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Geometrically bounding 3–manifolds, volume and Betti numbers

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A hyperbolic 3–manifold is geometrically bounding if it is the only boundary of a totally geodesic hyperbolic 4–manifold. According to previous results of Long and Reid (2000) and Meyerhoff and Neumann (1992), geometrically bounding closed hyperbolic 3–manifolds are very rare. Assume the value $v \approx 4.3062\ldots$ for the volume of the regular right-angled hyperbolic dodecahedron P in $\mathbb{H}^3$. For each positive integer n and each odd integer k in $[1, 5n+3]$, we construct a closed hyperbolic 3–manifold M with $\beta^1(M) = k$ and $\mathrm{vol}(M) = 16nv$ which bounds a totally geodesic hyperbolic 4–manifold. In particular, for every positive odd integer k, there are infinitely many geometrically bounding 3–manifolds whose first Betti numbers are k. The proof exploits the real toric manifold theory over a sequence of stacking dodecahedra, together with some results obtained by Kolpakov, Martelli and Tschantz (2015).

57R90, 57M50, 57S25

1 Introduction

1.1 Geometrically bounding 3–manifolds

There is a well-known result given by Rohlin in 1951, saying that any closed orientable 3–manifold is null-cobordant (see, for example, Corollary 2.5 of [18]), whereas for higher dimensions, it remains an open problem to say which closed n–manifolds can bound $(n+1)$–manifolds. Farrell and Zdravkovska [7] conjectured that every almost flat n–manifold bounds an $(n+1)$–manifold; see also Davis and Fang [5]. This conjecture is far from being solved. Farrell and Zdravkovska also conjectured in the same paper that every flat n–manifold M is the cusp section of a one-cusped hyperbolic $(n+1)$–manifold. However, Long and Reid [11] refuted this stronger conjecture by showing that if M is the cusp section of a one-cusped hyperbolic $4n$–manifold, its η–invariant $\eta(M)$ must be an integer.

If a hyperbolic n–manifold M is the unique totally geodesic boundary of a hyperbolic $(n+1)$–manifold N, we say that M *bounds geometrically* or M is a *geometrically bounding* n–manifold. In this context, Long and Reid [11] studied what kinds of 3–manifolds bound geometrically; Ratcliffe and Tschantz [16] provided some cosmological motivations for studying geometrically bounding 3–manifolds. In general, it is not a trivial task to look for geometrically bounding 3–manifolds, since only few explicit hyperbolic 4–manifolds are known. Moreover, Long and Reid showed in [11] that if a closed hyperbolic 3–manifold M is geometrically bounding, its η–invariant $\eta(M)$ must be an integer. This, together with the result of Meyerhoff and Neumann [13] that the set of η–invariants of all hyperbolic 3–manifolds is dense in $\mathbb{R}$, shows that geometrically bounding 3–manifolds are very rare in the set of hyperbolic 3–manifolds. To the best of our knowledge, the following question remains open:

Question 1.1 Given a closed hyperbolic 3–manifold M with η–invariant $\eta(M) \in \mathbb{Z}$, is there a totally geodesic hyperbolic 4–manifold N with $\partial N = M$?

By Jorgensen–Thurston's Dehn surgery theory [23], we know that there are only finitely many (possibly zero) hyperbolic 3–manifolds with a given volume x. More precisely, if we consider the function

$$f(x) = \sup\{n \mid \text{there are } n \text{ different hyperbolic 3–manifolds with volume } v \le x\},$$

then Jorgensen–Thurston theory implies that $f(x)$ is finite. Furthermore, Millichap [14] showed that $f(x)$ grows at least factorially.

In this paper, we consider instead the number of geometrically bounding 3–manifolds with a given volume. That is, we focus on the function

$$f_b(x) = \sup\{n \mid \text{there are } n \text{ different geometrically bounding 3–manifolds with volume } v \le x\}.$$

Building on Kolpakov, Martelli and Tschantz [9] and real toric manifold theory, we prove the following:

Theorem 1.2 *Assume that $v \approx 4.3062\ldots$ is the volume of the regular right-angled hyperbolic dodecahedron in $\mathbb{H}^3$. Then, for each positive integer n and each odd integer k in $[1, 5n+3]$, there is a closed hyperbolic 3–manifold M with $\beta^1(M) = k$ and $\mathrm{vol}(M) = 16nv$ that bounds a totally geodesic hyperbolic 4–manifold.*

Therefore, we construct some families $\mathcal{F}_n$, $n \geq 1$, of closed hyperbolic 3–manifolds having the following special features:

- They all *bound geometrically*, ie for any n, each manifold in $\mathcal{F}_n$ is the connected geodesic boundary of a compact hyperbolic 4–manifold.
- Each manifold in $\mathcal{F}_n$ can be decomposed into $16n$ right-angled dodecahedra. The set $\mathcal{F}_n$ contains manifolds with first Betti numbers $1, 3, 5, \dots, 5n+3$. In particular, $\mathcal{F}_n$ contains at least n elements.

This implies that the above-defined function $f_b(x)$ grows at least linearly. Moreover, we have a corollary of Theorem 1.2 as follows.

Corollary 1.3 *For every positive odd number k, there are infinitely many geometrically bounding 3–manifolds whose first Betti numbers are k.*

We refer to the paper of Ratcliffe and Tschantz [17] for counting the number of totally geodesic hyperbolic 4–manifolds with the same 3–manifold M as boundary, and to Chu and Kolpakov [4] and Slavich [19; 20] for other topics regarding geometrically bounding hyperbolic manifolds. Also see the recent paper by Kolpakov, Reid and Slavich [10] for problems related to geodesically embedding hyperbolic manifolds. However, we emphasize that being geometrically bounding is a more subtle property than being geodesically embedding.

1.2 Real toric manifolds

Small covers, also known as Coxeter orbifold coverings, have been studied by Davis and Januszkiewicz [6], see also Vesnin [24]. They are a class of n–manifolds which admit locally standard $\mathbb{Z}_2^n$–actions, such that the orbit spaces are n–dimensional simple polytopes. The algebraic and topological properties of a small cover are closely related to the combinatorics of the orbit polytope and to the coloring on the codimension-one faces of that polytope. For example, the mod 2 Betti numbers $\beta_i^{(2)}$ of a small cover M over the polytope L is equal to h_i, where $h = (h_0, h_1, \dots, h_n)$ is the h–vector of the polytope L; see [6].

Those manifolds admitting locally standard $\mathbb{Z}_2^k$–actions are usually referred to as *real toric manifolds* and form a wider class. Given an n–dimensional simple polytope L, we can define a map $\lambda\colon \mathcal{F} \to \mathbb{Z}_2^k$ that satisfies certain conditions, where $\mathcal{F}$ is the set of codimension-one faces of L. Furthermore, by the equivalence relation determined by the map λ, we can construct a smooth closed manifold $M(L, \lambda)$. See Section 2.1 for more details.

For instance, we may color the four codimension-one faces of a tetrahedron by e_1, e_2, e_3 and $e_1+e_2+e_3$, where e_1, e_2 and e_3 are the standard basis of $\mathbb{Z}_2^3$. From the construction mentioned in the previous paragraph, we construct the closed orientable 3–manifold $\mathbb{RP}^3$. Note that a tetrahedron admits a unique right-angled spherical structure. We thus naturally obtain a unique spherical structure on $\mathbb{RP}^3$ by inheriting spherical structures from the four tetrahedral copies.

In the rest of this section, we assume that P is the regular right-angled hyperbolic dodecahedron in $\mathbb{H}^3$ with twelve 2–dimensional facets, and nP is the polytope obtained by stacking n copies of P. It is obvious that nP has 12 pentagonal facets and $5n-5$ hexagonal facets. See Section 2.3 for more details.

Given a $\mathbb{Z}_2^3$–coloring λ over the polytope nP, we generate the *natural* $\mathbb{Z}_2^4$–coloring δ on nP in the following manner. Suppose $\{e_1,e_2,e_3,e_4\}$ is the standard basis of $\mathbb{Z}_2^4$. For each facet F of nP, if $\lambda(F)=\sum_{i=1}^3 x_ie_i$ with $x_i=1$ or 0, we take $\delta(F)=\sum_{i=1}^4 x_ie_i$, where $x_4=1+\sum_{i=1}^3 x_i \mod 2$. A $\mathbb{Z}_2^3$–coloring λ is called *nonorientable* if the corresponding 3–manifold $M(nP,\lambda)$ is nonorientable. Furthermore, if the 3–manifold $M(nP,\lambda)$ is nonorientable, then its natural $\mathbb{Z}_2^4$–coloring δ is called the *natural* $\mathbb{Z}_2^4$*–extension* of λ. It can be shown that $M(nP,\delta)$ is the orientable double cover of $M(nP,\lambda)$ when $M(nP,\lambda)$ is nonorientable. Our main technical theorem is the following.

Theorem 1.4 *For each positive integer n and each odd integer k in $[1,5n+3]$, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ on the polytope nP such that the first Betti number of the orientable 3–manifold $M(nP,\delta)$ is k, where δ is the natural $\mathbb{Z}_2^4$–extension of λ.*

From Theorem 1.4, given a positive integer n and an odd integer k in $[1,5n+3]$, there exists an orientable 3–manifold $M(nP,\delta)$ whose first Betti number is exactly k. Moreover, we conjecture that there is no coloring on nP leading to an orientable manifold $M(nP,\delta)$ with first Betti number not an odd integer $k\le 5n+3$. The converse has been checked numerically, but has not been proved rigorously yet.

Proof of Theorem 1.2 For a nonorientable $\mathbb{Z}_2^3$–coloring λ on the polytope nP, there is a natural $\mathbb{Z}_2^4$–extension δ on nP. Both $M(nP,\delta)$ and $M(nP,\lambda)$ are 3–manifolds and $M(nP,\delta)$ is the orientable double cover of $M(nP,\lambda)$. See Proposition 2.11 in Section 2.4 for more details.

Next, we want to show that $M(nP,\delta)$ is geometrically bounding. First, we use Proposition 2.9 in [9] to extend the $\mathbb{Z}_2^4$–coloring δ on the 3–dimensional polytope nP to a $\mathbb{Z}_2^5$–coloring ε on the 4–dimensional polytope nE. Here, nE is a 4–dimensional polytope obtained by stacking n copies of the hyperbolic right-angled 120–cell E. Then $M(nE,\varepsilon)$ is an orientable hyperbolic 4–manifold in which $M(nP,\lambda)$ can be embedded. Second, since $M(nP,\delta)$ is the orientable double cover of $M(nP,\lambda)$, it admits a fixed-point-free orientation-reversing involution. We may thus apply Corollary 9 of [12]. By cutting $M(nE,\varepsilon)$ along the hypersurface $M(nP,\lambda)$ and applying completion, we can obtain a totally geodesic hyperbolic 4–manifold with boundary $M(nP,\delta)$. Now, Theorem 1.2 follows from Theorem 1.4. $\square$

Outline of the paper

In Section 2, we provide some preliminaries on the algebraic theory of real toric manifolds. In Section 3, we prove Lemma 3.1, which is the key element of the main theorem. In Sections 4 and 5, we prove Theorem 1.4 for the cases of even and odd n, respectively.

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2 Preliminaries

In this section, we list some facts concerning real toric manifolds and introduce the 3–dimensional right-angled hyperbolic polytope nP. Proofs, details, and definitions can be found in [1]. For the sake of brevity, we write *n–polytope* instead of *n–dimensional polytope*, and by *facet* we mean a face of codimension one. An n–polytope is called simple if every r–face belongs to exactly $n-r$ facets.

2.1 Real toric manifolds

Given a simple n–polytope L, let $\mathcal{F}(L)=\{F_1,F_2,\dots,F_m\}$ be its set of facets. Let us define the $\mathbb{Z}_2^k$*–coloring characteristic function*, $n\le k\le m$, as a function

$$\lambda\colon \mathcal{F}(L)=\{F_1,F_2,\dots,F_m\}\to\mathbb{Z}_2^k$$

that satisfies the *nonsingularity condition*. That is, $\lambda(F_{i_1}), \lambda(F_{i_2}), \dots, \lambda(F_{i_n})$ generate a subgroup of $\mathbb{Z}_2^k$ which is isomorphic to $\mathbb{Z}_2^n$ when the n facets $F_{i_1}, F_{i_2}, \dots, F_{i_n}$ share a common vertex. The binary matrix $\Lambda_{(n\times m)} = (\lambda(F_1), \lambda(F_2), \dots, \lambda(F_m))$ is called the *characteristic matrix* of λ.

Then, we can construct a smooth manifold $M(L,\lambda) := L \times \mathbb{Z}_2^k/\sim$, called a *real toric manifold over the polytope P*, by the equivalence relation

$$(x, g_1) \sim (y, g_2) \quad \Longleftrightarrow \quad \begin{cases} x = y \text{ and } g_1 = g_2 & \text{if } x \in \operatorname{Int} L, \\ x = y \text{ and } g_1^{-1} g_2 \in G_f & \text{if } x \in \partial L, \end{cases}$$

where $f = F_{i_1} \cap \cdots \cap F_{i_{n-r}}$ is the unique face of codimension $n-r$ that contains x as an interior point, and G_f is the subgroup generated by $\lambda(F_{i_1}), \lambda(F_{i_2}), \dots, \lambda(F_{i_{n-r}})$. The notation $M(L,\lambda)$ also highlights that each real toric manifold corresponds to a pair $\{(L,\lambda)\}$ that is made of a polytope and a characteristic function. For brevity, we refer to the colorings when the polytope is given instead of talking about both colorings and manifolds. When $k = m$, $M(L,\lambda)$ is known as the *real moment-angle manifold* over the polytope L, which admits a natural $\mathbb{Z}_2^m$–action. If $k = n$, then the corresponding manifold is called a *small cover*. By the four color theorem, we know that small covers can always be realized over any 3–dimensional simple polytope.

