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On bordered theories for Khovanov homology

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We describe how to formulate Khovanov's functor-valued invariant of tangles in the language of bordered Heegaard Floer homology. We then give an alternate construction of Lawrence Roberts' type D and type A structures in Khovanov homology, and his algebra $\mathcal{B}\Gamma_n$, in terms of Khovanov's theory of modules over the ring H^n. We reprove invariance and pairing properties of Roberts' bordered modules in this language. Along the way, we obtain an explicit generators-and-relations description of H^n which may be of independent interest.

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1 Introduction

We consider two tangle theories for Khovanov homology which are inspired by the bordered Heegaard Floer homology of Lipshitz, Ozsváth and Thurston [5]. The first theory is a reformulation of Khovanov's functor-valued invariant [4] in the bordered language. The second theory was introduced by Lawrence Roberts in [11; 12].

These bordered Khovanov theories share the same basic structure. Each assigns a differential bigraded algebra $\mathcal{B}$ to a collection of $2n$ points on the line $\{0\}\times\mathbb{R}$ in the plane $\mathbb{R}\times\mathbb{R}$. To a tangle diagram T_1 in $\mathbb{R}_{\geq 0}\times\mathbb{R}$ with $2n$ endpoints on $\{0\}\times\mathbb{R}$, these theories assign a (left) type D structure $\widehat{D}_{T_1}$ over $\mathcal{B}$. The definitions of type D structures, and other elements of the algebra of bordered Floer homology, will be given in Section 2.

To a tangle diagram T_2 in $\mathbb{R}_{\leq 0}\times\mathbb{R}$ with $2n$ endpoints on $\{0\}\times\mathbb{R}$, bordered theories assign a (right) type A structure (ie an $\mathcal{A}_\infty$–module) $\widehat{A}_{T_2}$ over $\mathcal{B}$. There is a natural pairing operation between type D and type A structures over $\mathcal{B}$ called the box tensor product, denoted $\boxtimes$ (or $\boxtimes_{\mathcal{B}}$ when $\mathcal{B}$ is unclear). If T_2T_1 denotes the link diagram obtained by concatenating T_2 and T_1 horizontally, bordered theories compute the Khovanov complex $\mathrm{CKh}(T_2T_1)$ using the following pairing formula:

$$\mathrm{CKh}(T_2T_1)\cong\widehat{A}_{T_2}\boxtimes_{\mathcal{B}}\widehat{D}_{T_1}.$$

In Section 3, we will obtain a bordered theory with the above structure by taking $\mathcal{B}$ to be Khovanov's arc algebra H^n from [4], viewed as a differential bigraded algebra with the differential and one of the two gradings identically equal to zero. The type D and

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type A structures $\widehat{D}_{T_1}$ and $\widehat{A}_{T_2}$ will be referred to as $\widehat{D}(T_1)$ and $\widehat{A}(T_2)$ in this setting. Both come from Khovanov's tangle invariants $[T_i]^{\mathrm{Kh}}$, which are chain complexes of projective graded H^n modules up to homotopy equivalence.

1.0.1 Theorem (Theorem 3.2.1) *After multiplying the intrinsic gradings on $\widehat{A}(T_2)\boxtimes_{H^n}\widehat{D}(T_1)$ by -1,*

$$\mathrm{CKh}(T_2T_1)\cong\widehat{A}(T_2)\boxtimes_{H^n}\widehat{D}(T_1).$$

Roberts [11; 12] has a different construction of a bordered theory for Khovanov homology, including a differential bigraded algebra $\mathcal{B}\Gamma_n$ as well as type D and type A structures for tangles. The goal of Section 5 and Section 6 is to construct Roberts' theory using Khovanov's theory. The basic idea is to refine Khovanov's proofs of the existence and invariance of his tangle invariants by splitting the equations involved into subequations, each of which holds individually.

The construction of Roberts' theory from Khovanov's is not straightforward or trivial; the combinatorics is quite involved. Moreover, at various points we take our inspiration directly from [11; 12] rather than from abstract algebraic definitions. In particular, see Remark 5.3.5 below. While it would be interesting to search for the most general or natural possible explanation for the connection between these two theories, we do not pursue this goal here.

We take the first step toward relating Roberts' and Khovanov's theories in Section 4. In Section 4.1, we discuss quadratic and linear-quadratic algebras following Polishchuk and Positselski [10]. In Section 4.2, we show that H^n may be viewed as a linear-quadratic algebra.

1.0.2 Theorem (Theorem 4.2.1) *With the set of generators specified at the beginning of Section 4.2, H^n is a linear-quadratic algebra.*

This theorem allows us to write H^n as the quotient of the tensor algebra on the specified generators by an ideal generated by certain explicitly given relations, which are listed in items (1)–(4) of the proof of Theorem 4.2.1. See Corollary 4.2.7 for a more precise statement.

A combinatorial lemma about noncrossing partitions, Lemma 4.2.4, is needed to prove Theorem 1.0.2. While Theorem 1.0.2 is not necessary for the remainder of the paper, Lemma 4.2.4 is important for Section 5. Proofs of Lemma 4.2.4 were found by Dömötör Pálvölgyi [9] and independently by Aaron Potechin in a private email communication. This lemma, and Theorem 1.0.2, may be of interest to readers independently of the other constructions in this paper.

In Section 4.3, we consider a notion of Polishchuk and Positselski [10] of quadratic duality for linear-quadratic algebras. In Section 4.5, we discuss a bordered-algebra version of this duality using type DD bimodules. Generalized Koszul duality between two algebras $\mathcal{B}$ and $\mathcal{B}'$ in bordered Floer homology is defined (see Lipshitz, Ozsváth and Thurston [6]) by the existence of a quasi-invertible rank-one type DD bimodule over $\mathcal{B}$ and $\mathcal{B}'$. The algebras used in Lipshitz, Ozsváth and Thurston's construction have interesting Koszul self-duality properties. However, it seems that no such properties hold for H^n. Viewing H^n as a linear-quadratic algebra, we will see in Section 4.4 that its quadratic dual is infinite-dimensional, whereas H^n is always finitely generated over $\mathbb{Z}$.

One could ask whether the duality between H^n and this infinite-dimensional algebra is a (generalized) Koszul duality; one could also explore related theories in which everything stays finite-dimensional. We will take the second option here.

In Section 5, we will outline an alternate construction, based on H^n, of Roberts' algebra $\mathcal{B}\Gamma_n$. We define an algebra $\mathcal{B} = \mathcal{B}_R(H^n)$, and we show in Proposition 5.1.7 that the algebra $\mathcal{B}$ is linear-quadratic. The proof is very similar to the proof of Theorem 4.2.1 asserting that H^n is linear-quadratic, and it also uses Lemma 4.2.4 in an essential way. We deduce that $\mathcal{B}$ is isomorphic to the subalgebra $\mathcal{B}_R\Gamma_n$ of $\mathcal{B}\Gamma_n$ generated by right-pointing generators $\overrightarrow{e}$.

The quadratic dual $\mathcal{B}^!$ of $\mathcal{B}$ is closely related to the subalgebra $\mathcal{B}_L\Gamma_n$ of $\mathcal{B}\Gamma_n$ generated by left-pointing generators $\overleftarrow{e}$. In more detail, a mirroring operation m is defined on certain algebras in Definition 5.2.6. We will see in Proposition 5.2.8 that $\mathcal{B}_L\Gamma_n$ is a quotient of the mirror $m(\mathcal{B}^!)$ of $\mathcal{B}^!$ by certain additional relations, listed in that proposition. As Remark 5.2.3 points out, $\mathcal{B}^!$ is finitely generated for idempotent reasons.

In Section 5.3, we define a product algebra $m(\mathcal{B}^!) \odot \mathcal{B}$ of $m(\mathcal{B}^!)$ and $\mathcal{B}$. We may describe Roberts' full algebra $\mathcal{B}\Gamma_n$ as a quotient of $m(\mathcal{B}^!) \odot \mathcal{B}$.

1.0.3 Theorem (Corollary 5.3.4) *$\mathcal{B}\Gamma_n$ is isomorphic to the quotient of $m(\mathcal{B}^!) \odot \mathcal{B}$ by the extra relations on $m(\mathcal{B}^!)$ listed in Proposition 5.2.8.*

The duality properties of $\mathcal{B}\Gamma_n$ and $m(\mathcal{B}^!) \odot \mathcal{B}$ seem more promising than those of H^n. In Proposition 5.3.6 we define a rank-one type DD bimodule over $m(\mathcal{B}^!) \odot \mathcal{B}$ and its mirror version $m(m(\mathcal{B}^!) \odot \mathcal{B})$. Conjecture 5.3.9 predicts that this DD bimodule is quasi-invertible and thus yields a Koszul duality. By taking quotients of the type DD algebra outputs, we can obtain a related rank-one DD bimodule over $\mathcal{B}\Gamma_n$ and its mirror version $m(\mathcal{B}\Gamma_n)$. Thus, we could also ask if Conjecture 5.3.9 is true with $m(\mathcal{B}^!) \odot \mathcal{B}$ replaced by $\mathcal{B}\Gamma_n$. A proof of either conjecture would establish that, with regard to

Koszul duality, Roberts' bordered theory (or the version over $m(\mathcal{B}^!) \odot \mathcal{B}$) has closer formal parallels with bordered Floer homology than Khovanov's H^n theory does.

In Section 6, we show how to obtain type A and type D structures over $m(\mathcal{B}^!) \odot \mathcal{B}$ from chain complexes of graded projective H^n–modules satisfying certain algebraic conditions.

1.0.4 Theorem *The following constructions are well-defined:*

- *Let M be a chain complex of projective graded right H^n–modules satisfying the algebraic condition C_{module} of Definition 6.1.1. To M we may associate a type A structure $\widehat{A}(M)$ over $m(\mathcal{B}^!) \odot \mathcal{B}$.*
- *Let N be a chain complex of projective graded left H^n–modules satisfying the condition C_{module} of Definition 6.3.3. To N we may associate a type D structure $\widehat{D}(N)$ over $m(\mathcal{B}^!) \odot \mathcal{B}$.*

Theorem 1.0.4 is a summary of Definition 6.2.4, Proposition 6.2.5 and Definition 6.3.7 (as well as the definitions and propositions preceding them).

The chain complexes $[T_i]^{\text{Kh}}$ associated to tangles by Khovanov satisfy C_{module}, so Theorem 1.0.4 gives us type A and type D structures $\widehat{A}([T_2]^{\text{Kh}})$ and $\widehat{D}([T_1]^{\text{Kh}})$ over $m(\mathcal{B}^!) \odot \mathcal{B}$. By Proposition 6.2.6, the extra relations of Theorem 1.0.3 act as zero on the type A structure $\widehat{A}([T_2]^{\text{Kh}})$, so we get a type A structure over the quotient algebra $\mathcal{B}\Gamma_n$. We may also take quotients of the algebra outputs of the type D structure $\widehat{D}([T_1]^{\text{Kh}})$ to get a type D structure over $\mathcal{B}\Gamma_n$.

1.0.5 Theorem (Proposition 6.2.7 and Proposition 6.3.10) *The type A structure $\widehat{A}([T_2]^{\text{Kh}})$ over $\mathcal{B}\Gamma_n$, and the type D structure $\widehat{D}([T_1]^{\text{Kh}})$ over $\mathcal{B}\Gamma_n$, are isomorphic to the type A and D structures Roberts associates to T_2 and T_1 in [11; 12].*

We show that the pairing of the bordered modules over $m(\mathcal{B}^!) \odot \mathcal{B}$ agrees with the tensor product of the original chain complexes over H^n.

1.0.6 Theorem (Proposition 6.4.1) *Given M and N as in Theorem 1.0.4, we have*

$$\widehat{A}(M) \boxtimes_{m(\mathcal{B}^!)\odot\mathcal{B}} \widehat{D}(N) \cong M \otimes_{H^n} N,$$

after multiplying the intrinsic gradings on $M \otimes_{H^n} N$ by -1.

By Proposition 6.4.3, the pairing $\boxtimes$ is the same over $m(\mathcal{B}^!) \odot \mathcal{B}$ and its quotient $\mathcal{B}\Gamma_n$. Thus, we get an alternate proof that the pairing of Roberts' type D and type A structures computes Khovanov homology.

Finally, in Section 6.5 and Section 6.6 we show that the homotopy types of $\widehat{A}([T_2]^{\mathrm{Kh}})$ and $\widehat{D}([T_1]^{\mathrm{Kh}})$, as type A and type D structures over $m(\mathcal{B}^!) \odot \mathcal{B}$, are invariants of the tangles underlying the diagrams T_1 and T_2.

1.0.7 Theorem (Corollary 6.5.21 and Corollary 6.6.9) *Performing a Reidemeister move on T_2 or T_1 yields a homotopy equivalence between the corresponding type A structures $\widehat{A}([T_2]^{\mathrm{Kh}})$ or type D structures $\widehat{D}([T_1]^{\mathrm{Kh}})$ over $m(\mathcal{B}^!) \odot \mathcal{B}$.*

With the help of Proposition 6.5.22, we also obtain an alternate proof that Roberts' type A and type D structures over $\mathcal{B}\Gamma_n$ are homotopy-invariant under Reidemeister moves.

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2 Some bordered algebra

The standard reference for the algebra of bordered Floer homology is Lipshitz, Ozsváth and Thurston [7]. We will use only a subset of the full algebraic machinery; however, we will work with coefficients in $\mathbb{Z}$ rather than $\mathbb{Z}/2\mathbb{Z}$. For this sign lift, we will follow the conventions of Roberts in [11; 12].

2.1 Differential graded algebras and modules

2.1.1 Convention Unless otherwise specified, all algebras and modules discussed in this paper will be assumed to be finitely generated over $\mathbb{Z}$.

The following is the notion of differential graded algebra which will be most useful for us; we will not need to use more general $\mathcal{A}_\infty$–algebras. In this paper, the coefficient ring R is always a direct product of finitely many copies of $\mathbb{Z}$.

2.1.2 Definition A *differential bigraded algebra*, or *dg algebra*, is a bigraded unital associative algebra $\mathcal{B}$ over a coefficient ring R, equipped with an R–bilinear differential

μ_1 which is homogeneous of degree $(0,+1)$ with respect to the bigrading. The two gradings on a dg algebra will be called the intrinsic and homological gradings (in that order). Thus, the differential should preserve the intrinsic grading and increase the homological grading by 1.

The differential must satisfy the following Leibniz rule:

$$\mu_1(xy) = (-1)^{\deg_h y}(\mu_1(x))y + x(\mu_1(y)),$$

where $\deg_h$ denotes the homological degree, for elements x and y of $\mathcal{B}$ which are homogeneous with respect to the homological grading. The coefficient ring R is required to coincide with the summand $\mathcal{B}_{0,0}$ of $\mathcal{B}$ in bigrading $(0,0)$.

2.1.3 Definition Suppose $R = \mathbb{Z}^{\times k}$. The elements $e_1 = (1,0,\dots,0),\dots,e_k = (0,\dots,0,1)$ will be called the *minimal*, or *elementary*, idempotents of $\mathcal{B}$. The coefficient ring R will also be referred to as the *idempotent ring* of $\mathcal{B}$. For each elementary idempotent e_i, there is a left R–module $Re_i \simeq \mathbb{Z}$.

2.1.4 Remark The usual convention in bordered Floer homology is to have the differential decrease the homological grading by 1; we have chosen to reverse this convention since the differentials in Khovanov homology increase homological grading by 1.

2.1.5 Remark Bordered Floer homology requires more general gradings by a (possibly nonabelian) group G and a distinguished element λ in the center of G. We use here only the special case where G is the abelian group $\mathbb{Z}^2$ and λ is $(0,1)$.

When dealing with bigraded algebras or modules, we will use the following degree shift convention: if $X = \bigoplus_{i,j} X_{i,j}$ is any type of bigraded object, then $X[m,n]$ is the same type of bigraded object, and the summand of $X[m,n]$ in bigrading (i,j) is $X_{i-m,j-n}$.

Since we are working over $\mathbb{Z}$, the following notation will also be useful, following Roberts [11; 12]. If X is any type of bigraded object, then $|\mathrm{id}|\colon X \to X$ is defined by multiplication by $(-1)^{\deg_h}$, where $\deg_h$ denotes the homological degree. Similarly, $|\mathrm{id}|^j\colon X \to X$ is defined by multiplication by $(-1)^{j\deg_h}$, and $|\mathrm{id}|^{j\otimes k}$ is the k–fold tensor product of $|\mathrm{id}|^j$. In this notation, if μ_2 denotes the multiplication on a dg algebra $\mathcal{B}$, then the Leibniz rule for the differential μ_1 on $\mathcal{B}$ can be written as

$$\mu_1 \circ \mu_2 = \mu_2 \circ (\mu_1 \otimes |\mathrm{id}|) + \mu_2 \circ (\mathrm{id} \otimes \mu_1).$$

2.1.6 Definition A *left differential bigraded module*, or *left dg module*, over a dg algebra $\mathcal{B}$, is a bigraded left $\mathcal{B}$–module M equipped with a differential d of bidegree

$(0,+1)$, such that the Leibniz rule

$$d\circ m = m\circ(\mu_1\otimes|\mathrm{id}|)+m\circ(\mathrm{id}\otimes d)$$

is satisfied, where $m\colon \mathcal{B}\otimes_R M\to M$ is the action of $\mathcal{B}$ on M and μ_1 is the differential on $\mathcal{B}$.

2.1.7 Definition A *right differential bigraded module*, or *right dg module*, over a dg algebra $\mathcal{B}$, is a bigraded right $\mathcal{B}$–module M equipped with a differential d of bidegree $(0,+1)$, such that the Leibniz rule

$$d\circ m = m\circ(d\otimes|\mathrm{id}|)+m\circ(\mathrm{id}\otimes\mu_1)$$

is satisfied, where $m\colon M\otimes_R\mathcal{B}\to M$ is the action of $\mathcal{B}$ on M and μ_1 is the differential on $\mathcal{B}$.

If M is a right dg module and M' is a left dg module over $\mathcal{B}$, then we can take the tensor product of M and M' over $\mathcal{B}$ to produce a chain complex of graded abelian groups, or equivalently a differential bigraded $\mathbb{Z}$–module.

2.1.8 Definition Let M be a right dg module and M' be a left dg module over $\mathcal{B}$. The differential on the tensor product $M\otimes_{\mathcal{B}}M'$ is defined to be

$$d_{M\otimes_{\mathcal{B}}M'} := d_M\otimes|\mathrm{id}_{M'}|+\mathrm{id}_M\otimes d_{M'}.$$

2.2 Type D structures

2.2.1 Definition Let $\mathcal{B}$ be a differential bigraded algebra over R as in Definition 2.1.2. Let μ_1 and μ_2 denote the differential and multiplication on $\mathcal{B}$, respectively.

A *type D structure* over $\mathcal{B}$ is, firstly, a bigraded left R–module $\widehat{D}$ which is isomorphic to a finite direct sum of R–modules $Re_{i_\alpha}[j_\alpha,k_\alpha]$, where the e_{i_α} are elementary idempotents of $\mathcal{B}$ (all in bigrading $(0,0)$) and $[j_\alpha,k_\alpha]$ is a grading shift. The module $\widehat{D}$ should be equipped with a bigrading-preserving R–linear map

$$\delta\colon \widehat{D}\to(\mathcal{B}\otimes_R\widehat{D})[0,-1],$$

such that

$$(\mu_1\otimes|\mathrm{id}|)\circ\delta+(\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta)\circ\delta=0.$$

2.2.2 Remark The condition that $\widehat{D}=\bigoplus_\alpha Re_{i_\alpha}[j_\alpha,k_\alpha]$ would be unnecessary if R were a direct product of copies of $\mathbb{Z}/2\mathbb{Z}$, rather than $\mathbb{Z}$. But over $\mathbb{Z}$, we want to exclude cases like $\mathcal{B}=R=\mathbb{Z}$, $\widehat{D}=\mathbb{Z}/2\mathbb{Z}$, $\delta=0$ from being valid type D structures. The reason for this restriction is that we want Proposition 2.2.3 below, which is true over $\mathbb{Z}/2\mathbb{Z}$, to hold over $\mathbb{Z}$ as well.

2.2.3 Proposition *If $(\widehat{D},\delta)$ is a type D structure over $\mathcal{B}$, then $\mathcal{B}\otimes_R \widehat{D}$ is a projective left dg $\mathcal{B}$–module when equipped with the differential*

$$d := \mu_1\otimes|\mathrm{id}| + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta),$$

where μ_1 and μ_2 denote the differential and multiplication on $\mathcal{B}$, respectively.

Proof First, since $\widehat{D}$ (as an R–module) is a direct sum of R–modules $Re_{i_\alpha}[j_\alpha,k_\alpha]$, $\mathcal{B}\otimes_R\widehat{D}$ is a direct sum of $\mathcal{B}$–modules $\mathcal{B}e_{i_\alpha}[j_\alpha,k_\alpha]$. These are each projective because they are summands of grading shifts of $\mathcal{B}$: if $R=\mathbb{Z}^{\times k}$, we have $\mathcal{B}=\bigoplus_{i=1}^k \mathcal{B}e_i$ as left $\mathcal{B}$–modules. Thus, $\mathcal{B}\otimes_R\widehat{D}$ is a projective $\mathcal{B}$–module.

Before showing that $d^2=0$, we check that d satisfies the Leibniz rule. The action of the algebra $\mathcal{B}$ on $\mathcal{B}\otimes_R\widehat{D}$ is given by the following map:

$$m := \mu_2\otimes\mathrm{id}\colon \mathcal{B}\otimes_R(\mathcal{B}\otimes_R\widehat{D}) = (\mathcal{B}\otimes_R\mathcal{B})\otimes_R\widehat{D}\to\mathcal{B}\otimes_R\widehat{D}.$$

We want to show that $d\circ m = m\circ(\mu_1\otimes|\mathrm{id}|)+m\circ(\mathrm{id}\otimes d)$, as maps from $\mathcal{B}\otimes_R(\mathcal{B}\otimes_R\widehat{D})$ to $(\mathcal{B}\otimes_R\widehat{D})$. We can write out the left side:

$$\begin{aligned}
d\circ m &= (\mu_1\otimes|\mathrm{id}| + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta))\circ(\mu_2\otimes\mathrm{id})\\
&= (\mu_1\circ\mu_2)\otimes|\mathrm{id}| + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta)\circ(\mu_2\otimes\mathrm{id})\\
&= (\mu_2\circ(\mu_1\otimes|\mathrm{id}|))\otimes|\mathrm{id}| + (\mu_2\circ(\mathrm{id}\otimes\mu_1))\otimes|\mathrm{id}|\\
&\qquad + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta)\circ(\mu_2\otimes\mathrm{id})\\
&= (\mu_2\otimes\mathrm{id})\circ(\mu_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|) + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_1\otimes|\mathrm{id}|)\\
&\qquad + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta)\circ(\mu_2\otimes\mathrm{id}).
\end{aligned}$$

Meanwhile, the right side is this:

$$\begin{aligned}
&m\circ(\mu_1\otimes|\mathrm{id}|)+m\circ(\mathrm{id}\otimes d)\\
&\quad= (\mu_2\otimes\mathrm{id})\circ(\mu_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|) + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes d)\\
&\quad= (\mu_2\otimes\mathrm{id})\circ(\mu_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|) + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes(\mu_1\otimes|\mathrm{id}| + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta)))\\
&\quad= (\mu_2\otimes\mathrm{id})\circ(\mu_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|) + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_1\otimes|\mathrm{id}|)\\
&\qquad\qquad + (\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes((\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta))).
\end{aligned}$$

The first two terms on the left side cancel with those on the right side, and we only need show that

$$(\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta)\circ(\mu_2\otimes\mathrm{id}) = (\mu_2\otimes\mathrm{id})\circ\big(\mathrm{id}\otimes((\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta))\big).$$

This identity follows since

$$\begin{aligned}(\mu_2 \otimes \mathrm{id}) \circ \big(\mathrm{id} \otimes ((\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta))\big) &= (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \delta) \\ &= (\mu_2 \otimes \mathrm{id}) \circ (\mu_2 \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \delta) \\ &= (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \circ (\mu_2 \otimes \mathrm{id}).\end{aligned}$$

Now suppose $a \otimes x$ is a generator of $\mathcal{B} \otimes_R \widehat{D}$; we want to show $d^2(a \otimes x) = 0$. We may write $a \otimes x$ as $m(a, 1 \otimes x)$ and apply the Leibniz rule

$$d(a \otimes x) = (-1)^{\deg_h x} m(\mu_1(a), 1 \otimes x) + m(a, \delta(x)),$$

so

$$\begin{aligned}d^2(a \otimes x) &= (-1)^{\deg_h x} d(m(\mu_1(a), 1 \otimes x)) + d(m(a, \delta(x))) \\ &= (-1)^{\deg_h x} m(\mu_1(a), \delta(x)) + (-1)^{\deg_h x + 1} m(\mu_1(a), \delta(x)) + m(a, d(\delta(x))).\end{aligned}$$

The first two terms cancel each other, so it suffices to show that $d(\delta(x)) = 0$. Writing out d, this equation amounts to

$$(\mu_1 \otimes |\mathrm{id}|) \circ \delta + (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \circ \delta = 0.$$

This is exactly the type D structure relation. □

The following propositions will be useful in the description of Khovanov's functor-valued invariant as a bordered theory.

2.2.4 Proposition *Let $\mathcal{B}$ be a dg algebra over R. Suppose that $\mathcal{B}$ is concentrated in homological degree 0 (it may have nontrivial intrinsic gradings). Then a dg module over $\mathcal{B}$ is the same as a chain complex of singly graded $\mathcal{B}$–modules with $\mathcal{B}$–linear grading-preserving differential maps.*

Proof Since $\mathcal{B}$ is concentrated in homological degree 0, the differential on $\mathcal{B}$ must be zero. Let M be a dg module over $\mathcal{B}$, with summand $M_{j,k}$ in bigrading (j,k). Then, for each homological grading k, the summand $\bigoplus_j M_{j,k}$ of M is preserved when multiplying by $\mathcal{B}$; it is a singly graded $\mathcal{B}$–module. Define a chain complex with chain module $C_k = \bigoplus_j M_{j,k}$. The differential $C_k \to C_{k+1}$ is the differential on M; it is $\mathcal{B}$–linear by the Leibniz rule, since $\mathcal{B}$ has no differential.

In the other direction, taking direct sums over chain modules yields a map from chain complexes to dg modules. These operations are inverse to each other. □

The next proposition involves isomorphisms of type D structures; see Definition 6.6.1 for the basic definitions.

2.2.5 Proposition *Let $\mathcal{B}$ be a dg algebra over R. Suppose that $\mathcal{B}$ is concentrated in homological degree 0, and that all intrinsic gradings of $\mathcal{B}$ are nonnegative. Then a type D structure over $\mathcal{B}$ is the same, up to isomorphism, as a chain complex of singly graded projective left $\mathcal{B}$–modules with $\mathcal{B}$–linear grading-preserving differential maps.*

Proof Given a type D structure $\widehat{D}$ over $\mathcal{B}$, Proposition 2.2.3 shows that $\mathcal{B}\otimes_R \widehat{D}$ is a dg module over $\mathcal{B}$, or equivalently a chain complex of graded left $\mathcal{B}$–modules by Proposition 2.2.4. In fact, each term of the chain complex is projective, since it is a direct sum of modules $\mathcal{B}e_{i_\alpha}[j_\alpha, k_\alpha]$.

Conversely, suppose $\cdots \to C_k \to C_{k+1} \to \cdots$ is a chain complex of graded projective left $\mathcal{B}$–modules. Since each C_k is assumed to be finitely generated, it may be written as a direct sum of indecomposable graded projective left R–modules $C_{k,\alpha}$. By Khovanov [4, Lemma 1 of Section 2.5], which assumes that the intrinsic gradings of $\mathcal{B}$ are nonnegative, we see that each $C_{k,\alpha}$ is isomorphic to $\mathcal{B}e_{i_{k,\alpha}}[j_{k,\alpha}]$ for some uniquely determined elementary idempotent $e_{i_{k,\alpha}}$ and grading shift $j_{k,\alpha}$. Define $\widehat{D}$ as a bigraded R–module to be the direct sum, over all k and α, of $Re_{i_{k,\alpha}}[j_{k,\alpha}, k]$.

We may identify $\bigoplus_k C_k$ with $\mathcal{B}\otimes_R \widehat{D}$, since $\bigoplus_k C_k = \bigoplus_{k,\alpha} \mathcal{B}e_{i_{k,\alpha}}[j_{k,\alpha}, k]$ and $\widehat{D} = \bigoplus_{k,\alpha} Re_{i_{k,\alpha}}[j_{k,\alpha}, k]$. Let d denote the differential on the dg module $\bigoplus_k C_k$. Then the type D operation $\delta\colon \widehat{D} \to \mathcal{B}\otimes_R \widehat{D}$ is obtained by restricting d to $\widehat{D} \cong 1\otimes_R \widehat{D} \subset \mathcal{B}\otimes_R \widehat{D}$. It has the correct grading properties because d does.

Since d satisfies the Leibniz rule, we may write $d = \mu_1 \otimes |\mathrm{id}| + (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta)$. Thus, the type D relations for δ are equivalent to $d \circ \delta = 0$, which holds because δ is a restriction of d.

Finally, we show the two constructions given above are inverses up to isomorphism. Suppose we start with a chain complex $\cdots \to C_k \to C_{k+1} \to \cdots$, decompose each C_k as $\bigoplus_\alpha C_{k,\alpha}$ and take the corresponding type D structure $\widehat{D}$. Then $\mathcal{B}\otimes_R \widehat{D}$ is clearly isomorphic to the dg module associated to $\cdots \to C_k \to C_{k+1} \to \cdots$. On the other hand, suppose we start with a type D structure $\widehat{D}$ and then obtain a type D structure by decomposing $\mathcal{B}\otimes_R \widehat{D}$ into indecomposable projectives. The resulting type D structure has the same number of generators as $\widehat{D}$, with the same idempotents and bigradings. However, the type D operation may be different: in decomposing $\mathcal{B}\otimes_R \widehat{D}$, we may have incorporated a change of basis.

From Definition 6.6.1, we see that if $\widehat{D}$ and $\widehat{D}'$ are two type D structures such that $\mathcal{B}\otimes_R \widehat{D}$ and $\mathcal{B}\otimes_R \widehat{D}'$ are isomorphic as dg modules, then $\widehat{D}$ and $\widehat{D}'$ are isomorphic as type D structures. The required isomorphisms of type D structures may be obtained by restricting the isomorphisms of dg modules. Thus, any type D structure obtained by decomposing $\mathcal{B}\otimes_R \widehat{D}$ as above is isomorphic to $\widehat{D}$. □

2.3 Type A structures and pairing

2.3.1 Definition Let $\mathcal{B}$ be a dg algebra over R as in Definition 2.1.2. Let μ_1 and μ_2 denote the differential and multiplication on $\mathcal{B}$, respectively.

A *type A structure* $\widehat{A}$ over $\mathcal{B}$, synonymous with $\mathcal{A}_\infty$*–module* over $\mathcal{B}$, is a bigraded right R–module $\widehat{A}$, finitely generated over R as usual by Convention 2.1.1, together with R–linear bigrading-preserving maps $m_i\colon \widehat{A}\otimes_R \mathcal{B}^{\otimes(i-1)} \to \widehat{A}[0, i-2]$, $i \in \mathbb{Z}_{\geq 1}$, satisfying

$$\sum_{i+j=n+1} (-1)^{j(i+1)} m_i \circ (m_j \otimes |\mathrm{id}|^{j\otimes(i-1)}) + (-1)^{n+1} \sum_{k=1}^{n-1} m_n \circ (\mathrm{id}^{\otimes k} \otimes \mu_1 \otimes |\mathrm{id}|^{\otimes(n-k-1)}) + \sum_{k=1}^{n-2} (-1)^k m_{n-1} \circ (\mathrm{id}^{\otimes k} \otimes \mu_2 \otimes \mathrm{id}^{\otimes(n-k-2)}) = 0$$

for every $n \geq 1$. The type A structure $\widehat{A}$ is called *strictly unital* if $m_2(-, 1) = \mathrm{id}_{\widehat{A}}$ and $m_n = 0$ for $n > 2$ when any of the algebra inputs to m_n is 1.

2.3.2 Example If M is a (right) dg $\mathcal{B}$–module, then M is a strictly unital type A structure over $\mathcal{B}$ with $m_i = 0$ for $i \neq 1, 2$. If M is an ordinary bigraded module over $\mathcal{B}$, with no differential, then M is a strictly unital type A structure with $m_i = 0$ for $i \neq 2$.

2.3.3 Remark We will only need to work with type A structures which come from dg modules as in Example 2.3.2. Thus, all our type A structures will be strictly unital, so we will omit mention of this condition in what follows. However, although our type A structures will have no nontrivial higher action terms, we will eventually need to work with $\mathcal{A}_\infty$*–morphisms* between these type A structures. We will need to consider morphisms which do have nontrivial higher $\mathcal{A}_\infty$–terms; see Section 6.5.

Given a type D structure $(\widehat{D}, \delta)$ and a type A structure $(\widehat{A}, \{m_i \mid i \geq 1\})$ over $\mathcal{B}$, with either $\widehat{D}$ or $\widehat{A}$ operationally bounded in an appropriate sense, the natural way to pair them is known as the box tensor product. It yields a differential bigraded abelian group $\widehat{A} \boxtimes \widehat{D}$. We will not worry about boundedness in this paper since all type D and type A structures under consideration are bounded. See Lipshitz, Ozsváth and Thurston [7] for more details and algebraic properties of $\boxtimes$ over $\mathbb{Z}/2\mathbb{Z}$. The material below follows Roberts [11]; we include proofs for completeness.

To define $\boxtimes$, the following notation will be useful.

2.3.4 Definition Let $(\widehat{D},\delta)$ be a type D structure over $\mathcal{B}$. The map $\delta^k\colon \widehat{D}\to\mathcal{B}^k\otimes_R\widehat{D}$ is

$$\delta^k := (\overbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}^{k-1}\otimes\delta)\circ\cdots\circ(\mathrm{id}\otimes\delta)\circ\delta,$$

where δ is applied k times. In particular, $\delta=\delta^1$.

2.3.5 Definition [11, Definition 79] $\widehat{A}\boxtimes\widehat{D}$, as a bigraded abelian group, is the tensor product $\widehat{A}\otimes_R\widehat{D}$. The differential on $\widehat{A}\boxtimes\widehat{D}$ is

$$\partial^{\boxtimes} := \sum_{n=1}^{\infty}(m_n\otimes|\mathrm{id}|^n)\circ(\mathrm{id}\otimes\delta^{n-1}).$$

Since we are implicitly assuming boundedness, only finitely many terms of the sum are nonzero.

2.3.6 Proposition [11, Theorem 80] *The operator $\partial^{\boxtimes}$, as defined in Definition 2.3.5, satisfies*

$$(\partial^{\boxtimes})^2=0.$$

Proof In this proof, when referring to identity operators, we will use subscripts to explicitly indicate which identity operators we mean.

First, note that as maps from $\widehat{A}\otimes_R\mathcal{B}^{j-1}\otimes_R\widehat{D}$ to $\widehat{A}\otimes_R\mathcal{B}^{i-1}\otimes_R\widehat{D}$, we have

$$(\mathrm{id}_{\widehat{A}}\otimes\delta^{i-1})\circ(m_j\otimes|\mathrm{id}_{\widehat{D}}|^j)=(-1)^{j(i+1)}(m_j\otimes|\mathrm{id}_{\mathcal{B}^{i-1}\otimes\widehat{D}}|^j)\circ(\mathrm{id}_{\widehat{A}\otimes\mathcal{B}^{j-1}}\otimes\delta^{i-1}).$$

This identity is immediate over $\mathbb{Z}/2\mathbb{Z}$, and we need only verify that the signs are right. On the right side of the equality, we have $|\mathrm{id}_{\mathcal{B}^{i-1}\otimes\widehat{D}}|^j$ which is computed from the homological degree of an output of δ^{i-1}. Since δ increases homological degree by 1, δ^{i-1} increases homological degree by $i-1$. Thus, compared with the left side, the right side has an extra factor of $(-1)^{j(i-1)}=(-1)^{j(i+1)}$.

Thus,

$$\begin{aligned}(\partial^{\boxtimes})^2 &= \sum_{n\geq1}\sum_{i+j=n+1}(m_i\otimes|\mathrm{id}_{\widehat{D}}|^i)\circ(\mathrm{id}_{\widehat{A}}\otimes\delta^{i-1})\circ(m_j\otimes|\mathrm{id}_{\widehat{D}}|^j)\circ(\mathrm{id}_{\widehat{A}}\otimes\delta^{j-1})\\ &= \sum_{n\geq1}\sum_{i+j=n+1}(-1)^{j(i+1)}(m_i\otimes|\mathrm{id}_{\widehat{D}}|^i)\circ(m_j\otimes|\mathrm{id}_{\mathcal{B}^{i-1}}|^j\otimes|\mathrm{id}_{\widehat{D}}|^j)\\ &\qquad\circ(\mathrm{id}_{\widehat{A}}\otimes\mathrm{id}_{\mathcal{B}^{j-1}}\otimes\delta^{i-1})\circ(\mathrm{id}_{\widehat{A}}\otimes\delta^{j-1})\\ &= \sum_{n\geq1}\sum_{i+j=n+1}(-1)^{j(i+1)}((m_i\circ(m_j\otimes|\mathrm{id}_{\mathcal{B}^{i-1}}|^j))\otimes|\mathrm{id}_{\widehat{D}}|^{n+1})\circ(\mathrm{id}_{\widehat{A}}\otimes\delta^{n-1})\end{aligned}$$

$$= \sum_{n\geq 1}\Bigg(\Bigg((-1)^n \sum_{k=1}^{n-1} m_n \circ (\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{B}^{k-1}} \otimes \mu_1 \otimes |\mathrm{id}_{\mathcal{B}^{n-k-1}}|) \\ - \sum_{k=1}^{n-2} (-1)^k m_{n-1} \circ (\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{B}^{k-1}} \otimes \mu_2 \otimes \mathrm{id}_{\mathcal{B}^{n-k-2}})\Bigg) \otimes |\mathrm{id}_{\widehat{D}}|^{n+1}\Bigg) \\ \circ (\mathrm{id}_{\widehat{A}} \otimes \delta^{n-1}),$$

where the type A relations for $\widehat{A}$ were used in the final equality. It remains to show that the derivative terms

$$\sum_{n\geq 1}\Bigg(\Bigg((-1)^n \sum_{k=1}^{n-1} m_n \circ (\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{B}^{k-1}} \otimes \mu_1 \otimes |\mathrm{id}_{\mathcal{B}^{n-k-1}}|)\Bigg) \otimes |\mathrm{id}_{\widehat{D}}|^{n+1}\Bigg) \circ (\mathrm{id}_{\widehat{A}} \otimes \delta^{n-1})$$

are equal to the multiplication terms

$$\sum_{n\geq 1}\Bigg(\Bigg(\sum_{k=1}^{n-2} (-1)^k m_{n-1} \circ (\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\mathcal{B}^{k-1}} \otimes \mu_2 \otimes \mathrm{id}_{\mathcal{B}^{n-k-2}})\Bigg) \otimes |\mathrm{id}_{\widehat{D}}|^{n+1}\Bigg) \circ (\mathrm{id}_{\widehat{A}} \otimes \delta^{n-1}).$$

For a fixed $n \geq 1$ and $1 \leq k \leq n-1$, we claim that the derivative term is equal to

$$(-1)^{k+1}(m_n \otimes |\mathrm{id}_{\widehat{D}}|^n) \circ \big(\mathrm{id}_{\widehat{A}} \otimes \big((\mathrm{id}_{\mathcal{B}^{n-2}} \otimes \delta) \circ \cdots \\ \circ (\mathrm{id}_{\mathcal{B}^{k-1}} \otimes \mu_1 \otimes |\mathrm{id}_{\widehat{D}}|) \circ (\mathrm{id}_{\mathcal{B}^{k-1}} \otimes \delta) \circ \cdots \circ (\mathrm{id}_{\mathcal{B}} \otimes \delta) \circ \delta\big)\big).$$

Over $\mathbb{Z}/2\mathbb{Z}$, this equality follows from expanding out δ^{n-1}. To see that the formula holds over $\mathbb{Z}$, note that when $k = n-1$, the sign in front of the above expression is $(-1)^n$, in agreement with the original expression for the derivative term. Each time k is decreased by 1, the sign should flip because $|\mathrm{id}_{\widehat{D}}|$ occurs after one fewer instance of δ in the second expression, compared to the original.

Now,

$$(\mathrm{id}_{\mathcal{B}^{k-1}} \otimes \mu_1 \otimes |\mathrm{id}_{\widehat{D}}|) \circ (\mathrm{id}_{\mathcal{B}^{k-1}} \otimes \delta) = \mathrm{id}_{\mathcal{B}^{k-1}} \otimes ((\mu_1 \otimes |\mathrm{id}_{\widehat{D}}|) \circ \delta) \\ = -\mathrm{id}_{\mathcal{B}^{k-1}} \otimes ((\mu_2 \otimes \mathrm{id}_{\widehat{D}}) \circ (\mathrm{id}_{\mathcal{B}} \otimes \delta) \circ \delta)$$

by the type D relations for $\widehat{D}$. Thus, the sum of the derivative terms is

$$\sum_{n\geq 1}\sum_{k=1}^{n-1}(-1)^k(m_n \otimes |\mathrm{id}_{\widehat{D}}|^n) \circ \big(\mathrm{id}_{\widehat{A}} \otimes \big((\mathrm{id}_{\mathcal{B}^{n-2}} \otimes \delta) \circ \cdots \\ \circ (\mathrm{id}_{\mathcal{B}^k} \otimes \delta) \circ (\mathrm{id}_{\mathcal{B}^{k-1}} \otimes \mu_2 \otimes \mathrm{id}_{\widehat{D}}) \circ (\mathrm{id}_{\mathcal{B}^k} \otimes \delta) \circ (\mathrm{id}_{\mathcal{B}^{k-1}} \otimes \delta) \circ \cdots \circ \delta\big)\big) \\ = \sum_{n\geq 1}\sum_{k=1}^{n-1}(-1)^k(m_n \otimes |\mathrm{id}_{\widehat{D}}|^n) \circ (\mathrm{id}_{\widehat{A}\otimes\mathcal{B}^{k-1}} \otimes \mu_2 \otimes \mathrm{id}_{\mathcal{B}^{n-k-1}\otimes\widehat{D}}) \circ (\mathrm{id}_{\widehat{A}} \otimes \delta^n).$$

Since the $n=1$ multiplication term is zero, and $|\mathrm{id}_{\widehat{D}}|^{n+1} = |\mathrm{id}_{\widehat{D}}|^{n-1}$, the sum of the multiplication terms is

$$\sum_{n\geq 2}\sum_{k=1}^{n-2}(-1)^k(m_{n-1}\otimes|\mathrm{id}_{\widehat{D}}|^{n-1})\circ(\mathrm{id}_{\widehat{A}\otimes\mathcal{B}^{k-1}}\otimes\mu_2\otimes\mathrm{id}_{\mathcal{B}^{n-k-2}\otimes\widehat{D}})\circ(\mathrm{id}_{\widehat{A}}\otimes\delta^{n-1})$$
$$=\sum_{n\geq 1}\sum_{k=1}^{n-1}(-1)^k(m_n\otimes|\mathrm{id}_{\widehat{D}}|^{n})\circ(\mathrm{id}_{\widehat{A}\otimes\mathcal{B}^{k-1}}\otimes\mu_2\otimes\mathrm{id}_{\mathcal{B}^{n-k-1}\otimes\widehat{D}})\circ(\mathrm{id}_{\widehat{A}}\otimes\delta^{n}).$$

The sums of the derivative and multiplication terms agree, proving that $(\partial^{\boxtimes})^2=0$. □

2.3.7 Proposition [5, Example 2.2.7] *Let $\mathcal{B}$ be a dg algebra over R as in Definition 2.1.2. Let $\widehat{D}$ be a type D structure over $\mathcal{B}$, and let $\widehat{A}$ be a right dg module over $\mathcal{B}$. Then $\widehat{A}\boxtimes\widehat{D}$ and $\widehat{A}\otimes_{\mathcal{B}}(\mathcal{B}\otimes_R\widehat{D})$ are isomorphic as differential bigraded abelian groups.*

For completeness, and since we are working over $\mathbb{Z}$, we will give a proof of this proposition.

Proof of Proposition 2.3.7 As bigraded abelian groups,

$$\widehat{A}\otimes_{\mathcal{B}}(\mathcal{B}\otimes_R\widehat{D})\cong(\widehat{A}\otimes_{\mathcal{B}}\mathcal{B})\otimes_R\widehat{D}\cong\widehat{A}\otimes_R\widehat{D}.$$

Thus, $\widehat{A}\otimes_{\mathcal{B}}(\mathcal{B}\otimes_R\widehat{D})$ and $\widehat{A}\boxtimes\widehat{D}$ have the same underlying group; we must verify that the differentials agree.

The differential on $\widehat{A}\otimes_{\mathcal{B}}(\mathcal{B}\otimes_R\widehat{D})$ may be written as

$$\begin{aligned}d_{\widehat{A}}\otimes(|\mathrm{id}_{\mathcal{B}}|\otimes|\mathrm{id}_{\widehat{D}}|)+\mathrm{id}_{\widehat{A}}\otimes(\mu_1\otimes|\mathrm{id}_{\widehat{D}}|+(\mu_2\otimes\mathrm{id}_{\widehat{D}})\circ(\mathrm{id}_{\mathcal{B}}\otimes\delta))&\\ =d_{\widehat{A}}\otimes(|\mathrm{id}_{\mathcal{B}}|\otimes|\mathrm{id}_{\widehat{D}}|)+\mathrm{id}_{\widehat{A}}\otimes(\mu_1\otimes|\mathrm{id}_{\widehat{D}}|)&\\ +(\mathrm{id}_{\widehat{A}}\otimes(\mu_2\otimes\mathrm{id}_{\widehat{D}}))\circ(\mathrm{id}_{\widehat{A}}\otimes(\mathrm{id}_{\mathcal{B}}\otimes\delta)).&\end{aligned}$$

Regrouping the parentheses, we get

$$(d_{\widehat{A}}\otimes|\mathrm{id}_{\mathcal{B}}|)\otimes|\mathrm{id}_{\widehat{D}}|+(\mathrm{id}_{\widehat{A}}\otimes\mu_1)\otimes|\mathrm{id}_{\widehat{D}}|+((\mathrm{id}_{\widehat{A}}\otimes\mu_2)\otimes\mathrm{id}_{\widehat{D}})\circ((\mathrm{id}_{\widehat{A}}\otimes\mathrm{id}_{\mathcal{B}})\otimes\delta).$$

Identifying $\widehat{A}$ with $\widehat{A}\otimes_{\mathcal{B}}\mathcal{B}$, the differential on $\widehat{A}$ becomes $d_{\widehat{A}}\otimes|\mathrm{id}_{\mathcal{B}}|+\mathrm{id}_{\widehat{A}}\otimes\mu_1$. Similarly, the algebra multiplication $m\colon\widehat{A}\otimes\mathcal{B}\to\widehat{A}$ becomes

$$\mathrm{id}_{\widehat{A}}\otimes\mu_2\colon(\widehat{A}\otimes_{\mathcal{B}}\mathcal{B})\otimes\mathcal{B}\to(\widehat{A}\otimes_{\mathcal{B}}\mathcal{B}).$$

Thus, we can identify the above formula for the differential on $\widehat{A}\otimes_{\mathcal{B}}(\mathcal{B}\otimes_R\widehat{D})\cong\widehat{A}\otimes_R\widehat{D}$ with

$$d_{\widehat{A}}\otimes|\mathrm{id}_{\widehat{D}}|+(m\otimes\mathrm{id}_{\widehat{D}})\circ(\mathrm{id}_{\widehat{A}}\otimes\delta).$$

This is also the differential on $\widehat{A}\boxtimes\widehat{D}$. □

3 Khovanov's functor-valued invariant as a bordered theory

We will assume some familiarity with Khovanov's paper [4]. Here we briefly introduce some useful conventions and notation.

Khovanov's arc algebra H^n has one $\mathbb{Z}$–grading. We will view H^n as a differential bigraded algebra, concentrated in homological degree 0 and with no differential. The usual $\mathbb{Z}$–grading on H^n becomes the intrinsic component of the bigrading. The intrinsic gradings of H^n are nonnegative. Thus, both Proposition 2.2.4 and Proposition 2.2.5 apply to H^n as a dg algebra.

The component of H^n in degree 0 (or, with our conventions, in bidegree $(0,0)$) will be denoted $\mathcal{I}_n$ and referred to as the idempotent ring of H^n. It is isomorphic to $\mathbb{Z}^{\times(C_n)}$, where C_n is the n^{th} Catalan number. The elementary idempotents of H^n are the idempotents 1_a described by [4, Section 2.4]. The index a runs over elements of the set B^n of crossingless matchings of $2n$ points; since this set will be important later, we recall its definition here.

3.0.1 Definition Let P be a set of $2n$ distinct points $p_1, \dots, p_{2n}$ on the line $\{0\}\times\mathbb{R} \subset \mathbb{R}\times\mathbb{R}$, ordered from top to bottom. A *crossingless matching* a of P is a partition of P into n pairs of points, such that there exists an embedding of n arcs $[0,1]^{\sqcup n}$ disjointly into $\mathbb{R}_{\geq 0}\times\mathbb{R}$ with each arc connecting a pair of points matched in a. The set of crossingless matchings of $2n$ points will be denoted B^n (different choices of P yield canonical bijections between the relevant sets B^n).

