

AG
T

*Algebraic & Geometric
Topology*

Volume 23 (2023)

Convexity in hierarchically hyperbolic spaces

JACOB RUSSELL
DAVIDE SPRIANO
HUNG CONG TRAN



Convexity in hierarchically hyperbolic spaces

JACOB RUSSELL
 DAVIDE SPRIANO
 HUNG CONG TRAN

Hierarchically hyperbolic spaces (HHSs) are a large class of spaces that provide a unified framework for studying the mapping class group, right-angled Artin and Coxeter groups, and many 3-manifold groups. We investigate strongly quasiconvex subsets in this class and characterize them in terms of their contracting properties, relative divergence, the coarse median structure, and the hierarchical structure itself. Along the way, we obtain new tools to study HHSs, including two new equivalent definitions of hierarchical quasiconvexity and a version of the bounded geodesic image property for strongly quasiconvex subsets. Utilizing our characterization, we prove that the hyperbolically embedded subgroups of hierarchically hyperbolic groups are precisely those that are almost malnormal and strongly quasiconvex, producing a new result in the case of the mapping class group. We also apply our characterization to study strongly quasiconvex subsets in several specific examples of HHSs. We show that while many commonly studied HHSs have the property that every strongly quasiconvex subset is either hyperbolic or coarsely covers the entire space, right-angled Coxeter groups exhibit a wide variety of strongly quasiconvex subsets.

20F65, 20F67

1. Introduction	1168
2. Coarse geometry	1177
3. Divergence of contracting subsets	1181
4. Hierarchically hyperbolic spaces	1186
5. Constructing hulls with hierarchy paths	1199
6. Characterization of strongly quasiconvex subsets in HHSs	1207
7. Strongly quasiconvex subsets in familiar examples	1227
8. Hyperbolically embedded subgroups of HHGs	1238
Appendix. Subsets with arbitrary reasonable lower relative divergence	1243
References	1245

1 Introduction

From Gromov's original work on hyperbolic groups to the resolution of the virtual Haken conjecture, quasiconvex subsets have played a central role in the study of hyperbolic metric spaces and groups; see Agol [2], Gromov [31; 32] and Wise [55]. A subset Y is *quasiconvex* if every geodesic based on Y is contained in a fixed neighborhood of Y . A central feature of quasiconvex subsets of hyperbolic spaces is their quasi-isometry invariance, ie the image of a quasiconvex subset of a hyperbolic space under a quasi-isometry is quasiconvex.

Outside of hyperbolic spaces, quasiconvexity fails to be a quasi-isometry invariant. However, a strengthening of this definition to require "quasiconvexity with respect to quasigeodesics" and not just geodesics is sufficient to ensure quasi-isometry invariance. A subset Y of a quasigeodesic metric space X is *strongly quasiconvex* if every quasigeodesic based in Y is contained in a bounded neighborhood of Y , where the radius of the neighborhood is determined by the quasigeodesic constants. Strong quasiconvexity provides a "coarsification" of the classical definition of a convex subset that ensures that the image of a strongly quasiconvex subset under a quasi-isometry will be strongly quasiconvex, regardless of the geometry of the ambient space. Strongly quasiconvex subsets are therefore an avenue to study the geometry of any space up to quasi-isometry.

The study of strongly quasiconvex geodesics in nonhyperbolic spaces (often called *Morse geodesics*) has been a vibrant and fruitful area of research over the last decade; for example, Arzhantseva, Cashen, Gruber and Hume [6], Charney and Sultan [19], Druţu, Mozes and Sapir [22] and Ol'shanskii, Osin and Sapir [44]. Recently, considerable interest has arisen in understanding general strongly quasiconvex subsets in nonhyperbolic spaces.

The third author studied strongly quasiconvex subsets and subgroups in [54] and showed that many important properties of quasiconvex subsets in hyperbolic spaces persist for strongly quasiconvex subsets of any geodesic metric space. These results have found applications in understanding the cell stabilizers of groups acting on CAT(0) cube complexes — see Groves and Manning [33] — and the splittings of groups over codimension 1 subgroups — see Petrosyan [46]. Using the name Morse instead of strongly quasiconvex, Genevois studied strongly quasiconvex subsets of CAT(0) cube complexes in [28] and Kim studied strongly quasiconvex subgroups of the mapping class groups in [38]. Strongly quasiconvex subgroups that are also hyperbolic were introduced by Durham and Taylor as *stable* subgroups [25] and have received considerable

study; for a sampling see Abbott, Behrstock and Durham [1], Antolín, Mj, Sisto and Taylor [3], Aougab, Durham and Taylor [4], Behrstock [7], and Koberda, Mangahas and Taylor [39].

In this paper, we are primarily interested in understanding the strongly quasiconvex subsets of hierarchically hyperbolic spaces (HHSs). Introduced by Behrstock, Hagen and Sisto in [9] and refined in [10], examples of hierarchically hyperbolic spaces include hyperbolic spaces, the mapping class group of a surface, Teichmüller space with either the Weil–Petersson or Teichmüller metrics, many cocompactly cubulated groups, and the fundamental groups of 3–manifolds without Nil or Sol components. Important consequences of hierarchical hyperbolicity include a Masur–Minsky style distance formula [10], a quadratic isoperimetric inequality [10], restrictions on morphisms from higher rank lattices (Haettel [34]), a largest acylindrical action on a hyperbolic space [1], rank-rigidity and Tits alternative theorems (Durham, Hagen and Sisto [24]), control over the top-dimensional quasiflats (Behrstock, Hagen and Sisto [11]), and bounds on the asymptotic dimension (Durham, Hagen and Sisto [8]). The definition and much of the theory of hierarchically hyperbolic spaces is inspired by the Masur–Minsky subsurface projection machinery for the mapping class group. Our investigation is therefore a natural extension of the problem purposed by Farb in [27, Problem 2.3.8] to study convexity in the mapping class group.

Heuristically, a hierarchically hyperbolic space consists of a metric space X with an associated collection of hyperbolic spaces \mathfrak{S} , such that for each space Z in \mathfrak{S} , there is a projection map $X \rightarrow Z$. The philosophy of hierarchically hyperbolic spaces is that one can study the coarse geometry of X by studying the projection of X to each of the spaces in \mathfrak{S} . In this paper, we shall consider hierarchically hyperbolic spaces satisfying the *bounded domain dichotomy*; a minor regularity condition requiring every space in \mathfrak{S} to have either infinite or uniformly bounded diameter. The bounded domain dichotomy simplifies the statements and proofs of our results while being satisfied by all of the examples of hierarchically hyperbolic spaces given above and more broadly by all hierarchically hyperbolic groups.

Equivalent conditions to being strongly quasiconvex The main goal of this paper is to provide several equivalent conditions for a subset of a hierarchically hyperbolic space to be strongly quasiconvex. A major theme is that several different notions of convexity that coincide with being quasiconvex in a hyperbolic space, coincide with being strongly quasiconvex in a hierarchically hyperbolic spaces. One such notion of convexity is

that of contracting subsets. A subset $Y \subseteq X$ of a quasigeodesic space is *contracting* if there exists a coarsely Lipschitz retraction $r: X \rightarrow Y$ under which large balls far from Y have images with uniformly bounded diameter. Being contracting generalizes the strong contracting behavior of the closest point projection onto a convex subset of the hyperbolic plane. In general, strongly quasiconvex subsets are not contracting (see Example 3.8); however these two notions of convexity tend to agree in the presence of nonpositive curvature. Indeed, it is a classical fact that a subset of a hyperbolic space is strongly quasiconvex if and only if it is contracting; the same is true for subsets of a CAT(0) cube complex [19; 28]. The first of our equivalent condition is to extend these results to hierarchically hyperbolic spaces.

Theorem 1.1 (strongly quasiconvex and contracting are equivalent) *Let X be a hierarchically hyperbolic space with the bounded domain dichotomy. A subset $Y \subseteq X$ is strongly quasiconvex if and only if Y is contracting.*

In [6], a different notion of contracting subset is considered, and it is shown that a subset of a geodesic metric space is strongly quasiconvex if and only if the subset is *sublinearly contracting*. Example 3.8 demonstrates that our definition of contracting (Definition 2.10) is strictly stronger than sublinear contracting, but the two notions agree in the setting of hierarchically hyperbolic spaces. Another key difference between our definition of contracting and that in [6] is that we do not require the contracting map $r: X \rightarrow Y$ to be the closest point projection, but allow for any coarsely Lipschitz retraction that has the contracting property. This has the advantage of turning contracting into a quasi-isometry invariant directly from the definition and is crucial in allowing us to utilize a naturally occurring retraction map in hierarchically hyperbolic spaces that is far more tractable than the closest point projection.

The third notion of convexity considered is *hierarchical quasiconvexity*, which is specific to hierarchically hyperbolic spaces. Introduced in [10] by Behrstock, Hagen and Sisto, hierarchically quasiconvex subsets have played a central role in the study of hierarchically hyperbolic space [8; 10; 11]. Notably, a hierarchical quasiconvex subset of an HHS is itself an HHS. While hierarchically quasiconvex subsets are not always strongly quasiconvex, we classify precisely when the two concepts agree. Strongly quasiconvex subsets are exactly the hierarchically quasiconvex subsets that satisfy the *orthogonal projection dichotomy* (Definition 6.2), which describes how the projections of a strongly quasiconvex subset to each of the associated hyperbolic spaces must look.

Theorem 1.2 (strongly quasiconvex subsets are hierarchically hyperbolic) *Let X be a hierarchically hyperbolic space with the bounded domain dichotomy. A subset $Y \subseteq X$ is strongly quasiconvex if and only if Y is hierarchically quasiconvex and has the orthogonal projection dichotomy. In particular, if $Y \subseteq X$ is strongly quasiconvex, then Y is hierarchically hyperbolic.*

Theorem 1.2 is truly the central result of this paper as it explains how the strongly quasiconvex subsets interact with the projections defining the hierarchically hyperbolic structure of the ambient space. Further, this characterization is complete as the theorem fails whenever any of the hypotheses are weakened; see Remark 6.14.

In [1], Abbott, Behrstock and Durham give several equivalent conditions for quasi-geodesics in a hierarchically hyperbolic space to be strongly quasiconvex and for a map from a quasigeodesic space Y into a hierarchically hyperbolic space to be a stable embedding; see Proposition 2.8. Theorems 1.1 and 1.2 generalize these results to general strongly quasiconvex subsets and do not require the hypothesis of unbounded products utilized by Abbott, Behrstock and Durham. This generalization to all strongly quasiconvex subsets is essential to our applications in Sections 7 and 8.

Part of the proof of Theorem 1.2 involves studying hierarchically quasiconvex hulls in hierarchically hyperbolic spaces. The *hierarchically quasiconvex hull* of a subset Y is (coarsely) the smallest hierarchically quasiconvex set containing Y . We show that the hull of any subset of a hierarchically hyperbolic space can be constructed using special quasigeodesics called hierarchy paths (see Theorem 5.2 for the precise statement).

Theorem 1.3 (constructing hulls with hierarchy paths) *If Y is a subset of a hierarchically hyperbolic space X , then the hierarchically quasiconvex hull of Y can be constructed in a uniformly finite number of steps by iteratively connecting points by hierarchy paths.*

This construction is reminiscent of the construction of quasiconvex hulls in hyperbolic spaces by connecting pairs of points by geodesics and is similar to the join construction of hulls in coarse median spaces presented by Bowditch in [16]. The main purpose of Theorem 1.3 in this article is to establish that hierarchically quasiconvex subsets are exactly the subsets that are “quasiconvex with respect to hierarchy paths”. However, we expect this construction to have further applications in the study of hierarchically hyperbolic spaces. Indeed, Hagen and Petyt have used this construction to build quasi-isometries from some hierarchically hyperbolic groups to cube complexes [35], and

in Section 5.1 we apply Theorem 1.3 to provide a characterization of hierarchical quasiconvexity in terms of the coarse median structure on a hierarchically hyperbolic space. This later result allows us to conclude that, in the setting of hierarchically hyperbolic spaces, the coarse median hull constructed in [16] is coarsely equal to the hierarchically quasiconvex hull; extending [16, Lemma 7.3] from finite to arbitrary subsets.

Charney and Sultan proved that strongly quasiconvex geodesics in a CAT(0) space are characterized by having at least quadratic *lower divergence* [19]. The third author introduced a generalization of lower divergence to all subsets [53] and studied its relationship with strong quasiconvexity [54]. If Y is a subset of the quasigeodesic space X , the *lower relative divergence of X with respect to Y* (or the divergence of Y in X) is a family of functions that measures how efficiently one can travel in X while avoiding Y . Building on the work in [54], we establish the following.

Theorem 1.4 (contracting subsets have at least quadratic divergence) *Let X be a quasigeodesic metric space. If $Y \subseteq X$ is contracting, then the lower relative divergence of X with respect to Y is at least quadratic. Further, if X is a hierarchically hyperbolic space with the bounded domain dichotomy, then the lower relative divergence of X with respect to Y is at least quadratic if and only if Y is strongly quasiconvex (equivalently if and only if Y is contracting).*

Since the lower relative divergence of X with respect to Y agrees with Charney and Sultan's lower divergence when Y is a geodesic in X , Theorem 1.4 proves that strongly quasiconvex geodesics (aka Morse geodesics) in hierarchically hyperbolic spaces with the bounded domain dichotomy are also characterized by having at least quadratic lower divergence.

After proving Theorems 1.1 through 1.4, we establish several HHS analogues of the "bounded geodesic image property" of quasiconvex subsets of hyperbolic spaces. One of these analogues is the following.

Theorem 1.5 *Let Y be a strongly quasiconvex subset of a hierarchically hyperbolic space X with the bounded domain dichotomy. There is a contracting map $g_Y : X \rightarrow Y$ such that for each $\lambda \geq 1$ there exists a constant $r_\lambda > 0$ such that, for all $x, y \in X$, if $d(g_Y(x), g_Y(y)) > r_\lambda$, then any λ -hierarchy path from x to y must intersect the r_λ -neighborhood of Y .*

Strongly quasiconvex subsets in specific examples After characterizing the strongly quasiconvex subsets of hierarchically hyperbolic spaces, we apply our results to study the strongly quasiconvex subsets of some of the most common examples of hierarchically hyperbolic spaces: the mapping class group, Teichmüller space, right-angled Artin and Coxeter groups, and the fundamental groups of graph manifolds.

It has been shown that strongly quasiconvex subgroups of the mapping class group [38], right-angled Artin groups with connected defining graph [28; 54], and certain \mathcal{CFS} right-angled Coxeter groups (Nguyen and Tran [43]) are either hyperbolic or finite-index. We give sufficient conditions for a hierarchically hyperbolic space to have the property that all its strongly quasiconvex subsets are either hyperbolic or coarsely cover the entire space; see Proposition 7.2. Applying this criteria to specific examples yields a new, unified proof of the work of Kim, Genevois, Nguyen and Tran as well as the following new results for Teichmüller space, graph manifolds, and a class of right-angled Coxeter groups that we call strongly \mathcal{CFS} .

Corollary 1.6 *The following HHSs have the property that every strongly quasiconvex subset is either hyperbolic or coarsely covers the entire space:*

- (a) *The Teichmüller space of a finite-type surface with the Teichmüller metric.*
- (b) *The Teichmüller space of a finite-type surface of complexity at least 6 with the Weil–Petersson metric.*
- (c) *The mapping class group of an oriented, connected, finite type surface.*
- (d) *A right-angled Artin group with connected defining graph*
- (e) *A right-angled Coxeter group with strongly \mathcal{CFS} defining graph.*
- (f) *The fundamental group of a nongeometric graph manifold.*

In particular, if H is a strongly quasiconvex subgroup in any of the groups (c)–(f), then H is either stable or finite-index.

Stable subgroups of the mapping class group and right-angled Artin groups have been studied extensively and have several interesting equivalent characterizations including convex cocompactness in the mapping class group and purely loxodromic in right-angled Artin groups [25; 39].

We also use HHS theory and Theorem 1.2 to give a new proof of [54, Theorem 1.11] and [28, Proposition 4.9] characterizing when a special subgroup of a right-angled

Coxeter group is strongly quasiconvex. We then utilize this characterization, along with a construction of Behrstock, to demonstrate the large variety of different strongly quasiconvex subsets that can be found in the class of \mathcal{CFS} right-angled Coxeter groups.

Theorem 1.7 *Every right-angled Coxeter group is an infinite-index strongly quasiconvex subgroup of some \mathcal{CFS} right-angled Coxeter group.*

Hyperbolically embedded subgroups As a final application of our characterization of strongly quasiconvex subsets, we study the hyperbolically embedded subgroups of hierarchically hyperbolic groups. Hyperbolically embedded subgroups are generalizations of peripheral subgroups in relatively hyperbolic groups (see Dahmani, Guirardel and Osin [20]) and are a key component of studying acylindrically hyperbolic groups, a large class of groups exhibiting hyperbolic-like behavior (see Osin [45]). Work of Dahmani, Guirardel and Osin [20] and Sisto [50] showed that if a finite collection of subgroups $\{H_i\}$ is hyperbolically embedded in a finitely generated group G , then $\{H_i\}$ is an almost malnormal collection and each H_i is strongly quasiconvex. While the converse of this statement is false in general (see the beginning of Section 8 for a counterexample), the converse does hold in the case of hyperbolic groups — see Bowditch [13, Theorem 7.11] — and cocompactly cubulated groups [28, Theorem 6.31]. We prove the converse in the setting of hierarchically hyperbolic groups.

Theorem 1.8 (characterization of hyperbolically embedded subgroups) *Let G be a hierarchically hyperbolic group. A finite collection of subgroups $\{H_i\}$ is hyperbolically embedded in G if and only if $\{H_i\}$ is an almost malnormal collection and each H_i is strongly quasiconvex.*

By [38, Theorem A], an infinite-index subgroup of the mapping class group of a surface is strongly quasiconvex if and only if it is convex cocompact (this fact can also be deduced from Corollary 1.6). Thus, as a specific case of Theorem 1.8, we have the following new result for the mapping class group.

Corollary 1.9 *If S is an oriented, connected, finite-type surface of complexity at least 2 and $\{H_i\}$ is a finite collection of subgroups of the mapping class group of S then the following are equivalent:*

- $\{H_i\}$ is hyperbolically embedded.
- $\{H_i\}$ is an almost malnormal collection and each H_i is strongly quasiconvex.
- $\{H_i\}$ is an almost malnormal collection and each H_i is convex cocompact.

1.1 Open questions

We believe that strongly quasiconvex subgroups are a rich area of study with many interesting open questions both in the setting of hierarchically hyperbolic groups and beyond. In light of Theorem 1.1, it is natural to wonder which results for strongly quasiconvex subgroups of hyperbolic groups can be extended to strongly quasiconvex subgroups of hierarchically hyperbolic groups (or even finitely generated groups). As a starting point, one may aim to extend work of Gromov [31], Arzhantseva [5], and Gitik [30] on combination theorems for strongly quasiconvex subgroups of hyperbolic groups.

Question 1 *Prove combination theorems for strongly quasiconvex subgroups of hierarchically hyperbolic groups (or even finitely generated groups). In particular, investigate conditions guaranteeing that the subgroup generated by two strongly quasiconvex subgroups, Q_1 and Q_2 , is strongly quasiconvex and isomorphic to $Q_1 *_{Q_1 \cap Q_2} Q_2$.*

As strongly quasiconvex subsets are invariant under quasi-isometry, they have the potential to play an important role in the quasi-isometric classification of hierarchically hyperbolic spaces. The following would be an interesting first step in this direction.

Question 2 *Provide necessary conditions for an HHS to have the property that all its strongly quasiconvex subsets are either hyperbolic or coarsely cover the entire space. Using defining graphs, characterize all right-angled Coxeter groups whose strongly quasiconvex subsets are hyperbolic or coarsely cover the entire group.¹*

Looking beyond hierarchically hyperbolic spaces, we wonder about the possibilities of understanding strongly quasiconvex subsets in other spaces with a notion of nonpositive curvature. Specifically we ask the following.

Question 3 *For what other spaces are strongly quasiconvex subsets contracting (in the sense of Definition 2.10)?*

Some of the first spaces one could consider are CAT(0) spaces, coarse median spaces, and the outer automorphism groups of free groups. Sultan [52] shows that strongly quasiconvex geodesics in CAT(0) spaces are always contracting. We conjecture the

¹The case of right-angled Coxeter groups has been resolved by Genevois [29].

same holds for all strongly quasiconvex subsets of a CAT(0) space.² A possible starting point for coarse median spaces could be the recently posted paper [16], in which Bowditch constructs hulls for subsets of coarse median spaces and produces a number of results similar to our work in Section 5.

Our proof of Theorem 1.8 rests strongly upon the equivalence between strongly quasiconvex and contracting subsets. One may then presume that any group that is an answer to Question 3 is also an answer for the following question.

Question 4 *For what other finitely generated groups are almost malnormal, strongly quasiconvex subgroups hyperbolically embedded?*

A long-standing open question in the study of quasiconvex subgroups of hyperbolic group is whether or not finitely generated, almost malnormal subgroups of hyperbolic groups must be quasiconvex. Accordingly, we ask the analogous question for the larger class of hierarchically hyperbolic groups.

Question 5 *Are finitely generated, almost malnormal subgroups of hierarchically hyperbolic groups strongly quasiconvex?*

Outline

In Section 2, we begin with the basic definitions and properties of strongly quasiconvex subsets and the related notions of stability and contracting subsets of general quasigeodesic spaces. In Section 3, we define lower relative divergence and study the relationship between contracting subsets, strongly quasiconvex subsets, and lower relative divergence in any quasigeodesic space. We move on to hierarchically hyperbolic spaces in Section 4, where we give the definition of an HHS and detail the relevant tools and constructions we will need from the theory. In Section 5, we explain how to construct hierarchically quasiconvex hulls using hierarchy paths. As applications of this construction, we give a characterization of hierarchically quasiconvex sets in terms of the coarse median structure on the HHS and prove that strongly quasiconvex subsets are also hierarchically quasiconvex. In Section 6, we state and prove our equivalent characterizations of strongly quasiconvex subsets, finishing the proofs of Theorems 1.1, 1.2, and 1.4. The remaining sections are devoted to applications of this characterization. We give a generalization of the bounded geodesic image property

²This conjecture has been confirmed by Cashen [18].

for strongly quasiconvex subsets in Section 6.3, study strongly quasiconvex subsets in specific examples in Section 7, and characterize hyperbolically embedded subgroups of HHGs in Section 8.

Acknowledgements We gratefully acknowledge Mark F Hagen for suggesting the strategy to attack Theorem 5.2; Mark V Sapir for an example of an almost malnormal strongly quasiconvex subgroup that is not hyperbolically embedded; and Johanna Mangahas for suggesting the relation between hierarchical quasiconvexity and coarse median quasiconvexity in Section 5.1. We thank Brian Bowditch, Tai-Danae Bradley, Heejoung Kim, Chris Hruska, and Dan Beryne for their comments on early versions of this paper. We are also grateful to Kevin Schreve for pointing out an error in the first version of this paper. Russell and Spriano thank the organizers of YGGT 2018 and GAGTA 2018 where some of the work on this paper was completed. They also give special thanks to their respective advisors, Jason Behrstock and Alessandro Sisto, for their ongoing support and their many helpful comments on early drafts of this paper. Finally, we would like to thank the referees for a number of comments that improved this paper.

2 Coarse geometry

2.1 Quasigeodesic spaces, conventions, and notation

This paper focuses on understanding the geometry of metric spaces up to quasi-isometry. While many of the metric spaces we are interested in applying our results to are geodesic metric spaces, many of the subspaces we will be studying will be quasigeodesic, but not geodesic metric spaces. Thus, we will almost always assume our metric spaces are *quasigeodesic metric spaces*.

Definition 2.1 A metric space X is a (K, L) -*quasigeodesic metric space* if for all $x, y \in X$ there exists a (K, L) -quasigeodesic $\gamma: [a, b] \rightarrow X$ with $\gamma(a) = x$ and $\gamma(b) = y$.

Given a (K, L) -quasigeodesic metric space X , we can construct a geodesic metric space quasi-isometric to X as follows: fix an ϵ -separated net $N \subseteq X$ and connect a pair of points $x, y \in N$ by an edge of length $d(x, y)$ if $d(x, y) < 2\epsilon$. The resulting metric graph will be quasi-isometric to X . Since ϵ can be chosen to depend only on K and L , this graph can be constructed such that the quasi-isometry constants will also depend only on K and L . When convenient, we will exploit this fact to reduce proofs to the geodesic case.

A particularly important collection of metric spaces in geometric group theory is the class of δ -hyperbolic metric spaces, introduced by Gromov in [31; 32]. While δ -hyperbolic spaces are usually required to be geodesic, the following is a direct extension of the definition to the setting of quasigeodesic metric spaces.

Definition 2.2 A (K, L) -quasigeodesic metric space is δ -hyperbolic if for every (K, L) -quasigeodesic triangle the δ -neighborhood of the union of any two of the edges contains the third.

Gromov's four-point condition can also be used to define a hyperbolic quasigeodesic metric space; however as shown in [21, Example 11.36], this definition fails to be a quasi-isometry invariant if the spaces are not geodesic. In contrast, Definition 2.2 is a quasi-isometry invariant among quasigeodesic spaces. In particular, using the "guessing geodesic" criterion, from [42, Theorem 3.15] or [15, Theorem 3.1], one can show that a quasigeodesic space is hyperbolic in the sense of Definition 2.2 if and only if it is quasi-isometric to a geodesic metric space that is hyperbolic in the usual sense.

When referring to a property defined by a parameter (eg δ -hyperbolic), we will often suppress that parameter when its specific value is not needed. To reduce the proliferation of additive and multiplicative constants throughout this paper, we will adopt the following notation.

Notation 2.3 Let A, B, K, L be real numbers. We write

$$A \preceq_{K,L} B \quad \text{if } A \leq KB + L.$$

If $A \preceq_{K,L} B$ and $B \preceq_{K,L} A$, we write $A \asymp_{K,L} B$.

We say two subsets of a metric space K -coarsely coincide if their Hausdorff distance is at most K .

2.2 Strong quasiconvexity, contracting, and stability

The primary notion of convexity we will consider is the following notion of strong quasiconvexity.

Definition 2.4 (strongly quasiconvex subset) A subset Y of a quasigeodesic metric space X is *strongly quasiconvex* if there is a function $Q : [1, \infty) \times [0, \infty) \rightarrow [0, \infty)$ such that for every (K, L) -quasigeodesic γ with endpoints in Y , we have $\gamma \subseteq N_{Q(K,L)}(Y)$. We call the function Q the *convexity gauge* for Y .

It follows directly from the definition that strong quasiconvexity is a quasi-isometry invariant in the following sense.

Lemma 2.5 *Let X and Z be a quasigeodesic metric spaces and $f: X \rightarrow Z$ be a (K, L) -quasi-isometry. If Y is a Q -strongly quasiconvex subset of X , then $f(Y)$ is a Q' -strongly quasiconvex subset of Z , with Q' depending only on Q, K and L .*

In the setting of hyperbolic spaces, strong quasiconvexity is equivalent to the weaker condition of quasiconvexity.

Definition 2.6 A subset Y of a geodesic metric space X is *quasiconvex* if there exists $D \geq 0$ such that for any geodesic γ with endpoints on Y , we have $\gamma \subseteq N_D(Y)$. We call the constant D the *convexity constant* for Y .

If Y is a Q -strongly quasiconvex subset of the (K, L) -quasigeodesic space X , then any two points in Y can be joined by a (K, L) -quasigeodesic in X that lies uniformly close to Y . Thus Y equipped with the metric inherited from X will be a (K', L') -quasigeodesic metric space where K' and L' depend only on K, L , and Q . For the rest of the paper, when discussing geometric properties (such as hyperbolicity) of a strongly quasiconvex subset, we shall implicitly do so with respect to the metric inherited from the ambient space. In particular, if $f: X \rightarrow Z$ is a quasi-isometry between quasigeodesic spaces and Y is a strongly quasiconvex subset of X , then Y is quasi-isometric to $f(Y)$.

In [25], Durham and Taylor introduced the following related notion of convexity.

Definition 2.7 A quasi-isometric embedding Φ from a quasigeodesic metric space Y into a quasigeodesic metric space X is a *stable embedding* if there is a function $R: [1, \infty) \times [0, \infty) \rightarrow [0, \infty)$ such that if α and β are two (K, L) -quasigeodesics of X with the same endpoints in $\Phi(Y)$, then $d_{\text{Haus}}(\alpha, \beta) \leq R(K, L)$.

While the images of stable embeddings maintain many of the features of quasiconvex subsets of hyperbolic spaces, the definition is highly restrictive. In particular, as the next proposition records, stable embeddings must always be onto hyperbolic subsets.

Proposition 2.8 *Let $\Phi: Y \rightarrow X$ be a quasi-isometric embedding from a quasigeodesic metric space Y to a quasigeodesic metric space X . Then Φ is a stable embedding if and only if Y is hyperbolic and $\Phi(Y)$ is strongly quasiconvex. In particular, if Y is a*

strongly quasiconvex subset of X , then the inclusion $i : Y \hookrightarrow X$ is a stable embedding if and only if Y is hyperbolic with respect to the metric inherited from X .