Example 2.1 Define a $\mathbb{Z}_2^3$–coloring characteristic function λ on the right-angled spherical triangle $\triangle^2$ as shown in Figure 1. Namely, the characteristic function is

$$\begin{gathered} \lambda \colon \{\{a,b\},\{b,c\},\{a,c\}\} \to \{(1,0,0),(0,1,0),(0,0,1)\}, \\ (a,b) \mapsto (1,0,0), \\ (b,c) \mapsto (0,1,0), \\ (a,c) \mapsto (0,0,1), \end{gathered}$$

where $(1,0,0) = e_1$, $(0,1,0) = e_2$ and $(0,0,1) = e_3$ are the standard basis vectors of $\mathbb{Z}_2^3$.

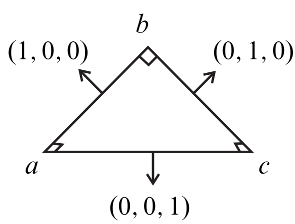


Figure 1: The coloring in Example 2.1.

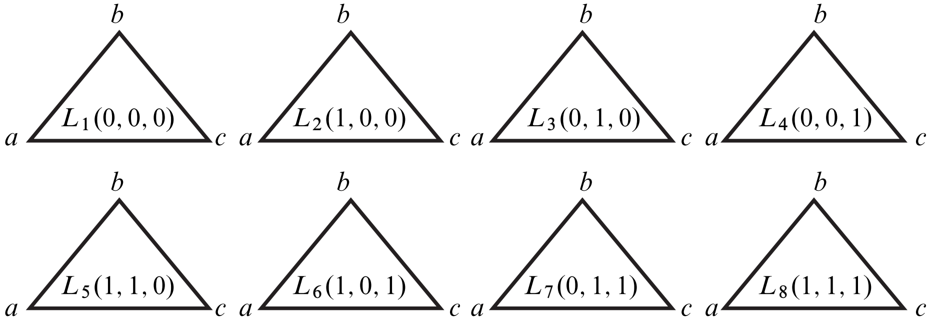


Figure 2: The eight polytopes $\triangle^2 \times \mathbb{Z}_3^2$ of Example 2.1.

Now, we have eight copies of the polytope, namely $\triangle^2 \times \mathbb{Z}_2^3$, as shown in Figure 2.

By the equivalence relation

$$(p, g_1) \sim (q, g_2) \iff \begin{cases} p = q, \\ g_1 - g_2 \in \{(1,0,0), (0,1,0), (0,0,1)\}, \end{cases}$$

we can finally obtain the manifold $M(\triangle^2, \lambda) \approx \mathbb{S}^2$ as shown in Figure 3, which inherits a spherical structure from the eight copies of right-angled triangles. □

In order to keep notation concise, we regard every $\mathbb{Z}_2^*$–color as a binary number and encode it with an integer. For example in the $\mathbb{Z}_2^3$–coloring case, we can use 1, 2, 3, 4, 5, 6 and 7 to represent the seven colors $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$, respectively. Then, a characteristic matrix can also be viewed

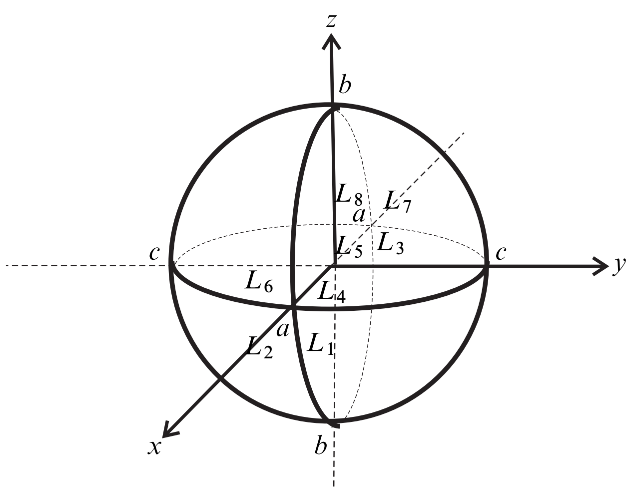


Figure 3: The real toric manifold $M(\triangle^2, \lambda)$.

as a characteristic vector. For example, the characteristic matrix of the $\mathbb{Z}_2^3$–coloring characteristic function in Example 2.1 is

$$\Lambda_{(3\times 3)} = (\lambda(\mathcal{F}_1), \lambda(\mathcal{F}_2), \lambda(\mathcal{F}_3)) = (\lambda(a,b), \lambda(b,c), \lambda(a,c)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the corresponding characteristic vector C is $(1,2,4)$. The characteristic function λ, characteristic matrix Λ, and the characteristic vector C can be constructed from each other easily; the characteristic vector C represents the most concise form.

2.2 Cohomology of real toric manifolds

Davis and Januszkiewicz [6] formulated how to calculate the $\mathbb{Z}_2$–coefficient cohomology groups of a small cover from the polytope and characteristic function. In 2013, Cai [2] suggested a method to calculate the $\mathbb{Z}$–coefficient cohomology groups of a real moment-angle manifold. Based on the results of Cai, Suciu and Trevisanon [21; 22] on rational homology groups of real toric manifolds, Choi and Park [3] obtained a formula for the cohomology groups of real toric manifolds. This can also be viewed as a combinatorial version of the Hochster theorem [8].

Since the dual of the boundary of a simple polytope L is a simplicial complex K (see eg [1]), the definition of real toric manifolds introduced above has a dual version. By substituting the facet set $\mathcal{F}(L)$ with the vertex set $\mathcal{V}$ of the simplicial complex K, we can define the *characteristic function* λ on K, namely

$$\lambda\colon \mathcal{V}(K) = \{v_1, v_2, \dots, v_m\} \to \mathbb{Z}_2^k.$$

The nonsingularity condition changes as follows: if for n vertices $v_{i_1}, v_{i_2}, \dots, v_{i_n}$ the convex hull $\operatorname{conv}\{v_{i_1}, v_{i_2}, \dots, v_{i_n}\}$ is a facet of K, the images $\lambda(v_{i_1}), \lambda(v_{i_2}), \dots, \lambda(v_{i_n})$ shall generate a subgroup isomorphic to $\mathbb{Z}_2^n$. For the sake of brevity, we denote the linear space $\mathbb{Z}_2^{|\mathcal{V}|}$ by $\mathbb{Z}_2^{\mathcal{V}}$. In addition, we can identify $\mathbb{Z}_2^{\mathcal{V}}$ with the power set $2^{\mathcal{V}}$ in the canonical way, where $\varnothing$ corresponds to the identity element and multiplication to the symmetric difference. Namely, we have a map $\varphi\colon \mathbb{Z}_2^{\mathcal{V}} \to 2^{\mathcal{V}}$. Denote by K_ω the full subcomplex of $K = (\partial L)^*$ obtained by restricting to $\omega \subseteq \mathcal{V}$. Then every full subcomplex K_ω of K, where $\omega \subseteq \mathcal{V}$, is identified with an element of $\mathbb{Z}_2^{\mathcal{V}}$.

Let λ be a $\mathbb{Z}_2^k$–coloring characteristic function. Denote by $\operatorname{row}\Lambda$ the *row space* of the characteristic matrix Λ. The following Choi–Park theorem shows that the cohomology group of a real toric manifold $M(L,\lambda)$ is the direct sum of the cohomology groups of some full subcomplexes of the dual polytope $K = (\partial L)^*$. The full subcomplexes are determined by the characteristic function.

Theorem 2.2 (Choi–Park [3]) *Assume G is the coefficient ring $\mathbb{Q}$ or $\mathbb{Z}_q$ for a positive odd integer q. There is an additive isomorphism*

$$H^p(M(L,\lambda);G) \cong \bigoplus_{\varphi^{-1}(\omega)\in \text{row}\,\Lambda} \widetilde{H}^{p-1}(K_\omega;G),$$

where Λ is the characteristic matrix of λ.

We use β^i to denote the rank of $H^i(M(L,\lambda);\mathbb{Q})$, called the i^{th} Betti number of $M(L,\lambda)$; and use $\widetilde{\beta}^0$ to denote the rank of $\widetilde{H}^0(K_\omega;\mathbb{Q})$, called the reduced zeroth Betti number of K_ω. For the purpose of this paper, we only need the following result.

Corollary 2.3 *For a simple polytope L,*

$$\beta^1(M(L,\lambda);\mathbb{Q}) = \sum_{\varphi^{-1}(\omega)\in \text{row}\,\Lambda} \widetilde{\beta}^0(K_\omega;\mathbb{Q}),$$

where Λ is the characteristic matrix of λ.

By means of Corollary 2.3, we can calculate the first Betti number of a real toric manifold using the combinatorial information of the orbit polytope and the row space of its characteristic matrix. In the following, we show a simple example.

Example 2.4 Calculate the first Betti number of the Klein bottle $S = M(L,\lambda)$.

Figure 4, left, is a colored 2–dimensional square L, whereas Figure 4, right, is its dual $K = (\partial L)^*$, with its vertices colored accordingly.

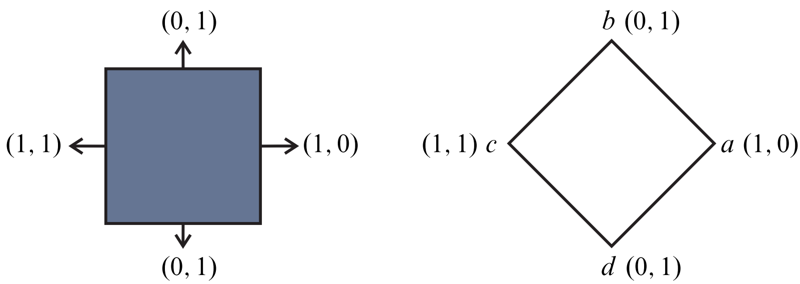


Figure 4: The colored square for Example 2.4.

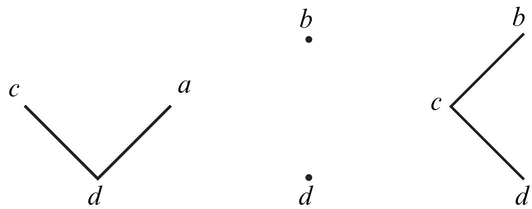


Figure 5: Left to right, the subcomplexes K_{ω_i}, $2 \le i \le 4$.

Thus, the row space is

$$\operatorname{row}\Lambda = \langle (1,0,1,0),(0,1,1,1)\rangle = \{(0,0,0,0),(1,0,1,1),(0,1,0,1),(1,1,1,0)\}.$$

For $\omega_1 = (0,0,0,0)$, $K_{\omega_1} = \varnothing$.

For $\omega_2 = (1,0,1,1)$, then K_{ω_2} is as shown in Figure 5, left. So $\widetilde{\beta}^0(K_{\omega_2}) = 0$.

For $\omega_3 = (0,1,0,1)$, then K_{ω_3} is as shown in Figure 5, center. So $\widetilde{\beta}^0(K_{\omega_3}) = 1$.

For $\omega_4 = (1,1,1,0)$, then K_{ω_4} is as shown in Figure 5, right. So $\widetilde{\beta}^0(K_{\omega_4}) = 0$.

By Corollary 2.3, we have

$$\beta^1(S) = \widetilde{\beta}^0(K_{\omega_1}) + \widetilde{\beta}^0(K_{\omega_2}) + \widetilde{\beta}^0(K_{\omega_3}) + \widetilde{\beta}^0(K_{\omega_4}) = 0+0+1+0 = 1,$$

which coincides with the well-known result of rational homology groups of the Klein bottle. □

2.3 The 3–polytopes nP

In the following, we assume that P is the regular right-angled dodecahedron in $\mathbb{H}^3$ with twelve 2–dimensional facets. We use nP to denote the stacking of n copies of P, ie the polytope made of n dodecahedra in a row; see Figures 6, 7 and 12. The simplicial complex nK is the dual of the boundary of nP. For each polytope nP with $n \geqslant 2$, there are $n+3$ layers of facets of nP: the first and the last layers are pentagons, the second and the $(n+2)^{\text{nd}}$ layers consist of five pentagons, and each layer from the third to the $(n+1)^{\text{st}}$ is made of five hexagons. There is no hexagonal layer in $1P$, and the polytope nP has $5n+7$ facets in total. All the polytopes nP, with $n \in \mathbb{Z}_+$, are right-angled hyperbolic polytopes. In addition, we call the i–layer of a colored 3–polytope nP a *brick*, where $2 \le i \le n+1$ and $n \ge 2$. The symbols nP and nK are used throughout the paper with this meaning, unless stated otherwise.

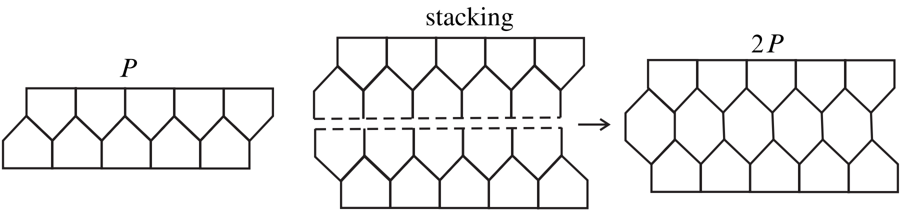


Figure 6: Build up the polytope $2P$ by stacking.

Definition 2.5 Given a polytope L with m facets, we define $X(L) = (a_{ij})_{m\times m}$ to be the *adjacency matrix* of L, where

$$a_{ij} = \begin{cases} 1 & \text{if } F_i \cap F_j \text{ for } F_i, F_j \in \mathcal{F}(L) \text{ is an } (n-2)\text{–face of } L \text{ or } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.6 A simple polytope L is called a flag polytope if every collection of pairwise intersecting facets has a nonempty intersection.