3.0.2 Remark The set B^n is also in bijection with the set NC_n of noncrossing partitions of n points. A noncrossing partition a of a set Q of n points $q_1, \dots, q_n$ on $\{0\}\times\mathbb{R}$ (ordered from top to bottom again) is defined to be any partition of Q into k disjoint subsets, such that there exists an embedding of k acyclic graphs disjointly into $\mathbb{R}_{\geq 0}\times\mathbb{R}$, with each graph bounding one of the k subsets of $\{q_1, \dots, q_n\}$ comprising a.

To go from a crossingless matching a of $2n$ points $p_1, \dots, p_{2n}$ to a noncrossing partition a' of n points $q_1, \dots, q_n$, checkerboard color the half-plane $\mathbb{R}_{\geq 0}$ with respect to some embedding of arcs representing a, such that the unbounded region of the half-plane is colored white. Without loss of generality, we may put the point q_i on the line $\{0\}\times\mathbb{R}$ between the points p_{2i-1} and p_{2i}. In the noncrossing partition a', two points q_i and q_j are placed in the same subset if they can be connected in $\mathbb{R}_{\geq 0}\times\mathbb{R}$ by a path through the black region of the checkerboard coloring. The skeleton of the black region provides the planar graphs which verify that a' is a noncrossing partition.

On the other hand, given a noncrossing partition a' of n points, one can pick an embedding of graphs representing a', and fatten each graph to obtain a planar surface.

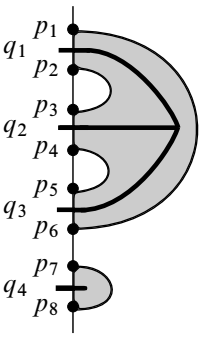

Figure 1: Bijection between crossingless matchings on $2n$ points and non-crossing partitions on n points

The boundary of this surface is a crossingless matching of $2n$ points. These two constructions are inverse to each other; see Figure 1 for an illustration.

Let T be an oriented tangle diagram in the half-plane, with $2n$ endpoints, and assume we have chosen an ordering of the crossings of T. Khovanov's construction assigns a bounded chain complex of finitely generated projective graded H^n–modules, with H^n–linear differential maps, to T. We will call this complex $[T]^{\mathrm{Kh}}$; we will often view $[T]^{\mathrm{Kh}}$ as a dg H^n–module using Proposition 2.2.4. If T lies in $\mathbb{R}_{\geq 0} \times \mathbb{R}$, then $[T]^{\mathrm{Kh}}$ is a left dg module; if T lies in $\mathbb{R}_{\leq 0} \times \mathbb{R}$, then $[T]^{\mathrm{Kh}}$ is a right dg module.

If T_1 is an oriented tangle diagram in $\mathbb{R}_{\geq 0} \times \mathbb{R}$, Proposition 2.2.5 gives us an isomorphism class of type D structures $\widehat{D}(T_1)$ over H^n, such that

$$[T_1]^{\mathrm{Kh}} \cong H^n \otimes_{\mathcal{I}_n} \widehat{D}(T_1).$$

As we will see in Section 3.1, Khovanov's construction of $[T_1]^{\mathrm{Kh}}$ naturally gives us an explicit type D structure $\widehat{D}(T_1)$ with this property.

If T_2 is an oriented tangle diagram in $\mathbb{R}_{\leq 0} \times \mathbb{R}$, we will simply take the type A structure $\widehat{A}(T_2)$ of T_2 to be the right dg module $[T_2]^{\mathrm{Kh}}$. Suppose T_2 and T_1 have consistent orientations; put them together to obtain an oriented link diagram L. Order the crossings of L so that those of T_1 come before those of T_2 and let $\mathrm{CKh}(L)$ be the Khovanov complex of L. Khovanov shows in [4] that

$$\mathrm{CKh}(L) \cong [T_2]^{\mathrm{Kh}} \otimes_{H^n} [T_1]^{\mathrm{Kh}},$$

after multiplying the intrinsic gradings on $[T_2]^{\mathrm{Kh}} \otimes_{H^n} [T_1]^{\mathrm{Kh}}$ by -1. By Proposition 2.3.7, we have

$$\mathrm{CKh}(L) \cong \widehat{A}(T_2) \boxtimes \widehat{D}(T_1),$$

as in bordered Floer homology, after applying the same intrinsic-grading reversal to $\widehat{A}(T_2) \boxtimes \widehat{D}(T_1)$. We will summarize this discussion more formally below in Theorem 3.2.1.

3.0.3 Remark The reversal of the gradings here comes from Khovanov's choice [4, pages 672–673] to make H^n positively rather than negatively graded. It is only a convention; one could define the basic generators of H^n to live in degrees -1 and -2, rather than 1 and 2, and then no grading reversal would be necessary.

3.0.4 Remark Up to isomorphism, the bigraded chain complex $\mathrm{CKh}(L)$ does not depend on the ordering of the crossings. Indeed, suppose we reverse the ordering of two adjacent crossings i and $i+1$. Then an isomorphism

$$F\colon (\mathrm{CKh}(L),\ \text{first ordering}) \to (\mathrm{CKh}(L),\ \text{second ordering})$$

can be defined, on the summand of $\mathrm{CKh}(L)$ corresponding to a vertex ρ of the cube of resolutions, to be $F := (-1)^{f(\rho)} \cdot \mathrm{id}$, where $f(\rho) := 1$ if ρ resolves crossings i and $i+1$ both as 1, rather than 0, and $f(\rho) := 0$ otherwise.

The same argument applies unchanged to the tangle complexes $[T]^{\mathrm{Kh}}$: the isomorphism type of $[T]^{\mathrm{Kh}}$ does not depend on the ordering of the crossings.

3.0.5 Remark Khovanov avoids having to choose an ordering of the crossings by using the skew-commutative cubes formalism. We will not do this here, but we will usually suppress mention of the choice of ordering of the crossings.

3.1 Type D structures

In this section we give a concrete definition of $\widehat{D}(T_1)$. First, we recall some properties of H^n and Khovanov's dg modules $[T]^{\mathrm{Kh}}$. This section has some overlap with the author's PhD thesis [8, Sections 4.2–4.3].

The algebra H^n has an additive basis β, over $\mathbb{Z}$, consisting of elements which we will denote $((W(a)b), \sigma)$. Here, a and b are elements of B^n, the set of crossingless matchings of $2n$ points, and the operation W mirrors the matching from the right half-plane to the left half-plane. The horizontal concatenation $W(a)b$ is a collection of disjoint circles in $\mathbb{R}^2$. The remaining datum σ consists of a choice of sign, $+$ or $-$, on each of these circles. For $a \in B^n$, the idempotent 1_a is $(W(a)a, \text{all plus})$. Multiplication in H^n is defined using minimal cobordisms and a two-dimensional topological quantum field theory; we refer the reader to Khovanov [4] for a precise definition.

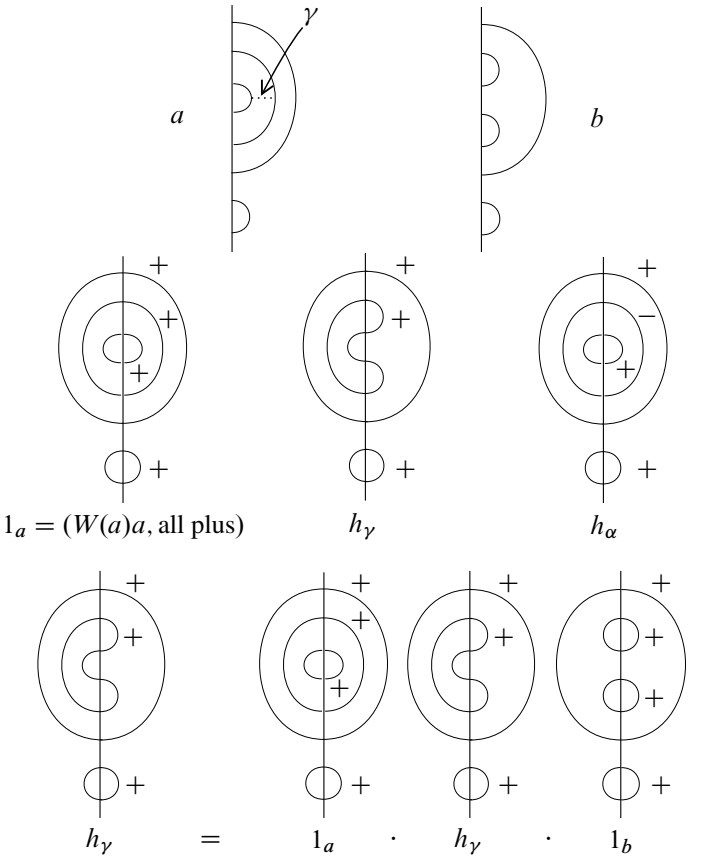

Figure 2: Top line: crossingless matchings a and b, with b obtained by surgering a bridge γ in a. Second line: the idempotent element $(W(a)a, \text{all plus})$ and the multiplicative generators $h_\gamma = (W(a)b, \text{all plus})$ and $h_\alpha = (W(a)a, \text{ minus on } W(\alpha)\alpha)$. Third line: left and right idempotents of h_γ.

Below we use the notion of a bridge of a crossingless matching; see Roberts [12, Definition 8]. The dotted arc γ in the first line of Figure 2 is a bridge of the crossingless matching a.

Certain of the basis elements $((W(a)b), \sigma)$ form a natural set of multiplicative generators for H^n. These generators come in two forms: the first are elements $h_\gamma = (W(a)b, \text{all plus})$, where $a \in B^n$, the element $b \in B^n$ is obtained from a by surgering one pair of arcs along a bridge γ, and all circles of $W(a)b$ are labeled $+$. The other generators are elements $h_\alpha = (W(a)a, \text{ minus on } W(\alpha)\alpha)$, where $a \in B^n$ and all circles of $W(a)a$ are labeled $+$ except one circle, $W(\alpha)\alpha$ for some arc α of a, which is labeled $-$. Each generator h_γ and h_α has a unique left idempotent and right

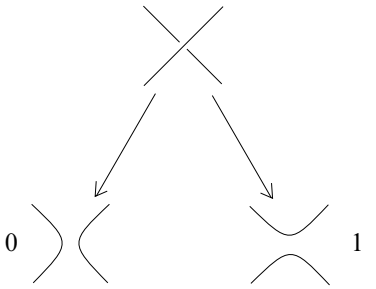

Figure 3: Zero- and one-resolutions of a crossing

idempotent in $\mathcal{I}_n$. We will denote the set of multiplicative generators $\{h_\gamma, h_\alpha\}$ as β_{mult}; it is a subset of β. Some examples are shown in Figure 2.

3.1.1 Proposition *The elements h_γ and h_α of β_{mult} generate H^n multiplicatively.*

Proof We have not defined the multiplication on H^n here, so we will only sketch the proof. It suffices to show that any element of the form $(W(a)b)$, all plus) may be written as a product of h_γ generators; the rest of the elements of β may then be obtained using these elements and h_α generators.

The element $(W(a)b)$, all plus) may be identified with a disjoint union of disks embedded in $D^2 \times I$ with boundary restricting to a on $D^2 \times \{0\}$, b on $D^2 \times \{1\}$, and $2n$ straight lines on $(\partial D^2) \times I$ (in other words, a cobordism from a to b). Here we identify crossingless matchings in the right half-plane with crossingless matchings in D^2; see Figure 5 below.

We may assume (after an isotopy if necessary) that the I–coordinate of $D^2 \times I$ gives a Morse function on the disjoint union of disks which has only index-1 critical points, each of which occurs at a distinct value of the I–coordinate. In such a configuration, the disjoint union of disks can be viewed as a composition of elementary saddle cobordisms beginning at a and ending at b. Each saddle cobordism corresponds to a generator h_γ. Furthermore, the composition of the saddle cobordisms in this sense agrees with the result of multiplying the elements h_γ in H^n using minimal cobordisms; see [8, Figure 4.3] for an illustration of this fact. Thus, we may write $(W(a)b)$, all plus) as a product of generators h_γ. □

Let T be an oriented tangle diagram in $\mathbb{R}_{\geq 0} \times \mathbb{R}$. To specify a generator x_i of $[T]^{\text{Kh}}$, we first specify a resolution ρ_i of all crossings of T; we can view ρ_i as a function from the set of crossings to the two-element set $\{0, 1\}$ (see Figure 3). If T_{ρ_i} denotes the diagram T with the crossings resolved according to ρ_i, then T_{ρ_i} consists of a

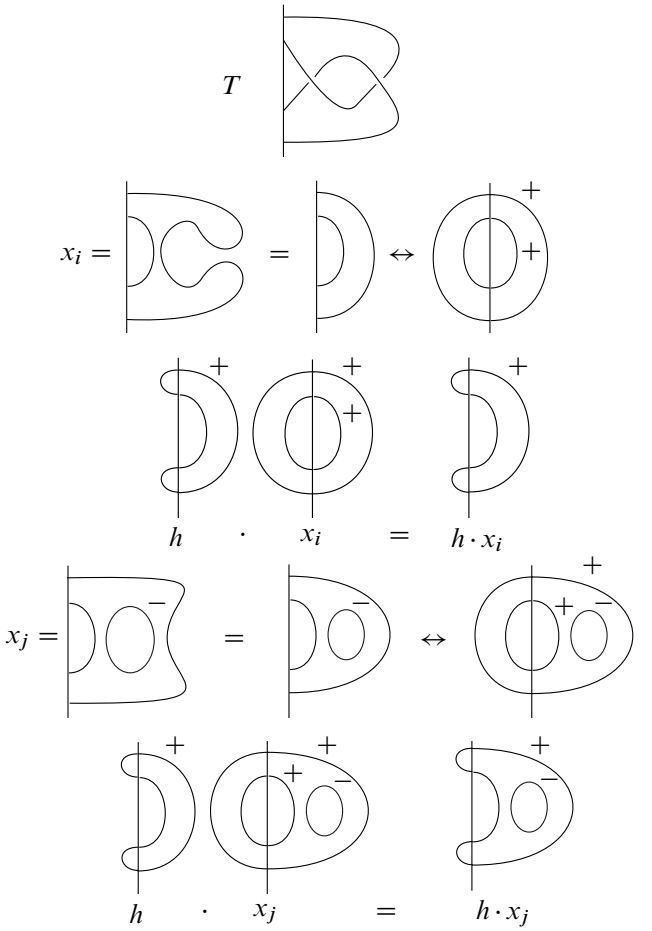

Figure 4: Some generators x_i and $h \cdot x_i$ of $[T]^{\mathrm{Kh}}$

crossingless matching of $2n$ points together with some circles contained in $\mathbb{R}_{>0} \times \mathbb{R}$. Following Roberts [12], these circles will be called free circles. The remaining data needed to specify x_i are a choice of $+$ or $-$ on each free circle.

Identify x_i with the diagram obtained by gluing the mirror of the crossingless-matching part of T_{ρ_i} to the left side of T_{ρ_i} and labeling all resulting circles with $+$. Then $[T]^{\mathrm{Kh}}$ has a $\mathbb{Z}$–basis consisting of elements $h \cdot x_i$, where the right idempotent of h agrees with the crossingless-matching part of T_{ρ_i}. (By multiplying h with x_i in Khovanov's usual minimal-cobordism sense, one obtains the basis for $[T]^{\mathrm{Kh}}$ described in [4].) See Figure 4 for an illustration of the generators x_i and basis elements $h \cdot x_i$.

The remainder of this section may also be found in the author's thesis [8, Section 4.3.1], with minor modifications.

3.1.2 Definition [8, Definition 4.3.1] Let T_1 be an oriented tangle diagram in $\mathbb{R}_{\geq 0}\times\mathbb{R}$, with n_+ positive crossings and n_- negative crossings, and let $0\leq r\leq n_++n_-$. Define $\widehat{D}(T_1)$ to be generated as an (intrinsically) graded abelian group, in homological degree $r-n_-$, by the generators $1\cdot x_i$ of $([T_1]^{\mathrm{Kh}})_{r-n_-}$, where $([T_1]^{\mathrm{Kh}})_{r-n_-}$ is the chain space of $[T_1]^{\mathrm{Kh}}$ in degree $r-n_-$. These generators have the same crossingless matching on the left and right sides of $\{0\}\times\mathbb{R}$, and all circles touching the boundary line have a $+$ sign. With this definition, $\widehat{D}(T_1)$ is an $\mathcal{I}_n$–submodule of $[T_1]^{\mathrm{Kh}}$.

3.1.3 Proposition [8, Proposition 4.3.2] *As H^n–modules,*

$$[T_1]^{\mathrm{Kh}}\cong H^n\otimes_{\mathcal{I}_n}\widehat{D}(T_1),$$

with $\widehat{D}(T_1)$ as defined in Definition 3.1.2.

Proof This follows from Definition 3.1.2. □

Let $\iota:=\iota_{\widehat{D}(T_1)}$ (the inclusion of $\widehat{D}(T_1)$ into $[T_1]^{\mathrm{Kh}}$) and let d be the differential on $[T_1]^{\mathrm{Kh}}$. Let μ denote the multiplication on H^n.

3.1.4 Definition [8, Definition 4.3.3] The type D differential δ on $\widehat{D}(T_1)$ is defined by restricting the differential d to the $\mathcal{I}_n$–submodule $\widehat{D}(T_1)$ of $[T_1]^{\mathrm{Kh}}$:

$$\delta:=\widehat{D}(T_1)\xrightarrow{\iota}[T_1]^{\mathrm{Kh}}\xrightarrow{d}[T_1]^{\mathrm{Kh}}\cong H^n\otimes_{\mathcal{I}_n}\widehat{D}(T_1).$$

It is an $\mathcal{I}_n$–linear map because ι and d are.

Lemma 3.1.5 and Proposition 3.1.6 below follow from the proof of Proposition 2.2.5, but we give short justifications to keep this section self-contained.

3.1.5 Lemma [8, Lemma 4.3.4] *Under the identification $[T_1]^{\mathrm{Kh}}\cong H^n\otimes_{\mathcal{I}_n}\widehat{D}(T_1)$ from Proposition 3.1.3, we have*

$$d=(\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta),$$

where $\mu_2\colon H^n\otimes H^n\to H^n$ is the algebra multiplication.

Proof Let $h\cdot\iota(x)$ denote a generator of $[T_1]^{\mathrm{Kh}}$. Then, by the Leibniz property for $[T_1]^{\mathrm{Kh}}$, we have $d(h\cdot\iota(x))=h\cdot d\iota(x)$, since H^n has no differential. But since $d\iota(x)=\delta(x)$, we can conclude that

$$d(h\cdot\iota(x))=(\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta)(h\cdot\iota(x)).$$ □

3.1.6 Proposition [8, Proposition 4.3.5] *$(\widehat{D}(T_1),\delta)$ satisfies the type D relations:*

$$(\mu_1\otimes|\mathrm{id}|)\circ\delta+(\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta)\circ\delta=0.$$

Proof There is no differential on H^n, so the μ_1 term is zero. For the other term, if x is a generator of $\widehat{D}(T_1)$, then

$$((\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \circ \delta)(x) = (d) \circ (d \circ \iota)(x) = 0,$$

since $d^2 = 0$ on $[T_1]^{\mathrm{Kh}}$. □

In Definition 3.1.2 and Definition 3.1.4, we constructed $\widehat{D}(T_1)$ as an $\mathcal{I}_n$–submodule of $[T]^{\mathrm{Kh}}$ following the proof of Proposition 2.2.5. Thus, the isomorphism class of $\widehat{D}(T_1)$ agrees with the isomorphism class of type D structures obtained from $[T]^{\mathrm{Kh}}$ by using Proposition 2.2.5.

3.2 Type A structures and pairing

Let T_2 be an oriented tangle diagram in $\mathbb{R}_{\leq 0} \times \mathbb{R}$. Since dg modules are special cases of type A structures, Proposition 2.2.4 tells us that the right dg module $[T_2]^{\mathrm{Kh}}$ is a valid example of a type A structure over H^n. We define $\widehat{A}(T_2)$ to be $[T_2]^{\mathrm{Kh}}$.

3.2.1 Theorem (Theorem 1.0.1) *Let T_1 and T_2 be oriented tangle diagrams in $\mathbb{R}_{\geq 0} \times \mathbb{R}$ and $\mathbb{R}_{\leq 0} \times \mathbb{R}$, respectively, with orderings chosen of the crossings of T_1 and T_2. Assume that T_1 and T_2 have consistent orientations, so that their horizontal concatenation is an oriented link diagram L in $\mathbb{R}^2$. Order the crossings of L such that those of T_1 come before those of T_2. Then*

$$\mathrm{CKh}(L) \cong \widehat{A}(T_2) \boxtimes \widehat{D}(T_1),$$

after multiplying the intrinsic gradings on $\widehat{A}(T_2) \boxtimes \widehat{D}(T_1)$ by -1.

Proof Since, up to a grading reversal, $\mathrm{CKh}(L) \cong [T_2]^{\mathrm{Kh}} \otimes_{H^n} [T_1]^{\mathrm{Kh}}$, which is the same as $\widehat{A}(T_2) \otimes_{H^n} (H^n \otimes_{\mathcal{I}_n} \widehat{D}(T_1))$, this proposition follows from Proposition 2.3.7, Remark 3.0.3, Lemma 3.1.5 and Khovanov's results from [4]. □

4 Quadratic and linear-quadratic algebras and duality

4.1 Quadratic and linear-quadratic algebras

We now consider a method of describing algebras using explicit generators and relations. It will be important for the following sections where we relate the bordered Khovanov theory discussed above with Roberts' constructions in [11; 12]. The definitions and basic properties of quadratic and linear-quadratic algebras here all follow Polishchuk and Positselski [10], with some minor modifications.

Let $\mathcal{B}$ be a unital associative algebra over a ring R, where $R \cong \mathbb{Z}e_1 \times \cdots \times \mathbb{Z}e_k$ as above. We will not assume $\mathcal{B}$ is graded; however, we will assume $\mathcal{B}$ comes equipped with an *augmentation*, ie an algebra homomorphism ϵ from $\mathcal{B}$ to the coefficient ring R. The algebras of interest to us have a grading of some form, and R is the degree-zero summand. Such an algebra has a natural augmentation given by projection onto this summand.

Suppose $b_1, \dots, b_m$ is a set of multiplicative generators of $\mathcal{B}$, each in the kernel $\mathcal{B}_+$ of ϵ. We may assume that for each b_i, there is a unique idempotent e_j such that $e_j b_i = b_i$ and $e'_j b_i = 0$ for $j' \neq j$. Indeed, if $e_j b_i = 0$ for all j, then $b_i = 0$ so b_i is irrelevant as a generator, and if $e_{j_a} b_i$ were nonzero for multiple indices a, we could remove b_i from the list of generators and add each of the nonzero elements $e_{j_a} b_i$ to the list. So we may assume $e_j b_i \neq 0$ for exactly one j, and then $b_i = 1 b_i = (\sum_{j'} e_{j'}) b_i = e_j b_i$. The idempotent e_j will be called the left idempotent of b_i and denoted $e_L(b_i)$.

Similarly, we may further assume that for each b_i, there exists a unique right idempotent $e_R(b_i)$ such that $b_i e_R(b_i) = b_i$ and $b_i e_j = 0$ for $e_j \neq e_R(b_i)$.

Let V be the free $\mathbb{Z}$–module spanned by $\{b_1, \dots, b_m\}$. The assumptions above equip V with left and right module structures over R. The statement that the b_i generate $\mathcal{B}$ multiplicatively means that $\mathcal{B}$ is isomorphic to $T(V)/J$, where

$$T(V) = \bigoplus_{n \geq 0} T^n(V) = R \oplus V \oplus (V \otimes_R V) \oplus (V \otimes_R V \otimes_R V) \oplus \cdots$$

and J is the kernel of the natural map $T(V) \to \mathcal{B}$ sending a string of generators to their product in $\mathcal{B}$. As above, we may assume that each generator of the ideal J has unique left and right idempotents.

4.1.1 Definition The augmented algebra $\mathcal{B}$, with its choice of generators, is a *quadratic algebra* if the ideal of relations $J \subset T(V)$ is generated multiplicatively by its intersection with $T^2(V) = V \otimes_R V$. In other words,

$$J = T(V) \cdot I \cdot T(V),$$

where $I := J \cap (V \otimes_R V)$. Note that J always contains $T(V) \cdot I \cdot T(V)$, so $\mathcal{B}$ is a quadratic algebra if $J \subset T(V) \cdot I \cdot T(V)$.

4.1.2 Remark If $\mathcal{B}$ is a quadratic algebra, then $\mathcal{B}$ obtains a grading by word-length in the generators $\{b_1, \dots, b_m\}$.

4.1.3 Remark Let $\mathcal{B}$ be a quadratic algebra. At various points it will be helpful to work with the generators and relations of $\mathcal{B}$ more explicitly. Following Polishchuk and Positselski [10, Chapter 4.1], choose an ordering of the multiplicative generators $\{b_1, \dots, b_m\}$; we may assume that $b_i < b_j$ when $i < j$. Use this order to put a lexicographic ordering on monomials in these generators: the leftmost factor in a product is defined to be the most significant part.

Let Q denote the set of quadratic monomials in the b_i. Then Q can be naturally partitioned into two subsets Q_1 and Q_2: Q_1 consists of the monomials which cannot be written in $\mathcal{B}$ as sums of lesser monomials with respect to the lexicographic order, and Q_2 consists of the monomials which can. If $b_i b_j \in Q_2$, then

$$b_i b_j = \sum_{(i',j')<(i,j)} c_{i,j;i',j'} b_{i'} b_{j'}$$

and the coefficients $c_{i,j;i',j'}$ are uniquely determined if we require that $c_{i,j;i',j'} = 0$ for $b_{i'} b_{j'}$ in Q_2. By [10, Lemma 1.1 of Chapter 4.1], a set of generators for the quadratic relation ideal $I = J \cap T^2(V)$ of $\mathcal{B}$ is obtained by taking

$$I_{i,j} := b_i b_j - \sum_{(i',j')<(i,j)} c_{i,j;i',j'} b_{i'} b_{j'}$$

for all (i, j) such that $b_i b_j$ is in Q_2.

4.1.4 Definition [10, Chapter 5.1] The augmented algebra $\mathcal{B}$, with its choice of generators, is a *linear-quadratic algebra* if the ideal of relations $J \subset T(V)$ is generated multiplicatively by its intersection with $T^1(V) \oplus T^2(V)$. In other words, writing $J_2 := J \cap (V \oplus (V \otimes_R V))$, $\mathcal{B}$ is linear-quadratic if

$$J = T(V) \cdot J_2 \cdot T(V),$$

or equivalently

$$J \subset T(V) \cdot J_2 \cdot T(V).$$

We will furthermore assume that $J \cap V = 0$ so that there are no linear redundancies among the chosen generators. (As always, we assume that $V \subset \mathcal{B}_+$.)

4.1.5 Remark If $\mathcal{B}$ is a linear-quadratic algebra, we get a word-length filtration on $\mathcal{B}$ rather than a grading. An element of $\mathcal{B}$ has filtration level at most k if it is a sum of products of word-length at most k in the generators b_i.

The following definitions and propositions will be used in Section 4.3 to define quadratic duality for linear-quadratic algebras. They can all be found in [10, Chapter 5.1].

4.1.6 Definition Let $\mathcal{B}$ be a linear-quadratic algebra, so that $\mathcal{B} \cong T(V)/J$ with $J \subset T(V) \cdot J_2 \cdot T(V)$. The quadratic algebra $\mathcal{B}^{(0)}$ is defined as

$$\mathcal{B}^{(0)} := T(V)/(T(V) \cdot I \cdot T(V)),$$

where $I \subset T^2(V)$ is defined as the image of $J_2 \subset (V \oplus T^2(V))$ under the projection $(V \oplus T^2(V)) \to T^2(V)$ onto the second summand.

Every generator r of I is the image of some generator $v \oplus r$ of J_2, where $v \in V$. Furthermore, if $v \oplus r$ and $v' \oplus r$ were both in J_2 with $v \neq v'$, then $(v - v') \oplus 0$ would be a nonzero element of $J_2 \cap V$, contradicting the assumption that $J_2 \cap V = 0$. Thus, the following definition makes sense.

4.1.7 Definition The function $\varphi\colon I \to V$ is defined by sending a generator $r \in I$ to the unique element $\varphi(r)$ of V such that $\varphi(r) \oplus r$ is in J_2.

4.1.8 Proposition *The map φ respects the left and right R–actions on I and V.*

Proof Suppose e is the left idempotent of r (which exists without loss of generality). Then $e(\varphi(r) \oplus r)$ is in J_2, and $e(\varphi(r) \oplus r) = e\varphi(r) \oplus er = e\varphi(r) \oplus r$, so by the uniqueness above, $\varphi(r) = e\varphi(r)$. If e' is any idempotent not equal to e, then $e'(\varphi(r) \oplus r)$ is still in J_2, but now this expression equals $e'\varphi(r) \oplus 0$. Since $J_2 \cap V = 0$, we must have $e'\varphi(r) = 0$ for $e' \neq e$. Thus, φ respects the left R–action on I and V. The right action is analogous. □

Let φ^{12} denote $\varphi \otimes \mathrm{id}_V\colon I \otimes_R V \to V \otimes_R V$ and let φ^{23} denote $\mathrm{id}_V \otimes \varphi\colon V \otimes_R I \to V \otimes_R V$.

4.1.9 Proposition [10, Chapter 5.1, Proposition 1.1] *The map*

$$\varphi^{12} - \varphi^{23}\colon (V \otimes_R I) \cap (I \otimes_R V) \to (V \otimes_R V)$$

has image contained in I, and

$$\varphi \circ (\varphi^{12} - \varphi^{23}) = 0.$$

Proof The definition of φ implies that the image of the map

$$\varphi \oplus \iota\colon I \to (V \oplus (V \otimes_R V))$$

is contained in J_2, where ι denotes the inclusion map of I into $V \otimes_R V$. Thus, the map

$$(\varphi \oplus \iota) \otimes \mathrm{id}_V\colon I \otimes_R V \to (V \oplus (V \otimes_R V)) \otimes_R V = (V \otimes_R V) \oplus (V \otimes_R V \otimes_R V)$$

has image contained in J. On the other hand, $(\varphi \oplus \iota) \otimes \mathrm{id}_V$ is equal to the map

$$\varphi^{12} \oplus (\iota \otimes \mathrm{id}_V)\colon I \otimes_R V \to (V \otimes_R V) \oplus (V \otimes_R V \otimes_R V).$$

Thus, $\varphi^{12} \oplus (\iota \otimes \mathrm{id}_V)$ has image contained in J. By the same reasoning, $\varphi^{23} \oplus (\mathrm{id}_V \otimes \iota)$ has image contained in J as well.

If x is an element of $(V \otimes_R I) \cap (I \otimes_R V) \subset V^{\otimes 3}$, then we can apply $\varphi^{12} \oplus (\iota \otimes \mathrm{id}_V)$ and $\varphi^{23} \oplus (\mathrm{id}_V \otimes \iota)$ to x, producing two elements $\varphi^{12}(x) \oplus x$ and $\varphi^{23}(x) \oplus x$ of J. Subtracting, the x terms cancel and $\varphi^{12}(x) - \varphi^{23}(x)$ is also in J.

Since $\varphi^{12}(x) - \varphi^{23}(x)$ is an element of both J and $V \otimes_R V$, it is also an element of $J_2 = J \cap (V \oplus (V \otimes_R V))$. The corresponding element of I under the projection from J_2 to I is the same element $\varphi^{12}(x) - \varphi^{23}(x)$.

Hence we can conclude that $\varphi^{12} - \varphi^{23}$ has image contained in I, so it makes sense to postcompose this map with φ. Furthermore, the image of $\varphi^{12} - \varphi^{23}$ is contained not just in I but in J_2 and so $\varphi(\varphi^{12} - \varphi^{23}) = 0$. □

4.2 Khovanov's arc algebra as a linear-quadratic algebra

In this section we present a combinatorial result, Lemma 4.2.4, whose proof was found by Pálvölgyi [9] and independently by Potechin in an email correspondence. Besides being important for the constructions in Section 5 and Section 6, it will yield an explicit generators-and-relations description of H^n in Corollary 4.2.7. This description is not necessary, strictly speaking, for Section 5 and Section 6, but it may be of interest independently.

Let V be the free $\mathbb{Z}$–module spanned by the generators h_γ and h_α of β_{mult} as defined in Section 3.1. The idempotent ring $R = \mathcal{I}_n$ of H^n has both left and right actions on V. We may write H^n as $T(V)/J$ for some ideal J of $T(V)$.

4.2.1 Theorem (Theorem 1.0.2) *With the generators $\{h_\gamma, h_\alpha\}$ and the augmentation coming from its grading, H^n is a linear-quadratic algebra.*

This theorem will be proved using Lemma 4.2.4. We begin with some background.

Recall from Remark 3.0.2 that the elementary idempotents of H^n are in bijection with the set NC_n of noncrossing partitions of n points. In fact, NC_n has a natural partial ordering: suppose p and q are elements of NC_n. Then $p \le q$ if p is a refinement of q. In other words, $p \le q$ if each of the subsets comprising p is contained in one of the subsets comprising q (recall that p and q are collections of subsets of a set of n points).

As a poset, NC_n is a lattice: any two noncrossing partitions have a unique least upper bound and a unique greatest lower bound, although we will not make use of this property.

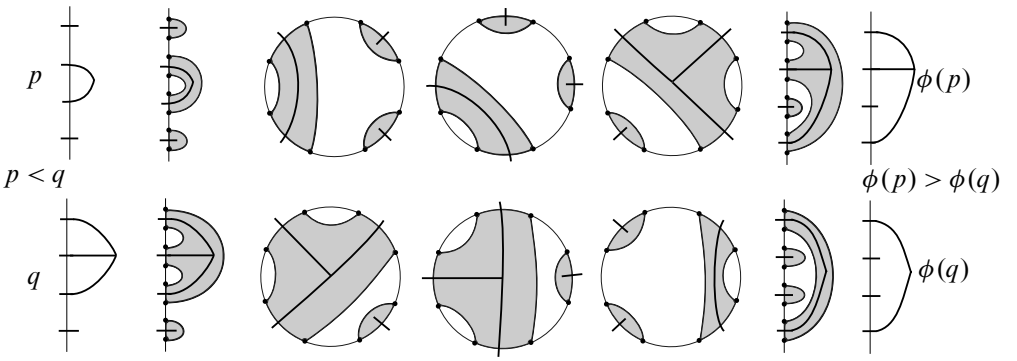

Figure 5: The order-reversing automorphism ϕ of NC_n (when read from left to right), or its inverse ϕ' (when read from right to left)

The dual of a partially ordered set is defined by reversing the order relations. It is a standard fact that the poset NC_n is self-dual.

4.2.2 Proposition NC_n *is order-isomorphic to the poset obtained by reversing all the order relations on* NC_n.

Proof We want to define a bijection $\phi\colon \mathrm{NC}_n \to \mathrm{NC}_n$ such that $p < q$ if and only if $\phi(p) > \phi(q)$. Let p be a noncrossing partition. Pick an embedding of acyclic graphs in the half-plane representing p; as in Remark 3.0.2, thicken these graphs to get planar surfaces embedded in the half-plane. Color the interiors of these surfaces black; then the half-plane is divided into black and white regions. Identify the half-plane with the disk and rotate the disk counterclockwise through an angle of π/n. Swap the colors of the regions and identify the disk back with the half plane. The skeleton of the new black region represents the noncrossing partition $\phi(p)$. This procedure is illustrated in Figure 5.

One may verify that ϕ, defined in this way, reverses the order relations. Finally, ϕ has an inverse whose definition is the same as for ϕ, except that the rotation is clockwise. □

Associated to the partial order on NC_n is a Hasse diagram G_n, which is a directed graph whose vertices are the elements of NC_n and which has an edge from p to q precisely when $p < q$ and there exists no vertex q' with $p < q' < q$.

We will view G_n as an undirected graph, ignoring the orientations on edges. For any two vertices p, q of G_n connected by an edge, there is a generator h_γ of H^n with left idempotent p and right idempotent q, and all the h_γ generators of H^n are of this form. Note that the existence of such h_γ does not depend on the ordering of p and q.

If $h_\gamma = (W(p)q, \text{all plus})$, where γ is a bridge on the crossingless matching p, then there is also a dual bridge $\gamma^\dagger$ on q; see Roberts [12, Definition 10]. Doing surgery on q along the bridge $\gamma^\dagger$ gives p, and the generator $h_{\gamma^\dagger} = (W(q)p, \text{ all plus})$ has left idempotent q and right idempotent p.

Monomials in the generators h_γ either correspond to paths in G_n, or are zero for idempotent reasons. We will be especially concerned with paths of minimum length.

4.2.3 Definition Let p and q be vertices of G_n. The graph $G_{p,q}$ has one vertex for each minimal-length path, or geodesic, α from p to q in G_n. If α and β are two vertices of $G_{p,q}$, they are connected by an edge when α and β differ in exactly one vertex of G_n (viewing paths in G_n as sequences of vertices of G_n).

The proof of the following lemma was found by Dömötör Pálvölgyi and posted as an answer to a question on MathOverflow [9]; independently, another proof was found by Aaron Potechin and shared with the author privately in an email correspondence.

4.2.4 Lemma (Potechin [9]) *Let G_n denote the Hasse diagram of NC_n, viewed as an undirected graph. Let p, q be vertices of G_n and define $G_{p,q}$ as in Definition 4.2.3. Then $G_{p,q}$ is a connected graph.*

Proof First, note that as partitions of a set of n points, either q contains a singleton part or the dual $\phi(q)$ of q contains a singleton part. Indeed, out of the n points, consider a minimal pair of points which are matched in q. (We consider a pair to be minimal if there is no pair of points, also matched in q, nested inside the first pair.) If there are any points nested inside the minimal pair, then these points must be singletons by minimality, so q contains a singleton. On the other hand, if there are no points nested inside, then one can see from Figure 5 that $\phi(q)$ contains a singleton.

In the latter case, we can use Proposition 4.2.2 to reduce to the former case: if $G_{\phi(p),\phi(q)}$ is connected, then so is $G_{p,q}$. Thus, we may assume without loss of generality that q contains a singleton part, say $\{m\}$ where m is one of the n points on the line.

We will induct on both n and the distance between p and q; if this distance is 2 or less, or if $n \leq 2$, there is nothing to prove.

Consider two minimal-length paths α and β from p to q in G_n. Since p is a partition of the n points on the line, the point m must be contained in one of the partitioning subsets which comprise p, say S. If S contains only m, then for each vertex along either α or β, the point m must be in a singleton set; otherwise α or β would not have minimal length (the length could be reduced by removing the steps that connect

and disconnect m from the other points on the line). Hence we may ignore m and view α and β as paths in G_{n-1}. By induction, α may be modified one vertex at a time to produce β, and we may reintroduce the singleton point m without issue.

On the other hand, suppose S contains additional points as well as m. Then we may modify both α and β, one vertex at a time, to get paths α' and β' from p to q such that the first step of both α' and β' separates m from the other points in S. To do this, find the first step along α or β after which m is an isolated point and commute this step to the beginning of α or β by changing the path one vertex at a time.

Now let p' denote the partition p with the point m isolated from S. Both α' and β' start by moving from p to p' and then along a minimal-length path (say α'' or β'') from p' to q. Since the distance from p' to q is one less than the distance from p to q, we may conclude by induction that α'' may be modified one vertex at a time to obtain β''. Thus, the same is true for α' and β' and hence for α and β as well. □

4.2.5 Remark If p is the minimal element of NC_n with respect to the partial ordering, and q is the maximal element, then elements of $G_{p,q}$ are maximal chains in NC_n and Lemma 4.2.4 is a well-known result; see Bessis [2, Proposition 1.6.1], as well as Adin and Roichman [1] for more properties of $G_{p,q}$ in this case. Lemma 4.2.4 can be viewed as a generalization of this result to a setting in which p and q may not necessarily be comparable in the partial ordering.

Proof of Theorem 4.2.1 We want to show that $J \subset T(V) \cdot J_2 \cdot T(V)$. We start by exhibiting elements in the intersection $J_2 = J \cap (V \oplus (V \otimes_R V))$; we will let $\bar{J}_2$ denote the ideal generated by these elements. Recall that the notion of a bridge for a crossingless matching was defined in Roberts [12, Definition 8], and the dual bridge $\gamma^\dagger$ for a bridge γ was defined in [12, Definition 10].

(1) Whenever γ and η are two bridges which can be drawn without intersection on the same crossingless matching, the element $h_\gamma h_{\eta'} - h_\eta h_{\gamma'}$ is in $\bar{J}_2$, for the natural choices of η' and γ'.

(2) Whenever γ is a bridge and α is an arc such that h_γ and h_α have the same left idempotent, and neither of the endpoints of γ lies on the arc α, the element $h_\gamma h_{\alpha'} - h_\alpha h_\gamma$ is in $\bar{J}_2$ for the natural choice of α'. If one of the endpoints of the bridge γ lies on the arc α, then there are two natural choices for α'; for each of these choices, $h_\gamma h_{\alpha'} - h_\alpha h_\gamma$ is an element of $\bar{J}_2$.

(3) Whenever α_1 and α_2 are distinct arcs in the same crossingless matching, so that h_α and $h_{\alpha'}$ have the same left idempotent, the element $h_{\alpha_1} h_{\alpha_2} - h_{\alpha_2} h_{\alpha_1}$ is in $\bar{J}_2$. Furthermore, for every arc α, the element h_α^2 is in $\bar{J}_2$.

(4) Finally, if γ is any bridge, such that h_γ has left idempotent e_L and right idempotent e_R, the element $h_\gamma h_{\gamma^\dagger} - h_{\alpha_1} - h_{\alpha_2}$ is in $\bar{J}_2$, where α_1 and α_2 are the arcs containing the endpoints of γ.

To show that $J \subset T(V)\cdot J_2 \cdot T(V)$, it suffices to show that $J \subset T(V)\cdot \bar{J}_2 \cdot T(V)$, since $\bar{J}_2 \subset J_2$. Actually, $\bar{J}_2 = J_2$ (see Remark 4.2.6 below), but we will not need this fact in the current proof.

Let r be an arbitrary element of J. We may assume without loss of generality that r has a unique left idempotent e_L and right idempotent e_R. Since J is an ideal of the tensor algebra $T(V)$, r may be written as a linear combination of monomials in the generators h_γ and h_α. Let

$$r = \sum_i n_i (h_{i,1} \cdots h_{i,l_i}),$$

where $n_i \in \mathbb{Z}$ and each $h_{i,j}$ is one of the generators h_γ or h_α.

Consider one of the monomial summands $m_i = h_{i,1} \cdots h_{i,l_i}$ of r. After adding elements of $T(V)\cdot \bar{J}_2 \cdot T(V)$ to this monomial, we may assume that all the h_γ generators among the $h_{i,j}$ come before (ie with lower j than) the h_α generators. The necessary relation elements come from item (2) above. Let m_i' denote the monomial obtained by modifying m_i in this way.

Write m_i' as $m_{\gamma,i} \cdot m_{\alpha,i}$, where $m_{\gamma,i}$ is a product of h_γ generators and $m_{\alpha,i}$ is a product of h_α generators. Each $m_{\gamma,i}$ has left idempotent e_L and right idempotent e_R. We may view e_L and e_R as vertices of G_n, the undirected Hasse diagram of NC_n, and to the monomial $m_{\gamma,i}$ we may associate a path $p(m_{\gamma,i})$ in G_n from e_L to e_R.

We claim that we may further modify m_i' by adding elements of $T(V)\cdot \bar{J}_2 \cdot T(V)$ until $p(m_{\gamma,i})$ is a minimal-length path between e_L and e_R. Indeed, suppose $p(m_{\gamma,i})$ is a path of nonminimal length. Write $m_{\gamma,i} = h_{\gamma_1} \cdots h_{\gamma_k}$. Then there exists a minimal index $2 \le j \le k$ such that $h_{\gamma_1} \cdots h_{\gamma_{j-1}}$ corresponds to a path of minimal length in G_n but $h_{\gamma_1} \cdots h_{\gamma_j}$ does not.

Let $e_R(h_{\gamma_j})$ denote the right idempotent of h_{γ_j}. By assumption, the distance between e_L and $e_R(h_{\gamma_{j-1}})$ in G_n is $j-1$, but the distance between e_L and $e_R(h_{\gamma_j})$ is $j-2$ rather than j. Indeed, this distance must be less than j. It cannot be less than $j-2$, or the distance between e_L and $e_R(h_{\gamma_{j-1}})$ would be less than $j-1$. The distance between e_L and $e_R(h_{\gamma_j})$ also cannot be $j-1$, because each edge in G_n connects two noncrossing partitions whose number of parts differs by one modulo two. Hence this distance must be $j-2$.

Thus, there exists some monomial $h_{\gamma_1'}\cdots h_{\gamma_{j-2}'}$ corresponding to a path in G_n from e_L to $e_R(h_{\gamma_j})$. Appending $h_{\gamma_j}^\dagger$ to this monomial, we get $h_{\gamma_1'}\cdots h_{\gamma_{j-2}'}\cdot h_{\gamma_j}^\dagger$, which corresponds to a path of length $j-1$ in G_n between e_L and $e_R(h_{\gamma_{j-1}})$. By assumption, the distance between e_L and $e_R(h_{\gamma_{j-1}})$ is $j-1$, so $h_{\gamma_1'}\cdots h_{\gamma_{j-2}'}\cdot h_{\gamma_j}^\dagger$ corresponds to a minimal-length path in G_n.

We now have two minimal-length paths in G_n between e_L and $e_R(h_{\gamma_{j-1}})$, namely $\alpha = p(h_{\gamma_1}\cdots h_{\gamma_{j-1}})$ and $\beta = p(h_{\gamma_1'}\cdots h_{\gamma_{j-2}'}\cdot h_{\gamma_j}^\dagger)$. By Lemma 4.2.4, we may modify α one vertex at a time to obtain β. Such modifications correspond, on the level of monomials, to adding relation terms obtained from item (1) above.

Thus, we may modify $m_{\gamma,i}$, which equals $h_{\gamma_1}\cdots h_{\gamma_j-1}\cdot h_{\gamma_j}\cdots h_{\gamma_k}$, by adding terms in $T(V)\cdot\bar{J}_2\cdot T(V)$ to obtain $h_{\gamma_1'}\cdots h_{\gamma_{j-2}'}\cdot h_{\gamma_j}^\dagger\cdot h_{\gamma_j}\cdots h_{\gamma_k}$. Inside this monomial is $h_{\gamma_j}^\dagger\cdot h_{\gamma_j}$, which may be replaced with a sum of h_α terms using the relation terms in item (4) above. As before, these h_α terms may be commuted to the right side of m_i' using item (2).

After this modification, we have strictly reduced the length of $m_{\gamma,i}$ in the factorization of m_i' as $m_{\gamma,i}\cdot m_{\alpha,i}$. If the new $m_{\gamma,i}$ still does not represent a minimal-length path $p(m_{\gamma,i})$ in G_n, we can repeat the same procedure and eventually it will terminate.

At this point, we have shown that we can modify our original $r=\sum_i n_i(m_i)$ by adding terms in $T(V)\cdot\bar{J}_2\cdot T(V)$, until each m_i is a monomial factorizable as $m_{\gamma,i}\cdot m_{\alpha,i}$ with $m_{\alpha,i}$ a monomial in the generators h_α and $m_{\gamma,i}$ a monomial in the generators h_γ representing a minimal-length path in G_n. The starting and ending points of all these paths are the same, namely the left and right idempotents of r. Thus, by Lemma 4.2.4 and the relations from item (1), we may do further modifications until all of the m_{γ_i} are the same monomial m_γ and we have

$$r = m_\gamma \sum_i n_i(m_{\alpha,i}) \quad \text{modulo } T(V)\cdot\bar{J}_2\cdot T(V).$$

Let r' denote the right side of the above equality; r' is an element of $T(V)$ and we want to show that $r'=0$ modulo $T(V)\cdot\bar{J}_2\cdot T(V)$.

The monomial m_γ represents an element of H^n of the form $(W(a)b, \text{all plus})$, where $a:=e_L$ is the left idempotent of r and $b:=e_R$ is the right idempotent. The signs are all plus because m_γ corresponds to a path of minimal length. Indeed, by Proposition 3.1.1, $(W(a)b, \text{all plus})$ can be written as a product of h_γ generators. If m_γ represented a sum of basis elements $(W(a)b,\sigma)$ in H^n with σ not all plus, then the length of m_γ would be at least two plus the length of the product expansion of $(W(a)b, \text{all plus})$. This claim follows because the length of a monomial m_γ in the generators h_γ is equal to the grading of the corresponding element of H^n (assuming this element is

nonzero). But then the product expansion of $(W(a)b, \text{all plus})$ would correspond to a shorter-length path from a to b than $p(m_\gamma)$, a contradiction. Finally, if m_γ were zero in H^n, then we could write $m_\gamma = \tilde{m}_\gamma \tilde{\tilde{m}}_\gamma$, where $\tilde{m}_\gamma$ is nonzero in H^n but becomes zero when multiplied on the right by the leftmost factor of $\tilde{\tilde{m}}_\gamma$. For this to be true, $\tilde{m}_\gamma$ must represent a sum of basis elements $(W(a)\tilde{b}, \sigma)$ with σ not all plus, where $\tilde{b}$ is the right idempotent of $\tilde{m}_\gamma$. By the above argument, we can obtain a shorter path than $p(\tilde{m}_\gamma)$ from a to $\tilde{b}$. Appending $p(\tilde{\tilde{m}}_\gamma)$, we get a shorter path from a to b than $p(m_\gamma)$, a contradiction.

