1), I(i)^2]_{\mathsf{Alg}} \to [I(i-1), I(i)]_{\mathsf{Alg}} \to [I(i-1), I(i)^1_2]_{\mathsf{Alg}},$$

and there are identifications

$$[I(i-1), I(i)^1_2]_{\mathsf{Alg}} \simeq [I(i-1), z(\mathrm{TQ}(I(i)))]_{\mathsf{Alg}} \simeq [\mathrm{TQ}(I(i-1)), \mathrm{TQ}(I(i))]_{\mathsf{Mod}}.$$

It follows that since $\mathrm{TQ}(f(i))$ is null, $f(i)$ lifts to $\tilde{f}(i)\colon I(i-1) \to I(i)^2$. Then Theorem 3.9 gives maps

$$\tilde{f}(i)_{2^{i-1}}\colon I(i-1)^{2^{i-1}} \to I(i)^{2^i}.$$

Now let $\tilde{f}$ be the composite of these s maps: $\tilde{f} = \tilde{f}(s)_{2^{s-1}} \circ \cdots \circ \tilde{f}(1)$. □

The theorem, combined with Proposition 3.5, has the following corollary.

Corollary 3.12 *Suppose that R and $\mathcal{O}$ are connective and $J \in \mathsf{Alg}_{\mathcal{O}}(R)$ is $(c-1)$–connected. If $f\colon I \to J$ has* AQ*–filtration s, then $f_*\colon \pi_*(I) \to \pi_*(J)$ will be zero for $* < 2^s c$.*

For more results in this spirit, see [9].

4 Deferred proofs

In this section we prove Propositions 2.6, 2.8, and 2.9 and Theorem 2.11. When $R = S$, so that our algebras just have the underlying structure of an S–module, these results can be deduced from the second author's work, specifically [14, Theorem 1.1]. The case of a general R requires a suitable generalization of that result, which we state as Theorem 4.4.

4.1 The homotopical behavior of the composition product

Fixing a commutative S–algebra R, it is useful to generalize the context slightly.

Notation 4.1 Let $\mathcal{P}$ be an operad in R–mod, ie a monoid object for the monoidal structure $\circ_R$ in $\mathsf{Sym}(R)$ defined just as in (1) but with $\wedge$ replaced by $\wedge_R$. We then denote by $\mathsf{Mod}^r_{\mathcal{P}}$, $\mathsf{Mod}^l_{\mathcal{P}}$, and $\mathsf{Alg}_{\mathcal{P}}$ the associated categories of left modules, right modules, and algebras over $\mathcal{P}$ in $\mathsf{Sym}(R)$. We endow $\mathsf{Mod}^l_{\mathcal{P}}$, and $\mathsf{Alg}_{\mathcal{P}}$ with the model structure as in Theorem 2.2.[5]

Remark 4.2 As noted in Remark 2.1, there are identifications

$$\mathsf{Alg}_{R\wedge\mathcal{O}} \simeq \mathsf{Alg}_{\mathcal{O}}(R) \simeq \mathsf{Alg}_{R\wedge\mathcal{O}}(S)$$

and

$$\mathsf{Mod}^l_{R\wedge\mathcal{O}} \simeq \mathsf{Mod}^l_{\mathcal{O}}(R) \simeq \mathsf{Mod}^l_{R\wedge\mathcal{O}}(S).$$

By contrast, one only has a proper inclusion of categories

$$\mathsf{Mod}^r_{R\wedge\mathcal{O}} \subset \mathsf{Mod}^r_{R\wedge\mathcal{O}}(S),$$

where $\mathsf{Mod}^r_{R\wedge\mathcal{O}}$ is the category of right $R\wedge\mathcal{O}$–modules in $\mathsf{Sym}(R)$, and $\mathsf{Mod}^r_{R\wedge\mathcal{O}}(S)$ is the category of right $R\wedge\mathcal{O}$–modules in $\mathsf{Sym}(S)$.

To see the reason for this, assume for simplicity that $\mathcal{O}(1) = S$. Then if $N \in \mathsf{Mod}^r_{R\wedge\mathcal{O}}(S)$, $N(n)$ will be a right $\Sigma_n \wr R$–module. But unwinding definitions reveals that, for any $N \in \mathsf{Mod}^r_{R\wedge\mathcal{O}}$, this $\Sigma_n \wr R$–module structure on $M(n)$ must be one pulled back along the canonical ring map $\Sigma_n \wr R \to \Sigma_n \times R$.