For a flag polytope, all of the information about the intersection of its facets is included in the adjacency matrix. As can be easily checked, the polytope nP is a flag polytope for every n. In order to obtain more unified adjacency matrices $X(nP)$, $n \in \mathbb{Z}_+$, we order the facets of the polytope nP in the following manner. The first and the last layer are labeled as 1 and $5n+7$, respectively, while the facets in between are labeled layer by layer. For even layers, we start from the middle and order the rest by left-right double siding, whereas for odd layers, we adopt a right-left double siding. We illustrate the labeling manner on the polytope $5P$ in Figure 7, where the double sidings of even and odd layers are displayed by the arrow-lines on the second and third layers, respectively.

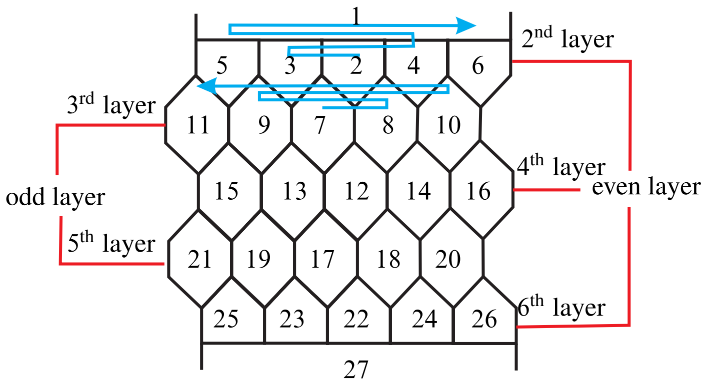


Figure 7: Facet ordering of the polytope $5P$.

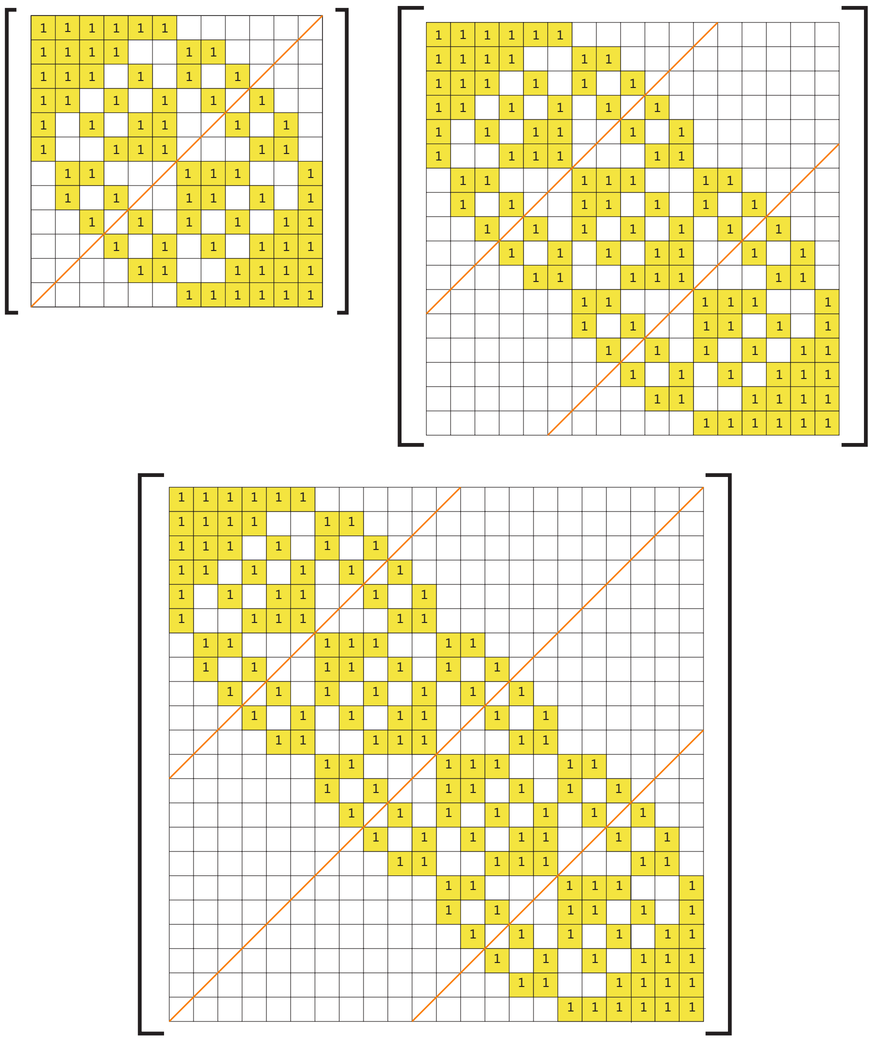


Figure 8: The adjacency matrices of the polytopes P, $2P$ and $3P$ are given at top left, top right and bottom, respectively.

Using this ordering, we obtain more unified increasing patterns of the adjacency matrices. We display some of them in Figure 8 (the omitted entries are zeros).

2.4 Orientability of real toric manifolds

H Nakayama and Y Nishimura discussed the orientability of small covers in [15]. Below we quote their main theorem.

Theorem 2.7 (Nakayama–Nishimura [15]) *For a simple n–dimensional polytope L, and for a basis $\{e_1, \dots, e_n\}$ of $\mathbb{Z}_2^n$, a homomorphism $\epsilon\colon \mathbb{Z}_2^n \to \mathbb{Z}_2 = \{0,1\}$ is defined by $\epsilon(e_i) = 1$ for each $i = 1, \dots, n$. A small cover $M(L,\delta)$ is orientable if and only if there exists a basis $\{e_1, \dots, e_n\}$ of $\mathbb{Z}_2^n$ such that the image of $\epsilon\delta$ is $\{1\}$.*

The techniques used in proving Theorem 2.7 are actually suitable for all real toric manifolds, not just for small covers. Corollary 2.3 with rational coefficients implies this conclusion as well. The n^{th} Betti number of a real toric manifold $M(L,\delta)$ over the n–polytope L is 1 if and only if there is an element in the row space of the characteristic matrix of δ with all entries equal to 1.

Corollary 2.8 (Nakayama–Nishimura [15] and Choi–Park [3]) *For a simple n–dimensional polytope L, the real toric manifold $M(L,\delta)$ is orientable if and only if there is a basis such that the sum of every column of the characteristic matrix Λ of δ is* $1 \bmod 2$.

In particular, the four vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$, which are the binary forms of 1, 2, 4 and 7, are the only four elements in $\mathbb{Z}_2^3$ whose entry sums are $1 \bmod 2$. These four vectors are called *orientable colors*. The three colors left are $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$, which are the binary forms of 3, 5 and 6. An *orientable basis* in $\mathbb{Z}_2^3$ is defined to be a basis in $\mathbb{Z}_2^3$ that consists of three linearly independent orientable colors. In particular, the standard basis in $\mathbb{Z}_2^3$, ie $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, is an orientable basis. If the small cover $M(L,\lambda)$ is orientable, then there exists an orientable basis such that all the colors of λ are orientable. Note that, for an orientable color, the number of entries with value 1 is always odd. In other words, when changing from one orientable basis to another orientable one, we actually add or remove an even number of 1s from the previous characteristic matrix to form the new one. Hence the parity of the number of 1s in each column is preserved under different orientable bases. Therefore, we have the following corollary.

Corollary 2.9 *Given a 3–polytope nP with facets ordered as required in Section 2.3, we fix the colors on first three facets to be $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Suppose $(1,2,4,a_1,\dots,a_m)$ is a characteristic vector of nP. Then the corresponding small cover is nonorientable if there is some $a_i \in \{3,5,6\}$.*

Starting from a $\mathbb{Z}_2^3$–coloring λ on the polytope nP, we can obtain $2^m - 1$ $\mathbb{Z}_2^4$–colorings on nP by adding a nonzero fourth row to the $3 \times m$ characteristic matrix Λ of λ

as shown:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \\ * & * & * & \cdots & * \end{pmatrix},$$

where $m = 5n+7$ and $* \in \{0, 1\}$. Those characteristic functions are called the *extensions* of λ, and they naturally satisfy the nonsingularity condition.

Definition 2.10 A $\mathbb{Z}_2^3$–coloring λ on the polytope nP is *admissible* if there is a $\mathbb{Z}_2^4$–coloring extension of λ, denoted by δ, such that $M(nP, \lambda)$ is nonorientable and $M(nP, \delta)$ is the orientable double cover of $M(nP, \lambda)$.

Along with some basic facts about the fundamental group of a double cover we have the following proposition. It is valid for any polytope and we are now interested in the case of polytope nP.

Proposition 2.11 *A $\mathbb{Z}_2^3$–coloring λ over a simple 3–dimensional polytope nP is admissible if $M(nP, \lambda)$ is nonorientable.*

Proof Because $M(nP, \lambda)$ is nonorientable, at least one column of its characteristic matrix Λ has an even sum. Therefore, we can add a nonzero fourth row to the characteristic matrix Λ to obtain a $\mathbb{Z}_2^4$–coloring extension of λ, denoted by δ, satisfying that the sum of all its columns are odd. By Corollary 2.8, $M(nP, \delta)$ is orientable.

Let $W(nP)$ be the Coxeter group of nP and $\theta\colon \mathcal{F}(L) = \{F_1, F_2, \dots, F_m\} \to \mathbb{Z}_2^m$ be the map that sends each F_i to e_i. Now we have the diagram

$$\begin{tikzcd} W(nP) \arrow[r, "l"] & \mathbb{Z}_2^m \arrow[r, "\widehat{\delta}"] \arrow[dr, "\widehat{\lambda}"'] & \mathbb{Z}_2^4 \arrow[d, "p"] \\ & & \mathbb{Z}_2^3 \end{tikzcd}$$

where l is the abelianization, p is the natural projection of $\mathbb{Z}_2^4$ to $\mathbb{Z}_2^3$ that keeps only the first three coordinates, and $\widehat{\lambda}$ and $\widehat{\delta}$ are the maps induced by the characteristic functions λ and δ, ie $\lambda = \widehat{\lambda} \circ \theta$ and $\delta = \widehat{\delta} \circ \theta$. It is easy to check that the triangular circuit commutes, namely, $p \circ \widehat{\delta} = \widehat{\lambda}$.

By [6, Corollary 4.5], $\pi_1(M(nP, \lambda)) = \ker(\widehat{\lambda} \circ l) = \ker(p \circ \widehat{\delta} \circ l)$ and $\pi_1(M(nP, \delta)) = \ker(\widehat{\delta} \circ l)$. Thus $M(nP, \delta)$ is an orientable double cover of $M(nP, \lambda)$. □

The $\mathbb{Z}_2^4$–coloring δ on the polytope nP in Proposition 2.11 is called an *admissible extension* of λ or a *natural* $\mathbb{Z}_2^4$*–coloring associated to* λ (also referred to as the *natural* $\mathbb{Z}_2^4$*–extension* of λ for short). We use the symbols λ and δ with this meaning in the rest of the paper, unless stated otherwise. Moreover, by Corollary 2.3, the Betti numbers of the orientable manifold recovered by the natural $\mathbb{Z}_2^4$–extension δ can be easily computed, as we are going to show in Example 2.12.

Example 2.12 Let us calculate the Betti numbers of some orientable real toric manifold $M(P,\delta)$.

We show in Figure 9, left, a plane figure of the dodecahedron P whose facets are ordered in the "double siding" manner introduced in Section 2.3. In Figure 9, right, is the dual simplicial complex $K=(\partial P)^*$ with its 12 vertices labeled correspondingly.

Color the polytope P with the characteristic vector $v=(1,2,4,5,3,7,7,3,5,4,2,1)$ and denote the corresponding characteristic function by λ. Then we have a $\mathbb{Z}_2^3$–coloring characteristic matrix

$$\Lambda=\begin{pmatrix}0&0&1&1&0&1&1&0&1&1&0&0\\0&1&0&0&1&1&1&1&0&0&1&0\\1&0&0&1&1&1&1&1&1&0&0&1\end{pmatrix}_{3\times 12}.$$

By Corollary 2.9, λ is nonorientable. The characteristic matrix Δ of its admissible extension δ is

$$\Delta=\begin{pmatrix}&&&&&\Lambda&&&&&&\\0&0&0&1&1&0&0&1&1&0&0&0\end{pmatrix}_{4\times 12}.$$

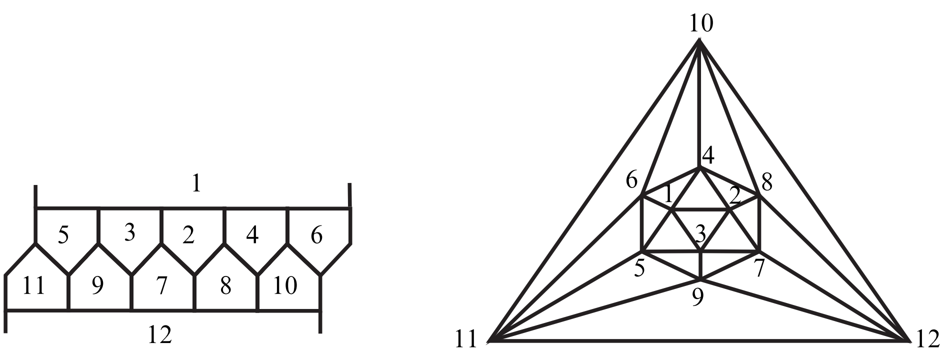


Figure 9: The facet-ordered polytope P, left, and its dual simplicial complex $K=(\partial P)^*$, right.

The row space of Δ is given by

$$\text{row}\,\Delta = \big\langle(0,0,1,1,0,1,1,0,1,1,0,0),(0,1,0,0,1,1,1,1,0,0,1,0),\\ (1,0,0,1,1,1,1,1,1,0,0,1),(0,0,0,1,1,0,0,1,1,0,0,0)\big\rangle.$$

For each $\omega_i \in \text{row}\,\Lambda$, we calculate its reduced 0^{th} Betti number in Tables 1–2.