Now we use the fact that $r \in J$, or in other words that $r = 0$ as an element of H^n. The same holds for r', since $T(V) \cdot \bar{J}_2 \cdot T(V)$ is a subset of J. The summand ${}_a(H^n)_b$ is a free abelian group with a basis element for every assignment of signs σ to the circles of $W(a)b$. Saying that $r' = 0$ in H^n means that the coefficient of r' on each of these basis elements is zero. In other words, for each assignment of signs σ to the circles of $W(a)b$, the sum of the terms $n_i m_\gamma m_{\alpha,i}$ of r' corresponding to σ is zero in H^n.

We will show that for a fixed σ, the terms $n_i m_\gamma m_{\alpha,i}$ such that $m_\gamma m_{\alpha,i}$ equals $(W(a)b, \sigma)$ in H^n actually sum to zero modulo the relation terms from items (2) and (3) above. There may also be some terms $m_\gamma m_{\alpha,i}$ which are already zero in H^n and thus which represent no basis element $(W(a)b, \sigma)$ of H^n. We will deal with these terms at the end.

Suppose $m_\gamma h_\alpha = m_\gamma h_{\alpha'} = (W(a)b, \sigma)$ in H^n, where m_γ corresponds to a minimal-length path; here α and α' are arcs in b which lie on the same circle in $W(a)b$, and σ assigns $-$ to this circle while assigning $+$ to all other circles of $W(a)b$. Then we may use relations from item (2) to write both $m_\gamma h_\alpha$ and $m_\gamma h_{\alpha'}$ as $h_{\tilde{\alpha}} m_\gamma$, where $\tilde{\alpha}$ is any arc in the left idempotent a of m_γ which, in $W(a)b$, lies in the same circle as α and α'. This generalization of the item (2) relations is true by induction on the length of γ.

Now, for a more general sum of terms $m_\gamma m_{\alpha,i}$ all representing $(W(a)b, \sigma)$ in H^n, we can use the above modifications to replace each of the monomials $m_{\alpha,i}$ with the same monomial m_α. We do this by picking, for example, $m_\alpha = m_{\alpha,i_1}$, and then for $i \neq i_1$, we move each factor of $m_{\alpha,i}$ to the left and back to the right so that it becomes identical to the factor appearing in m_{α,i_1}. After doing this for all i, we use relations from item (3) to replace each $m_{\alpha,i}$ with m_α.

For a fixed σ, let N_σ be the sum of the n_i such that $m_\gamma m_{\alpha,i}$ represents $(W(a)b, \sigma)$ in H^n. By the above paragraph, the sum of the terms $n_i m_\gamma m_{\alpha,i}$ of r' with $m_\gamma m_{\alpha,i}$ representing $(W(a)b, \sigma)$ in H^n is equivalent to $N_\sigma m_\gamma m_\alpha$ modulo $T(V) \cdot \bar{J}_2 \cdot T(V)$. We see that $N_\sigma m_\gamma m_\alpha = 0$ in H^n. But since $m_\gamma m_\alpha$ is the basis element of ${}_a(H^n)_b$ corresponding to σ, we can conclude that $N_\sigma = 0$. Thus, the sum of the terms $n_i m_\gamma m_{\alpha,i}$ of r' under consideration is equal to zero modulo $T(V) \cdot \bar{J}_2 \cdot T(V)$.

Finally, some of the terms $m_\gamma m_{\alpha,i}$ may not represent any $(W(a)b,\sigma)$ in H^n; this happens if and only if $m_\gamma m_{\alpha,i}$ is zero in H^n. In this case, by the above logic, we can use relations from items (2) and (3) to rearrange $m_\gamma m_{\alpha,i}$ until it has h_α^2 somewhere, for some generator h_α. Thus, these terms $m_\gamma m_{\alpha,i}$ are in $T(V)\cdot\bar{J}_2\cdot T(V)$ by item (3) above.

Starting with $r\in J$ above, we have successively modified r using linear-quadratic relations until we obtained zero. Hence

$$J\subset T(V)\cdot\bar{J}_2\cdot T(V)\subset T(V)\cdot J_2\cdot T(V)$$

and so H^n is a linear-quadratic algebra. □

4.2.6 Remark In fact, one can show by analyzing the grading possibilities case-by-case that the linear-quadratic relations listed above in (1)–(4) are a full set of generators for J_2. In other words, $\bar{J}_2=J_2$.

We get a description of H^n in terms of generators and relations.

4.2.7 Corollary *Let V denote the free $\mathbb{Z}$–module spanned by the degree-1 generators h_γ and the degree-2 generators h_α of H^n, with left and right actions of $R=\mathcal{I}_n\cong\mathbb{Z}^{C_n}$ on V given by multiplication in H^n. Then*

$$H^n\cong T(V)/(T(V)\cdot J_2\cdot T(V)),$$

where the tensor products in $T(V)$ are over R, and $J_2=\bar{J}_2$ is generated by the explicit relations given above in items (1)–(4) of the proof of Theorem 4.2.1.

4.2.8 Remark All of the generators of J_2 listed in items (1)–(4) of the proof of Theorem 4.2.1 are homogeneous with respect to the intrinsic grading on H^n. Thus, Corollary 4.2.7 also gives us a description of H^n as a graded algebra. This grading differs from the word-length filtration which H^n acquires as a linear-quadratic algebra by Remark 4.1.5, even on the basic multiplicative generators: h_α has intrinsic degree 2 and word-length 1, while h_γ has intrinsic degree and word-length both equal to 1.

4.2.9 Remark Braden [3] gives a generators-and-relations description of a ring $A_{n,n}$ which has H^n as an idempotent truncation; see also Stroppel [13]. It would be interesting to compare Braden's generators and relations with the h_γ and h_α generators and relations discussed here; we have not tried to do this in any detail.

4.3 Quadratic duality

Next we discuss quadratic duality for quadratic and linear-quadratic algebras. The dual of a quadratic algebra $\mathcal{B}$ is another quadratic algebra $\mathcal{B}^!$. The dual of a linear-quadratic

algebra $\mathcal{B}$ is a quadratic algebra $\mathcal{B}^!$ with a differential. Even if $\mathcal{B}$ is finitely generated over $\mathbb{Z}$, following Convention 2.1.1, the algebra $\mathcal{B}^!$ might be infinitely generated over $\mathbb{Z}$. Accordingly, Convention 2.1.1 will not be taken to hold for dual algebras $\mathcal{B}^!$ in general. However, $\mathcal{B}^!$ will still be generated multiplicatively by a finite set of elements.

We review the relevant definitions from Polishchuk and Positselski [10, Chapters 1 and 5]. We start with the case of quadratic algebras and then discuss the modification needed for linear-quadratic algebras. Let $\mathcal{B}$, R, b_i, V and J be defined as in Section 4.1.

Let V^* denote $\mathrm{Hom}_{\mathbb{Z}}(V,\mathbb{Z})$. Since V is a free $\mathbb{Z}$–module, V^* is free of the same rank as V. If b_i is a generator of V, let b_i^* denote the corresponding generator of V^*. We define left and right actions of R on V^* by declaring that b_i^* has the same left and right idempotents as b_i.

4.3.1 Definition Let $\mathcal{B}$ be a quadratic algebra and write $\mathcal{B} = T(V)/J$ as in Section 4.1, with $I := J \cap T^2(V)$. The *quadratic dual* $\mathcal{B}^!$ of $\mathcal{B}$ is defined to be

$$\mathcal{B}^! := T(V^*)/(T(V^*)\cdot I^{\perp}\cdot T(V^*)),$$

where $I^{\perp}$ is the submodule of $T^2(V^*) = V^* \otimes_R V^*$ annihilating I via the natural action of $V^* \otimes_R V^*$ on $V \otimes_R V$.

4.3.2 Remark Let Q_1, Q_2 and b_i be as defined as in Remark 4.1.3 above. We have a relation $I_{i,j}$ in I for every monomial b_ib_j in Q_2. If b_ib_j is in Q_1 rather than Q_2, consider instead the dual monomial $b_i^*b_j^*$ in $\mathcal{B}^!$. We can define a relation in $I^{\perp}$ by

$$I^!_{i,j} := b_i^*b_j^* + \sum_{(i',j')>(i,j)} c^!_{i,j;i',j'}b_{i'}^*b_{j'}^*,$$

where $c^!_{i,j;i',j'}$ is only nonzero if $b_{i'}b_{j'}$ is in Q_2, in which case $c^!_{i,j;i',j'}$ is defined to be the coefficient $c_{i',j';i,j}$ of the $(i''=i,\, j''=j)$ term in the relation

$$I_{i',j'} = b_{i'}b_{j'} - \sum_{(i'',j'')<(i',j')} c_{i',j';i'',j''}b_{i''}b_{j''}.$$

The ideal $I^{\perp}$ is spanned by the relations $I^!_{i,j}$; like Remark 4.1.3, this follows from [10, Lemma 1.1 of Section 4.1].

We now extend quadratic duality to linear-quadratic algebras.

4.3.3 Definition [10, Chapter 5.4] Let $\mathcal{B}$ be a linear-quadratic algebra; recall that Definition 4.1.6 associates a quadratic algebra $\mathcal{B}^{(0)}$ to $\mathcal{B}$. The quadratic dual $\mathcal{B}^!$

of $\mathcal{B}$ is defined, as an algebra, to be the usual quadratic dual of $\mathcal{B}^{(0)}$. Since $\mathcal{B}^!$ is a quadratic algebra, it has a grading by word-length. We will interpret this grading as the homological grading for a differential μ_1 on $\mathcal{B}^!$.

We will first define μ_1 on the basis elements of V^* and extend to $\mathcal{B}^!$ with the Leibniz rule. We will use the map $\varphi\colon I \to V$ from Definition 4.1.7. Dualizing φ, we get $\varphi^*\colon V^* \to I^*$, where $I^* := \mathrm{Hom}_{\mathbb{Z}}(I, \mathbb{Z})$.

We claim that I^* is isomorphic to the degree-2 summand of $\mathcal{B}^!$. To see this, write the degree-2 summand of $\mathcal{B}^!$ as $T^2(V^*)/I^\perp = \mathrm{Hom}_{\mathbb{Z}}(V \otimes V, \mathbb{Z})/I^\perp$. There is a natural map Ξ from $\mathrm{Hom}_{\mathbb{Z}}(V \otimes V, \mathbb{Z})$ to I^* given by precomposing with the inclusion from I into $V \otimes V$. The map Ξ is surjective because any functional from I to $\mathbb{Z}$ may be extended to a functional from $V \otimes V$ to $\mathbb{Z}$. Indeed, using the conventions of Remark 4.1.3, the $\mathbb{Z}$–basis $\{I_{i,j} \mid b_i b_j \in Q_2\}$ for I may be extended to a $\mathbb{Z}$–basis $\{I_{i,j} \mid b_i b_j \in Q_2\} \cup Q_1$ for $V \otimes V$.

The kernel of Ξ, by definition, consists of those functionals on $V \otimes V$ which annihilate I. Thus, the kernel is the same as $I^\perp$. We can conclude that Ξ induces an isomorphism from the degree-2 summand of $\mathcal{B}^!$ to I^*.

Now, for a degree-1 element of $\mathcal{B}^!$, ie an element $v^* \in V^*$ dual to a basis element v of V, define $\mu_1(v^*)$ to be $\varphi^*(v^*)$. This is an element of I^* and thus a degree-2 element of $\mathcal{B}^!$.

We may extend μ_1 to a map from $\mathcal{B}^!$ to $\mathcal{B}^!$, homogeneous of degree $+1$, using the Leibniz rule

$$\mu_1(xy) = (-1)^{\deg x}\mu_1(x)y + x\mu_1(y).$$

Note that this Leibniz rule differs from the one used in Polishchuk and Positselski [10], to stay consistent with our earlier sign conventions.

4.3.4 Remark Suppose $\mathcal{B}$ is a linear-quadratic algebra with an intrinsic grading, whose augmentation map is induced from the grading. Suppose further that all the multiplicative generators b_i of $\mathcal{B}$ and the explicit generators $I_{i,j}$ of I from Remark 4.1.3 are homogeneous with respect to the intrinsic grading, and the map $\varphi\colon I \to V$ preserves intrinsic degree. For example, H^n satisfies these properties: the generators h_γ have intrinsic degree 1 and the generators h_α have intrinsic degree 2. Each term of each relation in items (1) and (4) of the proof of Theorem 4.2.1 has intrinsic degree 2. Those in item (2) have degree 3 and those in item (3) have degree 4. The map φ is only nonzero on relations from item (4), and sends elements of degree 2 to elements of degree 2.

With these assumptions, V^* has a natural intrinsic grading, namely the negative of the grading on V (so that the pairing of V^* with V is grading-preserving). The

generators of $I^{\perp}$ are homogeneous with respect to this grading; this can be seen from Remark 4.3.2. Thus, $\mathcal{B}^!$ acquires an intrinsic grading. Since $\varphi\colon I \to V$ preserves intrinsic grading, so does $\varphi^*\colon V^* \to I^*$, and hence the differential μ_1 on $\mathcal{B}^!$ preserves intrinsic grading. The intrinsic grading on $\mathcal{B}^!$ is different from the homological grading, which μ_1 increases by one. In summary, $\mathcal{B}^!$ may be viewed as a differential bigraded algebra with a $(0,+1)$ differential.

4.4 The dual of Khovanov's arc algebra

Theorem 4.2.1, Definition 4.3.3 and Remark 4.3.4 together give us a differential bigraded algebra $(H^n)^!$, which we will call the dual of H^n.

4.4.1 Example When $n=1$, $H^n = H^1$ is the algebra $\mathbb{Z}[x]/x^2$ over the idempotent ring $\mathcal{I}_1 = \mathbb{Z}$. The generator x has intrinsic degree 2. Thus, the dual $(H^1)^!$ is $\mathbb{Z}[x^*]$, where x^* has bidegree $(-2,1)$. The differential on $(H^1)^!$ is zero and $(H^1)^!$ is not finitely generated over $\mathbb{Z}$.

In general, $(H^n)^!$ is never finitely generated over $\mathbb{Z}$, since arbitrary powers of any generator h_α^* will be nonzero in $(H^n)^!$.

4.5 Type *DD* bimodules

We may relate the duality discussed in Section 4.3 with the type DD bimodules encountered in bordered Heegaard Floer homology; see Lipshitz, Ozsváth and Thurston [6], especially Section 8. First, we give a definition of these bimodules over $\mathbb{Z}$; as in Section 2, we do not cover the most general possible case.

Let $\mathcal{B}$ and $\mathcal{B}'$ be differential bigraded algebras over an idempotent ring $R = \Pi_i(\mathbb{Z}e_i)$. The case $\mathcal{B}' = \mathcal{B}^!$ will be important, so we will not assume that $\mathcal{B}'$ is finitely generated over $\mathbb{Z}$.

Over $\mathbb{Z}/2\mathbb{Z}$, the following is equivalent to Definition 2.2.55 of Lipshitz, Ozsváth and Thurston [7].

4.5.1 Definition A *type DD bimodule* over $\mathcal{B}$ and $\mathcal{B}'$ is, first of all, a bigraded free $\mathbb{Z}$–module $\widehat{DD}$ with left and right actions of R, such that $\widehat{DD}$ admits a $\mathbb{Z}$–basis consisting of grading-homogeneous elements with unique left and right idempotents among the e_i. Furthermore, $\widehat{DD}$ must be equipped with an R–bilinear map

$$\delta\colon \widehat{DD} \to \mathcal{B} \otimes_R \widehat{DD} \otimes_R (\mathcal{B}')^{\mathrm{op}}$$

of degree $(0,+1)$, such that the type DD structure relations

$$(\mu_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|)\circ\delta+(\mathrm{id}\otimes|\mathrm{id}|\otimes\mu_1)\circ\delta+(\mu_2\otimes\mathrm{id}\otimes\mu_2)\circ\sigma\circ(\mathrm{id}\otimes\delta\otimes\mathrm{id})\circ\delta=0$$

are satisfied, where μ_1 and μ_2 denote the differential and multiplication on $\mathcal{B}$ or $\mathcal{B}'$ as appropriate, and

$$\sigma\colon \mathcal{B}\otimes\mathcal{B}\otimes\widehat{DD}\otimes(\mathcal{B}')^{\mathrm{op}}\otimes(\mathcal{B}')^{\mathrm{op}}\to\mathcal{B}\otimes\mathcal{B}\otimes\widehat{DD}\otimes(\mathcal{B}')^{\mathrm{op}}\otimes(\mathcal{B}')^{\mathrm{op}}$$

is a sign flip which multiplies $b_1\otimes b_2\otimes x\otimes(b_3')^{\mathrm{op}}\otimes(b_4')^{\mathrm{op}}$ by $(-1)^{(\deg_h b_2)(\deg_h b_4')}$.

4.5.2 Remark The odd-seeming sign conventions reflect the fact that, while we write $\widehat{DD}$ with $\mathcal{B}$ on the left and $(\mathcal{B}')^{\mathrm{op}}$ on the right to make the notation more manageable, we really want to think of both $\mathcal{B}$ and $\mathcal{B}'$ being on the left of $\widehat{DD}$ when fixing sign conventions.

Of particular interest here are type DD bimodules with $\widehat{DD}=R$ as an R–bimodule. We will refer to these as *rank-one* DD bimodules, following the notation of [6, Section 8]. For a rank-one DD bimodule, we have

$$\mathcal{B}\otimes_R\widehat{DD}\otimes_R(\mathcal{B}')^{\mathrm{op}}\cong\mathcal{B}\otimes_R(\mathcal{B}')^{\mathrm{op}},$$

so we may rewrite the type DD structure relations as

$$(\mu_1\otimes|\mathrm{id}|)\circ\delta+(\mathrm{id}\otimes\mu_1)\circ\delta+(\mu_2\otimes\mu_2)\circ\sigma\circ(\mathrm{id}\otimes\delta\otimes\mathrm{id})\circ\delta=0,$$

where σ is now a map from $\mathcal{B}\otimes\mathcal{B}\otimes(\mathcal{B}')^{\mathrm{op}}\otimes(\mathcal{B}')^{\mathrm{op}}$ to itself.

When $\mathcal{B}$ is a linear-quadratic algebra with an intrinsic grading as in Remark 4.3.4, we can construct an associated rank-one DD bimodule over $\mathcal{B}$ and $\mathcal{B}^!$. Setting $\widehat{DD}=R$, we define $\delta\colon R\to\mathcal{B}\otimes_R(\mathcal{B}^!)^{\mathrm{op}}$ by

$$\delta(e):=\sum_i b_i\otimes(b_i^*)^{\mathrm{op}},$$

where e is one of the elementary idempotents and the sum runs over those multiplicative generators b_i of R which have left idempotent e. (These idempotent conditions will be implicit in what follows.) Note that δ has degree $(0,+1)$; it preserves the intrinsic grading, since the grading on $\mathcal{B}^!$ was defined to be the negative of that on $\mathcal{B}$, and it increases the homological grading by 1, since b_i has homological degree 0 while b_i^* has homological degree 1.

4.5.3 Proposition *The map δ, as defined above, satisfies the type DD structure relations.*

Proof First, let $e \in R$ be one of the elementary idempotents. Applying the term $(\mu_2 \otimes \mu_2) \circ \sigma \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta$ to e, we get

$$(\mu_2 \otimes \mu_2) \circ \sigma \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta(e) = \sum_{i,j} b_i b_j \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}},$$

where the sum runs over all pairs of multiplicative generators b_i, b_j of $\mathcal{B}$ with compatible idempotents, such that the left idempotent of b_i is e. Note that $\sigma = \mathrm{id}$ here, because the generators b_j all have homological degree zero.

In the notation of Remark 4.1.3, we may split the above sum as

$$\sum_{b_i b_j \in Q_1} b_i b_j \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}} + \sum_{b_i b_j \in Q_2} b_i b_j \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}}. \tag{4-1}$$

If $b_i b_j$ is in Q_1, then in $\mathcal{B}^!$, we may write $b_i^* b_j^*$ as

$$- \sum_{\substack{b_{i'} b_{j'} \in Q_2 \\ (i',j') > (i,j)}} c_{i',j';i,j} b_{i'}^* b_{j'}^*.$$

Thus,

$$\sum_{b_i b_j \in Q_1} b_i b_j \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}} = - \sum_{b_i b_j \in Q_1} b_i b_j \otimes \sum_{\substack{b_{i'} b_{j'} \in Q_2 \\ (i',j') > (i,j)}} c_{i',j';i,j} (b_{j'}^*)^{\mathrm{op}} (b_{i'}^*)^{\mathrm{op}}. \tag{4-2}$$

On the other hand, if $b_i b_j$ is in Q_2, then we may write $b_i b_j$ as

$$b_i b_j = \Bigg(\sum_{\substack{b_{i'} b_{j'} \in Q_1 \\ (i',j') < (i,j)}} c_{i,j;i',j'} b_{i'} b_{j'} \Bigg) - \varphi \Bigg(b_i b_j - \sum_{\substack{b_{i'} b_{j'} \in Q_1 \\ (i',j') < (i,j)}} c_{i,j;i',j'} b_{i'} b_{j'} \Bigg).$$

The expression $\varphi\big(b_i b_j - \sum_{b_{i'} b_{j'} \in Q_1, (i',j') < (i,j)} c_{i,j;i',j'} b_{i'} b_{j'}\big)$, or $\varphi(I_{i,j})$, denotes some linear combination of the multiplicative generators b_k of $\mathcal{B}$. Define integers $C_{i,j;k}$ by

$$\varphi \Bigg(b_i b_j - \sum_{\substack{b_{i'} b_{j'} \in Q_1 \\ (i',j') < (i,j)}} c_{i,j;i',j'} b_{i'} b_{j'} \Bigg) = \sum_k C_{i,j;k} b_k. \tag{4-3}$$

We have

$$\begin{aligned} &\sum_{b_i b_j \in Q_2} b_i b_j \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}} \\ &= \sum_{b_i b_j \in Q_2} \sum_{\substack{b_{i'} b_{j'} \in Q_1 \\ (i',j') < (i,j)}} c_{i,j;i',j'} b_{i'} b_{j'} \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}} - \sum_{b_i b_j \in Q_2} \Bigg(\sum_k C_{i,j;k} b_k \Bigg) \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}}. \end{aligned}$$

On the right side of this equation, the first term cancels with the first term

$$\sum_{b_i b_j \in Q_1} b_i b_j \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}}$$

of expression (4-1), by (4-2). Thus, we see that

$$(\mu_2 \otimes \mu_2) \circ \sigma \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta(e) = - \sum_{b_i b_j \in Q_2} \left(\sum_k C_{i,j;k} b_k \right) \otimes (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}}.$$

Now we consider the terms $(\mu_1 \otimes |\mathrm{id}|) \circ \delta(e)$ and $(\mathrm{id} \otimes \mu_1) \circ \delta(e)$. The first of these is zero, because $\mathcal{B}$ has no differential. The second may be written as

$$(\mathrm{id} \otimes \mu_1) \circ \delta(e) = \sum_k b_k \otimes (\varphi^*(b_k^*))^{\mathrm{op}}.$$

To compute $\varphi^*(b_k^*)$ as an element of I^*, ie a homomorphism from I to $\mathbb{Z}$, use (4-3) above: this homomorphism sends the generator

$$I_{i,j} = b_i b_j - \sum_{\substack{b_{i'} b_{j'} \in Q_1 \\ (i',j') < (i,j)}} c_{i,j;i',j'} b_{i'} b_{j'}$$

of I to the coefficient $C_{i,j;k} \in \mathbb{Z}$.

We want to view $\varphi^*(b_k^*)$ as an element of $\mathcal{B}^!$ of homological degree 2. To do this, following Definition 4.3.3, we pick any extension of $\varphi^*(b_k^*)$ to a functional from $V \otimes_R V$ to $\mathbb{Z}$, or in other words an element of $V^* \otimes_R V^*$, and then consider this element modulo the ideal $I^\perp$. Since $\{I_{i,j} \mid b_i b_j \in Q_2\} \cup Q_1$ is a $\mathbb{Z}$–basis for $V \otimes_R V$, we may extend $\varphi^*(b_k^*)$ to $V \otimes_R V$ by defining it to be zero on any $b_{i'} b_{j'}$ in Q_1.

This extended $\varphi^*(b_k^*)$ sends $b_i b_j \in Q_2$ to $C_{i,j;k}$, since it sends $I_{i,j}$ to $C_{i,j;k}$ and sends every $b_{i'} b_{j'} \in Q_1$ to zero. Thus,

$$\varphi^*(b_k^*) = \sum_{b_i b_j \in Q_2} C_{i,j;k} b_i^* b_j^*.$$

We conclude that

$$(\mathrm{id} \otimes \mu_1) \circ \delta(e) = \sum_k b_k \otimes \sum_{b_i b_j \in Q_2} C_{i,j;k} (b_j^*)^{\mathrm{op}} (b_i^*)^{\mathrm{op}},$$

canceling the remaining term of $(\mu_2 \otimes \mu_2) \circ \sigma \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta(e)$. This computation verifies that the type DD structure relations for δ are satisfied. □

We can also reverse the roles of $\mathcal{B}$ and $\mathcal{B}^!$: define $\delta'\colon R \to \mathcal{B}^! \otimes_R (\mathcal{B})^{\mathrm{op}}$ by

$$\delta'(e) := \sum_i b_i^* \otimes (b_i)^{\mathrm{op}},$$

where again the sum is over all multiplicative generators b_i with left idempotent e.

4.5.4 Proposition *The map δ' satisfies the type DD structure relations.*

Proof The proof is similar enough to the proof of Proposition 4.5.3 that we will omit it to save space. □

4.5.5 Definition The rank-one type DD bimodules constructed in Proposition 4.5.3 and Proposition 4.5.4 will be denoted ${}^{\mathcal{B}}K^{(\mathcal{B}^!)^{\mathrm{op}}}$ and ${}^{\mathcal{B}^!}K^{\mathcal{B}^{\mathrm{op}}}$ respectively.

4.5.6 Remark In [6, Section 8], Lipshitz, Ozsváth and Thurston define a notion of Koszul duality in the language of DD bimodules: two algebras $\mathcal{B}$ and $\mathcal{B}'$ are Koszul dual if there exists a rank-one DD bimodule over $\mathcal{B}$ and $\mathcal{B}'$ which is quasi-invertible (and such that the algebra outputs of the DD operation δ lie in the kernel of the augmentation maps on $\mathcal{B}$ and $\mathcal{B}'$; this technical condition is satisfied for all the bimodules we consider). We will not define the notion of quasi-invertibility precisely here; see [6], although they use $\mathbb{Z}/2\mathbb{Z}$ coefficients.

By Proposition 4.5.3, we get a type DD bimodule over H^n and $(H^n)^!$; Proposition 4.5.4 gives us a type DD bimodule over $(H^n)^!$ and H^n. It would be interesting to know whether these bimodules are quasi-invertible; if they were, then $(H^n)^!$ could be regarded as the Koszul dual of H^n in this generalized sense.

However, bordered Floer homology has even stronger duality properties: Theorem 13 of [6] asserts that the bordered surface algebra $\mathcal{A}(\mathcal{Z}, i)$ is Koszul dual to both $\mathcal{A}(\mathcal{Z}, -i)$ and $\mathcal{A}(\mathcal{Z}_*, i)$, where $\mathcal{Z}$ is a pointed matched circle and $\mathcal{Z}_*$ is another pointed matched circle constructed from $\mathcal{Z}$. This situation contrasts with that of H^n, where the quadratic dual algebra is infinitely generated and thus much larger than H^n itself. Below, we will see that Roberts' construction is able to avoid this issue.

5 Khovanov's algebra and Roberts' algebra

In this section we begin to discuss Roberts' bordered theory for Khovanov homology from [11; 12]. Roberts' bordered theory uses a differential bigraded algebra which is denoted $\mathcal{B}\Gamma_n$. This algebra is generated by some right-pointing generators $\overrightarrow{e}$ and left-pointing generators $\overleftarrow{e}$, modulo some explicitly given relations. The differential on $\mathcal{B}\Gamma_n$ is zero on all the right-pointing generators $\overrightarrow{e}$.

We start by defining an algebra $\mathcal{B}_R(H^n)$ using the structure of H^n, with its additive basis $\beta = \{(W(a)b, \sigma)\}$ and set of multiplicative generators $\beta_{\text{mult}} = \{h_\gamma, h_\alpha\}$. In Proposition 5.1.9, we show that $\mathcal{B}_R(H^n)$ is isomorphic to the subalgebra $\mathcal{B}_R\Gamma_n$ of $\mathcal{B}\Gamma_n$ generated by the right-pointing elements.

The subalgebra $\mathcal{B}_R\Gamma_n$ is a linear-quadratic algebra. Its quadratic dual, as defined in Section 4.3, is closely related to the subalgebra $\mathcal{B}_L\Gamma_n$ of $\mathcal{B}\Gamma_n$ generated by the left-pointing elements. More precisely, in Definition 5.2.6 we define a mirroring operation on algebras over $\mathcal{I}_\beta$, and in Proposition 5.2.8, we show that $\mathcal{B}_L\Gamma_n$ is a quotient of $m(\mathcal{B}_R(H^n)^!)$, the mirroring of the quadratic dual of $\mathcal{B}_R(H^n) \cong \mathcal{B}_R\Gamma_n$, by a few explicitly given extra relations.

Finally, in Section 5.3, we take a suitably defined product of $m(\mathcal{B}_R(H^n))^!$ and $\mathcal{B}_R(H_n)$, obtaining an algebra whose quotient by the same extra relations as above is $\mathcal{B}\Gamma_n$.

5.1 Right side of Roberts' algebra

As in Section 3.1, let β denote the $\mathbb{Z}$–basis $\{(W(a)b, \sigma)\}$ of H^n. As at the beginning of Section 3, let $\mathcal{I}_n$ denote the idempotent ring of H^n. The space $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$ of left $\mathcal{I}_n$–module maps from H^n to itself is a free $\mathbb{Z}$–module. A $\mathbb{Z}$–basis for $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$ has generators $e(h_1, h_2)$ for each pair $h_1 \in \beta, h_2 \in \beta$ such that h_1 and h_2 have the same left idempotent. Here, $e(h_1, h_2)$ is the homomorphism that sends h_1 to h_2 and sends all other basis elements in β to zero.

Note that $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$ has the structure of a ring, with multiplication given by composition. We will define a grading on $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$ which differs from the usual one by a factor of $-\frac{1}{2}$.

5.1.1 Definition Let $e(h_1, h_2)$ be a generator of $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$. The *degree* of $e(h_1, h_2)$ is defined to be $\frac{1}{2}(\deg h_1 - \deg h_2)$.

5.1.2 Remark This choice of grading has the advantage that it agrees with Roberts' choice, but it can also be justified on its own grounds. The factor of -1 comes from the fact that Khovanov, in [4], replaces the usual q–grading by its negative, to make H^n positively rather than negatively graded. We will see below (in the proof of Proposition 6.4.1 in particular) why the factor of $\frac{1}{2}$ is reasonable. Note that while this grading is now a $\frac{1}{2}\mathbb{Z}$–grading rather than a $\mathbb{Z}$–grading, it will always function as an intrinsic grading rather than a homological grading. Thus, it will have no effect on signs and we are free to use a $\frac{1}{2}\mathbb{Z}$–grading if desired.

The elements $e(h, h) \in \mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$, for $h \in \beta$, generate a subring which is isomorphic to a direct product of copies of $\mathbb{Z}$. We will denote this subring by $\mathcal{I}_\beta$; note that $\mathcal{I}_\beta$ is isomorphic to the idempotent ring of $\mathcal{B}\Gamma_n$, and hence of $\mathcal{B}_R\Gamma_n$ as well.

5.1.3 Definition Let $\mathcal{B}_R(H^n)$ denote the smallest subring of $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$ containing $\mathcal{I}_\beta$ and containing every $e(h_1, h_2)$ such that h_2 occurs as a nonzero term in the β–expansion of $h_1 \cdot h$, for some h in the set of multiplicative generators β_{mult}. We may view $\mathcal{B}_R(H^n)$ as an algebra over $\mathcal{I}_\beta$.

The algebra $\mathcal{B}_R(H^n)$ inherits an intrinsic grading from the grading on $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$ defined in Definition 5.1.1. The degree-0 summand of $\mathcal{B}_R(H^n)$ is its idempotent ring $\mathcal{I}_\beta$. The multiplicative generators $e(h_1, h_2)$ such that h_2 occurs as a nonzero term in the basis expansion of $h_1 \cdot h_\gamma$, for some γ, have degree $-\frac{1}{2}$, since h_γ has intrinsic degree 1. Those such that h_2 occurs as a nonzero term in the expansion of some $h_1 \cdot h_\alpha$ have degree -1, since h_α has degree 2.

A natural set of multiplicative generators for $\mathcal{B}_R(H^n)$ as an algebra over $\mathcal{I}_\beta$ is given in its definition, namely the elements $e(h_1, h_2)$ such that h_2 occurs as a nonzero term in the β–expansion of $h_1 \cdot h$, for some h in the set of multiplicative generators β_{mult}. If $h = h_\gamma$, the corresponding element of $\mathcal{B}_R(H^n)$ will be denoted $b_{\gamma;h_1,h_2}$. If $h = h_\alpha$, the corresponding element of $\mathcal{B}_R(H^n)$ will be denoted $b_{C;h_1,h_2}$, where if $h_1 = (W(a)b, \sigma)$, then C is the circle in $W(a)b$ containing α. Note that for a fixed h_1, all arcs α' which lie on the same circle C as α in $W(a)b$ yield the same generator $b_{C;h_1,h_2}$ of $\mathcal{B}_R(H^n)$.

5.1.4 Remark We use notation with subscripts, such as $b_{\gamma;h_1,h_2}$ or $b_{C;h_1,h_2}$, to refer to elements of $\mathcal{B}_R(H^n)$. We also use b, without any subscripts, to refer to a crossingless matching. Below, if γ is a bridge on b, we will let $b(\gamma)$ denote the crossingless matching obtained by surgery on b along γ.

There are no linear relations among the generators $b_{\gamma;h_1,h_2}$ and $b_{C;h_1,h_2}$. The generators are homogeneous with intrinsic degree $-\frac{1}{2}$ or -1, so they are in the kernel of the augmentation map on $\mathcal{B}_R(H^n)$ (which is the projection onto the degree-0 summand). The left idempotent of each generator of the form $b_{\gamma;h_1,h_2}$ and $b_{C;h_1,h_2}$ is $e(h_1, h_1)$; the right idempotent is $e(h_2, h_2)$. For compactness of notation, we will identify each elementary idempotent $e(h, h) \in \mathcal{I}_\beta$ with the corresponding element $h \in \beta$. Thus, we say that the left idempotent of $b_{\gamma;h_1,h_2}$ and $b_{C;h_1,h_2}$ is h_1 and the right idempotent is h_2.

We will show that $\mathcal{B}_R(H^n)$, with the set of generators $b_{\gamma;h_1,h_2}$ and $b_{C;h_1,h_2}$, is a linear-quadratic algebra. The proof will closely follow that of Theorem 4.2.1.

Let V be the free $\mathbb{Z}$–module spanned by the generators of $\mathcal{B}_R(H^n)$; as discussed above, the idempotent ring $\mathcal{I}_\beta$ has left and right actions on V. We may write $\mathcal{B}_R(H^n) = T(V)/J$ for some ideal J of $T(V)$. Let $J_2 := J \cap (T^1(V) \oplus T^2(V))$. We identify a set of generators for J_2.

5.1.5 Proposition *The ideal J_2 of linear-quadratic relations of $\mathcal{B}_R(H^n)$ is generated by the following relations:*

(1) *Suppose γ and η are two bridges which can be drawn without intersection on the same crossingless matching b; let η' denote the bridge on $b(\gamma)$ corresponding to η, where $b(\gamma)$ is the crossingless matching resulting from surgery on γ. Define γ' similarly. For any choice of (a,σ), let $h_1 = (W(a)b,\sigma) \in \beta$. If the generators $b_{\gamma;h_1,h_2}$ and $b_{\eta';h_2,h_3}$ exist in $\mathcal{B}_R(H^n)$ for some h_2, h_3 in β, we get a relation*

$$b_{\gamma;h_1,h_2} b_{\eta';h_2,h_3} - b_{\eta;h_1,\tilde{h}_2} b_{\gamma';\tilde{h}_2,h_3} \in J_2,$$

where $\tilde{h}_2$ is any element of β such that $b_{\eta;h_1,\tilde{h}_2}$ and $b_{\gamma';\tilde{h}_2,h_3}$ exist in $\mathcal{B}_R(H^n)$.

(2) *Suppose γ is a bridge on a crossingless matching b. For any choice of (a,σ), let $h_1 = (W(a)b,\sigma) \in \beta$. Let C be any circle in $W(a)b$. Let C' be any circle in $W(a)(b(\gamma))$ which corresponds to C under surgery on γ; if the endpoints of γ do not both lie on C, then C' is unique, and otherwise there are two choices for C'. If the generators $b_{\gamma;h_1,h_2}$ and $b_{C';h_2,h_3}$ exist in $\mathcal{B}_R(H^n)$ for some h_2, h_3 in β, we get a relation*

$$b_{\gamma;h_1,h_2} b_{C';h_2,h_3} - b_{C;h_1,\tilde{h}_2} b_{\gamma;\tilde{h}_2,h_3} \in J_2,$$

where $\tilde{h}_2$ is any element of β such that $b_{C;h_1,\tilde{h}_2}$ and $b_{\gamma;\tilde{h}_2,h_3}$ exist; note that $\tilde{h}_2$ is uniquely determined by C and h_1.

(3) *For any choice of (a,b,σ), let $h_1 = (W(a)b,\sigma) \in \beta$. Let C_1 and C_2 be two circles in $W(a)b$. If the generators $b_{C_1;h_1,h_2}$ and b_{C_2,h_2,h_3} exist in $\mathcal{B}_R(H^n)$ for some h_2, h_3 in β, we get a relation*

$$b_{C_1;h_1,h_2} b_{C_2;h_2,h_3} - b_{C_2;h_1;\tilde{h}_2} b_{C_1;\tilde{h}_2,h_3} \in J_2,$$

where $\tilde{h}_2$ is any element of β such that $b_{C_2;h_1;\tilde{h}_2}$ and $b_{C_1;\tilde{h}_2,h_3}$ exist. As above, $\tilde{h}_2$ is uniquely determined by C_2 and h_1.

(4) *Finally, suppose γ is any bridge on a crossingless matching b. Recall from Section 4.2 or Roberts* [12, Definition 10] *that γ has a dual bridge $\gamma^\dagger$. For any choice of (a,σ), let $h_1 = (W(a)b,\sigma) \in \beta$. If the generators $b_{\gamma;h_1,h_2}$ and $b_{\gamma^\dagger;h_2,h_3}$ exist in $\mathcal{B}_R(H^n)$ for some h_2, h_3 in β, then h_3 differs from h_1 by switching the sign of one circle of $W(a)b$ from plus to minus. Let C denote this circle. We get a relation*

$$b_{\gamma;h_1,h_2} b_{\gamma^\dagger;h_2,h_3} - b_{C;h_1,h_3} \in J_2.$$

Proof Since $\mathcal{B}_R(H^n)$ is an intrinsically graded algebra, if we have a relation in J_2, then each of its grading-homogeneous parts must also be in J_2. Thus, we may analyze

J_2 one degree at a time. Since the generators of $\mathcal{B}_R(H^n)$ have intrinsic degree $-\frac{1}{2}$ or -1, and we are trying to identify the linear-quadratic relations among them, we may assume these relations have intrinsic degree -1, $-\frac{3}{2}$ or -2. The case of intrinsic degree $-\frac{1}{2}$ is excluded since any such relation would be a linear dependency among the generators of $\mathcal{B}_R(H^n)$.

The relations of intrinsic degree -1 may be sums of quadratic monomials in the degree $-\frac{1}{2}$ generators $b_{\gamma;h_1,h_2}$ of $\mathcal{B}_R(H^n)$ and linear monomials in the degree -1 generators $b_{C;h_1,h_2}$. Analyzing the possible cases, we get the relations of items (1) and (4) above.

The relations of intrinsic degree $-\frac{3}{2}$ are sums of quadratic monomials, each involving one degree $-\frac{1}{2}$ generator $b_{\gamma;h_1,h_2}$ and one degree -1 generator $b_{C;h_1,h_2}$. These relations are generated by the relations of item (2) above.

Finally, the relations of degree -2 are sums of quadratic monomials in the degree -2 generators $b_{C;h_1,h_2}$. They are generated by the relations of item (3) above. □

5.1.6 Remark As in Section 4.2, Proposition 5.1.5 is not actually needed to prove Proposition 5.1.7. We could instead introduce $\bar{J}_2$, generated by the relations in Proposition 5.1.5, and show that $J \subset T(V) \cdot \bar{J}_2 \cdot T(V)$.

5.1.7 Proposition *With J and J_2 defined as above, we have*

$$J = T(V) \cdot J_2 \cdot T(V).$$

Thus, $\mathcal{B}_R(H^n)$ is a linear-quadratic algebra.

Proof We want to show that $J \subset T(V) \cdot J_2 \cdot T(V)$. As in Theorem 4.2.1 above, it suffices to show that for a general element r of J, one may successively add to r elements of the ideal generated by the relation elements listed in items (1)–(4) of Proposition 5.1.5, until one obtains zero.

Let r be an arbitrary element of J. We may assume without loss of generality that r has a unique left idempotent and right idempotent. Since J is an ideal of the tensor algebra $T(V)$, r may be written as a linear combination of monomials in the generators $b_{\gamma;h_1,h_2}$ and $b_{C;h_1,h_2}$. Let

$$r = \sum_i n_i (b_{i,1} \cdots b_{i,l_i}),$$

where $n_i \in \mathbb{Z}$ and each $b_{i,j}$ is one of the generators $b_{\gamma;h_1,h_2}$ or $b_{C;h_1,h_2}$.

Consider one of the monomial summands $m_i = b_{i,1} \cdots b_{i,l_i}$ of r. After adding elements of $T(V) \cdot J_2 \cdot T(V)$ to this monomial, we may assume that all the $b_{\gamma;h_1,h_2}$ generators

among the $b_{i,j}$ come before (ie with lower j than) the $b_{C;h_1,h_2}$ generators. The necessary relations come from item (2) of Proposition 5.1.5. Let m_i' denote the monomial obtained by modifying m_i in this way.

Write m_i' as $m_{\gamma,i} \cdot m_{C,i}$, where $m_{\gamma,i}$ is a product of $b_{\gamma;h_1,h_2}$ generators and $m_{C,i}$ is a product of $b_{C;h_1,h_2}$ generators. Let $h \in \beta$ be the left idempotent of $m_{\gamma,i}$ and let $h_i' \in \beta$ be the right idempotent of $m_{\gamma,i}$. Note that h does not depend on i, since h is the left idempotent of our original relation term r.

Viewing h and h_i' as elements of H^n, let $e \in \mathcal{I}_n$ denote the right idempotent of h. Let $e' \in \mathcal{I}_n$ denote the right idempotent of h_i', which does not depend on i since the monomial $m_{C,i}$ is a product of b_C generators. As in Theorem 4.2.1, e and e' are vertices of G_n, the undirected Hasse diagram of NC_n. To the monomial $m_{\gamma,i}$, we can associate a path $p(m_{\gamma,i})$ from e to e' in G_n.

We claim that we may further modify m_i' such that $p(m_{\gamma,i})$ is a minimal-length path between e and e' as vertices of G_n. Indeed, suppose $m_{\gamma,i}$ corresponds to a path of nonminimal length between e and e'. Write $m_{\gamma,i} = b_{\gamma_1;h_1,h_2} \cdots b_{\gamma_k;h_k,h_{k+1}}$. Then there exists a minimal index $2 \le j \le k$ such that $b_{\gamma_1;h_1,h_2} \cdots b_{\gamma_{j-1};h_{j-1},h_j}$ corresponds to a path ϕ of minimal length in G_n but $b_{\gamma_1;h_1,h_2} \cdots b_{\gamma_j;h_j,h_{j+1}}$ does not.

Let $e_R(h_j) \in \mathcal{I}_n$ denote the right idempotent of h_j. Then $e_R(h_j)$ is a vertex of G_n and the distance in G_n between e and $e_R(h_j)$ is $j-1$. However, the distance between e and $e_R(h_{j+1})$ is $j-2$ rather than j; the argument is the same as in the proof of Theorem 4.2.1. Thus, there exists a path $\tilde{\psi}$ in G_n, of length $j-2$, from e to $e_R(h_{j+1})$. Appending $e_R(h_j)$ to the end of the path $\tilde{\psi}$, we get a path ψ in G_n, of length $j-1$, between e and $e_R(h_j)$. By assumption, ψ is a minimal-length path.

We now have two minimal-length paths ϕ and ψ between e and $e_R(h_j)$. The path ϕ corresponds to the monomial $b_{\gamma_1;h_1,h_2} \cdots b_{\gamma_{j-1};h_{j-1},h_j}$. The path ψ corresponds to a monomial $b_{\gamma_1';h_1,h_2'} \cdots b_{\gamma_{j-2}';h_{j-2}',h_{j-1}'} \cdot b_{\gamma_j^\dagger;h_{j-1}',h_j}$, and we have $e_R(h_{j-1}') = e_R(h_{j+1})$.

By Lemma 4.2.4, we may modify ϕ one vertex at a time to obtain ψ. Such modifications can be mirrored on the level of monomials by adding relation terms obtained from item (1) of Proposition 5.1.5. Thus, we may modify $m_{\gamma,i}$, which equals

$$b_{\gamma_1;h_1,h_2} \cdots b_{\gamma_{j-1};h_{j-1},h_j} \cdot b_{\gamma_j;h_j,h_{j+1}} \cdots b_{\gamma_k;h_k,h_{k+1}},$$

by adding terms in $T(V) \cdot J_2 \cdot T(V)$ to obtain

$$b_{\gamma_1';h_1,h_2'} \cdots b_{\gamma_{j-2}';h_{j-2}',h_{j-1}'} \cdot b_{\gamma_j^\dagger;h_{j-1}',h_j} \cdot b_{\gamma_j;h_j,h_{j+1}} \cdots b_{\gamma_k;h_k,h_{k+1}}.$$

Inside this monomial is $b_{\gamma_j^\dagger;h_{j-1}',h_j} \cdot b_{\gamma_j;h_j,h_{j+1}}$, which may be replaced with a $b_{C;h_{j-1}',h_{j+1}}$ term using the relation terms in item (4) of Proposition 5.1.5. As before, this b_C term may be commuted to the right side of m_i'.

After this modification, we have strictly reduced the length of $m_{\gamma,i}$ in the factorization of m_i' as $m_{\gamma,i} \cdot m_{C,i}$. If the new $m_{\gamma,i}$ still does not represent a minimal-length path in G_n, we can repeat the same procedure and eventually it will terminate.

At this point, we have shown that we can modify our original $r = \sum_i n_i(m_i)$ by adding terms in $T(V) \cdot J_2 \cdot T(V)$, until each m_i is a monomial factorizable as $m_{\gamma,i} \cdot m_{C,i}$ with $m_{\gamma,i}$ representing a minimal-length path in G_n. The starting and ending vertices of all these paths are the same. Thus, by Lemma 4.2.4 and the relations from item (1) of Proposition 5.1.5, we may do further modifications until all of the $m_{\gamma,i}$ are the same monomial m_γ and we have

$$r = m_\gamma \sum_i n_i(m_{C,i}) \quad \text{modulo } T(V) \cdot J_2 \cdot T(V).$$

Since r was assumed to have unique left and right idempotents in $\mathcal{I}_\beta$, the set of circles C involved in each term $m_{C,i}$ of the above expression must be the same. Thus, using relations from item (3) of Proposition 5.1.5, we may rewrite each $m_{C,i}$ as the same monomial m_C. Then

$$r = N \cdot m_\gamma m_C \quad \text{modulo } T(V) \cdot J_2 \cdot T(V),$$

where $N = \sum n_i$.

Finally, we use the fact that $r \in J$, or in other words that $r = 0$ as an element of $\mathcal{B}_R(H^n)$. This condition implies that $N \cdot m_\gamma m_C$ must also be in J, since it differs from r by an element of $T(V) \cdot J_2 \cdot T(V)$ which is contained in J.

Note that $\mathcal{B}_R(H^n)$ is a subring of $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$; the element $m_\gamma m_C$ may be identified with the left R–linear map from H^n to H^n which sends e to e' and sends all other elements of β to zero, where e and e' here are the left and right idempotents of $m_\gamma m_C$. If $N \cdot m_\gamma m_C$ is zero in $\mathcal{B}_R(H^n)$, then it is zero in $\mathrm{Hom}_{\mathcal{I}_n}(H^n, H^n)$, implying that N must be zero.

In other words, starting with $r \in J$ above, we have shown that $r = 0$ modulo $T(V) \cdot J_2 \cdot T(V)$. Hence $J \subset T(V) \cdot J_2 \cdot T(V)$, so $B_R(H^n)$ is a linear-quadratic algebra. □

Now we can see that $\mathcal{B}_R(H^n)$ is isomorphic to $\mathcal{B}_R\Gamma_n$. First, we define the latter algebra more precisely.

