Algebraic & Geometric Topology 17 (2017) 1771–1783

Equivariant corks

DAVE AUCKLY
HEE JUNG KIM
PAUL MELVIN
DANIEL RUBERMAN

For any finite subgroup G of $SO(4)$, we construct a contractible 4–manifold C with a G–action on its boundary that can be embedded in a closed 4–manifold so that cutting C out and regluing using distinct elements of G will always yield distinct smooth 4–manifolds. If we simply require G to be a subgroup of the mapping class group of the boundary, then such examples exist for groups that cannot act on any homology sphere.

57M99; 57R55

0 Introduction

A *cork* is a smooth, compact, contractible 4–manifold with an involution on its boundary that does not extend to a diffeomorphism of the full manifold. Akbulut [1] discovered this phenomenon for the classical Mazur manifold $\mathbb{W}$ [18] with the boundary involution τ shown in Figure 1, proving that $\mathbb{W}$ embeds in a 4–manifold X so that the result of removing $\mathbb{W}$ and regluing it using τ is not diffeomorphic to X.

This operation is called *cork twisting*, and it is now known (see Curtis, Freedman, Hsiang and Stong [9] and Matveyev [17]) that any two smooth, closed, simply connected 4–manifolds that are homeomorphic differ by a single cork twist. It is not known whether the same cork can be used in all situations, ie whether there exists a *universal cork*; it is indeed conceivable, though unlikely, that the Mazur cork is universal.

The property that the cork twist τ is an *involution* is interesting, indeed inherent in most constructions of corks to date, but it is not clear that it is fundamental to the

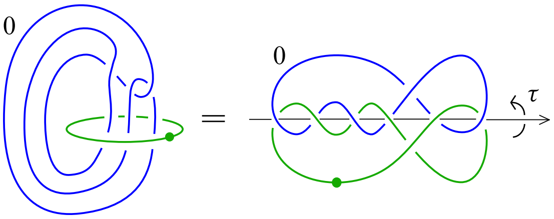

Figure 1: The Mazur cork

Published: 17 July 2017
DOI: 10.2140/agt.2017.17.1771

relation between cork twists and other smooth 4–manifold constructions. It is therefore natural to ask whether cutting and gluing by higher order diffeomorphisms of the boundary of a contractible submanifold of a 4–manifold can change the underlying smooth structure. In this note, we give an affirmative answer, producing examples of embeddings of contractible 4–manifolds with twists of arbitrary finite order that alter the ambient smooth structure; it follows that none of those twists extend over the contractible manifold. A different construction of such nonextending twists was given in a recent preprint of Tange [19].

In fact we show more: for suitable finite groups G, there exist contractible 4–manifolds with effective G–actions on the boundary that embed in closed 4–manifolds so that twists corresponding to distinct elements of G yield distinct smooth structures. We call such a gadget an *equivariant cork*, or G–cork if we want to specify the group.

Theorem A *There exist G–corks for any finite subgroup G of* $\mathrm{SO}(4)$.

If the action of G on S^3 is free, then the action of G on the boundary of the cork constructed in the theorem is free; this seems to be a new phenomenon, even for $G = \mathbb{Z}_2$. The notion of an equivariant cork can be extended to a *weak equivariant cork* where the relevant group is a subgroup of the mapping class group of the boundary; see the end of Section 1 for details. In the final section of the paper, we give an example of a *weak G–cork* in this sense, where G is a group that does not act effectively on any homology 3–sphere.

Theorem B *There are groups G that do not act effectively on any homology sphere, but for which there exist weak equivariant G–corks.*

The boundaries of the corks constructed in the proof of Theorem A are reducible. In a sequel we will prove the following theorem, using rather different techniques from those in the current paper.

Theorem C *Given an oriented 3–manifold Y with an effective, orientation-preserving, smooth action of a finite group G, there is an equivariant invertible $\mathbb{Z}[\pi_1(Y)]$–homology cobordism from it to a hyperbolic manifold.*

As in Akbulut and Ruberman [2], this immediately implies:

Corollary D *For any given finite subgroup G of* $\mathrm{SO}(4)$, *there exists a G–cork with hyperbolic boundary.*

Some experimentation with SnapPy [8] suggests that the simplest corks in Tange's paper [19] have hyperbolic boundaries, but a proof in general would require different techniques.

Acknowledgements The construction of G–corks and their hyperbolization were worked out by the authors at the American Institute of Mathematics (AIM) at our SQuaRE meeting in July 2015. We thank AIM for its support for this and future endeavors. Our results were announced at the 2016 Joint Mathematics Meeting; see [6].

Kim was supported by NRF grants 2015R1D1A1A01059318 and BK21 PLUS SNU Mathematical Sciences Division. Ruberman was supported by the NSF grants DMS-1105234 and DMS-1506328, and by NSF FRG grant 1065827.