From Tables 1–2 and Corollary 2.3, we have

$$\beta^1(M(P,\delta);Q) = \sum_{i=1}^{16}\widetilde{\beta}^0(K_{\omega_i};\mathbb{Q}) = \beta^2(M(P,\delta);\mathbb{Q}) \\ = \sum_{i=1}^{16}\widetilde{\beta}^1(K_{\omega_i};\mathbb{Q}) = 7. \qquad \square$$

For an orientable 3–manifold $M(nP,\delta)$, by Poincaré duality we have $\beta^0(M(nP,\delta)) = \beta^3(M(nP,\delta)) = 1$ and $\beta^1(M(nP,\delta)) = \beta^2(M(nP,\delta))$. So β^1 is the only thing we need in order to determine the free part of $H^*(M(nP,\delta))$. By Corollary 2.3, $\beta^1(M(nP,\delta))$ is equal to the sum of the reduced zeroth Betti numbers of the 16 full subcomplexes k_{ω_i} of the simplicial complex $nK = (\partial(nP))^*$. Each subcomplex k_{ω_i} corresponds to a nonzero vector in the row space row Δ.

3 The key lemma

The purpose of this section is to prove Lemma 3.1, which is the key element in proving Theorem 1.4. We want to find a special family of admissible $\mathbb{Z}_2^3$–colorings over the polytope nP. According to the correspondence discussed in Section 2, we construct a family of orientable 3–manifolds $M(nP,\delta)$.

Lemma 3.1 *For every positive even integer n, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP such that $\beta^1(M(nP,\delta)) = n+1$, where δ is the natural associated $\mathbb{Z}_2^4$–coloring extension of λ.*

Proof We first prove the special case in which $n = 2$. We use the notation $a_1 = 1$, $S_1 = (24247)$ and $S_2S_1 = (35716\ 24247)$. By $[a_1 S_1 \underline{S_2 S_1} a_1]$, we mean the colored polytope $2P$ shown in Figure 10. The corresponding characteristic vector C is

$$(1,2,4,4,2,7,7,1,5,6,3,2,4,4,2,7,1).$$

It can be checked with little effort that the nonsingularity condition holds at every vertex. We call S_i, $1 \le i \le 2$, a *brick* and a_i, which represents the first or the last

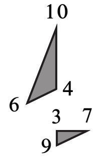
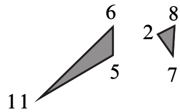
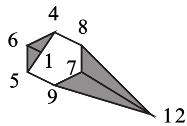
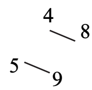
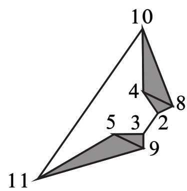
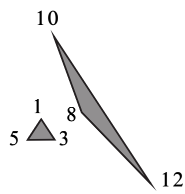
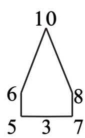
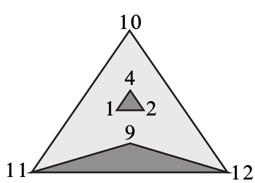
i	ω_i	K_{ω_i}	$\widetilde{\beta}^0(K_{\omega_i})$	$\beta^1(K_{\omega_i})$	$\beta^2(K_{\omega_i})$
1	$(0,0,1,1,0,1,1,0,1,1,0,0)$		1		
2	$(0,1,0,0,1,1,1,1,0,0,1,0)$		1		
3	$(1,0,0,1,1,1,1,1,1,0,0,1)$		0	1	
4	$(0,0,0,1,1,0,0,1,1,0,0,0)$		1		
5	$(0,1,1,1,1,0,0,1,1,1,1,0)$		0	1	
6	$(1,0,1,0,1,0,0,1,0,1,0,1)$		1		
7	$(0,0,1,0,1,1,1,1,0,1,0,0)$		0	1	
8	$(1,1,0,1,0,0,0,0,1,0,1,1)$		1		

Table 1: The values of $\widetilde{\beta}^0(K_{\omega_i})$ for $i = 1, \dots, 8$.

colored facet, an *affix*. They are used for building the coloring. The symbols S_i and a_i are used with this meaning in the rest of the paper unless stated otherwise.

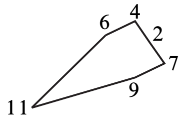
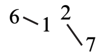
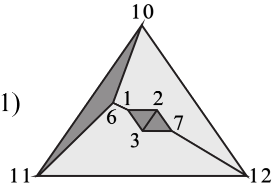
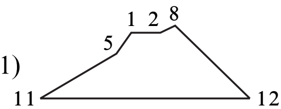
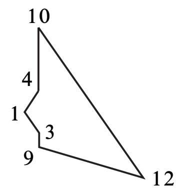
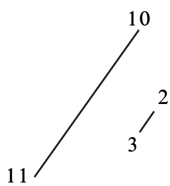
i	ω_i	K_{ω_i}	$\widetilde{\beta}^0(K_{\omega_i})$	$\beta^1(K_{\omega_i})$	$\beta^2(K_{\omega_i})$
9	$(0,1,0,1,0,1,1,0,1,0,1,0)$		0	1	
10	$(1,0,0,0,0,1,1,0,0,0,0,1)$		1		
11	$(1,1,1,0,0,1,1,0,0,1,1,1)$		0	1	
12	$(1,1,0,0,1,0,0,1,0,0,1,1)$		0	1	
13	$(1,0,1,1,0,0,0,0,1,1,0,1)$		0	1	
14	$(0,1,1,0,0,0,0,0,0,1,1,0)$		1		
15	$(0,0,0,0,0,0,0,0,0,0,0,0)$	$\varnothing$	no contribution to $\beta^1(M(P,\delta))$		
16	$(1,1,1,1,1,1,1,1,1,1,1,1)$	$\cong \mathbb{S}^2$	0	0	1

Table 2: The values of $\widetilde{\beta}^0(K_{\omega_i})$ for $i = 9, \dots, 16$.

Let us denote by λ the characteristic function of C. Corollary 2.9 and Proposition 2.11 imply that λ is admissible, and we denote by δ its natural $\mathbb{Z}_2^4$–extension. It follows that $M(2P,\lambda)$ is nonorientable, and $M(2P,\delta)$ is the orientable double cover of $M(2P,\lambda)$. The characteristic matrix Δ of the coloring δ is

$$\begin{pmatrix} 0&0&1&1&0&1&1&0&1&1&0&0&1&1&0&1&0\\ 0&1&0&0&1&1&1&0&0&1&1&1&0&0&1&1&0\\ 1&0&0&0&0&1&1&1&1&0&1&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0 \end{pmatrix}. \tag{3-1}$$

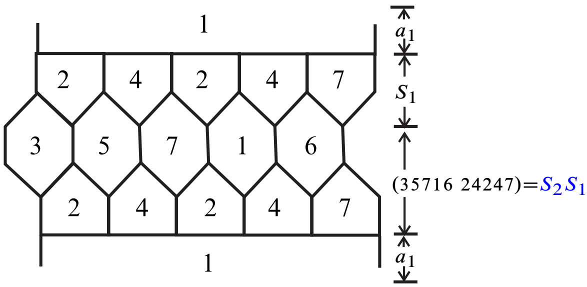


Figure 10: Colored polytope $2P$.

Then, the row space row Δ is given by

$$
\begin{pmatrix}
0&0&1&1&0&1&1&0&1&1&0&0&1&1&0&1&0\\
0&1&0&0&1&1&1&0&0&1&1&1&0&0&1&1&0\\
1&0&0&0&0&1&1&1&1&0&1&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
0&1&1&1&1&0&0&0&1&0&1&1&1&1&1&0&0\\
1&0&1&1&0&0&0&1&0&1&1&0&1&1&0&0&1\\
0&0&1&1&0&1&1&0&0&0&1&0&1&1&0&1&0\\
1&1&0&0&1&0&0&1&1&1&0&1&0&0&1&0&1\\
0&1&0&0&1&1&1&0&1&0&0&1&0&0&1&1&0\\
1&0&0&0&0&1&1&1&0&1&0&0&0&0&0&1&1\\
1&1&1&1&1&1&1&1&0&0&0&1&1&1&1&1&1\\
1&1&0&0&1&0&0&1&0&0&1&1&0&0&1&0&1\\
1&0&1&1&0&0&0&1&1&0&0&0&1&1&0&0&1\\
0&1&1&1&1&0&0&0&0&1&0&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{pmatrix}. \tag{3-2}
$$

By Corollary 2.3, we can calculate $\beta^1(M(2P,\delta))$ through its 15 nonempty full subcomplexes K_ω. Since $\widetilde{\beta}^0=\widetilde{\beta}_0$, the reduced zeroth Betti number of each K_ω is equal to the number of connected components of K_ω minus one.

For every i^{th} row $\omega_i(\Delta)=(w_{i1},\dots,w_{ij},\dots,w_{im})$ of the row space row Δ, where $m=5n+7$ is the number of facets of nP and $1\le i\le 2^4-1$, we define

$$\omega_i^*(\Delta):=\{j \mid 1\leqslant j\leqslant m \text{ and } \omega_{ij}=1, \text{ where } \omega_{ij}\in \text{row } \Delta\}.$$

Then define $X(nP,\omega_i(\Delta))$ to be the submatrix of $X(nP)$ obtained by selecting the q^{th} rows and q^{th} columns as q varies in $\omega_i^*(\Delta)$.

For example, pick the first row $\omega_1(\Delta)=(0,0,1,1,0,1,1,0,1,1,0,0,1,1,0,1,0)$ of the row space row Δ shown in matrix (3-2); then $\omega_1^*(\Delta)=(3,4,6,7,9,10,13,14,16)$.

$\omega_1(\Delta)$		0	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	0
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
1	2	1	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0
0	3	1	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0
0	4	1	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0
1	5	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0
1	6	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0
1	7	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0
0	8	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0
0	9	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0
1	10	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0
1	11	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0
1	12	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1
0	13	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1
0	14	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	1
1	15	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	1
1	16	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	1
0	17	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1

$\omega_1(\Delta)$		1	1	1	1	1	1	1	1	1
		3	4	6	7	9	10	13	14	16
1	3	1	0	0	1	1	0	0	0	0
1	4	0	1	1	0	0	1	0	0	0
1	6	0	1	1	0	0	1	0	0	0
1	7	1	0	0	1	1	0	1	0	0
1	9	1	0	0	1	1	0	1	0	0
1	10	0	1	1	0	0	1	0	1	1
1	13	0	0	0	1	1	0	1	0	0
1	14	0	0	0	0	0	1	0	1	1
1	16	0	0	0	0	0	1	0	1	1

$\omega_1^{(1)}(\Delta) = (3, 7, 9, 13)$

$\omega_1^{(2)}(\Delta) = (4, 6, 10, 14, 16)$

$\boxed{\widetilde{\beta}^0(K_{\omega_1(\Delta)}) = 1}$

Figure 11: The computation of $\widetilde{\beta}^0(K_{\omega_1(\Delta)})$. Left: $X(2P)$. Right: $X(2P, \omega_1(\Delta))$.

Let us consider the submatrix $X(2P, \omega_1(\Delta))$ which is obtained from the adjacency matrix $X(2P)$ by selecting the rows and columns set by $\omega_1^*(\Delta)$. By examining this matrix, it is obvious that there are two connected components. Use the notation $\omega_j^{(i)}(\Delta)$ to denote the vertex set of the i^{th} connected component of the full subcomplex $K_{\omega_j(\Delta)}$. Then, we have $\omega_1^{(1)}(\Delta) = (3, 7, 9, 13)$ and $\omega_1^{(2)}(\Delta) = (4, 6, 10, 14, 16)$; therefore, $\widetilde{\beta}^0(K_{\omega_1(\Delta)}) = 1$. The procedure is illustrated in Figure 11.

Likewise, we can calculate all of the $\widetilde{\beta}^0(K_{\omega_i(\Delta)})$, $1 \leq i \leq 15$, and the computation for $i = 2, 3, \dots, 7$ is illustrated in (A) and (B) of Figures 16–21 in the online supplement. Finally, we obtain $\beta^1(M(2P, \delta))$=3, as shown in the second line in Table 3. This completes the proof of Lemma 3.1 for the case $n = 2$.

From the results above, it follows that the first Betti numbers increase by a constant factor if the reduced 0^{th} Betti numbers $\widetilde{\beta}^0$ of the full subcomplexes corresponding to $\omega_i(\Delta)$ increase by a constant factor for $1 \leq i \leq 15$. Since the reduced Betti number $\widetilde{\beta}^0(K_{\omega_i(\Delta)})$ is obtained through the matrix $X(2P, \omega_i(\Delta))$, we only need to guarantee that matrices $X(nP, \omega_i)$ for $n = 2, 4, 6, \dots$ change with a certain pattern for all $1 \leq i \leq 15$. Notice that such a submatrix is completely determined by the adjacency matrix and the coloring of the polytope.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Betti number
$\widetilde{\beta}^0(K_{\omega_i}(\Delta))$	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	$\beta^1(M(2P,\delta)) = 3$
$\widetilde{\beta}^0(K_{\omega_i}(\Delta^1))$	1	1	0	1	0	1	1	0	0	0	0	0	0	0	0	$\beta^1(M(4P,\delta^1)) = 5$
$\widetilde{\beta}^0(K_{\omega_i}(\Delta^2))$	1	1	0	2	0	2	1	0	0	0	0	0	0	0	0	$\beta^1(M(6P,\delta^2)) = 7$
$\widetilde{\beta}^0(K_{\omega_i}(\Delta^3))$	1	1	0	3	0	3	1	0	0	0	0	0	0	0	0	$\beta^1(M(8P,\delta^3)) = 9$
$\widetilde{\beta}^0(K_{\omega_i}(\Delta^4))$	1	1	0	4	0	4	1	0	0	0	0	0	0	0	0	$\beta^1(M(10P,\delta^4)) = 11$
$\widetilde{\beta}^0(K_{\omega_i}(\Delta^5))$	1	1	0	5	0	5	1	0	0	0	0	0	0	0	0	$\beta^1(M(12P,\delta^5)) = 13$

Table 3: The computation of the first Betti number.