5.1.8 Definition Let $\mathcal{B}\Gamma_n$ be Roberts' algebra from [11; 12]. Let $\mathcal{B}_R\Gamma_n$ be the subalgebra of $\mathcal{B}\Gamma_n$ spanned over $\mathcal{I}_\beta$ by those generators $\overrightarrow{e}$ with right pointing arrows ($\mathcal{B}\Gamma_n$ also has some generators $\overleftarrow{e}$ with left pointing arrows). The subalgebra $\mathcal{B}_R\Gamma_n$

has no differential. It inherits a bigrading from $\mathcal{B}\Gamma_n$ (see [12, Definition 19]); the homological grading is identically zero on $\mathcal{B}_R\Gamma_n$.

5.1.9 Proposition *$\mathcal{B}_R(H^n) \cong \mathcal{B}_R\Gamma_n$ as bigraded algebras over $\mathcal{I}_\beta$.*

Proof An examination of the subset of Roberts' algebra relations in [12] which involve only right-pointing generators shows that they correspond with the relations listed in Proposition 5.1.5 under the (bigrading-preserving) identification of the generators b of $\mathcal{B}_R(H^n)$ with the generators $\overrightarrow{e}$ of $\mathcal{B}_R\Gamma_n$. Thus, this proposition follows from Proposition 5.1.7. □

5.2 Left side of Roberts' algebra

5.2.1 Definition Let $\mathcal{B}_L\Gamma_n$ be the subalgebra of $\mathcal{B}\Gamma_n$ spanned over $\mathcal{I}_\beta$ by those generators $\overleftarrow{e}$ with left pointing arrows. The bigrading and differential on $\mathcal{B}\Gamma_n$ give us a bigrading and differential on $\mathcal{B}_L\Gamma_n$.

We will see that $\mathcal{B}_L\Gamma_n$ may be identified, after a mirroring operation defined in Definition 5.2.6, with a quotient of the quadratic dual $(\mathcal{B}_R(H^n))^!$ of $\mathcal{B}_R(H^n)$ by a few explicitly given extra relations.

First, we analyze the dual algebra $(\mathcal{B}_R(H^n))^!$. As an algebra, it is the quadratic dual of $(\mathcal{B}_R(H^n))^{(0)}$. We may write $\mathcal{B}_R(H^n)$ as $T(V)/J$, where if

$$J_2 := J \cap (T^1(V) \oplus T^2(V))$$

then we have

$$J = T(V) \cdot J_2 \cdot T(V).$$

Let I denote the image of J_2 under the projection map $T^1(V) \oplus T^2(V) \to T^2(V)$ onto the second summand. Then

$$(\mathcal{B}_R(H^n))^{(0)} \cong T(V)/I,$$
$$(\mathcal{B}_R(H^n))^! \cong T(V^*)/I^\perp.$$

The ideal J_2 is generated explicitly by the relations listed in Proposition 5.1.5. We may discard the linear parts of these relations, and keep the quadratic parts, to get a set of generators for I. These generators have a simple form: if r is a generating relation in I, then r is either a single quadratic monomial or a difference of two quadratic monomials.

Define a graph G whose vertices are all quadratic monomials appearing with nonzero coefficient in some relation $r \in I$. Two monomials v and $\tilde{v}$ are connected by an edge in G if $v - \tilde{v}$ is in I.

5.2.2 Proposition *The graph G is a disjoint union of isolated points, line segments (two points connected by an edge and disconnected from the rest of G), triangles (three points, all connected, and disconnected from the rest of G), and tetrahedra (four points, all connected, and disconnected from the rest of G).*

Proof Define a graph G' with the same vertices as G; two monomials v and $\tilde{v}$ are connected by an edge in G' if $v-\tilde{v}$ is the quadratic part of one of the explicit relations (1)–(4) listed in Proposition 5.1.5. We will determine the structure of G' by looking at the quadratic parts of the relations (1)–(4); G may be obtained from G' by replacing each connected component of G' with a complete graph on the same number of vertices.

First, the isolated points in G' are quadratic monomials of the form

$$b_{\gamma;h_1,h_2}b_{\gamma^\dagger,h_2,h_3};$$

these are the quadratic parts of relations from item (4) of Proposition 5.1.5.

Next we look at relations from item (1) of Proposition 5.1.5, which have no linear parts and are already quadratic (the same applies to items (2) and (3); only the relations from item (4) have linear parts). Some of the line segments in G' come from relation terms

$$b_{\gamma;h_1,h_2}b_{\eta';h_2,h_3}-b_{\eta;h_1,\tilde{h}_2}b_{\gamma';\tilde{h}_2,h_3},$$

where γ and η are two bridges which can be drawn on the same crossingless matching without intersection, such that $\eta\in B_d(L,\gamma)$ in the notation of [12, Proposition 11]. Roberts' L corresponds to our $W(a)b$.

Other line segments in G' come from the same relations when $\eta\in B_o(L,\gamma)$, in every case except when γ splits a plus-labeled circle and η' joins the two newly formed circles into a new minus-labeled circle. For notations like $\mathcal{B}_o(L,\gamma)$ and $\mathcal{B}_d(L,\gamma)$, see Roberts [12, Proposition 11].

Line segments in G' also come from relations $b_{\gamma;h_1,h_2}b_{C;h_2,h_3}-b_{C;h_1,\tilde{h}_2}b_{\gamma;\tilde{h}_2,h_3}$ of item (2) of Proposition 5.1.5 when the circle C is disjoint from the support of γ and from relations $b_{C_1;h_1,h_2}b_{C_2;h_2,h_3}-b_{C_2;h_1;\tilde{h}_2}b_{C_1;\tilde{h}_2,h_3}$ of item (3) of Proposition 5.1.5, where C_1 and C_2 are two distinct circles labeled $+$ in h_1.

The remaining relations from item (2) of Proposition 5.1.5 give configurations of three vertices in G' connected by two edges. We get triangles in G which connect triples

$$\{b_{\gamma;h_1,h_2}b_{C;h_2,h_3},b_{C_1;h_1,\tilde{h}_2}b_{\gamma;\tilde{h}_2,h_3},b_{C_2;h_1,\tilde{\tilde{h}}_2}b_{\gamma;\tilde{\tilde{h}}_2,h_3}\},$$
$$\{b_{C;h_1,h_2}b_{\gamma;h_2,h_3},b_{\gamma;h_1,\tilde{h}_2}b_{C_1;\tilde{h}_2,h_3},b_{\gamma;h_1,\tilde{\tilde{h}}_2}b_{C_2;\tilde{\tilde{h}}_2,h_3}\}$$

when the circle C is not disjoint from the support of γ. The rest of the triangles in G connect triples

$$\{b_{\gamma_1;h_1,h_2}b_{*;h_2,h_3}, b_{\gamma_2;h_1,\tilde{h}_2}b_{*;\tilde{h}_2,h_3}, b_{\gamma_3;h_1,\tilde{\tilde{h}}_2}b_{*;\tilde{\tilde{h}}_2,h_3}\}$$

when $\gamma_i \in B_s(L,\gamma_j)$ for $1 \le i, j \le 3$, using relations from item (1) of Proposition 5.1.5.

We have accounted for the quadratic parts of all relations from items (2)–(4) of Proposition 5.1.5, as well as most of the relations from item (1). The remaining relations from item (1) give rise to squares of four vertices in G' and thus to tetrahedra in G. These four-vertex components exist whenever we have two bridges γ and η, with $\eta \in B_o(L,\gamma)$, such that γ splits a plus-labeled circle and η' joins the newly formed circles into a minus-labeled circle. In such cases, we have four quadratic monomials which are all equal modulo the relation terms in I (and thus are connected in G). These can be written as $b_{\gamma;h_1,h_2}b_{\eta';h_2,h_3}$, $b_{\gamma;h_1,\tilde{h}_2}b_{\eta';\tilde{h}_2,h_3}$, $b_{\eta;h_1,\tilde{\tilde{h}}_2}b_{\gamma';\tilde{\tilde{h}}_2,h_3}$ and $b_{\eta;h_1,\tilde{\tilde{h}}_2}b_{\gamma';\tilde{\tilde{h}}_2,h_3}$. □

Order the set of generators $b_{\gamma;h_1,h_2}$ and $b_{C;h_1,h_2}$ of $\mathcal{B}_R(H^n)$ such that the $b_{\gamma;h_1,h_2}$ come before the $b_{C;h_1,h_2}$ in the ordering. Using G, the generators of I may be summarized as follows: for every connected component of G, there exists a minimal vertex v. For all other vertices $\tilde{v}$ in the same component of v, there exists a relation $\tilde{v}-v$ in I, and if v is a singleton, then v is also a relation in I. These relations are a set of generators for I as in Remark 4.1.3.

We may use the reasoning of Remark 4.3.2 to identify a set of generators for $I^{\perp}$. For any quadratic monomial in the generators b which does not appear as a vertex of G, the corresponding monomial in the generators b^* is an element of I^*. Isolated points of G do not give generators of I^*. For every line segment in G between vertices v and $\tilde{v}$, let v^* and $(\tilde{v})^*$ denote the corresponding monomials in the generators b^*. Then $v^* + (\tilde{v})^*$ is an element of I^*. For every triangle in G with a minimal vertex v and two nonminimal vertices $\tilde{v}$ and $\tilde{\tilde{v}}$, let v^*, $(\tilde{v})^*$ and $(\tilde{\tilde{v}})^*$ denote the corresponding monomials in the generators b^*. Then

$$v^* + (\tilde{v})^* + (\tilde{\tilde{v}})^*$$

is an element of $I^{\perp}$. Finally, for every tetrahedron in G with a minimal vertex v and three nonminimal vertices $\tilde{v}$, $\tilde{\tilde{v}}$ and $\tilde{\tilde{\tilde{v}}}$, let v^*, $(\tilde{v})^*$, $(\tilde{\tilde{v}})^*$ and $(\tilde{\tilde{\tilde{v}}})^*$ denote the corresponding monomials in the generators b^*. Then

$$v^* + (\tilde{v})^* + (\tilde{\tilde{v}})^* + (\tilde{\tilde{\tilde{v}}})^*$$

is an element of $I^{\perp}$. The above-listed elements generate $I^{\perp}$.

We may also compute the action of the map φ on the generators of I, using the relations from item (4) of Proposition 5.1.5. For every generator of I of the form $b_{\gamma;h_1,h_2}b_{\gamma^\dagger;h_2,h_3} - b_{C;h_1,h_3}$, we have

$$\varphi(b_{\gamma;h_1,h_2}b_{\gamma^\dagger;h_2,h_3}) = -b_{C;h_1,h_3}.$$

The map φ sends all other generators of I to zero. Thus, dualizing φ, we have

$$\varphi^*(b^*_{C;h_1,h_3}) = -\sum_i b^*_{\gamma_i;h_1,h_{2,i}} b^*_{\gamma_i^\dagger;h_{2,i},h_3},$$

where the sum runs over all bridges γ_i on the right crossingless matching of h_1 which have an endpoint on C, as well as all compatible $h_{2,i}$.

Finally, $(\mathcal{B}_R(H^n))^!$ is bigraded; the generators $b^*_{\gamma;h_1,h_2}$ have degree $\left(\frac{1}{2},1\right)$ since $b_{\gamma;h_1,h_2}$ has degree $\left(-\frac{1}{2},0\right)$, and the generators $b^*_{C;h_1,h_2}$ have degree $(1,1)$ since $b_{C;h_1,h_2}$ has degree $(-1,0)$. Here, the first index denotes the intrinsic degree, and the second index denotes the homological degree (this is the reverse of Roberts' convention). The generators of $\mathcal{B}_R(H^n)$ are all placed in homological degree 0.

5.2.3 Remark While the quadratic dual of an algebra which is finitely generated over $\mathbb{Z}$ (like $\mathcal{B}_R\Gamma_n$) may in general be infinitely generated over $\mathbb{Z}$, the algebra $(\mathcal{B}_R(H^n))^!$ is finitely generated over $\mathbb{Z}$. In fact, the relations on the algebra are irrelevant for this property: $T(V^*)$ is already finitely generated over $\mathbb{Z}$, since the structure of the idempotents only allows monomials of a certain length in the generators of V^* to be nonzero. The same is true for $T(V)$.

We can now relate $(\mathcal{B}_R(H^n))^!$ with $\mathcal{B}_L\Gamma_n$. To do this, we need to define a mirroring operation for modules and bimodules over the idempotent ring $\mathcal{I}_\beta$ of $(\mathcal{B}_R(H^n))^!$ and $\mathcal{B}_L\Gamma_n$.

5.2.4 Definition Let X be any module or bimodule over the idempotent ring $\mathcal{I}_\beta$. The *mirror* of X, denoted $m(X)$, is the module or bimodule whose actions of $\mathcal{I}_\beta$ are the actions on X, precomposed with the map from $\mathcal{I}_\beta$ to $\mathcal{I}_\beta$ which mirrors each elementary idempotent across the line $\{0\}\times\mathbb{R}$. Note that $m(m(X)) = X$.

5.2.5 Example Suppose X is a left module over $\mathcal{I}_\beta$. Let $x \in X$ and let $m(x)$ denote the corresponding element of $m(X)$. Let $h = (W(a)b, \sigma) \in \mathcal{I}_\beta$; then

$$h \cdot m(x) := m(m(h)\cdot x),$$

where $m(h)$ is $(W(b)a, m(\sigma))$ and $m(\sigma)$ is the same labeling of circles as σ, mirrored across $\{0\}\times\mathbb{R}$. See Figure 6 for an illustration.

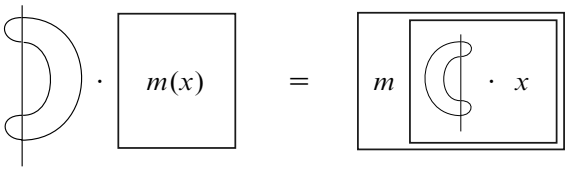

Figure 6: The mirror of a left module over $\mathcal{I}_\beta$

We will have definitions related to Definition 5.2.4 for $\mathcal{I}_\beta$–modules and bimodules with more structure. Here, we are concerned with algebras.

5.2.6 Definition Let $\mathcal{B}$ be a differential bigraded algebra over the idempotent ring $\mathcal{I}_\beta$. The *mirror* of $\mathcal{B}$, denoted $m(\mathcal{B})$, is the same differential bigraded ring as $\mathcal{B}$. As an algebra, the left and right actions of $\mathcal{I}_\beta$ are mirrored as in Definition 5.2.4. The map from $\mathcal{B}$ to $m(\mathcal{B})$ sending $b \in \mathcal{B}$ to $m(b) \in m(\mathcal{B})$ is an isomorphism of rings (but not of algebras); its inverse is the analogously defined map from $m(\mathcal{B})$ to $m(m(\mathcal{B})) = \mathcal{B}$. To avoid confusion with other uses of the letter m, we will refer to both of these mirroring maps as mirr: we have

$$\text{mirr: } \mathcal{B} \to m(\mathcal{B}) \quad \text{and} \quad \text{mirr: } m(\mathcal{B}) \to m(m(\mathcal{B})) = \mathcal{B}.$$

5.2.7 Remark The mirroring operation for algebras commutes with quadratic duality: if $\mathcal{B}$ is a linear-quadratic algebra over $\mathcal{I}_\beta$, then

$$m(\mathcal{B}^!) = (m(\mathcal{B}))^!.$$

Thus, we can write either of these algebras as $m(\mathcal{B})^!$. Mirroring also commutes with taking the opposite algebra: we have $m(\mathcal{B}^{\mathrm{op}}) = (m(\mathcal{B}))^{\mathrm{op}}$, so we can write either of these algebras as $m(\mathcal{B})^{\mathrm{op}}$.

5.2.8 Proposition *$\mathcal{B}_L\Gamma_n$ is isomorphic to the quotient of $m(\mathcal{B}_R(H^n))^!$ by the following extra relations. Let the graph G be defined as above; for each tetrahedron in G, the only relation in $m(\mathcal{B}_R(H^n))^!$ involving the vertices of the tetrahedron is that the sum of all its vertices is zero. The algebra $\mathcal{B}_L\Gamma_n$ imposes extra relations among the vertices of each tetrahedron. Recall that tetrahedra in G arise when we have two bridges γ and η, with $\eta \in B_o(L, \gamma)$, such that γ splits a plus-labeled circle and η' joins the newly formed circles into a minus-labeled circle. The vertices of the corresponding tetrahedron are, following the discussion above:*

- $a := m(b^*_{\gamma;m(h_1),m(h_2)})m(b^*_{\eta';m(h_2),m(h_3)})$,
- $b := m(b^*_{\gamma;m(h_1),m(\tilde{h}_2)})m(b^*_{\eta';m(\tilde{h}_2),m(h_3)})$,
- $c := m(b^*_{\eta;m(h_1),m(\tilde{\tilde{h}}_2)})m(b^*_{\gamma';m(\tilde{\tilde{h}}_2),m(h_3)})$,
- $d := m(b^*_{\eta;m(h_1),m(\tilde{\tilde{h}}_2))}m(b^*_{\gamma';m(\tilde{\tilde{h}}_2),m(h_3)})$.

Whereas the algebra $m(\mathcal{B}_R(H^n))^!$ imposes only the relation $a+b+c+d=0$, the algebra $\mathcal{B}_L\Gamma_n$ imposes the relations

- $a+c=0$,
- $a+d=0$,
- $b+c=0$,
- $b+d=0$ *(this relation also follows from the previous three).*

From these relations, $a+b+c+d=0$ may be deduced, as well as relations for the two remaining edges of the tetrahedron:

- $a-b=0$,
- $c-d=0$.

Proof Consider the map from $m(\mathcal{B}_R(H^n))^!$ to $\mathcal{B}_L\Gamma_n$ sending $m(b^*_{\gamma;m(h_1),m(h_2)})$ to $\overleftarrow{e}_{\gamma;h_1,h_2}$ and sending $m(b^*_{C;m(h_1),m(h_2)})$ to $\overleftarrow{e}_{C;h_1,h_2}$. By examining the subset of Roberts' relations from [12] involving only left-pointing generators, and comparing with the relations for $m(\mathcal{B}_R(H^n))^!$ above, we see that this is a well-defined surjective bigrading-preserving map whose kernel is generated by the extra relations listed in the statement of this proposition. These extra anticommutation relations can be found in Roberts' algebra as a subset of the relations (21), case (2) [12, page 98].

After mirroring, the formula above for φ^* agrees with Roberts' definition [12, Proposition 25], of the differential on $\mathcal{B}\Gamma_n$ (or equivalently on $\mathcal{B}_L\Gamma_n$, since the differential of any right-pointing generator $\overrightarrow{e}$ of $\mathcal{B}\Gamma_n$ is zero). Since both the differential on $m(\mathcal{B}_R(H^n))^!$ and the differential on $\mathcal{B}_L\Gamma_n$ are defined by the same formula on the degree-1 generators and extended formally to the full algebras by the Leibniz rule, we can conclude that the differential on $\mathcal{B}_L\Gamma_n$ agrees with the differential on $m(\mathcal{B}_R(H^n))^!$ after quotienting the latter algebra by the extra relations. □

5.3 The full algebra

In the following, $\mathcal{B}$ will denote $\mathcal{B}_R(H^n)\cong\mathcal{B}_R\Gamma_n$ unless otherwise specified.

The goal of this section is to construct a product algebra $m(\mathcal{B})^!\odot\mathcal{B}$ and identify $\mathcal{B}\Gamma_n$ with a quotient of $m(\mathcal{B})^!\odot\mathcal{B}$. We also want to construct a rank-one DD bimodule for $m(\mathcal{B})^!\odot\mathcal{B}$ using rank-one DD bimodules for $m(\mathcal{B})^!$ and $\mathcal{B}$.

By Proposition 4.5.3 and Proposition 4.5.4, we have rank-one type DD bimodules which we may refer to as ${}^{\mathcal{B}}K^{(\mathcal{B}^!)^{\mathrm{op}}}$ and ${}^{\mathcal{B}^!}K^{\mathcal{B}^{\mathrm{op}}}$. Like in Definition 5.2.6 above, we can extend the mirroring operation of Definition 5.2.4 to these bimodules.

5.3.1 Definition Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be differential bigraded algebras over the idempotent ring $\mathcal{I}_\beta$ and let (K,δ) be a type DD bimodule over $\mathcal{B}_1$ and $\mathcal{B}_2$. The *mirrored DD bimodule* $(m(K),\delta')$ is defined as follows: as an $(\mathcal{I}_\beta,\mathcal{I}_\beta)$–bimodule, $m(K)$ is the mirror of K as defined in Definition 5.2.4. As in Definition 5.2.6, denote the natural map from K to $m(K)$ or $m(K)$ to K by mirr. The DD operation on $m(K)$ is

$$\delta' := m(K) \xrightarrow{\text{mirr}} K \xrightarrow{\delta} \mathcal{B}_1 \otimes K \otimes (\mathcal{B}_2)^{\text{op}} \xrightarrow{\text{mirr}\otimes\text{mirr}\otimes\text{mirr}} m(\mathcal{B}_1)\otimes m(K)\otimes m(\mathcal{B}_2)^{\text{op}}.$$

Applying Definition 5.3.1 to the bimodules ${}^{\mathcal{B}}K^{(\mathcal{B}^!)^{\text{op}}}$ and ${}^{\mathcal{B}^!}K^{\mathcal{B}^{\text{op}}}$, we get rank-one DD bimodules which we will denote ${}^{m(\mathcal{B})}K^{m(\mathcal{B}^!)^{\text{op}}}$ and ${}^{m(\mathcal{B})^!}K^{m(\mathcal{B})^{\text{op}}}$.

We will focus on the DD bimodules ${}^{\mathcal{B}}K^{(\mathcal{B}^!)^{\text{op}}}$ and ${}^{m(\mathcal{B})^!}K^{m(\mathcal{B})^{\text{op}}}$. Let δ_1 and δ_2 denote the corresponding maps $\delta_1\colon \mathcal{I}_\beta \to \mathcal{B}\otimes_{\mathcal{I}_\beta}(\mathcal{B}^!)^{\text{op}}$ and $\delta_2\colon \mathcal{I}_\beta \to m(\mathcal{B})^!\otimes_{\mathcal{I}_\beta} m(\mathcal{B})^{\text{op}}$.

The set of multiplicative generators of $m(\mathcal{B})^!\odot\mathcal{B}$ will be the union of the generator sets of $\mathcal{B}$ and $m(\mathcal{B})^!$; there will be inclusion maps from $\mathcal{B}$ and $m(\mathcal{B})^!$ into $m(\mathcal{B})^!\odot\mathcal{B}$. Similarly, there will be inclusion maps from $m(\mathcal{B})$ and $\mathcal{B}^! = m(m(\mathcal{B}))^!$ into $m(m(\mathcal{B})^!\odot\mathcal{B})$, and thus maps from $m(\mathcal{B})^{\text{op}}$ and $(\mathcal{B}^!)^{\text{op}}$ into $m(m(\mathcal{B})^!\odot\mathcal{B})^{\text{op}}$. The algebra $m(\mathcal{B})^!\odot\mathcal{B}$ will be defined such that, when δ_1 and δ_2 are postcomposed with these inclusion maps, their sum

$$\delta_1+\delta_2\colon \mathcal{I}_\beta \to (m(\mathcal{B})^!\odot\mathcal{B})\otimes_{\mathcal{I}_\beta}(m(m(\mathcal{B})^!\odot\mathcal{B}))^{\text{op}}$$

satisfies the type DD structure relations.

Let $V_{\mathcal{B}}$ (respectively $V_{m(\mathcal{B})^!}$) denote the free $\mathbb{Z}$–module spanned by the multiplicative generators of $\mathcal{B}$ (respectively $m(\mathcal{B})^!$). Then $V_{\mathcal{B}}$ and $V_{m(\mathcal{B})^!}$ have left and right actions of $\mathcal{I}_\beta$, and we may write $\mathcal{B} = T(V_{\mathcal{B}})/J_{\mathcal{B}}$ and $m(\mathcal{B})^!$ as $T(V_{m(\mathcal{B})^!})/J_{m(\mathcal{B})^!}$.

Define V_{full}, as a bigraded free $\mathbb{Z}$–module, to be $V_{\mathcal{B}}\oplus V_{m(\mathcal{B})^!}$. The actions of $\mathcal{I}_\beta$ on the summands of V_{full} give V_{full} an $\mathcal{I}_\beta$–bimodule structure.

We will define $m(\mathcal{B})^!\odot\mathcal{B}$ to be $T(V_{\text{full}})/J_{\text{full}}$, for some ideal J_{full} of $T(V_{\text{full}})$. We will define J_{full} with an explicit set of linear-quadratic generators, which will agree with Roberts' relations involving both left-pointing and right-pointing elements of $\mathcal{B}\Gamma_n$.

We can start by analyzing $T^1(V_{\text{full}})\oplus T^2(V_{\text{full}})$, which is equal to

$$\begin{aligned}&(V_{\mathcal{B}}\oplus V_{m(\mathcal{B})^!})\oplus((V_{\mathcal{B}}\oplus V_{m(\mathcal{B})^!})\otimes(V_{\mathcal{B}}\oplus V_{m(\mathcal{B})^!}))\\ &\quad= V_{\mathcal{B}}\oplus V_{m(\mathcal{B})^!}\oplus(V_{\mathcal{B}}\otimes V_{\mathcal{B}})\oplus(V_{\mathcal{B}}\otimes V_{m(\mathcal{B})^!})\oplus(V_{m(\mathcal{B})^!}\otimes V_{\mathcal{B}})\oplus(V_{m(\mathcal{B})^!}\otimes V_{m(\mathcal{B})^!}).\end{aligned}$$

Thus, $T^1(V_{\text{full}})\oplus T^2(V_{\text{full}})$ is the direct sum of

$$T^1(V_{\mathcal{B}})\oplus T^2(V_{\mathcal{B}}),\quad T^1(V_{m(\mathcal{B})^!})\oplus T^2(V_{m(\mathcal{B})^!}),$$

and two more summands

$$V_{\mathcal{B}} \otimes V_{m(\mathcal{B})^!}, \quad V_{m(\mathcal{B})^!} \otimes V_{\mathcal{B}}.$$

The ideal J_{full} will be generated multiplicatively by

$$J_{\mathcal{B}} \cap (T^1(V_{\mathcal{B}}) \oplus T^2(V_{\mathcal{B}})), \quad J_{m(\mathcal{B})^!} \cap (T^1(V_{m(\mathcal{B})^!}) \oplus T^2(V_{m(\mathcal{B})^!})),$$

and some extra relations

$$J_{\text{extra}} \subset (V_{\mathcal{B}} \otimes V_{m(\mathcal{B})^!}) \oplus (V_{m(\mathcal{B})^!} \otimes V_{\mathcal{B}}).$$

5.3.2 Definition $J_{\text{extra}} \subset (V_{\mathcal{B}} \otimes V_{m(\mathcal{B})^!}) \oplus (V_{m(\mathcal{B})^!} \otimes V_{\mathcal{B}})$ is defined additively by the following relations:

(1) For two bridge generators $b_{\gamma;h_1,h_2}$ and $m(b^*_{\eta';m(h_2),m(h_3)})$, the "commutation relation"

$$b_{\gamma;h_1,h_2} m(b^*_{\eta';m(h_2),m(h_3)}) - m(b^*_{\eta;m(h_1),m(\tilde{h}_2)}) b_{\gamma';\tilde{h}_2,h_3}$$

is in J_{extra} for any $\tilde{h}_2 \in \beta$ such that $m(b^*_{\eta;m(h_1),m(\tilde{h}_2)})$ and $b_{\gamma';\tilde{h}_2,h_3}$ exist. The bridges γ' and η are uniquely determined. We will call such relations *commutation relations* even though they do not exactly express that two elements commute.

(2) Any generator $b_{C;h_1,h_2}$ of $\mathcal{B} = \mathcal{B}_R(H^n)$ for $n > 1$ can be written as a product of bridge generators $b_\gamma b_{\gamma^\dagger}$. Thus, by the commutation relations above, $b_{C;h_1,h_2}$ must also commute with bridge generators $m(b^*_{\eta;m(h_2),m(h_3)})$: the relation

$$b_{C;h_1,h_2} m(b^*_{\eta;m(h_2),m(h_3)}) - m(b^*_{\eta;m(h_1),m(\tilde{h}_2)}) b_{C;\tilde{h}_2,h_3}$$

must be in J_{extra} for any $\tilde{h}_2 \in \beta$ such that $m(b^*_{\eta;m(h_1),m(\tilde{h}_2)})$ and $b_{C;\tilde{h}_2,h_3}$ exist.

(3) For a bridge generator $b_{\gamma;h_1,h_2}$ and a decoration generator $m(b^*_{C;m(h_2),m(h_3)})$ in which the circle C is disjoint from the circles involved in surgery on γ, we also impose commutation relations:

$$b_{\gamma;h_1,h_2} m(b^*_{C;m(h_2),m(h_3)}) - m(b^*_{C;m(h_1),m(\tilde{h}_2)}) b_{\gamma;\tilde{h}_2,h_3}$$

must be in J_{extra}, for the unique $\tilde{h}_2 \in \beta$ such that $m(b^*_{C;m(h_1),m(\tilde{h}_2)})$ and $b_{\gamma;\tilde{h}_2,h_3}$ exist.

(4) For two disjoint circles C and C', we again have commutation relations:

$$b_{C;h_1,h_2} m(b^*_{C';m(h_2),m(h_3)}) - m(b^*_{C';m(h_1),m(\tilde{h}_2)}) b_{C;\tilde{h}_2,h_3}$$

must be in J_{extra}, for the unique $\tilde{h}_2 \in \beta$ such that $m(b^*_{C';m(h_1),m(\tilde{h}_2)})$ and $b_{C;\tilde{h}_2,h_3}$ exist.

(5) Finally, let γ be a bridge and let C be one of the circles involved in surgery on γ. When γ joins C' and C'' to form C,

$$b_{\gamma;h_1,h_2} m(b^*_{C;m(h_2),m(h_3)}) - m(b^*_{C';m(h_1),m(\tilde{h}_2)}) b_{\gamma;\tilde{h}_2,h_3} - m(b^*_{C'';m(h_1),m(\tilde{\tilde{h}}_2)}) b_{\gamma;\tilde{\tilde{h}}_2,h_3}$$

is in J_{extra} for the unique $\tilde{h}_2 \in \beta$ and $\tilde{\tilde{h}}_2 \in \beta$ such that the relevant generators exist, and when γ splits C to form C' and C'',

$$b_{\gamma;h_1,\tilde{h}_2} m(b^*_{C';m(\tilde{h}_2),m(h_3)}) + b_{\gamma;h_1,\tilde{\tilde{h}}_2} m(b^*_{C'';m(\tilde{\tilde{h}}_2),m(h_3)}) - m(b^*_{C;m(h_1),m(h_2)}) b_{\gamma;h_2,h_3}$$

is in J_{extra} for the unique $\tilde{h}_2 \in \beta$ and $\tilde{\tilde{h}}_2 \in \beta$ such that the relevant generators exist.

5.3.3 Definition The ideal J_{full} is defined by

$$J_{\text{full}} := T(V_{\text{full}}) \cdot \big((J_{\mathcal{B}} \cap (T^1(V_{\mathcal{B}}) \oplus T^2(V_{\mathcal{B}}))) \oplus \big(J_{m(\mathcal{B})^!} \cap (T^1(V_{m(\mathcal{B})^!}) \oplus T^2(V_{m(\mathcal{B})^!})) \big) \oplus J_{\text{extra}} \big) \cdot T(V_{\text{full}}).$$

The differential bigraded algebra $m(\mathcal{B})^! \odot \mathcal{B}$ is defined by

$$m(\mathcal{B})^! \odot \mathcal{B} := T(V_{\text{full}})/J_{\text{full}},$$

with a differential induced from the differential on $m(\mathcal{B})^!$. The differential of any generator of $\mathcal{B}$ is declared to be zero.

5.3.4 Corollary *$\mathcal{B}\Gamma_n$, as a differential bigraded algebra, is the quotient of the algebra $m(\mathcal{B})^! \odot \mathcal{B}$ by the same additional relations as specified in Proposition 5.2.8. These relations involve only quadratic monomials with two generators of $m(\mathcal{B})^!$.*

Proof The relations in J_{extra} are modeled on Roberts' relations for $\mathcal{B}\Gamma_n$ in [12] involving quadratic monomials with one left-pointing and one right-pointing generator; see [12, Section 2.3]. □

5.3.5 Remark The relations in Definition 5.3.2 were chosen to match Roberts' quadratic relations involving a left-pointing and a right-pointing generator. A more general formulation of the product operation $\odot$ would be desirable. The best we can do now is to say that the relations in J_{extra} have an additional motivation beyond lining up with Roberts' relations: with these relations, $\delta_1 + \delta_2$ defines a valid rank-one DD bimodule over the product algebra, as we see below in Proposition 5.3.6.

From the definition of the product algebra $m(\mathcal{B})^! \odot \mathcal{B}$, there are natural inclusion maps of $\mathcal{B}$ and $m(\mathcal{B})^!$ into the product. Also, by Definition 5.2.6, we have a mirror algebra $m(m(\mathcal{B})^! \odot \mathcal{B})$; both $m(\mathcal{B})$ and $\mathcal{B}^! = m(m(\mathcal{B})^!)$ have natural inclusion maps into $m(m(\mathcal{B})^! \odot \mathcal{B})$.

We may view the type DD map

$$\delta_1 \colon \mathcal{I}_\beta \to \mathcal{B} \otimes_{\mathcal{I}_\beta} (\mathcal{B}^!)^{\mathrm{op}} = \mathcal{B} \otimes_{\mathcal{I}_\beta} m(m(\mathcal{B})^!)^{\mathrm{op}}$$

as a map from $\mathcal{I}_\beta$ to $(m(\mathcal{B})^! \odot \mathcal{B}) \otimes_{\mathcal{I}_\beta} (m(m(\mathcal{B})^! \odot \mathcal{B}))^{\mathrm{op}}$, using the inclusion maps from $\mathcal{B}$ into $m(\mathcal{B})^! \odot \mathcal{B}$ and from $m(m(\mathcal{B})^!)^{\mathrm{op}}$ into $(m(m(\mathcal{B})^! \odot \mathcal{B}))^{\mathrm{op}}$. Similarly, we may view

$$\delta_2 \colon \mathcal{I}_\beta \to m(\mathcal{B})^! \otimes (m(\mathcal{B}))^{\mathrm{op}}$$

as a map from $\mathcal{I}_\beta$ to $(m(\mathcal{B})^! \odot \mathcal{B}) \otimes_{\mathcal{I}_\beta} (m(m(\mathcal{B})^! \odot \mathcal{B}))^{\mathrm{op}}$ using the inclusion maps from $m(\mathcal{B})^!$ into $m(\mathcal{B})^! \odot \mathcal{B}$ and from $m(\mathcal{B})^{\mathrm{op}}$ into $(m(m(\mathcal{B})^! \odot \mathcal{B}))^{\mathrm{op}}$.

5.3.6 Proposition *The map $\delta_1 + \delta_2 \colon \mathcal{I}_\beta \to (m(\mathcal{B})^! \odot \mathcal{B}) \otimes_{\mathcal{I}_\beta} (m(m(\mathcal{B})^! \odot \mathcal{B}))^{\mathrm{op}}$ satisfies the type DD structure relations.*

Proof Many of the type DD structure terms cancel since δ_1 and δ_2 individually satisfy the type DD relations. In particular, all terms of type $(\mu_1 \otimes |\mathrm{id}|) \circ \delta$ and $(\mathrm{id} \otimes \mu_1) \circ \delta$ are accounted for and we are left with terms of type $(\mu_2 \otimes \mu_2) \circ \sigma \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta$.

The remaining terms of type $(\mu_2 \otimes \mu_2) \circ \sigma \circ (\mathrm{id} \otimes \delta \otimes \mathrm{id}) \circ \delta$ are those in which one of the applications of δ uses δ_1 and the other uses δ_2. These are

$$-b_i \cdot m(b_j^*) \otimes m(b_j)^{\mathrm{op}} \cdot (b_i^*)^{\mathrm{op}} = -b_i \cdot m(b_j^*) \otimes m(m(b_i^*) \cdot b_j)^{\mathrm{op}}, \tag{5-1}$$

referred to as terms of type (5-1), as well as

$$m(b_j^*) \cdot b_i \otimes (b_i^*)^{\mathrm{op}} \cdot m(b_j)^{\mathrm{op}} = m(b_j^*) \cdot b_i \otimes m(b_j \cdot m(b_i^*))^{\mathrm{op}}, \tag{5-2}$$

referred to as terms of type (5-2), where b_i and b_j^* run over all generators of $\mathcal{B}$ and $\mathcal{B}^!$, respectively, with compatible idempotents. Note that the negative signs in the terms of type (5-1) come from the sign-flip operator σ.

The commutation relations among the relations defining $m(\mathcal{B})^! \odot \mathcal{B}$ ensure that all the above terms cancel, except for potentially two sets of terms.

The first set S_1 of terms includes those terms $-b_i \cdot m(b_j^*) \otimes m(m(b_i^*) \cdot b_j)^{\mathrm{op}}$ of type (5-1) in which both b_i and b_j are among the generators b_γ, and such that the product $b_i \cdot m(b_j^*)$ corresponds to splitting a circle C on the right into two circles C_1 and C_2 and then joining C_1 to C_2 again on the left to produce a new circle C_3. It also includes those terms $m(b_j^*) \cdot b_i \otimes m(b_j \cdot m(b_i^*))^{\mathrm{op}}$ of type (5-2) with b_i and b_j

among the generators b_γ such that $m(b_j^*)\cdot b_i$ corresponds to splitting a circle C on the left into two circles C_1 and C_2 and then joining C_1 and C_2 again on the right to produce a new circle C_3.

The second set S_2 consists of those terms of type (5-1) or (5-2) in which b_i is one of the generators b_γ and b_j is one of the generators b_C, where C is one of the circles involved in surgery on γ.

For all terms except those in S_1 and S_2, commutation relations may be applied to the type (5-1) term $-b_i\cdot m(b_j^*)\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}$ uniquely to cancel with a unique corresponding type (5-2) term $m(b_{j'}^*)\cdot b_{i'}\otimes m(b_{j'}\cdot m(b_{i'}^*))^{\mathrm{op}}$.

First, we show that the terms in S_1 sum to zero. If the type (5-1) term $-b_i\cdot m(b_j^*)\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}$ is in S_1, then there are two terms $m(b_{j'}^*)\cdot b_{i'}\otimes m(b_{j'}\cdot m(b_{i'}^*))^{\mathrm{op}}$ and $m(b_{j''}^*)\cdot b_{i''}\otimes m(b_{j''}\cdot m(b_{i''}^*))^{\mathrm{op}}$ of type (5-2) in S_1 which have the same left and right idempotents as $-b_i\cdot m(b_j^*)\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}$. By the commutation relations, both these terms are equal to $b_i\cdot m(b_j^*)\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}$.

Furthermore, there is one other type (5-1) term $-b_{i'''}\cdot m(b_{j'''}^*)\otimes m(m(b_{i'''}^*)\cdot b_{j'''})^{\mathrm{op}}$ of S_1 which is equal to both $-m(b_{j'}^*)\cdot b_{i'}\otimes m(b_{j'}\cdot m(b_{i'}^*))^{\mathrm{op}}$ and $-m(b_{j''}^*)\cdot b_{i''}\otimes m(b_{j''}\cdot m(b_{i''}^*))^{\mathrm{op}}$ by the commutation relations. Hence it is equal to $-b_i\cdot m(b_j^*)\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}$ as well. These four terms are the only terms in S_1 with the same idempotents as $-b_i\cdot m(b_j^*)\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}$, and their sum is zero. Thus, the terms in S_1 sum to zero.

Now we show that the terms in S_2 sum to zero. If b_i is a b_γ generator and b_j is a b_C generator with C involved in surgery on γ, then suppose first that γ joins two circles C_1 and C_2 to produce C. By item (5) of Definition 5.3.2, we have $b_i\cdot m(b_j^*)=m(b_j'^*)\cdot b_i'+m(b_j''^*)\cdot b_i''$, where b_j' and b_j'' are the generators b_{C_1} and b_{C_2}, and b_i' and b_i'' are the appropriate b_γ generators.

Thus, if $-b_i\cdot m(b_j^*)\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}$ is the corresponding term of type (5-1), we have

$$\begin{aligned}-b_i\cdot m(b_j^*)&\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}\\ &=-m(b_j'^*)\cdot b_i'\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}-m(b_j''^*)\cdot b_i''\otimes m(m(b_i^*)\cdot b_j)^{\mathrm{op}}\\ &=-m(b_j'^*)\cdot b_i'\otimes m(b_j'\cdot m(b_i'^*))^{\mathrm{op}}-m(b_j''^*)\cdot b_i''\otimes m(b_j''\cdot m(b_i''^*))^{\mathrm{op}},\end{aligned}$$

where in the last step we use commutation relations from item (2) of Definition 5.3.2. The two resulting terms cancel the two relevant terms of type (5-2). The case when γ splits a circle, rather than joining two circles, is analogous, so the terms in S_2 sum to zero.

Hence all the relation terms cancel and $(\delta_1 + \delta_2)$ satisfies the type DD structure relations. □

A general property of type D structures and type DD bimodules over an algebra $\mathcal{B}$ is they give induced type D or DD structures over any quotient of $\mathcal{B}$.

5.3.7 Proposition *Let $\mathcal{B}$ be a differential bigraded algebra; let J be a bigrading-homogeneous ideal of $\mathcal{B}$ which is preserved by the differential on $\mathcal{B}$. Let $\pi\colon \mathcal{B} \to \mathcal{B}/J$ denote the quotient projection map. Let $(\widehat{D}, \delta)$ be a type D structure over $\mathcal{B}$; then $\widehat{D}$ descends to a type D structure over $\mathcal{B}/J$, with structure operation*

$$\widehat{D} \xrightarrow{\delta} \mathcal{B} \otimes \widehat{D} \xrightarrow{\pi \otimes \mathrm{id}} (\mathcal{B}/J) \otimes \widehat{D}.$$

Similarly, if $\mathcal{B}'$ and J' are another algebra and ideal satisfying the same conditions as $\mathcal{B}$ and J, and $(\widehat{DD}, \delta)$ is a type DD bimodule over $\mathcal{B}$ and $\mathcal{B}'$, then $\widehat{DD}$ descends to a type DD bimodule over $\mathcal{B}/J$ and $\mathcal{B}'/J'$, with structure operation

$$\widehat{DD} \xrightarrow{\delta} \mathcal{B} \otimes \widehat{DD} \otimes (\mathcal{B}')^{\mathrm{op}} \xrightarrow{\pi \otimes \mathrm{id} \otimes (\pi')^{\mathrm{op}}} (\mathcal{B}/J) \otimes \widehat{DD} \otimes (\mathcal{B}'/J')^{\mathrm{op}}.$$

Proof This is a simple consequence of the type D and type DD structure relations. It is also a special case of induction of scalars for type D structures as defined by Lipshitz, Ozsváth and Thurston [7, Section 2.4.2]. □

We know that $\mathcal{B}\Gamma_n$ is a quotient of $m(\mathcal{B})^! \odot \mathcal{B}$, and it follows that $m(\mathcal{B}\Gamma_n)^{\mathrm{op}}$ is a quotient of $m(m(\mathcal{B})^! \odot \mathcal{B})^{\mathrm{op}}$.

5.3.8 Corollary *The map $\mathcal{I}_\beta \to \mathcal{B}\Gamma_n \otimes_{\mathcal{I}_\beta} m(\mathcal{B}\Gamma_n)^{\mathrm{op}}$ obtained by postcomposing $\delta_1 + \delta_2$ with the tensor product of the quotient projections from $m(\mathcal{B})^! \odot \mathcal{B}$ and $m(m(\mathcal{B})^! \odot \mathcal{B})^{\mathrm{op}}$ onto $\mathcal{B}\Gamma_n$ and $m(\mathcal{B}\Gamma_n)^{\mathrm{op}}$ satisfies the type DD structure relations.*

Thus, we have rank-one type DD bimodules

$${}^{m(\mathcal{B})^! \odot \mathcal{B}}K^{m(m(\mathcal{B})^! \odot \mathcal{B})^{\mathrm{op}}} \quad \text{and} \quad {}^{\mathcal{B}\Gamma_n}K^{m(\mathcal{B}\Gamma_n)^{\mathrm{op}}}.$$

5.3.9 Conjecture *Either or both of the DD bimodules ${}^{m(\mathcal{B})^! \odot \mathcal{B}}K^{m(m(\mathcal{B})^! \odot \mathcal{B})^{\mathrm{op}}}$ and ${}^{\mathcal{B}\Gamma_n}K^{m(\mathcal{B}\Gamma_n)^{\mathrm{op}}}$ are quasi-invertible. Hence, either or both of the algebras $m(\mathcal{B})^! \odot \mathcal{B}$ and $\mathcal{B}\Gamma_n$ are Koszul dual to their mirrors, $m(m(\mathcal{B})^! \odot \mathcal{B})$ and $m(\mathcal{B}\Gamma_n)$, in the generalized sense of Lipshitz, Ozsváth and Thurston* [6].

A proof of the above conjecture would provide a nice parallel between Roberts' theory and bordered Floer homology. In bordered Floer homology, the rank-one DD bimodule corresponding to the identity Heegaard diagram of a pointed matched circle has a quasi-inverse, namely the type AA bimodule associated to this diagram. See [6] for more on the Koszul duality structure of bordered Floer homology, including an additional duality relating the algebras of a pointed matched circle $\mathcal{Z}$ and a dual pointed matched circle $\mathcal{Z}_*$.

6 Khovanov's modules and Roberts' modules

In this section, we relate Roberts' type D and type A structures from [11; 12] to the type D and type A structures over H^n from Section 3, or equivalently to Khovanov's dg modules $[T]^{\mathrm{Kh}}$ which contain the same information. In Section 6.2, we show that given a chain complex of projective graded right H^n–modules satisfying an algebraic condition C_{module} defined below in Definition 6.1.1, we may construct a differential bigraded right module over $m(\mathcal{B})^! \odot \mathcal{B}$. Applied to Khovanov's tangle complex $[T]^{\mathrm{Kh}}$, which satisfies C_{module}, this module over $m(\mathcal{B})^! \odot \mathcal{B}$ descends to a module over $\mathcal{B}\Gamma_n$; in other words, the relations of Proposition 5.2.8 act as zero on the $m(\mathcal{B})^! \odot \mathcal{B}$–module. The resulting $\mathcal{B}\Gamma_n$–module agrees with Roberts' type A structure.

In Section 6.3, given a chain complex of projective graded left H^n–modules satisfying the algebraic condition C_{module} for left modules defined in Definition 6.3.3, we construct a type D structure over $m(\mathcal{B})^! \odot \mathcal{B}$. We do this by, first, reflecting the chain complex of left H^n–modules to get a complex of right modules (this operation will be defined in Definition 6.3.1). Then we take the associated type A structure over $m(\mathcal{B})^! \odot \mathcal{B}$, tensor with ${}^{m(\mathcal{B})^! \odot \mathcal{B}}K^{m(m(\mathcal{B})^! \odot \mathcal{B})^{\mathrm{op}}}$, the DD bimodule from the end of Section 5.3, to get a type D structure over $m(m(\mathcal{B})^! \odot \mathcal{B})$, and finally mirror this type D structure to get a type D structure over $m(\mathcal{B})^! \odot \mathcal{B}$. We may quotient the algebra outputs of this type D structure by the relations from Proposition 5.2.8 to get a type D structure over $\mathcal{B}\Gamma_n$, which agrees with the one constructed by Roberts when one starts with the complex $[T]^{\mathrm{Kh}}$.

Given two chain complexes, one of projective graded left H^n–modules and one of projective graded right H^n–modules, their tensor product over H^n is a chain complex with an additional grading, or equivalently a differential bigraded $\mathbb{Z}$–module. In Section 6.4, we show that this tensor product agrees with the box tensor product of the type D and type A structures over $m(\mathcal{B})^! \odot \mathcal{B}$ associated to the two complexes in Section 6.3 and Section 6.2, assuming these complexes satisfy C_{module}. The type A structure is always an ordinary right $m(\mathcal{B})^! \odot \mathcal{B}$–module; if it descends to a $\mathcal{B}\Gamma_n$–module, then the box

tensor products of the type A and type D structures over $m(\mathcal{B})^! \odot \mathcal{B}$ and $\mathcal{B}\Gamma_n$ agree. Applying these constructions to Khovanov's chain complexes $[T]^{\mathrm{Kh}}$ of H^n–modules, we get an alternate proof that the pairing of Roberts' type D and type A structures recovers the original Khovanov complex.

In Section 6.5, we show that chain homotopy equivalences of complexes of H^n–modules, satisfying appropriate algebraic conditions, give $\mathcal{A}_\infty$–homotopy equivalences of the corresponding type A structures over $m(\mathcal{B})^! \odot \mathcal{B}$. Reidemeister moves on tangle diagrams yield chain homotopy equivalences of complexes of H^n–modules, as shown by Khovanov in [4]. These Reidemeister-move homotopy equivalences satisfy the right conditions, and the $\mathcal{A}_\infty$–homotopy equivalences associated to them descend to $\mathcal{A}_\infty$–homotopy equivalences of type A structures over $\mathcal{B}\Gamma_n$. This reasoning yields an alternate proof that Roberts' type A structures are tangle invariants up to $\mathcal{A}_\infty$–homotopy equivalence.

