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A homology-valued invariant for trivalent fatgraph spines

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We introduce an invariant for trivalent fatgraph spines of a once-bordered surface, which takes values in the first homology of the surface. This invariant is a secondary object coming from two 1–cocycles on the dual fatgraph complex, one introduced by Morita and Penner in 2008, and the other by Penner, Turaev and the author in 2013. We present an explicit formula for this invariant and investigate its properties. We also show that the mod 2 reduction of the invariant is the difference of two naturally defined spin structures on the surface.

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1 Introduction

Let $\Sigma_{g,1}$ be a once-bordered C^∞–surface of genus $g > 0$, and let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$ relative to the boundary. It is known that the Teichmüller space $\mathcal{T}(\Sigma_{g,1})$ of $\Sigma_{g,1}$ has an $\mathcal{M}_{g,1}$–equivariant ideal simplicial decomposition; see Penner [21]. Taking its dual, one obtains a contractible CW complex $\widehat{\mathcal{G}}(\Sigma_{g,1})$ on which $\mathcal{M}_{g,1}$ acts freely and properly discontinuously. This CW complex is called the dual fatgraph complex of $\Sigma_{g,1}$, since its cells are indexed by fatgraph spines of $\Sigma_{g,1}$, which are graphs embedded in the surface satisfying some conditions. Each 0–cell of $\widehat{\mathcal{G}}(\Sigma_{g,1})$ corresponds to a trivalent fatgraph spine, and by contracting nonloop edges we obtain higher-dimensional cells. In particular, each oriented 1–cell of $\widehat{\mathcal{G}}(\Sigma_{g,1})$ corresponds to a flip (or a Whitehead move) between trivalent fatgraph spines of $\Sigma_{g,1}$.

This combinatorial structure of the Teichmüller space has a number of applications to the cohomology of the mapping class group and the moduli space of Riemann surfaces. See eg Harer [4; 5], Harer and Zagier [6], Penner [20] and Kontsevich [12].

Recently, mainly motivated by the theory of the Johnson homomorphisms (see Johnson [7; 9] and Morita [16]), several authors considered 1–cocycles on $\widehat{\mathcal{G}}(\Sigma_{g,1})$ with coefficients in various $\mathcal{M}_{g,1}$–modules. In 2008, Morita and Penner [18] first gave such a 1–cocycle $j \in Z^1(\widehat{\mathcal{G}}(\Sigma_{g,1}); \Lambda^3 H)$, where $\Lambda^3 H$ is the third exterior power of the first homology group $H = H_1(\Sigma_{g,1}; \mathbb{Z})$. (In fact, they worked with a once-punctured surface, but their construction works for $\Sigma_{g,1}$ as well.) Being a 1–cocycle on $\widehat{\mathcal{G}}(\Sigma_{g,1})$, the cocycle j associates an element of $\Lambda^3 H$ to each flip. Fixing a trivalent

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fatgraph spine of $\Sigma_{g,1}$, one obtains from j a twisted 1–cocycle on $\mathcal{M}_{g,1}$. Morita and Penner proved that its cohomology class in $H^1(\mathcal{M}_{g,1}; \Lambda^3 H)$ is six times the extended first Johnson homomorphism $\widetilde{k}$ discovered by Morita [17]. Similar constructions are also considered by Bene, Kawazumi and Penner [1] for the second and higher Johnson homomorphisms, by Massuyeau [14] for Morita's refinement [16] of the higher Johnson homomorphisms, and by Kuno, Penner and Turaev [13] for the Earle class $k \in H^1(\mathcal{M}_{g,1}; H)$.

We emphasize that these cocycles on $\widehat{\mathcal{G}}(\Sigma_{g,1})$ are all explicit and simple. In this way, the Johnson homomorphisms and related objects extend *canonically* to the *Ptolemy groupoid*, the combinatorial fundamental path groupoid of $\widehat{\mathcal{G}}(\Sigma_{g,1})$; see Bene, Kawazumi and Penner [1].

It is interesting that there are many ways of constructing cocycle representatives for the cohomology classes such as $\widetilde{k}$ and k, and that each construction reflects its own viewpoint for studying the mapping class group. It can happen that two cocycles constructed differently give the same cohomology class. In such a case, it is quite natural to compare these cocycles and to expect a secondary object in the background.

We will compare the Morita–Penner cocycle j and the cocycle $m \in Z^1(\widehat{\mathcal{G}}(\Sigma_{g,1}); H)$ which is related to k and considered in Kuno, Penner and Turaev [13]. Contracting the coefficients by using the intersection pairing on H, one has a natural homomorphism

$$C\colon Z^1(\widehat{\mathcal{G}}(\Sigma_{g,1}); \Lambda^3 H) \to Z^1(\widehat{\mathcal{G}}(\Sigma_{g,1}); H).$$

Let $j' = C \circ j$. It turns out that there is an $\mathcal{M}_{g,1}$–equivariant 0–cochain $\xi \in C^0(\widehat{\mathcal{G}}(\Sigma_{g,1}); H)$ such that $2j' - m = \delta\xi$ (Proposition 3.1). The 0–cochain ξ assigns an element $\xi_G \in H$ to each trivalent fatgraph spine $G \subset \Sigma_{g,1}$.

We will study the secondary object ξ_G as an H–valued invariant for trivalent fatgraph spines $G \subset \Sigma_{g,1}$. First of all, Theorem 3.4 gives an explicit formula for ξ_G. Based on this formula, we show in Theorem 5.2 that ξ_G is nontrivial. At the present moment, we do not have a full understanding of the topological meaning of the invariant ξ_G. In Theorem 6.7, we give a partial result in this direction by relating the mod 2 reduction of ξ_G to two naturally defined spin structures on $\Sigma_{g,1}$. It would be interesting to seek for or to find an obstruction to an extension of ξ_G to fatgraph spines which are not necessarily trivalent. In view of the fact that trivalent fatgraph spines correspond to maximal-dimensional simplices of the ideal simplicial decomposition of $\mathcal{T}(\Sigma_{g,1})$, this is related to finding a $\mathcal{M}_{g,1}$–equivariant function on $\mathcal{T}(\Sigma_{g,1})$ which takes values in $H \otimes_{\mathbb{Z}} \mathbb{R} = H_1(\Sigma_{g,1}; \mathbb{R})$.

This paper is organized as follows. In Section 2, we first review the dual fatgraph complex and in particular describe its 2–skeleton. Then we recall the 1–cocycles j

from [18] and m from [13]. Also, we correct an error in [13] about the evaluation of m. In Section 3, we show the existence and uniqueness of ξ, and then present an explicit formula for ξ_G (Theorem 3.4). In Section 4, we show a certain gluing formula for ξ_G, and then the behavior of ξ_G under a special kind of flip. The latter result makes it possible to define ξ_G for a trivalent fatgraph spine G of a *punctured* surface. In Section 5, we discuss the nontriviality of ξ_G based on its explicit formula. In Section 6, we construct two spin structures on $\Sigma_{g,1}$ from each trivalent fatgraph spine $G \subset \Sigma_{g,1}$. Then we prove that their difference coincides with the mod 2 reduction of ξ_G. Along the way we give a combinatorial description of spin structures on $\Sigma_{g,1}$ (Theorem 6.2), which seems to be new. In the appendix, we consider another spin structure coming from a naturally defined nonsingular vector field on $\Sigma_{g,1}$.

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2 Fatgraph complex and cocycles

We fix some notation about graphs. By a graph we mean a finite CW complex of dimension one. For a graph G, we denote by $V(G)$ the set of vertices of G, by $E(G)$ the set of edges of G, and by $E^{\mathrm{ori}}(G)$ the set of oriented edges of G. For $v \in V(G)$, we denote by $E^{\mathrm{ori}}_v(G)$ the set of oriented edges pointing toward v. The number of elements of $E^{\mathrm{ori}}_v(G)$ is called the *valency* of v. For $e \in E^{\mathrm{ori}}(G)$, we denote by $\bar{e} \in E^{\mathrm{ori}}(G)$ the edge e with reversed orientation. A *fatgraph* is a graph G endowed with a cyclic ordering to $E^{\mathrm{ori}}_v(G)$ about each $v \in V(G)$.

Let $\Sigma_{g,1}$ be a compact connected oriented C^∞–surface of genus $g > 0$ with one boundary component. We fix two distinct points p and q on the boundary $\partial\Sigma_{g,1}$.

Definition 2.1 An embedding $\iota\colon G \hookrightarrow \Sigma_{g,1}$ of a fatgraph G into $\Sigma_{g,1}$ is called a *fatgraph spine* of $\Sigma_{g,1}$ if the following conditions are satisfied:

(1) The map ι is a homotopy equivalence.

(2) For any $v \in V(G)$, the cyclic ordering given to $E^{\mathrm{ori}}_v(G)$ is compatible with the orientation of $\Sigma_{g,1}$.

(3) We have $\iota(G) \cap \partial\Sigma_{g,1} = \{p\}$ and $\iota^{-1}(p)$ is a unique univalent vertex of G. The other vertices have valencies greater than 2.

The unique edge connected to $\iota^{-1}(p)$ is called the *tail* of G. We consider fatgraph spines up to isotopies relative to $\partial\Sigma_{g,1}$. If there is no danger of confusion, we identify G with $\iota(G)$, and write G instead of $\iota\colon G \hookrightarrow \Sigma_{g,1}$. We denote by $V^{\mathrm{int}}(G)$ the set of nonunivalent vertices of G. We say that G is *trivalent* if the valency of any nonunivalent vertex of G is 3.

Fatgraph spines appear naturally in the combinatorial description of the Teichmüller space of a punctured or bordered surface. This was first shown for punctured surfaces by Harer and Mumford [5] and Thurston from the holomorphic point of view based on a work by Strebel [24], and by Penner [19] and Bowditch and Epstein [2] from the point of view of hyperbolic geometry.