1 Preliminaries and statement of results

In this section, we lay the groundwork for our proof of the existence of equivariant corks. Most of the ideas discussed here are well known, but since we will use "corks" in a broader sense than usual, and employ cork twists on multiple copies of boundary sums of embedded copies of the Mazur cork, we must give careful definitions of the relevant notions.

Corks and boundary equivalence

Extending the usual terminology, a *cork* will refer to any pair (C, g) where C is a smooth, compact, contractible 4–manifold, and g is an *arbitrary* diffeomorphism of ∂C. In particular, g need not be an involution, nor even of finite order, and C need not be Stein (as is often assumed; see Akbulut and Yasui [3]). But if g *is* a *special involution* (meaning orientation preserving with nonempty fixed point set, as with the Mazur twist τ) then we also refer to (C, g) as a *special* 2*–cork*.

In general, we call a cork (C, g) *trivial* if g extends to a diffeomorphism of C (it always extends to a homeomorphism by Freedman [11]) and *nontrivial* otherwise; with this convention, (B^4, g) is a trivial cork for any g, whereas the Mazur cork $(\mathbb{W}, \tau)$ is nontrivial. These notions induce an equivalence relation on corks associated with the same underlying manifold: (C, g) and (C, h) are *boundary equivalent* if and only if $(C, g^{-1}h)$ is trivial, ie $g^{-1}h$ extends over C.

Boundary sums of corks

The *boundary sum* operation $\natural$ is well defined on boundary equivalence classes of corks, as follows: Given corks (C_1, g_1) and (C_2, g_2), choose (for $i = 1, 2$) diffeomorphisms h_i isotopic (and thus boundary equivalent) to g_i that are the identity on 3–balls $B_i \subset \partial C_i$. Form $C_1 \natural C_2$ by identifying the C_i along the B_i so that h_1 and h_2 glue together to form $h_1 \sharp h_2$. The result

$$(C_1, g_1) \natural (C_2, g_2) := (C_1 \natural C_2, g_1 \sharp g_2)$$

may depend on the choices of h_i and B_i, but its boundary equivalence class does not. Note however that $\natural$ *is* well defined for special 2–corks *without* imposing boundary equivalence; just choose the B_i to be g_i–invariant 3–balls centered at fixed points, and then $g_1 \sharp g_2$ is a well-defined involution, independent of the choices up to equivariant diffeomorphism.

Cork embeddings

A *cork embedding* of (C, g) in a 4–manifold X is a smooth embedding $e\colon C \hookrightarrow X$ together with the induced map $\overline{g} = ege^{-1}$ on the boundary of its image $\overline{C} = e(C)$. The associated *cork twist* X^e_g is obtained by removing $\overline{C}$ from X and regluing using $\overline{g}$:

$$X^e_g = (X - \operatorname{int} \overline{C}) \cup_{\overline{g}} \overline{C}.$$

The embedding is *trivial* if X^e_g is diffeomorphic to X, and it is otherwise *nontrivial* or *effective*; note that this definition depends on both e and g. Thus the nontriviality of (C, g) can be verified by producing a nontrivial embedding, rather than trying to show directly that g does not extend smoothly across C.

Note that the definition of boundary equivalence of cork maps is compatible with the use of such maps in changing smooth structures, because the result of twisting by g is the same as the result of twisting by h when $g^{-1}h$ extends across C. Conversely, given any nontrivial cork (C, g), Akbulut and Ruberman [2] construct a pair of absolutely exotic structures on a contractible manifold related by twisting (C, g). It follows that for any two boundary inequivalent diffeomorphisms g and h, there is a 4–manifold X and an embedding $e\colon C \hookrightarrow X$ such that X^e_g is not diffeomorphic to X^e_h. Akbulut has made a similar observation.

Boundary sums of cork embeddings

Given any pair of embeddings $e_i\colon C_i \hookrightarrow X$ (for $i = 1, 2$) of corks (C_i, g_i) with disjoint images $\overline{C}_i = e_i(C_i)$ and induced boundary maps $\overline{g}_i\colon \partial\overline{C}_i \to \partial\overline{C}_i$, both twists can be performed simultaneously to produce the 4–manifold

$$X^{e_1e_2}_{g_1g_2} = (X - \operatorname{int}(\overline{C}_1 \sqcup \overline{C}_2)) \cup_{\overline{g}_1 \sqcup \overline{g}_2} (\overline{C}_1 \sqcup \overline{C}_2).$$

Alternatively, $\overline{C}_1$ and $\overline{C}_2$ can be joined by an embedded 1–handle in X, the thickening of an arc α in $X - \operatorname{int}(\overline{C}_1 \sqcup \overline{C}_2)$ from $\overline{C}_1$ to $\overline{C}_2$. The result is an embedding $e_1 \natural e_2$ of the single cork $(C_1, g_1) \natural (C_2, g_2) = (C_1 \natural C_2, g_1 \sharp g_2)$ (where, as noted above, the map $g_1 \sharp g_2$ is only defined up to boundary equivalence unless the g_i are special involutions) whose cork twist is independent of α. Indeed, it is readily seen that the single cork twist $X^{e_1 \natural e_2}_{g_1 \sharp g_2}$ is diffeomorphic to the pair of cork twists $X^{e_1e_2}_{g_1g_2}$.