In Section 6.6, we do the same for the type D structures over $m(\mathcal{B})^! \odot \mathcal{B}$. All homotopy equivalences of type D structures over $m(\mathcal{B})^! \odot \mathcal{B}$ descend to homotopy equivalences of type D structures over the quotient $\mathcal{B}\Gamma_n$. Thus, we obtain an alternate proof that Roberts' type D structures are tangle invariants up to homotopy equivalence.

6.1 Preliminaries

Let M be a differential bigraded projective right H^n–module, or equivalently a chain complex of projective graded H^n–modules by Proposition 2.2.4 or a right type D structure over H^n by the appropriate analogue of Proposition 2.2.5. Recall that in accordance with Convention 2.1.1, such an M is assumed to be finitely generated over $\mathbb{Z}$. Let $\{x_i \mid i \in S\}$ be bigrading-homogeneous elements of M, where S is some finite index set, such that

$$M \cong \bigoplus_i x_i H^n$$

as right H^n–modules and each summand $x_i H^n$ is isomorphic to eH^n for some elementary idempotent e of H^n via an isomorphism sending x_i to e. The idempotent e associated to x_i will be denoted $e(x_i)$.

We will use notation from Section 3.1: β will denote the usual $\mathbb{Z}$–basis of H^n and β_{mult} will denote the subset of β consisting of the multiplicative generators h_γ and h_α of H^n. We will further subdivide β_{mult} into β_γ, consisting of h_γ generators, and β_α, consisting of h_α generators. Recall that for $h \in \beta$, $e_L(h) \in \mathcal{I}_n$ is the left idempotent of h and $e_R(h)$ is the right idempotent of h.

The module M has a $\mathbb{Z}$–basis given by

$$\{x_i \cdot h_1 \mid i \in S, h_1 \in \beta, e_L(h_1) = e(x_i)\}.$$

We define integer coefficients $c_{i,j}$ and $\tilde{c}_{i,j;h'}$ for $i, j \in S$ and $h' \in \beta$ by expanding out the differential of each x_i:

$$d(x_i) = \sum_{\substack{j \in S \\ \deg x_j = \deg x_i + (0,1)}} c_{i,j} x_j + \sum_{\substack{j \in S,\, h' \in \beta,\, \deg h' \neq 0 \\ \deg x_j + \deg h' = \deg x_i + (0,1)}} \tilde{c}_{i,j;h'} x_j \cdot h'.$$

6.1.1 Definition M, together with the set of generators $\{x_i\}$, satisfies the algebraic condition C_{module} if $\tilde{c}_{i,j;h'} = 0$ unless $h' \in \beta_{\text{mult}}$.

This condition is satisfied for Khovanov's tangle complexes $[T]^{\text{Kh}}$; the natural choice of $\{x_i\}$ was described in Definition 3.1.2. By slight abuse of notation, we will speak of M satisfying C_{module}, but the choice of $\{x_i\}$ was necessary to define the coefficients $c_{i,j}$ and $\tilde{c}_{i,j;h'}$.

For any M satisfying C_{module}, we can write $d(x_i)$ as

$$\begin{aligned} d(x_i) = \sum_{\substack{j \in S \\ \deg x_j = \deg x_i + (0,1)}} c_{i,j} x_j + \sum_{\substack{j \in S, h' \in \beta_\gamma \\ \deg x_j = \deg x_i + (-1,1)}} \tilde{c}_{i,j;h'} x_j \cdot h' \\ + \sum_{\substack{j \in S, h' \in \beta_\alpha \\ \deg x_j = \deg x_i + (-2,1)}} \tilde{c}_{i,j;h'} x_j \cdot h'. \end{aligned}$$

This is an expansion of $d(x_i)$ in the $\mathbb{Z}$–basis of M.

Thus, if $x_i \cdot h_1$ is a basis element of M, we have

$$\begin{aligned} d(x_i \cdot h_1) = \sum_{\substack{j \in S \\ \deg x_j = \deg x_i + (0,1)}} c_{i,j} x_j \cdot h_1 + \sum_{\substack{j \in S, h' \in \beta_\gamma \\ \deg x_j = \deg x_i + (-1,1)}} \tilde{c}_{i,j;h'} x_j \cdot h' h_1 \\ + \sum_{\substack{j \in S, h' \in \beta_\alpha \\ \deg x_j = \deg x_i + (-2,1)}} \tilde{c}_{i,j;h'} x_j \cdot h' h_1. \end{aligned}$$

However, this is not necessarily a basis expansion of $d(x_i \cdot h_1)$, because the elements $h' h_1 \in H^n$ are not necessarily elements of the basis β. Instead, we may define integer coefficients $\tilde{\tilde{c}}$ by

$$h' h_1 := \sum_{h_2 \in \beta} \tilde{\tilde{c}}_{h' h_1; h_2} h_2$$

and thus

$$d(x_i \cdot h_1) = \sum_{\substack{j \in S \\ \deg x_j = \deg x_i + (0,1)}} c_{i,j} x_j \cdot h_1 + \sum_{\substack{j \in S, h' \in \beta_\gamma, h_2 \in \beta \\ \deg x_j = \deg x_i + (-1,1)}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} x_j \cdot h_2 + \sum_{\substack{j \in S, h' \in \beta_\alpha, h_2 \in \beta \\ \deg x_j = \deg x_i + (-2,1)}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} x_j \cdot h_2.$$

This is a basis expansion of $d(x_i \cdot h_1)$ in the $\mathbb{Z}$–basis of M.

6.1.2 Proposition *Suppose M satisfies C_{module}. The equation $d^2 = 0$ on M gives rise to the following five sets of equations involving the coefficients $c_{i,j}$, $\tilde{c}_{i,j,h'}$ and $\tilde{\tilde{c}}_{h'h_1;h_2}$:*

(1) *For all x_i and x_k with $\deg x_k = \deg x_i + (0,2)$, we have*

$$\sum_j c_{i,j} c_{j,k} = 0.$$

(2) *For all $x_i \cdot h_1$ and $x_k \cdot h_3$ with $\deg x_k = \deg x_i + (-1,2)$, we have*

$$\sum_{j, h' \in \beta_\gamma} (\tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_3} c_{j,k} + c_{i,j} \tilde{c}_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h_3}) = 0.$$

(3) *For all $x_i \cdot h_1$ and $x_k \cdot h_3$ with $\deg x_k = \deg x_i + (-2,2)$, we have*

$$\sum_{j, h' \in \beta_\alpha} (\tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_3} c_{j,k} + c_{i,j} \tilde{c}_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h_3}) + \sum_{\substack{j, h' \in \beta_\gamma, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{c}_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3} = 0.$$

(4) *For all $x_i \cdot h_1$ and $x_k \cdot h_3$ with $\deg x_k = \deg x_i + (-3,2)$, we have*

$$\sum_{\substack{j, h' \in \beta_\gamma, \\ h'' \in \beta_\alpha, h_2 \in \beta}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{c}_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3} + \sum_{\substack{j, h' \in \beta_\alpha, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{c}_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3} = 0.$$

(5) *For all $x_i \cdot h_1$ and $x_k \cdot h_3$ with $\deg x_k = \deg x_i + (-4,2)$, we have*

$$\sum_{\substack{j, h' \in \beta_\alpha, \\ h'' \in \beta_\alpha, h_2 \in \beta}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{c}_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3} = 0.$$

Proof This follows from writing out $d^2(x_i \cdot h_1)$ as a sum of basis elements $x_k \cdot h_3$, using the above basis expansion for $d(x_i \cdot h_1)$, and then grouping the $x_k \cdot h_3$ according to the intrinsic degree of x_k relative to x_i. □

6.1.3 Example Suppose $M = [T]^{\mathrm{Kh}}$, where T is an oriented tangle diagram in $\mathbb{R}_{\leq 0} \times \mathbb{R}$ with an ordering of its crossings. We will analyze the generators x_i and coefficients $\tilde{c}_{i,j;h'}$ and $c_{i,j}$. To specify a generator x_i of $[T]^{\mathrm{Kh}}$, we first specify a resolution ρ_i of all crossings of T; we can view ρ_i as a function from the set of crossings to the two-element set $\{0,1\}$. If T_{ρ_i} denotes the diagram T with the crossings resolved according to ρ_i, then T_{ρ_i} consists of a left crossingless matching of $2n$ points together with some free circles contained in $\mathbb{R}_{<0} \times \mathbb{R}$. The remaining data needed to specify x_i are a choice of $+$ (plus) or $-$ (minus) on each free circle; the crossingless-matching part of T_{ρ_i} is closed up symmetrically and all resulting circles are labeled plus. Then $[T]^{\mathrm{Kh}}$ has a $\mathbb{Z}$–basis consisting of elements $x_i \cdot h$, where the left crossingless matching of h agrees with the matching obtained from T_{ρ_i} by discarding the free circles.

Let S denote the set of x_i specified above. The basis expansion defining $c_{i,j}$ and $\tilde{c}_{i,j;h'}$ is

$$d(x_i) = \sum_{\substack{j \in S \\ \deg x_j = \deg x_i + (0,1)}} c_{i,j} x_j + \sum_{\substack{j \in S, h' \in \beta_\gamma \\ \deg x_j = \deg x_i + (-1,1)}} \tilde{c}_{i,j;h'} x_j \cdot h' + \sum_{\substack{j \in S, h' \in \beta_\alpha \\ \deg x_j = \deg x_i + (-2,1)}} \tilde{c}_{i,j;h'} x_j \cdot h'.$$

The coefficients $c_{i,j}$ and $\tilde{c}_{i,j;h'}$ can only be nonzero when the resolution ρ_j of x_j differs from the resolution ρ_i of x_i only at one crossing, to which ρ_i assigns 0 and ρ_j assigns 1. Let $\#_1(i,j)$ denote the number of 1–resolutions of crossings in x_i among those crossings that, in the ordering on crossings, occur earlier than the crossing being changed when going from x_i to x_j.

Changing the crossing to get from x_i to x_j has several possible effects:

(1) The crossing change could join two free circles or split a free circle. In this case, $c_{i,j}$ is $(-1)^{\#_1(i,j)}$ and all $\tilde{c}_{i,j;h'}$ are zero.

(2) The crossing change could join a free circle in x_i, labeled $+$, with an arc of x_i. Alternatively, it could split a new free circle, labeled $-$ in x_j, off an arc of x_i. In both these cases, $c_{i,j}$ is $(-1)^{\#_1(i,j)}$ and all $\tilde{c}_{i,j;h'}$ are zero.

(3) The crossing change could join a free circle in x_i, labeled $-$, with an arc α of x_i. Alternatively, it could split a new free circle, labeled $+$ in x_j, off an arc α of x_i. In both these cases, $c_{i,j}$ is zero and $\tilde{c}_{i,j;h'}$ is only nonzero for one value of h'. If a denotes the crossingless matching of x_i (or of x_j), then when $h' = (W(a)a, \text{ minus on } \alpha)$, we have $\tilde{c}_{i,j;h'} = (-1)^{\#_1(i,j)}$; all other $\tilde{c}_{i,j;h'}$ are zero.

(4) Finally, the crossing change could surger two arcs of x_i, changing the crossingless matching. Again, $c_{i,j} = 0$ and $\tilde{c}_{i,j;h'}$ is nonzero for a unique h'. Let a_i and a_j denote the crossingless matchings of x_i and x_j respectively. Then if $h' = (W(a_j)a_i, \text{all plus})$, we have $\tilde{c}_{i,j;h'} = (-1)^{\#_1(i,j)}$; all other $\tilde{c}_{i,j;h'}$ are zero.

Note that $[T]^{\mathrm{Kh}}$ satisfies C_{module} with the elements $x_i = x_i \cdot 1$ as generators.

6.2 Type A structures

As in Section 5.3, let $\mathcal{B}$ denote $\mathcal{B}_R(H^n) \cong \mathcal{B}_R\Gamma_n$. Recall that $\mathcal{B}_L\Gamma_n$ is the quotient of $m(\mathcal{B})^!$ by the extra relations listed in Proposition 5.2.8.

Let M be a differential bigraded projective right H^n module as at the beginning of Section 6.1; assume that M satisfies the algebraic condition C_{module} of Definition 6.1.1 for a set of generators $\{x_i \mid i \in S\}$. We first define a type A structure $\widehat{A}(M)_{m(\mathcal{B})^!}$ over $m(\mathcal{B})^!$. Then we formally extend $\widehat{A}(M)$ to a type A structure $\widehat{A}(M)_{m(\mathcal{B})^! \odot \mathcal{B}}$ over $m(\mathcal{B})^! \odot \mathcal{B}$.

6.2.1 Definition As a $\mathbb{Z}$–module, $\widehat{A}(M)$ is defined to be M. A $\mathbb{Z}$–basis for M is given by $\{x_i \cdot h_1 \mid i \in S, h_1 \in \beta, e_L(h_1) = e(x_i)\}$, where $\{x_i \mid i \in S\}$ is the designated set of generators of M. For $\widehat{A}(M)$, we label the same basis elements as

$$\{X_{x_i \cdot h_1} \mid i \in S, h_1 \in \beta, e_L(h_1) = e(x_i)\}.$$

The idempotent ring of $m(\mathcal{B})^!$ is $\mathcal{I}_\beta$; let $h_2 \in \beta$ be an elementary idempotent of $m(\mathcal{B})^!$. Multiplying $X_{x_i \cdot h_1}$ by h_2 gives $X_{x_i \cdot h_1}$ if $h_2 = h_1$ and zero otherwise.

Suppose that the generator x_i has bigrading (j,k) as an element of M, and h has grading j' (or bigrading $(j',0)$) as an element of H^n. Then, as an element of $\widehat{A}(M)$, the bigrading of $X_{x_i \cdot h_1}$ is defined to be

$$\deg_{\widehat{A}(M)}(X_{x_i \cdot h}) := \left(-j - \tfrac{1}{2}j', k\right).$$

The algebra $m(\mathcal{B})^!$ acts on $\widehat{A}(M)$ on the right; we will use m_2 to denote this action (not to be confused with m here, which means *mirror*). Let $m(b^*_{*;m(h_1),m(h_2)})$ denote either $m(b^*_{\gamma;m(h_1),m(h_2)})$ or $m(b^*_{C;m(h_1),m(h_2)})$. We define

$$m_2(X_{x_i \cdot h_1}, m(b^*_{*;m(h_1),m(h_2)})) := \sum_j \sum_{h' \in \beta_{\text{mult}}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} X_{x_j \cdot h_2}.$$

If $m(b^*_{*;m(h_1),m(h_2)}) = m(b^*_{\gamma;m(h_1),m(h_2)})$, then $\tilde{\tilde{c}}_{h'h_1;h_2}$ is only nonzero for one value of h', namely $h' = (W(a')a, \text{all plus})$, where $h_1 = (W(a)b, \sigma)$ and $h_2 = (W(a')b, \sigma')$.

For this value of h', $\tilde{\tilde{c}}_{h'h_1;h_2}$ is 1. Thus,

$$m_2(X_{x_i\cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)})) = \sum_j \tilde{c}_{i,j;h'} X_{x_j\cdot h_2}. \tag{6-1}$$

If $m(b^*_{*;m(h_1),m(h_2)}) = m(b^*_{C;m(h_1),m(h_2)})$, then $\tilde{\tilde{c}}_{h'h_1;h_2}$ will be nonzero for any h'_α which equals $(W(a)a,$ minus on $\alpha)$, where $h_1 = (W(a)b, \sigma)$ and α is any arc of a which is part of the circle C in $W(a)a$. For h' equal to one of the h'_α, $\tilde{\tilde{c}}_{h'h_1;h_2}$ is 1, and for all other h', $\tilde{\tilde{c}}_{h'h_1;h_2}$ is zero. Thus,

$$m_2(X_{x_i\cdot h_1}, m(b^*_{C;m(h_1),m(h_2)})) = \sum_j \sum_{\text{left arcs } \alpha \text{ of } C} \tilde{c}_{i,j;h'_\alpha} X_{x_j\cdot h_2}. \tag{6-2}$$

Note that m_2 is bigrading-preserving; this follows from the degree conditions on x_i and x_j in the basis expansion of $d(x_i \cdot h_1)$ given in Section 6.1 above.

We then extend m_2 to an action of $m(\mathcal{B})^!$ on $\widehat{A}(M)$ by imposing the associativity relation

$$m_2 \circ (\mathrm{id} \otimes \mu_2) := m_2 \circ (m_2 \otimes \mathrm{id}),$$

where μ_2 is the algebra multiplication on $m(\mathcal{B})^!$. Below we will verify that this algebra action is well-defined. Finally, $\widehat{A}(M)$ has a differential m_1 given by

$$m_1(X_{x_i\cdot h_1}) = \sum_j c_{i,j} X_{x_j\cdot h_1}.$$

6.2.2 Proposition *The action of $m(\mathcal{B})^!$ on $\widehat{A}(M)$ is well-defined and associative. Thus, $\widehat{A}(M)$ is a right module over $m(\mathcal{B})^!$.*

Proof The action is associative by definition, once we show that it is well-defined. We may write $\mathcal{B}^!$ as $T(V^*_{\mathcal{B}})/I^\perp$; thus,

$$m(\mathcal{B})^! = T(m(V^*_{\mathcal{B}}))/m(\mathcal{I}^\perp),$$

where the mirrors of the $\mathcal{I}_\beta$–bimodules $V^*_{\mathcal{B}}$ and $\mathcal{I}^\perp$ are defined as in Definition 5.2.4. Now, Definition 6.2.1, extended by associativity, gives us a map

$$\widehat{A}(M) \otimes_{\mathcal{I}_\beta} T(m(V^*_{\mathcal{B}})) \to \widehat{A}(M).$$

We want to show that if $m(r^*)$ is a generator of $m(I^\perp)$, then multiplying any $X_{x_i\cdot h_1}$ by $m(r^*)$ gives zero.

The generators $m(r^*)$ of $m(I^\perp)$ are quadratic in the $m(b^*_{*;m(h_1),m(h_2)})$ and they have intrinsic degree either 1, $\frac{3}{2}$ or 2. For those $m(r^*)$ of intrinsic degree 2, the equations in item (5) of Proposition 6.1.2 above imply that $m(r^*)$ acts as zero on any $X_{x_i\cdot h_1}$.

For those $m(r^*)$ of intrinsic degree $\frac{3}{2}$, the equations in item (4) of Proposition 6.1.2 similarly imply that $m(r^*)$ acts as zero on $\widehat{A}(M)$.

The generators $m(r^*)$ of $m(I^{\perp})$ which have intrinsic degree 1 are sums of either one, two, three or four terms $m(b^*_{\gamma})m(b^*_{\gamma'})$ with all coefficients $+1$. For a fixed $m(r^*)$, let $m(h_1) \in \mathcal{I}_\beta$ denote its left idempotent and let $m(h_3) \in \mathcal{I}_\beta$ denote its right idempotent. The element h_3 of β has degree 2 more than h_1, as elements of H^n with its intrinsic grading, and h_3 differs from h_1 by two surgeries on its left crossingless matching. In particular, the left crossingless matchings of h_1 and h_3 are different; this follows from inspection of the generators $m(r^*)$ of intrinsic degree 1 which actually appear in $m(I^{\perp})$. Monomials of the form $m(b^*_{\gamma})m(b^*_{\gamma^\dagger})$ do not appear as terms of these generators.

For any generators of $\widehat{A}(M)$ of the form $X_{x_i \cdot h_1}$ and $X_{x_k \cdot h_3}$, where h_1 and h_3 are as above, with $\deg x_k = \deg x_i + (-2, 2)$ as elements of M, the equations from item (3) of Proposition 6.1.2 become

$$\sum_{\substack{j, h' \in \beta_\gamma, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h' h_1; h_2} \tilde{c}_{j,k;h''} \tilde{\tilde{c}}_{h'' h_2; h_3} = 0;$$

the terms involving $h' \in \beta_\alpha$ vanish for these choices of h_1 and h_3. These equations imply that all generators $m(r^*)$ of $m(I^{\perp})$ of intrinsic degree 1 act as zero on $\widehat{A}(M)$. Thus, the algebra action m_2 of $m(\mathcal{B})^!$ on $\widehat{A}(M)$ is well-defined. □

6.2.3 Proposition *The differential m_1 on $\widehat{A}(M)$ satisfies $m_1^2 = 0$, and the Leibniz rule*

$$m_1 \circ m_2 = m_2 \circ (m_1 \otimes |\mathrm{id}|) + m_2 \circ (\mathrm{id} \otimes \mu_1)$$

is satisfied, where μ_1 is the differential on $m(\mathcal{B})^!$. Thus, $\widehat{A}(M)$ is a differential bigraded right module over $m(\mathcal{B})^!$ and hence a type A structure over $m(\mathcal{B})^!$.

Proof First, $m_1^2 = 0$ by the equations in item (1) of Proposition 6.1.2.

We want to show that the Leibniz rule is satisfied for $\widehat{A}(M)$. Since the action of $m(\mathcal{B})^!$ on $\widehat{A}(M)$ is associative, and μ_1 satisfies its own Leibniz rule, it suffices to show that

$$m_1 \circ m_2(X_{x_i \cdot h_1}, m(b^*_{\gamma; m(h_1), m(h_3)})) = -m_2 \circ (m_1(X_{x_i \cdot h_1}) \otimes m(b^*_{\gamma; m(h_1), m(h_3)}))$$

and

$$\begin{aligned} &m_1 \circ m_2(X_{x_i \cdot h_1}, m(b^*_{C; m(h_1), m(h_3)})) \\ &= -m_2 \circ (m_1(X_{x_i \cdot h_1}) \otimes m(b^*_{C; m(h_1), m(h_3)})) + m_2(X_{x_i \cdot h_1} \otimes \mu_1(m(b^*_{C; m(h_1), m(h_3)}))). \end{aligned}$$

Note that in the relation involving $m(b^*_\gamma)$, the μ_1 term vanishes.

The first of these two equations follows from item (2) of Proposition 6.1.2. For the second equation, note that for a fixed $h_1 \in \beta$, the only h_3 such that $m(b^*_{C;m(h_1),m(h_3)})$ is a generator of $m(\mathcal{B})^!$ are those $h_3 \in \beta$ which differ only from h_1 by changing the sign of one circle C from plus to minus. We have

$$\mu_1(m(b^*_{C;m(h_1),m(h_3)})) = -\sum_{h_2\in\beta} m(b^*_{\gamma;m(h_1),m(h_2)})m(b^*_{\gamma^\dagger;m(h_2),m(h_3)}),$$

where the sum is implicitly over those $h_2 \in \beta$ such that the generators $m(b^*_{\gamma;m(h_1),m(h_2)})$ and $m(b^*_{\gamma^\dagger;m(h_2),m(h_3)})$ exist.

For such h_1 and h_3, consider generators $X_{x_i\cdot h_1}$ and $X_{x_k\cdot h_3}$ such that $\deg x_k = \deg x_i + (-2,2)$; these are the only $X_{x_k\cdot h_3}$ which may appear in the basis expansion of the left or right side of the second equation above. Applying the equations in item (3) of Proposition 6.1.2 to $X_{x_i\cdot h}$ and $X_{x_k\cdot h_3}$, we see that the second equation above holds. Thus, the Leibniz rule on $\widehat{A}(M)_{m(\mathcal{B})^!}$ is satisfied. □

Now we formally add actions of $\mathcal{B}$ to $\widehat{A}(M)$, to make it a type A structure over $m(\mathcal{B})^! \odot \mathcal{B}$ rather than just over $m(\mathcal{B})^!$.

6.2.4 Definition The type A operation m_2 on $\widehat{A}(M)^{m(\mathcal{B})^!\odot\mathcal{B}}$ is defined as in Definition 6.2.1 on the generators of $m(\mathcal{B})^!$. On the generators of $\mathcal{B}$, it is defined by

$$m_2(X_{x_i\cdot h_1}, b_{\gamma;h_1,h_2}) := X_{x_i\cdot h_2} \quad \text{and} \quad m_2(X_{x_i\cdot h_1}, b_{C;h_1,h_2}) := X_{x_i\cdot h_2}.$$

Note that these actions are bigrading-preserving. To define the action of an arbitrary element of $m(\mathcal{B})^! \odot \mathcal{B}$ on $\widehat{A}(M)$, we impose associativity of the action. Below we check that this definition respects the relations on $m(\mathcal{B})^! \odot \mathcal{B}$.

6.2.5 Proposition *The action m_2 of $m(\mathcal{B})^! \odot \mathcal{B}$ on $\widehat{A}(M)$ is well-defined; with this action and the differential $m_1\colon \widehat{A}(M) \to \widehat{A}(M)$ from Definition 6.2.1, $\widehat{A}(M)$ is a differential bigraded module (hence type A structure) over $m(\mathcal{B})^! \odot \mathcal{B}$.*

Proof First we need to check that

$$m_2\colon \widehat{A}(M) \otimes_{\mathcal{I}_\beta} (m(\mathcal{B})^! \odot \mathcal{B}) \to \widehat{A}(M)$$

is well-defined. Recall that the relation ideal J_{full} of $m(\mathcal{B})^! \odot \mathcal{B}$ was defined to be

$$J_{\text{full}} := T(V_{\text{full}}) \cdot \Big(\big(J_{\mathcal{B}} \cap (T^1(V_{\mathcal{B}}) \oplus T^2(V_{\mathcal{B}}))\big) \\ \oplus \big(J_{m(\mathcal{B})^!} \cap (T^1(V_{m(\mathcal{B})^!}) \oplus T^2(V_{m(\mathcal{B})^!}))\big) \oplus J_{\text{extra}}\Big) \cdot T(V_{\text{full}}).$$

By Proposition 6.2.2, the generators of $J_{m(\mathcal{B})^!} \cap (T^1(V_{m(\mathcal{B})^!}) \oplus T^2(V_{m(\mathcal{B})^!}))$ act as zero on $\widehat{A}(M)$. It is immediate from the definition of the action of $\mathcal{B}$ on $\widehat{A}(M)$ that the generators of $J_{\mathcal{B}} \cap (T^1(V_{\mathcal{B}}) \oplus T^2(V_{\mathcal{B}}))$ act as zero on $\widehat{A}(M)$.

Thus, to show that m_2 is well-defined, it remains to show that the generators of J_{extra} act as zero on $\widehat{A}(M)$. These generators are listed in items (1)–(5) of Definition 5.3.2.

- Consider a relation

$$b_{\gamma;h_1,h_2} m(b^*_{\eta';m(h_2),m(h_3)}) - m(b^*_{\eta;m(h_1),m(\tilde{h}_2)}) b_{\gamma';\tilde{h}_2,h_3}$$

from item (1) of Definition 5.3.2. Write $h_1 = (W(a_1)b_1, \sigma_1)$ and let $X_{x_i \cdot h_1}$ be a generator of $\widehat{A}(M)$. Multiplying $X_{x_i \cdot h_1}$ by $b_{\gamma;h_1,h_2}$, we get $X_{x_i \cdot h_2}$ where $h_2 = (W(a_1)b_2, \sigma_2)$. Multiplying $X_{x_i \cdot h_2}$ by $m(b^*_{\eta';m(h_2),m(h_3)})$, with $h_3 = (W(a_2)b_2, \sigma_3)$, by (6-1) we get

$$\sum_j \tilde{c}_{i,j;h'} X_{x_j \cdot h_3},$$

where $h' \in \beta_{\text{mult}}$ is $(W(a_2)a_1, \text{all plus})$.

On the other hand, if we first multiply $X_{x_i \cdot h_1}$ by $m(b^*_{\eta;m(h_1),m(\tilde{h}_2)})$, by (6-1) we get

$$\sum_j \tilde{c}_{i,j;h'} X_{x_j \cdot \tilde{h}_2},$$

where h' is also $(W(a_2)a_1, \text{all plus})$.

If we multiply this result by $b_{\gamma';\tilde{h}_2,h_3}$, we get

$$\sum_j \tilde{c}_{i,j;h'} X_{x_j \cdot h_3}.$$

Thus, generators of J_{extra} from item (1) of Definition 5.3.2 act as zero on $\widehat{A}(M)$.

- For relations

$$b_{C;h_1,h_2} m(b^*_{\eta;m(h_2),m(h_3)}) - m(b^*_{\eta;m(h_1),m(\tilde{h}_2)}) b_{C;\tilde{h}_2,h_3}$$

from item (2) of Definition 5.3.2, the argument is essentially the same. If $h_1 = (W(a_1)b_1, \sigma_1)$ and $\tilde{h}_2 = (W(a_2)b_1, \sigma_2')$, then h' is again $(W(a_2)a_1, \text{ all plus })$. We still use (6-1).

- Consider a relation

$$b_{\gamma;h_1,h_2} m(b^*_{C;m(h_2),m(h_3)}) - m(b^*_{C;m(h_1),m(\tilde{h}_2)}) b_{\gamma;\tilde{h}_2,h_3}$$

from item (3) of Definition 5.3.2. Let $X_{x_i\cdot h_1}$ be a generator of $\widehat{A}(M)$ and write $h_1 = (W(a_1)b_1, \sigma_1)$. Multiplying $X_{x_i\cdot h_1}$ by $b_{\gamma;h_1,h_2}$, we get $X_{x_i\cdot h_2}$ where $h_2 = (W(a_1)b_2, \sigma_2)$. Multiplying $X_{x_i\cdot h_2}$ by $m(b^*_{C;m(h_2),m(h_3)})$, with $h_3 = (W(a_1)b_2, \sigma_3)$, by (6-2) we get

$$\sum_j \sum_{\text{left arcs } \alpha \text{ of } C} \tilde{c}_{i,j;h'_\alpha} X_{x_j\cdot h_3},$$

where h'_α is $(W(a_1)a_1, \text{ minus on } \alpha)$.

On the other hand, if we first multiply $X_{x_i\cdot h_1}$ by $m(b^*_{C;m(h_1),m(\tilde{h}_2)})$, by (6-2) we get

$$\sum_j \sum_{\text{left arcs } \alpha \text{ of } C} \tilde{c}_{i,j;h'_\alpha} X_{x_j\cdot \tilde{h}_2},$$

where h'_α is again $(W(a_1)a_1, \text{ minus on } \alpha)$.

If we multiply this result by $b_{\gamma;\tilde{h}_2,h_3}$, we get

$$\sum_j \sum_{\text{left arcs } \alpha \text{ of } C} \tilde{c}_{i,j;h'_\alpha} X_{x_j\cdot h_3}.$$

Thus, generators of J_{extra} from item (3) of Definition 5.3.2 act as zero on $\widehat{A}(M)$.

- For relations

$$b_{C;h_1,h_2} m(b^*_{C';m(h_2),m(h_3)}) - m(b^*_{C';m(h_1),m(\tilde{h}_2)}) b_{C;\tilde{h}_2,h_3}$$

from item (4) of Definition 5.3.2, the argument is the same as for relations from item (3).

- Finally, consider a relation

$$b_{\gamma;h_1,h_2} m(b^*_{C;m(h_2),m(h_3)}) - m(b^*_{C';m(h_1),m(h'_2)}) b_{\gamma;h'_2,h_3} - m(b^*_{C'';m(h_1),m(\tilde{\tilde{h}}_2)}) b_{\gamma;\tilde{\tilde{h}}_2,h_3}$$

from item (5) of Definition 5.3.2, in the case where γ joins two circles C' and C'' to produce C. Write $h_1 = (W(a_1)b_1, \sigma_1)$ and let $X_{x_i\cdot h_1}$ be a generator of $\widehat{A}(M)$. Multiplying $X_{x_i\cdot h_1}$ by $b_{\gamma;h_1,h_2}$, we get $X_{x_i\cdot h_2}$ where $h_2 = (W(a_1)b_2, \sigma_2)$. Multiplying $X_{x_i\cdot h_2}$ by $m(b^*_{C;m(h_2),m(h_3)})$, with $h_3 = (W(a_1)b_2, \sigma_3)$, by (6-2) we get

$$\sum_j \sum_{\text{left arcs } \alpha \text{ of } C} \tilde{c}_{i,j;h'_\alpha} X_{x_j\cdot h_3},$$

where h'_α is $(W(a_1)a_1, \text{ minus on } \alpha)$.

On the other hand, if we first multiply $X_{x_i\cdot h_1}$ by $m(b^*_{C';m(h_1),m(\tilde{h}_2)})$, we get

$$\sum_j \sum_{\text{left arcs } \alpha \text{ of } C'} \tilde{c}_{i,j;h'_\alpha} X_{x_j\cdot \tilde{h}_2},$$

where h'_α is again $(W(a_1)a_1,$ minus on $\alpha)$. Multiplying by $b_{\gamma';\tilde{h}_2,h_3}$, we get

$$\sum_j \sum_{\text{left arcs } \alpha \text{ of } C'} \tilde{c}_{i,j;h'_\alpha} X_{x_j \cdot h_3}.$$

Finally, if we multiply $X_{x_i \cdot h_1}$ by $m(b^*_{C'';m(h_1),m(\tilde{\tilde{h}}_2)})$, we get

$$\sum_j \sum_{\text{left arcs } \alpha \text{ of } C''} \tilde{c}_{i,j;h'_\alpha} X_{x_j \cdot \tilde{\tilde{h}}_2}.$$

Multiplying this result by $b_{\gamma;\tilde{\tilde{h}}_2,h_3}$, we get

$$\sum_j \sum_{\text{left arcs } \alpha \text{ of } C''} \tilde{c}_{i,j;h'_\alpha} X_{x_j \cdot h_3}.$$

Now, since γ was assumed to join the circles C' and C'' to produce C via a bridge on the right side of $\{0\} \times \mathbb{R}$, the set of left arcs α of C is the disjoint union of the sets of left arcs of C' and C''. Thus, the relation

$$b_{\gamma;h_1,h_2} m(b^*_{C;m(h_2),m(h_3)}) - m(b^*_{C';m(h_1),m(\tilde{h}_2)}) b_{\gamma;\tilde{h}_2,h_3} - m(b^*_{C'';m(h_1),m(\tilde{\tilde{h}}_2)}) b_{\gamma;\tilde{\tilde{h}}_2,h_3}$$

acts as zero on $\widehat{A}(M)$. For relations of the form

$$b_{\gamma;h_1,\tilde{h}_2} m(b^*_{C';m(\tilde{h}_2),m(h_3)}) + b_{\gamma;h_1,\tilde{\tilde{h}}_2} m(b^*_{C'';m(\tilde{\tilde{h}}_2),m(h_3)}) - m(b^*_{C;m(h_1),m(h_2)}) b_{\gamma;h_2,h_3},$$

where γ splits C into C' and C'', the argument is analogous. Thus, generators of J_{extra} from item (5) of Definition 5.3.2 act as zero on $\widehat{A}(M)$.

At this point, we have shown that the action m_2 of $m(\mathcal{B})^! \odot \mathcal{B}$ on $\widehat{A}(M)$ is well-defined. It is associative by definition. To show that the Leibniz rule is satisfied, it suffices by associativity to check it on the generators of $\mathcal{B}$ and of $m(\mathcal{B})^!$, and we have already done this for the generators of $m(\mathcal{B})^!$ in Proposition 6.2.2.

Let $X_{x_i \cdot h_1}$ be a generator of $\widehat{A}(M)$ and let $b_{\gamma;h_1,h_2}$ be a generator of $\mathcal{B}$. Then

$$m_1 \circ m_2(X_{x_i \cdot h_1}, b_{\gamma;h_1,h_2}) = m_1(X_{x_i \cdot h_2}) = \sum_j c_{i,j} X_{x_j \cdot h_2},$$

while

$$m_2(m_1 \otimes |\mathrm{id}|)(X_{x_i \cdot h_1}, b_{\gamma;h_1,h_2}) = m_2\left(\sum_j c_{i,j} X_{x_j \cdot h_1}, b_{\gamma;h_1,h_2}\right) = \sum_j c_{i,j} X_{x_j \cdot h_2},$$

and the $m_2 \circ (\mathrm{id} \otimes \mu_1)$ term is zero because $\mu_1 = 0$ on $\mathcal{B}$. The argument is unchanged for generators $b_{C;h_1,h_2}$. Thus, the Leibniz rule

$$m_1 \circ m_2 = m_2 \circ (m_1 \otimes |\mathrm{id}|) + m_2 \circ (\mathrm{id} \otimes \mu_1)$$

holds and $\widehat{A}(M)$ is a differential bigraded right module over $m(\mathcal{B})^! \odot \mathcal{B}$. □

Now we will consider the case $M = [T]^{\mathrm{Kh}}$, where T is a tangle diagram in $\mathbb{R}_{\leq 0} \times \mathbb{R}$. Recall from Section 6.1 that $[T]^{\mathrm{Kh}}$ satisfies C_{module}, so we get a type A structure $\widehat{A}([T]^{\mathrm{Kh}})_{m(\mathcal{B})^! \odot \mathcal{B}}$. This type A structure descends to a type A structure over the quotient algebra $\mathcal{B}\Gamma_n$.

6.2.6 Proposition *The extra relations from Proposition 5.2.8 act as zero on the $m(\mathcal{B})^! \odot \mathcal{B}$–module $\widehat{A}([T]^{\mathrm{Kh}})$ defined above in Definition 6.2.4. Thus, $\widehat{A}([T]^{\mathrm{Kh}})$ descends to a differential bigraded right module over the quotient algebra $\mathcal{B}\Gamma_n$ of $m(\mathcal{B})^! \odot \mathcal{B}$ by these relations.*

Proof Since the relations from Proposition 5.2.8 involve only quadratic monomials in the generators $m(b^*_{\gamma;m(h_1),m(h_2)})$ of $m(\mathcal{B})^!$, with no generators from $\mathcal{B}$ appearing, it suffices to show that these relations act as zero on the $m(\mathcal{B})^!$–module $\widehat{A}([T]^{\mathrm{Kh}})$ defined in Definition 6.2.1.

Consider a tetrahedron in the graph G of Proposition 5.2.8, with vertices a, b, c and d as labeled in that proposition. We will show that the relation term $a + c$ acts as zero on $\widehat{A}([T]^{\mathrm{Kh}})$; the proofs for the remaining extra relation terms are exactly analogous.

We may write out

$$a = m(b^*_{\gamma;m(h_1),m(h_2)})m(b^*_{\eta';m(h_2),m(h_3)}),$$
$$c = m(b^*_{\eta;m(h_1),m(\tilde{\tilde{h}}_2)})m(b^*_{\gamma';m(\tilde{\tilde{h}}_2),m(h_3)}),$$

as in Proposition 5.2.8. Suppose we have two generators of $\widehat{A}([T]^{\mathrm{Kh}})$ of the form $X_{x_{00}\cdot h_1}$ and $X_{x_{11}\cdot h_3}$. Here, T may have more than two crossings, but for two designated crossings, x_{00} has the zero-resolution at both and x_{11} has the one-resolution at both (and x_{00} and x_{11} agree at all other crossings). We assume that changing x_{00} to x_{10} has the effect of surgery on γ, while changing x_{00} to x_{01} has the effect of surgery on η.

To show that $a + c$ acts as zero on $\widehat{A}([T]^{\mathrm{Kh}})$, it suffices to show $m_2(X_{x_{00}\cdot h_1}, a + c)$ has zero coefficient on $X_{x_{11}\cdot h_3}$. We can compute the coefficient of $m_2(X_{x_{00}\cdot h_1}, a)$

and $m_2(X_{x_{00}\cdot h_1}, c)$ on $X_{x_{11}\cdot h_3}$ using associativity and (6-1). We have, ignoring terms which do not contribute to a coefficient on $X_{x_{11}\cdot h_3}$,

$$\begin{aligned} m_2(X_{x_{00}\cdot h_1}, a) &= \tilde{c}_{00,10;h'} m_2(X_{x_{10}\cdot h_2}, m(b^*_{\eta';m(h_2),m(h_3)})) \\ &= \tilde{c}_{00,10;h'}\tilde{c}_{10,11;h''} X_{x_{11}\cdot h_3} \end{aligned}$$

for two uniquely determined elements h' and h'' of β_γ. Similarly, the coefficient of $m_2(X_{x_{00}\cdot h_1}, c)$ on $X_{x_{11}\cdot h_3}$ is $\tilde{c}_{00,01;h'''}\tilde{c}_{01,11;h''''}$, for two further uniquely defined elements h''' and h'''' of β_γ.

By item (4) of Example 6.1.3, we have the following coefficients:

- $\tilde{c}_{00,10;h'} = (-1)^{\#_1(00,10)}$,
- $\tilde{c}_{10,11;h''} = (-1)^{\#_1(10,11)}$,
- $\tilde{c}_{00,01;h'''} = (-1)^{\#_1(00,01)}$,
- $\tilde{c}_{01,11;h''''} = (-1)^{\#_1(01,11)}$.

Recall that $\#_1(i, j)$ denotes the number of 1–resolutions of crossings in x_i among those crossings which occur earlier than the changed crossing (going from x_i to x_j) in the ordering on crossings of T (which is implicitly assumed, as usual, to be part of the choice of T).

Since

$$(-1)^{\#_1(00,10)}(-1)^{\#_1(10,11)} + (-1)^{\#_1(00,01)}(-1)^{\#_1(01,11)} = 0,$$

we can conclude that the coefficient of $m_2(X_{x_{00}\cdot h_1}, a + c)$ on $X_{x_{11}\cdot h_3}$ is zero, for all possible pairs $X_{x_{00}\cdot h_1}$ and $X_{x_{11}\cdot h_3}$. Thus, the extra relation terms of Proposition 5.2.8 act as zero on $\widehat{A}([T]^{\mathrm{Kh}})$. □

6.2.7 Proposition *Roberts' type A structure from* [11] *agrees with* $\widehat{A}([T]^{\mathrm{Kh}})$, *the module over* $\mathcal{B}\Gamma_n$ *constructed in Proposition 6.2.6.*

Proof First, $\widehat{A}([T]^{\mathrm{Kh}})$ has the same $\mathbb{Z}$–basis, with the same bigradings and action of the idempotent ring $\mathcal{I}_\beta$, as Roberts' type A structure. We can use the data of Example 6.1.3 to check that the differentials m_1 agree and that the algebra actions m_2 agree under the identification of $\mathcal{B}\Gamma_n$ with a quotient of $m(\mathcal{B})^! \odot \mathcal{B}$.

For the differentials, we have $m_1(X_{x_i\cdot h}) = \sum_j (-1)^{\#_1(i,j)} X_{x_j\cdot h}$, where the sum is over those x_j related to x_i by crossing changes from items (1) or (2) of Example 6.1.3. This formula also gives Roberts' differential $m_1 = d_{\mathrm{APS}}$ as specified in [11, Section 3.3].

It suffices to check that the algebra actions m_2 agree when multiplying by the generators of $\mathcal{B}$ and $m(\mathcal{B})^!$. First, for a generator $X_{x_i\cdot h_1}$ of $\widehat{A}([T]^{\mathrm{Kh}})$ and a generator $b_{\gamma;h_1,h_2}$ of $\mathcal{B}$, we have

$$m_2(X_{x_i\cdot h_1}, b_{\gamma;h_1,h_2}) = X_{x_i\cdot h_2},$$

agreeing with Roberts' definition of the action of $\overrightarrow{e}_{\gamma;h_1,h_2}$ in item 5 of his definition of m_2 [11, Section 4]. Similarly, our action of $b_{C;h_1,h_2}$ is the same as Roberts' action of $\overrightarrow{e}_{C;h_1,h_2}$, defined in item 2 of his definition of m_2.

For a generator $m(b^*_{\gamma;m(h_1),m(h_2)})$ of $m(\mathcal{B})^!$, we have

$$m_2(X_{x_i\cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)})) = \sum_j \tilde{c}_{i,j;h'} X_{x_j\cdot h_2},$$

where if $h_1 = (W(a_1)b_1, \sigma_1)$ and $h_2 = (W(a_2)b_1, \sigma_2)$, then $h' = (W(a_2)a_1, \text{all plus})$, and the coefficient $\tilde{c}_{i,j;h'}$ equals zero or $(-1)^{\#_1(i,j)}$ according to Example 6.1.3(4). Thus,

$$m_2(X_{x_i\cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)})) = \sum_j (-1)^{\#_1(i,j)} X_{x_j\cdot h_2},$$

where the sum is over the subset of j making $\tilde{c}_{i,j;h'}$ nonzero. This algebra action agrees with the action of $\overleftarrow{e}_{\gamma;h_1,h_2}$ as defined in item 4 of Roberts' definition of m_2.

Finally, for a generator $m(b^*_{C;m(h_1),m(h_2)})$ of $m(\mathcal{B})^!$, we have

$$m_2(X_{x_i\cdot h_1}, m(b^*_{C;m(h_1),m(h_2)})) = \sum_j \sum_{\text{left arcs } \alpha \text{ of } C} \tilde{c}_{i,j;h'_\alpha} X_{x_j\cdot h_2},$$

where if $h_1 = (W(a)b, \sigma_1)$ and $h_2 = (W(a)b, \sigma_2)$, then $h'_\alpha = (W(a)a, \text{ minus on } \alpha)$, and the coefficient $\tilde{c}_{i,j;h'}$ equals zero or $(-1)^{\#_1(i,j)}$ according to item (3) above. Thus,

$$m_2(X_{x_i\cdot h_1}, m(b^*_{C;m(h_1),m(h_2)})) = \sum_j (-1)^{\#_1(i,j)} X_{x_j\cdot h_2},$$

where the sum is over the subset of j making some $\tilde{c}_{i,j;h'_\alpha}$ nonzero (note that, given such j, the element h'_α is uniquely determined). This algebra action agrees with the action of $\overleftarrow{e}_{C;h_1,h_2}$ as defined in item 3 of Roberts' definition of m_2. □

6.3 Type D structures

Given a chain complex M of projective graded right H^n–modules satisfying C_{module}, in Definition 6.2.4 we defined a type A structure $\widehat{A}(M)$ over $m(\mathcal{B})^! \odot \mathcal{B}$. For a tangle diagram T, $\widehat{A}([T]^{\mathrm{Kh}})$ descends to a type A structure over $\mathcal{B}\Gamma_n$ by Proposition 6.2.6, which agrees with Roberts' type A structure by Proposition 6.2.7.

Now suppose N is a chain complex of graded projective left H^n–modules. We want to define condition C_{module} for left modules, and for N satisfying this condition, we want to define a type D structure $\widehat{D}(N)$ over $m(\mathcal{B})^! \odot \mathcal{B}$.

To do this, we will first define an operation called *reflection* which, when applied to N, yields a complex of right modules.

6.3.1 Definition Let N be a chain complex of projective graded left H^n–modules. Viewing N as a differential bigraded projective left H^n–module, write

$$N = \bigoplus_i H^n \cdot y_i[j_i, k_i],$$

with H^n acting by left multiplication. Then we may define a differential bigraded projective right H^n–module $r(N)$, called the *reflection* of N, as

$$r(N) := \bigoplus_i r(y_i)[j_i, k_i] \cdot H^n,$$

where the $r(y_i)$ are formal reflections of the y_i, with the same idempotents as the y_i, and H^n acts by right multiplication. Let refl denote the map from N to $r(N)$ such that

$$\text{refl}(h \cdot y_i) = r(y_i) \cdot m(h),$$

where $m(h)$ is defined as in Example 5.2.5. The inverse of refl: $N \to r(N)$ is refl: $r(N) \to r(r(N)) = N$ (using a simple generalization of the above definition which reflects right modules to left modules rather than left modules to right modules). If m_1 denotes the differential on N, then the differential on $r(N)$ is $\text{refl} \circ m_1 \circ \text{refl}$.

6.3.2 Remark Although the geometric content of both Definition 6.3.1 and (variants of) Definition 5.2.4 is just the reflection across the line $\{0\} \times \mathbb{R}$, the algebraic consequences of this reflection are different in Definition 6.3.1. Whereas in Definition 5.2.4, left modules remain left modules and right modules remain right modules under mirroring, in Definition 6.3.1 left modules are sent to right modules and vice-versa.

6.3.3 Definition A chain complex N of projective graded left H^n–modules satisfies the condition C_{module} for a generating set $\{y_i\}$ if and only if $r(N)$ satisfies the condition C_{module} as defined in Definition 6.1.1 for the generating set $\{r(y_i)\}$.

If N satisfies C_{module}, then we can take the box tensor product of $\widehat{A}(r(N))$ and the type DD bimodule ${}^{m(\mathcal{B})^! \odot \mathcal{B}}K^{m(m(\mathcal{B})^! \odot \mathcal{B})^{\text{op}}}$ to get a (left) type D structure over $m(m(\mathcal{B})^! \odot \mathcal{B})$. Below we define this tensor product precisely. It is a slight modification of Definition 2.3.5; we will not give the definition in the fullest possible generality. See [7, Definition 2.3.9] for a more general definition using $\mathbb{Z}/2\mathbb{Z}$ coefficients.

6.3.4 Definition Let $\mathcal{B}$ be a differential bigraded algebra over an idempotent ring $\mathcal{I}$. Let $\widehat{A}$ be a differential bigraded right module over $\mathcal{B}$. Assume $\widehat{A}$ is free as a $\mathbb{Z}$–module, with a $\mathbb{Z}$–basis consisting of elements which are grading-homogeneous and have a unique right idempotent.