In this paper, we work mainly with the once-bordered surface $\Sigma_{g,1}$. For definiteness, let us define the Teichmüller space $\mathcal{T}(\Sigma_{g,1})$ as the space of Riemannian metric on $\Sigma_{g,1}$ of constant Gaussian curvature -1 with geodesic boundary, modulo pullback of the metric by self-diffeomorphisms of $\Sigma_{g,1}$ fixing q which are isotopic to the identity relative to q. Let $\mathcal{M}_{g,1}$ be the mapping class group of $\Sigma_{g,1}$ relative to $\partial\Sigma_{g,1}$. Namely, $\mathcal{M}_{g,1}$ is the group of self-diffeomorphisms of $\Sigma_{g,1}$ fixing the boundary $\partial\Sigma_{g,1}$ pointwise, modulo isotopies fixing $\partial\Sigma_{g,1}$ pointwise. Note that $\mathcal{M}_{g,1}$ is identified with the group of connected components of the group of self-diffeomorphisms of $\Sigma_{g,1}$ fixing q. Then pullback of the metric induces an action of $\mathcal{M}_{g,1}$ on $\mathcal{T}(\Sigma_{g,1})$. This action is known to be free and properly discontinuous.

Theorem 2.2 (Penner [21]) *There is an $\mathcal{M}_{g,1}$–equivariant ideal simplicial decomposition of $\mathcal{T}(\Sigma_{g,1})$ with the following properties:*

- *Each simplex corresponds to a fatgraph spine of $\Sigma_{g,1}$.*
- *The face relation between simplices corresponds to the contraction of a nonloop edge of a fatgraph spine.*

Let $\widehat{\mathcal{G}}(\Sigma_{g,1})$ be the *dual* of this ideal simplicial decomposition. This is an honest CW complex of dimension $4g-2$. We call $\widehat{\mathcal{G}}(\Sigma_{g,1})$ the *dual fatgraph complex* of $\Sigma_{g,1}$. Note that there is a natural cellular action of the mapping class group $\mathcal{M}_{g,1}$ on $\widehat{\mathcal{G}}(\Sigma_{g,1})$. In fact, there is an $\mathcal{M}_{g,1}$–equivariant deformation retract of $\mathcal{T}(\Sigma_{g,1})$ onto $\widehat{\mathcal{G}}(\Sigma_{g,1})$; see [22].

The 2–skeleton of $\widehat{\mathcal{G}}(\Sigma_{g,1})$ is described as follows:

- Each 0–cell corresponds to a trivalent fatgraph spine of $\Sigma_{g,1}$.
- Each 1–cell corresponds to a fatgraph spine G, where G has a unique 4–valent vertex and the other nonunivalent vertices have valency 3.

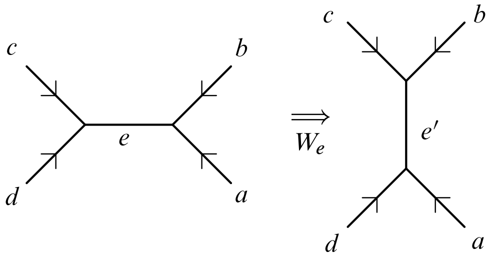

Figure 1: The flip along e applied to G (left) produces G' (right)

- Each *oriented* 1–cell corresponds to a *flip* (or a *Whitehead move*) between trivalent fatgraph spines. Here, if e is a nontail edge of a trivalent fatgraph spine, collapsing e and expanding the resulting 4–valent vertex to the unique distinct direction, one produces another trivalent fatgraph spine. We call this move a flip along e, and denote it by W_e. See Figure 1. If G' is obtained from G by a flip $W = W_e$, we write it as $G \xrightarrow{W} G'$. There is a natural bijection from $E(G)$ to $E(G')$ which restricts to an obvious identification of $E(G) \setminus \{e\}$ with $E(G') \setminus \{e'\}$. For this reason, we often use the same letter for edges of G and G' corresponding to each other via this bijection.
- Each 2–cell corresponds to a fatgraph spine G, where either G has a unique 5–valent vertex and the other nonunivalent vertices have valency 3, or G has two 4–valent vertices and the other nonunivalent vertices have valency 3.

Let G and G' be trivalent fatgraph spines. Since $\widehat{\mathcal{G}}(\Sigma_{g,1})$ is connected, there is a finite sequence of flips

$$G = G_0 \xrightarrow{W_1} G_1 \xrightarrow{W_2} G_2 \xrightarrow{W_3} \cdots \xrightarrow{W_m} G_m = G'$$

from G to G'. This sequence is not uniquely determined, but any two such sequences are related to each other by the following three types of relations among flips:

(1) **Involutivity relation** $W_{e'} \circ W_e = 1$ in the notation of Figure 1.

(2) **Commutativity relation** $W_{e_1} \circ W_{e_2} = W_{e_2} \circ W_{e_1}$ if e_1 and e_2 share no vertices.

(3) **Pentagon relation** $W_{f_4} \circ W_{g_3} \circ W_{f_2} \circ W_{g_1} \circ W_f = 1$ in the notation of Figure 2.

Here, we read composition of flips from right to left. The relations (2) and (3) come from the boundaries of 2–cells of $\widehat{\mathcal{G}}(\Sigma_{g,1})$.

There is a construction of twisted 1–cocycles on the mapping class group using the fatgraph complex appeared first in [18]. Let M be a (left) $\mathcal{M}_{g,1}$–module. By definition, a cellular 1–cochain c on $\widehat{\mathcal{G}}(\Sigma_{g,1})$ with values in M is an assignment of an element

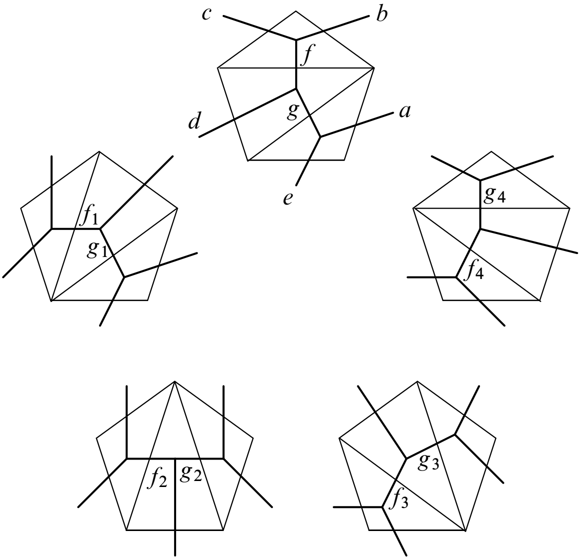

Figure 2: Pentagon relation

of M to each flip W satisfying $c(W_{e'}) = -c(W_e)$ for any pair of flips W_e and $W_{e'}$ as in Figure 1. Such a c is a 1–cocycle if it satisfies both the commutative equation

$$c(W_{e_1}) + c(W_{e_2}) = c(W_{e_2}) + c(W_{e_1}),$$

where e_1 and e_2 are any edges on a trivalent fatgraph spine sharing no vertices, and the pentagon equation

$$c(W_{f_4}) + c(W_{g_3}) + c(W_{f_2}) + c(W_{g_1}) + c(W_f) = 0$$

for any 5–tuple of flips as in Figure 2.

Now we assume that c is a 1–cocycle and is $\mathcal{M}_{g,1}$–equivariant in the sense that $\varphi \cdot c(W) = c(\varphi W)$ for any flip W and $\varphi \in \mathcal{M}_{g,1}$. Fix a trivalent fatgraph spine G. For $\varphi \in \mathcal{M}_{g,1}$, taking a sequence of flips

$$G = G_0 \xrightarrow{W_1} G_1 \xrightarrow{W_2} G_2 \xrightarrow{W_3} \cdots \xrightarrow{W_m} G_m = \varphi(G)$$

from G to $\varphi(G)$, we set

$$c_G(\varphi) := \sum_{i=1}^{m} c(W_i) \in M.$$

Since c is a 1–cocycle, this value does not depend on the choice of the sequence. The map $c_G\colon \mathcal{M}_{g,1} \to M$ is a twisted 1–cocycle. In fact, for $\varphi, \psi \in \mathcal{M}_{g,1}$, take a sequence of flips from G to $\varphi(G)$, and one from G to $\psi(G)$. Then the first sequence followed by application of φ to the second is a sequence of flips from G to $\varphi\psi(G)$. Since c is $\mathcal{M}_{g,1}$–equivariant, we obtain the cocycle condition

$$c_G(\varphi\psi) = c_G(\varphi) + \varphi \cdot c_G(\psi).$$

It is easy to see that the cohomology class $[c_G] \in H^1(\mathcal{M}_{g,1}; M)$ does not depend on the choice of G.

Here we record an elementary fact which will be used later.

Lemma 2.3 *Let M be an $\mathcal{M}_{g,1}$–module and suppose that c is an $\mathcal{M}_{g,1}$–equivariant cellular 1–cocycle on $\widehat{\mathcal{G}}(\Sigma_{g,1})$ with values in M. Then for any trivalent fatgraph spine G and any $\varphi, \psi \in \mathcal{M}_{g,1}$, we have*

$$c_G(\psi) + c_{\psi(G)}(\varphi) = c_G(\varphi) + \varphi \cdot c_G(\psi).$$

Proof Consider a sequence of flips from G to $\psi(G)$ and one from $\psi(G)$ to $\varphi\psi(G)$. The composition of these sequences is a sequence from G to $\varphi\psi(G)$, and thus we obtain $c_G(\varphi\psi) = c_G(\psi) + c_{\psi(G)}(\varphi)$. On the other hand, by the cocycle condition for c_G, we have $c_G(\varphi\psi) = c_G(\varphi) + \varphi \cdot c_G(\psi)$. □

We denote by $H = H_1(\Sigma_{g,1}; \mathbb{Z})$ the first integral homology group of $\Sigma_{g,1}$. Before giving examples of $\mathcal{M}_{g,1}$–equivariant cellular 1–cochains on $\widehat{\mathcal{G}}(\Sigma_{g,1})$, we recall from [18] homology markings for edges of fatgraph spines. Let G be a (not necessarily trivalent) fatgraph spine of $\Sigma_{g,1}$. For $e \in E^{\mathrm{ori}}(G)$, there is an oriented simple loop $\hat{e}$ on $\Sigma_{g,1}$ satisfying the following two conditions:

- The loop $\hat{e}$ intersects G once transversely at the middle point of e.
- The ordered pair of the velocity vectors of $\hat{e}$ and e at their intersection is compatible with the orientation of $\Sigma_{g,1}$.