To state our main technical theorem, we need the following construction.

Definition 4.3 Suppose given a map $f_1\colon M \to N$ in $\mathsf{Mod}^r_{\mathcal{P}}$ and a map $f_2\colon A \to B$ in $\mathsf{Mod}^l_{\mathcal{P}}$. Let $(M \circ_{\mathcal{P}} B) \vee_{M\circ_{\mathcal{P}} A} (N \circ_{\mathcal{P}} A)$ be the pushout of the diagram

$$\begin{tikzcd} M \circ_{\mathcal{P}} A \arrow[r, "f_1 \circ_{\mathcal{P}} A"] \arrow[d, "M \circ_{\mathcal{P}} f_2"'] & N \circ_{\mathcal{P}} A \\ M \circ_{\mathcal{P}} B & \end{tikzcd}$$

in $\mathsf{Sym}(R)$, and then define the *pushout circle product* of f_1 and f_2, to be the natural map

$$f_1 \square^{\circ_{\mathcal{P}}} f_2\colon (M \circ_{\mathcal{P}} B) \vee_{M\circ_{\mathcal{P}} A} (N \circ_{\mathcal{P}} A) \to N \circ_{\mathcal{P}} B.$$

[5] Note that we do not need to equip $\mathsf{Mod}^r_{\mathcal{P}}$ with a model structure.

Theorem 4.4 *Suppose $f_2\colon A \to B$ is a cofibration between cofibrant objects in $\mathsf{Mod}^l_{\mathcal{P}}$. If a $f_1\colon M \to N$ in $\mathsf{Mod}^r_{\mathcal{P}}$ is an underlying positive cofibration in $\mathsf{Sym}(R)$, then so is*

$$f_1 \square^{\circ_{\mathcal{P}}} f_2\colon (M \circ_{\mathcal{P}} B) \vee_{M \circ_{\mathcal{P}} A} (N \circ_{\mathcal{P}} A) \to N \circ_{\mathcal{P}} B.$$

Furthermore, this map will be a weak equivalence if either f_1 or f_2 is a weak equivalence.

When $R = S$, this theorem nearly coincides with [14, Theorem 1.1], and we defer the proof in the general case to the Appendix. For the purpose of proving results stated in Section 2, we will just need the following corollary.

Corollary 4.5 *Let $\mathcal{O}$ be an operad in S–mod. Suppose $f_2\colon I \to J$ is a cofibration between cofibrant objects in $\mathsf{Alg}_{\mathcal{O}}(R)$. If a map $f_1\colon M \to N$ in $\mathsf{Mod}^r_{\mathcal{O}}(S)$ is an underlying positive cofibration in $\mathsf{Sym}(S)$, then*

$$f_1 \square^{\circ_{\mathcal{O}}} f_2\colon (M \circ_{\mathcal{O}} J) \vee_{M \circ_{\mathcal{O}} I} (N \circ_{\mathcal{O}} I) \to N \circ_{\mathcal{O}} J$$

will be a positive cofibration in R–mod.

Furthermore, this map will be a weak equivalence if either f_1 or f_2 is a weak equivalence.

Proof Since the functor $R \wedge \_\colon \mathsf{Sym}(S) \to \mathsf{Sym}(R)$ sends positive cofibrations and trivial cofibrations in $\mathsf{Sym}(S)$ respectively to positive cofibrations and trivial cofibrations in $\mathsf{Sym}(R)$, the result follows immediately from Theorem 4.4 applied to $\mathcal{P} = R \wedge \mathcal{O}$, $R \wedge f_1$ and f_2. Note that the positive model structure on $\mathsf{Sym}(R)$ restricts on level 0 to the positive model structure on R–mod. $\square$

4.2 Proofs of results from Section 2

Proof of Proposition 2.6 If f_1 is the map $* \to \mathcal{O}$, and $f_2\colon I \to J$ is map in $\mathsf{Alg}_{\mathcal{O}}(R)$, then $f_1 \square^{\circ_{\mathcal{O}}} f_2$ is just the map $f_2\colon I \to J$, now viewed as a map in R–mod.