Let $\mathcal{B}'$ be another differential bigraded algebra over $\mathcal{I}$ and let $\widehat{DD}$ be a rank-one type DD bimodule over $\mathcal{B}$ and $\mathcal{B}'$ with DD operation $\delta_{DD}\colon \mathcal{I} \to \mathcal{B} \otimes_{\mathcal{I}} (\mathcal{B}')^{\mathrm{op}}$. The type D structure $\widehat{A} \boxtimes \widehat{DD}$ over $\mathcal{B}'$, as a $\mathbb{Z}$–module, is

$$\widehat{A} \boxtimes \widehat{DD} := \widehat{A} \otimes_{\mathcal{I}} \widehat{DD} = \widehat{A} \otimes_{\mathcal{I}} \mathcal{I} = \widehat{A}.$$

The idempotent ring $\mathcal{I}$ has a right action on $\widehat{A}$, which we will view instead as a left action (since $\mathcal{I}$ is commutative, we may view right actions as left actions and vice-versa). Since $\widehat{DD}$ is a rank-one DD bimodule, the left and right actions of $\mathcal{I}$ on $\widehat{DD}$ are the same. There is a bigrading on $\widehat{A} \boxtimes \widehat{DD}$ inherited from that on $\widehat{A}$ (recall that $\widehat{DD}$ is contained in bigrading $(0,0)$).

The type D operation $\delta^{\boxtimes}\colon \widehat{A} \boxtimes \widehat{DD} \to \mathcal{B}' \otimes_{\mathcal{I}} (\widehat{A} \boxtimes \widehat{DD})$ is defined by

$$\delta^{\boxtimes} := 1 \otimes m_1 + \xi \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD})\colon \widehat{A} \to \mathcal{B}' \otimes_{\mathcal{I}} \widehat{A},$$

where m_1 and m_2 are the type A operations on $\widehat{A}$, and $\xi\colon \widehat{A} \otimes_{\mathcal{I}} (\mathcal{B}')^{\mathrm{op}} \to \mathcal{B}' \otimes_{\mathcal{I}} \widehat{A}$ is defined by

$$\xi(X \otimes (b')^{\mathrm{op}}) := (-1)^{(\deg_h X)(\deg_h b')} b' \otimes X.$$

More precisely, the second summand is the composition

$$\widehat{A} \xrightarrow{\mathrm{id}\otimes\delta_{DD}} \widehat{A} \otimes_{\mathcal{I}} \mathcal{B} \otimes_{\mathcal{I}} (\mathcal{B}')^{\mathrm{op}} \xrightarrow{m_2\otimes\mathrm{id}} \widehat{A} \otimes_{\mathcal{I}} (\mathcal{B}')^{\mathrm{op}} \xrightarrow{\xi} \mathcal{B}' \otimes_{\mathcal{I}} \widehat{A}.$$

The map $\delta^{\boxtimes}$ has bidegree $(0,+1)$.

6.3.5 Proposition *$(\widehat{A} \boxtimes \widehat{DD}, \delta^{\boxtimes})$ is a well-defined type D structure over $\mathcal{B}'$.*

Proof First, since $\widehat{A}$ was assumed to have a $\mathbb{Z}$–basis $\{X_i\}$, with each X_i grading-homogeneous and having unique idempotents, the same is true for $\widehat{A} \boxtimes \widehat{DD} \cong \widehat{A}$.

To verify the type D structure relations, we must show that

$$(\mu_1 \otimes |\mathrm{id}|) \circ \delta^{\boxtimes} + (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta^{\boxtimes}) \circ \delta^{\boxtimes} = 0.$$

Substituting in the definition of $\delta^{\boxtimes}$ and simplifying some terms, we want to show that

$$\begin{aligned}(6\text{-}3)\quad &(\mu_1\otimes|\mathrm{id}|)\circ\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})\\ &+(\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\xi)\circ(\mathrm{id}\otimes m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\delta_{DD})\circ(1\otimes m_1)\\ &+(\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes 1\otimes m_1)\circ\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})\\ &+(\mu_2\otimes\mathrm{id})\circ\big(\mathrm{id}\otimes(\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD}))\big)\\ &\qquad\circ(\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD}))=0.\end{aligned}$$

We claim that we may rewrite the final term on the left side of (6-3) as

$$(6\text{-}4)\quad \xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_2\otimes\mu_2)\circ(\mathrm{id}\otimes\sigma)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\delta_{DD}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD}),$$

where

$$\sigma\colon \mathcal{B}\otimes\mathcal{B}\otimes(\mathcal{B}')^{\mathrm{op}}\otimes(\mathcal{B}')^{\mathrm{op}}\to\mathcal{B}\otimes\mathcal{B}\otimes(\mathcal{B}')^{\mathrm{op}}\otimes(\mathcal{B}')^{\mathrm{op}}$$

was defined in Definition 4.5.1.

To verify that term (6-4) is equal to the final term of (6-3), let X be a generator of $\widehat{A}$. Write $\delta_{DD}(1)=\sum_i b_i\otimes(b_i')^{\mathrm{op}}$. We have

$$\begin{aligned}&(\mu_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes(\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})))\circ(\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD}))(X)\\ &\qquad=\sum_{i,j}(-1)^{(\deg_h b_i')(\deg_h(Xb_i))+(\deg_h b_j')(\deg_h(Xb_ib_j))}(b_i'b_j')\otimes(Xb_ib_j).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_2\otimes\mu_2)\circ(\mathrm{id}\otimes\sigma)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\delta_{DD}\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})(X)\\ &\qquad=\sum_{i,j}(-1)^{\deg_h b_j\,\deg_h(b_i')+\deg_h(Xb_ib_j)\,\deg_h(b_i'b_j')}(b_i'b_j')\otimes(Xb_ib_j).\end{aligned}$$

A direct computation, using the additivity of $\deg_h$ under algebra multiplication, verifies that the signs in these expressions are equal.

Now, we may write term (6-4) as

$$\xi\circ(m_2\otimes\mathrm{id})\circ\big(\mathrm{id}\otimes((\mu_2\otimes\mu_2)\circ\sigma\circ(\mathrm{id}\otimes\delta_{DD}\otimes\mathrm{id})\circ\delta_{DD})\big).$$

Using the type DD bimodule relations for δ, we may replace

$$(\mu_2\otimes\mu_2)\circ\sigma\circ(\mathrm{id}\otimes\delta_{DD}\otimes\mathrm{id})\circ\delta_{DD}$$

with

$$-(\mu_1\otimes|\mathrm{id}|)\circ\delta_{DD}-(\mathrm{id}\otimes\mu_1)\circ\delta_{DD}.$$

Then term (6-4) is equal to

$$-\xi\circ(m_2\otimes\mathrm{id})\circ\big(\mathrm{id}\otimes((\mu_1\otimes|\mathrm{id}|)\circ\delta_{DD})\big)-\xi\circ(m_2\otimes\mathrm{id})\circ\big(\mathrm{id}\otimes((\mathrm{id}\otimes\mu_1)\circ\delta_{DD})\big)$$
$$=-\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_1\otimes|\mathrm{id}|)\circ(\mathrm{id}\otimes\delta_{DD})-\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu_1)\circ(\mathrm{id}\otimes\delta_{DD}).$$

The term

$$-\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu_1)\circ(\mathrm{id}\otimes\delta_{DD})$$

above cancels the first term

$$(\mu_1\otimes|\mathrm{id}|)\circ\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})$$

of the terms in (6-3), whose sum we are trying to show is zero. The remaining terms of (6-3) are, after some simplification,

- $-\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_1\otimes|\mathrm{id}|)\circ(\mathrm{id}\otimes\delta_{DD})$,
- $\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})\circ m_1$,
- $(\mathrm{id}\otimes m_1)\circ\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})$.

The final of these may be written as

$$\xi\circ(m_1\otimes|\mathrm{id}|)\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})=\xi\circ((m_1\circ m_2)\otimes|\mathrm{id}|)\circ(\mathrm{id}\otimes\delta_{DD}).$$

We may use the Leibniz rule on $\widehat{A}$ to replace $m_1\circ m_2$ with $m_2\circ(m_1\otimes|\mathrm{id}|)+m_2\circ(\mathrm{id}\otimes\mu_1)$. Thus, the final of the three remaining terms is equal to

$$\xi\circ(m_2\otimes\mathrm{id})\circ(m_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|)\circ(\mathrm{id}\otimes\delta_{DD})+\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_1\otimes|\mathrm{id}|)\circ(\mathrm{id}\otimes\delta_{DD}).$$

The second of these summands cancels with the first of the other three remaining terms listed above, so it remains to show that

$$\xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD})\circ m_1+\xi\circ(m_2\otimes\mathrm{id})\circ(m_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|)\circ(\mathrm{id}\otimes\delta_{DD})=0.$$

This follows from the equation

$$(m_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|)\circ(\mathrm{id}\otimes\delta_{DD})=-(\mathrm{id}\otimes\delta_{DD})\circ m_1.$$

Indeed, since all generators of $\widehat{DD}\cong\mathcal{I}$ have bigrading $(0,0)$, the element $\delta(1)$ has homological degree 1. □

Applying this construction to $\widehat{A}=\widehat{A}(r(N))$ with $\widehat{DD}={}^{m(\mathcal{B})^!\odot\mathcal{B}}K^{m(m(\mathcal{B})^!\odot\mathcal{B})^{\mathrm{op}}}$, which is a type DD bimodule over $m(\mathcal{B})^!\odot\mathcal{B}$ and $m(m(\mathcal{B})^!\odot\mathcal{B})$, we get a type D structure $\mathcal{A}(r(N))\boxtimes{}^{m(\mathcal{B})^!\odot\mathcal{B}}K^{m(m(\mathcal{B})^!\odot\mathcal{B})^{\mathrm{op}}}$ over $m(m(\mathcal{B})^!\odot\mathcal{B})$. We can then apply another mirroring operation, analogous to Definition 5.3.1 and in the spirit of Definition 5.2.4, to get a type D structure $\widehat{D}(N)$ over $m(\mathcal{B})^!\odot\mathcal{B}$.

6.3.6 Definition Let $\mathcal{B}$ be a differential bigraded algebra over the idempotent ring $\mathcal{I}_\beta$ and let $(\widehat{D}, \delta)$ be a type D structure over $\mathcal{B}$. The *mirrored type D structure* $(m(\widehat{D}), \delta')$ is defined as follows: as an $\mathcal{I}_\beta$–module, $m(\widehat{D})$ is the mirror of $\widehat{D}$ as defined in Definition 5.2.4. As usual, denote the natural map from $\widehat{D}$ to $m(\widehat{D})$ or $m(\widehat{D})$ to $\widehat{D}$ by mirr. The type D operation on $m(\widehat{D})$ is the following:

$$\delta' = m(\widehat{D}) \xrightarrow{\text{mirr}} \widehat{D} \xrightarrow{\delta} \mathcal{B} \otimes \widehat{D} \xrightarrow{\text{mirr} \otimes \text{mirr}} m(\mathcal{B}) \otimes m(\widehat{D}).$$

6.3.7 Definition Let N be a chain complex of graded projective left H^n–modules satisfying the algebraic condition C_{module} of Definition 6.3.3. The type D structure $\widehat{D}(N)$ over $m(\mathcal{B})^! \odot \mathcal{B}$ is defined to be

$$\widehat{D}(N) := m\big(\widehat{A}(r(N)) \boxtimes^{m(\mathcal{B})^! \odot \mathcal{B}} K^{m(m(\mathcal{B})^! \odot \mathcal{B})^{\text{op}}}\big).$$

6.3.8 Definition If N is a chain complex N of graded left projective H^n–modules satisfying C_{module}, the type D structure $\widehat{D}(N)$ over $\mathcal{B}\Gamma_n$ associated to N is induced from the type D structure $\widehat{D}(N)$ over $m(\mathcal{B})^! \odot \mathcal{B}$ defined in Definition 6.3.7, using Proposition 5.3.7.

For convenience, we describe the type D operation δ on $\widehat{D}(N)$ explicitly from the differential d_N on N. Let $\{h_1 \cdot y_i \mid e(y_i) = e_R(h_1)\}$ be the $\mathbb{Z}$–basis for N corresponding to the designated generators $\{y_i\}$ of N. By the condition C_{module}, we may expand $d_N(y_i)$ in this basis as

$$d_N(y_i) = \sum_j c'_{i,j}\, y_j + \sum_{j, h' \in \beta_{\text{mult}}} \tilde{c}'_{i,j;h'} h' \cdot y_j.$$

Then we have

$$d_N(h_1 \cdot y_i) = \sum_j c'_{i,j} h_1 \cdot y_j + \sum_{j, h' \in \beta_{\text{mult}}, h_2 \in \beta} \tilde{c}'_{i,j;h'} \tilde{\tilde{c}}_{h_1 h';h_2} h_2 \cdot y_j.$$

We let

$$\{Y_{h_1 \cdot y_i} \mid e(y_i) = e_R(h_1)\}$$

denote the $\mathbb{Z}$–basis of $\widehat{D}(N)$ corresponding to the $\mathbb{Z}$–basis $\{h_1 \cdot y_i\}$ of N.

6.3.9 Proposition *Defining the coefficients $c'_{i,j}$ and $\tilde{c}'_{i,j;h'}$ as above, and $\tilde{\tilde{c}}_{h_1 h';h_2}$ as in Section 6.1, the type D structure operation δ on $\widehat{D}(N)$ has a basis expansion given by*

$$\delta(Y_{h_1\cdot y_i}) = \sum_j c'_{i,j} Y_{h_1\cdot y_j} + \sum_{j,h'\in\beta_{\mathrm{mult}},h_2\in\beta} \tilde{c}'_{i,j;h'}\tilde{\tilde{c}}_{h_1h';h_2} b_{*;h_1,h_2}\otimes Y_{h_2\cdot y_j} + \sum_{\substack{h_2\in\beta \text{ such that}\\ m(b^*_{*;m(h),m(h_2)}) \text{ exists}}} (-1)^{\deg_h y_i} m(b^*_{*;m(h_1),m(h_2)})\otimes Y_{h_2\cdot y_i},$$

where $b_{*;h_1,h_2}$ means $b_{\gamma;h_1,h_2}$ or $b_{C;h_1,h_2}$ as appropriate.

Proof The operation δ is defined as the mirror of the type D structure operation $\delta^{\boxtimes}$ on $\widehat{A}(r(N))\boxtimes {}^{m(\mathcal{B})^!\odot\mathcal{B}}K^{m(m(\mathcal{B})^!\odot\mathcal{B})^{\mathrm{op}}}$, which in turn is defined as

$$\delta^{\boxtimes} := 1\otimes m_1 + \xi\circ(m_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\delta_{DD}).$$

Here, m_1 is the differential on $\widehat{A}(r(N))$ and δ_{DD} is the type DD operation on ${}^{m(\mathcal{B})^!\odot\mathcal{B}}K^{m(m(\mathcal{B})^!\odot\mathcal{B})^{\mathrm{op}}}$. Write an arbitrary generator of $\widehat{A}(r(N))$ as $X_{m(y_i)\cdot m(h_1)}$, and we have

$$m_1(X_{m(y_i)\cdot m(h_1)}) = \sum_j c'_{i,j} X_{m(y_j)\cdot m(h_1)},$$

$$m_2(X_{m(y_i)\cdot m(h_1)}, m(b^*_{*;h_1,h_2})) = \sum_{j,h'\in\beta_{\mathrm{mult}}} \tilde{c}'_{i,j;h}\tilde{\tilde{c}}_{h_1h';h_2} X_{m(y_j)\cdot m(h_2)},$$

$$m_2(X_{m(y_i)\cdot m(h_1)}, b_{*;m(h_1),m(h_2)}) = X_{m(y_i)\cdot m(h_2)}.$$

Here, $b_{*;m(h_1),m(h_2)}$ stands for either $b_{\gamma;m(h_1),m(h_2)}$ or $b_{C;m(h_1),m(h_2)}$ as appropriate, and similarly for $m(b^*_{*;h_1,h_2})$. Also note that we have $\tilde{\tilde{c}}_{h_1h';h_2} = \tilde{\tilde{c}}_{m(h')m(h_1);m(h_2)}$.

Thus,

$$\delta^{\boxtimes}(X_{m(y_i)\cdot m(h_1)}) = \sum_j c'_{i,j} X_{m(y_j)\cdot m(h_1)} + \sum_{j,h'\in\beta_{\mathrm{mult}},h_2\in\beta} \tilde{c}'_{i,j;h'}\tilde{\tilde{c}}_{h_1h';h_2} m(b_{*;h_1,h_2})\otimes X_{m(y_j)\cdot m(h_2)} + \sum_{\substack{h_2\in\beta \text{ such that}\\ m(b^*_{*;m(h_1),m(h_2)}) \text{ exists}}} (-1)^{\deg_h y_i} m(m(b^*_{*;m(h_1),m(h_2)}))\otimes X_{m(y_i)\cdot m(h_2)}.$$

Taking the mirror of this formula, we get

$$\delta(Y_{h_1\cdot y_i}) = \sum_j c'_{i,j} Y_{h_1\cdot y_j} + \sum_{j,h'\in\beta_{\mathrm{mult}},h_2\in\beta} \tilde{c}'_{i,j;h'}\tilde{\tilde{c}}_{h_1h';h_2} (b_{*;h_1,h_2})\otimes Y_{h_2\cdot y_j} + \sum_{\substack{h_2\in\beta \text{ such that}\\ m(b^*_{*;m(h_1),m(h_2)}) \text{ exists}}} (-1)^{\deg_h y_i} m(b^*_{*;m(h_1),m(h_2)})\otimes Y_{h_2\cdot y_i}. \ \square$$

When N is Khovanov's complex $[T]^{\mathrm{Kh}}$ for a tangle diagram T in $\mathbb{R}_{\geq 0} \times \mathbb{R}$, the induced type D structure $\widehat{D}([T]^{\mathrm{Kh}})$ over $\mathcal{B}\Gamma_n$ is the same as Roberts' type D structure from [12].

6.3.10 Proposition *$\widehat{D}([T]^{\mathrm{Kh}})$, as defined in Definition 6.3.8, agrees with the type D structure over $\mathcal{B}\Gamma_n$ which Roberts associates to T.*

Proof Roberts' type D structure is defined as a bigraded $\mathbb{Z}$–module in Definition 32 of [12]. As such, it agrees with $\widehat{D}([T]^{\mathrm{Kh}})$, and the action of the idempotent ring $\mathcal{I}_\beta$ is the same on both; Roberts defines the action of the idempotent ring at the end of Section 3.2 of [12].

Lastly, the type D operation δ on $\widehat{D}([T]^{\mathrm{Kh}})$ has an explicit form given in Proposition 6.3.9 above. The coefficients $c'_{i,j}$ and $\tilde{c}'_{i,j;h'}$ are either $(-1)^{\#_1(i,j)}$ or zero, just like the coefficients $c_{i,j}$ and $\tilde{c}_{i,j;h'}$ described in Example 6.1.3. Recall that the coefficient $\tilde{\tilde{c}}_{h_1 h';h_2}$ is either one or zero; this was also pointed out in Definition 6.2.1. By comparison, δ agrees with Roberts' type D operation defined at the beginning of [12, Section 5]. □

6.4 Pairing

Let M be a complex of graded projective right H^n–modules and let N be a complex of graded projective left H^n–modules, satisfying the algebraic conditions C_{module} of Definition 6.1.1 and Definition 6.3.3. The natural way to pair M and N and get a chain complex over $\mathbb{Z}$ is to take the tensor product $M \otimes_{H^n} N$. However, we could also use Definition 6.2.4 to construct a type A structure $\widehat{A}(M)$ and use Definition 6.3.7 to construct a type D structure $\widehat{D}(N)$, both over $m(\mathcal{B})^! \odot \mathcal{B}$, and then take their box tensor product. This produces the same chain complex as $M \otimes_{H^n} N$, after a reversal of the intrinsic grading.

6.4.1 Proposition *As differential bigraded $\mathbb{Z}$–modules, $\widehat{A}(M) \boxtimes_{m(\mathcal{B})^! \odot \mathcal{B}} \widehat{D}(N)$ is isomorphic to the module obtained from $M \otimes_{H^n} N$ by multiplying all intrinsic gradings on $M \otimes_{H^n} N$ by -1.*

Proof Let $\{x_i \cdot h_1 \mid e(x_i) = e_L(h_1)\}$ and $\{h_1 \cdot y_i \mid e(y_i) = e_R(h_1)\}$ be the $\mathbb{Z}$–bases for M and N, respectively, corresponding to the sets of designated generators $\{x_i\}$ of M and $\{y_i\}$ of N. Then a $\mathbb{Z}$–basis for $M \otimes_{H^n} N$ is $\{x_i \cdot h_1 \cdot y_j\}$ (we will suppress the idempotent conditions).

Write the differentials on M and N as d_M and d_N. As an element of $M \otimes_{H^n} N$, the differential of $x_i \cdot h_1 \cdot y_j$ is

$$\partial^{\otimes}(x_i \cdot h_1 \cdot y_j) = (-1)^{\deg_h y_j}(d_M(x_i) \cdot h_1 \cdot y_j) + (x_i \cdot h_1 \cdot d_N(y_j)).$$

If we expand out d_M as in Section 6.1 and d_N as in the discussion preceding Proposition 6.3.9, we may write this as

$$(6\text{-}5)\quad (-1)^{\deg_h y_j}\Bigg(\sum_k c_{i,k}(x_k\cdot h_1\cdot y_j)+\sum_{k,h'\in\beta_{\mathrm{mult}},h_2\in\beta}\tilde{c}_{i,k;h'}\tilde{\tilde{c}}_{h'h_1;h_2}(x_k\cdot h_2\cdot y_j)\Bigg)$$
$$+\sum_l c'_{j,l}(x_i\cdot h_1\cdot y_l)+\sum_{l,h'\in\beta_{\mathrm{mult}},h_2\in\beta}\tilde{c}'_{j,l;h'}\tilde{\tilde{c}}_{h_1h';h_2}(x_i\cdot h_2\cdot y_l).$$

Now, as a bigraded $\mathbb{Z}$–module, $\widehat{A}(M)\boxtimes_{m(\mathcal{B})^!\odot\mathcal{B}}\widehat{D}(N)$ is defined as $\widehat{A}(M)\otimes_{\mathcal{I}_\beta}\widehat{D}(N)$. A $\mathbb{Z}$–basis for $\widehat{A}(M)$ (respectively $\widehat{D}(N)$) is given by $\{X_{x_i\cdot h_1}\}$ (respectively $\{Y_{h_1\cdot y_i}\}$). A generator $X_{x_i\cdot h_1}$ of $\widehat{A}(M)$ has the same idempotent in $\mathcal{I}_\beta$ as a generator $Y_{h_2\cdot y_i}$ of $\widehat{D}(N)$ if and only if $h_1=h_2$.

Thus, $\widehat{A}(M)\otimes_{\mathcal{I}_\beta}\widehat{D}(N)$ has a $\mathbb{Z}$–basis consisting of all elements $X_{x_i\cdot h_1}\otimes Y_{h_1\cdot y_j}$, which is in bijection with the basis $\{x_i\cdot h_1\cdot y_j\}$ for $M\otimes_{H^n}N$. The bigradings agree on these two $\mathbb{Z}$–modules after negating the intrinsic gradings on $M\otimes_{H^n}N$: note that for the intrinsic grading on $\widehat{A}(M)\otimes_{\mathcal{I}_\beta}\widehat{D}(N)$, the grading of h_1 in $X_{x_i\cdot h_1}\otimes Y_{h_1\cdot y_j}$ is counted twice with coefficient $-\frac{1}{2}$, while for the intrinsic grading on $M\otimes_{H^n}N$, the grading of h in $x_i\cdot h_1\cdot y_j$ is counted once with coefficient 1. This explains the factor of $\frac{1}{2}$ in Definition 5.1.1.

It remains to show that the differential $\partial^\otimes$ on $M\otimes_{H^n}N$ agrees with the differential $\partial^\boxtimes$ on $\widehat{A}(M)\boxtimes_{m(\mathcal{B})^!\odot\mathcal{B}}\widehat{D}(N)$. We will use m_1 and m_2 to denote the differential and algebra action on $\widehat{A}(M)$ and δ to denote the type D operation on $\widehat{D}(N)$. Applying $\partial^\boxtimes$ to a generator $X_{x_i\cdot h_1}\otimes Y_{h_1\cdot y_j}$, we get

$$(-1)^{\deg_h(Y_{h_1\cdot y_j})}(m_1(X_{x_i\cdot h_1}))\otimes Y_{h_1\cdot y_j}+(m_2\otimes\mathrm{id})\circ(X_{x_i\cdot h_1}\otimes\delta(Y_{h_1\cdot y_j})).$$

Because H^n is concentrated in homological degree zero,

$$\deg_h(Y_{h_1\cdot y_j})=\deg_h(h_1\cdot y_j)=\deg_h(y_j).$$

Thus, the first term of $\partial^\boxtimes(X_{x_i\cdot h_1}\otimes Y_{h_1\cdot y_j})$ is

$$(-1)^{\deg_h y_j}\sum_k c_{i,k}X_{x_k\cdot h_1}\otimes Y_{h_1\cdot y_j},$$

which agrees with the first term of expression (6-5) for $\partial^\otimes(x_i\cdot h_1\cdot y_j)$ under the bijection between basis elements.

By Proposition 6.3.9, the other term of $\partial^{\boxtimes}(X_{x_i\cdot h_1}\otimes Y_{h_1\cdot y_j})$ can be expanded out as

$$(m_2\otimes\mathrm{id})\circ\Bigg(X_{x_i\cdot h_1}\otimes\Bigg(\sum_l c'_{j,l}Y_{h_1\cdot y_l}+\sum_{l,h'\in\beta_{\mathrm{mult}},h_2\in\beta}\tilde{c}'_{j,l;h'}\tilde{\tilde{c}}_{h_1h';h_2}b_{*;h_1,h_2}\otimes Y_{h_2\cdot y_l}\\ +\sum_{\substack{l,h_2\in\beta \text{ such that}\\ m(b^*_{*;m(h_1),m(h_2)}) \text{ exists}}}(-1)^{\deg_h y_j}m(b^*_{*;m(h_1),m(h_2)})\otimes Y_{h_2\cdot y_j}\Bigg)\Bigg),$$

where $b_{*;h_1,h_2}$ denotes either $b_{\gamma;h_1,h_2}$ or $b_{C;h_1,h_2}$ and $m(b^*_{*;m(h_1),m(h_2)})$ denotes either $m(b^*_{\gamma;m(h_1),m(h_2)})$ or $m(b^*_{C;m(h_1),m(h_2)})$. This expansion gives us three remaining terms of $\partial^{\boxtimes}(X_{x_i\cdot h_1}\otimes Y_{h_1\cdot y_j})$. The first of these is

$$\sum_l c'_{j,l}X_{x_i\cdot h_1}\otimes Y_{h_1\cdot y_l},$$

which agrees with the third term of expression (6-5) under the bijection between basis elements. The second is

$$\sum_{l,h'\in\beta_{\mathrm{mult}},h_2\in\beta}\tilde{c}'_{j,l;h'}\tilde{\tilde{c}}_{h_1h';h_2}X_{x_i\cdot h_2}\otimes Y_{h_2\cdot y_l},$$

which agrees with the fourth term of expression (6-5). Finally, the remaining term of $\partial^{\boxtimes}(X_{x_i\cdot h_1}\otimes Y_{h_1\cdot y_j})$ is

$$(-1)^{\deg_h y_j}\sum_{k,h'\in\beta_{\mathrm{mult}},h_2\in\beta}\tilde{c}_{i,k;h'}\tilde{\tilde{c}}_{h'h_1;h_2}X_{x_k\cdot h_2}\otimes Y_{h_2\cdot y_j},$$

which agrees with the second term of expression (6-5). Thus, after reversing the intrinsic gradings on $M\otimes_{H^n}N$, we conclude that $M\otimes_{H^n}N$ is isomorphic to $\widehat{A}(M)\otimes_{m(\mathcal{B})^!\odot\mathcal{B}}\widehat{D}(N)$ as differential bigraded $\mathbb{Z}$–modules. □

6.4.2 Remark The negation of the intrinsic gradings on $M\otimes_{H^n}N$ is done for the same reason as in Remark 3.0.3.

6.4.3 Proposition *Let $\mathcal{B}$ be a differential bigraded algebra and let J be a bigrading-homogeneous ideal of $\mathcal{B}$ which is preserved by the differential on $\mathcal{B}$. Let $\widehat{D}$ be a type D structure over $\mathcal{B}$ and let $\widehat{A}$ be a differential bigraded right $\mathcal{B}$–module which descends to a module over $\mathcal{B}/J$. By Proposition 5.3.7, $\widehat{D}$ automatically descends to a type D structure over $\mathcal{B}/J$, and we have*

$$\widehat{A}\boxtimes_{\mathcal{B}}\widehat{D}\cong\widehat{A}\boxtimes_{\mathcal{B}/J}\widehat{D}.$$

Proof This follows immediately from Definition 2.3.5. □

6.4.4 Corollary *Let T_1 and T_2 be oriented tangle diagrams in $\mathbb{R}_{\geq 0}\times\mathbb{R}$ and $\mathbb{R}_{\leq 0}\times\mathbb{R}$ respectively, with orderings chosen of the crossings of T_1 and T_2. Assume that T_1 and T_2 have consistent orientations, so that their horizontal concatenation is an oriented link diagram L in $\mathbb{R}^2$. Order the crossings of L such that those of T_1 come before those of T_2. Then*

$$\mathrm{CKh}(L) \cong \widehat{A}([T_2]^{\mathrm{Kh}}) \boxtimes_{m(\mathcal{B})^!\odot\mathcal{B}} \widehat{D}([T_1]^{\mathrm{Kh}}) \cong \widehat{A}([T_2]^{\mathrm{Kh}}) \boxtimes_{\mathcal{B}\Gamma_n} \widehat{D}([T_1]^{\mathrm{Kh}}).$$

Proof This is a corollary of Proposition 6.4.1, Proposition 6.4.3 and Khovanov's results from [4]. □

Identifying $\widehat{A}([T_2]^{\mathrm{Kh}})$ with Roberts' type A structure over $\mathcal{B}\Gamma_n$ by Proposition 6.2.7, and identifying $\widehat{D}([T_1]^{\mathrm{Kh}})$ with Roberts' type D structure over $\mathcal{B}\Gamma_n$ by Proposition 6.3.10, we obtain an alternate proof of Roberts [11, Proposition 36].

6.5 Equivalences of type A structures

We start by defining $\mathcal{A}_\infty$–morphisms. The following definition is general enough for our purposes, although it is not the most general definition possible. A more general definition is given in Roberts [11, Definition 26]; our sign conventions are the same as Roberts'.

6.5.1 Definition Let $\mathcal{B}$ be a differential bigraded algebra with idempotent ring $\mathcal{I}$. Let $\widehat{A}$ and $\widehat{A}'$ be differential bigraded right modules over $\mathcal{B}$ with differentials m_1, m_1' and algebra actions m_2, m_2' respectively. An *$\mathcal{A}_\infty$–morphism* F from $\widehat{A}$ to $\widehat{A}'$ is a collection

$$F_n\colon \widehat{A}\otimes_{\mathcal{I}}\mathcal{B}^{\otimes(n-1)} \to \widehat{A}'[0,n-1]$$

of bigrading-preserving $\mathcal{I}$–linear maps satisfying the compatibility condition

$$\begin{aligned} m_1'\circ F_n + (-1)^n m_2'\circ(F_{n-1}\otimes|\mathrm{id}|^n) &= F_{n-1}\circ(m_2\otimes \mathrm{id}^{\otimes(n-2)}) + (-1)^{n+1}F_n\circ(m_1\otimes|\mathrm{id}|^{\otimes(n-1)}) \\ &\quad + (-1)^{n+1}\sum_{k=1}^{n-1} F_n\circ(\mathrm{id}^{\otimes k}\otimes\mu_1\otimes|\mathrm{id}|^{\otimes(n-k-1)}) \\ &\quad + \sum_{k=1}^{n-2}(-1)^k F_{n-1}\circ(\mathrm{id}^{\otimes k}\otimes\mu_2\otimes|\mathrm{id}|^{\otimes(n-k-2)}) \end{aligned}$$

for all $n\geq 1$. Recall that $|\mathrm{id}|^n$ and $|\mathrm{id}|^{\otimes n}$ mean different things; see Section 2.1.

6.5.2 Remark Lipshitz, Ozsváth and Thurston also require all $\mathcal{A}_\infty$–morphisms to satisfy a unitality condition: all algebra inputs must be in the kernel $\mathcal{B}_+$ of the augmentation of $\mathcal{B}$ for the corresponding $\mathcal{A}_\infty$–morphism term to be nonzero. See [7, Remark 2.2.21]. We will not discuss this condition further because it is satisfied for all $\mathcal{A}_\infty$–morphisms and homotopies that we consider.

6.5.3 Example For an $\mathcal{A}_\infty$–morphism F with only F_1 and F_2 nonzero, the condition of Definition 6.5.1 is nontrivial only for $n = 1$, 2 and 3. The $n = 1$ condition is

$$m_1' \circ F_1 = F_1 \circ m_1,$$

the $n = 2$ condition is

$$m_1' \circ F_2 + m_2' \circ (F_1 \otimes \mathrm{id}) = F_1 \circ m_2 - F_2 \circ (m_1 \otimes |\mathrm{id}|) - F_2 \circ (\mathrm{id} \otimes \mu_1),$$

and the $n = 3$ condition is

$$-m_2' \circ (F_2 \otimes |\mathrm{id}|) = F_2 \circ (m_2 \otimes \mathrm{id}) - F_2 \circ (\mathrm{id} \otimes \mu_2).$$

Let (M, d_M) and $(M', d_{M'})$ be two chain complexes of graded projective right H^n–modules satisfying the algebraic condition C_{module} of Definition 6.1.1 for generating sets $\{x_i\}$ and $\{x_i'\}$ respectively. Let $f\colon M \to M'$ be a bigrading-preserving H^n–linear map such that $d_{M'} f = f d_M$; as shorthand, we will say "let f be a chain map from M to M'". We first show that certain chain maps f induce $\mathcal{A}_\infty$–morphisms of type A structures $\widehat{A}(M) \to \widehat{A}(M')$ over $m(\mathcal{B})^! \odot \mathcal{B}$.

Let $\{x_i \cdot h_1 \mid e(x_i) = e_L(h_1)\}$ be the $\mathbb{Z}$–basis for M corresponding to the generating set $\{x_i\}$ and let $\{x_i' \cdot h_1\}$ be the analogous $\mathbb{Z}$–basis for M' (we will suppress the idempotent conditions). We may expand $f(x_i)$ in the basis for M':

$$f(x_i) = \sum_j f_{i,j} x_j' + \sum_{j,h' \in \beta, \deg h' \neq 0} \tilde{f}_{i,j;h'} x_j' \cdot h'.$$

6.5.4 Definition The chain map f satisfies the algebraic condition C_{morphism} for the generating sets $\{x_i\}$ and $\{x_j'\}$ if $\tilde{f}_{i,j;h'}$ is only nonzero when $h' \in \beta_{\text{mult}}$.

For a chain map f satisfying C_{morphism}, we may write

$$f(x_i) = \sum_j f_{i,j} x_j' + \sum_{j,h' \in \beta_{\text{mult}}} \tilde{f}_{i,j;h'} x_j' \cdot h'.$$

Thus, a basis expansion for $f(x_i \cdot h_1)$ is

$$f(x_i \cdot h_1) = \sum_j f_{i,j} x'_j \cdot h_1 + \sum_{j,h' \in \beta_{\text{mult}}, h_2 \in \beta} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} x'_j \cdot h_2.$$

Since M and M' satisfy C_{module}, we also have

$$d_M(x_i \cdot h_1) = \sum_j c_{i,j}(x_j \cdot h_1) + \sum_{j,h' \in \beta_{\text{mult}}, h_2 \in \beta} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2}(x_j \cdot h_2),$$
$$d_{M'}(x'_i \cdot h_1) = \sum_j c'_{i,j}(x'_j \cdot h_1) + \sum_{j,h' \in \beta_{\text{mult}}, h_2 \in \beta} \tilde{c}'_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2}(x'_j \cdot h_2).$$

6.5.5 Proposition *Suppose the chain map f satisfies C_{morphism}. The equation $d_{M'} f = f d_M$ gives us the following equations in the coefficients $f_{i,j}$, $\tilde{f}_{i,j;h'}$, $c_{i,j}$, $\tilde{c}_{i,j;h'}$, $c'_{i,j}$ and $\tilde{c}'_{i,j,h'}$:*

(1) *For all generators x_i of M and x'_k of M',*

$$\sum_j f_{i,j} c'_{j,k} = \sum_j c_{i,j} f_{j,k}.$$

(2) *For all generators $x_i \cdot h_1$ of M and $x'_k \cdot h_3$ of M',*

$$\sum_{j,h' \in \beta_\gamma} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_3} c'_{j,k} + \sum_{j,h' \in \beta_\gamma} f_{i,j} \tilde{c}'_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h_3}$$
$$= \sum_{j,h' \in \beta_\gamma} c_{i,j} \tilde{f}_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h_3} + \sum_{j,h' \in \beta_\gamma} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_3} f_{j,k}.$$

(3) *For all generators $x_i \cdot h_1$ of M and $x'_k \cdot h_3$ of M',*

$$\sum_{j,h' \in \beta_\alpha} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_3} c'_{j,k} + \sum_{j,h' \in \beta_\alpha} f_{i,j} \tilde{c}'_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h_3}$$
$$+ \sum_{\substack{j,h' \in \beta_\gamma, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{c}'_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3}$$
$$= \sum_{j,h' \in \beta_\alpha} c_{i,j} \tilde{f}_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h+3} + \sum_{j,h' \in \beta_\alpha} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_3} f_{j,k}$$
$$+ \sum_{\substack{j,h' \in \beta_\gamma, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{f}_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3}.$$

(4) *For all generators* $x_i \cdot h_1$ *of* M *and* $x'_k \cdot h_3$ *of* M',

$$\sum_{\substack{j,h'\in\beta_\gamma,\\ h''\in\beta_\alpha,h_2\in\beta}} \tilde{f}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}\tilde{c}'_{j,k;h''}\tilde{\tilde{c}}_{h''h_2;h_3} + \sum_{\substack{j,h'\in\beta_\alpha,\\ h''\in\beta_\gamma,h_2\in\beta}} \tilde{f}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}\tilde{c}'_{j,k;h''}\tilde{\tilde{c}}_{h''h_2;h_3}$$
$$= \sum_{\substack{j,h'\in\beta_\gamma,\\ h''\in\beta_\alpha,h_2\in\beta}} \tilde{c}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}\tilde{f}_{j,k;h''}\tilde{\tilde{c}}_{h''h_2;h_3} + \sum_{\substack{j,h'\in\beta_\alpha,\\ h''\in\beta_\gamma,h_2\in\beta}} \tilde{c}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}\tilde{f}_{j,k;h''}\tilde{\tilde{c}}_{h''h_2;h_3}.$$

(5) *For all generators* $x_i \cdot h_1$ *of* M *and* $x'_k \cdot h_3$ *of* M',

$$\sum_{\substack{j,h'\in\beta_\alpha,\\ h''\in\beta_\alpha,h_2\in\beta}} \tilde{f}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}\tilde{c}'_{j,k;h''}\tilde{\tilde{c}}_{h''h_2;h_3} = \sum_{\substack{j,h'\in\beta_\alpha,\\ h''\in\beta_\alpha,h_2\in\beta}} \tilde{c}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}\tilde{f}_{j,k;h''}\tilde{\tilde{c}}_{h''h_2;h_3}.$$

Proof The proof is very similar to that of Proposition 6.1.2 and will be omitted. Note that explicitly writing the degree conditions in the sums is unnecessary, since the relevant products of coefficients are always zero unless the degree conditions are satisfied. In Proposition 6.1.2, we chose to write out the degree conditions for clarity. □

6.5.6 Definition Suppose (M, d_M) and $(M', d_{M'})$ satisfy C_{module} and $f\colon M \to M'$ is a chain map satisfying C_{morphism}. Define the first component $\widehat{A}(f)_1$ of an $\mathcal{A}_\infty$–morphism $\widehat{A}(f)\colon \widehat{A}(M) \to \widehat{A}(M')$ of type A structures over $m(\mathcal{B})^! \odot \mathcal{B}$ by

$$\widehat{A}(f)_1(X_{x_i\cdot h_1}) := \sum_j f_{i,j} X_{x'_j\cdot h_1}.$$

The map $\widehat{A}(f)_1\colon \widehat{A}(M) \to \widehat{A}(M')$ respects the right actions of the idempotent ring $\mathcal{I}_\beta$, and it is bigrading-preserving because f is.

If $\widehat{A}(f)_1$ were the only nonzero component of $\widehat{A}(f)$, then $\widehat{A}(f)$ would be an ordinary chain map between differential bigraded $m(\mathcal{B})^! \odot \mathcal{B}$–modules. However, $\widehat{A}(f)_2$ will also be nonzero in general; thus, we must deal with genuine higher $\mathcal{A}_\infty$–terms when working with these morphisms. The component

$$\widehat{A}(f)_2\colon \widehat{A}(M) \otimes_{\mathcal{I}_\beta} m(\mathcal{B})^! \odot \mathcal{B} \to \widehat{A}(M')[0,1]$$

of $\widehat{A}(f)$ is defined on the generators $X_{x_i\cdot h_1}$ of $\widehat{A}(M)$ and $m(b^*_{*;m(h_1),m(h_2)})$ of $m(\mathcal{B})^! \odot \mathcal{B}$ by

$$\widehat{A}(f)_2(X_{x_i\cdot h_1}, m(b^*_{*;m(h_1),m(h_2)})) := \sum_{j,h'\in\beta_{\text{mult}}} \tilde{f}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2} X_{x'_j\cdot h_2},$$

where $m(b^*_{*;m(h_1),m(h_2)})$ denotes $m(b^*_{\gamma;m(h_1),m(h_2)})$ or $m(b^*_{C;m(h_1),m(h_2)})$ as appropriate. Any action of the form $\widehat{A}(f)_2(X_{x_i\cdot h_1}, b_{*;h_1,h_2})$ is defined to be zero.

Suppose that the algebra input $m(b^*_{*;m(h_1),m(h_2)})$ equals $m(b^*_{\gamma;m(h_1),m(h_2)})$. Let $h_1 = (W(a)b, \sigma)$ and $h_2 = (W(a')b, \sigma')$; then $\tilde{\tilde{c}}_{h'h_1;h_2}$ is only nonzero for one value of h', namely $h' = (W(a')a, \text{all plus})$. For this value of h', $\tilde{\tilde{c}}_{h'h_1;h_2}$ is 1. Thus,

$$\widehat{A}(f)_2(X_{x_i\cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)})) = \sum_j \tilde{f}_{i,j;h'} X_{x'_j\cdot h_2}. \tag{6-6}$$

Now suppose the algebra input is equal to $m(b^*_{C;m(h_1),m(h_2)})$. As before, write h_1 as $h_1 = (W(a)b, \sigma)$. In this case, $\tilde{\tilde{c}}_{h'h_1;h_2}$ will be nonzero for any h'_α which equals $(W(a)a, \text{ minus on } \alpha)$, where α is any arc of a which is part of the circle C in $W(a)a$. For h' equal to one of the h'_α, we have $\tilde{\tilde{c}}_{h'h_1;h_2} = 1$, and for all other h', $\tilde{\tilde{c}}_{h'h_1;h_2}$ is zero. Thus,

$$\widehat{A}(f)_2(X_{x_i\cdot h_1}, m(b^*_{C;m(h_1),m(h_2)})) = \sum_j \sum_{\text{left arcs } \alpha \text{ of } C} \tilde{f}_{i,j;h'_\alpha} X_{x'_j\cdot h_2}. \tag{6-7}$$

Writing $m(\mathcal{B})^! \odot \mathcal{B}$ as $T(V_{\text{full}})/J_{\text{full}}$ as in Section 5.3, the above formulas define a map

$$\widehat{A}(f)_2\colon \widehat{A}(M) \otimes_{\mathcal{I}_\beta} V_{\text{full}} \to \widehat{A}(M')[0,1].$$

We can extend to a map

$$\widehat{A}(f)_2\colon \widehat{A}(M) \otimes_{\mathcal{I}_\beta} T(V_{\text{full}}) \to \widehat{A}(M')[0,1]$$

which is defined as the sum, over all $n \geq 2$, of the maps

$$\sum_{k=1}^{n-1} m'_2 \circ (m'_2 \otimes \mathrm{id}) \circ \cdots \circ (m'_2 \otimes \mathrm{id}^{\otimes(k-2)}) \circ (\widehat{A}(f)_2 \otimes |\mathrm{id}|^{\otimes(k-1)}) \circ (m_2 \otimes \mathrm{id}^{\otimes k}) \circ \cdots \circ (m_2 \otimes \mathrm{id}^{\otimes(n-2)})$$

from $\widehat{A}(M) \otimes (V_{\text{full}})^{\otimes(n-1)}$ to $\widehat{A}(M')$. In Proposition 6.5.8, we show that $\widehat{A}(f)_2$ descends to a map

$$\widehat{A}(f)_2\colon \widehat{A}(M) \otimes_{\mathcal{I}_\beta} m(\mathcal{B})^! \odot \mathcal{B} \to \widehat{A}(M');$$

in Proposition 6.5.9 we verify that $\widehat{A}(f)_1$ and $\widehat{A}(f)_2$ together satisfy the conditions to form an $\mathcal{A}_\infty$–morphism $\widehat{A}(f)$.

6.5.7 Example The $n = 2$ summand of $\widehat{A}(f)_2\colon \widehat{A}(M) \otimes_{\mathcal{I}_\beta} T(V_{\text{full}}) \to \widehat{A}(M')$ is simply $\widehat{A}(f)_2$, the map from $\widehat{A}(M) \otimes_{\mathcal{I}_\beta} V_{\text{full}}$ to $\widehat{A}(M')$ defined above. The $n = 3$ summand of $\widehat{A}(f)_2\colon \widehat{A}(M) \otimes_{\mathcal{I}_\beta} T(V_{\text{full}}) \to \widehat{A}(M')$, or in other words the definition of $\widehat{A}(f)_2$ when the algebra input is a quadratic monomial in the generators of V_{full}, is

$$\widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id}) + m'_2 \circ (\widehat{A}(f)_2 \otimes |\mathrm{id}|),$$

where in this expression $\widehat{A}(f)_2$ again denotes the map from $\widehat{A}(M) \otimes_{\mathcal{I}_\beta} V_{\text{full}}$ to $\widehat{A}(M')$.

6.5.8 Proposition *Write $m(\mathcal{B})^! \odot \mathcal{B}$ as $T(V_{\text{full}})/J_{\text{full}}$ and let r be any element of J_{full}. The map*

$$\widehat{A}(f)_2(-,r)\colon \widehat{A}(M) \to \widehat{A}(M')$$

is identically zero. Thus, we get a well-defined map

$$\widehat{A}(f)_2\colon \widehat{A}(M) \otimes_{\mathcal{I}_\beta} m(\mathcal{B})^! \odot \mathcal{B} \to \widehat{A}(M')$$

which is linear with respect to the right actions of the idempotent ring $\mathcal{I}_\beta$ on $\widehat{A}(M) \otimes_{\mathcal{I}_\beta} m(\mathcal{B})^! \odot \mathcal{B}$ and $\widehat{A}(M')$, and which preserves the intrinsic grading and decreases the homological grading by one.

Proof First, $\widehat{A}(f)_2$ decreases the homological grading by 1 (and thus preserves homological grading when accounting for shifts), since $m(b^*_{*;m(h_1),m(h_2)})$ carries homological grading 1 and $f\colon M \to M'$ preserves homological grading. Also, $\widehat{A}(f)_2$ preserves the intrinsic grading. To see this, note that as elements of M and M', $x_i \cdot h_1$ and $x'_j \cdot h_2$ must have the same intrinsic grading whenever $x'_j \cdot h_2$ appears with nonzero coefficient in the basis expansion of $f(x_i \cdot h_1)$, because f preserves intrinsic grading. As elements of H^n, the intrinsic degree of h_2 is either one or two greater than that of h_1. Since, in $\widehat{A}(M)$ and $\widehat{A}(M')$, the intrinsic degrees of h_1 and h_2 are multiplied by $-\frac{1}{2}$ whereas the intrinsic degrees of x_i and x'_j are multiplied by -1, the element $X_{x'_j \cdot h_2}$ of $\widehat{A}(M')$ should have intrinsic degree which is $\frac{1}{2}$ or 1 greater than the intrinsic degree of $X_{x_i \cdot h_1} \in \widehat{A}(M)$. This extra $\frac{1}{2}$ or 1 is compensated exactly by the intrinsic degree of $m(b^*_{*;m(h_1),m(h_2)})$, which is $\frac{1}{2}$ for $m(b^*_{\gamma;m(h_1),m(h_2)})$ and 1 for $m(b^*_{C;m(h_1),m(h_2)})$.

To show that $\widehat{A}(f)_2(-,r)$ is zero for any $r \in J_{\text{full}}$, note first that Definition 6.5.6 implies that if we have elements r and r' of $T(V_{\text{full}})$ such that $m_2(-,r) = 0$, $m'_2(-,r') = 0$, $\widehat{A}(f)_2(-,r) = 0$ and $\widehat{A}(f)_2(-,r') = 0$, then $\widehat{A}(f)_2(-, r \cdot r') = 0$ as well.