Since the surface obtained from $\Sigma_{g,1}$ by cutting along G is a disk, the homotopy class of such an $\hat{e}$ is unique. We define $\mu(e) \in H$ to be the homology class of $\hat{e}$ and call it the *homology marking* of e. The map $\mu\colon E^{\mathrm{ori}}(G) \to H$ has the following properties:

(1) For any $e \in E^{\mathrm{ori}}(G)$, we have $\mu(\overline{e}) = -\mu(e)$.

(2) The set $\{\mu(e)\}_{e \in E^{\mathrm{ori}}(G)}$ generates H.

(3) For any $v \in V(G)$, we have

$$\sum_{e \in E_v^{\mathrm{ori}}(G)} \mu(e) = 0.$$

For example, in the notation of the left part of Figure 1, where we orient edges a, b, c, d as indicated, we have $\mu(a)+\mu(b)+\mu(c)+\mu(d)=0$.

In what follows, we consider $\mathcal{M}_{g,1}$–modules such as H and its third exterior power $\Lambda^3 H$. There is a twisted cohomology class $\widetilde{k} \in H^1\big(\mathcal{M}_{g,1}; \frac{1}{2}\Lambda^3 H\big)$ called the *extended first Johnson homomorphism* [17]. Here, $\frac{1}{2}\Lambda^3 H = \big\{\frac{1}{2}u \in \Lambda^3(H \otimes_{\mathbb{Z}} \mathbb{Q}) \mid u \in \Lambda^3 H\big\}$, where we take the canonical embedding $H \to H \otimes_{\mathbb{Z}} \mathbb{Q}$, $x \mapsto x \otimes 1$. This cohomology class has a fundamental importance in the study of the cohomology of the mapping class group; see [11].

Theorem 2.4 (Morita and Penner [18]) *Keep the notation in Figure 1. For the flip W_e, set*

$$j(W_e)=\mu(a)\wedge\mu(b)\wedge\mu(c) \in \Lambda^3 H.$$

Then j is an $\mathcal{M}_{g,1}$–equivariant 1–cocycle on $\widehat{\mathcal{G}}(\Sigma_{g,1})$, and $[j_G]=6\widetilde{k}$.

Using the intersection pairing $(\cdot)$ on the homology, we define an $\mathrm{Sp}(H)$–equivariant map

$$C\colon \Lambda^3 H \to H, \quad x\wedge y\wedge z \mapsto (x\cdot y)z+(y\cdot z)x+(z\cdot x)y$$

called the contraction. Morita [15] showed that if $g \geq 2$, the twisted cohomology group $H^1(\mathcal{M}_{g,1}; H)$ is infinite cyclic. As is remarked in [17], the element $k := C(2\widetilde{k})$ is a generator of this cohomology group. Since Earle [3] first gave a cocycle representative for k, we call k the *Earle class*; see [10]. The restriction of k to the Torelli subgroup of $\mathcal{M}_{g,1}$ is called the *Chillingworth homomorphism*; see [7, Section 5].

Theorem 2.5 (Kuno, Penner and Turaev [13]) *Keep the notation in Figure 1. For the flip W_e, set*

$$m(W_e)=\mu(a)+\mu(c) \in H.$$

Then m is an $\mathcal{M}_{g,1}$–equivariant 1–cocycle on $\widehat{\mathcal{G}}(\Sigma_{g,1})$, and $[m_G]=6k$.

Here we correct an error in [13]. Let $\varphi_{\mathrm{BP}}=\varphi$ be the torus BP map in [13, Figure 3], which was first considered in [18]. In [13, Lemma 1], it was asserted that $m(\varphi_{\mathrm{BP}})=4a$, but this is not true. More precisely, in the proof of the lemma, we computed the contribution of the second Dehn twist (5 flips) as $-4a$, but this should be corrected to $4a$.

Lemma 2.6 (correction of [13, Lemma 1]) *Let φ_{BP} be the torus BP map as above. Then $m(\varphi_{\mathrm{BP}})=12\mu(a)$.*

In [13, Theorem 6], it is asserted that $[m_G]=-2k$, but this should be corrected as in Theorem 2.5 above.

3 A secondary invariant

We consider the cocycle $j' = C \circ j$. For the flip W_e in the notation of Figure 1, we have

$$j'(W_e) = (a \cdot b)\mu(c) + (b \cdot c)\mu(a) + (c \cdot a)\mu(b) \in H.$$

Here and throughout the paper, we write eg $(a \cdot b)$ instead of $(\mu(a) \cdot \mu(b))$ for simplicity. By Theorems 2.4 and 2.5, for any trivalent fatgraph spine G, we have

$$2[j'_G] = [m_G] = 6k.$$

Therefore, there exists an element $\xi_G \in H$ such that $2j'_G - m_G = \delta\xi_G$. Here the symbol δ in the right-hand side means the coboundary map in the standard cochain complex of $\mathcal{M}_{g,1}$ with coefficients in H. Explicitly, we have $(\delta\xi_G)(\varphi) = \varphi \cdot \xi_G - \xi_G$ for any $\varphi \in \mathcal{M}_{g,1}$. Such a ξ_G is unique since only 0 is $\mathcal{M}_{g,1}$–invariant in H. We regard the collection $\xi = \{\xi_G\}_G$ as a cellular 0–cochain of $\widehat{\mathcal{G}}(\Sigma_{g,1})$ with coefficients in H.

Proposition 3.1 (1) *The 0–cochain ξ is $\mathcal{M}_{g,1}$–equivariant in the sense that $\xi_{\psi(G)} = \psi \cdot \xi_G$ for any $\psi \in \mathcal{M}_{g,1}$ and any trivalent fatgraph spine G.*

(2) *We have $2j' - m = \delta\xi$. Namely, for any flip $G \xrightarrow{W} G'$, we have $\xi_{G'} - \xi_G = 2j'(W) - m(W)$.*

Moreover, these two properties characterize ξ.

Proof (1) For simplicity we write $s = 2j' - m$. Take $\varphi \in \mathcal{M}_{g,1}$. Using $s_G(\varphi) = \delta\xi_G(\varphi) = \varphi \cdot \xi_G - \xi_G$, etc, we compute from Lemma 2.3 that

$$\begin{aligned} s_{\psi(G)}(\varphi) &= s_G(\varphi) + \varphi \cdot s_G(\psi) - s_G(\psi) \\ &= \varphi \cdot \xi_G - \xi_G + \varphi \cdot (\psi \cdot \xi_G - \xi_G) - (\psi \cdot \xi_G - \xi_G) \\ &= \varphi \cdot (\psi \cdot \xi_G) - \psi \cdot \xi_G \\ &= \delta(\psi \cdot \xi_G)(\varphi). \end{aligned}$$

This proves $s_{\psi(G)} = \delta(\psi \cdot \xi_G)$. By the uniqueness of $\xi_{\psi(G)}$, it follows that $\xi_{\psi(G)} = \psi \cdot \xi_G$.

(2) This follows from $s_G(\varphi) + \varphi \cdot s(W) = s(W) + s_{G'}(\varphi)$ analogously, and so we omit the details.

Finally, suppose that ξ^0 is an $\mathcal{M}_{g,1}$–equivariant 0–cochain satisfying $2j' - m = \delta\xi^0$. Then $\xi - \xi^0$ is an $\mathcal{M}_{g,1}$–equivariant 0–cocycle. This shows that $\eta := \xi(G) - \xi^0(G) \in H$ is independent of G and $\varphi \cdot \eta = \eta$ for any $\varphi \in \mathcal{M}_{g,1}$. Therefore η must be zero and $\xi^0 = \xi$. □

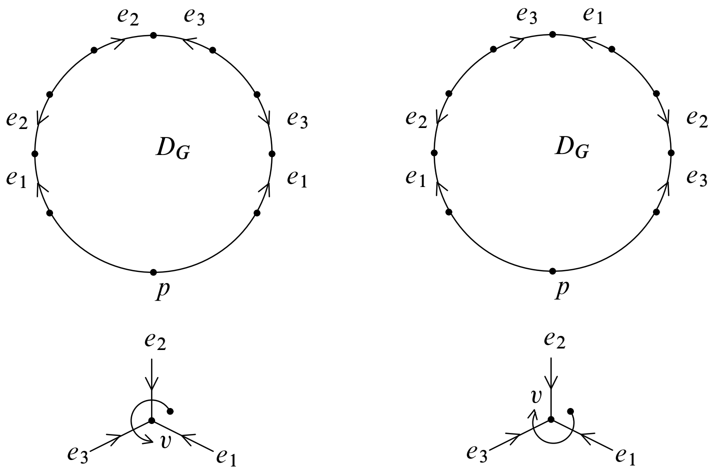

Figure 3: A vertex of type 1 (left) and a vertex of type 2 (right)

Let G be a trivalent fatgraph spine of $\Sigma_{g,1}$. We present an explicit formula for ξ_G. To begin with, we introduce a total ordering for $E^{\mathrm{ori}}(G)$. Note that if we cut $\Sigma_{g,1}$ along G, we obtain an oriented closed disk D_G.