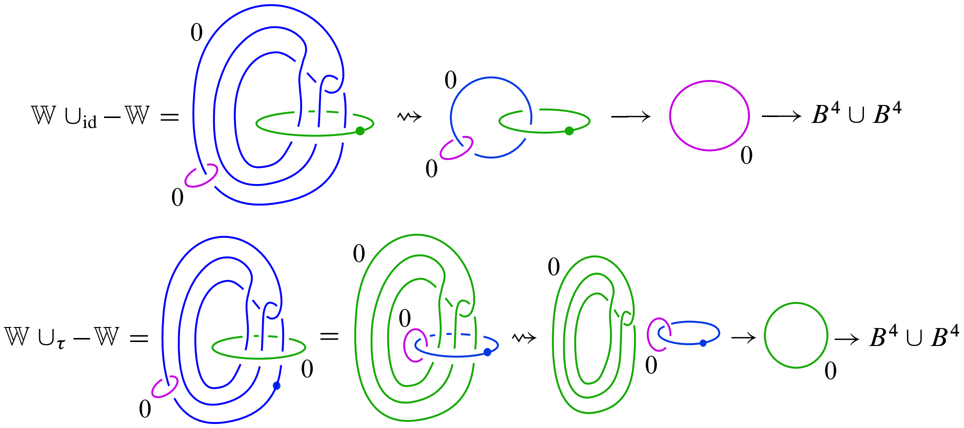

Figure 2: Trivial embedding of the Mazur cork in S^4

This process can be iterated to construct the *multiple cork twist* $X^{e_1\cdots e_n}_{g_1\cdots g_n}$ of a family $e_1, \dots, e_n$ of disjoint embeddings of corks $(C_1, g_1), \dots, (C_n, g_n)$ in X, or a single cork twist $X^{e_1\natural\cdots\natural e_n}_{g_1\sharp\cdots\sharp g_n}$ of an embedding of the boundary sum of the (C_i, g_i). Both twists produce the same smooth 4–manifold. This construction will play a key role in what follows.

Trivial cork embeddings

Most explicit corks (C, g) in the literature can be shown to have trivial embeddings in the 4–ball, and thus in every 4–manifold. In particular, it suffices to prove that the double $C \cup_{\mathrm{id}} -C$ and twisted double $C \cup_g -C$ are both diffeomorphic to the 4–sphere, often accomplished by an elementary Kirby calculus argument; cf Akbulut and Yasui [5, Section 2.6]. This is illustrated for the Mazur cork $(\mathbb{W}, \tau)$ in Figure 2, where the squiggly and straight arrows represent handle slides and cancellations, respectively, and as usual, the 3 and 4–handles are not drawn.

Equivariant corks

If G is a subgroup of the diffeomorphism group of ∂C with (C, g) nontrivial for all $g \neq 1$ in G, then (C, G) is called a *G–cork*. For cyclic G of finite order n, we refer to the corks (C, g) for generators g of G as *n–corks*. All explicit corks that have appeared in the literature prior to [19] are special 2–corks; recently, Gompf [13; 14] has shown how to construct $\mathbb{Z}$–corks.

There is a more general notion, which we call a *weakly equivariant cork*, in which the group G is a subgroup of the mapping class group of the boundary, ie the group of isotopy classes of diffeomorphisms. In this situation, it is more appropriate to use the relation of isotopy, rather than boundary equivalence, because the subgroup of diffeomorphisms of the boundary that extend across the cork need not be normal. Hence the set of boundary equivalent diffeomorphisms does not in general form a group in any natural way. In the last section, we give a construction of weakly equivariant corks for many groups G that are not subgroups of $\mathrm{SO}(4)$, and in fact that do not act effectively on any homology 3–sphere.

In general, if C is a cork with an effective G–action on ∂C, then an embedding $e\colon C \hookrightarrow X$ will be said to be *G–effective* if $X^e_{g_1}$ and $X^e_{g_2}$ are smoothly distinct for any $g_1 \neq g_2$ in G. Thus the existence of such embeddings shows that (C, G) is a G–cork. In this case, one has a G–action on the set of 4–manifolds $\{X^e_g \mid g \in G\}$ in the sense that $(X^e_{g_1})^{\bar{e}}_{g_2} = X^e_{g_1 g_2}$ for any two elements $g_1, g_2 \in G$, where $\bar{e}\colon C \to X^e_{g_1}$ is the obvious embedding induced by e.

For the reader's convenience, we repeat the statement of our main result:

Theorem A *There exist G–corks for any finite subgroup G of* $\mathrm{SO}(4)$.