Thus, we only need to show that $\widehat{A}(f)_2(-,r)$ is zero for the multiplicative generators r of J_{full}. These were defined to be the generators of

$$J_{\mathcal{B}} \cap (T^1(V_{\mathcal{B}}) \oplus T^2(V_{\mathcal{B}})), \quad J_{m(\mathcal{B})^!} \cap (T^1(V_{m(\mathcal{B})^!}) \oplus T^2(V_{m(\mathcal{B})^!})) \quad \text{and} \quad J_{\text{extra}}.$$

For generators in $J_{\mathcal{B}} \cap (T^1(V_{\mathcal{B}}) \oplus T^2(V_{\mathcal{B}}))$, there is nothing to show, since $\widehat{A}(f)_2(-,b)$ is zero for any $b \in V_{\mathcal{B}}$.

For the generators in $J_{m(\mathcal{B})^!} \cap (T^1(V_{m(\mathcal{B})^!}) \oplus T^2(V_{m(\mathcal{B})^!}))$, the proof closely follows the proof of Proposition 6.2.2. Write $J_{m(\mathcal{B})^!} \cap (T^1(V_{m(\mathcal{B})^!}) \oplus T^2(V_{m(\mathcal{B})^!}))$ as $m(I^\perp)$ as in that proof.

The generators $m(r^*)$ of $m(I^\perp)$ have intrinsic degree either 1, $\frac{3}{2}$ or 2, as in Section 5.2. For those $m(r^*)$ of intrinsic degree 2, the equations in item (5) of Proposition 6.5.5

above imply that $\widehat{A}(f)_2(X_{x_i\cdot h_1}, m(r^*)) = 0$ for any $X_{x_i\cdot h_1}$. For those $m(r^*)$ of intrinsic degree $\frac{3}{2}$, the equations in item (4) of Proposition 6.5.5 similarly imply that $\widehat{A}(f)_2(-, m(r^*))$ is zero.

The generators $m(r^*)$ of $m(I^\perp)$ which have intrinsic degree 1 are sums of either one, two, three or four terms $m(b^*_\gamma)m(b^*_{\gamma'})$ with all coefficients $+1$. For a fixed $m(r^*)$, let $h_1 \in \mathcal{I}_\beta$ denote its left idempotent and let $h_3 \in \mathcal{I}_\beta$ denote its right idempotent. The element h_3 of β has degree 2 more than h_1, as elements of H^n with its intrinsic grading, and h_3 differs from h_1 by two surgeries on its left crossingless matching. As in Proposition 6.2.2, the left crossingless matchings of h_1 and h_3 are different.

For any generators of $\widehat{A}(M)$ and $\widehat{A}(M')$ of the form $x_i \cdot h_1$ and $x'_k \cdot h_3$, where h_1 and h_3 are as above, the equations from item (3) of Proposition 6.5.5 become

$$\sum_{\substack{j,h'\in\beta_\gamma,\\ h''\in\beta_\gamma,h_2\in\beta}} \tilde{f}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}\tilde{c}'_{j,k;h''}\tilde{\tilde{c}}_{h''h_2;h_3} = \sum_{\substack{j,h'\in\beta_\gamma,\\ h''\in\beta_\gamma,h_2\in\beta}} \tilde{c}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}\tilde{f}_{j,k;h''}\tilde{\tilde{c}}_{h''h_2;h_3}.$$

The terms involving $h' \in \beta_\alpha$ vanish for these choices of h_1 and h_3. These equations imply that for all generators $m(r^*)$ of $m(I^\perp)$ of intrinsic degree 1, $\widehat{A}(f)_2(-, m(r^*))$ is zero.

Finally, the generators of J_{extra} are listed in items (1)–(5) of Definition 5.3.2. If r is one of these generators, the proof that the map $\widehat{A}(f)_2(-, r)$ is zero is similar to the proof of Proposition 6.2.5.

In more detail, consider a relation

$$r = b_{\gamma;h_1,h_2}m(b^*_{\eta';m(h_2),m(h_3)}) - m(b^*_{\eta;m(h_1),m(\tilde{h}_2)})b_{\gamma';\tilde{h}_2,h_3}$$

from item (1) of Definition 5.3.2. Write $h_1 = (W(a_1)b_1, \sigma_1)$ and let $X_{x_i\cdot h_1}$ be a generator of $\widehat{A}(M)$. By Example 6.5.7, we have

$$\begin{aligned}(6\text{-}8)\quad \widehat{A}(f)_2(X_{x_i\cdot h_1}, r) = \widehat{A}(f)_2\big(&m_2(X_{x_i\cdot h_1}, b_{\gamma;h_1,h_2}), m(b^*_{\eta';m(h_2),m(h_3)})\big)\\ &- m_2\big(\widehat{A}(f)_2(X_{x_i\cdot h_1}, m(b^*_{\eta;m(h_1),m(\tilde{h}_2)})), b_{\gamma';\tilde{h}_2,h_3}\big)\end{aligned}$$

Write h_2 as $(W(a_1)b_2, \sigma_2)$ and h_3 as $(W(a_2)b_2, \sigma_3)$. Let $h' = (W(a_2)a_1, \text{all plus})$, an element of β_γ. For the first term in (6-8) above, we first multiply $X_{x_i\cdot h_1}$ by $b_{\gamma;h_1,h_2}$ to get $X_{x_i\cdot h_2}$. Applying $\widehat{A}(f)_2(-, m(b^*_{\eta';m(h_2),m(h_3)}))$ to the element $X_{x_i\cdot h_2}$, by (6-6) we get

$$\sum_j \tilde{f}_{i,j;h'}X_{x_j\cdot h_3}.$$

For the second term on the right side of (6-8), we first compute

$$\widehat{A}(f)_2(X_{x_i\cdot h_1}, m(b^*_{\eta;m(h_1),m(\tilde{h}_2)})).$$

This expression equals

$$\sum_j \tilde{f}_{i,j;h'} X_{x_j \cdot \tilde{h}_2}$$

by (6-6) again, where h' is still equal to $(W(a_2)a_1, \text{all plus})$. If we multiply this result by $b_{\gamma';\tilde{h}_2,h_3}$, we get

$$\sum_j \tilde{f}_{i,j;h'} X_{x_j \cdot h_3}.$$

Thus, if a generator r of J_{extra} comes from item (1) of Definition 5.3.2, then the map $\widehat{A}(f)_2(-,r)\colon \widehat{A}(M) \to \widehat{A}(M')$ is zero.

For generators of J_{extra} from items (2)–(5) of Definition 5.3.2, the proof is analogous to that of Proposition 6.2.5 in the same way as above. We will leave the remaining cases to the reader. □

6.5.9 Proposition *$\widehat{A}(f)$ satisfies the $\mathcal{A}_\infty$–morphism compatibility conditions.*

Proof Since $\widehat{A}(f)_n$ is zero for $n > 2$, it suffices to show that the $n = 1$, $n = 2$ and $n = 3$ conditions listed in Example 6.5.3 hold. For the $n = 1$ condition, we want to show that

$$m'_1(\widehat{A}(f)_1(X_{x_i \cdot h_1})) = \widehat{A}(f)_1(m_1(X_{x_i \cdot h_1}))$$

for each generator $X_{x_i \cdot h_1}$ of $\widehat{A}(M)$. The left side is

$$m'_1\Big(\sum_j f_{i,j} X_{x_j \cdot h_1}\Big) = \sum_{j,k} f_{i,j} c'_{j,k} X_{x'_k \cdot h_1},$$

while the right side is

$$\widehat{A}(f)_1\Big(\sum_j c_{i,j} X_{x_j \cdot h_1}\Big) = \sum_{j,k} c_{i,j} f_{j,k} X_{x'_k \cdot h_1}.$$

These are equal by item (1) of Proposition 6.5.5.

We may write the $n = 3$ condition of Example 6.5.3 as

$$\widehat{A}(f)_2 \circ (\mathrm{id} \otimes \mu_2) = \widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id}) + m'_2 \circ (\widehat{A}(f)_2 \otimes |\mathrm{id}|). \tag{6-9}$$

In this form, it is clear from the definition of $\widehat{A}(f)_2$ in Definition 6.5.6 that this condition holds, generalizing Example 6.5.7.

For the $n = 2$ condition, we want to show that

$$\begin{aligned} m'_1 \circ \widehat{A}(f)_2 + m'_2 \circ (\widehat{A}(f)_1 \otimes \mathrm{id}) \\ = \widehat{A}(f)_1 \circ m_2 - \widehat{A}(f)_2 \circ (m_1 \otimes |\mathrm{id}|) - \widehat{A}(f)_2 \circ (\mathrm{id} \otimes \mu_1) \end{aligned} \tag{6-10}$$

as maps from $\widehat{A}(M)\otimes_{\mathcal{I}_\beta} m(\mathcal{B})^!\odot\mathcal{B}$ to $\widehat{A}(M')$.

We will first reduce to the case of proving the above equation when applied to elements of the form $X_{x_i\cdot h_1}\otimes b_{*;h_1,h_2}$ or $X_{x_i\cdot h_1}\otimes m(b^*_{*;m(h_1),m(h_2)})$, where $b_{*;h_1,h_2}$ and $m(b^*_{*;m(h_1),m(h_2)})$ are the multiplicative generators of $m(\mathcal{B})^!\odot\mathcal{B}$.

Claim *If b_1 and b_2 are two elements of $m(\mathcal{B})^!\odot\mathcal{B}$ such that the $n=2$ condition is satisfied both when the algebra input is b_1 and when it is b_2, then the $n=2$ condition is also satisfied when the algebra input is b_1b_2.*

Proof of claim We want to show that

$$\begin{aligned}(6\text{-}11)\quad m_1'\circ\widehat{A}(f)_2\circ(\mathrm{id}\otimes\mu_2)+m_2'\circ(\widehat{A}(f)_1\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_2)&\\ =\widehat{A}(f)_1\circ m_2\circ(\mathrm{id}\otimes\mu_2)-\widehat{A}(f)_2\circ(m_1\otimes|\mathrm{id}|)\circ(\mathrm{id}\otimes\mu_2)&\\ -\widehat{A}(f)_2\circ(\mathrm{id}\otimes\mu_1)\circ(\mathrm{id}\otimes\mu_2)&\end{aligned}$$

when the algebra input to these maps is $b_1\otimes b_2$, assuming the usual $n=2$ condition (6-10) holds when the algebra input is b_1 or b_2. In the proof of this claim, the algebra input to all maps will be assumed to be $b_1\otimes b_2$.

The left side of (6-11) can be rewritten as

$$\begin{aligned}(6\text{-}12)\qquad m_1'\circ m_2'\circ(\widehat{A}(f)_2\otimes|\mathrm{id}|)+m_1'\circ\widehat{A}(f)_2\circ(m_2\otimes\mathrm{id})&\\ +m_2'\circ(m_2'\otimes\mathrm{id})\circ(\widehat{A}(f)_1\otimes\mathrm{id}\otimes\mathrm{id})&\end{aligned}$$

using the $n=3$ consistency condition (6-9) for $\widehat{A}(f)$ and the associativity of the action of $m(\mathcal{B})^!\odot\mathcal{B}$ on $\widehat{A}(M')$. Call these terms LHS_1, LHS_2 and LHS_3. Now we may use the assumption that the $n=2$ consistency condition (6-10) holds when the algebra input is b_1 to write the third term LHS_3 of expression (6-12) as

$$\begin{aligned}m_2'\circ(\widehat{A}(f)_1\otimes\mathrm{id})\circ(m_2\otimes\mathrm{id})-m_2'\circ(m_1'\otimes|\mathrm{id}|)\circ(\widehat{A}(f)_2\otimes|\mathrm{id}|)&\\ -m_2'\circ(\widehat{A}(f)_2\otimes\mathrm{id})\circ(m_1\otimes|\mathrm{id}|\otimes\mathrm{id})&\\ -m_2'\circ(\widehat{A}(f)_2\otimes\mathrm{id})\circ(\mathrm{id}\otimes\mu_1\otimes\mathrm{id}).&\end{aligned}$$

Call these four terms LHS_{3a}, LHS_{3b}, LHS_{3c} and LHS_{3d}.

On the other hand, the right side of (6-11) can be rewritten as

$$\begin{aligned}(6\text{-}13)\qquad \widehat{A}(f)_1\circ m_2\circ(m_2\otimes\mathrm{id})-m_2'\circ(\widehat{A}(f)_2\otimes\mathrm{id})\circ(m_1\otimes|\mathrm{id}|\otimes\mathrm{id})&\\ -\widehat{A}(f)_2\circ(m_2\otimes\mathrm{id})\circ(m_1\otimes|\mathrm{id}|\otimes|\mathrm{id}|)&\\ -\widehat{A}(f)_2\circ(\mathrm{id}\otimes\mu_2)\circ(\mathrm{id}\otimes\mu_1\otimes|\mathrm{id}|)&\\ -\widehat{A}(f)_2\circ(\mathrm{id}\otimes\mu_2)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu_1)&\end{aligned}$$

using the $n=3$ consistency equation (6-9) for $\widehat{A}(f)$, the associativity of the action of $m(\mathcal{B})^! \odot \mathcal{B}$ on $\widehat{A}(M)$ and the Leibniz rule for the derivative μ_1 on $m(\mathcal{B})^! \odot \mathcal{B}$. Call these five terms RHS_1, RHS_2, RHS_3, RHS_4 and RHS_5. Using the $n=3$ consistency equation again, we can rewrite the term RHS_4 as

$$-m_2' \circ (\widehat{A}(f)_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_1 \otimes \mathrm{id}) - \widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_1 \otimes |\mathrm{id}|).$$

Call these two terms RHS_{4a} and RHS_{4b}. Similarly, we can rewrite the term RHS_5 as

$$-m_2' \circ (\widehat{A}(f)_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mu_1) - \widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mu_1).$$

Call these terms RHS_{5a} and RHS_{5b}. Using the assumption that the $n=2$ consistency condition (6-10) holds when the algebra input is b_2, we can write the term RHS_1 as

$$\begin{aligned} m_2' \circ (\widehat{A}(f)_1 \otimes \mathrm{id}) \circ (m_2 \otimes \mathrm{id}) + m_1' \circ \widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id}) \\ + \widehat{A}(f)_2 \circ (m_1 \otimes |\mathrm{id}|) \circ (m_2 \otimes \mathrm{id}) \\ + \widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mu_1). \end{aligned}$$

Call these four terms RHS_{1a}, RHS_{1b}, RHS_{1c} and RHS_{1d}.

After rewriting the left and right sides of (6-11) in this way, we want to show that

$$\begin{aligned} \mathrm{LHS}_1 + \mathrm{LHS}_2 + \mathrm{LHS}_{3a} + \mathrm{LHS}_{3b} + \mathrm{LHS}_{3c} + \mathrm{LHS}_{3d} \\ = \mathrm{RHS}_{1a} + \mathrm{RHS}_{1b} + \mathrm{RHS}_{1c} + \mathrm{RHS}_{1d} + \mathrm{RHS}_2 \\ + \mathrm{RHS}_3 + \mathrm{RHS}_{4a} + \mathrm{RHS}_{4b} + \mathrm{RHS}_{5a} + \mathrm{RHS}_{5b}. \end{aligned}$$

Several terms cancel:

- $\mathrm{LHS}_2 = \mathrm{RHS}_{1b}$,
- $\mathrm{LHS}_{3a} = \mathrm{RHS}_{1a}$,
- $\mathrm{LHS}_{3c} = \mathrm{RHS}_2$,
- $\mathrm{LHS}_{3d} = \mathrm{RHS}_{4a}$,
- $\mathrm{RHS}_{1d} + \mathrm{RHS}_{5b} = 0$,
- $\mathrm{RHS}_{1c} + \mathrm{RHS}_3 + \mathrm{RHS}_{4b} = 0$.

The final equality follows from the Leibniz rule for the differential m_1 on $\widehat{A}(M)$. Canceling corresponding terms between the sides, it remains to prove

$$\mathrm{LHS}_1 + \mathrm{LHS}_{3b} = \mathrm{RHS}_{5a}.$$

The Leibniz rule for the differential m_1' on $\widehat{A}(M')$ lets us rewrite $\mathrm{LHS}_1 + \mathrm{LHS}_{3b}$ as

$$m_2' \circ (\mathrm{id} \otimes \mu_1) \circ (\widehat{A}(f)_2 \otimes |\mathrm{id}|).$$

This term equals the remaining right-side term RHS_{5a}, because μ_1 increases homological grading by one. □

Thus, we have reduced to showing that the $n=2$ consistency condition (6-10) holds for $\widehat{A}(f)$ when the algebra input is either $b_{\gamma;h_1,h_2}$, $b_{C;h_1,h_2}$, $m(b^*_{\gamma;m(h_1),m(h_2)})$ or $m(b^*_{C;m(h_1),m(h_3)})$. If the input $b_{*;h_1,h_2}$ is $b_{\gamma;h_1,h_2}$ or $b_{C;h_1,h_2}$, the $\widehat{A}(f)_2$ terms in the $n=2$ equation are zero and we must show that

$$m'_2\circ(\widehat{A}(f)_1\otimes\mathrm{id})=\widehat{A}(f)_1\circ m_2$$

for these algebra inputs. If $X_{x_i\cdot h_1}$ is a generator of $\widehat{A}(M)$, then

$$m'_2\circ(\widehat{A}(f)_1\otimes\mathrm{id})(X_{x_i\cdot h_1},b_{*;h_1,h_2})=m'_2\Big(\sum_j f_{i,j}X_{x'_j\cdot h_1},b_{*;h_1,h_2}\Big)=\sum_j f_{i,j}X_{x'_j\cdot h_2},$$

while

$$\widehat{A}(f)_1\circ m_2(X_{x_i\cdot h_1},b_{*;h_1,h_2})=\widehat{A}(f)_1(X_{x_i\cdot h_2})=\sum_j f_{i,j}X_{x'_j\cdot h_2},$$

and these are equal.

Now let the algebra input be a generator $m(b^*_{\gamma;m(h_1),m(h_2)})$. The left side of the $n=2$ condition with this algebra input and module input $X_{x_i\cdot h}$ is

$$\sum_{j,k,h'\in\beta_\gamma}\tilde{f}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}c'_{j,k}X_{x'_k\cdot h_2}+\sum_{j,k,h'\in\beta_\gamma}f_{i,j}\tilde{c}'_{j,k;h'}\tilde{\tilde{c}}_{h'h_1;h_2}X_{x'_k\cdot h_2},$$

and the right side is

$$\sum_{j,k,h'\in\beta_\gamma}\tilde{c}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_2}f_{j,k}X_{x'_k\cdot h_2}+\sum_{j,k,h'\in\beta_\gamma}c_{i,j}\tilde{f}_{j,k;h'}\tilde{\tilde{c}}_{h'h_1;h_2}X_{x'_k\cdot h_2}.$$

These are equal by item (2) of Proposition 6.5.5.

Finally, let the algebra input be a generator $m(b^*_{C;m(h_1),m(h_3)})$. The left side of the $n=2$ condition with this algebra input and module input $X_{x_i\cdot h_1}$ is

$$\sum_{j,k,h'\in\beta_\alpha}\tilde{f}_{i,j;h'}\tilde{\tilde{c}}_{h'h_1;h_3}c'_{j,k}X_{x'_k\cdot h_3}+\sum_{j,k,h'\in\beta_\alpha}f_{i,j}\tilde{c}'_{j,k;h'}\tilde{\tilde{c}}_{h'h_1;h_3}X_{x'_k\cdot h_3}.$$

To compare with the right side of the $n=2$ condition, note that, as in the proof of Proposition 6.2.3, we have

$$\mu_1(m(b^*_{C;m(h_1),m(h_3)}))=-\sum_{h_2\in\beta}m(b^*_{\gamma;m(h_1),m(h_2)})m(b^*_{\gamma^\dagger;m(h_2),m(h_3)}),$$

where the sum is implicitly over those h_2 such that generators $m(b^*_{\gamma;m(h_1),m(h_2)})$ and $m(b^*_{\gamma^\dagger;m(h_2),m(h_3)})$ exist. Thus,

$$\begin{aligned}
&\widehat{A}(f)_2 \circ (\mathrm{id} \otimes \mu_1)(X_{x_i \cdot h_1}, m(b^*_{C;m(h_1),m(h_3)})) \\
&\quad = -\sum_{h_2} \widehat{A}(f)_2(X_{x_i \cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)}) m(b^*_{\gamma^\dagger;m(h_2),m(h_3)})) \\
&\quad = \sum_{h_2 \in \beta} m'_2 \circ (\widehat{A}(f)_2 \otimes \mathrm{id})(X_{x_i \cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)}) m(b^*_{\gamma^\dagger;m(h_2),m(h_3)})) \\
&\qquad - \sum_{h_2 \in \beta} \widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id})(X_{x_i \cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)}) m(b^*_{\gamma^\dagger;m(h_2),m(h_3)}))
\end{aligned}$$

by the $n = 3$ consistency condition (6-9); note that $\deg_h m(b^*_{\gamma^\dagger;m(h_2),m(h_3)}) = 1$. Expanding the above expression out, the top line is

$$\sum_{\substack{j,k,h' \in \beta_\gamma, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{c}'_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3} X_{x'_k \cdot h_3}$$

and the bottom line is

$$-\sum_{\substack{j,k,h' \in \beta_\gamma, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{f}_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3} X_{x'_k \cdot h_3}.$$

Thus, the right side of the $n = 2$ condition (6-10) with algebra input $m(b^*_{C;m(h_1),m(h_3)})$ and module input $X_{x_i \cdot h_1}$ is

$$\begin{aligned}
&\sum_{j,k,h' \in \beta_\alpha} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_3} f_{j,k} X_{x'_k \cdot h_3} + \sum_{j,k,h' \in \beta_\alpha} c_{i,j} \tilde{f}_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h_3} X_{x'_k \cdot h_3} \\
&\qquad - \sum_{\substack{j,k,h' \in \beta_\gamma, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{c}'_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3} X_{x'_k \cdot h_3} \\
&\qquad\qquad + \sum_{\substack{j,k,h' \in \beta_\gamma, \\ h'' \in \beta_\gamma, h_2 \in \beta}} \tilde{c}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} \tilde{f}_{j,k;h''} \tilde{\tilde{c}}_{h''h_2;h_3} X_{x'_k \cdot h_3}.
\end{aligned}$$

This is equal to the left side of the $n = 2$ condition with these inputs,

$$\sum_{j,k,h' \in \beta_\alpha} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_3} c'_{j,k} X_{x'_k \cdot h_3} + \sum_{j,k,h' \in \beta_\alpha} f_{i,j} \tilde{c}'_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h_3} X_{x'_k \cdot h_3},$$

by item (3) of Proposition 6.5.5. □

We now define the composition of two $\mathcal{A}_\infty$–morphisms as in [11, Definition 27].

6.5.10 Definition Define the *composition* of two $\mathcal{A}_\infty$–morphisms $F\colon \widehat{A} \to \widehat{A}'$ and $G\colon \widehat{A}' \to \widehat{A}''$ to be the $\mathcal{A}_\infty$–morphism $G \circ F$ with

$$(G \circ F)_n := \sum_{i+j=n+1} (-1)^{(i+1)(j+1)} G_i \circ (F_j \otimes |\mathrm{id}|^{(j+1)\otimes(i-1)}).$$

For the purposes of this section, we will only need to compose morphisms $\widehat{A}(f)$ and $\widehat{A}(g)$ such that either f or g satisfies a more restrictive condition than C_{morphism}.

6.5.11 Definition A chain map $f\colon M \to M'$ of complexes of graded projective right H^n–modules satisfies the algebraic condition $\widetilde{C}_{\mathrm{morphism}}$ for the generating sets $\{x_i\}$ of M and $\{x'_j\}$ of M' if it satisfies C_{morphism} of Definition 6.5.4 and furthermore $\tilde{f}_{i,j;h'} = 0$ for all $h' \in \beta_{\mathrm{mult}}$.

6.5.12 Proposition *Let $f\colon M \to M'$ and $g\colon M' \to M''$ be chain maps between complexes of graded projective right H^n modules, such that M, M' and M'' satisfy the algebraic condition C_{module} of Definition 6.1.1, while f and g satisfy the condition C_{morphism} and either f or g satisfies the condition $\widetilde{C}_{\mathrm{morphism}}$. Then $g \circ f$ satisfies C_{morphism} and*

$$\widehat{A}(g \circ f) = \widehat{A}(g) \circ \widehat{A}(f).$$

Proof By the conditions on f and g, the chain map $g \circ f$ satisfies the condition C_{morphism}, so $\widehat{A}(g \circ f)$ is a well-defined $\mathcal{A}_\infty$–morphism. We have

$$(g \circ f)_{i,k} = \sum_j f_{i,j} g_{j,k}.$$

If g satisfies $\widetilde{C}_{\mathrm{morphism}}$, then for $h' \in \beta_{\mathrm{mult}}$ we have

$$\widetilde{(g \circ f)}_{i,k;h'} = \sum_j \tilde{f}_{i,j;h'} g_{j,k},$$

while if f satisfies $\widetilde{C}_{\mathrm{morphism}}$, then

$$\widetilde{(g \circ f)}_{i,k;h'} = \sum_j f_{i,j} \tilde{g}_{j,k;h'}.$$

Let $X_{x_i \cdot h_1}$ be a generator of $\widehat{A}(M)$. We have

$$\widehat{A}(g \circ f)_1(X_{x_i \cdot h_1}) = \sum_{j,k} f_{i,j} g_{j,k} X_{x''_k \cdot h_1},$$

and this sum also equals $(\widehat{A}(g) \circ \widehat{A}(f))_1(X_{x_i \cdot h_1})$.

Let $b_{*;h_1,h_2}$ be a generator of $\mathcal{B} \subset m(\mathcal{B})^! \odot \mathcal{B}$. By the definition of the operation $f \mapsto \widehat{A}(f)$,

$$\widehat{A}(g \circ f)_2(X_{x_i \cdot h_1}, b_{*;h_1,h_2}) = 0 = (\widehat{A}(f) \circ \widehat{A}(g))_2(X_{x_i \cdot h_1}, b_{*;h_1,h_2}).$$

Finally, let $m(b^*_{*;m(h_1),m(h_2)})$ be a generator of $m(\mathcal{B})^! \subset m(\mathcal{B})^! \odot \mathcal{B}$. Suppose g satisfies $\widetilde{C}_{\text{morphism}}$. Then

$$\widehat{A}(g \circ f)_2(X_{x_i \cdot h_1}, m(b^*_{*;m(h_1),m(h_2)})) = \sum_{j,k,h' \in \beta_{\text{mult}}} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} g_{j,k} X_{x''_k \cdot h_2},$$

while

$$\begin{aligned}(\widehat{A}(g) \circ \widehat{A}(f))_2(X_{x_i \cdot h_1}, m(b^*_{*;m(h_1),m(h_2)})) &\\ &= (\widehat{A}(g)_1 \circ \widehat{A}(f)_2)(X_{x_i \cdot h_1}, m(b^*_{*;m(h_1),m(h_2)})) \\ &= \sum_{j,k,h' \in \beta_{\text{mult}}} \tilde{f}_{i,j;h'} \tilde{\tilde{c}}_{h'h_1;h_2} g_{j,k} X_{x''_k \cdot h_2}.\end{aligned}$$

The case when f satisfies $\widetilde{C}_{\text{morphism}}$ is analogous. Thus, $\widehat{A}(g \circ f)_2 = (\widehat{A}(g) \circ \widehat{A}(f))_2$. We have

$$\widehat{A}(g \circ f)_n = (\widehat{A}(g) \circ \widehat{A}(f))_n = 0$$

for all $n > 2$, so $\widehat{A}(g \circ f) = \widehat{A}(g) \circ \widehat{A}(f)$. □

Now we will consider homotopies. The following definition is a special case of [11, Definition 28].

6.5.13 Definition Let $\mathcal{B}$ be a differential bigraded algebra with idempotent ring $\mathcal{I}$. Let $\widehat{A}$ and $\widehat{A}'$ be differential bigraded right modules over $\mathcal{B}$. Let $F = \{F_n\}$ and $G = \{G_n\}$ be $\mathcal{A}_\infty$–morphisms from $\widehat{A}$ to $\widehat{A}'$. An $\mathcal{A}_\infty$–homotopy H between F and G is a collection

$$H_n \colon \widehat{A} \otimes_{\mathcal{I}} \mathcal{B}^{\otimes(n-1)} \to \widehat{A}'[0, n]$$

of bigrading-preserving $\mathcal{I}$–linear maps satisfying the relation

$$\begin{aligned}F_n - G_n = m'_1 \circ H_n &+ (-1)^{n-1} m'_2 \circ (H_{n-1} \otimes |\mathrm{id}|^{n-1}) \\ &+ (-1)^{n+1} H_n \circ (m_1 \otimes |\mathrm{id}|^{\otimes(n-1)}) + H_{n-1} \circ (m_2 \otimes \mathrm{id}^{\otimes(n-2)}) \\ &+ (-1)^{n+1} \sum_{k=1}^{n-1} H_n \circ (\mathrm{id}^{\otimes k} \otimes \mu_1 \otimes |\mathrm{id}|^{\otimes(n-k-1)}) \\ &+ \sum_{k=1}^{n-2} (-1)^k H_{n-1} \circ (\mathrm{id}^{\otimes k} \otimes \mu_2 \otimes \mathrm{id}^{\otimes(n-k-2)})\end{aligned}$$

for all $n \geq 1$.

6.5.14 Example Suppose H is an $\mathcal{A}_\infty$–homotopy between F and G with $H_n = 0$ for $n > 1$. Then the $n = 1$ $\mathcal{A}_\infty$–homotopy condition for H becomes

$$F_1 - G_1 = m_1' \circ H_1 + H_1 \circ m_1,$$

and the $n = 2$ homotopy condition becomes

$$F_2 - G_2 = -m_2' \circ (H_1 \otimes |\mathrm{id}|) + H_1 \circ m_2.$$

For $n > 2$, the homotopy condition is $F_n - G_n = 0$.

We will get homotopies $H = \widehat{A}(\psi)$ from certain homotopies ψ between chain maps f and g from a chain complex M of graded projective right H^n modules to another such complex M'; that is, H^n–linear maps $\psi\colon M \to M'$ of bidegree $(0, -1)$ satisfying

$$f - g = d_{M'}\psi + \psi d_M.$$

We will require that M and M' satisfy C_{module} for some generating sets $\{x_i\}$ and $\{x_j'\}$, and that f and g satisfy C_{morphism} for these generating sets. We will only need to consider homotopies ψ which satisfy the analogue of the more restrictive condition $\widetilde{C}_{\text{morphism}}$ on chain maps.

6.5.15 Definition A homotopy ψ as above satisfies the condition $\widetilde{C}_{\text{homotopy}}$ if, for all generators x_i of M,

$$\psi(x_i) = \sum_j \psi_{i,j} x_j'$$

is a basis expansion of $\psi(x_i)$ in the basis $\{x_j' \cdot h_1\}$ for M', for some integer coefficients $\psi_{i,j}$.

If ψ satisfies the condition $\widetilde{C}_{\text{homotopy}}$ (implying, in particular, that M and M' satisfy the condition C_{module} and f and g satisfy the condition C_{morphism}), then the homotopy relation $f - g = d_{M'}\psi + \psi d_M$ becomes the two sets of equations

$$f_{i,k} - g_{i,k} = \sum_j \psi_{i,j} c'_{j,k} + \sum_j c_{i,j} \psi_{j,k} \tag{6-14}$$

for all generators x_i of M and x_k' of M', and

$$\tilde{f}_{i,k;h'} - \tilde{g}_{i,k;h'} = \sum_j \psi_{i,j} \tilde{c}'_{j,k;h'} + \sum_j \tilde{c}_{i,j;h'} \psi_{j,k} \tag{6-15}$$

for all generators $x_i \in M$, $x_k' \in M'$ and $h' \in \beta_{\text{mult}}$.

6.5.16 Definition Suppose M and M' are chain complexes of graded projective right H^n–modules, satisfying the condition C_{module}, and f and g are chain maps from M to M' satisfying the condition C_{morphism}. Let ψ be an H^n–linear chain homotopy between f and g satisfying the condition $\widetilde{C}_{\text{homotopy}}$ defined above.

Define an $\mathcal{A}_\infty$–homotopy $\widehat{A}(\psi)$ between $\widehat{A}(f)$ and $\widehat{A}(g)$ by

$$\widehat{A}(\psi)_1(X_{x_i\cdot h_1}) := \sum_j \psi_{i,j} X_{x'_j\cdot h_1} \quad \text{and} \quad \widehat{A}(\psi)_n = 0$$

for $n > 1$.

6.5.17 Proposition *$\widehat{A}(\psi)$ is a valid $\mathcal{A}_\infty$–homotopy between $\widehat{A}(f)$ and $\widehat{A}(g)$.*

Proof First, $\widehat{A}(\psi)_1$ respects the right action of the idempotent ring $\mathcal{I}_\beta$ on $\widehat{A}(M)$ and $\widehat{A}(M')$, and $\widehat{A}(\psi)_1$ preserves intrinsic grading and decreases homological grading by one because ψ has the same properties.

By Example 6.5.14, the $n = 1$ condition is

$$\widehat{A}(f)_1 - \widehat{A}(g)_1 = m'_1 \circ \widehat{A}(\psi)_1 + \widehat{A}(\psi)_1 \circ m_1.$$

If $X_{x_i\cdot h_1}$ is a generator of $\widehat{A}(M)$, then

$$\begin{aligned}(\widehat{A}(f)_1 - \widehat{A}(g)_1)(X_{x_i\cdot h_1}) &= \sum_k (f_{i,k} - g_{i,k}) X_{x'_k\cdot h_1} \\ &= \sum_{j,k} (\psi_{i,j} c'_{j,k}) X_{x'_k\cdot h_1} + \sum_{j,k} (c_{i,j}\psi_{j,k}) X_{x'_k\cdot h_1}\end{aligned}$$

by (6-14), while

$$\begin{aligned}m'_1 \circ \widehat{A}(\psi)_1 X_{x_i\cdot h_1} &= \sum_{j,k} \psi_{i,j} c'_{j,k} X_{x'_k\cdot h_1}, \\ \widehat{A}(\psi)_1 \circ m_1 X_{x_i\cdot h_1} &= \sum_{j,k} c_{i,j}\psi_{j,k} X_{x'_k\cdot h_1}.\end{aligned}$$

Thus, the $n = 1$ condition is satisfied.

By Example 6.5.14, the $n = 2$ condition is

$$\widehat{A}(f)_2 - \widehat{A}(g)_2 = -m'_2 \circ (\widehat{A}(\psi)_1 \otimes |\mathrm{id}|) + \widehat{A}(\psi)_1 \circ m_2. \tag{6-16}$$

As in Proposition 6.5.9, we first reduce to the case where the algebra input is one of the generators $b_{*;h_1,h_2}$ or $m(b^*_{*;m(h_1),m(h_2)})$ of $m(\mathcal{B})^! \odot \mathcal{B}$.

Claim *If b_1 and b_2 are two elements of $m(\mathcal{B})^! \odot \mathcal{B}$ such that the $n=2$ homotopy condition* (6-16) *is satisfied both when the algebra input is b_1 and when it is b_2, then the $n=2$ homotopy condition is also satisfied when the algebra input is $b_1 b_2$.*

Proof of claim When the algebra input is $b_1 \otimes b_2$, we want to show that the maps

$$\widehat{A}(f)_2 \circ (\mathrm{id} \otimes \mu_2) - \widehat{A}(g)_2 \circ (\mathrm{id} \otimes \mu_2)$$

and

$$-m_2' \circ (\widehat{A}(\psi)_1 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \mu_2) + \widehat{A}(\psi)_1 \circ m_2 \circ (\mathrm{id} \otimes \mu_2)$$

take the same value. In the proof of this claim, the algebra input to all maps will be assumed to be $b_1 \otimes b_2$.

By Example 6.5.7 and the associativity of the algebra actions m_2 and m_2', we want to show the following equation (6-17), when the algebra input is $b_1 \otimes b_2$:

$$\begin{aligned}(6\text{-}17)\quad & m_2' \circ (\widehat{A}(f)_2 \otimes |\mathrm{id}|) + \widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id}) \\ & \qquad - m_2' \circ (\widehat{A}(g)_2 \otimes |\mathrm{id}|) - \widehat{A}(g)_2 \circ (m_2 \otimes \mathrm{id}) \\ & = -m_2' \circ (m_2' \otimes \mathrm{id}) \circ (\widehat{A}(\psi)_1 \otimes |\mathrm{id}| \otimes |\mathrm{id}|) + \widehat{A}(\psi)_1 \circ m_2 \circ (m_2 \otimes \mathrm{id}).\end{aligned}$$

Call the terms on the left side of (6-17) LHS_1, LHS_2, LHS_3 and LHS_4; call the terms on the right side RHS_1 and RHS_2. Using the $n=2$ homotopy condition for the algebra input b_1, the term RHS_1 can be written as

$$m_2' \circ (\widehat{A}(f)_2 \otimes |\mathrm{id}|) - m_2' \circ (\widehat{A}(g)_2 \otimes |\mathrm{id}|) - m_2' \circ (\widehat{A}(\psi)_1 \otimes |\mathrm{id}|) \circ (m_2 \otimes \mathrm{id}).$$

Call these terms RHS_{1a}, RHS_{1b} and RHS_{1c}. Using the $n=2$ homotopy condition for the algebra input b_2, the term RHS_2 can be written as

$$\widehat{A}(f)_2 \circ (m_2 \otimes \mathrm{id}) - \widehat{A}(g)_2 \circ (m_2 \otimes \mathrm{id}) + m_2' \circ (\widehat{A}(\psi)_1 \otimes |\mathrm{id}|) \circ (m_2 \otimes \mathrm{id}).$$

Call these terms RHS_{2a}, RHS_{2b} and RHS_{2c}. Then

- $\mathrm{LHS}_1 = \mathrm{RHS}_{1a}$,
- $\mathrm{LHS}_2 = \mathrm{RHS}_{2a}$,
- $\mathrm{LHS}_3 = \mathrm{RHS}_{1b}$,
- $\mathrm{LHS}_4 = \mathrm{RHS}_{2b}$,
- $\mathrm{RHS}_{1c} + \mathrm{RHS}_{2c} = 0$,

proving the claim. □

It remains to show that the $n = 2$ homotopy condition (6-16) is satisfied when the algebra input is one of the multiplicative generators of $m(\mathcal{B})^! \odot \mathcal{B}$. When the input is $b_{*;h_1,h_2}$, the left side of the $n = 2$ condition is zero, so we want to show that the right side is also zero. If $X_{x_i \cdot h_1}$ is a generator of $\widehat{A}(M)$, then the right side of the $n = 2$ condition with algebra input $b_{*;h_1,h_2}$ is

$$-\sum_j \psi_{i,j} X_{x'_j \cdot h_2} + \sum_j \psi_{i,j} X_{x'_j \cdot h_2},$$

which is zero as desired.

Finally, suppose the algebra input is $m(b^*_{*;m(h_1),m(h_2)})$ and let $X_{x_i \cdot h_1}$ be a generator of $\widehat{A}(M)$. The left side of the $n = 2$ condition applied to these inputs is

$$\sum_{k,h' \in \beta} (\tilde{f}_{i,k;h'} - \tilde{g}_{i,k;h'}) \tilde{\tilde{c}}_{h'h_1;h_2} X_{x'_k \cdot h_2},$$

which equals

$$\sum_{j,k,h' \in \beta} \psi_{i,j} \tilde{c}'_{j,k;h'} \tilde{\tilde{c}}_{h'h_1;h_2} X_{x'_k \cdot h_2} + \sum_{j,k,h' \in \beta} \tilde{c}_{i,j;h'} \psi_{j,k} \tilde{\tilde{c}}_{h'h_1;h_2} X_{x'_k \cdot h_2}$$

by (6-15). This expression is also equal to the right side of the $n = 2$ condition applied to these inputs, since $m(b^*_{m(h_1),m(h_2)})$ has homological degree 1. Thus, the $\mathcal{A}_\infty$–homotopy relations are satisfied for $\widehat{A}(\psi)$. □

6.5.18 Corollary *Let M and M' be chain complexes of graded projective right H^n–modules satisfying the algebraic condition C_{module}. Suppose there exist chain maps $f\colon M \to M'$ and $g\colon M' \to M$ satisfying the condition C_{morphism}, with either f or g satisfying the more restrictive condition $\widetilde{C}_{\text{morphism}}$, and chain homotopies ψ between $g \circ f$ and id_M and ψ' between $f \circ g$ and $\mathrm{id}_{M'}$, both satisfying the condition $\widetilde{C}_{\text{homotopy}}$.*

Then $\widehat{A}(M)$ and $\widehat{A}(M')$ are $\mathcal{A}_\infty$–homotopy equivalent type A structures over $m(\mathcal{B})^! \odot \mathcal{B}$.

Proof By Proposition 6.5.12,

$$\widehat{A}(g) \circ \widehat{A}(f) = \widehat{A}(g \circ f) \quad \text{and} \quad \widehat{A}(f) \circ \widehat{A}(g) = \widehat{A}(f \circ g).$$

By Proposition 6.5.17, $\widehat{A}(\psi)$ provides an $\mathcal{A}_\infty$–homotopy between $\widehat{A}(g) \circ \widehat{A}(f)$ and $\mathrm{id}_{\widehat{A}(M)} = \widehat{A}(\mathrm{id}_M)$, and $\widehat{A}(\psi')$ provides an $\mathcal{A}_\infty$–homotopy between $\widehat{A}(f) \circ \widehat{A}(g)$ and $\mathrm{id}_{\widehat{A}(M')} = \widehat{A}(\mathrm{id}_{M'})$. □

The case of interest to us is when $M = [T]^{\mathrm{Kh}}$ and $M' = [T']^{\mathrm{Kh}}$ for two oriented tangle diagrams T and T' in $\mathbb{R}_{\leq 0} \otimes \mathbb{R}$ which are related by a Reidemeister move. In [4],

Khovanov shows that $[T]^{\mathrm{Kh}}$ is chain homotopy equivalent to $[T']^{\mathrm{Kh}}$. In the following two propositions, we verify that the maps involved in these homotopy equivalences satisfy the relevant algebraic conditions.

6.5.19 Proposition *Let (M, d_M) be a chain complex of graded projective right H^n–modules. Assume the following conditions hold:*

(1) *M satisfies the algebraic condition C_{module} with respect to a set of generators $\{x_i\}$.*

(2) *$M \cong M_1 \oplus M_2$ as right H^n–modules; furthermore, M_1 is the submodule of M spanned over H^n by some subset of the x_i, while M_2 is the submodule spanned by the rest of the x_i.*

(3) *M_2 is a subcomplex of M. Write d_M in matrix form with respect to the direct sum decomposition as*

$$d_M = \begin{bmatrix} d_1 & 0 \\ d_{1,2} & d_2 \end{bmatrix}.$$

Note that $d_M^2 = 0$ is equivalent to the equations $d_1^2 = 0$, $d_{1,2} \circ d_1 + d_2 \circ d_{1,2} = 0$ and $d_2^2 = 0$.

(4) *There exists an H^n–linear map $\psi' \colon M_2 \to M_2$ of bidegree $(0, -1)$ with $\mathrm{id}_{M_2} = d_2\psi' + \psi' d_2$, and such that we may write, with integer coefficients $\psi'_{i,j}$,*

$$\psi'(x_i) = \sum_j \psi'_{i,j} x_j.$$

Among the equations implied by $d_M^2 = 0$ is $d_1^2 = 0$; thus (M_1, d_1) is a chain complex of graded projective right modules over H^n. Since (M, d_M) satisfies C_{module} for the generators $\{x_i\}$, (M_1, d_1) satisfies C_{module} for the appropriate subset of $\{x_i\}$. Define $f \colon (M, d_M) \to (M_1, d_1)$, $g \colon (M_1, d_1) \to (M, d_M)$ and $\psi \colon M \to M$ by the following matrix formulas:

$$f := \begin{bmatrix} \mathrm{id}_{M_1} & 0 \end{bmatrix}, \quad g := \begin{bmatrix} \mathrm{id}_{M_1} \\ -\psi' d_{1,2} \end{bmatrix}, \quad \psi := \begin{bmatrix} 0 & 0 \\ 0 & -\psi' \end{bmatrix}.$$

Then f and g are chain maps, $f \circ g = \mathrm{id}_{M_1}$ and $g \circ f - \mathrm{id}_M = d_M\psi + \psi d_M$. Furthermore, f satisfies the condition $\widetilde{C}_{\mathrm{morphism}}$, g satisfies the condition C_{morphism} and ψ satisfies the condition $\widetilde{C}_{\mathrm{homotopy}}$.

Proof Both f and g are bigrading-preserving and H^n–linear; ψ preserves the intrinsic grading and decreases the homological grading by one, because the same holds for ψ'. The map f is a chain map because it is the projection map onto a quotient

complex. To show that g is a chain map, we want to show that $g(d_1(x_i)) = d_M(g(x_i))$ for all x_i in M_1. We have

$$g(d_1(x_i)) = d_1(x_i) - \psi' d_{1,2} \circ d_1(x_i)$$

and

$$\begin{aligned} d_M(g(x_i)) &= d_M(x_i) - d_M(\psi' d_{1,2}(x_i)) \\ &= d_M(x_i) + \psi'(d_2 \circ d_{1,2})(x_i) - d_{1,2}(x_i) \\ &= d_1(x_i) - \psi'(d_{1,2} \circ d_1)(x_i). \end{aligned} \tag{6-18}$$

In the second line of (6-18), we use that

$$\begin{aligned} d_{1,2}(x_i) &= \mathrm{id}_{M_2}(d_{1,2}(x_i)) \\ &= d_2(\psi' d_{1,2}(x_i)) + \psi'(d_2 \circ d_{1,2})(x_i) \\ &= d_M(\psi' d_{1,2}(x_i)) + \psi'(d_2 \circ d_{1,2})(x_i) \end{aligned}$$

and in the third line of (6-18) we use the equation $d_2 \circ d_{1,2} + d_{1,2} \circ d_1 = 0$ from item (3) above. Thus, g is a chain map as well, and by definition, $f \circ g = \mathrm{id}_{M_2}$. To verify that ψ is a homotopy between $g \circ f$ and id_M, we can write out the terms of the relevant equation as matrices:

$$g \circ f = \begin{bmatrix} \mathrm{id}_{M_1} & 0 \\ -\psi' d_{1,2} & 0 \end{bmatrix}, \quad d_M \psi = \begin{bmatrix} 0 & 0 \\ 0 & -d_2 \psi' \end{bmatrix}, \quad \psi d_M = \begin{bmatrix} 0 & 0 \\ -\psi' d_{1,2} & -\psi' d_2 \end{bmatrix}.$$

Thus, the equation $g \circ f - \mathrm{id}_M = d_M \psi + \psi d_M$ holds.

By definition, f satisfies the condition $\widetilde{C}_{\text{morphism}}$. By item (4) above and the condition C_{module} for M, g satisfies C_{morphism}. By item (4), ψ satisfies $\widetilde{C}_{\text{homotopy}}$. □

6.5.20 Proposition *Let M be a chain complex of graded projective right H^n–modules. Assume the following conditions hold:*

(1) *M satisfies the algebraic condition C_{module} with respect to a set of generators $\{x_i\}$.*

(2) *$M \cong M_1 \oplus M_2$ as right H^n–modules, and M_1 is the submodule of M spanned over H^n by some subset of the x_i, say $M_1 = \{x_i \cdot h_1 \mid i \in S\}$. The submodule M_2 has a $\mathbb{Z}$–basis $\{z_i \cdot h_1 \mid i \notin S\}$, where*

$$z_i = x_i + \sum_{j \in S} \tau_{i,j} x_j + \sum_{j \in S, h' \in \beta_{\text{mult}}} \tilde{\tau}_{i,j;h'} x_j \cdot h'$$

for some integer coefficients $\tau_{i,j}$ and $\tilde{\tau}_{i,j;h'}$.

(3) *M_2 is a subcomplex of M. Write d_M in matrix form with respect to the direct sum decomposition:*

$$d_M = \begin{bmatrix} d_1 & 0 \\ d_{1,2} & d_2 \end{bmatrix}.$$

(4) *There exists an H^n–linear map $\psi'\colon M_2 \to M_2$ of bidegree $(0,-1)$ with $\mathrm{id}_{M_2} = d_2\psi' + \psi' d_2$ such that we may write*

$$\psi'(z_i) = \sum_j \psi'_{i,j} z_j$$

for some integer coefficients $\psi'_{i,j}$.

(5) *Write*

$$d_M(x_i) = \sum_j c_{i,j} x_j + \sum_{j,h' \in \beta_{\mathrm{mult}}} \tilde{c}_{i,j;h'} x_j \cdot h'.$$

For all indices $i \in S$, $j \notin S$, $k \notin S$ and elements h' of β_{mult}, we have $\tilde{c}_{i,j;h'}\psi'_{j,k} = 0$.

(6) *For all indices $i \in S$, $j \notin S$, $k \in S$ and elements h', h'' of β_{mult}, we have $\tilde{c}_{i,j;h'}\tau_{j,k} = 0$ and $\tilde{c}_{i,j;h'}\tilde{\tau}_{j,k;h''} = 0$.*

(7) *For all indices $i \notin S$, $j \notin S$, $k \in S$ and elements h' of β_{mult}, $\psi'_{i,j}\tau_{j,k} = 0$ and $\psi'_{i,j}\tilde{\tau}_{j,k;h'} = 0$.*

As in Proposition 6.5.19, we have $d_1^2 = 0$, so (M_1, d_1) is a chain complex. We may write $d_1 = d_M - d_{1,2}$. The right side of this equation has domain restricted to $M_1 \subset M$, and it takes values in M_1.