Definition 3.2 (1) For $e, e' \in E^{\mathrm{ori}}(G)$, we say $e \prec e'$ if the edge e occurs first when we go *clockwise* along the boundary of D_G from p.

(2) Let $e \in E^{\mathrm{ori}}(G)$. We say that e has the *preferred orientation* (or e is *preferably oriented*) if $e \prec \overline{e}$.

Note that any unoriented edge of G has a unique preferred orientation.

Let $v \in V^{\mathrm{int}}(G)$. We name the three elements of $E_v^{\mathrm{ori}}(G)$ as e_1, e_2 and e_3 so that

(1) $e_1 \prec e_2$ and $e_1 \prec e_3$, and

(2) the edge e_2 is next to e_1 in the cyclic ordering given to $E_v^{\mathrm{ori}}(G)$.

There are two possibilities for the ordering of e_i and its inverse $\overline{e}_i$, namely,

$$e_1 \prec \overline{e}_2 \prec e_2 \prec \overline{e}_3 \prec e_3 \prec \overline{e}_1 \quad \text{and} \quad e_1 \prec \overline{e}_2 \prec e_3 \prec \overline{e}_1 \prec e_2 \prec \overline{e}_3.$$

The vertex v is said to be *of type* 1 if the former case happens, and is said to be *of type* 2 otherwise. Figure 3 is an illustration of the situation.

We can count the number of vertices of type 1 and the number of type 2.

Proposition 3.3 *For any trivalent fatgraph spine G of $\Sigma_{g,1}$, the number of trivalent vertices of type 1 is $2g-1$, and that of type 2 is $2g$.*

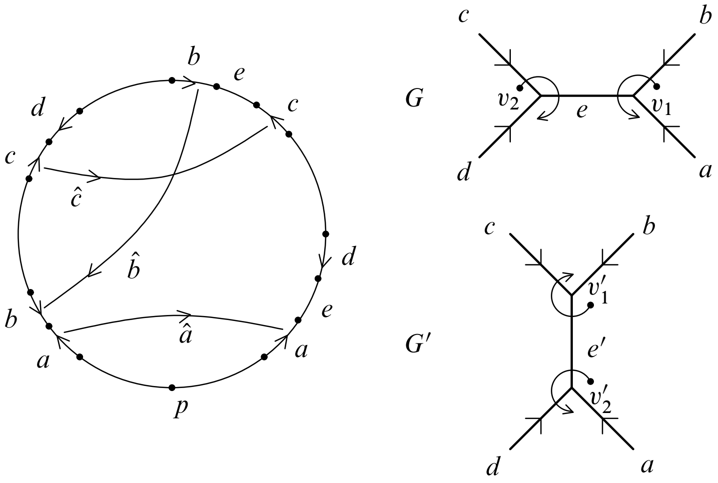

Figure 4: The case where $a \prec c \prec b \prec d$

Proof For $i = 1, 2$, let V_i be the number of trivalent vertices of type i. Since the number of trivalent vertices of G is $4g-1$, we have $V_1 + V_2 = 4g-1$. We observe that if a trivalent vertex v is of type i $(i = 1, 2)$, the number of preferably oriented edges toward v is i. Thus $V_1 + 2V_2$ is equal to the number of edges of G, ie $6g-1$. Hence we obtain $V_1 = 2g-1$ and $V_2 = 2g$. $\square$

We set

$$\begin{cases} e_v = e_2 \text{ and } f_v = e_3 & \text{if } v \text{ is of type 1,} \\ e_v = e_1 \text{ and } f_v = e_3 & \text{if } v \text{ is of type 2.} \end{cases}$$

Theorem 3.4 *We have*

$$\xi_G = \sum_v (\mu(e_v) - \mu(f_v)),$$

where the sum is taken over all trivalent vertices of G.

Proof We set $\xi_G^0 = \sum_v (\mu(e_v) - \mu(f_v))$ and consider the collection $\xi^0 = \{\xi_G^0\}_G$. Clearly, ξ^0 is $\mathcal{M}_{g,1}$–equivariant. By Proposition 3.1, it is sufficient to prove that $2j' - m = \delta\xi^0$.

Take the notation as in Figure 1. For example, assume that $a \prec c \prec b \prec d$. For simplicity, we write e instead of $\mu(e)$ for $e \in E^{\mathrm{ori}}(G)$. Then we can see from the left part of Figure 4 that $(a \cdot b) = (c \cdot a) = 0$ and $(b \cdot c) = 1$, and so $j'(W_e) = a$. Thus $2j'(W_e) - m(W_e) = 2a - (a+c) = a - c$. On the other hand, we can compute from

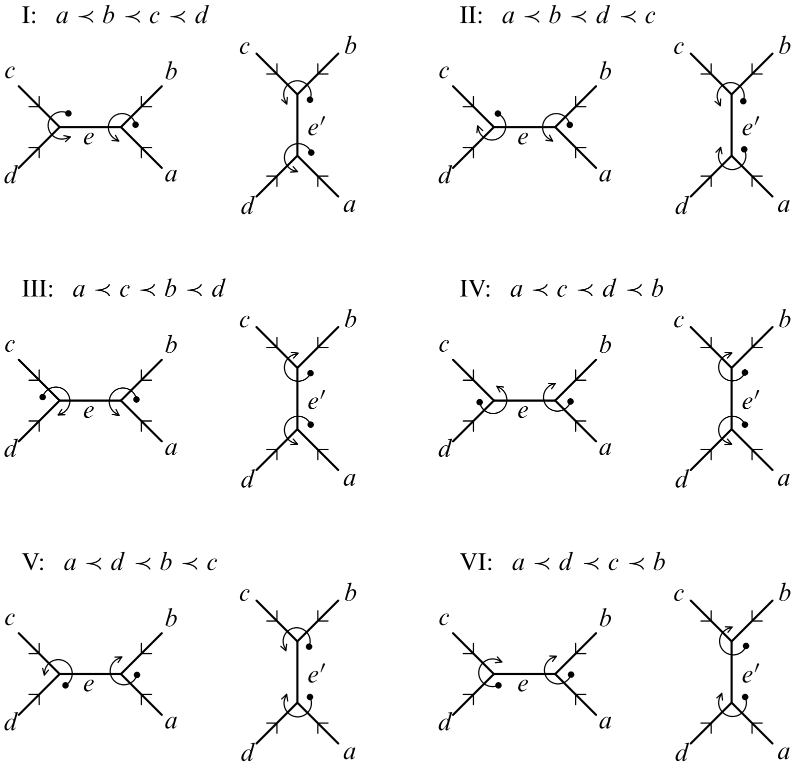

Figure 5: Situations near e

the right part of Figure 4 that

$$\begin{aligned}\xi^0_{G'} - \xi^0_G &= (e_{v'_1} - f_{v'_1}) + (e_{v'_2} - f_{v'_2}) - (e_{v_1} - f_{v_1}) - (e_{v_2} - f_{v_2}) \\ &= (a+d-c) + (b+c-d) - (b-(c+d)) - (c-(a+b)) \\ &= 2a+b+d = 2a+b+(-a-b-c) = a-c.\end{aligned}$$

We can compute similarly for other cases as well, and we obtain $2j'(W_e) - m(W_e) = \xi^0_{G'} - \xi^0_G$. (There are essentially six cases to consider; in each case in Figure 5, we may assume that G corresponds to the left picture.) Hence $2j' - m = \delta\xi^0$, as required. □

Example 3.5 Let G be the fatgraph as shown in Figure 6. We name edges as in the figure and give them the preferred orientation. For $1 \le i \le g$ and $1 \le j \le 3$, let $v_i^j \in V^{\mathrm{int}}(G)$ be the start point of e_i^j. For $1 \le i \le g-1$, let $v_i^4 \in V^{\mathrm{int}}(G)$ be the endpoint of e_i^4.

Since v_i^1 is of type 1, its contribution is $\mu(\overline{e}_i^1) - \mu(\overline{e}_i^4) = \mu(e_i^4) - \mu(e_i^1)$. Since v_i^2 is of type 2, its contribution is $\mu(e_i^1) - \mu(e_i^3)$. Since v_i^3 is of type 2, its contribution is

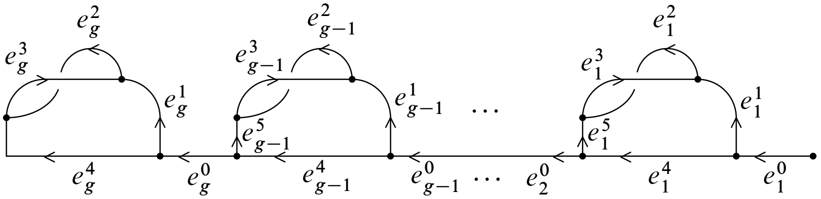

Figure 6: The fatgraph in Example 3.5

$\mu(e_i^2)-\mu(e_i^5)$. Here we understand that $e_g^5=e_g^4$. Since v_i^4 is of type 1, its contribution is $\mu(\overline{e}_i^5)-\mu(\overline{e}_{i+1}^0)=\mu(e_{i+1}^0)-\mu(e_i^5)$.

Moreover, we have $\mu(e_i^0)=0$, $\mu(e_i^1)+\mu(e_i^3)=\mu(e_i^2)$, and $\mu(e_i^4)=\mu(e_i^5)=-\mu(e_i^1)$. Using these relations, we obtain

$$\xi_G=\mu(e_g^1)+\sum_{i=1}^{g-1}2\mu(e_i^1).$$

4 Elementary properties

In this section, we record two elementary properties of ξ_G.

We first show a certain gluing formula. Let g and g' be positive integers, and suppose that we have two trivalent fatgraph spines $\iota\colon G\hookrightarrow\Sigma_{g,1}$ and $\iota'\colon G'\hookrightarrow\Sigma_{g',1}$. Fix $e\in E^{\mathrm{ori}}(G)$. Plugging the tail of G' in the right side of e, one produces a new fatgraph spine of $\Sigma_{g+g',1}$. A precise construction is as follows. Let v_e be the middle point of e.