If I is cofibrant in $\mathsf{Alg}_{\mathcal{O}}(R)$, then applying Corollary 4.5 to the map $f_2\colon * \to I$, shows that I will be cofibrant in R–mod.

Similarly, if $f_2\colon I \to J$ is a cofibration between cofibrant objects in $\mathsf{Alg}_{\mathcal{O}}(R)$, we learn that $f_2\colon I \to J$ is a cofibration in R–mod. $\square$

Proof of Proposition 2.8 For the first statement, we note that $B(M, \mathcal{O}, N)$ is the realization of the simplicial object $B_\bullet(M, \mathcal{O}, N)$, and thus will be cofibrant in $\mathsf{Sym}(S)$ if $B_\bullet(M, \mathcal{O}, N)$ is Reedy cofibrant in $\mathsf{Sym}(S)^{\Delta^{\mathrm{op}}}$. That this is true, under our hypotheses on M and N, is precisely the conclusion of [14, Theorem 1.6].

Proving the second statement is similar: one sees that $B_\bullet(M,\mathcal{O},I)$ is Reedy cofibrant in R–mod$^{\Delta^{\mathrm{op}}}$ by noting that the proof of [14, Theorem 1.6] (and in particular that of the auxiliary [14, Lemma 5.47]) goes through if one simply replaces the very last application of [14, Theorem 1.1] by an application of Corollary 4.5. □

Proof of Proposition 2.9 First note that by Corollary 4.5 the functor

$$M \circ_{\mathcal{O}} \_\colon \mathsf{Alg}_{\mathcal{O}}(R) \to R\text{–mod}$$

sends trivial cofibrations between cofibrant algebras to weak equivalences, and hence, by Ken Brown's lemma [6, Corollary 7.7.2], also preserves all weak equivalences between cofibrant algebras.

Hence, rewriting the map

$$B(M,\mathcal{O},I) \to M \circ_{\mathcal{O}} I$$

as

$$M \circ_{\mathcal{O}} (B(\mathcal{O},\mathcal{O},I) \to I)$$

one sees that it suffices to show that $B(\mathcal{O},\mathcal{O},I)$ is cofibrant in $\mathsf{Alg}_{\mathcal{O}}(R)$.

$B(\mathcal{O},\mathcal{O},I)$ is the realization of the simplicial algebra $B_\bullet(\mathcal{O},\mathcal{O},I)$, viewed as a simplicial object in R–mod. By [5, Proposition 6.11], this agrees with the realization of $B_\bullet(\mathcal{O},\mathcal{O},I)$, viewed as a simplicial object in $\mathsf{Alg}_{\mathcal{O}}(R)$. Thus it suffices to show that $B_\bullet(\mathcal{O},\mathcal{O},I)$ is Reedy cofibrant in $\mathsf{Alg}_{\mathcal{O}}(R)^{\Delta^{\mathrm{op}}}$.

Checking this involves showing that the latching maps for $B_\bullet(\mathcal{O},\mathcal{O},I)$ are cofibrations in $\mathsf{Alg}_{\mathcal{O}}(R)$. These depend only on $B_\bullet(\mathcal{O},\mathcal{O},I)$ together with its degeneracies, ie face maps can be ignored. From this perspective

$$B_\bullet(\mathcal{O},\mathcal{O},I) \simeq \mathcal{O} \circ B_\bullet(S(1),\mathcal{O},I),$$

where $S(1)$ is our notation for the unit symmetric sequence $(*,S,*,*,\dots)$ under $\circ$.

Hence, letting $\ell_n^{\mathcal{O}}$ and ℓ_n respectively denote the n^{th} latching map construction on $\mathbb{N}$–graded objects with degeneracies in $\mathsf{Alg}_{\mathcal{O}}(R)$ and R–mod, one has

$$\ell_n^{\mathcal{O}}(B_\bullet(\mathcal{O},\mathcal{O},I)) \simeq \mathcal{O} \circ \ell_n(B_\bullet(S(1),\mathcal{O},I)).$$

Since $\mathcal{O} \circ \_\colon R\text{–mod} \to \mathsf{Alg}_{\mathcal{O}}(R)$ is a left Quillen functor, $\ell_n^{\mathcal{O}}(B_\bullet(\mathcal{O},\mathcal{O},I))$ will be a cofibration in $\mathsf{Alg}_{\mathcal{O}}(R)$ if $\ell_n(B_\bullet(S(1),\mathcal{O},I))$ is a cofibration in R–mod. But the latter map *is* a cofibration, since it is a special case of the latching maps shown to be cofibrations in the proof of Proposition 2.8. □

Proof of Theorem 2.11(a) and (b) In this proof we focus on the identification $\mathsf{Alg}_{\mathcal{O}}(R) \simeq \mathsf{Alg}_{R\wedge\mathcal{O}}(S)$ so as to be able to directly apply [14, Theorem 1.1].