In [54, Proposition 4.3], the third author proves the above proposition for the case of geodesic spaces. The more general statement above follows immediately from the fact that a quasigeodesic space is always quasi-isometric to a geodesic space plus the fact that strong quasiconvexity, stability, and hyperbolicity are all quasi-isometry invariants.

One class of metric spaces we are particularly interested in are finitely generated groups equipped with a word metric. In this setting we are particularly interested in understanding the strongly quasiconvex and stable subgroups.

Definition 2.9 Let G be a finitely generated group equipped with a word metric from some finite generating set. A subgroup $H < G$ is a *strongly quasiconvex subgroup* of G if H is a strongly quasiconvex subset of G with respect to the word metric on G . A subgroup $H < G$ is a *stable subgroup* if H is a strongly quasiconvex subgroup and H is a hyperbolic group.

The above definition of stable subgroup is different than the one originally given in [25], but it is equivalent by Proposition 2.8.

If H is a strongly quasiconvex subgroup of G , then H is also finitely generated and undistorted in G . Further, since strongly quasiconvex is a quasi-isometry invariant, being a strongly quasiconvex or a stable subgroup is independent of the choice of finite generating set for G .

It is common in the literature to study various “contracting” properties of strongly quasiconvex subsets. We compare strongly quasiconvex subsets with the following notion of a contracting subset.

Definition 2.10 Let X be a quasigeodesic metric space and $Y \subseteq X$. A map $g : X \rightarrow Y$ is said to be (A, D) -contracting for some $A \in (0, 1]$ and $D \geq 1$ if

- (1) g is (D, D) -coarsely Lipschitz;
- (2) for any $y \in Y$, $d(y, g(y)) \leq D$;
- (3) for all $x \in X$, if we set $R = Ad(x, Y)$, then $\text{diam}(g(B_R(x))) \leq D$.

A subset Y is said to be (A, D) -contracting if there is an (A, D) -contracting map from X to Y .

The above definition is motivated by [40, Definition 2.2] and generalizes the usual definition of contracting in hyperbolic and CAT(0) spaces to include maps that are not the closest point projection. This is critical to our study of hierarchically hyperbolic spaces in Section 6 and allows quasi-isometry invariance to be established directly from the definition.

Lemma 2.11 *Let X and Z be quasigeodesic metric spaces and $f : X \rightarrow Z$ be a (K, L) -quasi-isometry. If Y is an (A, D) -contracting subset of X , then $f(Y)$ is an (A', D') -contracting subset of Z , where A' and D' depend only on A, D, K and L .*

In the setting of hyperbolic spaces, strongly quasiconvex subsets are contracting. The contracting map will be the following coarse closest point projection: if X is a δ -hyperbolic metric space and $Y \subseteq X$ is Q -strongly quasiconvex, then there exist K depending on δ and Q and a $(1, K)$ -coarsely Lipschitz map $p_Y : X \rightarrow Y$ such that for all $x \in X$, $d(x, p_Y(x)) \leq d(x, Y) + 1$. By an abuse of language, we will refer to p_Y as the *closest point projection* of X onto Y . For any Q -strongly quasiconvex subset Y of a δ -hyperbolic space, the map p_Y is $(1, D)$ -contracting where D depends only on Q and δ .

3 Divergence of contracting subsets

In this section we show that contracting subsets are always strongly quasiconvex. Without some negative curvature hypotheses, such as being hierarchically hyperbolic, the converse is not always true as we show in Example 3.8. Both of these statements are proved using lower relative divergence which was originally introduced by the third author in [53]. The lower relative divergence is a family of functions that measures how efficiently one can travel in X while avoiding a subset Y ; see Figure 1.

Definition 3.1 (lower relative divergence) Let X be a geodesic space and $Y \subseteq X$. For $r > 0$ we adopt the notation

- (1) $\partial N_r(Y) = \{x \in X \mid d(x, Y) = r\}$,
- (2) d_r is the induced path metric on $X - N_r(Y)$.

The *lower relative divergence of X with respect to Y* (or the *divergence of Y in X*), denoted by $\text{div}(X, Y)$, is the set of functions $\{\sigma_\rho^n\}$ defined as follows: For each $\rho \in (0, 1]$, integer $n \geq 2$ and $r \in (0, \infty)$, if there is no pair of $x_1, x_2 \in \partial N_r(Y)$

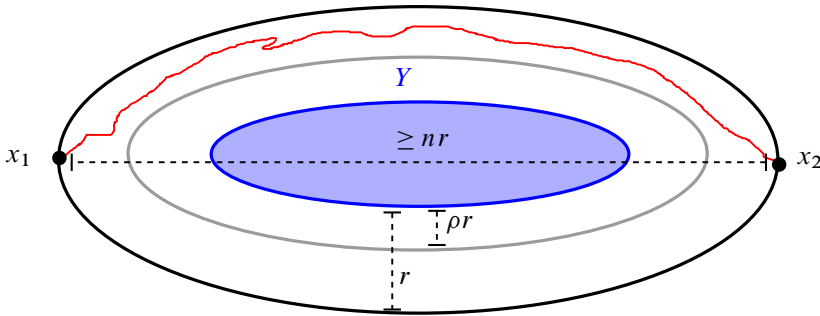


Figure 1: A sketch of a step in the construction of the function σ_ρ^n . The points $x_1, x_2 \in \partial N_r(Y)$ are at least nr far apart, so we measure the distance between x_1 and x_2 in the complement of the ρr -neighborhood of Y . We then take the infimum of these distances over all such pairs of points to obtain $\sigma_\rho^n(r)$.

such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \geq nr$, we define $\sigma_\rho^n(r) = \infty$. Otherwise, we define $\sigma_\rho^n(r) = \inf d_{\rho r}(x_1, x_2)$ where the infimum is taken over all $x_1, x_2 \in \partial N_r(Y)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \geq nr$.

The lower relative divergence is often characterized by how the asymptotics of the functions $\{\sigma_\rho^n\}$ compare to linear, polynomial and exponential functions. Such descriptions are described in detail in [53]. We will restrict our attention to the following two properties of $\text{div}(X, Y)$.

Definition 3.2 Let X be a geodesic metric space and $Y \subseteq X$.

The lower relative divergence of X with respect to Y is *completely superlinear* if there exists $n_0 \geq 3$ such that for every $\rho \in (0, 1]$ and $C > 0$ the set $\{r \in [0, \infty) \mid \sigma_\rho^{n_0}(r) \leq Cr\}$ is bounded.

The lower relative divergence of X with respect to Y is *at least quadratic* if there exists a positive integer M such that for every $\rho \in (0, 1]$ and $n \geq 2$ there exist $C > 0$ and $r_0 > 0$ such that $\sigma_\rho^{Mn}(r) > Cr^2$ for all $r > r_0$.

The properties of being completely superlinear and at least quadratic are preserved under quasi-isometry in the following sense.

Lemma 3.3 (consequence of [53, Proposition 4.9]) *Let $f: X \rightarrow Z$ be a quasi-isometry between geodesic spaces. If $Y \subseteq X$ and $W \subseteq Z$ with $d_{\text{Haus}}(f(Y), W) < \infty$,*

then $\text{div}(X, Y)$ is completely superlinear (resp. at least quadratic) if and only if $\text{div}(Z, W)$ is completely superlinear (resp. at least quadratic).

In [53], the lower relative divergence was defined only for geodesic ambient spaces; however the definition can be extended to include quasigeodesic metric spaces as follows.

Definition 3.4 (lower relative divergence in quasigeodesic spaces) Let X be a quasigeodesic space and $Y \subseteq X$. Let Z be a geodesic space and $f: X \rightarrow Z$ be a quasi-isometry. Then *the lower relative divergence of X with respect to Y* (or *the divergence of Y in X*), denoted by $\text{div}(X, Y)$, is the lower relative divergence of Z with respect to $f(Y)$.

We say $\text{div}(X, Y)$ is *completely superlinear* (resp. *at least quadratic*) if $\text{div}(Z, f(Y))$ is completely superlinear (resp. at least quadratic).

While the definition of $\text{div}(X, Y)$ in a quasigeodesic space depends on a choice of Z and f , $\text{div}(X, Y)$ being completely superlinear (resp. at least quadratic) is independent of this choice by Lemma 3.3. In fact, while it will not be relevant for the content of this paper, $\text{div}(X, Y)$ is independent of the choice of Z and f in a much stronger sense. In [53] the third author defined an equivalence relation \sim between the collections of functions used to define the lower relative divergence. If $f_1: X \rightarrow Z_1$ and $f_2: X \rightarrow Z_2$ are two quasi-isometries with Z_1 and Z_2 geodesic spaces, then by [53, Proposition 4.9], $\text{div}(Z_1, f_1(Y)) \sim \text{div}(Z_2, f_2(Y))$. Thus $\text{div}(X, Y)$ is well defined up to this notion of equivalence.

The following proposition shows that contracting subsets always have at least quadratic divergence.

Proposition 3.5 *If X is a quasigeodesic space and Y is a contracting subset of X , then the lower relative divergence of X with respect to Y is at least quadratic.*

Proof Since every quasigeodesic space is quasi-isometric to a geodesic metric space, Lemma 2.11 allows us to assume X is geodesic. Assume that Y is (A, D) -contracting and let $g: X \rightarrow Y$ be an (A, D) -contracting map. We first show that for all $x \in X$,

$$d(x, g(x)) \leq 2Dd(x, Y) + 4D.$$

Let $y \in Y$ be such that $d(x, y) \leq d(x, Y) + 1$. Then from the definition of (A, D) -contracting,

$$\begin{aligned} d(x, g(x)) &\leq d(x, y) + d(y, g(y)) + d(g(y), g(x)) \\ &\leq d(x, Y) + 1 + D + Dd(x, y) + D \\ &\leq (D + 1)d(x, Y) + 3D + 1 \\ &\leq 2Dd(x, Y) + 4D. \end{aligned}$$

Now, let $\{\sigma_\rho^n\}$ be the lower relative divergence of X with respect to Y . We claim that for each $n \geq 4D + 2$ and $\rho \in (0, 1]$,

$$\sigma_\rho^n(r) \geq \left(\frac{A\rho}{4D}\right)r^2 \quad \text{for each } r > 8D.$$

Let $r > 8D$, n be an integer greater than $4D + 2$, and $\rho \in (0, 1]$. If $\sigma_\rho^n(r) = \infty$, then the above inequality is true. Otherwise, let $x_1, x_2 \in \partial N_r(Y)$ be such that $d(x_1, x_2) \geq nr$ and $d_r(x_1, x_2) \leq \infty$. The distances $d(x_1, g(x_1))$ and $d(x_2, g(x_2))$ are bounded above by $2Dr + 4D$. Therefore,

$$d(g(x_1), g(x_2)) \geq d(x_1, x_2) - d(x_1, g(x_1)) - d(x_2, g(x_2)) \geq nr - 4Dr - 8D \geq r.$$

Let γ be a rectifiable path in $N_{\rho r}(Y)$ connecting x_1 and x_2 and $R = A\rho r/2$. There exist $t_0 < t_1 < t_2 < \dots < t_{m-1} < t_m$ such that $\gamma(t_0) = x_1$, $\gamma(t_m) = x_2$ and

$$\frac{1}{2}R \leq \ell(\gamma|_{[t_{i-1}, t_i]}) \leq R,$$

where $\ell(\cdot)$ denotes the length of a path. This implies

$$(1) \quad \ell(\gamma) = \sum_{i=1}^m \ell(\gamma|_{[t_{i-1}, t_i]}) \geq \frac{1}{2}mR.$$

Since g is an (A, D) -contracting map and $d(\gamma(t_{i-1}), \gamma(t_i)) < Ad(\gamma(t_{i-1}), Y)$, we have $d(g(\gamma(t_{i-1})), g(\gamma(t_i))) \leq D$ for each $1 \leq i \leq m$. Thus

$$(2) \quad d(g(x_1), g(x_2)) \leq \sum_{i=1}^m d(g(\gamma(t_{i-1})), g(\gamma(t_i))) \leq mD.$$

Since $d(g(x_1), g(x_2)) \geq r$, inequality (2) implies $m \geq r/D$. Combining this with inequality (1), we have

$$\ell(\gamma) \geq \frac{1}{2}mR \geq \left(\frac{A\rho}{4D}\right)r^2.$$

Therefore,

$$\sigma_\rho^n(r) \geq \left(\frac{A\rho}{4D}\right)r^2$$

for $n \geq 4D + 2$, $\rho \in (0, 1]$, and $r > 8D$. This implies that the lower relative divergence of X with respect to Y is at least quadratic. \square

In [53], the third author classified strongly quasiconvex subsets in terms of their lower relative divergence. This result continues to hold in the slightly more general setting of quasigeodesic spaces.

Theorem 3.6 [54, Theorem 3.1] *Let X be a quasigeodesic space and $Y \subseteq X$. Then Y is strongly quasiconvex if and only if the lower relative divergence of X with respect to Y is completely superlinear.*

Proof Since every quasigeodesic metric space is quasi-isometric to a geodesic metric space, the result follows immediately from [54, Theorem 1.5] when Y is infinite diameter. If $\text{diam}(A) = r_0 < \infty$, then for all $r > r_0$, $\partial N_{r_0}(Y) = \emptyset$ and thus $\sigma_\rho^n(r) = \infty$. Hence $\text{div}(X, Y)$ is completely superlinear and Y is strongly quasiconvex. \square

Proposition 3.5 and Theorem 3.6 combine to say that if a subset $Y \subseteq X$ is (A, D) -contracting, then Y is strongly quasiconvex. A direct proof of this result was shown by Sultan for the case of quasigeodesics, but the proof extends to any subset without modification [52, Lemma 3.3]. For completeness, we include a proof using the bound on the lower relative divergence of Y from Proposition 3.5.

Corollary 3.7 *Let X be a (K, L) -quasigeodesic space and $Y \subseteq X$. If Y is (A, D) -contracting, then Y is Q -strongly quasiconvex where Q is determined by A, D, K and L .*

Proof Let Y be a (A, D) -contracting subset of X . We first assume that X is a geodesic metric space. Let $\{\sigma_\rho^n\}$ be the lower relative divergence of X with respect to Y . The proof of Proposition 3.5 shows that for each $n \geq 4D + 2$ and $\rho \in (0, 1]$,

$$\sigma_\rho^n(r) \geq \left(\frac{A\rho}{4D}\right)r^2 \quad \text{for all } r > 8D.$$

Therefore, by fixing $n = n_0 = 4D + 3$ and $\rho = 1$,

$$\sigma_1^{n_0}(r) \geq \left(\frac{A}{4D}\right)r^2 \quad \text{for all } r > 8D.$$

If γ is a (λ, ϵ) -quasigeodesic with endpoints on Y , let $m = \inf\{B \in \mathbb{R} \mid \gamma \subseteq N_B(Y)\}$. The proof of [54, Proposition 3.1] establishes that if m is larger than a fixed constant

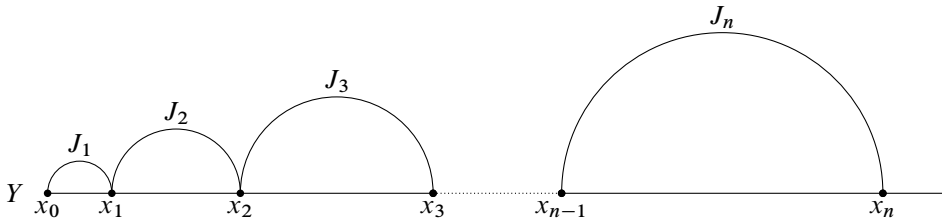


Figure 2: The space X of Example 3.8.

depending on λ and ϵ , then there exist constants C_0 and C_1 depending only on λ , ϵ and n_0 , such that $\sigma_1^{n_0}(C_0m) \leq C_1m$. Thus,

$$\left(\frac{A}{4D}\right)(C_0m)^2 \leq \sigma_1^{n_0}(C_0m) \leq C_1m,$$

and hence m is bounded by some constant depending only on λ , ϵ , A and D . Thus, there exists a function Q depending only on A and D such that Y is Q -strongly quasiconvex.

When X is a (K, L) -quasigeodesic space, there exist a geodesic metric space Z and a quasi-isometry $f : X \rightarrow Z$ with constants determined by K and L . The result follows from the geodesic case by Lemmas 2.5 and 2.11. \square

We finish this section by adapting [6, Example 3.4] to give a counterexample to the converse of Corollary 3.7.

Example 3.8 (strongly quasiconvex subsets need not be contracting) Let Y be a ray with initial point x_0 and let (x_n) be the sequence of points along Y such that for each $n \geq 1$ the distance between x_{n-1} and x_n is equal to n . We connect each pair (x_{n-1}, x_n) by an additional segment J_n of length $n^{3/2}$ as shown in Figure 2. Let X be the resulting geodesic space.

By Proposition A.2 the lower relative divergence of X with respect to Y is completely superlinear, but not at least quadratic — heuristically, $\text{div}(X, Y)$ behaves like $r^{3/2}$. So Y is strongly quasiconvex, but not contracting by Proposition 3.5 and Theorem 3.6.

4 Hierarchically hyperbolic spaces

We now recall the main definitions of hierarchically hyperbolic groups and spaces. The main references, where not specified, are [9; 10]. While we give the entire definition

of an HHS for completeness, we advise the reader that we shall only directly utilize axioms (1), (2), (3), (5), (8), and (10) of Definition 4.1 in the remainder of the paper.

Definition 4.1 (hierarchically hyperbolic space) Let \mathcal{X} be a quasigeodesic space. A *hierarchically hyperbolic space (HHS) structure* on \mathcal{X} consists of constants $E \geq \kappa_0 > 0$, an index set \mathfrak{S} , and a set $\{CW \mid W \in \mathfrak{S}\}$ of geodesic δ -hyperbolic spaces (CW, d_W) , such that the following conditions are satisfied.

(1) **Projections** For each $W \in \mathfrak{S}$, there exists a *projection* $\pi_W : \mathcal{X} \rightarrow 2^{CW}$ such that for all $x \in \mathcal{X}$, $\pi_W(x) \neq \emptyset$, and $\text{diam}(\pi_W(x)) < E$. Moreover, there exists a K such that each π_W is (K, K) -coarsely Lipschitz and $\pi_W(\mathcal{X})$ is K -quasiconvex in CW .

(2) **Nesting** \mathfrak{S} is equipped with a partial order \sqsubseteq , and either $\mathfrak{S} = \emptyset$ or \mathfrak{S} contains a unique \sqsubseteq -maximal element; when $V \sqsubseteq W$, we say V is *nested* in W . For each $W \in \mathfrak{S}$, we denote by \mathfrak{S}_W the set of $V \in \mathfrak{S}$ such that $V \sqsubseteq W$. Moreover, for all $V, W \in \mathfrak{S}$ with $V \not\sqsubseteq W$ there is a specified nonempty subset $\rho_W^V \subseteq CW$ with $\text{diam}_{CW}(\rho_W^V) \leq E$. There is also a *projection* $\rho_V^W : CW \rightarrow 2^{CV}$.

(3) **Orthogonality** \mathfrak{S} has a symmetric and antireflexive relation called *orthogonality*; we write $V \perp W$ when V and W are orthogonal. Whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$. Additionally, if $V \perp W$, then V and W are not \sqsubseteq -comparable.

(4) **Containers** For each $T \in \mathfrak{S}$ and each $U \in \mathfrak{S}_T$ for which $\{V \in \mathfrak{S}_T \mid V \perp U\} \neq \emptyset$, there exists a $W \in \mathfrak{S}_T - \{T\}$, such that whenever $V \perp U$ and $V \sqsubseteq T$, we have $V \sqsubseteq W$. We say W is a *container* for U in \mathfrak{S}_T .

(5) **Transversality and consistency** If $V, W \in \mathfrak{S}$ are not orthogonal and neither is nested in the other, then we say V and W are *transverse*, denoted by $V \pitchfork W$. If $V \pitchfork W$, then there are nonempty sets $\rho_W^V \subseteq CW$ and $\rho_V^W \subseteq CV$, each of diameter at most E , satisfying

$$\min\{d_W(\pi_W(x), \rho_W^V), d_V(\pi_V(x), \rho_V^W)\} \leq \kappa_0$$

for all $x \in \mathcal{X}$.

For $V, W \in \mathfrak{S}$ satisfying $V \sqsubseteq W$ and for all $x \in \mathcal{X}$,

$$\min\{d_W(\pi_W(x), \rho_W^V), \text{diam}_{CV}(\pi_V(x) \cup \rho_V^W(\pi_W(x)))\} \leq \kappa_0.$$

Finally, if $U \sqsubseteq V$, then $d_W(\rho_W^U, \rho_W^V) \leq \kappa_0$ whenever $W \in \mathfrak{S}$ satisfies either $V \not\sqsubseteq W$ or $V \pitchfork W$ and $W \not\sqsubseteq U$.

(6) **Finite complexity** There exists $n \geq 0$ such that any set of pairwise \sqsubseteq -comparable elements has cardinality at most n .

(7) **Large links** There exists $\zeta \geq 1$ such that the following holds. Let $W \in \mathfrak{S}$ and $x, x' \in \mathcal{X}$. There exist $\{U_i\}_{i=1, \dots, m} \subseteq \mathfrak{S}_W - \{W\}$ such that

$$m \leq \zeta d_W(\pi_W(x), \pi_W(x')) + \zeta$$

and for all $V \in \mathfrak{S}_W - \{W\}$, either $V \in \mathfrak{S}_{U_i}$ for some i , or $d_V(\pi_V(x), \pi_V(x')) < E$. Also, $d_W(\pi_W(x), \rho_W^{U_i}) \leq \zeta d_W(\pi_W(x), \pi_W(x')) + \zeta$ for each i .

(8) **Bounded geodesic image** For all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}_W - \{W\}$, and all geodesics γ of CW , either $\text{diam}(\rho_V^W(\gamma)) \leq E$ or $\gamma \cap N_E(\rho_V^W) \neq \emptyset$.

(9) **Partial realization** There exists a constant α with the following property. Let $\{V_j\}$ be a family of pairwise orthogonal elements of \mathfrak{S} , and let $p_j \in \pi_{V_j}(\mathcal{X}) \subseteq CV_j$. Then there exists $x \in \mathcal{X}$ such that

- $d_{V_j}(x, p_j) \leq \alpha$ for all j ;
- for each j and each $V \in \mathfrak{S}$ with $V_j \sqsubseteq V$, we have $d_V(x, \rho_V^{V_j}) \leq \alpha$;
- if $W \pitchfork V_j$ for some j , then $d_W(x, \rho_W^{V_j}) \leq \alpha$.

(10) **Uniqueness** For each $\kappa \geq 0$, there exists $\theta_\kappa = \theta_\kappa(\kappa)$ such that if $x, y \in \mathcal{X}$ and $d(x, y) \geq \theta_\kappa$, then there exists $V \in \mathfrak{S}$ such that $d_V(x, y) \geq \kappa$.

We will refer to the elements of the index set \mathfrak{S} as *domains* and use \mathfrak{S} to denote the entire HHS structure, including all the spaces, constants, projections and relations defined above. A quasigeodesic space \mathcal{X} is a *hierarchically hyperbolic space* (HHS) if it admits a hierarchically hyperbolic structure. We will use the pair $(\mathcal{X}, \mathfrak{S})$ to denote \mathcal{X} equipped with the hierarchically hyperbolic structure \mathfrak{S} .

If $(\mathcal{X}, \mathfrak{S})$ is a hierarchically hyperbolic space and $f: \mathcal{Y} \rightarrow \mathcal{X}$ is a quasi-isometry, then \mathfrak{S} is also an HHS structure for \mathcal{Y} where the projections maps are defined by $\pi_W \circ f$ for each $W \in \mathfrak{S}$.

Many of the key examples of hierarchically hyperbolic spaces are finitely generated groups where the Cayley graph admits an HHS structure. In the case where this structure is preserved by the group action, we will call those groups hierarchically hyperbolic groups.

Definition 4.2 (hierarchically hyperbolic groups) Let G be a finitely generated group. We say G is a *hierarchically hyperbolic group* (HHG) if:

- (1) G with the word metric from a finite generating set admits an HHS structure \mathfrak{S} .
- (2) There is a \sqsubseteq, \perp and \pitchfork preserving action of G on \mathfrak{S} by bijections such that \mathfrak{S} contains finitely many G orbits.
- (3) For each $W \in \mathfrak{S}$ and $g \in G$, there exists an isometry $g_W : CW \rightarrow C(gW)$ satisfying the following for all $V, W \in \mathfrak{S}$ and $g, h \in G$:
 - The map $(gh)_W : CW \rightarrow C(ghW)$ is equal to $g_{hW} \circ h_W : CW \rightarrow C(ghW)$.
 - For each $h \in G$, $g_W(\pi_W(h))$ and $\pi_{gW}(gh)$ E -coarsely coincide.
 - If $V \pitchfork W$ or $V \sqsubseteq W$, then $g_W(\rho_W^V)$ and ρ_{gW}^{gV} E -coarsely coincide.
 - If $V \sqsubseteq W$ and $p \in CW - N_E(\rho_W^V)$, then $g_W(\rho_W^V(p))$ and $\rho_{gW}^{gV}(g_W(p))$ E -coarsely coincide.

The HHS structure \mathfrak{S} satisfying (1)–(3) is called a hierarchically hyperbolic group (HHG) structure on G and we use (G, \mathfrak{S}) to denote a group G equipped with a specific HHG structure \mathfrak{S} .

Being a hierarchically hyperbolic group is independent of choice of generating set by virtue of being able to pass the HHG structure through a G -equivariant quasi-isometry. The reader may find it helpful to note that the conditions in (3) above can be summarized by saying the diagrams

$$\begin{array}{ccc}
 G & \xrightarrow{g} & G \\
 \downarrow \pi_W & & \downarrow \pi_{gW} \\
 CW & \xrightarrow{g_W} & C(gW)
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 CV & \xrightarrow{g_V} & C(gV) \\
 \downarrow \rho_W^V & & \downarrow \rho_{gW}^{gV} \\
 CW & \xrightarrow{g_W} & C(gW)
 \end{array}$$

coarsely commute whenever $V, U \in \mathfrak{S}$ are not orthogonal.

Notation 4.3 When writing distances in CW for some $W \in \mathfrak{S}$, we often simplify the notation by suppressing the projection map π_W , that is, given $x, y \in \mathcal{X}$ and $p \in CW$ we write $d_W(x, y)$ for $d_W(\pi_W(x), \pi_W(y))$ and $d_W(x, p)$ for $d_W(\pi_W(x), p)$. Note that when we measure distance between a pair of sets (typically both of bounded diameter) we are taking the minimum distance between the two sets. Given $A \subseteq \mathcal{X}$ and $W \in \mathfrak{S}$ we let $\pi_W(A)$ denote $\bigcup_{a \in A} \pi_W(a)$.

The guiding philosophy of hierarchically hyperbolic spaces is that one can “pull back” the hyperbolic geometry of the various CW ’s to obtain features of negative curvature

in the original space. The most prominent example of this philosophy is the following distance formula which allows distances in the main space \mathcal{X} to be approximated by distances in the hyperbolic spaces.

Theorem 4.4 (the distance formula; [10, Theorem 4.4]) *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Then there exists σ_0 such that, for all $\sigma \geq \sigma_0$, there exist $K \geq 1$ and $L \geq 0$ such that, for all $x, y \in \mathcal{X}$,*

$$d_{\mathcal{X}}(x, y) \asymp_{K,L} \sum_{U \in \mathfrak{S}} \{d_U(x, y)\}_\sigma,$$

where $\{N\}_\sigma = N$ if $N \geq \sigma$ and 0 otherwise.

The distance formula can be “distributed” over a sum of distances in the hyperbolic spaces as described in the next lemma.

Lemma 4.5 [48, Lemma 2.26] *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS and x_0, x_1, \dots, x_n be points in \mathcal{X} . If there exists $C \geq 1$ such that $\sum_{i=0}^{n-1} d_W(x_i, x_{i+1}) \asymp_{C,C} d_W(x_0, x_n)$ for all $W \in \mathfrak{S}$, then there exist K depending only on C, n , and $(\mathcal{X}, \mathfrak{S})$ such that*

$$\sum_{i=0}^{n-1} d_{\mathcal{X}}(x_i, x_{i+1}) \asymp_{K,K} d_{\mathcal{X}}(x_0, x_n).$$

Part of the content of Theorem 4.4 is that for any pair of points in an HHS, there is only a finite number of domains where that pair of points can have a large projection. More precisely, if $(\mathcal{X}, \mathfrak{S})$ is a hierarchically hyperbolic space, then a domain $W \in \mathfrak{S}$ is said to be σ -relevant for $x, y \in \mathcal{X}$ if $d_W(x, y) > \sigma$. We denote the set of all σ -relevant domains for $x, y \in \mathcal{X}$ by $\text{Rel}_\sigma(x, y)$. By Theorem 4.4, for all $\sigma \geq \sigma_0$, $\text{Rel}_\sigma(x, y)$ has finite cardinality. The relevant facts about $\text{Rel}_\sigma(x, y)$ that we will need are summarized in the following proposition.