(1) Take a small closed disk D_e in $\Sigma_{g,1}$ such that $\mathrm{Int}(D_e)\cap G=\varnothing$, the boundary ∂D_e intersects G once at v_e, and the center of D_e is on the right side of e with respect to the orientation of e.

(2) Glue $\Sigma_{g,1}\setminus\mathrm{Int}(D_e)$ with $\Sigma_{g',1}$ along the boundaries ∂D_e and $\partial\Sigma_{g',1}$ so that the univalent vertex of G' is identified with v_e.

(3) Let G'' be the union of the images of G and G' in the result of gluing.

The glued surface is diffeomorphic to $\Sigma_{g+g',1}$. We consider G'' as a trivalent fatgraph spine of $\Sigma_{g+g',1}$ by dividing e into two edges sharing the newly created trivalent vertex v_e. These two edges receive their orientation from e. We name them as $e_1,e_2\in E^{\mathrm{ori}}(G'')$ so that v_e is the endpoint of e_1. The edges e_1 and e_2 have the same homology marking as e.

A schematic figure of this construction is Figure 7. We call G'' the *gluing* of G and G' at e. Note that the inclusions $\Sigma_{g,1}\setminus\mathrm{Int}(D_e)\hookrightarrow\Sigma_{g+g',1}$ and $\Sigma_{g',1}\hookrightarrow\Sigma_{g+g',1}$ induce

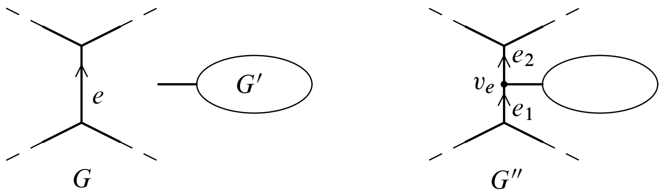

Figure 7: Gluing

a direct sum decomposition

$$H_1(\Sigma_{g+g',1};\mathbb{Z}) \cong H_1(\Sigma_{g,1};\mathbb{Z}) \oplus H_1(\Sigma_{g',1};\mathbb{Z}). \tag{4-1}$$

Proposition 4.1 (gluing formula) *Let G'' be the gluing of G and G' at e, as above. Then $\xi_{G''} = \xi_G + \mu(e) + \xi_{G'}$.*

Proof We have a natural identification $V^{\mathrm{int}}(G'') \cong V^{\mathrm{int}}(G) \sqcup \{v_e\} \sqcup V^{\mathrm{int}}(G')$. Observe that this identification respects the type of vertices. With the direct sum decomposition (4-1) in mind, we see that $V^{\mathrm{int}}(G)$ and $V^{\mathrm{int}}(G')$ contribute to $\xi_{G''}$ as ξ_G and $\xi_{G'}$, respectively.

We compute the contribution from v_e. Let $t' \in E^{\mathrm{ori}}_{v_e}(G'')$ be an edge coming from the tail of G'. The homology marking of t' is trivial. Then the contribution from v_e is $\mu(t') - \mu(\overline{e}_2) = \mu(e)$ if e has the preferred orientation, and is $\mu(e_1) - \mu(t') = \mu(e)$ otherwise. This completes the proof. □

We next show a formula describing how ξ_G changes under a special kind of flip. For a trivalent fatgraph spine $G \subset \Sigma_{g,1}$, we use the following notation:

- We denote by t the tail of G, and give it the preferred orientation.
- $e_1 \in E^{\mathrm{ori}}(G)$ is the oriented edge next to t in the total ordering given to $E^{\mathrm{ori}}(G)$.
- v_1 and v_2 are the start and end points of e_1, respectively.
- $b, c \in E^{\mathrm{ori}}_{v_2}(G)$ are the edges such that e_1, b and c are in this order in the cyclic ordering given to $E^{\mathrm{ori}}_{v_2}(G)$.

The situation is illustrated in Figure 8. We call the flip along (the unoriented edge underlying) e_1 the *tail slide* to G.

Proposition 4.2 (tail slide formula) *Let G' be the result of the tail slide to G. Then $\xi_{G'} = \xi_G + \mu(c)$.*

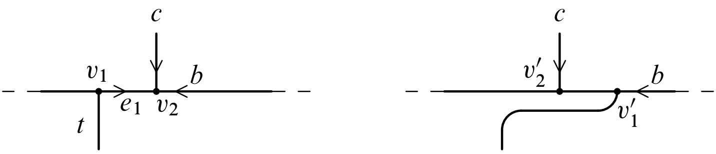

Figure 8: The tail slide applied to G (left) gives G' (right)

Proof We work with Figure 8. Suppose $b \prec c$ in $E^{\mathrm{ori}}(G)$. For simplicity, we write e instead of $\mu(e)$ for $e \in E^{\mathrm{ori}}(G)$. Then we compute

$$\begin{aligned}\xi_{G'} - \xi_G &= (e_{v_1'} - f_{v_1'}) + (e_{v_2'} - f_{v_2'}) - (e_{v_1} - f_{v_1}) - (e_{v_2} - f_{v_2}) \\ &= (b - (-b)) + (c - (-b - c)) - (b + c - (-b - c)) - (b - c) \\ &= c.\end{aligned}$$

The case where $c \prec b$ can be computed similarly. □

As an application of Proposition 4.2, we can extend the definition of our invariant to trivalent fatgraph spines of a *once-punctured* surface. Let Σ_g^1 be a surface obtained from $\Sigma_{g,1}$ by gluing a once-punctured disk along the boundaries. We regard $\Sigma_{g,1}$ as a subset of Σ_g^1. By definition, a fatgraph spine of Σ_g^1 is an embedding $\iota\colon G \hookrightarrow \Sigma_g^1$ of a fatgraph G into Σ_g^1 satisfying the first two conditions in Definition 2.1 (with $\Sigma_{g,1}$ replaced by Σ_g^1), and the condition that all vertices have valency greater than 2.

Let G be a trivalent fatgraph spine of Σ_g^1. By a suitable isotopy, we arrange that $G \subset \Sigma_{g,1}$. Let $e \in E^{\mathrm{ori}}(G)$. Take a simple arc ℓ on $\Sigma_{g,1}$ starting from p, reaching v_e from the right, and disjoint from $G \setminus \{v_e\}$. We say that such an arc ℓ is *admissible* for e. Regarding v_e as a newly created trivalent vertex, we can consider the union $\widetilde{G}(e,\ell) = G \cup \ell$ as a trivalent fatgraph spine of $\Sigma_{g,1}$. The arc ℓ becomes the tail of $\widetilde{G}(e,\ell)$.

Corollary 4.3 *Keep the notation as above. Then the element $\xi_{\widetilde{G}(e,\ell)} - \mu(e)$ does not depend on the choice of e and ℓ. In particular, for a trivalent fatgraph spine $G \subset \Sigma_g^1$, we can define $\xi_G \in H = H_1(\Sigma_{g,1};\mathbb{Z}) \cong H_1(\Sigma_g^1;\mathbb{Z})$ as*

$$\xi_G := \xi_{\widetilde{G}(e,\ell)} - \mu(e).$$

Proof Let ℓ^0 be another admissible arc for e. Then ℓ^0 is isotopic to the concatenation of some power of a simple based loop parallel to $\partial\Sigma_{g,1}$ and ℓ. This implies that $\widetilde{G}(e,\ell^0)$ is obtained from $\widetilde{G}(e,\ell)$ by application of some power of the Dehn twist along $\partial\Sigma_{g,1}$. Since the Dehn twist along $\partial\Sigma_{g,1}$ acts on H trivially, we have $\xi_{\widetilde{G}(e,\ell)} = \xi_{\widetilde{G}(e,\ell^0)}$. Hence $\xi_{\widetilde{G}(e,\ell)} - \mu(e)$ does not depend on the choice of ℓ.

Now, we can give a *cyclic* ordering to the set $E^{\mathrm{ori}}(G)$ in a way similar to that in the case where $G \subset \Sigma_{g,1}$ as in Definition 3.2. Suppose that $e, e' \in E^{\mathrm{ori}}(G)$ are consecutive in this cyclic ordering. Fix an admissible arc ℓ for e. Let v_0 be the vertex of G shared by e and e', and let $c \in E^{\mathrm{ori}}_{v_0}(G)$ be an edge other than e and $\overline{e}'$. We denote by e_0 an unoriented edge of $\widetilde{G}(e,\ell)$ with endpoints v_e and v_0.