Define $f\colon (M, d_M) \to (M_1, d_1)$, $g\colon (M_1, d_1) \to (M, d_M)$ and $\psi\colon M \to M$ by the following matrix formulas:

$$f := \begin{bmatrix} \mathrm{id}_{M_1} & 0 \end{bmatrix}, \quad g := \begin{bmatrix} \mathrm{id}_{M_1} \\ -\psi' d_{1,2} \end{bmatrix}, \quad \psi := \begin{bmatrix} 0 & 0 \\ 0 & -\psi' \end{bmatrix}.$$

Then f and g are chain maps, $f \circ g = \mathrm{id}_{M_1}$ and $g \circ f - \mathrm{id}_M = d_M\psi + \psi d_M$. Furthermore, (M_1, d_1) satisfies the condition C_{module} for the generators $\{x_i \mid i \in S\}$, f satisfies C_{morphism}, g satisfies $\widetilde{C}_{\mathrm{morphism}}$ and ψ satisfies $\widetilde{C}_{\mathrm{homotopy}}$. These last three conditions use the generators $\{x_i \mid i \in S\} \cup \{x_j \mid j \notin S\}$ for M and $\{x_i \mid i \in S\}$ for M_1.

Proof The proof that f and g are chain maps, and that ψ is a homotopy between f and g, is the same as in Proposition 6.5.19. Note that here, all the matrices are chosen with respect to the basis $\{x_i \cdot h_1 \mid i \in S\} \cup \{z_j \cdot h_1 \mid j \notin S\}$ of M, since this basis

is compatible with the direct sum decomposition $M \cong M_1 \oplus M_2$. For the algebraic conditions, we need to use the basis $\{x_i \cdot h_1 \mid i \in S\} \cup \{x_j \cdot h_1 \mid j \notin S\}$ instead, which is not compatible with the direct sum decomposition. Thus, we want to express d_1, f, g and ψ in this basis and show they satisfy C_{module}, C_{morphism}, $\widetilde{C}_{\text{morphism}}$ and $\widetilde{C}_{\text{homotopy}}$, respectively.

For $i \in S$, we may write

$$\begin{aligned} d_M(x_i) = \sum_{k \in S} c_{i,k} x_k + \sum_{k \in S, h' \in \beta_{\text{mult}}} \tilde{c}_{i,k;h'} x_k \cdot h' + \sum_{j \notin S} c_{i,j} z_j \\ - \sum_{j \notin S, k \in S} c_{i,j} \tau_{j,k} x_k - \sum_{j \notin S, k \in S, h' \in \beta_{\text{mult}}} c_{i,j} \tilde{\tau}_{j,k;h'} x_k \cdot h' \\ + \sum_{j \notin S, h' \in \beta_{\text{mult}}} \tilde{c}_{i,j;h'} z_j \cdot h'. \end{aligned}$$

The final two terms which would appear in this expression are zero by item (6) of the above assumptions. The third and sixth terms of this expression make up $d_{1,2}(x_i)$:

$$d_{1,2}(x_i) = \sum_{j \notin S} c_{i,j} z_j + \sum_{j \notin S, h' \in \beta_{\text{mult}}} \tilde{c}_{i,j;h'} z_j \cdot h'.$$

The rest of the terms make up $d_1(x_i)$:

$$\begin{aligned} d_1(x_i) = \sum_{k \in S} c_{i,k} x_k + \sum_{k \in S, h' \in \beta_{\text{mult}}} \tilde{c}_{i,k;h'} x_k \cdot h' \\ - \sum_{j \notin S, k \in S} c_{i,j} \tau_{j,k} x_k - \sum_{j \notin S, k \in S, h' \in \beta_{\text{mult}}} c_{i,j} \tilde{\tau}_{j,k;h'} x_k \cdot h'. \end{aligned}$$

From this formula we can see that (M_1, d_1) satisfies the condition C_{module} for the generators $\{x_i \mid i \in S\}$.

For x_i with $i \in S$, we have $f(x_i) = x_i$. For z_j with $j \notin S$, we have $f(z_j) = 0$. We may write this equation as

$$f\left(x_j + \sum_{k \in S} \tau_{j,k} x_k + \sum_{k \in S, h' \in \beta_{\text{mult}}} \tilde{\tau}_{j,k;h'} x_k \cdot h' \right) = 0,$$

or equivalently

$$f(x_j) = - \sum_{k \in S} \tau_{j,k} x_k - \sum_{k \in S, h' \in \beta_{\text{mult}}} \tilde{\tau}_{j,k;h'} x_k \cdot h'.$$

By this formula, we see that f satisfies condition C_{morphism}.

For the map g, we can write

$$\psi' \circ d_{1,2}(x_i) = \psi'\Big(\sum_{j\notin S} c_{i,j} z_j + \sum_{j\notin S, h'\in\beta_{\text{mult}}} \tilde{c}_{i,j;h'} z_j \cdot h'\Big) = \sum_{j\notin S, k\notin S} c_{i,j}\psi'_{j,k} z_k$$

by (5) above. Using (7), we can write this sum as

$$\sum_{j\notin S, k\notin S} c_{i,j}\psi'_{j,k} x_k.$$

Thus,

$$g(x_i) = x_i - \sum_{j\notin S, k\notin S} c_{i,j}\psi'_{j,k} x_k$$

and g satisfies condition $\widetilde{C}_{\text{morphism}}$.

Finally, if $x_i \in S$, then $\psi(x_i) = 0$ by definition. For $x_j \notin S$, we have

$$\begin{aligned}\psi(x_j) &= \psi\Big(z_j - \sum_{k\in S}\tau_{j,k}x_k - \sum_{k\in S, h'\in\beta}\tilde{\tau}_{j,k;h'}x_k\cdot h'\Big)\\ &= \psi(z_j) = -\sum_{k\notin S}\psi'_{j,k} z_k = -\sum_{k\notin S}\psi'_{j,k}x_k.\end{aligned}$$

The last equality follows from (7). Thus, ψ satisfies the condition $\widetilde{C}_{\text{homotopy}}$. □

6.5.21 Corollary *If T and T' are oriented tangle diagrams in $\mathbb{R}_{\le 0}\otimes\mathbb{R}$ which are related by a Reidemeister move, then $\mathcal{A}([T]^{\text{Kh}})$ and $\mathcal{A}([T']^{\text{Kh}})$ are $\mathcal{A}_\infty$–homotopy equivalent as type A structures over $m(\mathcal{B})^! \odot \mathcal{B}$.*

Proof When T and T' are related by an R1 move, Khovanov's homotopy equivalence between $[T]^{\text{Kh}}$ and $[T']^{\text{Kh}}$ from [4, Section 4.4] is of the type constructed in Proposition 6.5.19. This is most easily seen by looking at the top diagram of Figure 7. The map ψ' sends a generator x_i of the far-right rectangle to the corresponding generator x_j in the top rectangle, which topologically is x_i isotoped with a plus-labeled free circle added, times $(-1)^{\#_1(i,j)}$. One can check that the four conditions of Proposition 6.5.19 are satisfied.

When T and T' are related by an R2 move, Khovanov's homotopy equivalence from [4, Section 4.5] is a composition of a homotopy equivalence from Proposition 6.5.19 followed by one from Proposition 6.5.20. The relevant diagrams are the middle diagram of Figure 7 and the top diagram of Figure 8. The first homotopy equivalence is very similar to the R1 move, with ψ' defined analogously. For the second homotopy equivalence, ψ' isotopes a generator and deletes a minus-labeled free circle; it still

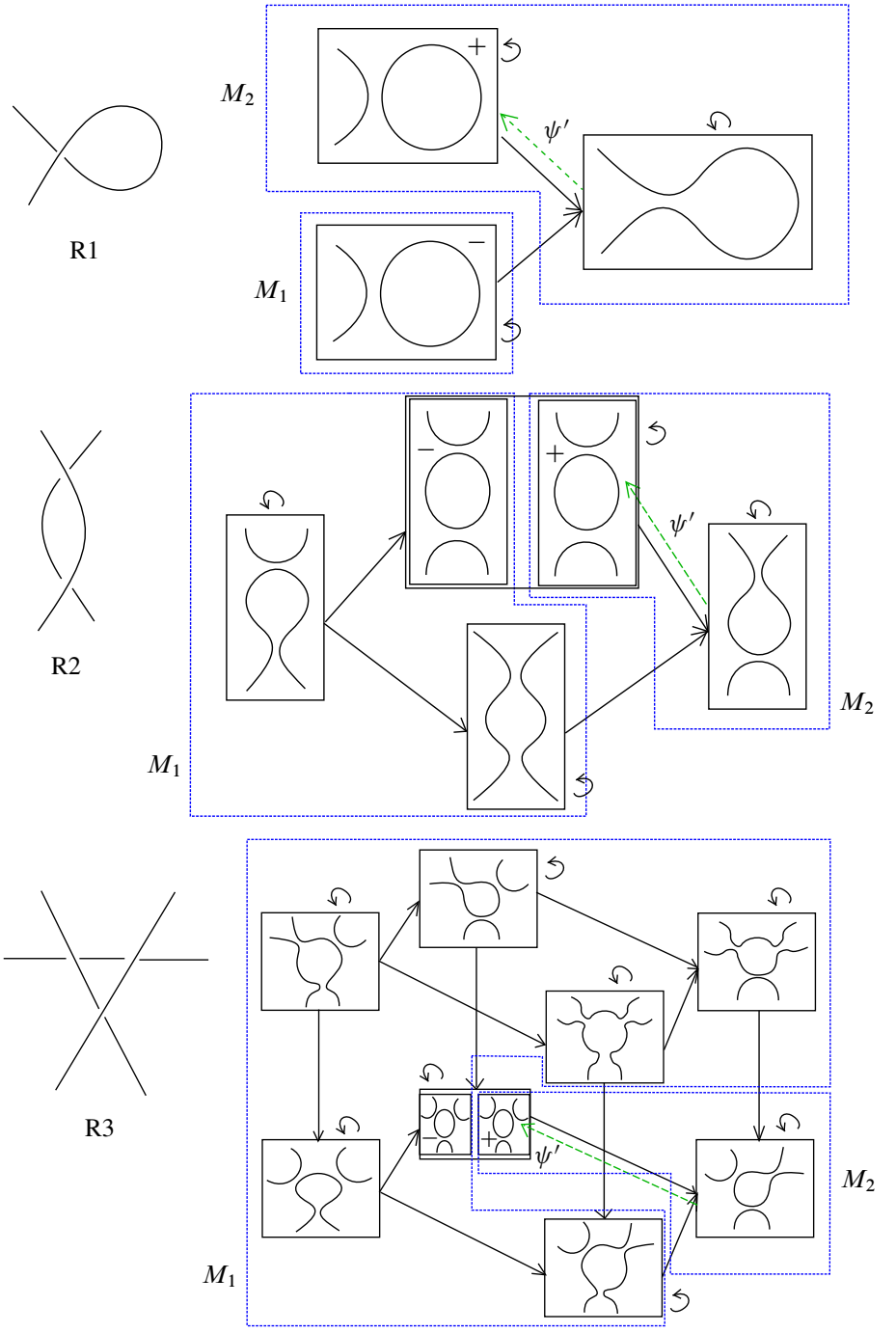

Figure 7: First step of R1, R2 and R3 moves

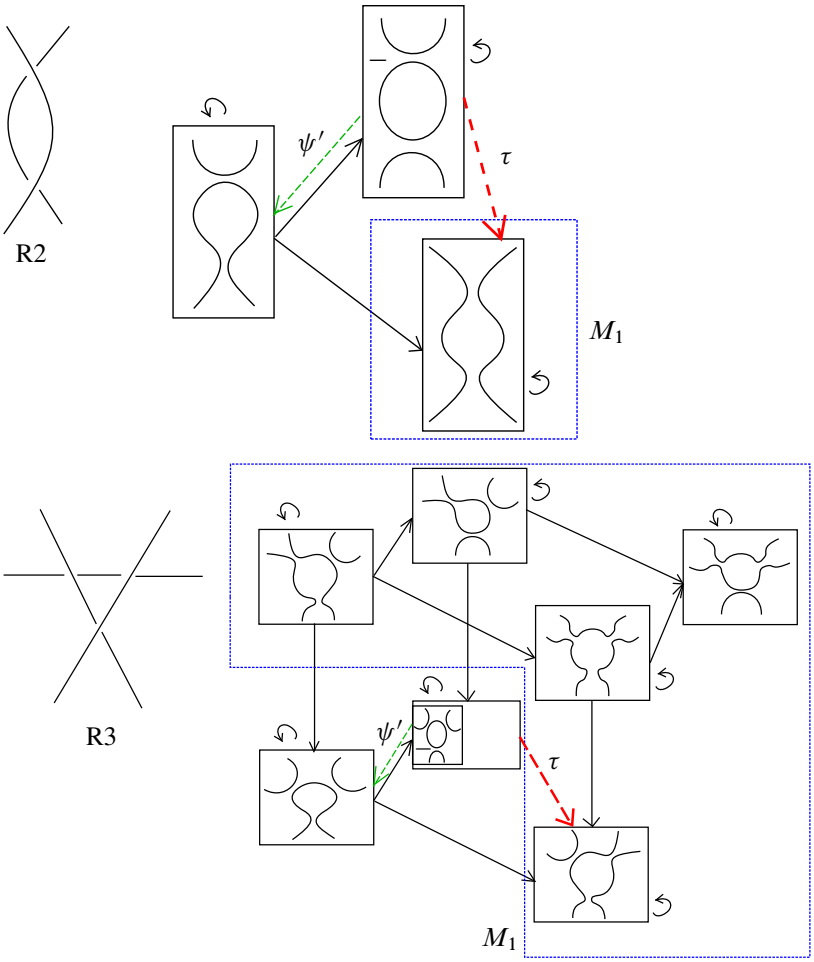

Figure 8: Second step of R2 and R3 moves

carries a coefficient of $(-1)^{\#_1(i,j)}$. The coefficients $\tau_{i,j}$ and $\tilde{\tau}_{i,j;h'}$ may be packaged into a map τ as shown in the diagram. This map is defined as ψ' postcomposed with the differential map from the left rectangle to the bottom rectangle.

For generators x_i in the left rectangle, z_i is x_i. For generators x_i in the top rectangle, z_i is $x_i + \tau(x_i)$; then M_2 is the subcomplex of $[T]^{\mathrm{Kh}}$ generated by the z_i. One can see from the diagram in Figure 8 that the conditions of Proposition 6.5.20 are satisfied. In particular, note that $\psi'(z_i) = \psi'(x_i)$ because ψ' is zero on generators from the bottom rectangle, so condition (4) is satisfied. We have $d_{1,2} = 0$, so conditions (5) and (6) hold automatically. Condition (7) holds because the arrows labeled ψ' and τ are not composable in the diagram.

Finally, when T and T' are related by an R3 move, Khovanov's homotopy equivalence from [4, Section 4.6] comes from doing a homotopy equivalence from Proposition 6.5.19 and then a homotopy equivalence from Proposition 6.5.20, to both $[T]^{\mathrm{Kh}}$ and $[T']^{\mathrm{Kh}}$. After these homotopy equivalences, the complexes are isomorphic. Thus, the full homotopy equivalence from $[T]^{\mathrm{Kh}}$ to $[T']^{\mathrm{Kh}}$ is a composition of four homotopy equivalences from Proposition 6.5.19 and Proposition 6.5.20.

The relevant diagrams are the bottom diagram of Figure 7 and the bottom diagram of Figure 8. The maps ψ' and τ are defined as in the $R2$ move. Again, one can check that the conditions of Proposition 6.5.19 are satisfied using Figure 7 and that the conditions of Proposition 6.5.20 are satisfied using Figure 8. This time, in the second step, $d_{1,2}$ is not zero. However, for one of the two arrows of Figure 8 contributing to $d_{1,2}$, all $\tilde{c}_{i,j;h'}$ are zero. The other arrow is not composable with the arrows labeled ψ' and τ. This suffices to show that conditions (5) and (6) of Proposition 6.5.20 hold. □

6.5.22 Proposition *If T and T' are oriented tangle diagrams in $\mathbb{R}_{\leq 0}\otimes\mathbb{R}$ which are related by a Reidemeister move, then the $\mathcal{A}_\infty$–homotopy equivalence between $\widehat{A}([T]^{\mathrm{Kh}})$ and $\widehat{A}([T']^{\mathrm{Kh}})$ of Corollary 6.5.21 descends to a $\mathcal{A}_\infty$–homotopy equivalence of type A structures over the quotient algebra $\mathcal{B}\Gamma_n$ of $m(\mathcal{B})^!\odot\mathcal{B}$.*

Proof For homotopy equivalences $\{f\colon M\to M_1, g\colon M_1\to M, \psi\colon M\to M\}$ coming from Proposition 6.5.19, when doing an R1 move, the first step of an R2 move, or the first or fourth step of an R3 move, we only need to show that $\widehat{A}(g)_2$ descends from a map

$$\widehat{A}(M_1)\otimes_{\mathcal{I}_\beta} m(\mathcal{B})^!\odot\mathcal{B}\to\widehat{A}(M)$$

to a map

$$\widehat{A}(M_1)\otimes_{\mathcal{I}_\beta}\mathcal{B}\Gamma_n\to\widehat{A}(M);$$

since f satisfies $\widetilde{C}_{\text{morphism}}$ and ψ satisfies $\widetilde{C}_{\text{homotopy}}$, we have $\widehat{A}(f)_2=0$ and $\widehat{A}(\psi)_2=0$.

As in Proposition 5.2.8, let a, b, c and d be vertices of a tetrahedron in the graph G. We want to show that $\widehat{A}(g)_2(-,a+c)=0$, $\widehat{A}(g)_2(-,a+d)=0$ and $\widehat{A}(g)_2(-,b+c)=0$. We will show only that $\widehat{A}(g)_2(-,a+c)=0$; by symmetry, the proof is the same for the other two extra relations.

Write

$$a=m(b^*_{\gamma;m(h_1),m(h_2)})m(b^*_{\eta';m(h_2),m(h_3)}),$$
$$c=m(b^*_{\eta;m(h_1),m(\tilde{h}_2)})m(b^*_{\gamma';m(\tilde{h}_2),m(h_3)}).$$

Let $X_{x_i\cdot h_1}$ be a generator of $\widehat{A}(M_1)$; we have $i \in S$, in the notation of Proposition 6.5.19. Then

$$\begin{aligned}(6\text{-}19)\quad &\widehat{A}(g)_2(X_{x_i\cdot h_1}, a)\\ &= (\widehat{A}(g)_2 \circ (m_2 \otimes \mathrm{id}))(X_{x_i\cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)}) \otimes m(b^*_{\eta';m(h_2),m(h_3)}))\\ &\quad + (m'_2 \circ (\widehat{A}(g)_2 \otimes |\mathrm{id}|))(X_{x_i\cdot h_1}, m(b^*_{\gamma;m(h_1),m(h_2)}) \otimes m(b^*_{\eta';m(h_2),m(h_3)}))\end{aligned}$$

using the $n=3$ consistency condition for the $\mathcal{A}_\infty$–morphism $\widehat{A}(g)_2$; see Example 6.5.3. The first term on the right side of (6-19) can be expanded out as

$$(6\text{-}20)\qquad -\sum_{i,j\in S,k,l\notin S} \tilde{c}_{i,j;h'_\gamma}\tilde{c}_{j,k;h''_{\eta'}}\psi'_{k,l}X_{x_l\cdot h_3},$$

where h'_γ and $h''_{\eta'}$ are determined by γ and η', while the second term on the right side of (6-19) can be expanded out as

$$(6\text{-}21)\qquad \sum_{i\in S,j,k,l\notin S} \tilde{c}_{i,j;h'_\gamma}\psi'_{j,k}\tilde{c}_{k,l;h''_{\eta'}}X_{x_l\cdot h_3}.$$

Similarly, we may write $\widehat{A}(g)_2(X_{x_i\cdot h_1}, c)$ as the sum of the expressions

$$(6\text{-}22)\qquad -\sum_{i,j\in S,k,l\notin S} \tilde{c}_{i,j;h'_\eta}\tilde{c}_{j,k;h''_{\gamma'}}\psi'_{k,l}X_{x_l\cdot h_3}$$

and

$$(6\text{-}23)\qquad \sum_{i\in S,j,k,l\notin S} \tilde{c}_{i,j;h'_\eta}\psi'_{j,k}\tilde{c}_{k,l;h''_{\gamma'}}X_{x_l\cdot h_3}.$$

We want to show that the expressions (6-20) and (6-23) sum to zero; the argument that expressions (6-21) and (6-22) sum to zero is very similar.

Indeed, generators of all complexes (M, d_M) and (M_1, d_1) under consideration come from generators of the Khovanov complex $[T]^{\mathrm{Kh}}$ of a tangle T, and by Remark 3.0.4 we may choose any ordering we like for the crossings of T. We will order the crossings of T such that the one, two or three crossings local to the Reidemeister move being performed come first in the ordering.

Now, to each quadruple $(i \in S, j \in S, k \notin S, l \notin S)$ giving rise to a nonzero term of expression (6-20), we may associate a pair of indices $(j' \notin S, k' \notin S)$, such that $\tilde{c}_{i,j';h'_\eta}\tilde{c}_{j',k';h''_{\gamma'}}\psi'_{k',l} \neq 0$. In fact, with the above ordering convention, we will have

$$\tilde{c}_{i,j;h'_\gamma}\tilde{c}_{j,k;h''_{\eta'}}\psi'_{k,l} = \tilde{c}_{i,j';h'_\eta}\psi'_{j',k'}\tilde{c}_{k',l;h''_{\gamma'}}.$$

To construct $(j' \notin S, k' \notin S)$ such that the above equation holds, first note that the only component of g relevant for $\widehat{A}(g)_2$ is $-\psi' \circ d_{1,2}$. Recall that the terms $\tilde{c}$, and thus also the terms ψ', are computed as in Example 6.1.3. A term like $\tilde{c}_{i,j;h'_\gamma}\tilde{c}_{j,k;h''_{\eta'}}\psi'_{k,l}$ corresponds to doing one step of d_1 by changing some crossing away from the local area (and thus higher in the ordering) from 0 to 1, then doing one step of $d_{1,2}$ by changing one of the local crossings from 0 to 1, and then finally doing one step of ψ' by changing a different local crossing from 1 to 0. The indices (j', k') and the corresponding term $\tilde{c}_{i,j';h'_\eta}\psi'_{j',k'}\tilde{c}_{k',l;h''_{\gamma'}}$ come from doing the $d_{1,2}$ and $-\psi'$ steps before changing the nonlocal crossing from 0 to 1. But, when changing the local crossings, the signs are the same for both terms because the local crossings occur at the beginning of the ordering. When changing the nonlocal crossing, the signs are also the same for both terms because doing $d_{1,2}$ and $-\psi$ on the local crossings does not increase or decrease the number of crossings with a 1–resolution ($d_{1,2}$ increases this number by 1 and then $-\psi'$ decreases it by 1).

The correspondence between quadruples $(i \in S, j \in S, k \notin S, l \notin S)$ such that $\tilde{c}_{i,j;h'_\gamma}\tilde{c}_{j,k;h''_{\eta'}}\psi'_{k,l}$ is nonzero and quadruples $(i \in S, j' \notin S, k' \notin S, l \notin S)$ such that $\tilde{c}_{i,j';h'_\eta}\psi'_{j',k'}\tilde{c}_{k',l;h''_{\gamma'}}$ is nonzero is bijective. Thus, expressions (6-20) and (6-23) sum to zero. Analogously, expressions (6-21) and (6-22) sum to zero.

We conclude that $\widehat{A}(g)_2(-, a+c) = 0$. By symmetry, we also have $\widehat{A}(g)_2(-, a+d) = 0$ and $\widehat{A}(g)_2(-, b+c) = 0$, so $\widehat{A}(g)$ descends to an $\mathcal{A}_\infty$–morphism of type A structures over $\mathcal{B}\Gamma_n$.

For homotopy equivalences $\{f\colon M \to M_1, g\colon M_1 \to M, \psi\colon M \to M\}$ coming from Proposition 6.5.20, the argument is similar enough that we will simply outline the differences with the above proof. Homotopy equivalences from Proposition 6.5.20 arise when doing the second step of an R2 move or the second or third step of an R3 move. For these equivalences, we only need to show that $\widehat{A}(f)_2$ descends from a map

$$\widehat{A}(M) \otimes_{\mathcal{I}_\beta} m(\mathcal{B})^! \odot \mathcal{B} \to \widehat{A}(M_1)$$

to a map

$$\widehat{A}(M) \otimes_{\mathcal{I}_\beta} \mathcal{B}\Gamma_n \to \widehat{A}(M_1)$$

since we automatically have $\widehat{A}(g)_2 = 0$ by condition $\widetilde{C}_{\text{morphism}}$ on g and $\widehat{A}(\psi)_2 = 0$ by condition $\widetilde{C}_{\text{homotopy}}$ on ψ.

The only terms of f in the basis expansion $\{x_i \cdot h_1 \mid i \in S\} \cup \{x_j \cdot h_1 \mid j \notin S\}$ of M which are relevant for $\widehat{A}(f)_2$ are the terms with coefficients $-\tilde{\tau}_{j,k;h'}$; see the proof of Proposition 6.5.20. These τ terms play a role analogous to the $-\psi' \circ d_{1,2}$ terms in the proof above for homotopy equivalences from Proposition 6.5.19. Indeed, a τ term corresponds to doing one step of ψ', by changing a local crossing from a 1 to

a 0, and then doing one step of d_M, by changing a local crossing from a 0 to a 1. Thus, an argument analogous to the one above shows that $\widehat{A}(f)_2(-, a+c) = 0$, and by symmetry that $\widehat{A}(f)_2(-, a+d) = 0$ and $\widehat{A}(f)_2(-, b+c) = 0$. Hence $\widehat{A}(f)$ descends to an $\mathcal{A}_\infty$–morphism of type A structures over $\mathcal{B}\Gamma_n$. □

Proposition 6.5.22 gives us an alternate proof of Roberts [11, Corollary 33].

6.6 Equivalences of type *D* structures

We first define morphisms and homotopies of type D structures with sign conventions following Roberts [12, Definition 37].

6.6.1 Definition Let $\mathcal{B}$ be a differential bigraded algebra with idempotent ring $\mathcal{I}$. Let $(\widehat{D}, \delta)$ and $(\widehat{D}', \delta')$ be type D structures over $\mathcal{B}$. A *morphism of type D structures* $F\colon \widehat{D} \to \widehat{D}'$ is a bigrading-preserving $\mathcal{I}$–linear map $F\colon \widehat{D} \to \mathcal{B} \otimes_{\mathcal{I}} \widehat{D}'$ satisfying the type D morphism relation

$$(\mu_1 \otimes |\mathrm{id}|) \circ F = (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes F) \circ \delta - (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta') \circ F.$$

The composition of two morphisms of type D structures $F\colon \widehat{D} \to \widehat{D}'$ and $G\colon \widehat{D}' \to \widehat{D}''$ is

$$G \circ F := (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes G) \otimes F,$$

a bigrading-preserving $\mathcal{I}$–linear map from $\widehat{D}$ to $\mathcal{B} \otimes_{\mathcal{I}} \widehat{D}''$ which also satisfies the type D morphism relation.

6.6.2 Definition Let $F\colon \widehat{D} \to \widehat{D}'$, $G\colon \widehat{D} \to \widehat{D}'$ be morphisms of type D structures over $\mathcal{B}$. A *homotopy of morphisms of type D structures* between F and G is a bigrading-preserving $\mathcal{I}$–linear map $H\colon \widehat{D} \to (\mathcal{B} \otimes_{\mathcal{I}} \widehat{D}')[0,1]$ satisfying

$$F - G = (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes H) \circ \delta + (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta') \circ H + (\mu_1 \otimes |\mathrm{id}|) \circ H.$$

If a homotopy exists between F and G, then F is said to be *homotopic* to G.

Two type D structures $\widehat{D}$ and $\widehat{D}'$ are *homotopy equivalent* if there exist type D structure morphisms $F\colon \widehat{D} \to \widehat{D}'$ and $G\colon \widehat{D}' \to \widehat{D}$, such that $G \circ F$ is homotopic to $\mathrm{id}_{\widehat{D}}$ and $F \circ G$ is homotopic to $\mathrm{id}_{\widehat{D}'}$.

6.6.3 Remark Suppose $\widehat{D}$ and $\widehat{D}'$ are homotopy equivalent type D structures over $\mathcal{B}$ and J is a bigrading-homogeneous ideal of $\mathcal{B}$ which is closed under the differential on $\mathcal{B}$. By Proposition 5.3.7, $\widehat{D}$ and $\widehat{D}'$ induce type D structures over $\mathcal{B}/J$. The induced type D structures are homotopy equivalent. Indeed, one may simply postcompose the

algebra outputs of F, G and H with the projection map from $\mathcal{B}$ onto $\mathcal{B}/J$, and all the relevant conditions are still satisfied.

Now let N be a chain complex of graded projective left H^n–modules. In Section 6.3, the type D structure $\widehat{D}(N)$ over $m(\mathcal{B})^! \odot \mathcal{B}$ was defined by applying the mirroring operation of Definition 6.3.6 to $\widehat{A}(r(N)) \boxtimes {}^{m(\mathcal{B})^! \odot \mathcal{B}} K^{m(m(\mathcal{B})^! \odot \mathcal{B})^{\mathrm{op}}}$. One can check that the mirroring operation of Definition 6.3.6 respects homotopy equivalences of type D structures. Thus, to show that $\widehat{D}([T]^{\mathrm{Kh}})$ is a tangle invariant up to homotopy equivalence, it would suffice to prove the following general result: if $\widehat{A}$ and $\widehat{A}'$ are type A structures over a differential bigraded algebra $\mathcal{B}$ which are free as $\mathbb{Z}$–modules, $\widehat{DD}$ is a type DD bimodule over $\mathcal{B}$ and another differential bigraded algebra $\mathcal{B}'$, and $\widehat{A}$ and $\widehat{A}'$ are $\mathcal{A}_\infty$–homotopy equivalent, then $\widehat{A} \boxtimes \widehat{DD}$ and $\widehat{A}' \boxtimes \widehat{DD}$ are homotopy equivalent as type D structures over $\mathcal{B}'$. Over $\mathbb{Z}/2\mathbb{Z}$, this is a standard property of the box tensor product; see [7, Lemma 2.3.13]. Here, we are working over $\mathbb{Z}$, but we will only need a simpler version of this result.

6.6.4 Definition Let $\mathcal{B}$ and $\mathcal{B}'$ be differential bigraded algebras over an idempotent ring $\mathcal{I}$. Let $\widehat{A}$ and $\widehat{A}'$ be differential bigraded right modules over $\mathcal{B}$ and let $(\widehat{DD}, \delta_{DD})$ be a rank-one type DD bimodule over $\mathcal{B}$ and $\mathcal{B}'$. Assume that $\widehat{A}$ and $\widehat{A}'$ are free as $\mathbb{Z}$–modules, with $\mathbb{Z}$–bases consisting of elements which are bigrading-homogeneous and have a unique right idempotent. Let δ and δ' denote the type D structure operations on $\widehat{A} \boxtimes \widehat{DD}$ and $\widehat{A}' \boxtimes \widehat{DD}$ respectively.

Let $F\colon \widehat{A} \to \widehat{A}'$ be an $\mathcal{A}_\infty$–morphism with $F_n = 0$ for $n > 2$. Define a morphism of type D structures $F \boxtimes \mathrm{id}_{DD}$ from $\widehat{A} \boxtimes \widehat{DD}$ to $\widehat{A}' \boxtimes \widehat{DD}$, or in other words a map

$$F \boxtimes \mathrm{id}_{DD}\colon (\widehat{A} \boxtimes \widehat{DD}) \to \mathcal{B}' \otimes_{\mathcal{I}} (\widehat{A}' \boxtimes \widehat{DD}),$$

by the formula

$$F \boxtimes \mathrm{id}_{DD} := 1 \otimes F_1 + \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}),$$

where we are identifying $\widehat{A} \boxtimes \widehat{DD}$ with $\widehat{A}$ and $\widehat{A}' \boxtimes \widehat{DD}$ with $\widehat{A}'$. Recall that

$$\xi\colon \widehat{A} \otimes_{\mathcal{I}} (\mathcal{B}')^{\mathrm{op}} \to \mathcal{B}' \otimes_{\mathcal{I}} \widehat{A}$$

was defined in Definition 6.3.4 and

$$\xi\colon \widehat{A}' \otimes_{\mathcal{I}} (\mathcal{B}')^{\mathrm{op}} \to \mathcal{B}' \otimes_{\mathcal{I}} \widehat{A}'$$

is defined analogously. The map $F \boxtimes \mathrm{id}_{DD}$ is bigrading-preserving and respects the actions of $\mathcal{I}$ on $\widehat{A} \boxtimes \widehat{DD}$ and $\widehat{A}' \boxtimes \widehat{DD}$.

6.6.5 Proposition *The map $F \boxtimes \mathrm{id}_{DD}$ defined in Definition 6.6.4 is a morphism of type D structures from $\widehat{A} \boxtimes \widehat{DD}$ to $\widehat{A}' \boxtimes \widehat{DD}$.*

Proof We want to show that

$$(\mu_1 \otimes |\mathrm{id}|) \circ (F \boxtimes \mathrm{id}_{DD}) = (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (F \boxtimes \mathrm{id}_{DD})) \circ \delta - (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta') \circ (F \boxtimes \mathrm{id}_{DD}). \tag{6-24}$$

The left side of (6-24) is

$$(\mu_1 \otimes |\mathrm{id}|) \circ \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) = -\xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mu_1) \circ (\mathrm{id} \otimes \delta_{DD}).$$

Using the DD bimodule relations for δ_{DD}, we may further rewrite this term as

$$\begin{aligned}&\xi \circ (F_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\quad + \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \mu_2 \otimes \mu_2) \circ (\mathrm{id} \otimes \sigma) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \delta_{DD} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}).\end{aligned}$$

Using the $n = 2$ and $n = 3$ $\mathcal{A}_\infty$ consistency conditions for F from Example 6.5.3, the sum of these two terms is

$$\begin{aligned}&\xi \circ (F_1 \otimes \mathrm{id}) \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\quad - \xi \circ (F_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes |\mathrm{id}| \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \circ m_1 \\ &\quad - \xi \circ (m_1' \otimes \mathrm{id}) \circ (F_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\quad - \xi \circ (m_2' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \circ F_1 \\ &\quad + \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (m_2 \otimes \mathrm{id} \otimes \mathrm{id}) \\ &\qquad \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mu_2) \circ \sigma \circ (\mathrm{id} \otimes \mathrm{id} \otimes \delta_{DD} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\quad + (|\mathrm{id}| \otimes \mathrm{id}) \circ \xi \circ (m_2' \otimes \mathrm{id}) \circ (F_2 \otimes |\mathrm{id}| \otimes \mathrm{id}) \\ &\qquad \circ (\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mu_2) \circ \sigma \circ (\mathrm{id} \otimes \mathrm{id} \otimes \delta_{DD} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}).\end{aligned}$$

We will refer to these six terms as LHS_1, LHS_2, LHS_3, LHS_4, LHS_5 and LHS_6.

The right side of (6-24) is

$$\begin{aligned}&1 \otimes (F_1 \circ m_1) + \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) \circ m_1 \\ &\quad + (\mathrm{id} \otimes F_1) \circ \xi \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\quad + (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \xi) \circ (\mathrm{id} \otimes F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \delta_{DD}) \\ &\qquad \circ \xi \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\quad - 1 \otimes (m_1' \circ F_1) \\ &\quad - \xi \circ (m_2' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \circ F_1 \\ &\quad - (\mathrm{id} \otimes m_1') \circ \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\quad - (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \xi) \circ (\mathrm{id} \otimes m_2' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \delta_{DD}) \\ &\qquad \circ \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}).\end{aligned}$$

We will refer to these eight terms as RHS_1 through RHS_8. We have the following:

- $\mathrm{RHS}_1 + \mathrm{RHS}_5 = 0$ by the $n = 1$ consistency conditions for F.
- $\mathrm{LHS}_1 = \mathrm{RHS}_3$ because F_1 is bigrading-preserving.
- $\mathrm{LHS}_2 = \mathrm{RHS}_2$ since
$$(\mathrm{id} \otimes |\mathrm{id}|) \circ \delta_{DD} = -(|\mathrm{id}| \otimes \mathrm{id}) \circ \delta_{DD}.$$
- $\mathrm{LHS}_3 = \mathrm{RHS}_7$.
- $\mathrm{LHS}_4 = \mathrm{RHS}_6$.

It remains to show that $\mathrm{LHS}_5 = \mathrm{RHS}_4$ and that $\mathrm{LHS}_6 = \mathrm{RHS}_8$. These claims follow from direct computation: let $\delta_{DD}(1) = \sum_i b_i \otimes (b_i')^{\mathrm{op}}$. The term LHS_5, when applied to a generator X of $\widehat{A}$, gives

$$\begin{aligned}\sum_{i,j} (-1)^{(\deg_h b_i')(\deg_h b_j)} (-1)^{\deg_h b_i' + \deg_h b_j'} (-1)^{(\deg_h X + \deg_h b_i + \deg_h b_j - 1)(\deg_h b_i' + \deg_h b_j')} \\ \cdot b_i' b_j' \otimes F_2(X b_i, b_j) \\ = \sum_{i,j} (-1)^{(\deg_h X)(\deg_h b_i' + \deg_h b_j') + (\deg_h b_i)(\deg_h b_j')} b_i' b_j' \otimes F_2(X b_i, b_j).\end{aligned}$$

To see that the second sum is equal to the first, use the fact that $\deg_h b_i + \deg_h b_i' = 1$ and $\deg_h b_j + \deg_h b_j' = 1$. In particular,

$$(-1)^{(\deg_h b_i)(\deg_h b_i')} = 1 \quad \text{and} \quad (-1)^{(\deg_h b_j)(\deg_h b_j')} = 1.$$

Applying the term RHS_4 to X gives

$$\begin{aligned}\sum_{i,j} (-1)^{(\deg_h X + \deg_h b_i)(\deg_h b_i')} (-1)^{\deg_h b_j'} (-1)^{(\deg_h X + \deg_h b_i + \deg_h b_j - 1)(\deg_h b_j')} \\ \cdot b_i' b_j' \otimes F_2(X b_i, b_j) \\ = \sum_{i,j} (-1)^{(\deg_h X)(\deg_h b_i' + \deg_h b_j') + (\deg_h b_i)(\deg_h b_j')} b_i' b_j' \otimes F_2(X b_i, b_j).\end{aligned}$$

Thus, $\mathrm{LHS}_5 = \mathrm{RHS}_4$.

Similarly, applying the term LHS_6 to X gives

$$\begin{aligned}\sum_{i,j} (-1)^{(\deg_h b_j)(\deg_h b_i')} (-1)^{\deg_h b_j} (-1)^{(\deg_h X + \deg_h b_i + \deg_h b_j - 1)(\deg_h b_i' + \deg_h b_j')} \\ \cdot (-1)^{\deg_h b_i' + \deg_h b_j'} b_i' b_j' \otimes F_2(X b_i, b_j) \\ = \sum_{i,j} -(-1)^{(\deg_h X)(\deg_h b_i' + \deg_h b_j') + (\deg_h b_i')(\deg_h b_j')} b_i' b_j' \otimes F_2(X b_i, b_j).\end{aligned}$$

Applying the term RHS_8 to X gives

$$-\sum_{i,j}(-1)^{\deg_h b_i'}(-1)^{(\deg_h X+\deg_h b_i-1)(\deg_h b_i')}(-1)^{(\deg_h X+\deg_h b_i+\deg_h b_j-1)(\deg_h b_j')} \cdot b_i'b_j' \otimes F_2(Xb_i, b_j)$$
$$= \sum_{i,j} -(-1)^{(\deg_h X)(\deg_h b_i'+\deg_h b_j')+(\deg_h b_i')(\deg_h b_j')} b_i'b_j' \otimes F_2(Xb_i, b_j).$$

Thus, $\mathrm{LHS}_6 = \mathrm{RHS}_8$, so $F \boxtimes \mathrm{id}_{DD}$ is a valid morphism of type D structures from $\widehat{A} \boxtimes \widehat{DD}$ to $\widehat{A'} \boxtimes \widehat{DD}$. □

6.6.6 Proposition *If F and G are $\mathcal{A}_\infty$–morphisms from $\widehat{A}$ to $\widehat{A}'$ as described in Definition 6.6.4, with $F_n, G_n = 0$ for $n > 2$ and either $F_2 = 0$ or $G_2 = 0$, then*

$$(G \circ F) \boxtimes \mathrm{id}_{DD} = (G \boxtimes \mathrm{id}_{DD}) \circ (F \boxtimes \mathrm{id}_{DD}).$$

Proof First, suppose $G_2 = 0$. Then $(G \circ F)_1 = G_1 \circ F_1$ and $(G \circ F)_2 = G_1 \circ F_2$. We have

$$\begin{aligned}(G \circ F) \boxtimes \mathrm{id}_{DD} &= 1 \otimes (G \circ F)_1 + \xi \circ ((G \circ F)_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &= 1 \otimes (G_1 \circ F_1) + \xi \circ (G_1 \otimes \mathrm{id}) \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(G \boxtimes \mathrm{id}_{DD}) \circ (F \boxtimes \mathrm{id}_{DD}) \\ &\quad = (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (G \boxtimes \mathrm{id}_{DD})) \circ (F \boxtimes \mathrm{id}_{DD}) \\ &\quad = (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes 1 \otimes G_1) \circ (1 \otimes F_1 + \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD})) \\ &\quad = 1 \otimes (G_1 \circ F_1) + (\mathrm{id} \otimes G_1) \circ \xi \circ (F_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}).\end{aligned}$$

This expression equals $(G \circ F) \boxtimes \mathrm{id}_{DD}$ because G_1 is bigrading-preserving.

Now suppose instead that $F_2 = 0$. Then $(G \circ F)_1$ is still $G_1 \circ F_1$, and $(G \circ F)_2 = G_2 \circ (F_1 \otimes \mathrm{id})$. We have

$$\begin{aligned}(G \circ F) \boxtimes \mathrm{id}_{DD} &= 1 \otimes (G \circ F)_1 + \xi \circ ((G \circ F)_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &= 1 \otimes (G_1 \circ F_1) + \xi \circ (G_2 \otimes |\mathrm{id}|) \circ (F_1 \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&(G \boxtimes \mathrm{id}_{DD}) \circ (F \boxtimes \mathrm{id}_{DD}) \\ &\quad = (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (G \boxtimes \mathrm{id}_{DD})) \circ (F \boxtimes \mathrm{id}_{DD}) \\ &\quad = (\mu_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes (1 \otimes G_1 + \xi \circ (G_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}))) \circ (1 \otimes F_1) \\ &\quad = 1 \otimes (G_1 \circ F_1) + \xi \circ (G_2 \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) \circ F_1,\end{aligned}$$

which equals $(G \circ F) \boxtimes \mathrm{id}_{DD}$ because

$$(F_1 \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) = (\mathrm{id} \otimes \delta_{DD}) \circ F_1. \qquad \square$$

6.6.7 Proposition *As in Proposition 6.6.6, let F and G be $\mathcal{A}_\infty$–morphisms from $\widehat{A}$ to $\widehat{A}'$ with $F_n, G_n = 0$ for $n > 2$ (here we do not require that either $F_2 = 0$ or $G_2 = 0$). Let H be an $\mathcal{A}_\infty$–homotopy between F and G with $H_n = 0$ for $n > 1$. Define*

$$H \boxtimes \mathrm{id}_{DD} := 1 \otimes H_1;$$

then $H \boxtimes \mathrm{id}_{DD}$ is a homotopy of type D morphisms between $F \boxtimes \mathrm{id}_{DD}$ and $G \boxtimes \mathrm{id}_{DD}$.

Proof Let δ and δ' denote the type D operations on $\widehat{A} \boxtimes \widehat{DD}$ and $\widehat{A}' \boxtimes \widehat{DD}$ respectively. We want to show that

$$F \boxtimes \mathrm{id}_{DD} - G \boxtimes \mathrm{id}_{DD} = (\mathrm{id} \otimes H_1) \circ \delta + \delta' \circ H_1;$$

the other term in the type D homotopy relations of Definition 6.6.2 is zero for this special type of H. Expanding out the left side, we want to show that

$$1 \otimes F_1 - 1 \otimes G_1 + \xi \circ ((F_2 - G_2) \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) = (\mathrm{id} \otimes H_1) \circ \delta + \delta' \circ H_1.$$

By Example 6.5.14, the $\mathcal{A}_\infty$–homotopy relations for H give us the following two equations:

$$\begin{aligned} F_1 - G_1 &= m_1' \circ H_1 + H_1 \circ m_1, \\ F_2 - G_2 &= -m_2' \circ (H_1 \otimes |\mathrm{id}|) + H_1 \circ m_2. \end{aligned}$$

Thus, the left side of the type D homotopy relation is

$$\begin{aligned} &1 \otimes (m_1' \circ H_1) + 1 \otimes (H_1 \circ m_1) \\ &\qquad + \xi \circ ((-m_2' \circ (H_1 \otimes |\mathrm{id}|) + H_1 \circ m_2) \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &= 1 \otimes (m_1' \circ H_1) + 1 \otimes (H_1 \circ m_1) - \xi \circ (m_2' \otimes \mathrm{id}) \circ (H_1 \otimes |\mathrm{id}| \otimes |\mathrm{id}|) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\qquad + \xi \otimes (H_1 \otimes |\mathrm{id}|) \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &= 1 \otimes (m_1' \circ H_1) + 1 \otimes (H_1 \circ m_1) + \xi \circ (m_2' \otimes \mathrm{id}) \circ (H_1 \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\qquad + (\mathrm{id} \otimes H_1) \circ \xi \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &= 1 \otimes (m_1' \circ H_1) + 1 \otimes (H_1 \circ m_1) + \xi \circ (m_2' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \circ H_1 \\ &\qquad + (\mathrm{id} \otimes H_1) \circ \xi \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}). \end{aligned}$$

On the other hand, using the definition of δ and δ' in Definition 6.3.4, the right side of the type D homotopy relation can be expanded out as

$$\begin{aligned} &1 \otimes (H_1 \circ m_1) + (\mathrm{id} \otimes H_1) \circ \xi \circ (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \\ &\qquad + 1 \otimes (m_1' \circ H_1) + \xi \circ (m_2' \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{DD}) \circ H_1 \end{aligned}$$

which is identical to the previous expression after rearranging terms. Thus, $H \boxtimes \mathrm{id}_{DD}$ is a valid type D homotopy between $F \boxtimes \mathrm{id}_{DD}$ and $G \boxtimes \mathrm{id}_{DD}$. □

6.6.8 Corollary *Let $\mathcal{B}$ and $\mathcal{B}'$ be differential bigraded algebras over an idempotent ring $\mathcal{I}$. Let $\widehat{A}$ and $\widehat{A}'$ be differential bigraded right modules over $\mathcal{B}$ and let $(\widehat{DD}, \delta_{DD})$ be a rank-one type DD bimodule over $\mathcal{B}$ and $\mathcal{B}'$. Assume that $\widehat{A}$ and $\widehat{A}'$ are free as $\mathbb{Z}$–modules, with $\mathbb{Z}$–bases consisting of elements which are grading-homogeneous and have a unique right idempotent.*

Suppose there exist $\mathcal{A}_\infty$–morphisms $F\colon \widehat{A} \to \widehat{A}'$ and $G\colon \widehat{A}' \to \widehat{A}$ with $F_n = 0$ and $G_n = 0$ for $n > 2$, and such that either $F_2 = 0$ or $G_2 = 0$. Furthermore, suppose that $G \circ F$ is $\mathcal{A}_\infty$–homotopic to $\mathrm{id}_{\widehat{A}}$ via an $\mathcal{A}_\infty$–homotopy H with $H_n = 0$ for $n > 1$, and $F \circ G$ is $\mathcal{A}_\infty$–homotopic to $\mathrm{id}_{\widehat{A}'}$ via another $\mathcal{A}_\infty$–homotopy H' with $H'_n = 0$ for $n > 1$.

Then the type D structures $\widehat{A} \boxtimes \widehat{DD}$ and $\widehat{A}' \boxtimes \widehat{DD}$ over $\mathcal{B}'$, defined in Definition 6.3.4, are homotopy equivalent.

Proof This follows from Proposition 6.6.5, Proposition 6.6.6 and Proposition 6.6.7, together with the fact that the box tensor product with id_{DD} on morphisms sends identity morphisms to identity morphisms. □

6.6.9 Corollary *If T and T' are oriented tangle diagrams in $\mathbb{R}_{\geq 0} \otimes \mathbb{R}$ which are related by a Reidemeister move, then $\widehat{D}([T]^{\mathrm{Kh}})$ and $\widehat{D}([T']^{\mathrm{Kh}})$ are homotopy equivalent as type D structures over $m(\mathcal{B})^! \odot \mathcal{B}$. Thus, they are also homotopy equivalent as type D structures over the quotient algebra $\mathcal{B}\Gamma_n$.*

Proof The first claim follows from Corollary 6.6.8 and the proof of Corollary 6.5.18, in which the $\mathcal{A}_\infty$–morphisms $F = \widehat{A}(f)$ and $G = \widehat{A}(g)$ and the $\mathcal{A}_\infty$–homotopy $H = \widehat{A}(\psi)$ used to realize the $\mathcal{A}_\infty$–homotopy equivalences satisfy the conditions of Corollary 6.6.8. The second claim follows from Remark 6.6.3 above. □

Corollary 6.6.9 gives us an alternate proof of Roberts [12, Theorem 46].

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