Proposition 4.6 [10, Lemma 2.2, Proposition 2.8, Lemma 2.14] *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and $E \geq 0$ be the maximum of all the constants in the HHS structure for $(\mathcal{X}, \mathfrak{S})$.*

- (1) *There exists $\chi > 0$ such that if $\mathfrak{U} \subseteq \mathfrak{S}$ does not contain a pair of transverse domains, then $|\mathfrak{U}| \leq \chi$.*
- (2) *If $\sigma \geq 100E$ and $x, y \in \mathcal{X}$, then the set $\text{Rel}_\sigma(x, y)$ can be partially ordered by*

$$U \leq V \iff U = V \text{ or } U \pitchfork V \text{ and } d_V(\rho_V^U, y) \leq \kappa_0.$$

- (3) If $\sigma \geq 100E$ and $x, y \in \mathcal{X}$, then there exists $n \leq \chi$ such that $\text{Rel}_\sigma(x, y)$ can be partitioned into n disjoint subsets $\mathcal{U}_1, \dots, \mathcal{U}_n$ where, for each i , \mathcal{U}_i is totally ordered with respect to the above ordering on $\text{Rel}_\sigma(x, y)$.

Hierarchically hyperbolic spaces contain a particularly nice class of quasigeodesics, called hierarchy paths. Even when considering a geodesic HHS, it is often preferable to work with hierarchy paths over geodesics.

Definition 4.7 (hierarchy path) For $\lambda \geq 1$, a (not necessarily continuous) path $\gamma: [a, b] \rightarrow \mathcal{X}$ is a λ -hierarchy path if

- (1) γ is a (λ, λ) -quasigeodesic,
- (2) for each $W \in \mathfrak{S}$, the path $\pi_W \circ \gamma$ is an unparametrized (λ, λ) -quasigeodesic.

Recall that a map $f: [a, b] \rightarrow X$ is an unparametrized (λ, λ) -quasigeodesic if there exists an increasing function $g: [0, \ell] \rightarrow [a, b]$ such that $g(0) = a$, $g(\ell) = b$, and $f \circ g$ is a (λ, λ) -quasigeodesic of X .

While not every quasigeodesic in an HHS is a hierarchy path, every pair of points can be connected by a hierarchy path as the next theorem describes.

Theorem 4.8 (existence of hierarchy paths; [10, Theorem 5.4]) *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. Then there exists a λ_0 such that any $x, y \in \mathcal{X}$ are joined by a λ_0 -hierarchy path.*

4.1 Hierarchical quasiconvexity and gate maps

In [10], Behrstock, Hagen and Sisto introduced *hierarchical quasiconvexity*, a notion of convexity unique to hierarchically hyperbolic spaces.

Definition 4.9 (hierarchical quasiconvexity; [10, Definition 5.1]) Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and $k: [0, \infty) \rightarrow [0, \infty)$. A subset $Y \subseteq \mathcal{X}$ is k -hierarchically quasiconvex if:

- (1) For all $U \in \mathfrak{S}$, the projection $\pi_U(Y)$ is a $k(0)$ -quasiconvex subspace of the δ -hyperbolic space CU .
- (2) For every $\kappa > 0$ and every point $x \in \mathcal{X}$ satisfying $d_U(x, Y) \leq \kappa$ for all $U \in \mathfrak{S}$, we have that $d_{\mathcal{X}}(x, Y) \leq k(\kappa)$.

While hierarchically quasiconvex subsets need not be strongly quasiconvex, they are “quasiconvex with respect to hierarchy paths”. That is, if $Y \subseteq \mathcal{X}$ is k -hierarchically quasiconvex then any λ -hierarchy path with endpoints in Y must stay uniformly close to Y . The existence of hierarchy paths (Theorem 4.8) therefore ensures that if Y is equipped with the induced metric from \mathcal{X} , then Y is also a quasigeodesic metric space with constants depending only on $(\mathcal{X}, \mathfrak{S})$ and k . In Section 5 we will prove that hierarchically quasiconvex subsets are actually characterized by this “quasiconvexity with respect to hierarchy paths”.

One of the key features of hierarchically quasiconvex subsets is that they are hierarchically hyperbolic spaces with the restriction of the HHS structure from the ambient space.

Theorem 4.10 [10, Proposition 5.6] *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and $Y \subseteq \mathcal{X}$ be k -hierarchically quasiconvex. Then (Y, \mathfrak{S}) is a hierarchically hyperbolic space, where Y is equipped with the induced metric from \mathcal{X} .*

The following lemma is a special case of the powerful realization theorem for hierarchically hyperbolic spaces; see [10, Theorem 3.1]. It is often useful when verifying that a subset is hierarchically quasiconvex.

Lemma 4.11 [10, Theorem 3.1, Lemma 5.3] *For each $R \geq 0$ there is a $\mu \geq 0$ such that the following holds. Let $Y \subseteq \mathcal{X}$ be such that $\pi_W(Y)$ is R -quasiconvex for each $W \in \mathfrak{S}$. Let $x \in \mathcal{X}$ and for each $W \in \mathfrak{S}$, let $p_W \in \pi_W(Y)$ satisfy $d_V(x, p_W) \leq d_W(x, Y) + 1$. Then there exists $p \in \mathcal{X}$ such that $d_W(p, p_W) \leq \mu$ for all $W \in \mathfrak{S}$.*

Given a subset $Y \subseteq \mathcal{X}$, there exists a *hierarchically quasiconvex hull* of Y which can be thought of as the coarsely smallest hierarchically quasiconvex subset of \mathcal{X} containing Y .

Definition 4.12 (hierarchically quasiconvex hull) For each set $Y \subseteq \mathcal{X}$ and $W \in \mathfrak{S}$, let $\text{hull}_{CW}(Y)$ denote the convex hull of $\pi_W(Y)$ in CW , ie the union of all CW -geodesics connecting pairs of points in $\pi_W(Y)$. Given $\theta \geq 0$, let $H_\theta(Y)$ be the set of all $p \in \mathcal{X}$ such that, for each $W \in \mathfrak{S}$, the set $\pi_W(p)$ lies at distance at most θ from $\text{hull}_{CW}(Y)$. Note that $Y \subseteq H_\theta(Y)$.

Lemma 4.13 [10, Lemma 6.2] *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS. There exists θ_0 such that for each $\theta \geq \theta_0$ there exists $k : [0, \infty) \rightarrow [0, \infty)$ such that for each $Y \subseteq \mathcal{X}$, the hull $H_\theta(Y)$ is k -hierarchically quasiconvex.*

In Section 5 we strengthen the analogy between hierarchically quasiconvex hulls and convex hulls in hyperbolic spaces, by showing that $H_\theta(Y)$ can be constructed by iteratively connecting points in Y by hierarchy paths.

One of the important properties of hierarchically quasiconvex subsets is the existence of a *gate map* which retracts the entire space onto the hierarchically quasiconvex subset. The gate map is a generalization to hierarchically hyperbolic spaces of the closest point projection, \mathfrak{p} , defined at the end of Section 2.

Lemma 4.14 (existence of coarse gates; [10, Lemma 5.5]) *If $(\mathcal{X}, \mathfrak{S})$ is a hierarchically hyperbolic space and $Y \subseteq \mathcal{X}$ is k -hierarchically quasiconvex and nonempty, then there exists a gate map $\mathfrak{g}_Y : \mathcal{X} \rightarrow Y$ such that*

- (1) \mathfrak{g}_Y is (K, K) -coarsely Lipschitz;
- (2) for all $y \in Y$, $d_{\mathcal{X}}(y, \mathfrak{g}_Y(y)) \leq K$;
- (3) for all $x \in \mathcal{X}$ and $U \in \mathfrak{S}$, $d_U(\mathfrak{g}_Y(x), \mathfrak{p}_{\pi_U(Y)}(\pi_U(x))) \leq K$;

where K depends only on k and \mathfrak{S} .

While the gate map need not be the closest point projection, it approximates the closest point projection with a multiplicative and additive error.

Lemma 4.15 [11, Lemma 1.27] *Let Y be a k -hierarchically quasiconvex subset of the HHS $(\mathcal{X}, \mathfrak{S})$ and $x \in \mathcal{X}$. If $y \in Y$ is a point such that $d_{\mathcal{X}}(x, y) \leq d_{\mathcal{X}}(x, Y) + 1$, then $d_{\mathcal{X}}(x, y) \asymp d_{\mathcal{X}}(x, \mathfrak{g}_Y(x))$ where the constants depend only on k and \mathfrak{S} .*

In the case of hierarchically hyperbolic groups, the gate is also coarsely equivariant.

Lemma 4.16 (coarse equivariance of gate maps) *Let (G, \mathfrak{S}) be a hierarchically hyperbolic group and let Y be a k -hierarchically quasiconvex subspace of G . There exists K depending on (G, \mathfrak{S}) and k such that, for every $g, x \in G$,*

$$d_G(\mathfrak{g}_Y(gx), \mathfrak{g}_Y(g\mathfrak{g}_Y(x))) \leq K.$$

Proof Since G acts on the disjoint union of the CW 's by isometries, Lemma 4.14 and the definition of HHG provide a uniform bound on $d_W(\pi_W(\mathfrak{g}_Y(gx)), \pi_W(\mathfrak{g}_Y(g\mathfrak{g}_Y(x))))$ for all $W \in \mathfrak{S}$, which depends only on \mathfrak{S} , k , and the choice of finite generating set for G . The result now follows from the distance formula (Theorem 4.4). \square

The following lemma explains the nice behavior of the gates of hierarchically quasiconvex sets onto each other. The lemma is stated in slightly more generality than presented in [11], but the more general statement is implicit in the proof of [11, Lemma 1.20]. The following notation will simplify the exposition.

Notation 4.17 If \mathfrak{S} is an HHS structure on a metric space \mathcal{X} and $\mathcal{H} \subseteq \mathfrak{S}$ we use \mathcal{H}^\perp to denote the set $\{W \in \mathfrak{S} \mid \forall H \in \mathcal{H}, H \perp W\}$. In particular, $\mathfrak{S}_U^\perp = \{W \in \mathfrak{S} \mid U \perp W\}$ for any $U \in \mathfrak{S}$. Note, if $\mathcal{H} = \emptyset$, then $\mathcal{H}^\perp = \mathfrak{S}$ as every domain in \mathfrak{S} would vacuously satisfy the condition of the set.

Theorem 4.18 (the bridge theorem; [11, Lemma 1.20]) *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and θ_0 be as in Lemma 4.13. For every k and $\theta \geq \theta_0$, there exist $k' : [0, \infty) \rightarrow [0, \infty)$ and $K_0 \geq 0$ such that, for any k -hierarchically quasiconvex sets A and B :*

- (1) $\mathfrak{g}_A(B)$ is k' -hierarchically quasiconvex.
- (2) The composition $\mathfrak{g}_A \circ \mathfrak{g}_B|_{\mathfrak{g}_A(B)}$ is bounded distance from the identity map $\mathfrak{g}_A(B) \rightarrow \mathfrak{g}_A(B)$.
- (3) For any $a \in \mathfrak{g}_A(B)$ and $b = \mathfrak{g}_B(a)$, we have a (K_0, K_0) -quasi-isometric embedding $f : \mathfrak{g}_A(B) \times H_\theta(a, b) \rightarrow \mathcal{X}$ with image $H_\theta(\mathfrak{g}_A(B) \cup \mathfrak{g}_B(A))$ such that $f(\mathfrak{g}_A(B) \times \{b\})$ K_0 -coarsely coincides with $\mathfrak{g}_B(A)$.

Let $K \geq K_0$ and $\mathcal{H} = \{U \in \mathfrak{S} : \text{diam}(\pi_U(\mathfrak{g}_A(B))) > K\}$.

- (4) For each $p, q \in \mathfrak{g}_A(B)$ and $t \in H_\theta(a, b)$,

$$\text{Rel}_K(f(p, t), f(q, t)) \subseteq \mathcal{H}.$$

- (5) For each $p \in \mathfrak{g}_A(B)$ and $t_1, t_2 \in H_\theta(a, b)$,

$$\text{Rel}_K(f(p, t_1), f(p, t_2)) \subseteq \mathcal{H}^\perp.$$

- (6) For each $p \in A, q \in B$,

$$\begin{aligned} & d(p, q) \\ & \asymp_{K_0, K_0} d(p, \mathfrak{g}_A(B)) + d(q, \mathfrak{g}_B(A)) + d(A, B) + d(\mathfrak{g}_{\mathfrak{g}_B(A)}(p), \mathfrak{g}_{\mathfrak{g}_B(A)}(q)). \end{aligned}$$

We name Theorem 4.18 the bridge theorem as one should think of the set

$$H_\theta(\mathfrak{g}_A(B) \cup \mathfrak{g}_B(A))$$

as a “bridge” between A and B : in order to efficiently travel between A and B one needs to always traverse this bridge. The bridge theorem, along with the construction of

the gate map and hulls produces the following fact about the set $H_\theta(\mathfrak{g}_A(B) \cup \mathfrak{g}_B(A))$ which we will need in Section 8.

Lemma 4.19 *For every k and $\theta \geq \theta_0$, there exists K such that for any k -hierarchically quasiconvex sets A and B , the sets $\mathfrak{g}_B(H_\theta(\mathfrak{g}_A(B) \cup \mathfrak{g}_B(A)))$ and $\mathfrak{g}_B(A)$ K -coarsely coincide.*

We finish this section by recalling the construction of *standard product regions* introduced in [9, Section 13] and studied further in [10]. For what follows, fix a hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$.

Definition 4.20 (nested partial tuple F_U) Recall $\mathfrak{S}_U = \{V \in \mathfrak{S} \mid V \sqsubseteq U\}$. Define F_U to be the set of tuples in $\prod_{V \in \mathfrak{S}_U} 2^{C^V}$ satisfying the conditions of Definition 4.1(5) for all $V, W \in \mathfrak{S}_U$ with $V \not\sqsubseteq W$.

Definition 4.21 (orthogonal partial tuple E_U) Recall $\mathfrak{S}_U^\perp = \{V \in \mathfrak{S} \mid V \perp U\}$. Define E_U to be the set of tuples in $\prod_{V \in \mathfrak{S}_U^\perp} 2^{C^V}$ satisfying the conditions of Definition 4.1(5) for all $V, W \in \mathfrak{S}_U^\perp$ with $V \not\sqsubseteq W$.

Definition 4.22 (product regions in \mathcal{X}) Let $U \in \mathfrak{S}$. There exists μ depending only on \mathfrak{S} such that for each $(a_V)_{V \in \mathfrak{S}_U} \in F_U$ and $(b_V)_{V \in \mathfrak{S}_U^\perp} \in E_U$, there exists $x \in \mathcal{X}$ such that for each $V \in \mathfrak{S}$:

- If $V \sqsubseteq U$, then $d_V(x, a_V) \leq \mu$.
- If $V \perp U$, then $d_V(x, b_V) \leq \mu$.
- If $V \pitchfork U$ or $U \sqsubseteq V$, then $d_V(x, \rho_V^U) \leq \mu$.

Thus there is a map $\phi_U: F_U \times E_U \rightarrow \mathcal{X}$, whose image is k -hierarchically quasiconvex where k only depends on \mathfrak{S} . We call $\phi_U(F_U \times E_U)$ the *product region for U* and denote it by P_U .

For any $e \in E_U$ and $f \in F_U$, the sets $\phi_U(F_U \times \{e\})$ and $\phi_U(\{f\} \times E_U)$ will also be hierarchically quasiconvex; thus E_U and F_U are quasigeodesic metric spaces when equipped with the subspace metric from $\phi_U(F_U \times \{e\})$ and $\phi_U(\{f\} \times E_U)$. While these metrics depend on the choice of e and f , the distance formula (Theorem 4.4) ensures that the different choices are all uniformly quasi-isometric.

The definition of the product regions ensure that they are not only uniformly hierarchically quasiconvex, but have easily described gate maps.

Lemma 4.23 [11, Section 5] *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS. There exists $k: [0, \infty) \rightarrow [0, \infty)$ such that for all $U \in \mathfrak{S}$, the product region \mathbf{P}_U is k -hierarchically quasiconvex. Moreover, there exists $K \geq 0$ depending only on $(\mathcal{X}, \mathfrak{S})$ such that for all $x \in \mathcal{X}$,*

- $d_V(\mathbf{g}_{\mathbf{P}_U}(x), x) \leq K$ if $V \sqsubseteq U$ or $V \perp U$,
- $d_V(\mathbf{g}_{\mathbf{P}_U}(x), \rho_V^U) \leq K$ if $V \pitchfork U$ or $U \sqsubset V$.

A version of our last result appeared as [10, Proposition 5.17]. However, that result contains an error in both its statement and its proof.³ We provide a corrected statement and proof.

Proposition 4.24 (active subpaths; corrected version of [10, Proposition 5.17]) *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS. There exist constants $D, \nu, \lambda \geq 1$ such that for all $x, y \in \mathcal{X}$, if $d_U(x, y) > D$ for some $U \in \mathfrak{S}$, then there exists a λ -hierarchy path $\gamma: [a, b] \rightarrow \mathcal{X}$ joining x and y that has a subpath $\alpha = \gamma|_{[a_1, b_1]}$ such that*

- (1) $\alpha \subseteq N_\nu(\mathbf{P}_U)$;
- (2) the diameters of $\pi_W(\gamma([a, a_1]))$ and $\pi_W(\gamma([b_1, b]))$ are both bounded by ν , for all $W \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$;
- (3) for any point $p \in \gamma([a, a_1])$ or $q \in \gamma([b_1, b])$,

$$d_{\mathcal{X}}(\mathbf{g}_{\mathbf{P}_U}(x), \mathbf{g}_{\mathbf{P}_U}(p)) \leq \nu \quad \text{and} \quad d_{\mathcal{X}}(\mathbf{g}_{\mathbf{P}_U}(y), \mathbf{g}_{\mathbf{P}_U}(q)) \leq \nu.$$

We call α the **active subpath** of γ for U .

Proof Let δ, E , and κ_0 be the constants appearing in the HHS structure \mathfrak{S} for \mathcal{X} . Let $x' = \mathbf{g}_{\mathbf{P}_U}(x)$ and $y' = \mathbf{g}_{\mathbf{P}_U}(y)$. Let $\lambda_0 \geq 1$ be the constant such that every pair of points in \mathcal{X} can be joined by a λ_0 -hierarchy path and μ be the constant from Definition 4.22. Both μ and λ_0 depend only on $(\mathcal{X}, \mathfrak{S})$.

Let γ_0, γ_1 , and γ_2 be λ_0 -hierarchy paths connecting the pairs (x, x') , (x', y') , and (y', y) respectively. Let $\gamma: [a, b] \rightarrow \mathcal{X}$ be the concatenation $\gamma_0 * \gamma_1 * \gamma_2$. We first verify that the path γ satisfies the requirements of the proposition with $\alpha = \gamma_1$ and then verify that γ is in fact a hierarchy path with constant depending only on the HHS $(\mathcal{X}, \mathfrak{S})$.

³The error in the proof of [10, Proposition 5.17] is the incorrect claim that $V \sqsubseteq U \implies \mathbf{P}_V \subseteq \mathbf{P}_U$. The error in the statement is that *all* hierarchy paths have the stated properties instead of there existing at least one hierarchy path with the stated properties.

For the first item, let $z \in \alpha = \gamma_1$. By Lemma 4.23, $\pi_W(\mathbf{g}_{P_U}(z))$ and $\pi_W(z)$ are uniformly close for all $W \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$. If $W \notin \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$, then $\pi_W(x')$, $\pi_W(y')$, and $\pi_W(\mathbf{g}_{P_U}(z))$ are all μ -close to ρ_W^U because x' , y' , and $\mathbf{g}_{P_U}(z)$ are all in P_U . Since $\pi_W \circ \gamma_1$ is an unparametrized λ_0 -quasigeodesic, $\pi_W(z)$ must also be uniformly close to ρ_W^U . Therefore, $d_W(\mathbf{g}_{P_U}(z), z)$ is uniformly bounded for all $W \notin \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$. Since $d_W(\mathbf{g}_{P_U}(z), z)$ is uniformly bounded for all $W \in \mathfrak{S}$, the distance formula (Theorem 4.4) provides $\nu_1 \geq 0$ such that $\gamma_1 \subseteq N_\nu(P_U)$.

For the second item, if $W \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$, then $d_W(x, x')$ and $d_W(y', y)$ are both uniformly bounded by Lemma 4.23. Since $\pi_W \circ \gamma_0$ and $\pi_W \circ \gamma_2$ are unparametrized (λ_0, λ_0) -quasigeodesics, there is a constant $\nu_2 \geq 0$ satisfying the second item.

We prove the third item for $p \in \gamma_0$ as the case $q \in \gamma_2$ is identical. By the second item, $d_W(x, p) \leq \nu_2$ for all $W \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$. Since $d_W(x, \mathbf{g}_{P_U}(x))$ and $d_W(p, \mathbf{g}_{P_U}(p))$ are uniformly bounded for all $W \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$ as well (Lemma 4.23), we have that $d_W(\mathbf{g}_{P_U}(x), \mathbf{g}_{P_U}(p))$ has a bound depending only on $(\mathcal{X}, \mathfrak{S})$ for all $W \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$. If instead $U \sqsubset W$ or $W \pitchfork U$, then $\pi_W(\mathbf{g}_{P_U}(x))$ and $\pi_W(\mathbf{g}_{P_U}(p))$ are both uniformly close to ρ_W^U as they are points in the product region P_U . Hence $d_W(\mathbf{g}_{P_U}(x), \mathbf{g}_{P_U}(p))$ is uniformly bounded for all $W \in \mathfrak{S}$. Thus, the distance formula provides $\nu_3 \geq 0$ depending only on \mathfrak{S} such that $d_{\mathcal{X}}(\mathbf{g}_{P_U}(x), \mathbf{g}_{P_U}(p)) \leq \nu_3$.

Set $\nu = \max\{\nu_1, \nu_2, \nu_3\}$. This depends only on $(\mathcal{X}, \mathfrak{S})$ since each of the ν_i depend only on $(\mathcal{X}, \mathfrak{S})$. It remains to show that γ is a hierarchy path with constant depending only on $(\mathcal{X}, \mathfrak{S})$. For this we need to assume that $d_U(x, y) > 10(E + \kappa_0)$.

We first show that $\pi_W \circ \gamma$ is a uniform unparametrized quasigeodesic for each $W \in \mathfrak{S}$.

- If $W \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$, then $\text{diam}(\pi_W(\gamma_0)) \leq \nu$, $\text{diam}(\pi_W(\gamma_2)) \leq \nu$, and $\pi_W \circ \gamma_1$ is an unparametrized (λ_0, λ_0) -quasigeodesic. Hence $\pi_W \circ \gamma$ is an unparametrized $(\lambda_0, \lambda_0 + 2\nu)$ -quasigeodesic.
- If $U \sqsubset W$, then by the bounded geodesic image axiom (8) any CW -geodesic from $\pi_W(x)$ to $\pi_W(y)$ must intersect the E -neighborhood of ρ_W^U . Since all of $\pi_W \circ \gamma_1$ is contained in $N_{\lambda_0(E+\mu)+\lambda_0}(\rho_W^U)$, the hyperbolicity of CW implies that both of the unparametrized quasigeodesics $\pi_W \circ \gamma_0$ and $\pi_W \circ \gamma_2$ are contained in a regular neighborhood of a CW -geodesic from $\pi_W(x)$ to $\pi_W(y)$. Thus $\pi_W \circ \gamma$ will be a unparametrized quasigeodesic with constants depending on λ_0, μ, E , and δ .
- If $W \pitchfork U$, then since $d_U(x, y) > 10(E + \kappa_0)$, the consistency axiom (5) ensures that at most one of $d_W(x, \rho_W^U)$ and $d_W(y, \rho_W^U)$ are larger than κ_0 . Without loss of

generality, assume $d_W(x, \rho_W^U) \leq \kappa_0$. Since $\pi_W(x')$ and $\pi_W(y')$ are μ -close to ρ_W^U and γ_0 and γ_1 are both λ_0 -hierarchy paths, the diameter of $\pi_W(\gamma_0) \cup \pi_W(\gamma_1)$ is at most

$$2\lambda_0(3E + \mu + \kappa_0) + 2\lambda_0.$$

This makes $\pi_W \circ \gamma$ an unparametrized $(\lambda_0, 2\lambda_0(3E + \mu + \kappa_0) + 3\lambda_0)$ -quasigeodesic.

The above shows that there exists $\lambda' \geq 1$ depending only on $(\mathcal{X}, \mathfrak{S})$ such that $\pi_W \circ \gamma$ is an unparametrized (λ', λ') -quasigeodesic for all $W \in \mathfrak{S}$.

Finally we show that $\gamma: [a, b] \rightarrow \mathcal{X}$ is a quasigeodesic with constants depending only on $(\mathcal{X}, \mathfrak{S})$. Let $t, s \in [a, b]$ and let $u = \gamma(t)$ and $v = \gamma(s)$. Since γ_0, γ_1 and γ_2 are all (λ_0, λ_0) -quasigeodesics, we can assume u and v do not lie in the same γ_i . Without loss of generality we have two cases.

In the first case, $u \in \gamma_0$ and $v \in \gamma_1$. Since $\pi_W \circ \gamma$ is a uniform unparametrized quasigeodesic, there exists $C \geq 1$ such that

$$d_W(u, x') + d_W(x', v) \asymp_{C,C} d_W(u, v)$$

for all $W \in \mathfrak{S}$. By Lemma 4.5, there is a $K \geq 1$ depending only on $(\mathcal{X}, \mathfrak{S})$ such that

$$d_{\mathcal{X}}(u, x') + d_{\mathcal{X}}(x', v) \asymp_{K,K} d_{\mathcal{X}}(u, v),$$

which implies

$$\frac{1}{\lambda_0 K} |t - s| - \frac{2\lambda_0}{K} - K \leq d_{\mathcal{X}}(\gamma(t), \gamma(s)) \leq \lambda_0 |t - s| + 2\lambda_0$$

because γ_0 and γ_1 are (λ_0, λ_0) -quasigeodesics.

The second case is when $u \in \gamma_0$ and $v \in \gamma_2$. The proof is the same as the first case using the fact that

$$d_W(u, x') + d_W(x', y') + d_W(y', v) \asymp d_W(u, v)$$

for all $W \in \mathfrak{S}$ instead. Hence γ is a quasigeodesic with constants depending only on $(\mathcal{X}, \mathfrak{S})$, as desired. □

4.2 Summary of constants

Before continuing we summarize the constants associated to the hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ that we will utilize frequently.

- δ is the hyperbolicity constant of CW for each $W \in \mathfrak{S}$.

- κ_0 is the consistency constant from axiom (5).
- E is the bound on projections in axioms (1), (5) and (8).
- σ_0 is the minimal threshold constant from the distance formula (Theorem 4.4).
- λ_0 is the constant such that any two points in \mathcal{X} can be joined by a λ_0 -hierarchy path (Theorem 4.8).
- χ is the constant from Proposition 4.6 which bounds the cardinality of any subset of \mathfrak{S} that does not contain a pair of transverse domains.
- θ_0 is the constant such that for all $\theta \geq \theta_0$ and $Y \subset \mathcal{X}$, $H_\theta(Y)$ is hierarchically quasiconvex (Lemma 4.13).

We can and shall assume that $E \geq \kappa_0$ and $E \geq \delta$. When we say that a quantity depends on \mathfrak{S} , we mean that it depends on any of the above constants.

5 Constructing hulls with hierarchy paths

In this section, we study hierarchically quasiconvex hulls in hierarchically hyperbolic spaces. The main result is Theorem 5.2 which says that the hierarchically quasiconvex hull can be constructed by iteratively connecting points with hierarchy paths. While our motivation for such a construction is to establish that strongly quasiconvex subsets are hierarchically quasiconvex (Proposition 5.7) we believe it will have many other applications. At the end of the section, we give an example of such an application by characterizing hierarchical quasiconvexity in terms of the coarse median structure on a hierarchically hyperbolic space.

Definition 5.1 (hierarchy path hull) Let Y be a subset of the hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$. Define $\mathcal{P}_\lambda^1(Y)$ to be the union of all λ -hierarchy paths between points in Y . Inductively define $\mathcal{P}_\lambda^n(Y) = \mathcal{P}_\lambda^1(\mathcal{P}_\lambda^{n-1}(Y))$ for all integers $n \geq 2$. For all $\lambda \geq \lambda_0$ and $n \geq 1$, $\mathcal{P}_\lambda^n(Y) \neq \emptyset$.

Theorem 5.2 (constructing hulls using hierarchy paths) Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and $N = 2\chi$, where χ is as in Proposition 4.6. There exist $\bar{\theta} \geq \theta_0$ and $\bar{\lambda} \geq \lambda_0$ depending only on \mathfrak{S} such that for all $\theta \geq \bar{\theta}$, $\lambda \geq \bar{\lambda}$ and $Y \subseteq \mathcal{X}$,

$$d_{\text{Haus}}(\mathcal{P}_\lambda^N(Y), H_\theta(Y)) < D$$

where D depends only on θ , λ , and \mathfrak{S} .

In a recent paper, Bowditch [16] independently constructs hulls in coarse median spaces in a similar manner to the construction in Definition 5.1. Hierarchically hyperbolic spaces are one of the primary examples of coarse median spaces and [16, Lemma 7.3] establishes a version of Theorem 5.2 for finite subsets of hierarchically hyperbolic spaces. At the end of this section we show that Bowditch's coarse median hull is coarsely equal to the hierarchical quasiconvex hull for any subset of an HHS. This is achieved by using Theorem 5.2 to give a new characterization of the hierarchical quasiconvexity in terms of the coarse median structure on a hierarchically hyperbolic space.