Let $\widetilde{G}'$ be the result of flip along e_0. Then $\widetilde{G}'$ can be identified with $\widetilde{G}(e',\ell')$, where ℓ' corresponds to the tail of $\widetilde{G}'$. By Proposition 4.2, we have $\xi_{\widetilde{G}(e',\ell')} = \xi_{\widetilde{G}(e,\ell)} + \mu(c)$. Since $\mu(c)+\mu(e)=\mu(e')$, we obtain $\xi_{\widetilde{G}(e',\ell')} - \mu(e') = \xi_{\widetilde{G}(e,\ell)} - \mu(e)$. Hence $\xi_{\widetilde{G}(e,\ell)} - \mu(e)$ does not depend on the choice of e either. □

5 Nontriviality and primitivity

Let us consider the mod 2 reduction of ξ_G:

$$\xi_G^2 := \xi_G \otimes (1 \bmod 2) \in H \otimes \mathbb{Z}_2 \cong H_1(\Sigma_{g,1};\mathbb{Z}_2).$$

Hereafter, $\equiv$ stands for an equality in $H \otimes \mathbb{Z}_2$. Since $\mu(\overline{e}) = -\mu(e) \equiv \mu(e) \in H \otimes \mathbb{Z}_2$ for any $e \in E^{\mathrm{ori}}(G)$, the homology marking μ induces a well-defined map $\mu^2\colon E(G) \to H \otimes \mathbb{Z}_2$. We call μ^2 the *mod 2 homology marking.*

Proposition 5.1 *Let G be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then we have*

$$\xi_G^2 = \sum_{e \in E(G)} \mu^2(e).$$

Proof Let $v \in V^{\mathrm{int}}(G)$. We work with Figure 3 and count preferably oriented edges toward v. By abuse of notation, we use the same letter for an oriented edge and its underlying unoriented edge. If v is of type 1, only e_1 has the preferred orientation. Since $\mu(e_1)+\mu(e_2)+\mu(e_3)=0$, we have

$$\mu(e_v) - \mu(f_v) = \mu(e_2) - \mu(e_3) \equiv \mu(e_1).$$

If v is of type 2, e_1 and e_3 have the preferred orientation and e_2 does not. Then we have

$$\mu(e_v) - \mu(f_v) = \mu(e_1) - \mu(e_3) \equiv \mu(e_1) + \mu(e_3).$$

Therefore, we have

$$\xi_G^2 = \sum_{v \in V^{\mathrm{int}}(G)} \begin{pmatrix} \text{sum of the mod 2 homology markings} \\ \text{of preferably oriented edges toward } v \end{pmatrix} = \sum_{e \in E(G)} \mu^2(e).$$

The last equality holds since any preferably oriented edge of G points to some trivalent vertex of G. □

Theorem 5.2 *Let G be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then the mod 2 reduction ξ_G^2 is nontrivial. In particular, we have $\xi_G \neq 0$.*

To prove this theorem, we need the following lemma.

Lemma 5.3 *Let G be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then G contains an edge cycle of odd length.*

Proof We introduce some terminology: a pair of consecutive oriented edges of G is called a *corner* of G. There are $3\,\#V^{\mathrm{int}}(G) = 3(4g-1)$ corners. We number them as $c_1, \dots, c_{3(4g-1)}$, so that c_1 contains the preferably oriented tail of G, and for each i, c_i and c_{i+1} share an oriented edge in common. There are $n_o := 6g-1$ odd-numbered corners, and $n_e := 6g-2$ even-numbered corners.

Since n_o and n_e are not divisible by 3, there exist distinct indices i and j with $1 \le i < j \le 3(4g-1)$ such that the corners c_i and c_j are around the same vertex and $i - j \equiv 1 \bmod 2$. We can write c_i and c_j as $c_i = (e_i, e'_i)$ and $c_j = (e_j, e'_j)$ with $e_i \prec e'_i$ and $e_j \prec e'_j$. Consider the edge cycle following consecutive oriented edges of G from e'_i to e_j. Since i and j have different parity, the length of this edge cycle must be odd. □

Proof of Theorem 5.2 By Lemma 5.3, G contains an edge cycle γ of odd length. By Proposition 5.1, the mod 2 intersection pairing of ξ_G^2 and γ is computed as

$$(\xi_G^2 \cdot \gamma) = \left(\sum_{e \in E(G)} \mu^2(e) \cdot \gamma \right) = (\text{length of } \gamma) = 1.$$

Therefore, $\xi_G^2 \neq 0$. □

Remark 5.4 As far as we observed, ξ_G seems to be a primitive element of H for any trivalent fatgraph spine $G \subset \Sigma_{g,1}$. Here, an element $x \in H$ is called *primitive* if there do not exist $m \in \mathbb{Z}$ and $y \in H$ such that $|m| \ge 2$ and $x = my$. This primitivity of ξ_G holds for $g \le 2$. In fact, there is only one combinatorial isomorphism class of trivalent fatgraph spines for $g = 1$, and there are 105 classes for $g = 2$. By a direct computation, we can show the primitivity of ξ_G for these cases. The case $g \ge 3$ remains open.

In the case of trivalent fatgraph spines of a once-punctured surface Σ_g^1, it can happen that $\xi_G = 0$. Two examples for $g = 2$ are given in Figure 9.

Let G be a trivalent fatgraph spine of Σ_g^1. A corner of G is a pair of consecutive oriented edges of G in the cyclic ordering given to $E^{\mathrm{ori}}(G)$ (see the proof of Corollary 4.3). Now we give labels α or β to each corner of G so that any pair of consecutive corners of G have distinct labels. Since the number of corners of G is even, this labeling is always possible and is determined once we choose the label of a fixed corner.

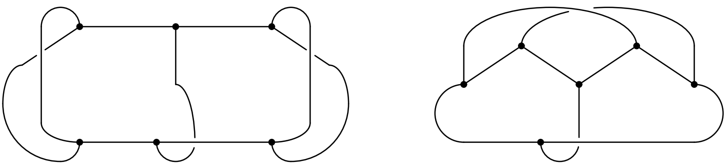

Figure 9: Trivalent fatgraph spines with $\xi_G = 0$

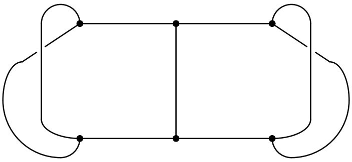

Figure 10: A balanced trivalent fatgraph spine with $\xi_G \neq 0$

We say that G is *balanced* if for any vertex of G, the three corners around the vertex have the same label. For example, trivalent fatgraph spines in Figure 9 and Figure 10 are balanced.

Theorem 5.5 *Let G be a trivalent fatgraph spine of Σ_g^1. Then the mod 2 reduction $\xi_G^2 = \xi_G \otimes (1 \bmod 2)$ is trivial if and only if G is balanced.*

Proof Pick a corner c of G and write it as $c = (e, e')$, where e' is next to e in the cyclic ordering given to $E^{\mathrm{ori}}(G)$. We give the label α to c and extend this labeling to all other corners as above. Take an admissible arc ℓ for e and set $\widetilde{G} = \widetilde{G}(e, \ell)$. The oriented edge e is split at the middle point v_e into two oriented edges. We name them as $e_1, e_2 \in E^{\mathrm{ori}}(\widetilde{G})$ so that v_e is the endpoint of e_1. We extend the labeling of corners of G to that of corners of $\widetilde{G}$ by giving α to $(e_1, \overline{\ell})$ and $(\overline{e}_2, \overline{e}_1)$, and β to (ℓ, e_2).

In view of Corollary 4.3, the condition $\xi_G^2 = 0$ is equivalent to $\xi_{\widetilde{G}}^2 = \mu^2(e_2)$. Furthermore, since the mod 2 homology markings $\{\mu^2(f)\}_{f \in E(\widetilde{G})}$ generate the mod 2 homology $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$, this condition is equivalent to the condition that $(\xi_{\widetilde{G}}^2 \cdot \mu^2(f)) = (\mu^2(e_2) \cdot \mu^2(f))$ for any $f \in E(\widetilde{G})$.

Assume that G is balanced. For any vertex of $\widetilde{G}$ other than v_e, the three corners about it are labeled by the same symbol. Let $f \in E(\widetilde{G})$. Let $\gamma(f)$ be the edge cycle following consecutive oriented edges of $\widetilde{G}$ from f to $\overline{f}$, where we give the preferred orientation to f. The mod 2 homology class $\mu^2(f)$ is represented by $\gamma(f)$. By the property of the labeling, the length of this edge cycle is odd if $f \prec \overline{e}_2 \prec \overline{f}$ (this also implies $f \neq e_2$), and is even otherwise. Note that the condition $f \prec \overline{e}_2 \prec \overline{f}$ is equivalent to $(\mu^2(e_2) \cdot \mu^2(f)) = 1$. Hence $(\xi_{\widetilde{G}}^2 \cdot \mu^2(f)) = (\text{the length of } \gamma(f)) = 1$ if and only if $(\mu^2(e_2) \cdot \mu^2(f)) = 1$. Therefore, $\xi_G^2 = 0$.

On the other hand, assume that $\xi_G^2 = 0$. Then for $f \in E(\widetilde{G})$, the length of $\gamma(f)$ is odd if and only if $f \prec \overline{e}_2 \prec \overline{f}$. Now we remove the tail from $\widetilde{G}$ and go back to G. Then $\gamma(f)$ is reduced to an edge cycle of G. Its length is 1 less than the length of $\gamma(f)$ if $f \prec \overline{e}_2 \prec \overline{f}$, and is the same as the length of $\gamma(f)$ otherwise. This implies that the reduced edge cycle of G has even length. Since f can be arbitrary, this shows that G is balanced. □

6 Mod 2 reduction and spin structures

In this section, we give a topological interpretation of the mod 2 reduction ξ_G^2. We start with the following description of the mod 2 homology of $\Sigma_{g,1}$.