Addenda (1) The proof will show that if $|G| = n$, then the boundary sum $\natural_{n^2}(\mathbb{W}, \tau)$ of n^2 copies of the Mazur cork can be given a G–cork structure that has G–effective embeddings in any blown-up elliptic surface $E(2k)\,\#\, m\overline{\mathbb{C}\mathrm{P}}^2$ for $k, m \geq n(n-1)/2$.

(2) More generally, if G is any finite group that acts effectively on the boundary of a compact, contractible submanifold of $\mathbb{R}^4$, then essentially the same proof shows that there is a G–cork with an effective embedding into a closed manifold; Theorem C can then be used to construct such corks with hyperbolic boundary.

2 Construction of equivariant corks

Our proof of Theorem A relies on the existence of certain embeddings e_i of the Mazur cork $(\mathbb{W}, \tau)$ in the blown-up Kummer surface

$$\mathbb{E} := E(2)\,\#\,\overline{\mathbb{C}\mathrm{P}}^2.$$

Here $E(2)$ is the minimal elliptic surface of Euler characteristic 24 (or Kummer surface; see for example [15]). The key input from Seiberg–Witten theory is the count of the number of basic classes in the associated cork twists $\mathbb{E}^{e_i}_\tau$.

Definition 2.1 Let X be a smooth, closed, simply connected 4–manifold. If $b_2^+(X)$ is odd and greater than 1, then $\mathcal{N}(X)$ will denote the number of Seiberg–Witten basic classes of X, and otherwise $\mathcal{N}(X) = 0$. For example, $\mathcal{N}(\mathbb{E}) = 2$ (the basic classes are $\pm\overline{\mathbb{C}\mathrm{P}}^1$).

Akbulut [1] established the nontriviality of $(\mathbb{W},\tau)$ by constructing a nontrivial embedding $e_0\colon \mathbb{W}\hookrightarrow\mathbb{E}$ with reducible cork twist $\mathbb{E}_\tau^{e_0}\cong 3\mathbb{C}\mathrm{P}^2\,\#\,20\overline{\mathbb{C}\mathrm{P}}^2$, so in particular, $\mathcal{N}(\mathbb{E}_\tau^{e_0})=0$. It was later observed [7] that such an embedding could be chosen with image in the complement $\mathbb{E}^\bullet$ of a *nucleus* in $\mathbb{E}$; see [12].

More recent work of Akbulut and Yasui [4] shows that $(\mathbb{W},\tau)$ has another nontrivial embedding $e_2\colon \mathbb{W}\hookrightarrow\mathbb{E}^\bullet$ with $\mathcal{N}(\mathbb{E}_\tau^{e_2})\neq 0$. The nontriviality of e_2 was proved by showing that $\mathbb{E}_\tau^{e_2}$ results from a rational blow-down of $\mathbb{E}$ [10], leaving $\mathcal{N}$ unchanged, followed by an honest blow-up, doubling $\mathcal{N}$, so $\mathcal{N}(\mathbb{E}_\tau^{e_2})=4$. (In particular, this follows from Theorem 4.1 for $p=2$, Proposition 5.1 for $n=1$ and $p_1=2$, and Lemma 6.6 in [4].)

As noted in the last section, $(\mathbb{W},\tau)$ also embeds *trivially* into any 4–manifold. Choose one such embedding $e_1\colon \mathbb{W}\hookrightarrow\mathbb{E}^\bullet$. Thus e_0, e_1 and e_2 are numbered so that $\mathcal{N}(\mathbb{E}_\tau^{e_i})=i\mathcal{N}(\mathbb{E})$. Only e_1 and e_2 are needed to prove the following key result, which is a strengthening of an analogous *noncompact* embedding theorem of Akbulut and Yasui [5, Theorem 1.5].

Lemma 2.2 *For each $n>0$, there exists a 2–cork $(\mathbb{S},\sigma)$ that has n disjoint embeddings $s_1,\dots,s_n$ in some closed 4–manifold X, with distinct cork twists*

$$X_\sigma^{s_1}\cong X,\ X_\sigma^{s_2},\dots,X_\sigma^{s_n}.$$

For example, the boundary sum $(\mathbb{S},\sigma)=\natural_n(\mathbb{W},\tau)$ has n such embeddings in the blown-up elliptic surface $X=E(2k)\,\#\,m\overline{\mathbb{C}\mathrm{P}}^2$ for any $k,m\geq n(n-1)/2$.