For part (a), note first that

$$F_M^R(I) = M \circ_{\mathcal{O}} B(\mathcal{O}, \mathcal{O}, I).$$

That $F_M^R(I)$ preserves weak equivalences in the I variable then follows from the proof of Proposition 2.9, where it was shown both that $B(\mathcal{O}, \mathcal{O}, I)$ is a cofibrant algebra and that $M \circ_{\mathcal{O}} \_\_$ preserves weak equivalences between cofibrant algebras.

To see that weak equivalences are also preserved in the M variable, one uses a similar argument: using the identifications of Remark 4.2 to change perspective to S–mod, one applies [14, Theorem 1.1] to any trivial cofibration $f_1\colon M \to N$ in $\mathsf{Mod}^r_{R\wedge\mathcal{O}}(S)$ and the map $f_2 = * \to B(\mathcal{O}, \mathcal{O}, I)$. One concludes that the functor sending M to $F_M^R(I)$ sends trivial cofibrations to weak equivalences. One now again uses Ken Brown's lemma.

The intuition behind part (b) comes from the observation that (1), the formula for the composition product $X \circ Y$ of symmetric sequences, is linear in the variable X. Our official proof goes as follows. Note that homotopy fibration sequences in $\mathsf{Alg}_{\mathcal{O}}(R)$ are detected by considering them as sequences in S–mod. Again using the identifications of Remark 4.2 to change perspectives, one immediately reduces to [14, Theorem 1.8] applied to the operad $R \wedge \mathcal{O}$ in S–modules. □

Appendix: Proof of Theorem 4.4

We now turn to the task of proving Theorem 4.4. If one just tries to redo all the work in [14] with a general commutative S–algebra R replacing occurrences of S, one finds that most of results generalize, with the key exception being the characterization of S cofibrations in [14, Proposition 3.9], which fails for general R (and, in particular, cofibrations in $\mathsf{Sym}(R)$ can not be detected by first forgetting the Σ_n–actions at each level). Here we take a somewhat blended approach: we use a string of arguments from [14] to ultimately reduce ourselves to a situation covered by [14, Theorem 1.1].

We begin by noting the following elementary lemma, a consequence of the fact that the positive model structure on $\mathsf{Sym}(R)$ is the projective structure induced from the positive model structure on $\mathsf{Sym}(S)$.

Lemma A.1 *A set of generating cofibrations for* $\mathsf{Sym}(R)$ *can be obtained by applying* $R \wedge \_\_$ *to a set of generating cofibrations for* $\mathsf{Sym}(S)$.

Let us remind ourselves of our goal. Given $f_1\colon M \to N$ in $\mathsf{Mod}^r_{\mathcal{P}}$ and $f_2\colon A \to B$ in $\mathsf{Mod}^l_{\mathcal{P}}$, we are considering the pushout corner map, in $\mathsf{Sym}(R)$, of the following commutative square:

$$
\begin{tikzcd}
M \circ_{\mathcal{P}} A \arrow[r, "M \circ_{\mathcal{P}} f_2"] \arrow[d, "f_1 \circ_{\mathcal{P}} A"] & M \circ_{\mathcal{P}} B \arrow[d, "f_1 \circ_{\mathcal{P}} B"] \\
N \circ_{\mathcal{P}} A \arrow[r, "N \circ_{\mathcal{P}} f_2"] & N \circ_{\mathcal{P}} B
\end{tikzcd}
\tag{2}
$$

By this we mean the map from the pushout of the upper left corner of the square to the lower right term.

We wish to show that if f_2 is a cofibration between cofibrant objects[6] in $\mathsf{Mod}^l_{\mathcal{P}}$, then if f_1 is a positive cofibration in $\mathsf{Sym}(R)$, so is the pushout corner map. Furthermore, in this situation, if either f_1 or f_2 is a weak equivalence, so is the pushout corner map.