The number of iterations of connecting pairs of points by hierarchy paths required by Theorem 5.2 is unlikely to be optimal. However, a simple example illustrates that the number of iteration required must increase with the maximal number of pairwise orthogonal domains. Consider the group \mathbb{Z}^n with the standard HHG structure. Let Y be the union of the positive halves of each of the coordinate axes. The hull $H_\theta(Y)$ then coarsely coincides with the positive orthant of \mathbb{Z}^n , but $\mathcal{P}_\lambda^m(Y)$ coarsely coincides with the set of points in the positive orthant where at most 2^m coordinates are nonzero. Thus, the number of iterations of $\mathcal{P}_\lambda^1(\cdot)$ required to achieve $H_\theta(Y)$ will be approximately $\log(n)$.

For the remainder of this section, let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space and $Y \subseteq \mathcal{X}$. Recall, there exist θ_0 and λ_0 such that for all $\theta \geq \theta_0$, $H_\theta(Y)$ is hierarchically quasiconvex (Lemma 4.13) and any two points in \mathcal{X} can be joined by a λ_0 -hierarchy path (Theorem 4.8).

The following lemma can be found in [10, Proposition 6.4.4] and says for sufficiently large θ , all hierarchically quasiconvex hulls coarsely coincide. We record the proof for completeness.

Lemma 5.3 [10, Proposition 6.4.4] *There exists $\bar{\theta} \geq \theta_0$ depending only on \mathfrak{S} , such that for all $\theta_1, \theta_2 \geq \bar{\theta}$,*

$$d_{\text{Haus}}(H_{\theta_1}(Y), H_{\theta_2}(Y)) \leq D,$$

where D depends on θ_1 and θ_2 .

Proof Without loss of generality, assume $\theta_0 < \bar{\theta} \leq \theta_1 < \theta_2$ with $\bar{\theta}$ to be determined below. By definition, $H_{\theta_1}(Y) \subseteq H_{\theta_2}(Y)$. Let $x \in H_{\theta_2}(Y)$. For each $U \in \mathfrak{S}$, $\pi_U(H_{\theta_0}(Y))$ is K -quasiconvex, where K depends on θ_0 and δ . Let y_U be the closest point projection of $\pi_U(x)$ onto $\pi_U(H_{\theta_0}(Y))$. By Lemma 4.11, there exist $y \in \mathcal{X}$

and θ' depending on θ_0 and \mathfrak{S} such that $d_U(\pi_U(y), y_U) \leq \theta'$. In particular, setting $\bar{\theta} = \theta_0 + \theta'$, we have $y \in H_{\bar{\theta}}(Y) \subseteq H_{\theta_1}(Y)$. To bound $d_{\mathcal{X}}(x, y)$, we will uniformly bound $d_U(x, y_U)$ in terms of θ_2 for every $U \in \mathfrak{S}$; the bound on $d_{\mathcal{X}}(x, y)$ will then follow from the distance formula (Theorem 4.4). By the definition of y_U we have $d_U(x, y_U) \leq d_U(x, \pi_U(H_{\theta_0}(Y))) + 1$. Since $\pi_U(H_{\theta_0}(Y))$ is quasiconvex, contains Y , and is contained in the θ_0 -neighborhood of $\text{hull}_{CU}(Y)$, there exists a D' depending only on \mathfrak{S} such that $\text{hull}_{CU}(Y) \subseteq N_{D'}(\pi_U(H_{\theta_0}(Y)))$. Since $d_U(x, \text{hull}_{CU}(Y)) \leq \theta_2$,

$$d_U(x, y_U) \leq d_U(x, \pi_U(H_{\theta_0}(Y))) + 1 \leq \theta_2 + D' + 1,$$

providing the result. □

For the remainder of this section, $\bar{\theta}$ will denote the constant from Lemma 5.3.

To prove Theorem 5.2 we will show for sufficiently large θ and λ , we can find $\theta' > \theta$ and $\lambda' > \lambda$ such that

$$\mathcal{P}_{\lambda'}^N(Y) \subseteq H_{\theta'}(Y) \quad \text{and} \quad H_{\theta}(Y) \subseteq \mathcal{P}_{\lambda}^N(Y).$$

Theorem 5.2 will then follow by applying Lemma 5.3. The inclusion $\mathcal{P}_{\lambda}^N(Y) \subseteq H_{\theta}(Y)$ is the following direct consequence of hierarchical quasiconvexity.

Lemma 5.4 *For each $\lambda, n \geq 1$, there exists $\theta \geq \bar{\theta}$ such that for any $Y \subseteq \mathcal{X}$,*

$$\mathcal{P}_{\lambda}^n(Y) \subseteq H_{\theta}(Y).$$

Proof The $n = 1$ case follows directly from the definition of $H_{\theta}(Y)$ and hierarchy paths. We can proceed by induction on n and assume there exists $\theta' \geq \bar{\theta}$ such that $\mathcal{P}_{\lambda}^{n-1}(Y) \subseteq H_{\theta'}(Y)$. Let $x \in \mathcal{P}_{\lambda}^n(Y)$. There exist $y_1, y_2 \in \mathcal{P}_{\lambda}^{n-1}(Y)$ such that x is on a λ -hierarchy path from y_1 to y_2 . For each $U \in \mathfrak{S}$, $\pi_U(y_i)$ is within θ' of $\text{hull}_{CU}(Y)$. Therefore, quasiconvexity of $\text{hull}_{CU}(Y)$ in CU guarantees there exists a θ depending only on λ and θ' (which in turn depends on n) such that $\pi_U(x)$ is within θ of $\text{hull}_{CU}(Y)$ and thus $x \in H_{\theta}(Y)$. □

The other inclusion, $H_{\theta}(Y) \subseteq \mathcal{P}_{\lambda}^N(Y)$, requires two main steps. First we prove that if $x \in H_{\theta}(Y)$, then there exists at most $2\chi + 1$ points, x_1, \dots, x_n , in Y such that $x \in H_{\theta'}(x_1, \dots, x_n)$ where θ' depends only on θ (Lemma 5.5). We then show that for any finite collection of points $x_1, \dots, x_n \in \mathcal{X}$, $H_{\theta'}(x_1, \dots, x_n) \subseteq \mathcal{P}_{\lambda}^{n-1}(x_1, \dots, x_n)$ where λ ultimately depends only on n and θ (Proposition 5.6). Together, these imply $H_{\theta}(Y) \subseteq \mathcal{P}_{\lambda}^{2\chi+1}(Y)$.

We start with the first step, which can be thought of a version of Carathéodory’s theorem for HHSs.

Lemma 5.5 *Let $Y \subseteq \mathcal{X}$, $\theta \geq \bar{\theta}$, and χ be as in Proposition 4.6. For each $x \in H_\theta(Y)$, there exist $x_1, \dots, x_{\ell+1} \in Y$, where $1 \leq \ell \leq 2\chi$, and θ' depending only on θ such that $x \in H_{\theta'}(x_1, \dots, x_{\ell+1})$.*

Proof Let $K = 100(E + 2\kappa_0 + \theta)$ and $x \in H_\theta(Y)$. If for all $y \in Y$, $\text{Rel}_K(x, y) = \emptyset$, then $x \in H_K(y)$ for each $y \in Y$. Thus we can assume there is $y \in Y$ such that $\text{Rel}_K(x, y) \neq \emptyset$.

As in Proposition 4.6, we can partition $\text{Rel}_K(x, y)$ in subsets $\mathcal{U}_1, \dots, \mathcal{U}_n$ where $n \leq \chi$. Further, for each i , all the elements of \mathcal{U}_i are pairwise transverse and are totally ordered with respect to the order $U \leq V$ if $d_U(\rho_U^V, y) \leq \kappa_0$. Let $U_{i,1} < \dots < U_{i,k_i}$ be the distinct domains in \mathcal{U}_i . For each i , there exist $a_i, b_i \in Y$ such that $\pi_{U_{i,1}}(x)$ is within θ of the $CU_{i,1}$ geodesic between a_i and b_i . If a_i and b_i project $2\kappa_0 + E$ close to y in $CU_{i,1}$, then $d_{U_{i,1}}(x, y) \leq \theta + 4\kappa_0 + 3E$ which contradicts $U_{i,1} \in \text{Rel}_K(x, y)$. Thus without loss of generality, $d_{U_{i,1}}(a_i, y) > 2\kappa_0 + E$ and in particular $d_{U_{i,1}}(a_i, \rho_{U_{i,1}}^{U_{i,j}}) > \kappa_0$ for all $j > 1$. The total order on \mathcal{U}_i and the consistency axiom (5) ensure that $d_{U_{i,j}}(x, a_i) \leq 2\kappa_0 + E$ for all $1 < j \leq k_i$. Thus for each $U_{i,j}$, x projects $\theta + 2\kappa_0 + E$ close to the $CU_{i,j}$ geodesic between a_i and b_i and $x \in H_K(y, a_1, \dots, a_n, b_1, \dots, b_n)$. □

Armed with Lemma 5.5, the next step is to prove that for a finite set of points, the hierarchical hull is contained in the path hull.

Proposition 5.6 *For each $\theta \geq \bar{\theta}$ and $n \geq 2$, there exists $\lambda \geq 1$ such that*

$$H_\theta(x_1, \dots, x_n) \subseteq \mathcal{P}_\lambda^{n-1}(x_1, \dots, x_n)$$

for any n distinct points $x_1, \dots, x_n \in \mathcal{X}$.

Proof We shall proceed by induction on n . First we will show the base case of $n = 2$.

Claim 1 (base case) *For each $\theta \geq \bar{\theta}$ there exists $\lambda \geq 1$ such that*

$$H_\theta(x, y) \subseteq \mathcal{P}_\lambda^1(x, y)$$

for each $x, y \in \mathcal{X}$.

Proof of Claim 1 Let $z \in H_\theta(x, y)$, $\gamma_0: [a, b] \rightarrow \mathcal{X}$ be a λ_0 -hierarchy path from x to z and $\gamma_1: [b, c] \rightarrow \mathcal{X}$ is a λ_0 -hierarchy path from z to y . We will show that

$\gamma = \gamma_0 * \gamma_1 : [a, c] \rightarrow \mathcal{X}$ is a λ -hierarchy path from x to y , where λ depends only on θ . By the definition of $H_\theta(x, y)$ and hyperbolicity of the CU 's we have that $\pi_U(\gamma)$ is an unparametrized (λ_1, λ_1) -quasigeodesic for each $U \in \mathfrak{S}$, where λ_1 depends only on θ . Therefore, it suffices to show that γ is a (λ, λ) -quasigeodesic in \mathcal{X} , where λ depends only on θ . That is, we need to prove for each $t, s \in [a, c]$,

$$|t - s| \asymp_{\lambda, \lambda} d_{\mathcal{X}}(\gamma(t), \gamma(s)).$$

Since γ_0 and γ_1 are both (λ_0, λ_0) -quasigeodesics, we can restrict ourselves to the case where $t \in [a, b)$ and $s \in (b, c]$. Let $u = \gamma(t)$ and $v = \gamma(s)$. Since $\pi_U(\gamma)$ is a uniform unparametrized quasigeodesic for each $U \in \mathfrak{S}$,

$$d_U(u, z) + d_U(z, v) \asymp_{C, C} d_U(u, v)$$

where $C \geq 1$ depends only on θ . Hence, Lemma 4.5 provides a constant $K \geq 1$ depending only on θ such that

$$d_{\mathcal{X}}(u, z) + d_{\mathcal{X}}(z, v) \asymp_{K, K} d_{\mathcal{X}}(u, v).$$

Since γ_0 and γ_1 are both (λ_0, λ_0) -quasigeodesics,

$$\frac{1}{\lambda_0 K} |t - s| - \frac{2\lambda_0}{K} - k \leq d_{\mathcal{X}}(\gamma(t), \gamma(s)) \leq \lambda_0 |t - s| + 2\lambda_0,$$

as desired. □

We now show the key fact for the inductive step, that the hull of n points can be obtained by taking the hull on $n - 1$ points, and then considering all the hierarchy paths between this smaller hull and the remaining point.

Claim 2 *Let $x_1, \dots, x_n \in \mathcal{X}$, for $n \geq 2$. If $x \in H_\theta(x_1, \dots, x_n)$ where $\theta \geq \bar{\theta}$, then there exist θ' and λ depending only on θ and $y \in H_{\theta'}(x_1, \dots, x_{n-1})$ such that x is on a λ -hierarchy path from x_n to y .*

Proof of Claim 2 For $1 \leq i \leq n$, let $A_i = \{x_1, \dots, x_i\}$. For each $U \in \mathfrak{S}$, $\pi_U(H_\theta(A_{n-1}))$ is R -quasiconvex where R depends only on θ . Let y_U be the closest point projection of $\pi_U(x)$ to $\pi_U(H_\theta(A_{n-1}))$, z_U be a point in $\text{hull}_{CU}(A_n)$ within θ of $\pi_U(x)$, and z'_U be the closest point projection of z_U to $\pi_U(H_\theta(A_{n-1}))$. By Lemma 4.11, there exist $y \in \mathcal{X}$ and a constant θ' depending on θ and δ such that $d_U(\pi_U(y), y_U) \leq \theta'$. Further, we can assume θ' is large enough that

- (1) $\theta' > \theta + \delta + R + 1$;
- (2) $y \in H_{\theta'}(A_{n-1})$;

- (3) for all $v, w \in CU$, if $d_U(v, w) < d_U(v, H_\theta(A_{n-1}))$, then the closest point projection of v and w to $\pi_U(H_\theta(A_{n-1}))$ are no more than θ' apart.

For each $U \in \mathfrak{S}$, let γ_U be a CU geodesic from $\pi_U(x_n)$ to $\pi_U(y)$. We will show that $d_U(x_n, \gamma_U)$ is uniformly bounded for each $U \in \mathfrak{S}$. If $d_U(y_U, z_U) \leq 5\theta'$, then $d_U(x, y_U) \leq 6\theta'$ which implies $d_U(x, \gamma_U) \leq 7\theta'$. Otherwise $d_U(y_U, z_U) > 5\theta'$ implies that $d_U(x, H_\theta(A_{n-1})) > d_U(x, z_U)$ and thus $d_U(y_U, z'_U) \leq \theta'$ by (3). This implies that $d_U(z_U, H_\theta(A_{n-1})) > 3\theta'$. Since $z_U \in \text{hull}_{CU}(A_n)$ and $z_U \notin H_\theta(A_{n-1})$, there exist $D \geq 0$ depending only on θ and $x_U \in \pi_U(A_{n-1})$ such that z_U is within D of any CU geodesic from $\pi_U(x_n)$ to x_U . Further, by increasing θ' , we can assume $D < \theta'$. Take a geodesic triangle with endpoints $\pi_U(x_n)$, y_U and x_U . Since $d_U(z_U, H_\theta(A_{n-1})) > 3\theta'$, it must be the case that z_U is within $2\theta'$ of any CU geodesic from $\pi_U(x_n)$ to y_U .

Thus there exists θ'' depending ultimately only on θ , such that $d_U(x, \gamma_U) \leq \theta''$ for all $U \in \mathfrak{S}$. Therefore $x \in H_{\theta''}(x_n, y)$ and the statement in Claim 2 follows from Claim 1. □

We now finish the proof of Proposition 5.6. Let $x \in H_\theta(x_1, \dots, x_n)$. Claim 2 shows that there exist a $\lambda' \geq 1$ and $\theta' \geq \bar{\theta}$ such that x is on a λ' -hierarchy path from x_n to a point in $H_{\theta'}(x_1, \dots, x_{n-1})$. By induction, there exists $\lambda \geq \lambda'$ such that $H_{\theta'}(x_1, \dots, x_{n-1}) \subseteq \mathcal{P}_\lambda^{n-2}(x_1, \dots, x_{n-1})$ and therefore $x \in \mathcal{P}_\lambda^{n-1}(x_1, \dots, x_n)$. □

We can now finish the proof of Theorem 5.2.

Proof of Theorem 5.2 Recall, we need to show that for all sufficiently large θ and λ , $H_\theta(Y)$ coarsely coincides with $\mathcal{P}_\lambda^N(Y)$ where $N = 2\chi$. First we will show that for all $\theta \geq \bar{\theta}$, there exists $\lambda \geq 1$ such that $H_\theta(Y) \subseteq \mathcal{P}_\lambda^N(Y)$.

Let $x \in H_\theta(Y)$ and let $x_1, \dots, x_{\ell+1}$ be the finite number of points in Y provided by Lemma 5.5. By Proposition 5.6, there exists λ depending on θ such that

$$x \in \mathcal{P}_\lambda^\ell(x_1, \dots, x_{\ell+1}) \subseteq \mathcal{P}_\lambda^\ell(Y) \subseteq \mathcal{P}_\lambda^N(Y).$$

Thus $H_\theta(Y) \subseteq \mathcal{P}_\lambda^N(Y)$.

Now, fix $\bar{\lambda} \geq \lambda_0$ such that $H_{\bar{\theta}}(Y) \subseteq \mathcal{P}_{\bar{\lambda}}^N(Y)$. If $\theta \geq \bar{\theta}$ and $\lambda \geq \bar{\lambda}$, then by Lemma 5.4 there exists $\theta' > \bar{\theta}$ such that

$$H_{\bar{\theta}}(Y) \subseteq \mathcal{P}_{\bar{\lambda}}^N(Y) \subseteq H_{\theta'}(Y).$$

The conclusion now follows by Lemma 5.3. □

The primary use of Theorem 5.2 in this paper is the following proof that hierarchically quasiconvex subsets are exactly the subsets that are “quasiconvex with respect to hierarchy paths”. From this it immediately follows that all strongly quasiconvex subsets are hierarchically quasiconvex.

Proposition 5.7 *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. A subset $Y \subseteq \mathcal{X}$ is k -hierarchically quasiconvex if and only if there exists a function $R: [1, \infty) \rightarrow [0, \infty)$ such that if γ is a λ -hierarchy path with endpoints on Y , then $\gamma \subseteq N_{R(\lambda)}(Y)$ where k and R each determines the other. In particular, if Y is Q -strongly quasiconvex, then Y is k -hierarchically quasiconvex where k is determined by Q .*

Proof The proof of the forward implication follows directly from the definition of hierarchical quasiconvexity and hierarchy path. Assume there exists a function $R: [1, \infty) \rightarrow [0, \infty)$ such that if γ is a λ -hierarchy path with endpoints in Y , then $\gamma \subseteq N_{R(\lambda)}(Y)$. The first condition of hierarchical quasiconvexity now follows from the existence of hierarchy paths (Theorem 4.8), the coarse Lipschitzness of the projection maps (axiom (1)), and the hyperbolicity of the CU 's. For the second condition, observe that the hypothesis implies there exists a bound on the Hausdorff distance between Y and $\mathcal{P}_\lambda^n(Y)$ depending only on R , n , and λ . Thus by Theorem 5.2, for each $\theta \geq \bar{\theta}$, there exists D_θ such that $d_{\text{Haus}}(H_\theta(Y), Y) \leq D_\theta$. Let $\kappa > 0$ and $x \in \mathcal{X}$ such that $d_U(x, Y) \leq \kappa$ for all $U \in \mathfrak{S}$. Thus $x \in H_\theta(Y)$ for each $\theta \geq \kappa + \bar{\theta}$. Let $k(\kappa) = D_{\bar{\theta} + \kappa}$. Then $d_{\mathcal{X}}(x, Y) \leq k(\kappa)$ and Y is hierarchically quasiconvex. \square

Remark 5.8 If \mathcal{X} is a hyperbolic space, there exist many HHS structures on \mathcal{X} ; see [51]. In this case, Proposition 5.7 recovers [51, Proposition 3.5], which states that a subset $Y \subseteq \mathcal{X}$ is quasiconvex if and only if Y is hierarchically quasiconvex in any of the HHS structures on \mathcal{X} .

5.1 Hulls and coarse medians

We now take a small detour from the main thrust of the paper to highlight an application of Theorem 5.2 and discuss the relation of our work in this section to the hulls in coarse median spaces constructed in [16].

In [14], Bowditch axiomatized the notion of a coarse center of three points in a metric space and defined *coarse median spaces* as metric spaces where every triple of points has such a coarse center. Bowditch observed that all hierarchically hyperbolic spaces

are coarse median spaces; see also [10, Theorem 7.3]. The salient property of the coarse median structure of an HHS is the following fact.

Lemma 5.9 (see proof of [10, Theorem 7.3]) *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space. There exist $\mu > 0$ and a map $m: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ with the property that for every $(x, y, z) \in \mathcal{X}^3$ and $U \in \mathfrak{S}$, the projection $\pi_U(m(x, y, z))$ is within μ of all three sides of any CU triangle with vertices $\pi_U(x)$, $\pi_U(y)$ and $\pi_U(z)$.*

We call the point $m(x, y, z)$ the *coarse center* of x , y and z . There is a natural notion of convexity for coarse median spaces, which we formulate in the hierarchically hyperbolic setting as follows.

Definition 5.10 (coarse median quasiconvexity) *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS. A subset Y of \mathcal{X} is said to be Q -median quasiconvex if for every $y, y' \in Y$ and $x \in \mathcal{X}$ we have $m(y, y', x) \in N_Q(Y)$.*

Behrstock, Hagen and Sisto showed that a hierarchically quasiconvex subset is median quasiconvex in [10, Proposition 7.12]. Using Theorem 5.2, we establish the converse.

Proposition 5.11 *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS and $Y \subseteq \mathcal{X}$. Y is k -hierarchically quasiconvex if and only if Y is Q -median quasiconvex where k and Q each determines the other.*

Proof Let Y be a Q -median quasiconvex subset of the HHS $(\mathcal{X}, \mathfrak{S})$ and γ be a λ -hierarchy path with endpoints $y_1, y_2 \in Y$. If $x \in \gamma$, then $d_U(x, m(y_1, y_2, x))$ is uniformly bounded in terms of λ and \mathfrak{S} for each $U \in \mathfrak{S}$. By the distance formula (Theorem 4.4), $d_{\mathcal{X}}(x, m(y_1, y_2, x))$ is also uniformly bounded. Since Y is median quasiconvex, this implies that there exist $R(\lambda)$ such that $d_{\mathcal{X}}(x, Y) \leq R(\lambda)$. In particular, $\gamma \subseteq N_{R(\lambda)}(Y)$ and Y is k -hierarchically quasiconvex, with k determined by Q , by Proposition 5.7. \square

If $Y \subseteq \mathcal{X}$, let $M(Y)$ denote the *coarse median hull* defined in [16, Proposition 6.2]. Proposition 5.11 implies the following corollary that extends [16, Lemma 7.3] in the special case of hierarchically hyperbolic spaces.

Corollary 5.12 *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS and $Y \subseteq \mathcal{X}$. For each $\theta \geq \theta_0$, there exists D depending only on θ and \mathfrak{S} such that*

$$d_{\text{Haus}}(H_{\theta}(Y), M(Y)) \leq D.$$

Proof Let $Y \subseteq \mathcal{X}$ and $\theta \geq \theta_0$. By Proposition 5.11, $H_\theta(Y)$ is Q_1 -median quasiconvex for some Q_1 depending on θ and \mathfrak{S} . By [16, Proposition 6.2] $M(Y)$ is Q_2 -median quasiconvex, where Q_2 depends only on \mathfrak{S} , and there exists D_1 depending on θ such that $M(Y) \subseteq N_{D_1}(H_\theta(Y))$. By Proposition 5.11, $M(Y)$ is k -hierarchically quasiconvex where k depends only on \mathfrak{S} . By the second condition in Definition 4.9, there exists D_2 depending on θ and \mathfrak{S} such that $H_\theta(Y) \subseteq N_{D_2}(M(Y))$. \square

6 Characterization of strongly quasiconvex subsets in HHSs

We now turn our attention to the main objective of this paper, characterizing the strongly quasiconvex subsets of hierarchically hyperbolic spaces. From now on we shall restrict our attention to HHSs with the *bounded domain dichotomy*; a minor regularity condition satisfied by all HHGs as well as Teichmüller space with either the Weil–Peterson or Teichmüller metric and the fundamental groups of 3-manifolds without Nil or Sol components.

Definition 6.1 (bounded domain dichotomy) A hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ has the B -bounded domain dichotomy if there exists $B > 0$ such that for all $U \in \mathfrak{S}$, if $\text{diam}(CU) > B$, then $\text{diam}(CU) = \infty$.

The key to characterizing the strongly quasiconvex subsets of hierarchically hyperbolic spaces is to determine what the projection of a strongly quasiconvex subset to each of the associated hyperbolic spaces looks like. The property that characterizes the projection of strongly quasiconvex subsets is the following orthogonal projection dichotomy.

Definition 6.2 (orthogonal projection dichotomy) For $B \geq 0$, a subset Y of an HHS $(\mathcal{X}, \mathfrak{S})$ has the B -orthogonal projection dichotomy if for all $U, V \in \mathfrak{S}$ with $U \perp V$, if $\text{diam}(\pi_U(Y)) > B$ then $CV \subseteq N_B(\pi_V(Y))$.

From now on, when we consider an HHS with the B_0 -bounded domain dichotomy and a subspace with the B -orthogonal projection dichotomy, we will assume that $B \geq B_0$.

We can now state our characterization of strongly quasiconvex subsets of hierarchically hyperbolic spaces with the bounded domain dichotomy.

Theorem 6.3 (characterization of strong quasiconvexity) *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with the bounded domain dichotomy and $Y \subseteq \mathcal{X}$. Then the following are equivalent:*

- (1) Y is an (A, D) -contracting subset.
- (2) The lower relative divergence of \mathcal{X} with respect to Y is at least quadratic.
- (3) The lower relative divergence of \mathcal{X} with respect to Y is completely superlinear.
- (4) Y is Q -strongly quasiconvex.
- (5) Y is k -hierarchically quasiconvex and has the B -orthogonal projection dichotomy.

Moreover, the pair (A, D) in part (1), the convexity gauge Q in part (4), and the pair (k, B) in part (5) each determine the other two.

The work in Section 3 showed that the implications

$$(1) \implies (2) \implies (3) \implies (4)$$

hold in any quasigeodesic space and that the pair (A, D) determines Q . Further, Proposition 5.7 showed that every Q -strongly quasiconvex subset of a hierarchically hyperbolic space is k -hierarchically quasiconvex with Q determining k . Thus in the next two subsections, we only need to prove:

- If Y is Q -strongly quasiconvex, then there exists $B > 0$ determined by Q such that Y has the B -orthogonal projection dichotomy (Section 6.1).
- If Y is k -hierarchically quasiconvex and has the B -orthogonal projection dichotomy, then Y is (A, D) -contracting where (A, D) is determined by (k, B) (Section 6.2).

Before beginning the proof, we record of the following corollary to Theorem 6.3 that allows us to characterize stable embeddings.

Corollary 6.4 *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with the bounded domain dichotomy and let $i : Y \rightarrow \mathcal{X}$ be a quasi-isometric embedding from a uniform quasigeodesic space Y to \mathcal{X} . The following are equivalent:*

- (1) i is a stable embedding.
- (2) $Z = i(Y)$ is hierarchically quasiconvex and there exists a $B > 0$ such that for all $U, V \in \mathfrak{S}$ with $U \perp V$, if $\text{diam}(\pi_U(Z)) > B$, then $\text{diam}(CV) < B$.

Proof By [11, Corollary 2.16], an HHS $(\mathcal{Z}, \mathfrak{T})$ is hyperbolic if and only if there exists B such that for all $U, V \in \mathfrak{T}$ with $U \perp V$, either $\text{diam}(\pi_U(\mathcal{Z})) < B$ or $\text{diam}(\pi_V(\mathcal{Z})) < B$.

By Proposition 2.8, i is a stable embedding if and only if the image $Z = i(Y)$ is strongly quasiconvex in \mathcal{X} and hyperbolic. The equivalence follows from these observations and the fact that hierarchically quasiconvex subsets inherit the hierarchy structure from the ambient space as described in [10, Proposition 5.6]. \square

Corollary 6.4 should be compared with [1, Corollary 6.2]. If $(\mathcal{X}, \mathfrak{S})$ has the extra assumption of unbounded products required in [1, Corollary 6.2], then Corollary 6.4 can be immediately improved to [1, Corollary 6.2]. However, Corollary 6.4 is a strict expansion of [1, Corollary 6.2] as many HHS structures do not have unbounded products. Naturally occurring HHS structures without unbounded products can be found in right angled Coxeter groups and the Weil–Petersson metric on Teichmüller space. We briefly describe these structures in Section 7.

6.1 Strongly quasiconvex subsets have orthogonal projection dichotomy

In this subsection, we provide the implication (4) to (5) in Theorem 6.3. Our focus will be on studying the following set of domains.

Definition 6.5 Define \mathfrak{S}^* to be the set of domains $U \in \mathfrak{S}$ such that $\text{diam}(CU) = \infty$ and there exists $V \in \mathfrak{S}_U^\perp$ such that $\text{diam}(CV) = \infty$.