Proof It suffices to prove the last statement. First consider the case $k=m=n^2$, and view $X=E(2n^2)\,\#\,n^2\overline{\mathbb{C}\mathrm{P}}^2$ as the fiber sum of n^2 copies of the blown-up Kummer surface $\mathbb{E}=E(2)\,\#\,\overline{\mathbb{C}\mathrm{P}}^2$ along regular torus fibers in a chosen nucleus. Denote the copies of $\mathbb{E}$ by $\mathbb{E}_{ij}$ for $1\leq i,j\leq n$. Choose an embedding e_{ij} of $(\mathbb{W},\tau)$ in each summand $\mathbb{E}_{ij}^\bullet$, with $e_{ij}=e_1$ if $i\leq j$ and $e_{ij}=e_2$ if $i>j$. For $1\leq i\leq n$, let s_i be the boundary sum $e_{i1}\natural\cdots\natural e_{in}$ of all the embeddings in the "i^{th} row". Then the s_i are distinct embeddings of $(\mathbb{S},\sigma)=\natural_n(\mathbb{W},\tau)$ and can be chosen with disjoint images by choosing the 1–handles that join the summands to be disjoint. Furthermore, s_i has $i-1$ nontrivial summands and $n-i+1$ trivial ones, and so $\mathcal{N}(X_\sigma^{s_i})=2^{i-1}\mathcal{N}(X)$. Since $\mathcal{N}(X)\neq 0$, the $X_\sigma^{s_i}$ are pairwise distinct.

Of course, one can be more efficient by using only the "nontrivial" copies of $\mathbb{E}$, ie $\mathbb{E}_{ij}$ for $i>j$, and putting all the trivial embeddings of the Mazur cork inside one of these. This handles the smallest case $k=m=n(n-1)/2$, and the fiber sum and blow-up formulas for Seiberg–Witten invariants show that k and m can be increased at will. □

Proof of Theorem A

Given a finite subgroup G of SO(4) of order n, apply Lemma 2.2 to produce n disjoint embeddings s_g of a cork $(\mathbb{S},\sigma)$ in a closed 4–manifold X, indexed by the elements of G, with distinct cork twists $X_\sigma^{s_g}$. Using these cork embeddings, we construct a G–cork $(\mathbb{T},G)$ and a G–effective embedding $t\colon \mathbb{T}\to X$, as follows.

The underlying contractible manifold $\mathbb{T}$ is the boundary sum $\natural_n\mathbb{S}$ of n copies of $\mathbb{S}$. To define the G action on $\partial\mathbb{T}$, it is convenient to represent $\mathbb{T}$ as a cork twist on a diffeomorphic copy $\overline{\mathbb{T}}$ of itself that supports a natural G–action, namely the equivariant boundary sum

$$\overline{\mathbb{T}} = B^4 \natural (G\times\mathbb{S})$$

taken along a *principal* orbit $\{b_g \mid g\in G\}$ of the linear G action on ∂B^4, where G acts on $G\times\mathbb{S}$ by left multiplication on the first factor and trivially on the second. In other words, $\overline{\mathbb{T}}$ is obtained from a disjoint union of the 4–ball and n copies $\mathbb{S}_g$ of $\mathbb{S}$ (indexed by $g\in G$) by adding 1–handles joining $b_g\in\partial B^4$ to $x_g\in\partial\mathbb{S}_g$, where the $x_g\in\partial\mathbb{S}_g$ correspond to a chosen point $x\in\partial\mathbb{S}$. The G action is linear on B^4, and permutes the copies of $\mathbb{S}_g$ by left multiplication on the subscript (since the boundary sum is along a principal orbit).

Now the embeddings s_g of $\mathbb{S}$ can be used to define an embedding

$$\overline{t}\colon \overline{\mathbb{T}}\hookrightarrow X$$

by identifying $\mathbb{S}_g$ with the image $s_g(\mathbb{S})$ in X, B^4 with a small 4–ball B disjoint from the $\mathbb{S}_g$, and the 1–handles joining B^4 to the $\mathbb{S}_g$ with embedded 1–handles.

To obtain $\mathbb{T}$, we twist a shrunken copy of the cork $1\times\mathbb{S}$ in $\overline{\mathbb{T}}$. To make this precise, recall that $\overline{\mathbb{T}}$ contains n copies $\mathbb{S}_g = g\times\mathbb{S}$ of $\mathbb{S}$, the images of the embeddings $e_g\colon \mathbb{S}\hookrightarrow\overline{\mathbb{T}}$ sending x to (g,x). Consider an embedding $s\colon\mathbb{S}\hookrightarrow\mathbb{S}$ that shrinks $\mathbb{S}$ inside itself; that is, s is the identity off of a boundary collar $\partial\mathbb{S}\times[0,1)$, and maps (x,t) to $(x,(t+1)/2)$ inside the collar. Then $e=e_1\circ s$ embeds $\mathbb{S}$ onto a shrunken copy of $\mathbb{S}_1$. We define $\mathbb{T}$ to be the cork twist associated with this embedding:

$$\mathbb{T} = \overline{\mathbb{T}}_\sigma^e.$$

Since the $\partial\mathbb{T}=\partial\overline{\mathbb{T}}$, there is still a G–action on $\partial\mathbb{T}$, and this defines our cork $(\mathbb{T},G)$. Note that $\mathbb{T}$ is actually diffeomorphic to $\overline{\mathbb{T}}$, and thus to $\natural_n\mathbb{S}$, since $\natural$ is a well defined operation, but for our purposes it is most convenient to describe $\mathbb{T}$ as a cork twist of $\overline{\mathbb{T}}$.