We will address this last statement at the end of the Appendix, and focus on the first statement. For this, we try to follow the proof of [14, Theorem 1.1], which is the case when $R = S$. The majority of the arguments in that proof are agnostic as to the category or model structure used — in particular, the filtrations of [14, Proposition 5.20] cover R–mod — with the exception of the two instances where [14, Theorems 1.2, 1.3] are used.

As in [14], we first assume that f_2 is a map between algebras, rather than more general left $\mathcal{P}$–modules. In this case, arguing as in [14, Section 5.4], one reduces to the case where $f_2\colon A \to B$ is the pushout of a generating cofibration. Using Lemma A.1, this means that f_2 is the lower horizontal map of a pushout in $\mathsf{Alg}_{\mathcal{P}}$ of the form

$$
\begin{tikzcd}
\mathcal{P} \circ_R (R \wedge X) \arrow[r] \arrow[d] & \mathcal{P} \circ_R (R \wedge Y) \arrow[d] \\
A \arrow[r, "f_2"] & B
\end{tikzcd}
$$

with $X \to Y$ a generating positive cofibration in S–mod.

The key is now to use the infinite filtration of the horizontal maps in (2) given by [14, Proposition 5.20]. (This key filtration is a generalization of similar filtrations appearing in [5, Proposition 5.10] and [2, proofs of Theorems 1.4 and 12.5].) Arguing as in [14, Section 5.4], one is reduced to studying the pushout corner maps of the following

[6] This is a bit redundant: if A is cofibrant, and f_2 is a cofibration, then B is necessarily cofibrant.

squares, for which we will shortly explain our notation:

$$
\begin{tikzcd}
M_A(r) \wedge_{R\times\Sigma_r} (R \wedge Q^r_{r-1}) \arrow[r] \arrow[d] & M_A(r) \wedge_{R\times\Sigma_r} (R \wedge Y^{\wedge r}) \arrow[d] \\
N_A(r) \wedge_{R\times\Sigma_r} (R \wedge Q^r_{r-1}) \arrow[r] & N_A(r) \wedge_{R\times\Sigma_r} (R \wedge Y^{\wedge r})
\end{tikzcd}
\tag{3}
$$

Firstly, if we view $X \to Y$ as a functor $\{\mathbf{0} \to \mathbf{1}\} \to S$–mod, we can smash this functor with itself r times, obtaining a cubical diagram $\{\mathbf{0} \to \mathbf{1}\}^{\times r} \to S$–mod. We let Q^r_{r-1} denote the colimit of this cube with the terminal object $\mathbf{1}^r$ removed; this comes with an evident map $Q^r_{r-1} \to Y^{\wedge r}$.

Secondly, as in [14, Definition 5.15], M_A denotes the $M \circ_{\mathcal{P}} \bigl(\mathcal{P} \coprod A\bigr)$, where the coproduct is taken in $\mathsf{Mod}^l_{\mathcal{P}}$.

We wish to show that the pushout corner map of (3) is a positive cofibration in R–mod. Since $X \to Y$ is a positive cofibration in S–mod, [14, Theorem 1.2] tells us that $Q^r_{r-1} \to Y^{\wedge r}$ is appropriately cofibrant in the category of S–modules with a Σ_r action.

If the map $M_A \to N_A$ were a generating positive cofibration in $\mathsf{Sym}(R)$, one would be able to pull a $R \wedge (-)$ factor out of the pushout corner map (by Lemma A.1), reducing to the S case, which in turn follows by applying [14, Theorems 1.2, 1.3] as in the proof of [14, Theorem 1.1].