As for the adjacency matrices, they do change in a uniform manner when using the facet ordering described in Section 2.3; see also Figure 22 in the online supplement for the facet ordering and adjacency matrix of the polytopes $2P$, $4P$ and $6P$.

As for the coloring, we duplicate the last two bricks of the colored $2P$ a total of $\frac{1}{2}n-1$ times to construct the desired coloring on nP, where n is a positive even integer equal to or greater than 2. It can be easily proved that the nonsingularity condition holds at every vertex. The colorings constructed this way on polytopes $4P$ and $6P$ are shown in Figure 12, lower left and lower right, respectively. The colorings are denoted by

$$[a_1 S_1 \underline{S_2 S_1}\,\underline{S_2 S_1} a_1] \quad \text{and} \quad [a_1 S_1 \underline{S_2 S_1}\,\underline{S_2 S_1}\,\underline{S_2 S_1} a_1].$$

Their characteristic functions are written λ^1 and λ^2, respectively, where the superscripts denote how many times the last two bricks $(S_2 S_2)$ of the coloring $[a_1 S_1 \underline{S_2 S_1} a_1]$ of λ are repeated. The repeated parts are highlighted in blue and underlined. The nonorientability of these $\mathbb{Z}_2^3$–colorings is guaranteed by Corollary 2.9. Moreover, we can obtain their natural $\mathbb{Z}_2^4$–extensions δ^1 and δ^2. By Proposition 2.11, the colorings δ^1 and δ^2 are admissible. That is, $M(4P,\delta^1)$ and $M(6P,\delta^2)$ are the orientable double covers of the nonorientable manifolds $M(4P,\lambda^1)$ and $M(6P,\lambda^2)$, respectively. We denote the characteristic matrices of δ^1 and δ^2 by Δ^1 and Δ^2. The three matrices row Δ, row Δ^1 and row Δ^2 are shown in Figure 23 of the online supplement. Since the coloring on nP is obtained by duplicating the last two bricks of the coloring $[a_1 S_1 \underline{S_2 S_1} a_1]$ on $2P$ a total of $\frac{1}{2}n-1$ times, the row space row Δ^i can be obtained from row space row Δ by duplicating its columns, from the 11^{th} to the second columns (counting from right to left), $\frac{1}{2}n-1$ times.

By the method outlined before, we also calculate $\beta^1(M((2+2i)P,\delta^i))$ for $i = 1, 2, \ldots, 5$, as shown in Table 3. We illustrate the calculation of $\widetilde{\beta}^0(K_{\omega_1}(\Delta^1))$ and $\widetilde{\beta}^0(K_{\omega_1}(\Delta^2))$ in Figures 13 and 14, respectively. See also panels (C)–(D) and (E)–(F)

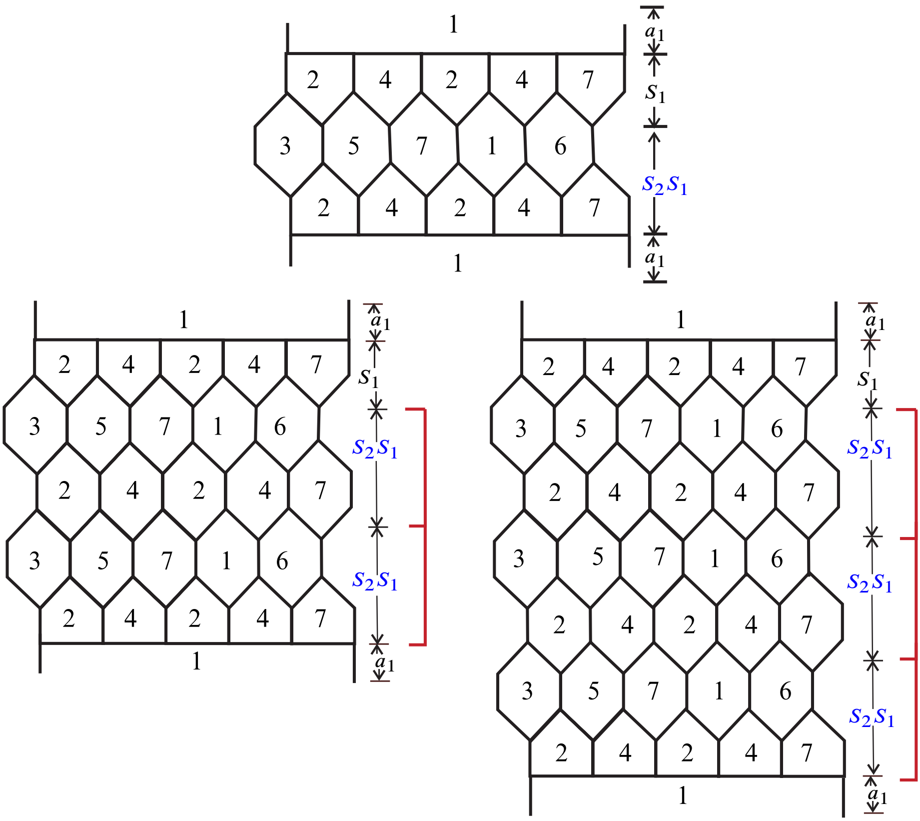


Figure 12: Top: The cololored polytope $2P$. Bottom left: The colored polytope $4P$. Bottom right: The colored polytope $6P$. Duplicate the last two bricks of the coloring $[a_1S_1\underline{S_2S_1}a_1]$ on $2P$ a total of $\frac{1}{2}n-1$ times to construct the desired coloring on nP.

in Figures 16–21 of the online supplement for the computation of $\widetilde{\beta}^0(K_{\omega_i}(\Delta^1))$ and $\widetilde{\beta}^0(K_{\omega_i}(\Delta^2))$ for $i=2,3,\dots,7$. The corresponding results are highlighted in blue in Table 3.

From Figure 11 and Table 3 we can see that the matrices $X(nP,\omega_i)$ for $n=2,4,6,\dots$ follow certain patterns for all $1\le i\le 15$. In order to guarantee that the sequence $\{\widetilde{\beta}^0(K_{\omega_i}(\Delta^t))\}$ with $t\in\mathbb{Z}_+$ is an arithmetic progression, we just need to guarantee that the first three items satisfy the relation of an arithmetic progression. For example, since $\widetilde{\beta}^0(K_{\omega_4}(\Delta))=0$, $\widetilde{\beta}^0(K_{\omega_4}(\Delta^1))=1$, $\widetilde{\beta}^0(K_{\omega_4}(\Delta^2))=2$ and the full subcomplex $K_{\omega_4}(\Delta^t)$ changes regularly as the colorings are obtained by duplicating t times the last two bricks of the colored $2P$ of $[a_1S_1\underline{S_2S_1}a_1]$, it follows that $\{\widetilde{\beta}^0(K_{\omega_i}(\Delta^t))\}$ with

$\omega_1(\Delta^1)$		0	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	0
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	1	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	3	1	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	4	1	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	5	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	6	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	7	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	8	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	9	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	10	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
1	11	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
1	12	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0
0	13	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0
0	14	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0
1	15	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0
1	16	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0
1	17	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0
0	18	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0
0	19	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0
1	20	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0
1	21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0
1	22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1
0	23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1
0	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	1
1	25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	1
1	26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	1
0	27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1

$\omega_1(\Delta^1)$		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
		3	4	6	7	9	10	13	14	16	17	19	20	23	24	26
1	3	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0
1	4	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0
1	6	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0
1	7	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0
1	9	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0
1	10	0	1	1	0	0	1	0	1	1	0	0	0	0	0	0
1	13	0	0	0	1	1	0	1	0	0	1	1	0	0	0	0
1	14	0	0	0	0	0	1	0	1	1	0	0	1	0	0	0
1	16	0	0	0	0	0	1	0	1	1	0	0	1	0	0	0
1	17	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0
1	19	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0
1	20	0	0	0	0	0	0	0	1	1	0	0	1	0	1	1
1	23	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0
1	24	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
1	26	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

$\omega_1^{(1)}(\Delta^1) = (3, 7, 9, 13, 17, 19, 23)$

$\omega_1^{(2)}(\Delta^1) = (4, 6, 10, 14, 16, 20, 24, 26)$

$$\boxed{\widetilde{\beta}^0(K_{\omega_1(\Delta^1)}) = 1}$$

Figure 13: The computation of $\widetilde{\beta}^0(K_{\omega_1(\Delta^1)})$. Top: $X(4P)$. Bottom: $X(4P, \omega_1(\Delta^1))$.

$\omega_1(\Delta^2)$		0	0	1	1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	1	1	0	0	1	1	0	1	1	0	1	1	0	0	1	1	0	1	0
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	2	1	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	3	1	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	4	1	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	5	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	6	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	7	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	8	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	9	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	10	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	11	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	12	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	13	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	14	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	15	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	16	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	17	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	18	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	19	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	20	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
0	21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
0	22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0
1	23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0
1	24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0
0	25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0	0	0	0	0	0
1	26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0	0	0	0	0	0
1	27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	0	0	0	0
0	28	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1	0	1	0	0	0
1	29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0
1	30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	0	0	1	0	1	0
0	31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0
0	32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1
1	33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1	0	1	0	1
1	34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	1
0	35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	1
1	36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	1	1
0	37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1

$\omega_1(\Delta^2)$		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
		3	4	6	7	9	10	13	14	16	17	19	20	23	24	26	27	29	30	33	34	36
1	3	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	4	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	6	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	7	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	9	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	10	0	1	1	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	13	0	0	0	1	1	0	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0
1	14	0	0	0	0	0	1	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0
1	16	0	0	0	0	0	1	0	1	1	0	0	1	0	0	0	0	0	0	0	0	0
1	17	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0
1	19	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0
1	20	0	0	0	0	0	0	0	1	1	0	0	1	0	1	1	0	0	0	0	0	0
1	23	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0	1	1	0	0	0	0
1	24	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1	0	0	0
1	26	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	1	0	0	0
1	27	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0
1	29	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	1	0	0
1	30	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0	1	1
1	33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0
1	34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
1	36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1

$\omega_1^{(1)}(\Delta^2) = (3, 7, 9, 13, 17, 19, 23, 27, 29, 33)$

$\omega_1^{(2)}(\Delta^2) = (4, 6, 10, 14, 16, 20, 24, 26, 30, 34, 36)$

$\boxed{\widetilde{\beta}^0(K_{\omega_1(\Delta^2)}) = 1}$

Figure 14: The computation of $\widetilde{\beta}^0(K_{\omega_1(\Delta^2)})$. Top: $X(6P)$. Bottom: $X(6P, \omega_1(\Delta^2))$.

$t \in \mathbb{Z}_+$ is an arithmetic progression. Namely, $\widetilde{\beta}^0(K_{\omega_4}(\Delta^3)) = 3$, $\widetilde{\beta}^0(K_{\omega_4}(\Delta^4)) = 4$, $\widetilde{\beta}^0(K_{\omega_4}(\Delta^5)) = 5, \dots$. As a consequence, if we want to prove that the whole Betti number sequence $\beta^1(M(nP, \delta^{\frac{1}{2}n}))$, where n is an even positive integer, is an arithmetic progression, we only need to verify that $\beta^1(M(4P, \delta^1)) - \beta^1(M(2P, \delta)) = \beta^1(M(6P, \delta^2)) - \beta^1(M(4P, \delta^1))$. Summarizing all these findings, we have the following proposition:

Proposition 3.2 *Let δ be a $\mathbb{Z}_2^4$–coloring over the polytope nP. For an arbitrary even number $s \geqslant n$, if*

$$\beta^1(M((n+2)P, \delta^{(1)})) - \beta^1(M(nP, \delta)) = \beta^1(M((n+4)P, \delta^{(2)})) - \beta^1(M((n+2)P, \delta^{(1)})),$$

we have

$$\beta^1(M(sP, \delta^{\frac{1}{2}(s-n)})) = \beta^1(M(nP, \delta)) + \tfrac{1}{2}(s-n)\big(\beta^1(M(n+1)P, \delta^1) - \beta^1(M(nP, \delta))\big),$$

where $\delta^{(t)}$ represents a $\mathbb{Z}_2^4$–coloring over the polytope $(n+2t)P$. The coloring vector of $\delta^{(t)}$ is obtained by duplicating the last two bricks of δ exactly t times.

By Proposition 3.2 and using the facts that $\beta^1(M(2P, \delta)) = 3$, $\beta^1(M(4P, \delta^1)) = 5$ and $\beta^1(M(6P, \delta^2)) = 7$, we can produce Table 4.

This concludes the proof of Lemma 3.1. □

4 Proof of Theorem 1.2 for *n* even

In this section, we prove Theorem 1.2 when n is even. It is similar to the proof of Lemma 3.1.

Lemma 4.1 *For any even positive number n, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP, such that, for its natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(nP, \delta)) = 5n - 3$.*

Proof Let $S_1 = (65372)$, $S_2S_3 = (72424\ 65372)$ and $a_1 = 1$. By the same idea of Lemma 3.1, we first construct a suitable nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope $2P$ as follows:

$$(1, 3, 5, 7, 6, 2, 4, 2, 2, 4, 7, 3, 5, 7, 6, 2, 1).$$

	$n=2$	$n=4$	$n=6$	$\cdots$	$n=2+2t, t\in\mathbb{N}$
1	1	1	1	$\cdots$	1
2	1	1	1	$\cdots$	1
3	0	0	0	$\cdots$	0
4	0	1	2	$\cdots$	t
5	0	0	0	$\cdots$	0
6	0	1	2	$\cdots$	t
7	1	1	1	$\cdots$	1
8	0	0	0	$\cdots$	0
9	0	0	0	$\cdots$	0
10	0	0	0	$\cdots$	0
11	0	0	0	$\cdots$	0
12	0	0	0	$\cdots$	0
13	0	0	0	$\cdots$	0
14	0	0	0	$\cdots$	0
15	0	0	0	$\cdots$	0
total β^1	3	5	7	$\cdots$	$3+2t=n+1$

Table 4: The values of β^1 in Lemma 3.1.