Now observe that the embedding $\overline{t}\colon\overline{\mathbb{T}}\hookrightarrow X$ above induces an embedding

$$t\colon\mathbb{T}\hookrightarrow X_\sigma^{s_1}$$

since $\mathbb{T} = \overline{\mathbb{T}}^{e}_{\sigma}$. Furthermore, twisting this embedding of $\mathbb{T}$ by an element $g \in G$ just transfers the cork twist from $\mathbb{S}_1$ to $\mathbb{S}_g$; that is,

$$(X^{s_1}_{\sigma})^{t}_{g} = X^{s_g}_{\sigma}.$$

Since the smooth 4–manifolds $X^{s_g}_{\sigma}$ are distinct for $g \in G$, this shows that t is a G–effective embedding, and so $(\mathbb{T}, G)$ is a G–cork. This completes the proof of Theorem A. □

Remark Even in the case $G = \mathbb{Z}_2$ this result can give something new. Applying the construction from Theorem A to the free $\mathbb{Z}_2$ action on S^3 extended across B^4 we get a 2–cork with free action on the boundary.

Proof of the addenda to Theorem A

The first addendum to the theorem follows from this proof by using $(\mathbb{S}, \sigma) = \natural_n(\mathbb{W}, \tau)$ and $X = E(2k) \# m\overline{\mathbb{C}\mathrm{P}}^2$, as provided by the lemma. Note that in the proof, $X^{s_1}_{\sigma}$ is diffeomorphic to X since s_1 is a trivial cork embedding, so t can be viewed as an embedding of $\natural_{n^2}\mathbb{W} \hookrightarrow X$.

With regard to the second addendum, if a finite group G acts on a compact contractible submanifold of $\mathbb{R}^4$, we may repeat the argument replacing B^4 by the contractible submanifold to produce a G–cork $\mathbb{T}$. To build a G–cork with hyperbolic boundary, let $\mathbb{U}$ be an invertible cobordism from $\partial\mathbb{T}$ to a hyperbolic 3–manifold M with inverse $\mathbb{V}$ as given by Theorem C. Then

$$\mathbb{T} \cup_{\partial\mathbb{T}} \mathbb{U} \ \subset\ \mathbb{T} \cup_{\partial\mathbb{T}} \mathbb{U} \cup_M \mathbb{V} \ \cong\ \mathbb{T},$$

and $\mathbb{T} \cup_{\partial\mathbb{T}} \mathbb{U}$ inherits a G action so twisting it via g has the same effect as twisting $\mathbb{T}$ since g extends across $\mathbb{V}$. □

Remark From the construction, we see that our G–corks are boundary-connected sums of Stein manifolds, and hence are Stein. In contrast to the argument in [19], this fact does not play any role in our verification that our corks are effective.

3 Weakly equivariant corks

In this section, we construct examples of weakly equivariant corks for certain finite groups that are not subgroups of $\mathrm{SO}(4)$. In fact, these groups cannot act on any homology sphere, so there are no corresponding equivariant corks. This will prove Theorem B.

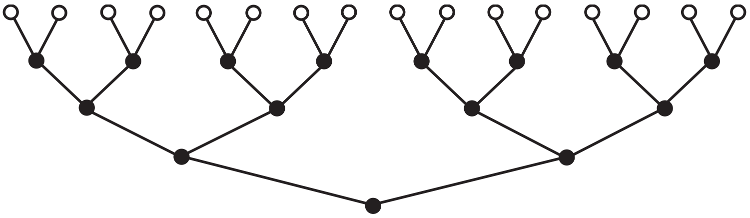

Figure 3: A weak C_2^4–cork

Proof of Theorem B

Fix $n \geq 4$, and let $G = C_2^n$, the product of n copies of the cyclic group C_2. It is known that G does not act effectively on any homology 3–sphere [20, Proposition 3]. In this proof, we show how to construct a nontrivial weak G–cork $\mathbb{V}$.

Apply Lemma 2.2 to get a 2–cork $(\mathbb{S},\sigma)$ with 2^n inequivalent embeddings s_g (for $g \in G$) in some 4–manifold X, meaning their cork twists $X_\sigma^{s_g}$ are 2^n distinct smooth 4–manifolds. For convenience, assume that $X_\sigma^1 \cong X$. For example, $\mathbb{S}$ could be the boundary sum of 2^n Mazur corks, with $X = E(2^{2n+1}) \# 2^{2n}\overline{\mathbb{C}\mathrm{P}}^2$; see the proof of Lemma 2.2.