For each $U \in \mathfrak{S}^*$ we have that both factors of the product region P_U have infinite diameter. In particular, if $\mathfrak{S}^* = \emptyset$ and \mathfrak{S} has the bounded domain dichotomy, then $(\mathcal{X}, \mathfrak{S})$ is hyperbolic by [11, Corollary 2.16]. Thus the intuition for restricting our attention to these domains is that the domains in \mathfrak{S}^* are the source of nonhyperbolic behavior in $(\mathcal{X}, \mathfrak{S})$.

The crucial step to proving strongly quasiconvex subsets have the orthogonal projection dichotomy is the following proposition that establishes a sort of orthogonal projection dichotomy for the product regions of domains in \mathfrak{S}^* .

Proposition 6.6 Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with the bounded domain dichotomy and $Y \subseteq \mathcal{X}$ be a Q -strongly quasiconvex subset. There is a constant $B_0 > 0$ depending on \mathfrak{S} and Q such that for all $B \geq B_0$ and $U \in \mathfrak{S}^*$,

$$\text{diam}(\pi_U(Y)) > B \implies P_U \subseteq N_B(\mathfrak{g}_{P_U}(Y)).$$

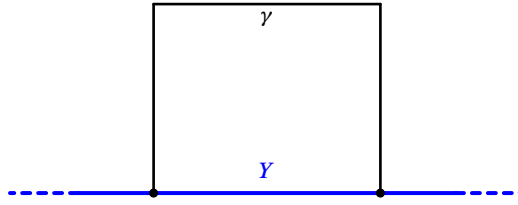


Figure 3: In \mathbb{R}^2 (equipped with the ℓ_1 -metric) consider Y to be the x -axis. Let γ be the $(3, 0)$ -quasigeodesic consisting of three sides of a square with the fourth side on Y . While the quasigeodesic constants do not change, increasing the distance between the endpoints of γ produces points of γ arbitrarily far away from Y .

Since U is in \mathfrak{S}^* , the product region \mathbf{P}_U coarsely coincides with the product of two infinite diameter metric spaces. The proof of Proposition 6.6 is therefore motivated by the situation described in Figure 3. Namely, if Y is a subset of the product of two infinite-diameter metric spaces, then either Y coarsely coincides with the whole product or there exists a quasigeodesic γ with endpoints on Y and fixed constants such that there are points of γ whose distance to Y is comparable to $\text{diam}(Y)$. Thus if Y is Q -strongly quasiconvex, then either Y has bounded diameter or it coarsely covers the entire product.

In Proposition 6.10, we prove that a similar situation holds for \mathbf{P}_U . We show if $\text{diam}(\pi_U(Y))$ is sufficiently large and Y does not coarsely coincide with \mathbf{P}_U , then we can find a uniform constant quasigeodesic with endpoints on $\mathfrak{g}_{\mathbf{P}_U}(Y)$ that contains points relatively far from $\mathfrak{g}_{\mathbf{P}_U}(Y)$. To utilize this to prove Proposition 6.6, we must promote this statement on $\mathfrak{g}_{\mathbf{P}_U}(Y)$ to a statement on Y . Specifically, we show that we can realize every quasigeodesic of \mathbf{P}_U with endpoints on $\mathfrak{g}_{\mathbf{P}_U}(Y)$ as a segment of a quasigeodesic with endpoints on Y , while maintaining uniform quasigeodesic constants (Lemma 6.11). This yields a quasigeodesic with endpoints on Y that contains a point x of \mathbf{P}_U such that $d_{\mathcal{X}}(x, \mathfrak{g}_{\mathbf{P}_U}(Y))$ is comparable with $\text{diam}(\mathfrak{g}_{\mathbf{P}_U}(Y))$. If Y is strongly quasiconvex, the bridge theorem (Theorem 4.18) implies that $d_{\mathcal{X}}(x, \mathfrak{g}_{\mathbf{P}_U}(Y))$ also provides a lower bound on the distance between x and Y . However, since Y is strongly quasiconvex, the distance between x and Y is uniformly bounded. Hence, if Y does not coarsely cover \mathbf{P}_U , we obtain that $\mathfrak{g}_{\mathbf{P}_U}(Y)$ must have bounded diameter which contradicts the assumption on $\text{diam}(\pi_U(Y))$.

We begin by describing a particularly nice class of paths in product spaces and show that they are quasigeodesics (Lemma 6.8).

Definition 6.7 (spiral path) Let X and Y be (K, L) -quasigeodesic metric spaces, and let $Z = X \times Y$ be equipped with the ℓ_1 -metric. A *spiral path* γ in Z is the concatenation $\gamma = \gamma_1 * \dots * \gamma_n$ of (K, L) -quasigeodesic of Z satisfying the following:

- Every γ_i is of the form $\eta \times c_y$ or $c_x \times \delta$ where η (resp. δ) is a (K, L) -quasigeodesic of X (resp. Y) and c_{x_0} (resp. c_{y_0}) is the constant function with value $x_0 \in X$ (resp. $y_0 \in Y$).
- For every i , if γ_i is constant on the X (resp. Y) factor of $Z = X \times Y$, then γ_{i+1} is constant on the Y (resp. X) component of $Z = X \times Y$.

A spiral path $\gamma = \gamma_1 * \dots * \gamma_n$ has *slope* N if for every $i \in \{1, \dots, n - 2\}$,

$$d(\gamma_{i+1}^+, \gamma_{i+1}^-) \geq Nd(\gamma_i^+, \gamma_i^-),$$

where γ_j^\pm are the endpoints of γ_j . Note that the distance between the endpoints of γ_n can be arbitrary.

Lemma 6.8 (spiral paths are quasigeodesics) *For each $K \geq 1$ and $L \geq 0$ there are constants K' and L' such that the following holds. Let X and Y be (K, L) -quasigeodesic metric spaces. If $\gamma = \gamma_1 * \dots * \gamma_n$ is a spiral path of slope $N > 4K^2$ in $Z = X \times Y$ such that the endpoints of γ_1 are at least $3K^2L + 1$ far apart, then γ is a (K', L') -quasigeodesic of $X \times Y$.*

The following proof is essentially the same as showing the logarithmic spiral in \mathbb{R}^2 is a quasigeodesic. However, as we were not able to find a sufficient reference in the literature, we have included it in the interest of completeness.

Proof Let $\gamma = \gamma_1 * \dots * \gamma_n : [a_0, a_n] \rightarrow Z$ be spiral path of slope $N > 4K^2$ and let $a_1 < \dots < a_n$ be points in $[a_0, a_n]$ such that $\gamma_i = \gamma|_{[a_{i-1}, a_i]}$.

Let $t_1, t_2 \in [a_0, a_n]$. We claim that

$$(3) \quad d(\gamma(t_1), \gamma(t_2)) \leq (K + 1)|t_2 - t_1| + 2L.$$

Since each γ_i is a (K, L) -quasigeodesic of Z for each i , we only need to consider the case where $t_1 \in [a_k, a_{k+1}]$ and $t_2 \in [a_j, a_{j+1}]$ with $j - k \geq 1$. By the choice on the distance between endpoints of γ_1 and the slope N ,

$$d(\gamma(a_{i-1}), \gamma(a_i)) > 3K^2L + 1,$$

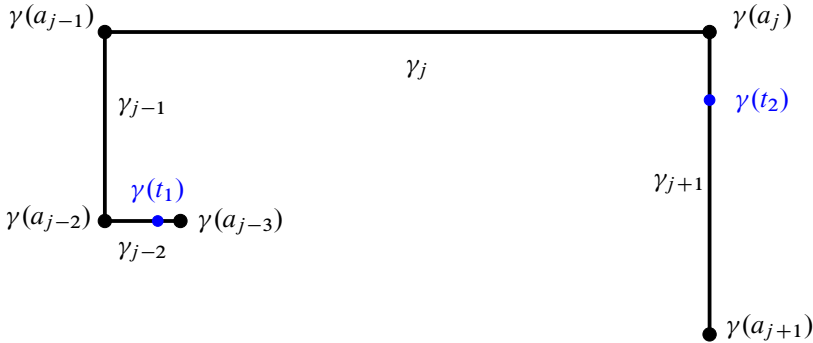


Figure 4

which implies $|a_i - a_{i-1}| > L$. Therefore,

$$|t_2 - t_1| \geq |a_j - a_{k+1}| \geq (j - k - 1)L.$$

Since each γ_i is (K, L) -quasigeodesic,

$$d(\gamma(t_1), \gamma(t_2)) \leq K|t_2 - t_1| + (j - k + 1)L \leq (K + 1)|t_2 - t_1| + 2L.$$

The remainder of the proof will show $|t_2 - t_1| \leq d(\gamma(t_1), \gamma(t_2))$.

For every i , $\gamma_i * \gamma_{i+1}$ is a $(K, 2L)$ -quasigeodesic of Z , so we only need to consider the case where $t_1 \in [a_k, a_{k+1}]$ and $t_2 \in [a_j, a_{j+1}]$ with $j - k \geq 2$ as in Figure 4.

We encourage the reader to refer to Figure 4 as they follow the remainder of the proof.

By the triangle inequality,

$$(4) \quad d(\gamma(t_2), \gamma(t_1)) \geq d(\gamma(t_2), \gamma(a_{j-1})) - d(\gamma(a_{j-1}), \gamma(t_1)).$$

The remainder of the proof has two parts. First we show that, $d(\gamma(t_2), \gamma(a_{j-1}))$ is much larger than $d(\gamma(a_{j-1}), \gamma(t_1))$, so

$$d(\gamma(t_2), \gamma(t_1)) \geq d(\gamma(t_2), \gamma(a_{j-1})) \geq |t_2 - a_{j-1}|.$$

We then finish by showing that $|t_2 - a_{j-1}| \geq |t_2 - t_1|$.

To simplify notation let $\ell(\gamma_i) = d(\gamma(a_{i-1}), \gamma(a_i))$. The slope condition then says $\frac{1}{N}\ell(\gamma_i) > \ell(\gamma_{i-1})$ for each $1 \leq i \leq n - 1$. Since $N > 4K^2$, we can iteratively apply the slope condition to get

$$(5) \quad \sum_{i=1}^{j-1} \ell(\gamma_i) \leq \left(\frac{1}{N^{j-2}} + \dots + \frac{1}{N} + 1 \right) \ell(\gamma_{j-1}) \leq 2\ell(\gamma_{j-1}) \leq \frac{2}{N}\ell(\gamma_j).$$

From the triangle inequality and the fact $|a_{k+1} - t_1| \leq |a_{k+1} - a_k|$,

$$\begin{aligned} d(\gamma(t_1), \gamma(a_{j-1})) &\leq d(\gamma(t_1), \gamma(a_{k+1})) + \sum_{i=k+2}^{j-1} \ell(\gamma_i) \\ &\leq K|a_{k+1} - a_k| + L + \sum_{i=k+2}^{j-1} \ell(\gamma_i) \\ &\leq K(K\ell(\gamma_{k+1}) + KL) + L + \sum_{i=k+2}^{j-1} \ell(\gamma_i) \\ &\leq K^2 \left(\sum_{i=k+1}^{j-1} \ell(\gamma_i) \right) + 2K^2L. \end{aligned}$$

Then by applying inequality (5),

$$d(\gamma(t_1), \gamma(a_{j-1})) \leq \left(\frac{2K^2}{N} \right) \ell(\gamma_j) + 2K^2L \leq \frac{1}{2}d(\gamma(t_2), \gamma(a_{j-1})) + 2K^2L.$$

Substituting this into inequality (4) produces

$$d(\gamma(t_2), \gamma(t_1)) \geq \frac{1}{2}d(\gamma(t_2), \gamma(a_{j-1})) - 2K^2L.$$

We can then use the fact that $\gamma_j * \gamma_{j+1}$ is a $(K, 2L)$ -quasigeodesic to obtain

$$\begin{aligned} (6) \quad d(\gamma(t_2), \gamma(t_1)) &\geq \frac{1}{2}d(\gamma(t_2), \gamma(a_{j-1})) - 2K^2L \\ &\geq \frac{1}{2} \left(\frac{1}{K}|t_2 - a_{j-1}| - 2L \right) - 2K^2L \\ &\geq \frac{1}{2K}|t_2 - a_{j-1}| - 3K^2L. \end{aligned}$$

We now show that $|t_2 - a_{j-1}| \geq |t_2 - t_1|$, which completes the proof by inequality (6). Since we required that $\ell(\gamma_1) > 3K^2L + 1$ and $N > 4K^2$, we have $\frac{1}{K}|a_i - a_{i-1}| > 2L$ for each i . This implies

$$\ell(\gamma_i) \geq \frac{1}{K}|a_i - a_{i-1}| - L > \frac{1}{2K}|a_i - a_{i-1}|.$$

In particular, using inequality (5) we obtain

$$\frac{2}{N}(K|a_j - a_{j-1}| + L) \geq \frac{2}{N}\ell(\gamma_j) \geq \sum_{i=1}^{j-1} \ell(\gamma_i) \geq \sum_{i=1}^{j-1} \frac{1}{2K}|a_i - a_{i-1}| \geq \frac{1}{2K}|a_{j-1} - t_1|.$$

Hence,

$$|a_{j-1} - t_1| \leq \frac{4K^2}{N}|a_j - a_{j-1}| + \frac{4KL}{N} \leq |a_j - a_{j-1}| + L$$

and we can conclude

$$\begin{aligned} |t_2 - t_1| &= |t_2 - a_j| + |a_j - a_{j-1}| + |a_{j-1} - t_1| \\ &\leq |t_2 - a_j| + 2|a_j - a_{j-1}| + L \\ &\leq 3|t_2 - a_{j-1}| + L. \end{aligned}$$

Combining this with inequalities (3) and (6), we obtain that there are constants K' and L' depending on K and L such that

$$\frac{1}{K'}(t_2 - t_1) - L' \leq d(\gamma(t_2), \gamma(t_1)) \leq K'(t_2 - t_1) + L'. \quad \square$$

For the remainder of this section $(\mathcal{X}, \mathfrak{S})$ will be an HHS with the bounded domain dichotomy and \mathfrak{S}^* is as in Definition 6.5. Recall, for each $U \in \mathfrak{S}$, the space $F_U \times E_U$ consists of tuples $a = (a_V)$, where $V \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$, and that P_U is defined as the image of $\phi_U : F_U \times E_U \rightarrow \mathcal{X}$. By restricting to a choice of factor, we can endow F_U and E_U with the subspace metric of their images under ϕ_U . While this relies on the choice of factor, the distance formula (Theorem 4.4) says any two choices result in uniformly quasi-isometric metric spaces. Given $a, b \in F_U \times E_U$ we use $d_V(a, b)$ to denote $d_V(a_V, b_V)$, where $V \in \mathfrak{S}_U \cup \mathfrak{S}_U^\perp$. If $U \in \mathfrak{S}^*$, then both F_U and E_U are infinite diameter and so we can apply Proposition 6.9 to build the desired quasigeodesic in P_U based on $\mathfrak{g}_{P_U}(Y)$.

Proposition 6.9 *Let $Y \subseteq \mathcal{X}$. There exist constants L' and r_0 , and functions*

$$f, g, h : [r_0, \infty) \rightarrow [0, \infty),$$

all depending only on \mathfrak{S} , such that $f(r), g(r), h(r) \rightarrow \infty$ as $r \rightarrow \infty$ and the following holds: for each $U \in \mathfrak{S}^$ and each $r \geq r_0$, if the r -neighborhood of $\phi_U^{-1}(\mathfrak{g}_{P_U}(Y))$ does not cover $F_U \times E_U$ and $\text{diam}(\pi_U(Y)) > f(r)$, then there exists a (L', L') -quasigeodesic η with endpoints $a, b \in \phi_U^{-1}(\mathfrak{g}_{P_U}(Y))$ such that η is not contained in the $g(r)$ -neighborhood of $\phi_U^{-1}(\mathfrak{g}_{P_U}(Y))$ and $d_U(a, b) > h(r)$.*

Proof Our approach is to construct a spiral path of sufficient slope in $F_U \times E_U$ and then apply Lemma 6.8 to conclude it is a quasigeodesic. Let $d(\cdot, \cdot)$ denote the ℓ_1 -distance in $F_U \times E_U$ and fix the following constants, which depend only on \mathfrak{S} :

- L such that F_U and E_U are (L, L) -quasigeodesic spaces.
- K such that π_U is (K, K) -coarsely Lipschitz.
- $N = 4L^2 + 1$ will be the slope of the spiral path we construct.

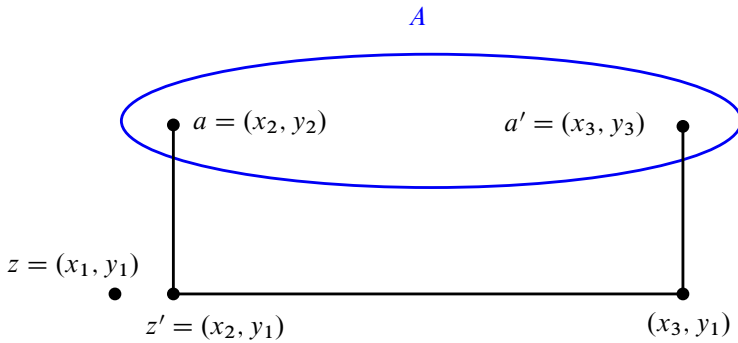


Figure 5: Spiral path constructed when $d_{F_U}(x_1, x_2) \leq \frac{1}{2}(r + 2L + 1)$.

Let $r > 10L^3 + 6$ and $A = \phi_U^{-1}(\mathfrak{g}_{P_U}(Y))$. Suppose that the r -neighborhood of A does not cover $F_U \times E_U$. Thus there exists a point $z = (x_1, y_1) \in F_U \times E_U$ such that $r \leq d(z, A) \leq r + 2L$. Let $a = (x_2, y_2)$ be a point of A such that $d(z, a) - 1 \leq d(z, A)$. We have $\min\{d_{F_U}(x_1, x_2), d_{E_U}(y_1, y_2)\} \leq \frac{1}{2}(r + 2L + 1)$. There are two cases depending on which of the two factors realizes the minimum.

Case 1 If $d_{F_U}(x_1, x_2)$ realizes the minimum, let $z' = (x_2, y_1)$ and $D_r = \frac{1}{2}(r - 2L - 1)$. Then $d(z', A) \geq d(z, A) - d(z, z') \geq D_r$, which implies $d(z', a) > 3L^3 + 1$ because $r > 10L^3 + 6$.

There exists $B_r > r$ such that for any pair of points u and v of F_U , if $d_U(u, v) \geq B_r$ then

$$d_{F_U}(u, v) \geq 2(r + 2L + 1)N.$$

We shall assume $\text{diam}(\pi_U(Y)) > 2B_r$, so there is a point $a' = (x_3, y_3)$ of A such that $d_U(x_2, x_3) \geq B_r$ and $d_{F_U}(x_2, x_3) > d_{E_U}(y_2, y_1)N$. We can now form a spiral path η of slope $N = 4L^2 + 1$ by connecting each sequential pair of points in the sequence

$$a = (x_2, y_2) - (x_2, y_1) - (x_3, y_1) - (x_3, y_3) = a'$$

by (L, L) -quasigeodesics. Since $d_{E_U}(y_2, y_1) > 3L^3 + 1$, η satisfies the hypothesis of Lemma 6.8 and is therefore an (L', L') -quasigeodesic for some L' determined by L .

Since $z' = (x_2, y_1)$ is at least D_r far from A , η has endpoints in A and is not contained in the D_r -neighborhood of A . Moreover, $d_U(a, a') \geq B_r$ and we get the claim with $f(r) = 2B_r$, $g(r) = D_r$, and $h(r) = B_r$

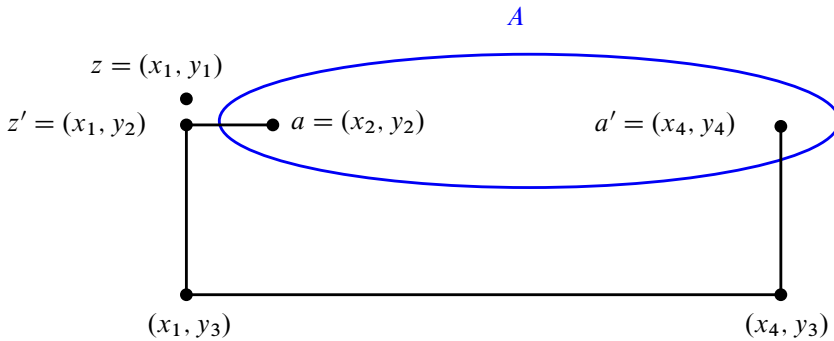


Figure 6: Spiral path constructed when $d_{E_U}(y_1, y_2) \leq \frac{1}{2}(r + 2L + 1)$.

Case 2 If $d_{E_U}(y_1, y_2)$ realizes the minimum, let $z' = (x_1, y_2)$. As before we have that $d(z', A) \geq D_r = \frac{1}{2}(r - 2L - 1)$, which implies $d(z', a) > 3L^3 + 1$. Let y_3 be a point of E_U such that

$$(r + 2L + 1)N \leq d_{E_U}(y_2, y_3) \leq 2(r + 2L + 1)N.$$

There exists $C_r > r$ such that for any pair of points u and v of F_U , if $d_U(u, v) \geq C_r$ then

$$d_{F_U}(u, v) \geq 2(r + 2L + 1)N^2.$$

We shall assume $\text{diam}(\pi_U(Y)) > 2C_r$, so there exists $a' = (x_4, y_4) \in A$ such that $d_U(x_1, x_4) > C_r$. This implies $d_{F_U}(x_1, x_4) > 2(r + 2L + 1)N^2$ and we can now form a spiral path η of slope $N = 4L^2 + 1$ by connecting each sequential pair of points in the sequence

$$a = (x_2, y_2) - (x_1, y_2) - (x_1, y_3) - (x_4, y_3) - (x_4, y_4) = a'$$

by an (L, L) -quasigeodesics.

As before, η satisfies the hypothesis of Lemma 6.8 and is therefore an (L', L') -quasigeodesic for some L' determined by L . The remaining claims follow as in the preceding case. \square

The distance formula makes the map $\phi_U: F_U \times E_U \rightarrow \mathcal{X}$ a uniform quasi-isometric embedding. Thus $\mathfrak{g}_{P_U}(Y)$ coarsely covers P_U if and only if $\phi_U^{-1}(\mathfrak{g}_{P_U}(Y))$ coarsely covers $F_U \times E_U$, Proposition 6.9 therefore allows us to immediately deduce the following result for $P_U \subseteq \mathcal{X}$.

Proposition 6.10 *Let $Y \subseteq \mathcal{X}$. There exist constants L' and r_0 , and functions*

$$f, g, h: [r_0, \infty) \rightarrow [0, \infty),$$

all depending only on \mathfrak{S} , such that $f(r), g(r), h(r) \rightarrow \infty$ as $r \rightarrow \infty$ and the following holds: for each $U \in \mathfrak{S}^$ and each $r \geq r_0$, if the r -neighborhood of $\mathfrak{g}_{\mathbf{P}_U}(Y)$ does not cover \mathbf{P}_U and $\text{diam}(\pi_U(Y)) > f(r)$, then there exists an (L', L') -quasigeodesic η with endpoints $a, b \in \mathfrak{g}_{\mathbf{P}_U}(Y)$ such that*

- (1) $\eta \subseteq \mathbf{P}_U$,
- (2) η is not contained in the $g(r)$ -neighborhood of $\mathfrak{g}_{\mathbf{P}_U}(Y)$,
- (3) $d_U(a, b) > h(r)$.

Proposition 6.10 furnishes a quasigeodesic η with endpoints in $\mathfrak{g}_{\mathbf{P}_U}(Y)$ that can be made as far from $\mathfrak{g}_{\mathbf{P}_U}(Y)$ as desired by increasing $\text{diam}(\pi_U(Y))$. However, to exploit the fact that Y is a strongly quasiconvex subset, we need the next lemma, which “promotes” η to a quasigeodesic with endpoints in Y .

Lemma 6.11 *There exists $D > 0$ such that if $x, y \in \mathcal{X}$ and $U \in \mathfrak{S}$, with $d_U(x, y) > D$ and η is a (k, c) -quasigeodesic contained in \mathbf{P}_U with endpoints $\mathfrak{g}_{\mathbf{P}_U}(x)$ and $\mathfrak{g}_{\mathbf{P}_U}(y)$, then there exists a (k', c') -quasigeodesic containing η and with endpoints x and y , where k' and c' depend only on λ and ϵ .*

Proof Let D and λ be as in Proposition 4.24. We further assume λ is large enough that every pair of points in \mathcal{X} can be joined by a λ -hierarchy path (Theorem 4.8).

Assume $d_U(x, y) > D$ and let $\tilde{\gamma}$ be the λ -hierarchy path connecting x and y provided by Proposition 4.24. Let α be the active subpath of $\tilde{\gamma}$ corresponding to U . Define x' (resp. y') to be the endpoint of α closest to x (resp. y) and $x'' = \mathfrak{g}_{\mathbf{P}_U}(x)$ (resp. $y'' = \mathfrak{g}_{\mathbf{P}_U}(y)$). If $\eta: [b, c] \rightarrow \mathbf{P}_U$ is any (k, c) -quasigeodesic in \mathbf{P}_U connecting x'' and y'' , let γ be the concatenation of $\tilde{\gamma} - \alpha$, any λ -hierarchy path from x' to x'' , η , and any λ -hierarchy path from y' to y'' . We will show that this path γ is a (k', c') -quasigeodesic where the constants depend only on k and c .

The distances $d_{\mathcal{X}}(x', \mathbf{P}_U)$ and $d_{\mathcal{X}}(y', \mathbf{P}_U)$ are uniformly bounded by Proposition 4.24. By Lemma 4.15, the distances $d_{\mathcal{X}}(x', \mathfrak{g}_{\mathbf{P}_U}(x'))$ and $d_{\mathcal{X}}(y', \mathfrak{g}_{\mathbf{P}_U}(y'))$ are uniformly bounded as well. Again by Proposition 4.24, $\mathfrak{g}_{\mathbf{P}_U}(x)$ coarsely coincides with $\mathfrak{g}_{\mathbf{P}_U}(x')$ and $\mathfrak{g}_{\mathbf{P}_U}(y)$ coarsely coincides with $\mathfrak{g}_{\mathbf{P}_U}(y')$. Thus there exists μ depending only on \mathfrak{S} such that $d_{\mathcal{X}}(x', x''), d_{\mathcal{X}}(y', y'') \leq \mu$.

Now, let γ_x (resp. γ_y) be the subset of γ from x to x'' (resp. y to y''). Since $d_{\mathcal{X}}(x', x'')$ and $d_{\mathcal{X}}(y', y'')$ are uniformly bounded by μ , γ_x and γ_y are both uniform quasigeodesics. By Lemma 4.15 and Proposition 4.24, there exists $K \geq 1$ depending on k, c , and \mathfrak{G} such that

- $d_{\mathcal{X}}(x', x''), d_{\mathcal{X}}(y', y'') \leq K$;
- $\text{diam}(\mathfrak{g}_{\mathbf{P}_U}(\gamma_x)), \text{diam}(\mathfrak{g}_{\mathbf{P}_U}(\gamma_y)) \leq K$;
- γ_x, γ_y and η are all (K, K) -quasigeodesics;
- for all $p \in \mathbf{P}_U$ and $q \in \mathcal{X}$, $d_{\mathcal{X}}(q, \mathfrak{g}_{\mathbf{P}_U}(q)) \leq Kd_{\mathcal{X}}(p, q) + K$.

Let $\gamma = \gamma_x * \eta * \gamma_y : [a, d] \rightarrow \mathcal{X}$ and $a < b < c < d$ such that $\gamma|_{[a,b]} = \gamma_x$, $\gamma|_{[b,c]} = \eta$ and $\gamma|_{[c,d]} = \gamma_y$. For $t, s \in [a, d]$, let $u = \gamma(t)$, $v = \gamma(s)$. We want to show $|t - s| \asymp d_{\mathcal{X}}(u, v)$ for some constants depending only on K . The only interesting cases are when u and v are in different components of $\gamma = \gamma_x * \eta * \gamma_y$, so without loss of generality, we have the following two cases.