This colored polytope $2P$ is denoted by $[a_1S_1\underline{S_2S_1}a_1]$. It follows from Corollary 2.9 and Proposition 2.11 that λ is nonorientable and admissible. Denote by δ the natural $\mathbb{Z}_2^4$–extension of λ. The 3–manifold $M(2P,\delta)$ is the orientable double cover of the nonorientable 3–manifold $M(2P,\lambda)$. By Corollary 2.3, we have $\beta^1(M(2P,\delta))=7$.

We repeat the last two bricks t times to construct a coloring over the polytope $(2+2t)P$, and denote its characteristic function by λ^t. In turn, the colored polytope $(2+2t)P$ is denoted by

$$[a_1S_1\underbrace{\underline{S_2S_1}\cdots\underline{S_2S_1}}_{t\text{ pairs}}a_1].$$

It can be easily checked that the nonsingularity condition holds at every vertex. Likewise, by Corollary 2.9 and Proposition 2.11, we can obtain an admissible extension δ^t of the nonorientable coloring λ^t. Moreover, $M((2+2t)P,\delta^t)$ is the orientable double cover of the nonorientable manifold $M((2+2t)P,\lambda^t)$. The Betti numbers of $(M(2P,\delta)$, $(M(4P,\delta^1)$ and $(M(6P,\delta^2)$ are shown in the second, third and fourth columns of Table 5. By Proposition 3.2 and using the facts that $\beta^1(M(2P,\delta))=7$, $\beta^1(M(4P,\delta^1))=17$ and $\beta^1(M(6P,\delta^2))=27$, we can deduce the last column of Table 5.

	$n=2$	$n=4$	$n=6$	$\cdots$	$n=2+2t,\ t\in\mathbb{N}$
1	0	0	0	$\cdots$	0
2	0	0	0	$\cdots$	0
3	2	4	6	$\cdots$	$2t+2$
4	1	2	3	$\cdots$	$t+1$
5	0	0	0	$\cdots$	0
6	1	3	5	$\cdots$	$2t+1$
7	0	0	0	$\cdots$	0
8	1	2	3	$\cdots$	$t+1$
9	0	0	0	$\cdots$	0
10	0	1	2	$\cdots$	t
11	0	0	0	$\cdots$	0
12	1	3	5	$\cdots$	$2t+1$
13	1	2	3	$\cdots$	$t+1$
14	0	0	0	$\cdots$	0
15	0	0	0	$\cdots$	0
total β^1	7	17	27	$\cdots$	$10t+7=5n-3$

Table 5: The values of β^1 for Lemma 4.1.

In other words, we may always find a nonorientable $\mathbb{Z}_2^3$–coloring λ such that its natural $\mathbb{Z}_2^4$–extension δ has $\beta^1(M(nP,\delta))=5n-3$. □

Lemma 4.2 *For any even positive integer n and any odd integer $k\in[5n-1,5n+3]$, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP such that, for its natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(nP,\delta))=k$.*

Proof We start at $n=2$ and construct suitable characteristic functions of the desired manifolds, whose first Betti numbers increase by $10t$ when the last pair of their coloring bricks are repeated t times. First, in Table 6 we prepare an affix and some bricks for constructing the coloring vectors needed.

Let λ_1^0, λ_1^1 and λ_1^2 be the three nonorientable $\mathbb{Z}_2^3$–coloring characteristic functions of the coloring vectors

$$[a_1S_1\underline{S_2S_1}a_1],\quad [a_1S_1\underline{S_2S_1}\,\underline{S_2S_1}a_1],\quad [a_1S_1\underline{S_2S_1}\,\underline{S_2S_1}\,\underline{S_2S_1}a_1]$$

over the polytopes $2P$, $4P$ and $6P$, respectively. Their characteristic vectors are

$(1,2,4,4,3,6,\underline{5,1,6,3,2,2,4,4,3,6},1)$,

$(1,2,4,4,3,6,\underline{5,1,6,3,2,2,4,4,3,6},\underline{5,1,6,3,2,2,4,4,3,6},1)$,

$(1,2,4,4,3,6,\underline{5,1,6,3,2,2,4,4,3,6},\underline{5,1,6,3,2,2,4,4,3,6},\underline{5,1,6,3,2,2,4,4,3,6},1)$.

to construct	affixes	brick	pair of bricks being repeated
λ_1^t	$a_1 = 1$	$S_1 = 34246$	$S_2S_1 = (26513\ 34246)$
λ_2^t, λ_3^t	$a_1 = 1$ $a_2 = 3$ $a_3 = 7$	$S_1 = (24246)$	$S_2S_3 = (73153\ 14245)$

Table 6: The affixes and bricks for constructing λ_1^t, λ_2^t and λ_3^t of Lemma 4.2.

It can be easily checked that the nonsingularity condition holds at every vertex. The natural associated $\mathbb{Z}_2^4$–extensions are denoted by δ_1^0, δ_1^1 and δ_1^2. By Corollary 2.3, we can calculate the first Betti numbers of those manifolds, namely $\beta^1(M(2P,\delta_1^0)) = 13$, $\beta^1(M(4P,\delta_1^1)) = 23$ and $\beta^1(M(6P,\delta_1^2)) = 33$. Thus, according to Proposition 3.2,

$$\beta^1(M((2+2t)P,\delta_1^t)) = 13+10t, \quad \text{where } t \in \mathbb{Z}_+. \tag{4-1}$$

Similarly, we describe the affixes and bricks for constructing λ_2^t and λ_3^t of Lemma 4.2 in Table 6.

Let us denote by λ_2^0, λ_2^1 and λ_2^2 the three nonorientable $\mathbb{Z}_2^3$–coloring characteristic functions of the following colored polytopes $2P$, $4P$ and $6P$:

$$[a_1S_1\underline{S_2S_3}a_2], \quad [a_1S_1\underline{S_2S_3}\,\underline{S_2S_3}a_2], \quad [a_1S_1\underline{S_2S_3}\,\underline{S_2S_3}\,\underline{S_2S_3}a_2],$$

and let λ_3^0, λ_3^1 and λ_3^2 be the $\mathbb{Z}_2^3$–coloring characteristic functions of the following colored polytopes $2P$, $4P$ and $6P$:

$$[a_1S_1\underline{S_2S_3}a_3], \quad [a_1S_1\underline{S_2S_3}\,\underline{S_2S_3}a_3], \quad [a_1S_1\underline{S_2S_3}\,\underline{S_2S_3}\,\underline{S_2S_3}a_3].$$

Their natural associated $\mathbb{Z}_2^4$–extensions are denoted as δ_2^0, δ_2^1, δ_2^2 and δ_3^0, δ_3^1, δ_3^2. The first Betti numbers of these manifolds, namely $\beta^1(M(2P,\delta_2^0))$, $\beta^1(M(4P,\delta_2^1))$, $\beta^1(M(6P,\delta_2^2))$ and $\beta^1(M(2P,\delta_3^0))$, $\beta^1(M(4P,\delta_3^1))$, $\beta^1(M(6P,\delta_3^2))$, are explicitly calculated to be 15, 25, 35 and 17, 27, 37, respectively.

Thus we have, for each $t \in \mathbb{Z}_+$,

$$\beta^1(M((2+2t)P,\delta_2^t)) = 15+10t, \tag{4-2}$$

$$\beta^1(M((2+2t)P,\delta_3^t)) = 17+10t. \tag{4-3}$$

Putting together the results in (4-1), (4-2) and (4-3), we have the proof of Lemma 4.2. □

Lemma 4.3 *For any even positive integer n and any odd integer $k \in [1, n-1]$, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP such that, for its natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(nP,\delta)) = k$.*

affixes	bricks	compatible pairs of bricks being repeated
$a_1 = 1$ $a_2 = 3$	$S_1 = (24247)$ $S_2 = (54241)$ $S_3 = (67172)$	$A_1 = (6717254241)$ $A_2 = (7317254241)$

Table 7: The affixes, bricks and compatible pairs for $\lambda^{(t_1,t_2)}$ of Lemma 4.3.

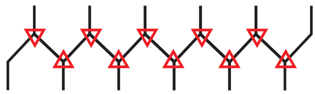


Figure 15: Compatible pair.

Proof We consider some affixes and bricks as described in Table 7. For the sake of brevity, we use the symbol A_i to denote a *compatible pair* of bricks, where "compatible" means the nonsingular condition is satisfied at all ten intersecting vertices of the two bricks as shown in Figure 15.

At first, we construct a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope $2P$, where the colored polytope is $[a_1 S_1 S_3 S_2 a_2]$. The nonorientability is guaranteed by Corollary 2.9. The natural $\mathbb{Z}_2^4$–extension of λ is denoted by δ. Let $\lambda^{(t_1,t_2)}$ be the $\mathbb{Z}_2^3$–coloring characteristic function of the colored polytope $2(t_1+t_2+1)P$,

$$[aS_1S_3S_2\underbrace{A_1,\ldots,A_1}_{t_1}\underbrace{A_2,\ldots,A_2}_{t_2}a_2].$$

It can be easily checked that the nonsingularity condition holds at every vertex. Moreover, $\delta^{(t_1,t_2)}$ is the natural $\mathbb{Z}_2^4$–extension of $\lambda^{(t_1,t_2)}$, which is also defined on the polytope $2(t_1+t_2+1)P$. In particular, $\lambda^{(0,0)} = \lambda$. The colored $2P$s corresponding to $\lambda^{(1,0)}$ and $\lambda^{(0,1)}$ are $[a_1S_1S_3S_2A_1a_2]$ and $[a_1S_1S_3S_2A_2a_2]$, respectively. In this case, the nonsingularity condition holds at every vertex. The calculated Betti numbers are given in Table 8.

By Proposition 3.2 and

$$\beta^1(M(2P,\delta^{(0,0)})) = 1,\quad \beta^1(M(4P,\delta^{(1,0)})) = 1,\quad \beta^1(M(6P,\delta^{(2,0)})) = 1,$$

$\beta^1(M(2P,\delta^{(0,0)})) = 1$	$\beta^1(M(4P,\delta^{(1,0)})) = 1$	$\beta^1(M(6P,\delta^{(2,0)})) = 1$	$\cdots$
	$\beta^1(M(4P,\delta^{(0,1)})) = 3$	$\beta^1(M(6P,\delta^{(1,1)})) = 3$	$\cdots$
		$\beta^1(M(6P,\delta^{(0,2)})) = 5$	$\cdots$

Table 8: The values of $\beta^1(M((2(t_1+t_2+2))P,\delta^{(t_1,t_2)}))$ in Lemma 4.3.

we have

$$\beta^1(M(2(t_1+t_2+1)P,\delta^{(t_1,t_2)}))=\beta^1(M(2(t_1+t_2+2)P,\delta^{(t_1+1,t_2)})). \tag{4-4}$$

Likewise, from

$$\beta^1(M(2P,\delta^{(0,0)}))=1,\quad \beta^1(M(4P,\delta^{(0,1)}))=3,\quad \beta^1(M(6P,\delta^{(0,2)}))=5,$$

we have

$$\beta^1(M(2(t_1+t_2+1)P,\delta^{(t_1,t_2)}))+2=\beta^1(M(2(t_1+t_2+2)P,\delta^{(t_1,t_2+1)})). \tag{4-5}$$

By (4-4) and (4-5), we obtain

$$\beta^1(M(nP,\delta^{(t,\frac{1}{2}n-1-t)}))=n-2t-1, \tag{4-6}$$

where n is even and $0\leqslant t\leqslant \frac{1}{2}n-1$, which completes the proof of Lemma 4.3. □

Lemma 4.4 *For any even positive integer n and any odd integer $k\in[n+3,5n-5]$, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP such that, for its natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(nP,\delta))=k$.*

Proof The considered affixes and bricks are described in Table 9.

First, we construct three nonorientable $\mathbb{Z}_2^3$–coloring characteristic functions $\widetilde{\lambda}^0$, $\widetilde{\lambda}^1$ and $\widetilde{\lambda}^2$ of polytopes $2P$, $4P$ and $6P$, respectively as below:

$$[a_1S_1A_3a_1],$$
$$[a_1S_1A_3A_3a_1],$$
$$[a_1S_1A_3A_3A_3a_1].$$

Their characteristic vectors are

$$(1,2,4,4,2,7,\underline{3,7,5,2,6,2,4,4,2,7},1),$$
$$(1,2,4,4,2,7,\underline{3,7,5,2,6,2,4,4,2,7},\underline{3,7,5,2,6,2,4,4,2,7},1),$$
$$(1,2,4,4,2,7,\underline{3,7,5,2,6,2,4,4,2,7},\underline{3,7,5,2,6,2,4,4,2,7},\underline{3,7,5,2,6,2,4,4,2,7},1).$$

affixes	brick	compatible pairs of bricks being repeated
$a_1=1, a_2=4$	$S_1=(24247)$	$A_1=(42472\ 71635)$ $A_2=(42472\ 37265)$ $A_3=(65372\ 24247)$ $A_4=(65372\ 71635)$

Table 9: The affixes, brick and compatible pairs for constructing $\lambda_i^{(t_1,t_2)}$ of Lemma 4.4.