Lemma 6.1 *Let G be a fatgraph spine of $\Sigma_{g,1}$. For $v \in V^{\mathrm{int}}(G)$, let $\{e_i^v\}_i$ be the set of unoriented edges of G having v as an endpoint. If there is an edge loop based at v, we count it twice. Then the mod 2 homology marking induces an isomorphism*

$$H_1(\Sigma_{g,1};\mathbb{Z}_2) \cong \bigoplus_{e\in E(G)} \mathbb{Z}_2 e \Big/ \sum_{v\in V^{\mathrm{int}}(G)} \mathbb{Z}_2\Big(\sum_i e_i^v\Big).$$

Proof Recall from Section 2 that we associate an oriented simple loop $\widehat{e}$ to each (oriented) edge e. In the proof of this lemma we forget the orientation of e and $\widehat{e}$. We can arrange that the simple loops $\{\widehat{e}\}_{e\in E(G)}$ share only one point $q \in \partial\Sigma_{g,1}$, and that if t is the tail of G then $\widehat{t} = \partial\Sigma_{g,1}$ with basepoint q. Then we obtain a cell decomposition of $\Sigma_{g,1}$ whose 1–cells coincide with $\{\widehat{e}\}_{e\in E(G)}$. Now the right-hand side of the assertion can be identified with the first mod 2 cellular homology group of this cell decomposition. □

Recall that a *spin structure* on $\Sigma_{g,1}$ is an element $w \in H^1(UT\Sigma_{g,1};\mathbb{Z}_2)$, where $UT\Sigma_{g,1}$ is the unit tangent bundle of $\Sigma_{g,1}$ (with respect to some Riemannian metric), such that the restriction of w to a fiber of the projection $UT\Sigma_{g,1} \to \Sigma_{g,1}$ is nontrivial. As Johnson [8] showed, the set of spin structures on $\Sigma_{g,1}$ is naturally identified with the set of quadratic forms on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$. Here, a map $q\colon H_1(\Sigma_{g,1};\mathbb{Z}_2) \to \mathbb{Z}_2$ is called a *quadratic form* on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$ if it satisfies

$$q(x+y) = q(x) + q(y) + (x\cdot y)$$

for any $x, y \in H_1(\Sigma_{g,1};\mathbb{Z}_2)$. The set of spin structures on $\Sigma_{g,1}$ is a torsor under the action of $H^1(\Sigma_{g,1};\mathbb{Z}_2)$. In other words, the difference between two quadratic forms on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$ can be written as a uniquely determined element of

$\mathrm{Hom}(H_1(\Sigma_{g,1};\mathbb{Z}_2),\mathbb{Z}_2) \cong H^1(\Sigma_{g,1};\mathbb{Z}_2)$. Note that, using the mod 2 intersection pairing, we have a natural isomorphism

$$H_1(\Sigma_{g,1};\mathbb{Z}_2) \cong \mathrm{Hom}(H_1(\Sigma_{g,1};\mathbb{Z}_2),\mathbb{Z}_2), \quad x \mapsto [y \mapsto (x\cdot y)]. \tag{6-1}$$

In what follows, G is a trivalent fatgraph spine of $\Sigma_{g,1}$. The following result gives an identification of certain $\mathbb{Z}_2$–valued functions on $E(G)$ with the set of quadratic forms on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$, thus with the set of spin structures on $\Sigma_{g,1}$ via Johnson's result stated above.

Theorem 6.2 *Let G be a trivalent fatgraph spine of $\Sigma_{g,1}$. Let $Q(G)$ be the set of maps $q\colon E(G)\to\mathbb{Z}_2$ such that, for any $v\in V^{\mathrm{int}}(G)$, the sum of values of q at the three edges having v as an endpoint is 0 if v is of type 1, and is 1 if v is of type 2. Then there is a natural bijection from $Q(G)$ to the set of quadratic forms on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$.*

Proof Given a map $q\colon E(G)\to\mathbb{Z}_2$, we extend q to a map from the free $\mathbb{Z}_2$–module generated by $E(G)$ by

$$q\Big(\sum_{e\in E(G)} m_e e\Big) := \sum_{e\in E(G)} m_e q(e) + \sum_{e\prec e'} m_e m_{e'}(\mu^2(e)\cdot\mu^2(e')), \tag{6-2}$$

for $m_e\in\mathbb{Z}_2$, $e\in E(G)$. Here $(\cdot)$ is the mod 2 intersection pairing and we give the preferred orientation to each element of $E(G)$. By a direct computation, we can check that, for any $x,y\in\bigoplus_{e\in E(G)}\mathbb{Z}_2 e$,

$$q(x+y) = q(x)+q(y)+(x\cdot y). \tag{6-3}$$

Here $(x\cdot y)$ is the mod 2 intersection pairing of the homology class determined by x and y through the isomorphism in Lemma 6.1.

We claim that if $q\in Q(G)$, then for any $v\in V^{\mathrm{int}}(G)$,

$$q(e_1^v+e_2^v+e_3^v)=0.$$

By (6-2), this condition is equivalent to the equality

$$\sum_{i=1}^{3} q(e_i^v) + (\mu^2(e_1^v)\cdot\mu^2(e_2^v)) + (\mu^2(e_1^v)\cdot\mu^2(e_3^v)) + (\mu^2(e_2^v)\cdot\mu^2(e_3^v)) = 0. \tag{6-4}$$

If v is of type 1, then $(\mu^2(e_i^v)\cdot\mu^2(e_j^v))=0$ for any $1\le i,j\le 3$. If v is of type 2, then $(\mu^2(e_i^v)\cdot\mu^2(e_j^v))=1$ for any $1\le i,j\le 3$ with $i\neq j$. See Figure 3. Therefore, the condition (6-4) is exactly equivalent to the condition for q being an element of $Q(G)$. This proves the claim.

By the claim, Lemma 6.1 and (6-3), it follows that the map q induces a quadratic form on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$. The above construction gives a map from $Q(G)$ to the set of quadratic forms on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$, and the inverse of this map is given by composing any quadratic form on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$ with the mod 2 homology marking $\mu^2\colon E(G)\to H_1(\Sigma_{g,1};\mathbb{Z}_2)$. □

We record how the set $Q(G)$ changes under a flip.

Proposition 6.3 *Let $W=W_e$ be a flip from G to G'. Then the bijection in Theorem 6.2 induces a bijection from $Q(G)$ to $Q(G')$, which maps a given $q\in Q(G)$ to the element $q'\in Q(G')$ defined as follows:*

- *For any edge f in $E(G')\setminus\{e'\}\cong E(G)\setminus\{e\}$, we have $q'(f)=q(f)$.*
- *We adopt the notation in Figure 5, and assume that in each case G and G' correspond to the left and right pictures, respectively. Then the value $q'(e')$ is given by the following formula:*

$$\begin{aligned}
\text{I:}\quad & q'(e')=q(b)+q(c)=q(a)+q(d),\\
\text{II:}\quad & q'(e')=q(b)+q(c)=q(a)+q(d)+1,\\
\text{III:}\quad & q'(e')=q(b)+q(c)+1=q(a)+q(d),\\
\text{IV:}\quad & q'(e')=q(b)+q(c)+1=q(a)+q(d),\\
\text{V:}\quad & q'(e')=q(b)+q(c)=q(a)+q(d)+1,\\
\text{VI:}\quad & q'(e')=q(b)+q(c)+1=q(a)+q(d)+1.
\end{aligned}$$

By a suitable replacement of labels of edges, one can similarly obtain a formula for q' in terms of q for the case where G and G' correspond to the right and left pictures, respectively, in each case in Figure 5.

Proof To prove the first condition, note that the mod 2 homology marking of f as an edge of $E(G)$ is the same as that of f as an edge of $E(G')$. The second condition follows from the first condition and the defining relation for elements of $Q(G')$. For example, in case VI, two endpoints of e' are of type 2, and hence we have $q'(b)+q'(c)+q'(e')=q'(a)+q'(d)+q'(e')=1$. □

Remark 6.4 The description of spin structures on $\Sigma_{g,1}$ given in Theorem 6.2 and how it changes under a flip as in Proposition 6.3 was pointed out by Robert Penner. Recently, Penner and Zeitlin [23] gave another natural description of spin structures on a punctured surface in terms of orientations on a trivalent fatgraph spine of the surface, and they also showed how it changes under a flip. In other words, Penner and Zeitlin gave a lift of the action of the mapping class group on the set of quadratic forms to the action of the Ptolemy groupoid, and the present construction gives another lift. It should be remarked that while their description works for any surfaces with multiple punctures, our description here is for a once (punctured/bordered) surface. It is an interesting question whether our description generalizes to any (punctured/bordered) surface.

In what follows, we denote by $E^+(G)$ the set of preferably oriented edges of G (see Definition 3.2), and let $E^-(G) := E^{\mathrm{ori}}(G) \setminus E^+(G)$.

Let $e \in E(G)$. We give e the preferred orientation and use the same letter e for the resulting element in $E^+(G)$. We define elements $q_G(e), \overline{q}_G(e) \in \mathbb{Z}_2$ by

$$\begin{aligned} q_G(e) &:= \#\{f \in E^+(G) \mid e \prec f \prec \overline{e}\} \bmod 2, \\ \overline{q}_G(e) &:= \#\{f \in E^-(G) \mid e \prec f \prec \overline{e}\} \bmod 2. \end{aligned}$$

Here # means the number of elements of a set.

Proposition 6.5 *The maps q_G and $\overline{q}_G$ are elements of $Q(G)$.*

Proof We consider the case of q_G only.

We work with Figure 3. Suppose that v is of type 1. Then e_1, $\overline{e}_2$ and $\overline{e}_3$ have the preferred orientation, and we have a disjoint union decomposition

$$\begin{aligned} &\{f \in E^+(G) \mid e_1 \prec f \prec \overline{e}_1\} \\ &\qquad = \{\overline{e}_2, \overline{e}_3\} \sqcup \{f \in E^+(G) \mid \overline{e}_2 \prec f \prec e_2\} \sqcup \{f \in E^+(G) \mid \overline{e}_3 \prec f \prec e_3\}. \end{aligned}$$

This implies that $q_G(e_1) = q_G(e_2) + q_G(e_3)$.

Suppose that v is of type 2. Then e_1, $\overline{e}_2$, and e_3 have the preferred orientation, and we have a disjoint union decomposition

$$\begin{aligned} &\{f \in E^+(G) \mid \overline{e}_2 \prec f \prec e_2\} \\ &\qquad = \big(\{f \in E^+(G) \mid e_1 \prec f \prec \overline{e}_1\} \setminus \{\overline{e}_2\}\big) \sqcup \{f \in E^+(G) \mid e_3 \prec f \prec \overline{e}_3\}. \end{aligned}$$

This implies that $q_G(e_2) = q_G(e_1) + q_G(e_3) + 1$. Therefore, $q_G \in Q(G)$. □

By Theorem 6.2, q_G and $\overline{q}_G$ induce quadratic forms on $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$. For simplicity, we use the same letter q_G and $\overline{q}_G$ for these quadratic forms. This construction of quadratic forms is $\mathcal{M}_{g,1}$–equivariant in the following sense.