Hence, by standard arguments, it suffices to show that $M_A \to N_A$ is a positive cofibration in $\mathsf{Sym}(R)$. This would follow from the special case of our theorem when f_2 has the form $i\colon \mathcal{P} \to \mathcal{P} \coprod A$, which would say that the pushout corner map of the middle square of the diagram

$$
\begin{tikzcd}
M \arrow[r, equal] \arrow[d, "f_1"] & M \circ_{\mathcal{P}} \mathcal{P} \arrow[r, "M\circ_{\mathcal{P}} i"] \arrow[d, "f_1\circ_{\mathcal{P}}\mathcal{P}"] & M \circ_{\mathcal{P}} \bigl(\mathcal{P} \coprod A\bigr) \arrow[r, equal] \arrow[d, "f_1\circ_{\mathcal{P}}(\mathcal{P}\coprod A)"] & M_A \arrow[d] \\
N \arrow[r, equal] & N \circ_{\mathcal{P}} \mathcal{P} \arrow[r, "N\circ_{\mathcal{P}} i"] & N \circ_{\mathcal{P}} \bigl(\mathcal{P} \coprod A\bigr) \arrow[r, equal] & N_A
\end{tikzcd}
\tag{4}
$$

is a positive cofibration in $\mathsf{Sym}(R)$.

Now we use our assumption that A is cofibrant in $\mathsf{Alg}_{\mathcal{P}}$, and basically proceed as before. The map i can be assumed to be an infinite composition of maps of the form $\mathcal{P} \coprod A_\beta \xrightarrow{\mathcal{P}\coprod i_\beta} \mathcal{P} \coprod A_{\beta+1}$, where i_β is the lower horizontal map of a pushout in $\mathsf{Alg}_{\mathcal{P}}$

of the form

$$\begin{tikzcd} \mathcal{P} \circ_R (R \wedge X_\beta) \arrow[r] \arrow[d] & \mathcal{P} \circ_R (R \wedge Y_\beta) \arrow[d] \\ A_\beta \arrow[r] & A_{\beta+1} \end{tikzcd}$$

with $X_\beta \to Y_\beta$ a generating positive cofibration in S–mod.

It suffices to show by induction on β that $N_{A_\beta} \coprod_{M_{A_\beta}} M_{A_{\beta+1}} \to N_{A_{\beta+1}}$ is a positive cofibration. Note that the induction hypothesis then implies $M_{A_\beta} \to N_{A_\beta}$ is a positive cofibration.

After a filtration argument as before, one is left needing to show that the pushout corner map in

$$\begin{tikzcd} M_{(\mathcal{P} \coprod A_\beta)}(r) \check{\wedge}_{R \times \Sigma_r} (R \wedge Q^r_{r-1}) \arrow[r] \arrow[d] & M_{(\mathcal{P} \coprod A_\beta)}(r) \check{\wedge}_{R \times \Sigma_r} (R \wedge Y_\beta^{\check{\wedge} r}) \arrow[d] \\ N_{(\mathcal{P} \coprod A_\beta)}(r) \check{\wedge}_{R \times \Sigma_r} (R \wedge Q^r_{r-1}) \arrow[r] & N_{(\mathcal{P} \coprod A_\beta)}(r) \check{\wedge}_{R \times \Sigma_r} (R \wedge Y_\beta^{\check{\wedge} r}) \end{tikzcd}$$

is a positive cofibration in $\mathsf{Sym}(R)$, where $\check{\wedge}$ denotes the smash product in $\mathsf{Sym}(R)$.

Using the obvious analogue of Lemma A.1 for R bisymmetric sequences and [14, Propositions 5.43, 5.44] (the analogues of [14, Theorems 1.2, 1.3] for $\mathsf{Sym}(S)$) just as in the argument following (3), one further reduces to just needing to show that $M_{\mathcal{P} \coprod A_\beta} \to N_{\mathcal{P} \coprod A_\beta}$ is a positive cofibration in $\mathsf{biSym}(R)$, the category of bisymmetric sequences of R–modules. (The notion of cofibration is defined by analogy with $\mathsf{Sym}(R)$.) But since [14, Proposition 5.19] identifies the (r,s) level of this map with $M_{A_\beta}(r+s) \to N_{A_\beta}(r+s)$, the result follows by our induction hypothesis.

To deal with the case of f_2 a general map of left modules one repeats the argument in the last paragraph of the proof of [14, Theorem 1.1].

Finally, the case where either f_1 or f_2 are additionally weak equivalences follows by using the identifications of Remark 4.2 to reduce the question to the S–mod level and then applying the monomorphism part of [14, Theorem 1.1].

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Department of Mathematics, University of Virginia
Charlottesville, VA 22904, United States

`njk4x@virginia.edu`, `lp2h@virginia.edu`

`http://pi.math.virginia.edu/Faculty/Kuhn/`

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