Case 1 Assume $t \in [a, b]$ and $s \in [b, c]$. Thus $u \in \gamma_x$ and $v \in \eta$, and

$$d_{\mathcal{X}}(u, v) \leq d_{\mathcal{X}}(u, x'') + d_{\mathcal{X}}(x'', v) \leq K|t - b| + K|b - s| + 2K \leq K|t - s| + 2K.$$

For the inequality $|t - s| \leq d_{\mathcal{X}}(u, v)$, our choice of K provides

$$d_{\mathcal{X}}(u, x'') \leq d_{\mathcal{X}}(u, \mathfrak{g}_{\mathbf{P}_U}(u)) + K \leq Kd_{\mathcal{X}}(u, v) + 2K.$$

By the triangle inequality $d_{\mathcal{X}}(v, x'') \leq d_{\mathcal{X}}(v, u) + d_{\mathcal{X}}(u, x'')$ and we derive the desired inequality as

$$\begin{aligned} |t - s| &= |t - b| + |b - s| \\ &\leq Kd_{\mathcal{X}}(u, x'') + Kd_{\mathcal{X}}(v, x'') + 2K \\ &\leq K^2d_{\mathcal{X}}(u, v) + K(d_{\mathcal{X}}(u, v) + d_{\mathcal{X}}(u, x'')) + 2K^2 + 2K \\ &\leq K^2d_{\mathcal{X}}(u, v) + Kd_{\mathcal{X}}(u, v) + K^2d_{\mathcal{X}}(u, v) + 4K^2 + 2K \\ &\leq 3K^2d_{\mathcal{X}}(u, v) + 6K^2. \end{aligned}$$

Case 2 Assume $t \in [a, b]$ and $s \in [c, d]$ so that $u \in \gamma_x$ and $v \in \gamma_y$. Further we can assume $u, v \in \tilde{\gamma}$, since otherwise the above proof holds by increasing the constants by $4K$. The inequality $d_{\mathcal{X}}(u, v) \leq |t - s|$ can be established by a nearly identical argument to the previous case. For the inequality $|t - s| \leq d_{\mathcal{X}}(u, v)$ we need to utilize the fact that $\tilde{\gamma}$ is a (λ_0, λ_0) -quasigeodesic. Thus, by increasing K , we can ensure that

- $d_{\mathcal{X}}(u, v) \asymp_{K,K} d_{\mathcal{X}}(u, x') + d_{\mathcal{X}}(x', y') + d_{\mathcal{X}}(y', v)$,

- $d_{\mathcal{X}}(x', y') \asymp_{1,2K} d_{\mathcal{X}}(x'', y'') \asymp_{K,K} |b - c|$,
- $d_{\mathcal{X}}(u, x') \asymp_{1,K} d_{\mathcal{X}}(u, x'') \asymp_{K,K} |t - b|$,
- $d_{\mathcal{X}}(v, y') \asymp_{1,K} d_{\mathcal{X}}(v, y'') \asymp_{K,K} |c - s|$.

We then have the calculation

$$\begin{aligned} |t - s| &= |t - b| + |b - c| + |c - s| \\ &\leq Kd_{\mathcal{X}}(u, x'') + Kd_{\mathcal{X}}(x'', y'') + Kd_{\mathcal{X}}(y'', v) + 3K \\ &\leq Kd_{\mathcal{X}}(u, x') + Kd_{\mathcal{X}}(x', y') + Kd_{\mathcal{X}}(y', v) + 7K^2 \\ &\leq K^2 d_{\mathcal{X}}(u, v) + 8K^2. \end{aligned} \quad \square$$

We can now provide the proof of Proposition 6.6.

Proof of Proposition 6.6 Let $Y \subseteq \mathcal{X}$ be Q -strongly quasiconvex and $U \in \mathfrak{S}$ such that $\text{diam}(CU) = \infty$ and there exists $V \in \mathfrak{S}_U^\perp$ with $\text{diam}(CV) = \infty$. Recall our goal is to show that there exists B depending on \mathfrak{S} and Q such that if $\text{diam}(\pi_U(Y)) > B$, then $\mathbf{P}_U \subseteq N_B(\mathfrak{g}_{\mathbf{P}_U}(Y))$. Begin by fixing the following constants that all depend only on \mathfrak{S} and Q :

- μ such that for all $x \in \mathcal{X}$, $d_U(x, \mathfrak{g}_{\mathbf{P}_U}(x)) < \mu$.
- D , the constant from Lemma 6.11.
- L' , the quasigeodesic constant from Proposition 6.10.
- k' , the quasigeodesic constant obtained by applying Lemma 6.11 to a (L', L') -quasigeodesic.
- K , the constant from the bridge theorem (Theorem 4.18) for Y and \mathbf{P}_U (recall Y is hierarchically quasiconvex by Proposition 5.7).

Let f, g and h be as in Proposition 6.10 and fix r be large enough that

$$g(r) > 2KQ(k', k') + K^2 + K \quad \text{and} \quad h(r) > D + 2\mu.$$

If $\mathbf{P}_U \subseteq N_r(\mathfrak{g}_{\mathbf{P}_U}(Y))$, then we are done. So for the purposes of contradiction, suppose that $\mathbf{P}_U \not\subseteq N_r(\mathfrak{g}_{\mathbf{P}_U}(Y))$ and that $\text{diam}(\pi_U(Y)) > f(r)$. Let η be the (L', L') -quasigeodesic provided by Proposition 6.10 and let $a_1, b_1 \in \mathfrak{g}_{\mathbf{P}_U}(Y)$ be the endpoints of η . Let $a_0, b_0 \in Y$ such that $\mathfrak{g}_{\mathbf{P}_U}(a_0) = a_1$ and $\mathfrak{g}_{\mathbf{P}_U}(b_0) = b_1$. Since

$$d_U(a_0, b_0) > d_U(a_1, b_1) - 2\mu > h(r) - 2\mu > D,$$

Lemma 6.11 produces a (k', k') -quasigeodesic γ with endpoints a_0 and b_0 and containing η where k' depending ultimately only on \mathfrak{S} . Since Y is Q -strongly quasiconvex, $\gamma \subseteq N_{Q(k', k')}(Y)$. By Proposition 6.10, there exists $x \in \eta$ such that $d_{\mathcal{X}}(x, \mathfrak{g}_{P_U}(Y)) > g(r)$. Let $y \in Y$ be such that $d_{\mathcal{X}}(x, y) - 1 \leq d_{\mathcal{X}}(x, Y)$. Then by the bridge theorem (Theorem 4.18) we have a contradiction,

$$Q(k', k') \geq d_{\mathcal{X}}(x, y) - 1 \geq \frac{1}{K} d_{\mathcal{X}}(x, \mathfrak{g}_{P_U}(Y)) - K - 1 > 2Q(k', k'). \quad \square$$

The following proposition uses Proposition 6.6 to finish the proof of the implication from (4) to (5) in Theorem 6.3.

Proposition 6.12 *If $(\mathcal{X}, \mathfrak{S})$ is an HHS with the bounded domain dichotomy and Y is a Q -strongly quasiconvex subset of \mathcal{X} , then there exists $B > 0$ depending only on Q and \mathfrak{S} such that Y has the B -orthogonal projection dichotomy.*

Proof Let $Y \subseteq \mathcal{X}$ be Q -strongly quasiconvex and $B' > 0$ be larger than the bounded domain dichotomy constant for \mathfrak{S} and the constant B_0 from Proposition 6.6. Let $U \in \mathfrak{S}$. If $U \notin \mathfrak{S}^*$, then by the bounded domain dichotomy, either $\text{diam}(CU) < B'$ or for all $V \in \mathfrak{S}_U^\perp$, $\text{diam}(CV) < B'$. In either case, the B' -orthogonal projection dichotomy is satisfied for U . Thus we can assume that $U \in \mathfrak{S}^*$, so $\text{diam}(CU) = \infty$ and there exists $V \in \mathfrak{S}_U^\perp$ with $\text{diam}(CV) = \infty$. Suppose $\text{diam}(\pi_U(Y)) > B'$. By Proposition 6.6, $P_U \subseteq N_{B'}(\mathfrak{g}_{P_U}(Y))$. For all $V \in \mathfrak{S}_U^\perp$, $\pi_V(P_U)$ uniformly coarsely covers CV , thus there exists $B \geq B'$ depending only on Q and \mathfrak{S} such that $CV \subseteq N_B(\pi_V(Y))$. \square

6.2 Contracting subsets in HHSs

We now finish the proof of Theorem 6.3 by showing that for hierarchically quasiconvex subsets, the orthogonal projection dichotomy implies that the gate map \mathfrak{g}_Y is contracting.

Proposition 6.13 *Let $(\mathcal{X}, \mathfrak{S})$ be a hierarchically hyperbolic space with the bounded domain dichotomy and $Y \subseteq \mathcal{X}$ be k -hierarchically quasiconvex. If Y has the B -orthogonal projection dichotomy, then the gate map $\mathfrak{g}_Y : \mathcal{X} \rightarrow Y$ is (A, D) -contracting, where A and D depend only on k, B , and \mathfrak{S} .*

Proof The gate map satisfies the first two condition in the definition of a contracting map by Lemma 4.14. It only remains to prove: *there exist $0 < A < 1$ and $D \geq 1$ depending only on k, B , and \mathfrak{S} , such that for all $x \in \mathcal{X}$, $\text{diam}(\mathfrak{g}_Y(B_R(x))) \leq D$ where $R = Ad(x, Y)$.*

Fix a point $x_0 \in \mathcal{X}$ with $d_{\mathcal{X}}(x_0, Y) \geq C_0$ and let $x \in \mathcal{X}$ be any point with

$$d_{\mathcal{X}}(x_0, x) < C_1 d_{\mathcal{X}}(x_0, Y)$$

for constants C_0 and C_1 to be determined below. We will prove that for each domain $U \in \mathfrak{S}$ the distance $d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x))$ is uniformly bounded, then the above will follow from the distance formula (Theorem 4.4).

We choose a “large” number L (we will clarify how large L is later). Let $K \geq 1$ be the coarse equality constant from the distance formula with thresholds L and $2L$. Take $C_0 > (2K + 1)K$ sufficiently large so there is $W \in \mathfrak{S}$ such that $d_W(x_0, \mathfrak{g}_Y(x_0)) > 2L$. Choose $C_1 < 1/(2K^2 + 1)$, ensuring that $d_{\mathcal{X}}(x_0, \mathfrak{g}_Y(x_0)) > (2K^2 + 1)d_{\mathcal{X}}(x_0, x)$. If $d_{\mathcal{X}}(x_0, x) \leq C_0$, then by the coarse Lipschitzness of the projections $d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x))$ is uniformly bounded by a number depending on C_0 for each $U \in \mathfrak{S}$. Therefore, we can assume that $d_{\mathcal{X}}(x_0, x) > C_0$. We claim that there is a $V \in \mathfrak{S}$ such that $d_V(x_0, \mathfrak{g}_Y(x_0)) > d_V(x_0, x) + L$.

Assume for the purpose of contradiction that $d_W(x_0, \mathfrak{g}_Y(x_0)) \leq d_W(x_0, x) + L$ for all $W \in \mathfrak{S}$. Therefore, $d_W(x_0, \mathfrak{g}_Y(x_0)) \geq 2L \implies d_W(x_0, x) \geq L$ and this implies

$$\{\{d_W(x_0, \mathfrak{g}_Y(x_0))\}\}_{2L} \leq 2\{\{d_W(x_0, x)\}\}_L$$

for all $W \in \mathfrak{S}$. Thus,

$$\begin{aligned} d_{\mathcal{X}}(x_0, \mathfrak{g}_Y(x_0)) &\leq K \sum_{W \in \mathfrak{S}} \{\{d_W(x_0, \mathfrak{g}_Y(x_0))\}\}_{2L} + K \\ &\leq 2K \sum_{W \in \mathfrak{S}} \{\{d_W(x_0, x)\}\}_L + K \\ &\leq 2K(Kd_{\mathcal{X}}(x_0, x) + K) + K \\ &\leq 2K^2 d_{\mathcal{X}}(x_0, x) + (2K + 1)K \\ &\leq 2K^2 d_{\mathcal{X}}(x_0, x) + C_0 \\ &\leq (2K^2 + 1)d_{\mathcal{X}}(x_0, x) \end{aligned}$$

which contradicts $C_1 < 1/(2K^2 + 1)$. Therefore, we can fix $V \in \mathfrak{S}$ such that

$$d_V(x_0, \mathfrak{g}_Y(x_0)) > d_V(x_0, x) + L.$$

The construction of the gate map and the hyperbolicity of CV ensure that, after enlarging L and shrinking C_1 if necessary, $d_V(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x)) < r$ where r depends only on k and \mathfrak{S} . The triangle inequality then gives us

$$d_V(x, \mathfrak{g}_Y(x_0)) > L \quad \text{and} \quad d_V(x, \mathfrak{g}_Y(x)) > L - r.$$

Now let $U \in \mathfrak{S}$. If $\text{diam}(\pi_U(Y)) \leq B$, then $d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x)) \leq B$ and we are done. Thus we can assume that $\text{diam}(\pi_U(Y)) > B$. If $U = V$, then the distance $d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x))$ is uniformly bounded above by the number r and we are done. We now consider the other possible cases depending on the relation between U and V .

Case 1 Suppose $V \sqsubseteq U$. If we choose L greater than $E + r$, then

$$d_V(x_0, \mathfrak{g}_Y(x_0)) > E \quad \text{and} \quad d_V(x, \mathfrak{g}_Y(x)) > E.$$

Thus by the bounded geodesic image axiom (8), the CU geodesics from $\pi_U(x_0)$ to $\pi_U(\mathfrak{g}_Y(x_0))$ and from $\pi_U(x)$ to $\pi_U(\mathfrak{g}_Y(x))$ must intersect $N_E(\rho_U^V)$. Therefore, the distance $d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x))$ is uniformly bounded due to the hyperbolicity of CU and the properties of the gate map (Lemma 4.14).

Case 2 Suppose $U \sqsubseteq V$. If some CV geodesic from $\pi_V(\mathfrak{g}_Y(x_0))$ to $\pi_V(\mathfrak{g}_Y(x))$ stays E -far from ρ_V^U , then by the bounded geodesic image axiom (8), $d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x)) \leq E$ and we are done. Therefore, we assume that all CV geodesics from $\pi_V(\mathfrak{g}_Y(x_0))$ to $\pi_V(\mathfrak{g}_Y(x))$ intersect $N_E(\rho_V^U)$. Since $d_V(x_0, \mathfrak{g}_Y(x_0)) > d_V(x_0, x) + L$, if there was also a CV geodesic from $\pi_V(x_0)$ to $\pi_V(x)$ that intersected $N_E(\rho_V^U)$ we would have

$$\begin{aligned} d_V(\mathfrak{g}_Y(x_0), \rho_V^U) &\geq d_V(\mathfrak{g}_Y(x_0), x_0) - d_V(x_0, \rho_V^U) \\ &> d_V(\mathfrak{g}_Y(x_0), x_0) - d_V(x_0, x) - 2E \\ &\geq L - 2E. \end{aligned}$$

However, $d_V(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x)) \leq r$ which implies $\pi_V(\mathfrak{g}_Y(x_0))$ lies in $N_{E+r}(\rho_V^U)$. Therefore, by assuming $L > 4E + r$ we can ensure that no CV geodesic from $\pi_V(x_0)$ to $\pi_V(x)$ intersects $N_E(\rho_V^U)$. Thus $d_U(x_0, x) < E$ by the bounded geodesic image axiom and it immediately follows that $d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x))$ is bounded by a constant depending on k and \mathfrak{S} .

Case 3 Suppose $U \not\sqsubseteq V$ and $V \not\sqsubseteq U$. Recall that we can assume $\text{diam}(\pi_U(Y)) > B$. Thus if $U \perp V$, we have $CV \subseteq N_B(\pi_V(Y))$ by the orthogonal projection dichotomy. However $d_V(x_0, \mathfrak{g}_Y(x_0)) > L$, so by Lemma 4.15 we can choose L large enough such that $\pi_V(x_0)$ does not lie in the B -neighborhood of $\pi_V(Y)$. Thus U and V cannot be orthogonal and hence $U \pitchfork V$.

Now assume $L > 2\kappa_0 + 3r + 2E + 1$. Then if $d_V(\mathfrak{g}_Y(x_0), \rho_V^U) \leq \kappa_0 + r + E$,

$$d_V(x_0, \rho_V^U) \geq d_V(x_0, \mathfrak{g}_Y(x_0)) - d_V(\mathfrak{g}_Y(x_0), \rho_V^U) - E \geq L - (\kappa_0 + r + E) - E > \kappa_0$$

and

$$d_V(x, \rho_V^U) \geq d_V(x, \mathfrak{g}_Y(x_0)) - d_V(\mathfrak{g}_Y(x_0), \rho_V^U) - E > L - (\kappa_0 + r + E) - E > \kappa_0.$$

Therefore, $d_U(x_0, \rho_U^V) < \kappa_0$ and $d_U(x, \rho_U^V) < \kappa_0$ by consistency (axiom (5)). This implies that $d_U(x_0, x) \leq 2\kappa_0 + E$ and thus $d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x))$ is bounded by a constant depending on k and \mathfrak{S} .

If instead $d_V(\mathfrak{g}_Y(x_0), \rho_V^U) > \kappa_0 + r + E$, then $d_V(\mathfrak{g}_Y(x), \rho_V^U) > \kappa_0$ since

$$d_V(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x)) < r.$$

By consistency, $d_U(\mathfrak{g}_Y(x_0), \rho_U^V) < \kappa_0$ and $d_U(\mathfrak{g}_Y(x), \rho_U^V) < \kappa_0$, which implies that

$$d_U(\mathfrak{g}_Y(x_0), \mathfrak{g}_Y(x)) \leq 2\kappa_0 + E. \quad \square$$

Remark 6.14 Both hypotheses on the subspace in Proposition 6.13 are in fact required. In the standard HHG structure of \mathbb{Z}^2 , the subgroup $\langle(1, 0)\rangle$ is hierarchically quasiconvex, but does not satisfy the orthogonal projection dichotomy. On the other hand, the subgroup $\langle(1, 1)\rangle$ has the orthogonal projection dichotomy, but is not hierarchically quasiconvex. Neither of these subsets are strongly quasiconvex and thus neither are contracting. Both of the above examples can even be made to be (nonstrongly) quasiconvex by choosing $\{(1,0), (1,1), (0,1)\}$ to be the generating set for \mathbb{Z}^2 .

6.3 A generalization of the bounded geodesic image property

As a first application of Theorem 6.3 — our characterization of strongly quasiconvex subsets — we show that strongly quasiconvex subspaces of HHSs satisfy a version of the bounded geodesic image property. First recall the bounded geodesic image property for quasiconvex subsets of hyperbolic spaces (not to be confused with the bounded geodesic image axiom of an HHS).

Proposition 6.15 (bounded geodesic image property for hyperbolic spaces) *Let Y be a K -quasiconvex subset of a geodesic δ -hyperbolic space X . Then there exists $r > 0$ (depending on δ and K) such that if $d(p_Y(x), p_Y(y)) > r$, then every geodesic connecting x and y must intersect the r -neighborhood of Y .*

In the case of strongly quasiconvex subsets of hierarchically hyperbolic space, we replace the closest point projection with the gate map and geodesics with hierarchy paths. Theorem 1.5 from the introduction will follow as a result of the following proposition, which is a version of the active subpath theorem (Proposition 4.24) for strongly quasiconvex subsets.

Proposition 6.16 *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with the bounded domain dichotomy and $Y \subseteq \mathcal{X}$ be a Q -strongly quasiconvex. For all $\lambda \geq 1$, there exist constants ν and D , depending on λ and Q , such that for all $x, y \in \mathcal{X}$, if $d_{\mathcal{X}}(\mathfrak{g}_Y(x), \mathfrak{g}_Y(y)) > D$ and $\gamma: [a, b] \rightarrow \mathcal{X}$ is a λ -hierarchy path joining x and y , then there is a subpath $\alpha = \gamma|_{[a_1, b_1]}$ of γ with*

- (1) $\alpha \subseteq N_{\nu}(Y)$,
- (2) the diameters of $\mathfrak{g}_Y(\gamma([a, a_1]))$ and $\mathfrak{g}_Y(\gamma([b_1, b]))$ both bounded by ν .

Proof By Theorem 6.3, Y is hierarchically quasiconvex and has the orthogonal domain dichotomy. In particular, $\pi_U(Y)$ is uniformly quasiconvex in CU for all $U \in \mathfrak{S}$. Let $x, y \in \mathcal{X}$ and γ be a λ -hierarchy path connecting x and y . Since γ is a (λ, λ) -quasigeodesic, we can choose

$$x = x_0, x_1, x_2, \dots, x_n = y$$

on γ such that the distances between x_i and x_{i+1} are all bounded by 2λ . We will show that there exist $0 \leq i_0 \leq j_0 \leq n$ such that:

- For $i = i_0$ or $i = j_0$, $d_{\mathcal{X}}(x_i, \mathfrak{g}_Y(x_i))$ is bounded by a constant depending only on Q, λ , and \mathfrak{S} .
- If $s < t < i_0$ or $j_0 < s < t$, then $d_{\mathcal{X}}(\mathfrak{g}_Y(x_s), \mathfrak{g}_Y(x_t))$ is bounded by a constant depending only on Q, λ , and \mathfrak{S} .

Since Y is strongly quasiconvex, once we have shown the above, the proposition will follow with α as the subsegment of γ between x_{i_0} and x_{j_0} .

For each $U \in \mathfrak{S}$, the projection π_U is uniformly coarsely Lipschitz, thus there is a λ' depending on $(\mathcal{X}, \mathfrak{S})$ and λ such that the distances $d_U(x_i, x_{i+1})$ are all bounded above by λ' .

By the hyperbolicity of each CU and the properties of gate map (Lemma 4.14), there are constants B and μ depending only on \mathfrak{S}, Q , and λ such that for each $V \in \mathfrak{S}$ satisfying $d_V(\mathfrak{g}_Y(x), \mathfrak{g}_Y(y)) > B$ there are $0 \leq I_V < J_V \leq n$ with the following properties:

- (1) $d_V(x_i, \mathfrak{g}_Y(x_i)) \leq \mu$ for $I_V \leq i \leq J_V$.
- (2) If $s < t < I_V$ or $J_V < s < t$, then $d_V(\mathfrak{g}_Y(x_s), \mathfrak{g}_Y(x_t)) < \mu$.
- (3) $d_V(x_{I_V}, x_{J_V}) \geq 10D$, where $D = 3(E + \mu + \kappa_0 + \lambda')$.

For future convenience, we can and shall assume B is large enough that $B > E$, $(\mathcal{X}, \mathfrak{S})$ has the B -bounded domain dichotomy, and Y has the B -orthogonal projection dichotomy. By the uniqueness axiom (10), there is a constant K depending on B and $(\mathcal{X}, \mathfrak{S})$ such that if $d_{\mathcal{X}}(\mathfrak{g}_Y(x), \mathfrak{g}_Y(y)) > K$, then the set $\mathcal{R} = \text{Rel}_B(\mathfrak{g}_Y(x), \mathfrak{g}_Y(y))$ is nonempty. Since for each $V \in \mathcal{R}$ we have $d_V(x_{I_V}, x_{J_V}) \geq 10D$ and each distance $d_V(x_i, x_{i+1})$ is bounded above by $\lambda' < D$, there are $I_V < i_V < j_V < J_V$ such that

$$(*) \quad D \leq d_V(x_{i_V}, x_{I_V}) \leq 2D \quad \text{and} \quad D \leq d_V(x_{j_V}, x_{J_V}) \leq 2D.$$

Let $i_0 = \min_{V \in \mathcal{R}} i_V$ and $j_0 = \max_{V \in \mathcal{R}} j_V$.

We first prove that for each s and t that are both less than i_0 or both greater than j_0 the distance $d_{\mathcal{X}}(\mathfrak{g}_Y(x_s), \mathfrak{g}_Y(x_t))$ is uniformly bounded by some constant depending only on \mathfrak{S} , Q and λ . We will provide the proof for the case s and t are both less than i_0 and the proof for the other case is essentially identical. Let $V \in \mathfrak{S}$. If $V \notin \mathcal{R}$, then $d_V(\mathfrak{g}_Y(x), \mathfrak{g}_Y(y)) \leq B$ which implies $\text{diam}(\pi_V(\mathfrak{g}_Y(\gamma)))$ is bounded by a constant that depends only on B , λ , Q and \mathfrak{S} . In particular, $d_V(\mathfrak{g}_Y(x_s), \mathfrak{g}_Y(x_t))$ is also uniformly bounded by this constant. When $V \in \mathcal{R}$, then s and t are both less than i_V . Therefore by item (2) above and (*) we have that $d_V(\mathfrak{g}_Y(x_s), \mathfrak{g}_Y(x_t))$ is bounded by a constant depending only on \mathfrak{S} , Q , and λ . By the distance formula (Theorem 4.4) the distance $d_{\mathcal{X}}(\mathfrak{g}_Y(x_s), \mathfrak{g}_Y(x_t))$ is therefore bounded by a constant that ultimately depends only on \mathfrak{S} , Q and λ .

We now prove that there exists v' depending on \mathfrak{S} , Q and λ such that for $i = i_0$ or $i = j_0$,

$$(**) \quad d_{\mathcal{X}}(x_i, \mathfrak{g}_Y(x_i)) \leq v'.$$

Again we only give the proof for the case of $i = i_0$ and the argument for the case $i = j_0$ is almost identical. By the distance formula, it is sufficient to check that we can uniformly bound $d_U(x_i, \mathfrak{g}_Y(x_i))$ for each $U \in \mathfrak{S}$.

Fix a domain $V \in \mathcal{R}$ such that $i = i_0 = i_V$. We shall show $d_U(x_i, \mathfrak{g}_Y(x_i))$ for all $U \in \mathfrak{S}$ by examining the four cases for how U can be related to V .

Case 1 Suppose $V \perp U$. Since Y has the B -orthogonal domain dichotomy,

$$V \in \mathcal{R} \implies CU \subseteq N_B(\pi_U(Y)).$$

Therefore by the properties of the gate map (Lemma 4.14), we have that $d_{\mathcal{X}}(x_i, \mathfrak{g}_Y(x_i))$ is uniformly bounded.

Case 2 Suppose $V \pitchfork U$. If $d_V(x_i, \rho_V^U) > \kappa_0 + \mu + E$, then

$$d_V(\mathfrak{g}_Y(x_i), \rho_V^U) > \kappa_0$$

and by the consistency axiom (5) and triangle inequality,

$$d_U(x_i, \mathfrak{g}_Y(x_i)) \leq 2\kappa_0 + E.$$

Now assume that $d_V(x_i, \rho_V^U) < \kappa_0 + \mu + E$. Since $D > \mu + E + \kappa_0$, $d_V(x_i, x_{I_V}) \geq D$, and $d_V(x_i, x_{J_V}) \geq D$, we have that x_{I_V} , $\mathfrak{g}_Y(x_{I_V})$, x_{J_V} and $\mathfrak{g}_Y(x_{J_V})$ all project at least κ_0 far from ρ_V^U in CV . Therefore, by the consistency axiom and triangle inequality,

$$d_U(x_{I_V}, \mathfrak{g}_Y(x_{I_V})) \leq 2\kappa_0 + E \quad \text{and} \quad d_U(x_{J_V}, \mathfrak{g}_Y(x_{J_V})) \leq 2\kappa_0 + E.$$

Thus, by the quasiconvexity of $\pi_U(Y)$ in CU and the properties of the gate map, the distance $d_U(x_i, \mathfrak{g}_Y(x_i))$ is bounded by a uniform constant determined by \mathfrak{S} , Q and λ .

Case 3 Suppose $U \sqsubseteq V$. Consider geodesics in CV connecting the projections of the pairs of points $(x_{I_V}, \mathfrak{g}_Y(x_{I_V}))$, $(x_i, \mathfrak{g}_Y(x_i))$ and $(x_{J_V}, \mathfrak{g}_Y(x_{J_V}))$. By the assumptions on I_V , i and J_V , at most one of these geodesics intersects $N_E(\rho_V^U)$. If such a geodesic is not the one connecting $\pi_V(x_i)$ and $\pi_V(\mathfrak{g}_Y(x_i))$, then we are done by the bounded geodesic image axiom (8). Otherwise, the bounded geodesic image axiom requires that $\pi_V(x_{I_V})$ and $\pi_V(x_{J_V})$ are contained in the $3E$ -neighborhood of $\pi_U(Y)$ in CU . By the quasiconvexity of $\pi_U(Y)$ in CU and the properties of the gate map, the distance $d_U(x_i, \mathfrak{g}_Y(x_i))$ is thus bounded by a uniform constant determined by \mathfrak{S} , Q and λ .

Case 4 Suppose $V \sqsubseteq U$. Recall that $\pi_U(\gamma)$ is a unparametrized quasigeodesic in CU , and let γ_0 be the subsegment of $\pi_U(\gamma)$ from x_{I_V} to x_i and γ_1 be the subsegment from x_i to x_{J_V} . By the bounded geodesic image axiom and (*), there exists $E' \geq E$ determined by \mathfrak{S} , such that both γ_0 and γ_1 intersect the E' -neighborhood of ρ_U^V . Since $\pi_U(\gamma)$ is an unparametrized (λ, λ) -quasigeodesic, there exists R depending on E' and λ such that $d_U(x_i, \rho_U^V) \leq R$. If α is some CU geodesic connecting $\mathfrak{g}_Y(x)$ and $\mathfrak{g}_Y(y)$, then α also intersects the E -neighborhood of ρ_U^V by the bounded geodesic image axiom. Therefore, by the quasiconvexity of $\pi_U(Y)$ in CU and the properties of the gate map, the distance $d_U(x_i, \mathfrak{g}_Y(x_i))$ is bounded by a uniform constant determined by \mathfrak{S} , Q and λ . □

Remark 6.17 The hypotheses of Proposition 6.16 cannot be relaxed by taking Y to be hierarchically quasiconvex instead of strongly quasiconvex. As a counterexample, one can consider \mathbb{Z}^2 with the standard HHG structure and let Y be the x -axis. As any horizontal line in \mathbb{Z}^2 is a hierarchy path, for any $D > 0$, there exists a hierarchy path γ where both $d_{\mathcal{X}}(\gamma, Y) > D$ and $\text{diam}(\mathfrak{g}_Y(\gamma)) > D$.