Proposition 6.6 *Let G be a trivalent fatgraph spine of $\Sigma_{g,1}$, and let $\varphi \in \mathcal{M}_{g,1}$. Then we have $q_{\varphi(G)} \circ \varphi_* = q_G$ and $\overline{q}_{\varphi(G)} \circ \varphi_* = \overline{q}_G$, where φ_* is the automorphism of $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$ induced by φ.*

Proof We consider the case of q_G only. Consider a homomorphism

$$\Phi\colon \bigoplus_{e\in E(G)} \mathbb{Z}_2 e \to \bigoplus_{e'\in E(\varphi(G))} \mathbb{Z}_2 e', \quad \Phi(e)=\varphi(e).$$

Since φ gives a combinatorial isomorphism from G to $\varphi(G)$, we have $q_{\varphi(G)}\circ\Phi = q_G$. Now Φ induces the map φ_* on the level of homology, and we conclude $q_{\varphi(G)}\circ\varphi_* = q_G$. □

Finally, we compute the difference between q_G and $\overline{q}_G$.

Theorem 6.7 *Under the isomorphism (6-1), we have*

$$q_G - \overline{q}_G = \xi_G^2.$$

Moreover, we have $q_G \neq \overline{q}_G$.

Proof For $e \in E(G)$, we have

$$\begin{aligned} q_G(e) - \overline{q}_G(e) &= q_G(e) + \overline{q}_G(e) \\ &= \#\{f \in E^{\mathrm{ori}}(G) \mid e \prec f \prec \overline{e}\} \bmod 2 \\ &= \Bigl(\sum_{f\in E(G)} \mu^2(f)\cdot\mu^2(e)\Bigr) = (\xi_G^2\cdot\mu^2(e)), \end{aligned}$$

where the last equality follows from Proposition 5.1. Since $\{\mu^2(e)\}_{e\in E(G)}$ generates $H_1(\Sigma_{g,1};\mathbb{Z}_2)$, we obtain $q_G - \overline{q}_G = \xi_G^2$. The second statement follows from Theorem 5.2. □

Appendix: A nonsingular vector field associated to a once-bordered trivalent fatgraph spine

Let G be a trivalent fatgraph spine of $\Sigma_{g,1}$. In this appendix, we define a nonsingular vector field $\mathcal{X}_G$ on $\Sigma_{g,1}$, and then consider the induced quadratic form on $H_1(\Sigma_{g,1};\mathbb{Z}_2)$. In particular, we discuss a relationship among this quadratic form, q_G and $\overline{q}_G$.

The following construction of $\mathcal{X}_G$ was communicated to the author by Gwénaël Massuyeau.

Let $\mathrm{Vect}(\Sigma_{g,1})$ be the homotopy set of nonsingular vector fields on $\Sigma_{g,1}$. In other words, $\mathrm{Vect}(\Sigma_{g,1})$ is the homotopy set of sections of the projection $\pi\colon UT\Sigma_{g,1}\to\Sigma_{g,1}$.

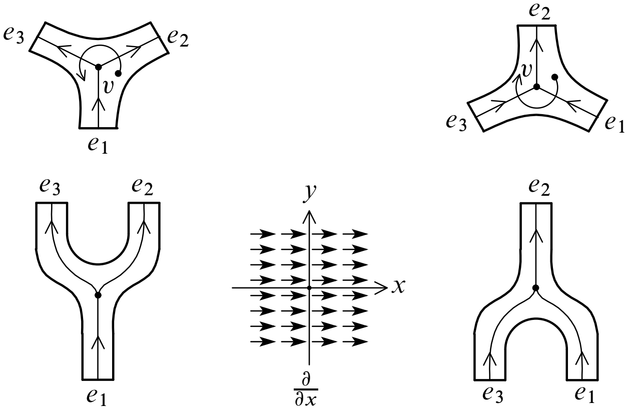

Figure 11: $\mathcal{X}_G$ on N_v. Left: a vertex of type 1. Right: a vertex of type 2.

For $\mathcal{X} \in \mathrm{Vect}(\Sigma_{g,1})$, the winding number

$$\mathrm{wind}_{\mathcal{X}}\colon \pi_1(UT\Sigma_{g,1}) \to \mathbb{Z}$$

is defined as follows. Let $\tilde{\gamma}\colon S^1 \to UT\Sigma_{g,1}$ be a (based) loop. For any $t \in S^1$, there exists a unique element $\Phi_t = \Phi(\mathcal{X}, \tilde{\gamma}, t) \in S^1 = U(1)$ such that $\mathcal{X}(\pi \circ \tilde{\gamma}(t))\Phi_t = \tilde{\gamma}(t)$. Then $\mathrm{wind}_{\mathcal{X}}(\tilde{\gamma})$ is defined to be the mapping degree of the map $S^1 \to S^1$, $t \mapsto \Phi_t$. The map $\mathrm{wind}_{\mathcal{X}}$ is a group homomorphism, and its mod 2 reduction

$$w_{\mathcal{X}} \in \mathrm{Hom}(\pi_1(UT\Sigma_{g,1}), \mathbb{Z}_2) \cong H^1(UT\Sigma_{g,1}; \mathbb{Z}_2)$$

is a spin structure on $\Sigma_{g,1}$.

Now we give the preferred orientation to any unoriented edge of G. Let $v \in V^{\mathrm{int}}(G)$. According to the type of v, we realize a small neighborhood N_v of v in the xy–plane as in Figure 11, and then restrict the horizontal vector field $\partial/\partial x$ to N_v. We extend the vector field on $\bigsqcup_v N_v$ thus obtained to a globally defined nonsingular vector field $\mathcal{X}_G$, so that outside $\bigsqcup_v N_v$, each trajectory of $\mathcal{X}_G$ is perpendicular to G.

Let $q_{\mathcal{X}_G}$ be the quadratic form on $H_1(\Sigma_{g,1}; \mathbb{Z}_2)$ corresponding to $w_{\mathcal{X}}$. Following Johnson [8], one can compute it as follows. Let γ be an oriented simple closed curve and consider its lift $\tilde{\gamma} = (\gamma, \dot{\gamma})$ to a loop in $UT\Sigma_{g,1}$ (here $\dot{\gamma}$ is the velocity vector of γ normalized to have unit length). Then

$$q_{\mathcal{X}_G}([\gamma]) = \mathrm{wind}_{\mathcal{X}_G}(\tilde{\gamma}) + 1 \bmod 2. \tag{A-1}$$

We apply this formula to $\gamma = \hat{e}$, where $e \in E^{\mathrm{ori}}(G)$. Assume that e has the preferred orientation. Let $L(e)$ be the set of corners (f, f') of G (see the proof of Lemma 5.3) such that

(1) $e \preceq f \prec f' \preceq \overline{e}$, and

(2) exactly one of f and f' have the preferred orientation.

Here, $e \preceq f$ means $e \prec f$ or $e = f$. For example, if v is a vertex of type 1 as in the left part of Figure 11, only $(\overline{e}_2, e_3)$ is an element of $L(e)$ among the three corners around v. Set $\lambda(e) = \#L(e)$.

Lemma A.1 *We have* $\mathrm{wind}_{\mathcal{X}_G}(\widetilde{\widehat{e}}) = (1-\lambda(e))/2$.

Proof Take a small regular neighborhood $N(G)$ of G; we may arrange that $\widehat{e}$ stays inside $N(G)$ throughout. Every time when $\widehat{e}$ goes through a common vertex of a member of $L(e)$, the velocity vector of $\widehat{e}$ rotates by an angle $-\pi$ with respect to $\mathcal{X}_G$. Also, when $\widehat{e}$ goes through the middle point of e, the velocity vector of $\widehat{e}$ rotates by an angle π with respect to $\mathcal{X}_G$. This proves the lemma. □

In particular, using the fact that $\lambda(e)$ is odd (since e has the preferred orientation and $\overline{e}$ does not), we have from (A-1) that

$$q_{\mathcal{X}_G}(e) = q_{\mathcal{X}_G}([\widehat{e}]) = \tfrac{1}{2}(1-\lambda(e)) + 1 \bmod 2 = \tfrac{1}{2}(1+\lambda(e)) \bmod 2.$$

Proposition A.2 *Let G be a trivalent fatgraph spine of $\Sigma_{g,1}$. Then the quadratic forms $q_{\mathcal{X}_G}$, q_G and $\overline{q}_G$ are distinct from each other.*

Proof By Theorem 6.7, it is sufficient to prove $q_{\mathcal{X}_G} \neq q_G$ and $q_{\mathcal{X}_G} \neq \overline{q}_G$.

Let $e_1 \in E^{\mathrm{ori}}(G)$ be the "last" preferably oriented edge. Namely, e_1 is the unique element such that e_1 has the preferred orientation and if $e_1 \prec f$ then f does not have the preferred orientation. We have $\lambda(e_1) = 1$ and $q_{\mathcal{X}_G}(e_1) = (1+1)/2 = 1$. On the other hand, since there are no preferably oriented edges f with $e_1 \prec f \prec \overline{e}_1$, we have $q_G(e_1) = 0$. Hence $q_{\mathcal{X}_G} \neq q_G$.

Let $e_2 \in E^{\mathrm{ori}}(G)$ be the unique element such that e_2 has the preferred orientation and if $f \prec \overline{e}_2$ then f has the preferred orientation. We have $\lambda(e_2) = 1$ and $q_{\mathcal{X}_G}(e_2) = 1$. On the other hand, since any edge $f \in E^{\mathrm{ori}}(G)$ with $e_2 \prec f \prec \overline{e}_2$ has the preferred orientation, $\overline{q}_G(e_2) = 0$. Hence $q_{\mathcal{X}_G} \neq \overline{q}_G$. □

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