As in the proof of Theorem A, we will define the cork $\mathbb{V}$ to be a suitable cork twist of a diffeomorphic copy $\overline{\mathbb{V}}$ of $\mathbb{V}$. To define $\overline{\mathbb{V}}$, consider a full binary tree T of height n, built from the bottom up, as shown in Figure 3 for the case $n = 4$. Thus T has one vertex at the root, two at the first level, four at the second level, etc. At the top there are 2^n vertices which can be indexed in a natural way by the elements of G (as explained below). To get $\overline{\mathbb{V}}$, replace the black dots by 4–balls, the white dots by copies of the cork $\mathbb{S}$ (referred to as the *leaves* of the cork) and the edges by 1–handles. Also choose an *equatorial* 3*–disk* D for each black 4–ball B that separates the 1–handle attached to B below D (if any) from the two attached above; D splits $\overline{\mathbb{V}}$ into two components with closures D^+ (locally above D) and D^- (locally below D).

Let $\tau_0, \dots, \tau_{n-1}$ denote the generators of the C_2 factors in $G = C_2^n$, and let τ_k act on $\overline{\mathbb{V}}$ by performing half Dehn twists on all the level k equatorial 3–disks. Here a *half Dehn twist* about such a disk D is the diffeomorphism of $\overline{\mathbb{V}}$ that leaves D^- fixed, sends a collar neighborhood $D \times [0,\pi]$ of D in D^+ to itself by the map $(x,\theta) \mapsto (\mathrm{rot}_\theta(x), \theta)$, and sends the rest of D^+ to itself in the obvious way, reversing the order of the leaves above D. Thus, for example, τ_0 reverses the order of all the leaves at the top, τ_1 independently reverses the orders of the first and second halves of the leaves, and so forth. Note that a *full Dehn twist* of a 4–manifold X can be defined in a similar way about *any* 3–disk D that is either properly embedded or embedded in ∂X. In either case one uses a collar $D \times [0, 2\pi]$ that restricts to a collar of ∂D in ∂X,

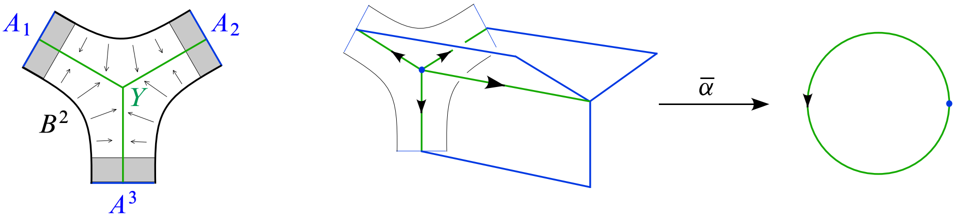

Figure 4: $A_i, Y \subset B^2$ when $n = 3$ (left) and the map $\overline{\alpha}\colon Y \times I \to S^1$ (right)

lying to the outside of D when $D \subset \partial X$; the shaded region in Figure 4 (left) illustrates how the collar meets the boundary in this latter case.

Now observe that τ_k is of order 2 in the mapping class group of $\overline{\mathbb{V}}$. This is clear for τ_0, since τ_0^2 is a full Dehn twist about the equatorial disk D_0 that untwists by an isotopy over the 4–ball D_0^- below it, and in general we claim that τ_k^2 is isotopic to τ_{k-1}^2. Indeed, the portion of $\overline{\mathbb{V}}$ lying between level $k-1$ and level k is a union of 4–balls, each containing exactly three equatorial 3–disks in its boundary. Thus it suffices to prove that a full twist about two of these disks is isotopic to a full twist about the third. Since $\pi_1 \mathrm{SO}(3) = \mathbb{Z}_2$, this is a consequence of the following elementary fact (cf [16, page 190]):

Lemma 3.1 *The composition δ of Dehn twists of a 4–ball B about any finite number of disjoint 3–disks $D_1, \dots, D_n$ in its boundary is isotopic to the identity, leaving the D_i fixed.*

Proof of the Lemma View $B = B^2 \times B^2$ and $D_i = A_i \times B^2$, where the A_i are disjoint arcs in ∂B^2. Let $r\colon B^2 \to Y$ be a deformation retraction that collapses each A_i to its midpoint a_i, where Y is the cone $0 * \{a_1, \dots, a_n\}$. Pictures of the arcs A_i and the graph Y in B^2, and an indication of the retraction r, are shown in Figure 4 (left) for the case $n = 3$, with collars corresponding to the shaded regions.

With this parametrization $B = B^2 \times B^2$, we can take

$$\delta(x, y) = (x, \mathrm{rot}_{\alpha(r(x))}(y)),$$

where $\alpha\colon Y \to S^1$ is a map of degree one on each edge $e_i = 0 * a_i$ of Y. Evidently, α extends to a map $\overline{\alpha}\colon Y \times I \to S^1$ that has degree one on each edge $e_i \times 0$ and $0 \times I$, and is constant on each edge $e_i \times 1$ and $a_i \times I$; see Figure 4 (right). This defines the desired isotopy δ_t from $\delta = \delta_0$ to the identity, rel the D_i, given by $\delta_t(x, y) = (x, \mathrm{rot}_{\overline{\alpha}(r(x),t)}(y))$. □

Continuing with the proof of Theorem B, it is clear that the action of the τ_k extends to an embedding of G in the mapping class group of $\overline{\mathbb{V}}$, and that distinct elements of G

carry the first leaf to distinct leaves. This gives a natural way to index the leaves of $\overline{\mathbb{V}}$ by the elements $g \in G$, according to where g carries the first leaf. Thus, for example, the last leaf is indexed by τ_0, while the $(2^{n-1})^{\text{st}}$ leaf is indexed by τ_1.