7 Strongly quasiconvex subsets in familiar examples

In this section, we utilize Theorem 6.3 to give descriptions of the strongly quasiconvex subsets in well studied examples of hierarchically hyperbolic spaces. We will begin by briefly discussing the HHS structure for the mapping class group, Teichmüller space, right-angled Artin and Coxeter groups, and graph manifolds. The descriptions are not complete as we only describe the parts of the HHS structure that we shall need to be able to apply the results from the general case. We direct the reader to the references provided alongside each example for complete details.

The mapping class group and Teichmüller space For the mapping class group, see [10; 41]; for the Teichmüller metric, see [23; 26; 47]; and for the Weil–Petersson metric, see [17].

Let S be an oriented, connected, finite-type surface with genus g and p punctures. The complexity of S is $\xi(S) = 3g - 3 + p$. Assume $\xi(S) \geq 1$ and let \mathcal{X} be the marking complex of S .

- **Index set** \mathfrak{S} will be the collection of isotopy classes of (not necessarily connected) essential subsurfaces of S excluding 3-punctured sphere, but including annuli.
- **Hyperbolic spaces** For each $U \in \mathfrak{S}$, CU will be the curve graph of U . The space CU will be infinite diameter if and only if U is connected. The projection maps, π_U , are the well studied subsurface projections of Masur and Minsky.
- **Relations** $U \perp V$ if U and V are disjoint and $U \sqsubseteq V$ if U is nested into V . If $U \sqsubseteq V$, then ρ_V^U will be the subset of curves in CV corresponding to ∂U .

The HHS structure for Teichmüller space with either metric is identical except for the annular domains of \mathfrak{S} . For the Teichmüller metric, modify the curve graphs of the annular domains by attaching a horoball. For the Weil–Petersson metric, the index set \mathfrak{S} simply excludes annuli. This difference in the treatment of annular domains accounts for all of the differences in the coarse geometry of the these three spaces.

RAAGs and RACGs [9] Let Γ be a finite simplicial graph and G_Γ be the associated right-angled Artin or right-angled Coxeter group equipped with a word metric from a finite generating set. For an induced subgraph $\Lambda \subseteq \Gamma$, $\text{link}(\Lambda)$ is the subgraph of $\Gamma - \Lambda$ induced by the vertices adjacent to every vertex in Λ and $\text{star}(\Lambda)$ be the induced

subgraph of $\text{link}(\Lambda) \cup \Lambda$. If Λ is an induced subgraph of Γ , then G_Λ is a subgroup of G_Γ . We call subgroups of this form the *special subgroups* of G_Γ . The following is an HHG structure on G_Γ .

- **Index set** For $g, h \in G_\Gamma$ and Λ a nonempty, induced subgraph of Γ , define the equivalence relation $gG_\Lambda \sim hG_\Lambda$ if $g^{-1}h \in G_{\text{star}(\Lambda)}$. Let \mathfrak{S} be defined as $\{gG_\Lambda\}/\sim$.
- **Hyperbolic spaces** $C[gG_\Lambda]$ can be obtained by starting with the coset gG_Λ and coning off each left coset of the special subgroups contained in gG_Λ . $C[gG_\Lambda]$ is infinite diameter if and only if G_Λ is infinite and Λ does not split as a join.
- **Relations** $[gG_{\Lambda'}] \sqsubseteq [gG_\Lambda]$ if $\Lambda' \subseteq \Lambda$ and $[gG_{\Lambda'}] \perp [gG_\Lambda]$ if $\Lambda \subseteq \text{link}(\Lambda')$ (and hence $\Lambda' \subseteq \text{link}(\Lambda)$). If $[gG_{\Lambda'}] \sqsubseteq [gG_\Lambda]$, then $\rho_{[gG_\Lambda]}^{[gG_{\Lambda}]}$ will be the subset $gG_{\Lambda'}$ in $C[gG_\Lambda]$.

Graph manifolds [10] Let M be a nongeometric graph manifold and \mathcal{X} be the universal cover of M . Since the fundamental group of every graph manifold is quasi-isometric to the fundamental group of a flip graph manifold, we will assume M is flip. Let T be Bass–Serre tree for M and X_v be the subspace of \mathcal{X} corresponding to a vertex $v \in T$. Each X_v is bi-Lipschitz to the product $R_v \times H_v$ where R_v is a copy of the real line and H_v is the universal cover of a hyperbolic surface with totally geodesic boundary. If v, w are adjacent vertices in T , then let $\partial_w H_v$ and $\partial_v H_w$ denote the boundary components of H_v and H_w such that $R_v \times \partial_w H_v$ is identified with $R_w \times \partial_v H_w$ in \mathcal{X} . Since M is flip, R_v is identified with $\partial_v H_w$. For each $v \in T$, let \hat{H}_v denote the spaced obtained from H_v after coning off each copy of $\partial_w H_v$ for each vertex w adjacent to v . The following is an HHS structure on \mathcal{X} .

- **Index set** For adjacent vertices $v, w \in T$, define $R_v \sim \partial_v H_w$ and then let $\mathfrak{S} = \{T, R_v, \partial_v H_w, \hat{H}_w\}/\sim$.
- **Hyperbolic spaces** Every element of \mathfrak{S} is a hyperbolic space, so we have $CU = U$ for all $U \in \mathfrak{S}$. The diameter of CU is infinite for all $U \in \mathfrak{S}$.
- **Relations** T is the \sqsubseteq -maximal domain and $[\partial_w H_v] \sqsubseteq \hat{H}_v$ for all w and v adjacent in T . For adjacent vertices $v, w \in T$, $\rho_T^{[R_v]} = \rho_T^{[\partial_w H_v]} = \{v, w\} \subset T$ and $\rho_{\hat{H}_v}^{[\partial_w H_v]}$ is the cone point for $\partial_w H_v$ in \hat{H}_v . For v and w adjacent in T , we have $[R_v] \perp \hat{H}_v$ and $[R_v] \perp [R_w]$ (recall $[\partial_w H_v] = [R_w]$).

Remark 7.1 When the manifold M is flip, the above describes an HHG structure on $\pi_1(M)$. However, if M is not flip, then the quasi-isometry from $\pi_1(M)$ to the

fundamental group of the flip graph manifold need not be equivariant and the above will be an HHS, but not an HHG structure on $\pi_1(M)$. See [10, Remark 10.2] for a discussion of the existence of HHG structures on 3–manifold groups.

In the case of right-angled Artin groups with connected defining graphs, Tran and Genevois independently showed that strongly quasiconvex subgroups are either finite-index or hyperbolic (and are actually free when they are hyperbolic) [28; 54]. The same result is shown for the mapping class group in [38] and for certain \mathcal{CFS} right-angled Coxeter groups in [43]. Based on these examples, one may conjecture that the strongly quasiconvex subsets of any not relatively hyperbolic, hierarchically hyperbolic space are either hyperbolic or coarsely cover the entire space. While [54] provides a counterexample to this conjecture in right-angled Coxeter groups, it nevertheless holds in many of the examples described above. In Proposition 7.2, we give sufficient conditions for every strongly quasiconvex subset of an HHS to be either hyperbolic or coarsely covering. We then unite and expand the work of Genevois, Kim, Nguyen and Tran by applying Proposition 7.2 to the mapping class group, Teichmüller space, right-angled Artin and Coxeter groups, and graph manifolds in Corollary 7.4.

Proposition 7.2 *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with the bounded domain dichotomy and let \mathfrak{S}^* be as defined in Definition 6.5. Assume the following two conditions hold:*

- (1) *For all $W \in \mathfrak{S} - \mathfrak{S}^*$ either CW has bounded diameter or the set*

$$\{\rho_W^V \mid V \in \mathfrak{S}^* \text{ with } V \pitchfork W \text{ or } V \sqsubseteq W\}$$

uniformly coarsely covers CW .

- (2) *For every $U, V \in \mathfrak{S}^*$ there exists a sequence $U = U_1, \dots, U_n = V$ of domains in \mathfrak{S}^* with $U_i \perp U_{i+1}$ for all $1 \leq i \leq n - 1$.*

Then, if $Y \subseteq \mathcal{X}$ is strongly quasiconvex, either Y is hyperbolic or some finite neighborhood of Y covers all of \mathcal{X} .

Proof Let $Y \subseteq \mathcal{X}$ be Q –strongly quasiconvex. By Theorem 6.3 there exists B , depending only on Q and \mathfrak{S} , such that Y has the B –orthogonal projection dichotomy. Further, we can assume B is large enough such that $(\mathcal{X}, \mathfrak{S})$ satisfies the B –bounded domain dichotomy. We will show that if Y is not hyperbolic, then for all $W \in \mathfrak{S}$ we have that CW is uniformly coarsely covered by $\pi_W(Y)$. Thus for all $x \in \mathcal{X}$ we will have that $d_W(x, \mathfrak{g}_Y(x))$ is uniformly bounded and therefore Y will coarsely cover \mathcal{X} by the distance formula (Theorem 4.4).

Suppose that Y is not hyperbolic. By Proposition 2.8, the inclusion map $i : Y \hookrightarrow X$ cannot be a stable embedding. Therefore by Corollary 6.4, there exists a domain $U \in \mathfrak{S}^*$ such that $\text{diam}(\pi_U(Y)) > B$. First we will show that for any domain $W \in \mathfrak{S}^*$, $CW \subseteq N_B(\pi_W(Y))$.

Let $W \in \mathfrak{S}^*$. By hypothesis, there exists a sequence $U = U_1, \dots, U_n = W$ of domains in \mathfrak{S}^* with $U_i \perp U_{i+1}$ for all $1 \leq i \leq n - 1$. Since Y has the B -orthogonal projection dichotomy and $\text{diam}(CU_i) = \infty$ for each $1 \leq i \leq n$, we have $CU_i \subseteq N_B(\pi_{U_i}(Y))$ for all $1 \leq i \leq n$. In particular, $CW \subseteq N_B(\pi_W(Y))$.

Now let $W \in \mathfrak{S} - \mathfrak{S}^*$ such that $\text{diam}(CW) = \infty$. We will show that $\pi_W(Y)$ uniformly coarsely covers CW by showing that for all $V \in \mathfrak{S}^*$ such that ρ_W^V is defined there exists $y \in Y$ such that $\pi_W(y)$ is uniformly close to ρ_W^V . First suppose $V \in \mathfrak{S}^*$ with $V \sqsubseteq W$. By the preceding paragraph, there exist $x, x' \in Y$ such that $d_V(x, x') > 100E$. If γ is a hierarchy path connecting x and x' , then $\pi_W(\gamma)$ is uniformly close to ρ_W^V by the bounded geodesic image axiom (8). Further, since Y is strongly quasiconvex there exists $y \in Y$ such that $d_W(\rho_W^V, \pi_W(y)) < B'$ where B' depends only on Q and \mathfrak{S} . If instead $V \in \mathfrak{S}^*$ and $V \pitchfork W$, then there exists $y \in Y$ such that $d_V(y, \rho_V^W) > \kappa_0$. Thus $d_W(y, \rho_W^V) \leq \kappa_0$ by the consistency axiom (5). Since the set

$$\{\rho_W^V \mid V \in \mathfrak{S}^* \text{ with } V \pitchfork W \text{ or } V \sqsubseteq W\}$$

uniformly coarsely covers CW by hypothesis, we have that $\pi_W(Y)$ uniformly coarsely covers all of CW as well.

Hence we have that for all $W \in \mathfrak{S}$, CW is uniformly coarsely covered by $\pi_W(Y)$ and so Y coarsely covers \mathcal{X} by the distance formula. □

Before continuing, we will take a brief detour to define a property of graphs that will be relevant to our study of right-angled Coxeter groups. Given a graph Γ , define Γ^4 as the graph whose vertices are induced 4-cycles of Γ . Two vertices in Γ^4 are adjacent if and only if the corresponding induced 4-cycles in Γ have two nonadjacent vertices in common. Given graphs Λ_1 and Λ_2 , recall that the join $\Lambda_1 * \Lambda_2$ is the graph obtained from $\Lambda_1 \sqcup \Lambda_2$ by adding an edge between each vertex of Λ_1 and each vertex of Λ_2 .

Definition 7.3 (constructed from squares) A graph Γ is *CF S* if $\Gamma = \Omega * K$, where K is a (possibly empty) clique and Ω is a nonempty subgraph such that Ω^4 has a connected component T where every vertex of Ω is contained in a 4-cycle that is a vertex of T . If Γ is *CF S* and Ω^4 is connected, then we say Γ is *strongly CF S*. If Γ is

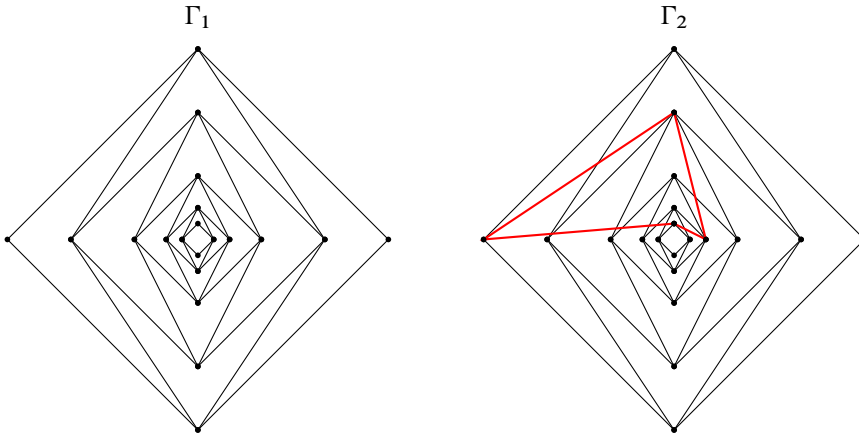


Figure 7: Two graphs Γ_1 and Γ_2 are both \mathcal{CFS} . However, graph Γ_1 is graph strongly \mathcal{CFS} but Γ_2 is not since the red induced 4-cycle in Γ_2 is not “connected” to any other induced 4-cycle in the graph.

(strongly) \mathcal{CFS} , then by abuse of language we will say that the right-angled Coxeter group G_Γ is (strongly) \mathcal{CFS} . See Figure 7 for examples of \mathcal{CFS} and strongly \mathcal{CFS} graphs.

Corollary 7.4 *The following HHSs have the property that every strongly quasiconvex subset is either hyperbolic or coarsely covers the entire space:*

- (a) *The Teichmüller space of a finite-type, oriented surface with the Teichmüller metric.*
- (b) *The Teichmüller space of a finite-type, oriented surface of complexity at least 6 with the Weil–Peterson metric.*
- (c) *The mapping class group of a finite-type, oriented surface.*
- (d) *A right-angled Artin group with connected defining graph.*
- (e) *A right-angled Coxeter group with strongly \mathcal{CFS} defining graph.*
- (f) *The fundamental group of a nongeometric graph manifold.*

In particular, if H is a strongly quasiconvex subgroup in any of the groups (c)–(f), then H is either stable or finite-index.

Proof All of the above examples have the bounded domain dichotomy. We shall show they satisfy the two hypotheses of Proposition 7.2.

Mapping class group and Teichmüller metric If $\xi(S) \leq 1$, then the mapping class group and Teichmüller space will both be hyperbolic; thus we can assume $\xi(S) \geq 2$. In this case, \mathfrak{S}^* is the set of all connected proper subsurfaces. Thus hypothesis (1) follows from the fact that every curve on the surface corresponds to the boundary curve of some connected subsurface. Given two subsurfaces U and V , a sequence satisfying hypothesis (2) is found by taking a path in CS connecting ∂U and ∂V .

Weil–Peterson \mathfrak{S}^* is the collection of all connected proper subsurfaces whose complement contains a subsurface of complexity at least 1. In particular, since the complexity is at least 6, \mathfrak{S}^* contains every subsurface of complexity 1. For every connected subsurface $W \notin \mathfrak{S}^*$, every curve on W corresponds to the boundary curve of some complexity 1 subsurface providing hypothesis (1). Hypothesis (2) follows from the observations that if $U \subset S$ is a subsurface of complexity 1 and α is a curve disjoint from U , then there exists $V \subseteq S$, a subsurface of complexity 1, such that $\alpha \subseteq \partial V$ and U is disjoint from V . Thus any path in CS can be promoted to a sequence of sequentially disjoint subsurfaces in \mathfrak{S}^* .

RAAGs \mathfrak{S}^* is the collection of $[gG_\Lambda]$ such that there exists $\Delta \subseteq \text{link}(\Lambda)$ where Λ and Δ are both nonempty and not joins. In particular, since Γ is connected, \mathfrak{S}^* contains all of the $[gG_\Lambda]$ where Λ is a single vertex. Hypothesis (1) follows from the fact that G_Λ acts cocompactly on its Cayley graph and the construction of $C[gG_\Lambda]$. For hypothesis (2), let $[g_1G_{\Lambda_1}], [g_2G_{\Lambda_2}] \in \mathfrak{S}^*$ and $m = |g_1^{-1}g_2|$. We shall proceed by induction on m . If $m = 0$, then $g_1 = g_2 = g$ and since Γ is connected, there is a sequence of vertices v_1, v_2, \dots, v_n such that v_i and v_{i+1} are adjacent for all $1 \leq i \leq n-1$ and $v_1 \in \text{link}(\Lambda_1), v_n \in \text{link}(\Lambda_2)$. Thus $[gG_{\Lambda_1}], [gG_{v_1}], \dots, [gG_{v_n}], [gG_{\Lambda_2}]$ is the required sequence.

If $m > 0$, then there exists $g_3 \in G_\Gamma$ such that $|g_1^{-1}g_3| = m - 1$ and $|g_3^{-1}g_2| = 1$. Let v be the vertex of Γ such that $g_3^{-1}g_2$ is either v or v^{-1} . By induction, there exist two sequences of elements of \mathfrak{S}^* ,

$$[g_1G_{\Lambda_1}] = U_1, U_2, \dots, U_n = [g_3G_v] \quad \text{and} \quad [g_2G_v] = V_1, V_2, \dots, V_k = [g_2G_{\Lambda_2}],$$

such that $U_i \perp U_{i+1}$ for $1 \leq i \leq n - 1$ and $V_i \perp V_{i+1}$ for all $1 \leq i \leq k - 1$. Since $[g_3G_v] = [g_2G_v]$,

$$[g_1G_{\Lambda_1}] = U_1, U_2, \dots, U_n, V_2, \dots, V_n = [g_2G_{\Lambda_2}]$$

is the required sequence.

RACGs Since Γ is strongly \mathcal{CFS} , we can write $\Gamma = \Omega * K$ where K is a clique (possibly empty) and Ω is a nonempty graph such that Ω^4 is connected and every vertex of Ω is contained a 4-cycle that is a vertex of Ω^4 . Since G_Ω is a finite-index subgroup of G_Γ , it suffices to prove that every strongly quasiconvex subset of G_Ω is either hyperbolic or coarsely covers G_Ω . We now prove that the standard HHG structure, \mathfrak{S} , on G_Ω satisfies the two hypotheses of Proposition 7.2. The argument will be similar to the case of right-angled Artin groups above.

We first observe that \mathfrak{S}^* is the collection of $[gG_\Lambda]$ such that there exists $\Delta \subseteq \text{link}(\Lambda)$ where Λ and Δ both have at least two points and they are not joins. In particular, \mathfrak{S}^* contains all domains $[gG_{\{a,b\}}]$ where a and b are two nonadjacent vertices of an induced 4-cycle. Hypothesis (1) follows from the fact that G_Λ acts cocompactly on its Cayley graph and the construction of $C[gG_\Lambda]$.

For hypothesis (2), let $[g_1G_{\Lambda_1}], [g_2G_{\Lambda_2}] \in \mathfrak{S}^*$ and $m = |g_1^{-1}g_2|$. We shall proceed by induction on m . We first assume that $m = 0$. Therefore, $g_1 = g_2 = g$. We note that for $i = 0$ or 1 there exists $\Delta_i \subseteq \text{link}(\Lambda_i)$ where Λ_i and Δ_i both contain at least two vertices and are not joins. Therefore, $\text{link}(\Lambda_i)$ contains a pair (u_i, v_i) of two nonadjacent vertices of some induced 4-cycle. Since Ω^4 is connected, there is a sequence of pairs of nonadjacent vertices $(u_1, v_1) = (a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) = (u_2, v_2)$ such that a_i and b_i are both adjacent to a_{i+1} and b_{i+1} for all $1 \leq i \leq n - 1$. Thus $[gG_{\Lambda_1}], [gG_{\{a_1,b_1\}}], \dots, [gG_{\{a_n,b_n\}}], [gG_{\Lambda_2}]$ is the required sequence.

If $m > 0$, then there exists $g_3 \in G_\Omega$ such that $|g_1^{-1}g_3| = m - 1$ and $|g_3^{-1}g_2| = 1$. Let v be the vertex of Ω such that $g_3^{-1}g_2 = v$. Since every vertex of Ω is contained in a four cycle that is a vertex of Ω^4 , there is a vertex w such that v and w are two nonadjacent vertices of an induced 4-cycle. By induction, there exist two sequences of elements of \mathfrak{S}^* ,

$$[g_1G_{\Lambda_1}] = U_1, U_2, \dots, U_n = [g_3G_{\{v,w\}}]$$

and

$$[g_2G_{\{v,w\}}] = V_1, V_2, \dots, V_k = [g_2G_{\Lambda_2}],$$

such that $U_i \perp U_{i+1}$ for $1 \leq i \leq n - 1$ and $V_i \perp V_{i+1}$ for all $1 \leq i \leq k - 1$. Since $[g_3G_{\{v,w\}}] = [g_2G_{\{v,w\}}]$,

$$[g_1G_{\Lambda_1}] = U_1, U_2, \dots, U_n, V_2, \dots, V_n = [g_2G_{\Lambda_2}]$$

is the required sequence.

Graph manifolds In this case, $\mathfrak{S}^* = \mathfrak{S} - \{T\}$ and hypothesis (1) is immediate from the facts that for every vertex $v \in T$ is an element of $\rho_T^{[R_v]}$ and every point in H_v is uniformly close to some boundary component $\partial_w H_v$. For hypothesis (2), consider $U, W \in \mathfrak{S}^*$. If $U = [R_u]$ and $W = [R_w]$, let v_1, \dots, v_n be a sequence of adjacent vertices in T such that v_1 is adjacent to u and v_n is adjacent to w . In this case the sequence $[R_u], [R_{v_1}], \dots, [R_{v_n}], [R_w]$ satisfies the hypothesis. If $U = [\hat{H}_u]$ or $W = [\hat{H}_w]$, the hypothesis is satisfied by adding $[\hat{H}_u]$ before $[R_u]$ or $[\hat{H}_w]$ after $[R_w]$ to $[R_u], [R_{v_1}], \dots, [R_{v_n}], [R_w]$ as needed. \square

In the setting of 2-dimensional right-angled Coxeter groups, Tran provided a characterization of the special strongly quasiconvex subgroups [54]. This characterization was expanded by Genevois to include all right-angled Coxeter groups in [28]. We provide a new proof of this characterization using Theorem 6.3.

Theorem 7.5 [28; 54] *Let Γ be a simplicial graph and Δ an induced subgraph of Γ . If G_Γ is the right-angled Coxeter group corresponding to Γ and G_Δ is the subgroup generated by the vertices of Δ , then the following conditions are equivalent.*

- (1) *The subgroup G_Δ is strongly quasiconvex in G_Γ .*
- (2) *If Δ contains two nonadjacent vertices of an induced 4-cycle σ , then Δ contains all vertices of σ .*

Proof Before we begin, we document a few additional facts we will need about the HHG structure on a right-angled Coxeter group. For any induced subgraph Λ , $\mathbf{P}_{[G_\Lambda]}$ is coarsely equal to the subgroup $G_\Lambda \times G_{\text{link}(\Lambda)}$ and G_Λ can be coarsely identified with $\mathbf{F}_{[G_\Lambda]}$. In particular, G_Λ is hierarchically quasiconvex, $\pi_U(G_\Lambda)$ uniformly coarsely covers CU for $U \subseteq [G_\Lambda]$, and $\pi_V(G_\Lambda)$ is uniformly bounded for all $V \not\subseteq [G_\Lambda]$. See [9] for full details on the HHG structure on right-angled Coxeter groups.

(1) \implies (2) Assume for a contradiction that G_Δ is strongly quasiconvex, but there is a 4-cycle σ with two pairs of nonadjacent vertices $\{a_1, a_2\}$ and $\{b_1, b_2\}$ such that $\{a_1, a_2\}$ is a subset of Δ and $\{b_1, b_2\}$ is not. We know that $U = [G_{\{a_1, a_2\}}]$ and $[G_{\{b_1, b_2\}}] = V$ are orthogonal domains. However, $\pi_U(G_\Delta)$ coarsely covers CU , but $\pi_V(G_\Delta)$ has uniformly bounded diameter which contradicts Theorem 6.3.

(2) \implies (1) As G_Δ is hierarchically quasiconvex, we only need to demonstrate that G_Δ satisfies the orthogonal projection dichotomy. Let B be a positive number such that (G_Γ, \mathfrak{S}) has the B -bounded domain dichotomy, $CW \subseteq N_B(\pi_W(G_\Delta))$ for all $W \subseteq [G_\Delta]$, and $\text{diam}(\pi_W(G_\Delta)) < B$ for all $W \not\subseteq [G_\Delta]$. If $\text{diam}(\pi_U(G_\Delta)) > B$, then it

must be the case that $U = [G_\Lambda]$ where $\Lambda \subseteq \Delta$ and Λ contains two nonadjacent vertices s and t . If $V \in \mathfrak{S}_V^\perp$, then $V = [G_{\Lambda'}]$ where $\Lambda' \subseteq \text{link}(\Lambda)$ and $\Lambda \subseteq \text{link}(\Lambda')$. If Λ' is a join or $\Lambda' = \{v\}$, then $\text{diam}(CV) \leq B$ and $CV \subseteq N_{2B}(\pi_V(G_\Delta))$. In the other case, we will show $\Lambda' \subseteq \Delta$.

If Λ' is not a join and contains at least two vertices, then for each vertex $v \in \Lambda'$ there exists a vertex $w \in \Lambda'$ that is not adjacent to v . Since $\Lambda \subseteq \text{link}(\Lambda')$, the vertices v, s, w and t form a 4-cycle. However, (2) then requires $v, w \in \Delta$. Hence, $\Lambda' \subseteq \Delta$ and $V = [G_{\Lambda'}] \subseteq [G_\Delta]$ implying $CV \subseteq N_B(\pi_V(G_\Delta))$. Thus G_Δ has the $2B$ -orthogonal projection dichotomy and we are finished by Theorem 6.3. \square

7.1 CFS right-angled Coxeter groups

Recently, Behrstock proposed the program of classifying all \mathcal{CFS} right-angled Coxeter groups up to quasi-isometry and commensurability. This was motivated by the genericity of \mathcal{CFS} right-angled Coxeter groups among random right-angled Coxeter groups as well as the fact that being \mathcal{CFS} is a necessary (but not sufficient) condition for a right-angled Coxeter group to be quasi-isometric to a right-angled Artin group; see [7].

In [7], Behrstock presented the first example of a \mathcal{CFS} right-angled Coxeter group that contains a one-ended stable subgroup answering outstanding questions about stable subgroups and quasi-isometries between right-angled Artin groups and right-angled Coxeter groups. Using Theorem 7.5, we can expand Behrstock’s construction to produce \mathcal{CFS} right-angled Coxeter groups that contain any other right-angled Coxeter group as a strongly quasiconvex subgroup. This shows that there is incredible diversity among the quasi-isometry types of \mathcal{CFS} right-angled Coxeter groups.

Proposition 7.6 *Any right-angled Coxeter group (resp. hyperbolic right-angled Coxeter group) is an infinite-index strongly quasiconvex subgroup (resp. stable subgroup) of a \mathcal{CFS} right-angled Coxeter group.*

Proof To prove the proposition we shall utilize a construction of certain \mathcal{CFS} graphs described in [7]. Let Ω_n be a graph with $2n$ vertices built in the following inductive way. Let Ω_1 be a pair of vertices a_1, b_1 with no edge between them. Given the graph Ω_{n-1} , we obtain the graph Ω_n by adding a new pair of vertices a_n and b_n to the graph Ω_{n-1} and adding four new edges, one connecting each of $\{a_{n-1}, b_{n-1}\}$ to each of $\{a_n, b_n\}$. In Figure 8, graph Γ_1 is exactly Ω_{13} . For each integer $m \geq 2$ there is a sufficiently large n such that the graph Ω_n contains m vertices whose pairwise distances are at least 3.