Also in this case, the nonsingularity condition holds at every vertex. Their natural associated $\mathbb{Z}_2^4$–colorings are denoted by $\widetilde{\delta}^0$, $\widetilde{\delta}^1$ and $\widetilde{\delta}^2$. By Corollary 2.3, we obtain that the first Betti numbers of the corresponding manifolds are 5, 15 and 25, respectively. Thus, we have

$$\beta^1(M((2+2t)P,\widetilde{\delta}^t)) = 5+10t \tag{4-7}$$

for each $t \in \mathbb{Z}_{\geq 0}$, where t is the number of times the last two bricks of $\widetilde{\delta}^0$ are repeated.

Next, we use $\lambda_i^{(t_1,t_2)}$ to represent the $\mathbb{Z}_2^3$–coloring characteristic function of coloring vector

$$[aS_1\underbrace{A_3,\dots,A_3}_{t_1}A_4\underbrace{A_1,\dots,A_1}_{t_2}A_ia_j]$$

over the polytope $2(t_1+t_2+2)P$. Here a_j is the affix element and $j = 2,1,1,2$ when $i = 1,2,3,4$, respectively. In particular, the coloring vector of $\lambda_i^{(0,0)}$ is $[aS_1A_4A_ia_j]$. The nonsingularity condition holds at every vertex. Moreover, $\delta_i^{(t_1,t_2)}$ is the natural associated $\mathbb{Z}_2^4$–extension of $\lambda_i^{(t_1,t_2)}$.

From

$$\begin{aligned}\beta^1(M(4P,\delta_i^{(0,0)})) &= 5+2i,\\ \beta^1(M(6P,\delta_i^{(0,1)})) &= 7+2i,\\ \beta^1(M(8P,\delta_i^{(0,2)})) &= 9+2i\end{aligned}$$

for $i = 1,2,3,4$, we have

$$\beta^1(M(2(t_1+t_2+2)P,\delta_i^{(t_1,t_2)}))+2 = \beta^1(M(2(t_1+t_2+3)P,\delta_i^{(t_1,t_2+1)}) \tag{4-8}$$

for $i = 1,2,3,4$. From

$$\begin{aligned}\beta^1(M(4P,\delta_i^{(0,0)})) &= 5+2i,\\ \beta^1(M(6P,\delta_i^{(1,0)})) &= 15+2i,\\ \beta^1(M(8P,\delta_i^{(2,0)})) &= 25+2i\end{aligned}$$

for $i = 1,2,3,4$, we have

$$\beta^1(M(2(t_1+t_2+2)P,\delta_i^{(t_1,t_2)}))+10 = \beta^1(M(2(t_1+t_2+3)P,\delta_i^{(t_1+1,t_2)})). \tag{4-9}$$

By (4-8) and (4-9) it follows that

$$\beta^1(M(nP,\delta_i^{(t,\frac{1}{2}n-2-t)})) = n+8t+2i+3 \tag{4-10}$$

for n even and $0 \leqslant t \leqslant \frac{1}{2}n-2$.

$2P$		$4P$		$6P$		$8P$		$10P$		$\cdots$
δ	β^1	δ	β^1	δ	β^1	δ	β^1	δ	β^1	$\cdots$
		$\delta_1^{(0,0)}$	7	$\delta_1^{(0,1)}$	9	$\delta_1^{(0,2)}$	11	$\delta_1^{(0,3)}$	13	$\cdots$
$\widetilde{\delta}^0$	5	$\delta_2^{(0,0)}$	9	$\delta_2^{(0,1)}$	11	$\delta_2^{(0,2)}$	13	$\delta_2^{(0,3)}$	15	$\cdots$
		$\delta_3^{(0,0)}$	11	$\delta_3^{(0,1)}$	13	$\delta_3^{(0,2)}$	15	$\delta_3^{(0,3)}$	17	$\cdots$
		$\delta_4^{(0,0)}$	13	$\delta_4^{(0,1)}$	15	$\delta_4^{(0,2)}$	17	$\delta_4^{(0,3)}$	19	$\cdots$
				$\delta_1^{(1,0)}$	17	$\delta_1^{(1,1)}$	19	$\delta_1^{(1,2)}$	21	$\cdots$
		$\widetilde{\delta}^1$	15	$\delta_2^{(1,0)}$	19	$\delta_2^{(1,1)}$	21	$\delta_2^{(1,2)}$	23	$\cdots$
				$\delta_3^{(1,0)}$	21	$\delta_3^{(1,1)}$	23	$\delta_3^{(1,2)}$	25	$\cdots$
				$\delta_4^{(1,0)}$	23	$\delta_4^{(1,1)}$	25	$\delta_4^{(1,2)}$	27	$\cdots$
						$\delta_1^{(2,0)}$	27	$\delta_1^{(2,1)}$	29	$\cdots$
				$\widetilde{\delta}^2$	25	$\delta_2^{(2,0)}$	29	$\delta_2^{(2,1)}$	31	$\cdots$
						$\delta_3^{(2,0)}$	31	$\delta_3^{(2,1)}$	33	$\cdots$
						$\delta_4^{(2,0)}$	33	$\delta_4^{(2,1)}$	35	$\cdots$
								$\delta_1^{(3,0)}$	37	$\cdots$
						$\widetilde{\delta}^3$	35	$\delta_2^{(3,0)}$	39	$\cdots$
								$\delta_3^{(3,0)}$	41	$\cdots$
								$\delta_4^{(3,0)}$	43	$\cdots$
								$\widetilde{\delta}^4$	45	$\cdots$

Table 10: The values of $\beta^1 = \beta^1(M(nP, \delta)$ for Lemma 4.4.

By (4-7) and (4-10), we finish the proof of Lemma 4.4. All of the Betti numbers of Lemma 4.4 are listed in Table 10. □

Now, using Lemmas 3.1 and 4.1–4.4, we complete the proof of Theorem 1.2 for n even.

5 Proof of Theorem 1.2 for n odd

In this section, we analogously prove Theorem 1.2 for odd n.

Lemma 5.1 *For any odd positive integer n, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP such that, for its natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(nP, \delta)) = n$.*

	$n=3$	$n=5$	$n=7$	$\cdots$	$n=3+2t, t\in\mathbb{N}$
1	1	1	1	$\cdots$	1
2	1	1	1	$\cdots$	1
3	0	0	0	$\cdots$	0
4	0	1	2	$\cdots$	t
5	0	0	0	$\cdots$	0
6	0	1	2	$\cdots$	t
7	1	1	1	$\cdots$	1
8	0	0	0	$\cdots$	0
9	0	0	0	$\cdots$	0
10	0	0	0	$\cdots$	0
11	0	0	0	$\cdots$	0
12	0	0	0	$\cdots$	0
13	0	0	0	$\cdots$	0
14	0	0	0	$\cdots$	0
15	0	0	0	$\cdots$	0
total β^1	3	5	7	$\cdots$	$3+2t=n$

Table 11: The values of $\beta^1(M(nP,\delta^t))$ for $n=3+2t$ in Lemma 5.1.

Proof We first prove the special case in which $n=3$. Consider bricks $S_1=(24247)$ and $S_2=(35716)$, and affixes $a_1=1$ and $a_2=4$. We construct a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope $3P$ whose coloring and characteristic vector are

$$[a_1S_1S_2S_1S_2a_2] \quad\text{and}\quad (1,2,4,4,2,7,7,1,5,6,3,2,4,4,2,7,7,1,5,6,3,4),$$

respectively.

By Corollary 2.3, $\beta^1(M(3P,\delta))=3$, where δ is the natural $\mathbb{Z}_2^4$–extension of λ. We repeat the last two bricks t times to construct a coloring over the polytope $(3+2t)P$, and denote its characteristic function by λ^t. It can be easily checked that the nonsingularity condition holds at every vertex. By Corollary 2.9 and Proposition 2.11, we obtain the admissible extension δ^t of the nonorientable λ^t. That is, $M((3+2t)P,\delta^t)$ is the orientable double cover of the nonorientable manifold $M((3+2t)P,\lambda^t)$. The progressions of corresponding Betti numbers are shown in Table 11.

This concludes the proof of Lemma 5.1. □

Lemma 5.2 *For any odd positive integer n and any odd integer $k\in[5n-9,5n+3]$, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP such that, for its natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(nP,\delta))=k$.*

affix	compatible pairs of bricks being repeated
$a_1 = 1$	$A_1 = (53726\ 71635)$ $A_2 = (24724\ 37265)$ $A_3 = (53726\ 74242)$

Table 12: An affix and compatible pairs of bricks for Lemma 5.2, I.

Proof We start at $n = 3$ and construct six suitable characteristic vectors whose corresponding manifolds' Betti numbers would increase by $10t$ when repeating the last pair of coloring bricks t times. An affix and some useful compatible pairs are described in Table 12.

For every $i = 0, 1, 2$, let λ_i^0, λ_i^1 and λ_i^2 be the three $\mathbb{Z}_2^3$–coloring characteristic functions of the three colorings over the polytopes $3P$, $5P$ and $7P$ as shown in Table 13. Here t represents how many times the last compatible pair of λ_i^0 is repeated. It can be checked with little effort that the nonsingularity condition holds at every vertex.

Let δ_i^t be the natural $\mathbb{Z}_2^4$–extensions of λ_i^t for $i = 0, 1, 2$. By Corollary 2.3, we may calculate the first Betti numbers of the manifolds corresponding to the coloring vectors in Table 13, namely

$$\beta^1(M(3P, \delta_0^0)) = 7, \quad \beta^1(M(5P, \delta_0^1)) = 17, \quad \beta^1(M(7P, \delta_0^2)) = 27,$$
$$\beta^1(M(3P, \delta_1^0)) = 9, \quad \beta^1(M(5P, \delta_1^1)) = 19, \quad \beta^1(M(7P, \delta_1^2)) = 29$$

and

$$\beta^1(M(3P, \delta_2^0)) = 11, \quad \beta^1(M(5P, \delta_2^1)) = 21, \quad \beta^1(M(7P, \delta_2^2)) = 31.$$

Therefore, according to Proposition 3.2, for each $t \in \mathbb{N}$,

$$\beta^1(M((3+2t)P, \delta_1^t)) = 7 + 10t, \tag{5-1}$$

$$\beta^1(M((3+2t)P, \delta_2^t)) = 9 + 10t, \tag{5-2}$$

$$\beta^1(M((3+2t)P, \delta_3^t)) = 11 + 10t. \tag{5-3}$$

i	$t = 0$	1	2
0	$[a_1 A_1 A_2 a_1]$	$[a_1 A_1 A_2 A_2 a_1]$	$[a_1 A_1 A_2 A_2 A_2 a_1]$
1	$[a_1 A_1 A_3 a_1]$	$[a_1 A_1 A_3 A_3 a_1]$	$[a_1 A_1 A_3 A_3 A_3 a_1]$
2	$[a_1 A_1 A_1 a_1]$	$[a_1 A_1 A_1 A_1 a_1]$	$[a_1 A_1 A_1 A_1 A_1 a_1]$

Table 13: The coloring vectors of λ_i^t in Lemma 5.2.

affixes	compatible pairs of bricks being repeated
$a_1 = 1, a_2 = 3$	$A_0 = (34246\ 26513)$ $A_1 = (31245\ 26416)$ $A_2 = (31245\ 16416)$ $A_3 = (31245\ 46452)$

Table 14: The affixes and compatible pairs for Lemma 5.2, II.

Similarly, we prepare the affixes and compatible pairs for constructing the desired characteristic function $\widetilde{\lambda}_i^t$ in Table 14.

For every $i = 0, 1, 2$, let $\widetilde{\lambda}_i^0$, $\widetilde{\lambda}_i^1$ and $\widetilde{\lambda}_i^2$ be the three $\mathbb{Z}_2^3$–coloring characteristic functions of the three colorings over the polytopes $3P$, $5P$ and $7P$ as shown in Table 15. Here t represents how many times the last compatible pair of $\widetilde{\lambda}_i^0$ is repeated. It can be easily checked that the nonsingularity condition holds at every vertex.

Let $\widetilde{\delta}_i^t$ be the natural $\mathbb{Z}_2^4$–extensions of $\widetilde{\lambda}_i^t$, for $i = 1, 2, 3$. By Corollary 2.3, we calculate the first Betti numbers of the manifolds corresponding to the coloring vectors in Table 15, namely

$$\beta^1(M(3P, \widetilde{\delta}_0^0)) = 13, \quad \beta^1(M(5P, \widetilde{\delta}_0^1)) = 23, \quad \beta^1(M(7P, \widetilde{\delta}_0^2)) = 33,$$
$$\beta^1(M(3P, \widetilde{\delta}_1^0)) = 15, \quad \beta^1(M(5P, \widetilde{\delta}_1^1)) = 25, \quad \beta^1(M(7P, \widetilde{\delta}_1^2)) = 35,$$

and

$$\beta^1(M(3P, \widetilde{\delta}_2^0)) = 17, \quad \beta^1(M(5P, \widetilde{\delta}_2^1)) = 27, \quad \beta^1(M(7P, \widetilde{\delta}_2^2)) = 37.$$

Thus, according to Proposition 3.2, for each $t \in \mathbb{N}$,

$$\beta^1(M((3+2t)P, \widetilde{\delta}_1^t)) = 13 + 10t, \tag{5-4}$$

$$\beta^1(M((3+2t)P, \widetilde{\delta}_2^t)) = 15 + 10t, \tag{5-5}$$

$$\beta^1(M((3+2t)P, \widetilde{\delta}_3^t)) = 17 + 10t. \tag{5-6}$$

Putting together the results in (5-1)–(5-6), we have the proof of Lemma 5.2. □

i	$t = 0$	1	2
0	$[a_1 A_0 A_1 a_2]$	$[a_1 A_0 A_1 A_1 a_2]$	$[a_1 A_0 A_1 A_1 A_1 a_2]$
1	$[a_1 A_0 A_2 a_2]$	$[a_1 A_0 A_2 A_2 a_2]$	$[a_1 A_0 A_2 A_2 A_2 a_2]$
2	$[a_1 A_0 A_3 a_2]$	$[a_1 A_0 A_3 A_3 a_2]$	$[a_1 A_0 A_3 A_3 A_3 a_2]$

Table 15: The coloring vectors of $\widetilde{\lambda}_i^t$ in Lemma 5.2.

affixes	compatible pairs of bricks being repeated
$a_1 = 1, a_2 = 4$	$A_1 = (24247\ 17532)$ $A_2 = (53176\ 17532)$ $A_3 = (53147\ 17532)$

Table 16: The affixes and compatible pairs for $\lambda^{(t_1,t_2)}$ of Lemma 5.3.