Now let $\mathbb{V}$ be the cork twist of $\overline{\mathbb{V}}$ along (a shrunken copy of) the first leaf. Then $\partial\mathbb{V}$ is naturally identified with $\partial\overline{\mathbb{V}}$, so there is an induced embedding of G in the mapping class group of $\partial\mathbb{V}$. To see that this defines a weak G–cork structure on $\mathbb{V}$, just choose an embedding $e\colon \mathbb{V} \hookrightarrow X$ that restricts to the embeddings s_g (for $g \in G$) on the leaves of $\mathbb{V}$. Then $X_g^e = X_\sigma^{s_g}$, and so X_g^e and X_h^e are not diffeomorphic unless $g = h$. □

References

[1] **S Akbulut**, *A fake compact contractible* 4*–manifold*, J. Differential Geom. 33 (1991) 335–356 MR

[2] **S Akbulut**, **D Ruberman**, *Absolutely exotic compact* 4*–manifolds*, Comment. Math. Helv. 91 (2016) 1–19 MR

[3] **S Akbulut**, **K Yasui**, *Corks, plugs and exotic structures*, J. Gökova Geom. Topol. 2 (2008) 40–82 MR

[4] **S Akbulut**, **K Yasui**, *Knotting corks*, J. Topol. 2 (2009) 823–839 MR

[5] **S Akbulut**, **K Yasui**, *Stein* 4*–manifolds and corks*, J. Gökova Geom. Topol. 6 (2012) 58–79 MR

[6] **D Auckly**, **H J Kim**, **P Melvin**, **D Ruberman**, *From tangles to equivariant hyperbolic corks*, AMS abstract (2016) `http://tinyurl.com/jmm2181abs1116-57-1143`

[7] **Ž Bižaca**, **R E Gompf**, *Elliptic surfaces and some simple exotic* $\mathbf{R}^4$*'s*, J. Differential Geom. 43 (1996) 458–504 MR

[8] **M Culler**, **N M Dunfield**, **J R Weeks**, *SnapPy, a computer program for studying the topology of* 3*–manifolds* `http://snappy.computop.org`

[9] **C L Curtis**, **M H Freedman**, **W C Hsiang**, **R Stong**, *A decomposition theorem for* h*–cobordant smooth simply-connected compact* 4*–manifolds*, Invent. Math. 123 (1996) 343–348 MR

[10] **R Fintushel**, **R J Stern**, *Rational blowdowns of smooth* 4*–manifolds*, J. Differential Geom. 46 (1997) 181–235 MR

[11] **M H Freedman**, *The topology of four-dimensional manifolds*, J. Differential Geom. 17 (1982) 357–453 MR

[12] **R E Gompf**, *Nuclei of elliptic surfaces*, Topology 30 (1991) 479–511 MR

[13] **R E Gompf**, *Infinite order corks*, preprint (2016) arXiv To appear in Geom. Topol.

[14] **R E Gompf**, *Infinite order corks via handle diagrams*, preprint (2016) arXiv

[15] **R E Gompf**, **A I Stipsicz**, *4–manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, Amer. Math. Soc., Providence, RI (1999) MR

[16] **H Hendriks**, *Applications de la théorie d'obstruction en dimension 3*, Bull. Soc. Math. France Mém. 53 (1977) 81–196 MR

[17] **R Matveyev**, *A decomposition of smooth simply-connected h–cobordant 4–manifolds*, J. Differential Geom. 44 (1996) 571–582 MR

[18] **B Mazur**, *A note on some contractible 4–manifolds*, Ann. of Math. 73 (1961) 221–228 MR

[19] **M Tange**, *Finite order corks*, preprint (2016) arXiv

[20] **B Zimmermann**, *On the classification of finite groups acting on homology 3–spheres*, Pacific J. Math. 217 (2004) 387–395 MR

Department of Mathematics, Kansas State University
Manhattan, KS 66506, United States

Department of Mathematical Sciences, Seoul National University
Seoul 151-747, South Korea

Department of Mathematics, Bryn Mawr College
Bryn Mawr, PA 19010, United States

Department of Mathematics, Brandeis University, MS 050
Waltham, MA 02454, United States

dav@math.ksu.edu, heejungorama@gmail.com, pmelvin@brynmawr.edu, ruberman@brandeis.edu

www.math.ksu.edu/~dav/, www.math.uga.edu/directory/hee-jung-kim, www.brynmawr.edu/math/people/melvin/, people.brandeis.edu/~ruberman/

Received: 20 April 2016 Revised: 14 September 2016