Lemma 5.3 *For any odd positive integer n and any odd integer $k \in [1, n-1]$, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP such that, for its natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(nP,\delta)) = k$.*

Proof We prepare some affixes and compatible pairs as described in Table 16.

At first, we construct a nonorientable $\mathbb{Z}_2^3$–coloring characteristic function λ, whose coloring vector is $[a_1 A_1 A_2 a_2]$, on the polytope $3P$, and denote its natural $\mathbb{Z}_2^4$–extension by δ. Let $\lambda^{(t_1,t_2)}$ be the $\mathbb{Z}_2^3$–coloring characteristic function of

$$[a_1 A_1 A_2 \underbrace{A_2, \dots, A_2}_{t_1}\ \underbrace{A_3, \dots, A_3}_{t_2} a_2]$$

over the polytope $(2(t_1+t_2)+3))P$. We use $\delta^{(t_1,t_2)}$ to denote the natural associated $\mathbb{Z}_2^4$–extension of $\lambda^{(t_1,t_2)}$. In particular, $\lambda^{(0,0)} = \lambda$. It can be easily checked that the nonsingularity condition holds at every vertex. The results of the calculations of the Betti numbers are reported in Table 17.

According to Proposition 3.2, the Betti number sequence would be an arithmetic progression if the first three numbers satisfy the relation of arithmetic progression.

From

$$\beta^1(M(3P,\delta^{(0,0)})) = 1, \quad \beta^1(M(5P,\delta^{(1,0)})) = 1, \quad \beta^1(M(7P,\delta^{(2,0)})) = 1,$$

we have

$$\beta^1\big(M\big((2(t_1+t_2)+3)P,\delta^{(t_1,t_2)}\big)\big) = \beta^1\big(M\big((2(t_1+t_2)+5)P,\delta^{(t_1+1,t_2)}\big)\big). \tag{5-7}$$

$\beta^1(M(3P,\delta^{(0,0)})) = 1$	$\beta^1(M(5P,\delta^{(1,0)})) = 1$	$\beta^1(M(7P,\delta^{(2,0)})) = 1$	$\cdots$
	$\beta^1(M(5P,\delta^{(0,1)})) = 3$	$\beta^1(M(7P,\delta^{(1,1)})) = 3$	$\cdots$
		$\beta^1(M(7P,\delta^{(0,2)})) = 5$	$\cdots$

Table 17: The values of $\beta^1\big(M\big((2(t_1+t_2)+3)P,\delta^{(t_1,t_2)}\big)\big)$ in Lemma 5.3.

From

$$\beta^1(M(3P,\delta^{(0,0)}))=1,\quad \beta^1(M(5P,\delta^{(0,1)}))=3,\quad \beta^1(M(7P,\delta^{(0,2)}))=5,$$

we have

$$\beta^1\big(M\big((2(t_1+t_2)+3)P,\delta^{(t_1,t_2)}\big)\big)+2=\beta^1\big(M\big((2(t_1+t_2)+5)P,\delta^{(t_1,t_2+1)}\big)\big). \tag{5-8}$$

By (5-7) and (5-8), we obtain

$$\beta^1(M(nP,\delta^{(t,\frac{1}{2}(n-3)-t)}))=n-2-2t, \tag{5-9}$$

for each n odd with $n\in\mathbb{Z}_{\geq 3}$ and $0\leqslant t\leqslant \frac{1}{2}(n-3)$.

This concludes the proof of Lemma 5.3. □

Lemma 5.4 *For any odd positive integer n and any odd integer $k\in[n+1,5n-9]$, there is a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope nP such that, for the natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(nP,\delta))=k$.*

Proof The affixes and compatible pairs of bricks considered are described in Table 18.

At first, we construct a nonorientable $\mathbb{Z}_2^3$–coloring λ over the polytope $3P$ whose coloring vector is $[a_1A_4A_1a_2]$. We denote by δ the natural associated $\mathbb{Z}_2^4$–extension. By calculation, we have

$$\beta^1(M(3P,\delta))=5. \tag{5-10}$$

We denote by λ_i^{t-1}, where $t\in\mathbb{Z}_{\geq 1}$ and $i=1,2,3,4$, the nonorientable $\mathbb{Z}_2^3$–coloring characteristic function λ on the polytope $(2t+3)P$ corresponding to coloring vector

$$[a_1A_4\underbrace{A_1,\dots,A_1}_{t}\;A_ia_j],$$

where a_j is an affix element and j is given by $2,1,1,2$ for $i=1,2,3,4$, respectively.

affixes	compatible pairs of bricks being repeated
$a_1=1, a_2=4$	$A_1=(42472\ 57163)$ $A_2=(42472\ 53726)$ $A_3=(65372\ 72424)$ $A_4=(65372\ 57163)$

Table 18: The affixes and compatible pairs for Lemma 5.4.

$3P$		$5P$		$7P$		$9P$		$11P$		$\cdots$
δ	β^1	δ	β^1	δ	β^1	δ	β^1	δ	β^1	$\cdots$
$\widetilde{\delta}$	5	δ_1^0	7	δ_1^1	9	δ_1^2	11	δ_1^3	13	$\cdots$
$\widetilde{\delta}^0$	5	δ_2^0	9	δ_2^1	11	δ_2^2	13	δ_2^3	15	$\cdots$
		δ_3^0	11	δ_3^1	13	δ_3^2	15	δ_3^3	17	$\cdots$
		δ_4^0	13	δ_4^1	15	δ_4^2	17	δ_4^3	19	$\cdots$
				$\delta_1^{(0,0)}$	17	$\delta_1^{(0,1)}$	19	$\delta_1^{(0,2)}$	21	$\cdots$
		$\widetilde{\delta}^1$	15	$\delta_2^{(0,0)}$	19	$\delta_2^{(0,1)}$	21	$\delta_2^{(0,2)}$	23	$\cdots$
				$\delta_3^{(0,0)}$	21	$\delta_3^{(0,1)}$	23	$\delta_3^{(0,2)}$	25	$\cdots$
				$\delta_4^{(0,0)}$	23	$\delta_4^{(0,1)}$	25	$\delta_4^{(0,2)}$	27	$\cdots$
						$\delta_1^{(1,0)}$	27	$\delta_1^{(1,1)}$	29	$\cdots$
				$\widetilde{\delta}^2$	25	$\delta_2^{(1,0)}$	29	$\delta_2^{(1,1)}$	31	$\cdots$
						$\delta_3^{(1,0)}$	31	$\delta_3^{(1,1)}$	33	$\cdots$
						$\delta_4^{(1,0)}$	33	$\delta_4^{(1,1)}$	35	$\cdots$
								$\delta_1^{(2,0)}$	37	$\cdots$
						$\widetilde{\delta}^3$	35	$\delta_2^{(2,0)}$	39	$\cdots$
								$\delta_3^{(2,0)}$	41	$\cdots$
								$\delta_4^{(2,0)}$	43	$\cdots$
								$\widetilde{\delta}^4$	45	$\cdots$

Table 19: The values of $\beta^1(M(nP,\delta))$, $n = 3, 5, 7, 9, 11, \ldots$, for Lemma 5.4.

In particular, λ_1^t is obtained by inserting $(t+1)$ copies of A_1 into the coloring vector of λ. We denote by δ_i^{t-1} the natural $\mathbb{Z}_2^4$–extension of λ_i^{t-1}. From

$$\beta^1(M(5P,\delta_i^0)) = 5+2i, \quad \beta^1(M(7P,\delta_i^1)) = 7+2i, \quad \beta^1(M(9P,\delta^2)) = 9+2i,$$

we have

$$\beta^1\bigl(M\bigl((2t+3)P,\delta_i^{t-1}\bigr)\bigr)+2 = \beta^1\bigl(M\bigl((2t+5)P,\delta_i^t\bigr)\bigr) \tag{5-11}$$

for $i = 1, 2, 3, 4$.

Next, we construct three nonorientable $\mathbb{Z}_2^3$–colorings $\widetilde{\lambda}^0$, $\widetilde{\lambda}^1$ and $\widetilde{\lambda}^2$ on the polytopes $3P$, $5P$, $7P$, whose coloring vectors are, respectively,

$$[a_1 A_1 A_3 a_1], \quad [a_1 A_1 A_3 A_3 a_1], \quad [a_1 A_1 A_3 A_3 A_3 a_1].$$

The natural $\mathbb{Z}_2^4$–extensions are denoted by $\widetilde{\delta}^0$, $\widetilde{\delta}^1$ and $\widetilde{\delta}^2$. By calculation, we have

$$\beta^1(M(3P,\widetilde{\delta}^0)) = 5, \quad \beta^1(M(5P,\widetilde{\delta}^1)) = 15, \quad \beta^1(M(7P,\widetilde{\delta}^2)) = 25.$$

For $t \in \mathbb{Z}_{\geq 1}$, we denote by $\widetilde{\lambda}^{t-1}$ the $\mathbb{Z}_2^3$–coloring characteristic function of

$$[a_1 A_1 \underbrace{A_3, \ldots, A_3}_{t} a_1]$$

over the polytope $(2t+1)P$ and write its natural $\mathbb{Z}_2^4$–extension as $\widetilde{\delta}^{t-1}$. Then we have, for each $t \in \mathbb{Z}_{\geq 1}$,

$$\beta^1(M((2t+1)P,\widetilde{\delta}^{t-1})) = 10t-5. \tag{5-12}$$

Let $\lambda_i^{(t_1-1,t_2)}$ denote the $\mathbb{Z}_2^3$–coloring characteristic function of the coloring vector

$$[a_1 A_1 \underbrace{A_3, \ldots, A_3}_{t_1} \; A_4 \; \underbrace{A_1, \ldots, A_1}_{t_2} \; A_i a_j]$$

over the polytope $(2(t_1+t_2)+5)P$, where a_j is an affix element and j is given by $2,1,1,2$ for $i = 1,2,3,4$, respectively. In particular, the coloring vector of $\lambda_i^{(0,0)}$ is $[aA_1A_3A_4A_ia_j]$. Also $\delta_i^{(t_1-1,t_2)}$ is the natural $\mathbb{Z}_2^4$–extension of $\lambda_i^{(t_1-1,t_2)}$ over the polytope $(2(t_1+t_2)+5)P$.

From

$$\beta^1(M(7P,\delta_i^{(0,0)})) = 5+2i,$$
$$\beta^1(M(9P,\delta_i^{(0,1)})) = 7+2i,$$
$$\beta^1(M(11P,\delta_i^{(0,2)})) = 9+2i$$

for $i = 1,2,3,4$, we have

$$\begin{aligned}\beta^1\bigl(M\bigl((2(t_1+t_2)+5)P,\delta^{(t_1-1,t_2)}\bigr)\bigr)+2 \\ = \beta^1\bigl(M\bigl((2(t_1+t_2)+7)P,\delta^{(t_1-1,t_2+1)}\bigr)\bigr)\end{aligned} \tag{5-13}$$

for each $t \in \mathbb{Z}_{\geq 1}$.

From

$$\beta^1(M(7P,\delta_i^{(0,0)})) = 5+2i,$$
$$\beta^1(M(9P,\delta_i^{(1,0)})) = 15+2i,$$
$$\beta^1(M(11P,\delta_i^{(2,0)})) = 25+2i$$

for $i = 1,2,3,4$, we have

$$\beta^1\bigl(M\bigl((2(t_1+t_2)+5)P,\delta_i^{(t_1-1,t_2)}\bigr)\bigr)+10 = \beta^1\bigl(M\bigl((2(t_1+t_2)+7)P,\delta_i^{(t_1,t_2)}\bigr)\bigr) \tag{5-14}$$

for each $t \in \mathbb{Z}_{\geq 1}$.

	λ	$\beta^1(M(P,\delta))$
1	(1, 2, 4, 4, 2, 7, 1, 7, 7, 5, 6, 4)	1
2	(1, 2, 4, 4, 2, 7, 7, 3, 1, 5, 4, 2)	3
3	(1, 2, 4, 4, 2, 7, 3, 5, 5, 6, 3, 1)	5
4	(1, 2, 4, 5, 2, 6, 3, 6, 5, 4, 3, 1)	7

Table 20: The $\mathbb{Z}_2^3$–colorings and β^1 of their natural $\mathbb{Z}_2^4$–extensions of Lemma 5.5.

By (5-13) and (5-14), we have

$$\beta^1(M(nP,\delta_i^{(t,\frac{1}{2}(n-1)-3-t)})) = n+2i+8t \tag{5-15}$$

for $n \in \mathbb{Z}_{\geq 7}^{\text{odd}}$ and $0 \leqslant t \leqslant \frac{1}{2}(n-7)$.

Putting together the results in (5-10)–(5-12) and (5-15), we complete the proof of Lemma 5.4. All the Betti numbers of Lemma 5.4 are listed in Table 19. □

Lemma 5.5 *For any odd integer $k \in [1,7]$, there is a nonorientable $\mathbb{Z}_2^3$–coloring over the dodecahedron P such that, for its natural associated $\mathbb{Z}_2^4$–coloring δ, we have $\beta^1(M(P,\delta)) = k$.*

Proof We report the required characteristic functions in Table 20 to conclude this lemma. □

Now, using Lemmas 5.1–5.5, we complete the proof of Theorem 1.2 for an odd n. Thus, together with Section 4, we finish the proof of Theorem 1.2.

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