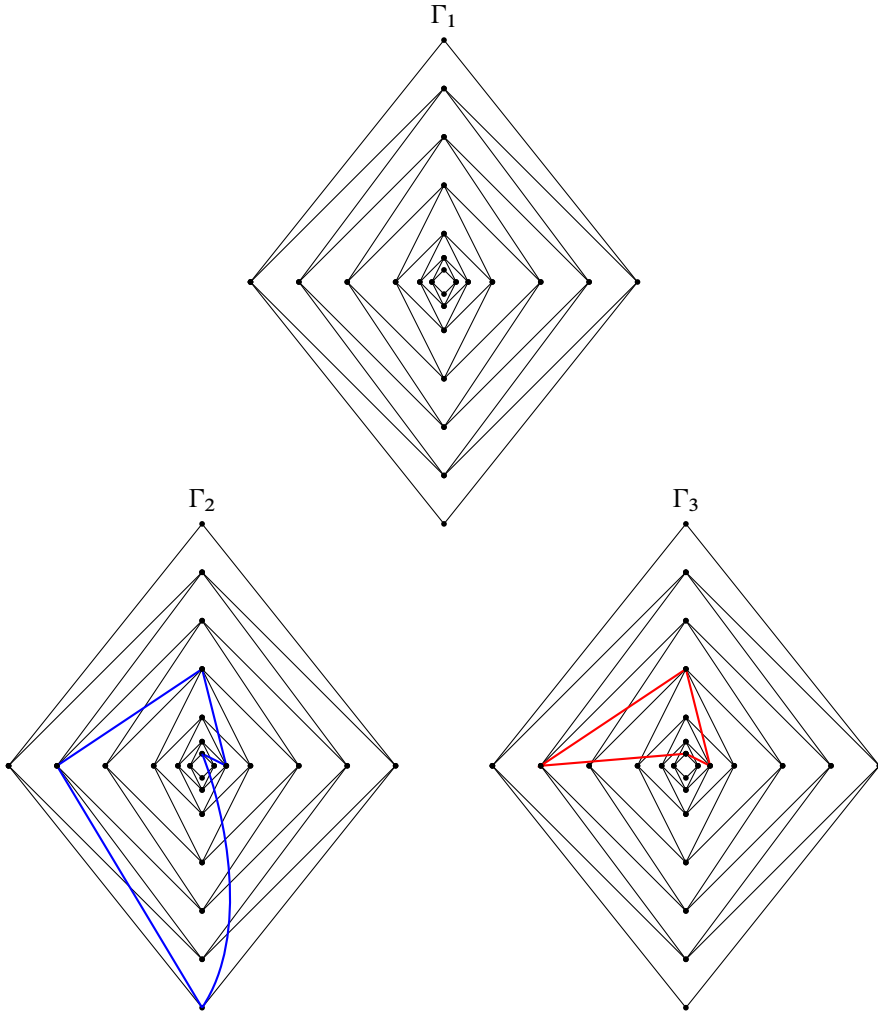


Figure 8: Three graphs Γ_1 , Γ_2 , and Γ_3 are all \mathcal{CFS} , but no pair of them are quasi-isometric.

Let G_Γ be an arbitrary right-angled Coxeter group. We will construct a \mathcal{CFS} right-angled Coxeter group G_Ω that contains G_Γ as a strongly quasiconvex subgroup. Let m be a number of vertices of Γ . Choose a positive integer n sufficient large so the graph Ω_n contains a set S of m vertices whose pairwise distance is at least 3. We glue the graphs Γ and Ω_n by identifying the vertex set of Γ to S . Let Ω be the resulting graph. In Figure 8, graph Γ_2 is an example of graph Ω when Γ is the 5-cycle graph and graph Γ_3 is another example of graph Ω when Γ is the 4-cycle graph.

	G_{Γ_1}	G_{Γ_2}	G_{Γ_3}
strongly \mathcal{CFS}	yes	yes	no
noncoarsely covering strongly quasiconvex subsets	all quasitrees	all hyperbolic; contains a one-ended stable subgroup	contains a strongly quasiconvex virtually \mathbb{Z}^2 subgroup
Morse boundary	totally disconnected	contains a circle	connectivity unknown
quasi-isometric to an RAAG	yes	no	no

Table 1: Note that Karrer has since shown that the Morse boundary of G_{Γ_3} is totally disconnected [37].

The graphs Ω and Ω_n have the same vertex set and $\Omega_n^4 \subset \Omega^4$. Thus Ω is a \mathcal{CFS} graph as Ω_n is a \mathcal{CFS} graph. Since the distance in Ω_n between any distinct vertices of S is at least 3, Γ is an induced subgraph of Ω with the property that if Γ contains two nonadjacent vertices of an induced 4-cycle σ , then Γ contains all vertices of σ . Therefore, G_Γ is a strongly quasiconvex subgroup of G_Ω by Theorem 7.5. If G_Γ is a hyperbolic group, then it is a stable subgroup of G_Ω . \square

In light of Proposition 7.6, we believe that strongly quasiconvex subgroups will play an important role in understanding the quasi-isometry classification of \mathcal{CFS} right-angled Coxeter groups. In particular, it suggests that the quasi-isometry classification of \mathcal{CFS} right-angled Coxeter groups may be no simpler than the quasi-isometry classification of all right-angled Coxeter groups.

We finish this section by illustrating the results of this section with three \mathcal{CFS} right-angled Coxeter groups whose quasi-isometry types can be distinguished utilizing their strongly quasiconvex subsets.

Example 7.7 Let Γ_1, Γ_2 , and Γ_3 be the graphs in Figure 8. All of the right-angled Coxeter groups $G_{\Gamma_1}, G_{\Gamma_2}$, and G_{Γ_3} are \mathcal{CFS} , but no pair of them are quasi-isometric. By [43], G_{Γ_1} is quasi-isometric to a right-angled Artin group with connected defining graph. Thus, all of G_{Γ_1} 's noncoarsely covering strongly quasiconvex subsets are quasitrees. However, G_{Γ_2} contains a one-ended hyperbolic strongly quasiconvex subgroup (induced by the blue 5-cycle) and G_{Γ_3} contain a virtually \mathbb{Z}^2 strongly quasiconvex subgroup (induced by the red 4-cycle). Table 1 summarizes some of the differences between $G_{\Gamma_1}, G_{\Gamma_2}$, and G_{Γ_3} .

8 Hyperbolically embedded subgroups of HHGs

In this section, we utilize Theorem 6.3 to prove the following classification of hyperbolically embedded subgroups of hierarchically hyperbolic groups. As our proof does not directly utilize the definition of hyperbolically embedded, we shall omit the definition here and direct the curious reader to [20].

Theorem 8.1 *Let G be a hierarchically hyperbolic group and let $\{H_i\}$ be a finite collection of subgroups. Then the following are equivalent:*

- (1) *The collection $\{H_i\}$ is hyperbolically embedded in G .*
- (2) *The collection $\{H_i\}$ is almost malnormal and each H_i is strongly quasiconvex.*

Combining work of Dahmani, Guirardel and Osin [20] and Sisto [50], the implication (1) \implies (2) holds for all finitely generated groups. To see that the converse does not hold in general, consider a nonvirtually cyclic lacunary hyperbolic group G where every proper subgroup is infinite cyclic and strongly quasiconvex — the existence of such a group is shown in [44, Theorem 1.12]. If I is a proper subgroup of G , then by [54, Theorem 1.2], I has finite index in its commensurator H . Thus H is a proper, infinite, almost malnormal, strongly quasiconvex subgroup of G . However, H cannot be hyperbolically embedded as G does not contain any nonabelian free subgroups and thus fails to be acylindrically hyperbolic; see [20; 45].

Despite this failure in general, Genevois showed that in the setting of CAT(0) cubical groups, (2) does imply (1) [28, Theorem 6.31]. Genevois employed a combination of the Bestvina–Bromberg–Fujiwara construction [12, Theorems A and B] with some work of Sisto [49, Theorems 6.3 and 6.4] that is summarized in the following sufficient conditions for a collection of subgroups to be hyperbolically embedded.

Theorem 8.2 [12; 49] *Let G be a finitely generated group and \mathcal{Z} be the collection of all (left) cosets of a finite collection of finitely generated subgroups $\{H_i\}$ in G . Fix a finite generating set S for G such that $H_i = \langle H_i \cap S \rangle$ for all i . Suppose for every $Z_1 \neq Z_2 \in \mathcal{Z}$ we are given a subset $\tau_{Z_1}(Z_2) \subseteq Z_1$ and for $Z_1, Z_2, Z_3 \in \mathcal{Z}$ define $d_{Z_3}^\tau(Z_1, Z_2) = \text{diam}_{Z_3}(\tau_{Z_3}(Z_1) \cup \tau_{Z_3}(Z_2))$. The collection $\{H_i\}$ is hyperbolically embedded in G if there exists $C > 0$ such that:*

- (P0) *For all $Z_1 \neq Z_2$, $\text{diam}(\tau_{Z_1}(Z_2)) \leq C$.*
- (P1) *For any triple $Z_1, Z_2, Z_3 \in \mathcal{Z}$ of distinct elements, at most one of the three numbers $d_{Z_1}^\tau(Z_2, Z_3)$, $d_{Z_2}^\tau(Z_1, Z_3)$ and $d_{Z_3}^\tau(Z_1, Z_2)$ is greater than C .*

(P2) For any $Z_1, Z_2 \in \mathcal{Z}$, the set

$$\{Z \in \mathcal{Z} \mid d_Z^\tau(Z_1, Z_2) > C\}$$

is finite.

(P3) For all $g \in G$, $d_{gZ_1}^\tau(gZ_2, gZ_3) = d_{Z_1}^\tau(Z_2, Z_3)$ for any $Z_1, Z_2, Z_3 \in \mathcal{Z}$.

As Genevois does in the cubical case, we shall show that an almost malnormal collection of strongly quasiconvex subgroups of an HHG satisfies (P0)–(P3) of Theorem 8.2. The bulk of that work is in Proposition 8.6, which we will state and prove after collecting a few preliminary lemmas.

Lemma 8.3 *Let $\{H_1, \dots, H_n\}$ be an almost malnormal collection of subgroups of a finitely generated group G and $B \geq 0$. For all $g_1, g_2 \in G$, if $g_1 H_i \neq g_2 H_j$, then $\text{diam}(N_B(g_1 H_i) \cap N_B(g_2 H_j))$ is finite.*

Proof The conclusion follows directly from [36, Proposition 9.4] and the definition of almost malnormal. \square

The next two lemmas tell us that a hierarchically quasiconvex subset coarsely intersects a strongly quasiconvex subset whenever the image under the gate map is large. Further, the diameter of this coarse intersection is proportional to the diameter of the gate. In addition to being key components in our proof of Theorem 8.1, these lemmas can also be interpreted as additional generalizations of the bounded geodesic image property of strongly quasiconvex subsets of hyperbolic spaces.

Lemma 8.4 *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with the bounded domain dichotomy, $A \subseteq \mathcal{X}$ be k -hierarchically quasiconvex subset, and $Y \subseteq \mathcal{X}$ be Q -strongly quasiconvex. There exists $r > 1$ depending on Q and k such that if $\text{diam}_{\mathcal{X}}(\mathfrak{g}_Y(A)) > r$, then $d_{\mathcal{X}}(a, \mathfrak{g}_Y(a)) < r$ for each $a \in \mathfrak{g}_A(Y)$.*

Proof By Proposition 5.7, there exists k' such that both A and Y are k' -hierarchically quasiconvex. Recall that for each point $x \in \mathcal{X}$ and $U \in \mathfrak{S}$, the distance in CU between $\mathfrak{g}_Y(x)$ and the closest point projection of $\pi_U(x)$ onto $\pi_U(Y)$ is uniformly bounded by some $\epsilon > 1$. Let $K \geq \epsilon$ be such that Y has the K -orthogonal projection dichotomy and that K is larger than the constant from the bridge theorem (Theorem 4.18) determined by k' . Define $\mathcal{H} = \{U \in \mathfrak{S} : \text{diam}(\pi_U(\mathfrak{g}_Y(A))) > 2K\}$. By the uniqueness axiom (10), there exists C such that if $\text{diam}(\mathfrak{g}_Y(A)) > C$, then $\mathcal{H} \neq \emptyset$. Assume $\text{diam}(\mathfrak{g}_Y(A)) > C$ and let $a \in \mathfrak{g}_A(Y)$. By (5) of the bridge theorem, $\text{Rel}_{2K}(a, \mathfrak{g}_Y(a)) \subseteq \mathcal{H}^\perp$. Suppose for the purposes of contradiction that $V \in \text{Rel}_{2K}(a, \mathfrak{g}_Y(a))$. Thus, there must exist $H \in \mathcal{H}$

with $V \perp H$. By Theorem 6.3, $CH \subseteq N_K(\pi_H(Y))$ and $CV \subseteq N_K(\pi_V(Y))$ which implies that $d_V(a, g_Y(a)) < K + \epsilon < 2K$. However, this contradicts $V \in \text{Rel}_{2K}(a, g_Y(a))$. Hence, $\text{Rel}_{2K}(a, g_Y(a)) = \emptyset$, and by the distance formula (Theorem 4.4), there exists K' depending only on K (and thus only on Q and κ_1) such that $d_X(a, g_Y(a)) < K'$. The conclusion follows by choosing $r = \max\{K', C\}$. \square

Lemma 8.5 *Let $(\mathcal{X}, \mathfrak{S})$ be an HHS with the bounded domain dichotomy, $A \subseteq \mathcal{X}$ be a k -hierarchically quasiconvex subset and $Y \subseteq \mathcal{X}$ be Q -strongly quasiconvex. There exists $r > 1$ depending on k and Q such that for all $D \geq r$ if $\text{diam}(g_Y(A)) > r$, then there exists $K \geq 1$ depending on k, D and Q such that*

$$\text{diam}(N_D(A) \cap N_D(Y)) \asymp_{1,K} \text{diam}(g_Y(A)).$$

Proof Let r be the constant given by Lemma 8.4 and suppose $\text{diam}(g_Y(A)) > r$. Thus, for $D \geq r$, $\text{diam}(N_D(A) \cap N_D(Y)) \neq \emptyset$. First consider $x, y \in N_D(A) \cap N_D(Y)$. Let $x', y' \in A$ be points such that $d_X(x, x') \leq D$ and $d_X(y, y') \leq D$. By Lemma 4.15 and the fact that $x, y \in N_D(Y)$, there exists K' depending on Q such that

$$d_X(x, g_Y(x')) \leq 4DK' \quad \text{and} \quad d_X(y, g_Y(y')) \leq 4DK'.$$

Hence,

$$d_X(x, y) \leq d_X(g_Y(x'), g_Y(y')) + 8DK',$$

which shows

$$\text{diam}(N_D(A) \cap N_D(Y)) \leq \text{diam}(g_Y(A)) + 8DK'.$$

For the inequality $\text{diam}(g_Y(A)) \leq \text{diam}(N_D(A) \cap N_D(Y))$, Lemma 8.4 provides $g_Y(g_A(Y)) \subseteq N_D(A) \cap N_D(Y)$ and the bridge theorem (Theorem 4.18) says there exists K'' depending on k and Q such that $g_Y(A) \subseteq N_{K''}(g_Y(g_A(Y)))$. Thus,

$$\text{diam}(g_Y(A)) \leq \text{diam}(g_Y(g_A(Y))) + 2K'' \leq \text{diam}(N_D(A) \cap N_D(Y)) + 2K''$$

and we are finished by choosing $K = \max\{2K'', 6DK' + 3K'\}$. \square

We now prove that the cosets of a collection of almost malnormal, strongly quasiconvex subgroups of an HHG satisfy (P0)–(P2) of Theorem 8.2 when $\tau_{Z_1}(Z_2)$ is defined by the gate map. This is the main tool for the proof of Theorem 8.1.

Proposition 8.6 *Let (G, \mathfrak{S}) be an HHG and $d(\cdot, \cdot)$ denote the distance in the word metric on G with respect to some fixed finite generating set. If $\{H_1, \dots, H_n\}$ is a collection of Q -strongly quasiconvex, almost malnormal subgroups of G and \mathcal{Z} is the collection of all left cosets of the H_i , then there exists $C > 0$ such that, for all distinct $Z_1, Z_2, Z_3 \in \mathcal{Z}$:*

- (1) $\text{diam}(\mathfrak{g}_{Z_1}(Z_2)) \leq C$.
- (2) If $d(\mathfrak{g}_{Z_3}(Z_1), \mathfrak{g}_{Z_3}(Z_2)) > C$, then

$$d(\mathfrak{g}_{Z_2}(Z_1), \mathfrak{g}_{Z_2}(Z_3)) < C \quad \text{and} \quad d(\mathfrak{g}_{Z_1}(Z_2), \mathfrak{g}_{Z_1}(Z_3)) < C.$$
- (3) $\{Z \in \mathcal{Z} \mid d(\mathfrak{g}_Z(Z_1), \mathfrak{g}_Z(Z_2)) > C\}$ has only a finite number of elements.

Proof We will prove each of the three assertions individually. Before beginning, we remind the reader that all hierarchically hyperbolic groups satisfy the bounded domain dichotomy and that every element of \mathcal{Z} is k -hierarchically quasiconvex for some k depending only on Q .

Assertion (1) There exists $C_1 > 0$ such that $\text{diam}(\mathfrak{g}_{Z_1}(Z_2)) \leq C_1$ for all $Z_1, Z_2 \in \mathcal{Z}$.

Proof Let $r > 1$ be the constant from Lemma 8.5 for Q and define

$$F = \{gH_i \in \mathcal{Z} \mid gH_i \cap B_r(e) \neq \emptyset\}$$

where $B_r(e)$ is the ball of radius r around the identity in G . Since F is a finite set, Lemma 8.3 provides a uniform number D_1 such that $\text{diam}(N_r(gH_i) \cap N_r(H_j)) \leq D_1$ for any distinct $gH_i, H_j \in F$. By Lemma 8.5, there exists D_2 depending on Q such that $\text{diam}(\mathfrak{g}_{H_j}(gH_i)) \leq D_2$ where $gH_i \neq H_j$ are elements in F .

We now prove that there is a uniform constant C_1 such that for each pair of distinct cosets g_1H_i and g_2H_j we have

$$\text{diam}(\mathfrak{g}_{g_1H_i}(g_2H_j)) \leq C_1.$$

If $\text{diam}(\mathfrak{g}_{g_1H_i}(g_2H_j)) \leq r$, then we are done. Otherwise, by Lemma 8.4, there are elements $h_i \in H_i$ and $h_j \in H_j$ such that $d_G(g_1h_i, g_2h_j) < r$. This implies that $h_i^{-1}g_1^{-1}g_2H_j$ is an element in F and $h_i^{-1}g_1^{-1}g_2H_j \neq H_i$. Therefore,

$$\text{diam}(\mathfrak{g}_{H_i}(h_i^{-1}g_1^{-1}g_2H_j)) \leq D_2.$$

Thus, by the coarse equivariance of the gate maps (Lemma 4.16), the diameter of $\mathfrak{g}_{g_1H_i}(g_2H_j)$ is bounded above by a uniform number C_1 . □

Assertion (2) There exists $C_2 > 0$ such that for all $Z_1, Z_2, Z_3 \in \mathcal{Z}$, if

$$d(\mathfrak{g}_{Z_3}(Z_1), \mathfrak{g}_{Z_3}(Z_2)) > C_2,$$

then

$$d(\mathfrak{g}_{Z_2}(Z_1), \mathfrak{g}_{Z_2}(Z_3)) < C_2 \quad \text{and} \quad d(\mathfrak{g}_{Z_1}(Z_2), \mathfrak{g}_{Z_1}(Z_3)) < C_2.$$

Proof Fix $\theta \geq \theta_0$. Let $Z_1, Z_2, Z_3 \in \mathcal{Z}$ and $B = H_\theta(\mathfrak{g}_{Z_2}(Z_1) \cup \mathfrak{g}_{Z_1}(Z_2))$. We remind the reader that they should view B as a bridge between Z_1 and Z_2 . Our goal is to show that there exists $b \in B$ such that $d(b, \mathfrak{g}_{Z_3}(b))$ is uniformly bounded. From this our conclusion will follow from the coarse Lipschitzness of the gate map.

By assertion (1), $\mathfrak{g}_{Z_3}(Z_1)$ and $\mathfrak{g}_{Z_3}(Z_2)$ are uniformly coarsely contained in $\mathfrak{g}_{Z_3}(B)$. Since the gate map is coarsely Lipschitz,

$$\text{diam}(\mathfrak{g}_{Z_3}(B)) \succeq d(\mathfrak{g}_{Z_3}(Z_1), \mathfrak{g}_{Z_3}(Z_2))$$

with constants depending only on Q . Let r be the constant from Lemma 8.4 with $A = B$ and $Y = Z_3$ and suppose $d(\mathfrak{g}_{Z_3}(Z_1), \mathfrak{g}_{Z_3}(Z_2))$ is large enough that $\text{diam}(\mathfrak{g}_{Z_3}(B)) > r$. By Lemma 8.4, there exists $b \in B$ such that $d(b, Z_3) < r$.

By Lemma 4.19, we have that $\mathfrak{g}_{Z_2}(Z_1)$ is uniformly coarsely equal to $\mathfrak{g}_{Z_2}(B)$ in particular $\mathfrak{g}_{Z_2}(b)$ is uniformly coarsely contained in $\mathfrak{g}_{Z_2}(Z_1)$. Since the gate maps are uniformly coarsely Lipschitz and $d(b, Z_3) < r$, we have that $d(\mathfrak{g}_{Z_2}(Z_3), \mathfrak{g}_{Z_2}(Z_1)) < C_2$. By switching the roles of Z_1 and Z_2 , we get $d(\mathfrak{g}_{Z_1}(Z_3), \mathfrak{g}_{Z_1}(Z_2)) < C_2$. \square

Assertion (3) *There exists $C_3 > 0$ such that for all $Z_1, Z_2 \in \mathcal{Z}$, the set*

$$\{Z \in \mathcal{Z} \mid d_X(\mathfrak{g}_Z(Z_1), \mathfrak{g}_Z(Z_2)) > C_3\}$$

has only a finite number of elements.

Proof Let $Z_1, Z_2 \in \mathcal{Z}$. Fix $\theta \geq \theta_0$ and let $B = H_\theta(\mathfrak{g}_{Z_2}(Z_1) \cup \mathfrak{g}_{Z_1}(Z_2))$. By the bridge theorem, we have that B is coarsely equals to the product of $\mathfrak{g}_{Z_1}(Z_2) \times H_\theta(a, b)$, where $a \in \mathfrak{g}_{Z_1}(Z_2)$ and $b = \mathfrak{g}_{Z_2}(a)$. By assertion (1), the gate $\mathfrak{g}_{Z_1}(Z_2)$ has uniformly bounded diameter. By Proposition 5.6, there exists $\lambda \geq \lambda_0$ such that $H_\theta(a, b)$ is contained in $\mathcal{P}_\lambda^1(a, b)$, the set of λ -hierarchy paths between a and b . Since the distance between a and b is finite, so is the diameter of $\mathcal{P}_\lambda^1(a, b)$. Therefore $H_\theta(a, b)$ has bounded diameter and so does the set $B = H_\theta(\mathfrak{g}_{Z_2}(Z_1) \cup \mathfrak{g}_{Z_1}(Z_2))$. Since G is locally finite, B can contain only a finite number of elements of G .

Let r be as in Lemma 8.4. Since $\mathfrak{g}_{Z_2}(Z_1), \mathfrak{g}_{Z_1}(Z_2) \subseteq B$, for any $Z \in \mathcal{Z}$ with $d(\mathfrak{g}_Z(Z_1), \mathfrak{g}_Z(Z_2))$ larger than r we have $\text{diam}(\mathfrak{g}_Z(B)) > r$. Thus every such Z intersects the r -neighborhood of B . By locally finiteness of G , we obtain that $N_r(B)$ contains a finite number of element of G . Since the elements of \mathcal{Z} are cosets of finitely many subgroups, every point of $N_r(B)$ can belong to uniformly finitely many elements of \mathcal{Z} , which concludes the proof of assertion (3). \square

Proposition 8.6 now holds by taking $C = \max\{C_1, C_2, C_3\}$. \square

We now have all the ingredients needed to give the proof of Theorem 8.1.

Proof of Theorem 8.1 Recall, we need to show that if G is a hierarchically hyperbolic group and $\{H_i\}$ a finite almost malnormal collection of strongly quasiconvex subgroups, then $\{H_i\}$ is hyperbolically embedded in G . In particular, we shall show that the left cosets of the H_i 's satisfy the requirements of Theorem 8.2. Since each H_i is a strongly quasiconvex subgroup of G , by [54, Theorem 1.2] we have that they are all finitely generated. Let S be a finite generating set for G such that for each i , $H_i \cap S$ generates H_i . As before, let \mathcal{Z} be the set of all left cosets of $\{H_i\}$. For every pair of distinct $Z_1, Z_2 \in \mathcal{Z}$ we want to define a set $\tau_{Z_1}(Z_2)$ that satisfies (P0)–(P3) of Theorem 8.2. If we define $\tau_{Z_1}(Z_2)$ as $\mathfrak{g}_{Z_1}(Z_2)$, Proposition 8.6 provides that (P0)–(P2) will be satisfied. However, since the gate maps are only coarsely equivariant, condition (P3) may not hold.

Thus, for $Z_1 \neq Z_2$ define

$$\tau_{Z_1}(Z_2) = \bigcup_{g \in G} g^{-1} \mathfrak{g}_{gZ_1}(gZ_2).$$

By construction we have that $\tau_{gZ_1}(gZ_2) = g(\tau_{Z_1}(Z_2))$ and thus (P3) holds. Since $\tau_{Z_1}(Z_2)$ and $\mathfrak{g}_{Z_1}(Z_2)$ uniformly coarsely coincide by the coarse equivariance of the gates maps (Lemma 4.16), (P0)–(P2) are satisfied as a corollary of Proposition 8.6. Hence, the collection $\{H_i\}$ is hyperbolically embedded in G by Theorem 8.2. \square

Our method of proof for Theorem 8.1 relies in a fundamental way upon the coarse equivariance of the gate map. If the group G has an HHS structure, but not an HHG structure, then the gate map need not be coarsely equivariant. In particular, Theorem 8.1 does not (currently) apply to the fundamental groups of nonflip graph manifolds and thus we have the following interesting case of Question 4.

Question 6 *If M is a nonflip graph manifold and $\{H_i\}$ is a finite, almost malnormal collection of strongly quasiconvex subgroups of $\pi_1(M)$, is $\{H_i\}$ hyperbolically embedded in $\pi_1(M)$?*

Appendix Subsets with arbitrary reasonable lower relative divergence

The proposition in this appendix utilizes the notion of asymptotic equivalence between families of functions. We will present the definition in the specific case we need and direct the reader to [53, Section 2] for the more general case.

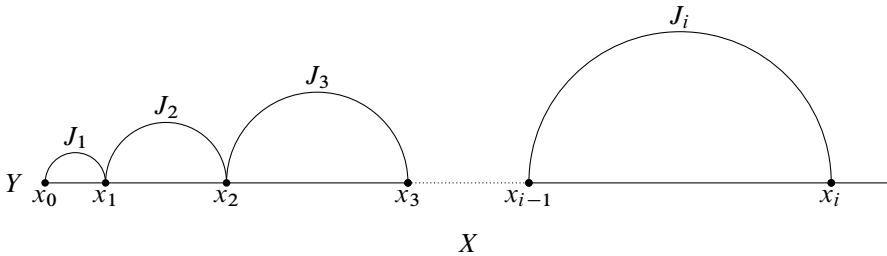


Figure 9: By controlling the length of each arc J_i we can get the desired lower relative divergence of the geodesic space X with respect to the subspace Y .

Definition A.1 Let f and g be two functions from $[0, \infty)$ to $[0, \infty)$. The function f is *dominated by the function* g if there are positive constants A, B, C and D such that $f(r) \leq Ag(Br) + Cr$ for all $r > D$. Two functions f and g are *equivalent* if f is dominated by g and vice versa.

Let X be a geodesic metric space and $\{\sigma_\rho^n\} = \text{div}(X, Y)$ be the lower relative divergence of X with respect to some subset $Y \subseteq X$. We say $\text{div}(X, Y)$ is *equivalent* to a function $f : [0, \infty) \rightarrow [0, \infty)$ if there exists $L \in (0, 1]$ and a positive integer M such that $\sigma_{L\rho}^{Mn}$ is equivalent to f for all $\rho \in (0, 1]$ and $n \geq 2$.

Proposition A.2 Let $f : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing function, and assume that there is a positive integer r_0 such that $f(r) \geq r$ for each $r > r_0$. There is a geodesic space X with a subspace Y such that the lower relative divergence $\text{div}(X, Y)$ is equivalent to f .

Proof Let Y be a ray with initial point x_0 . Let (x_i) be the sequence of points along Y such that for each $i \geq 1$ the distance $d_Y(x_{i-1}, x_i) = i$ and we connect each pair (x_{i-1}, x_i) by a segment J_i of length $f(i)$; see Figure 9. Let X be the resulting geodesic space and $\text{div}(X, Y) = \{\sigma_\rho^n\}$. We shall show that $\text{div}(X, Y)$ is equivalent to f .

We first prove that for all $n \geq 3$ and $\rho \in (0, 1]$, f dominates σ_ρ^n by showing that $\sigma_\rho^n(r) \leq f((n + 3)r)$ for each $r > r_0$. Let i_0 be a smallest integer that is greater or equal to $(n + 2)r$. Let x and y be two points in the segment J_{i_0} such that

$$d(x_{i_0-1}, x) = d(x_{i_0}, y) = r.$$

Both x and y belong to $\partial N_r(Y)$. Moreover, the subpath α of J_{i_0} connecting x and y lies outside the r -neighborhood of Y , and the length of α is exactly is $f(i) - 2r$.

Therefore, $d(x, y) = \min\{i_0 + 2r, f(i_0) - 2r\}$. Hence $d(x, y) \geq nr$ as

$$f(i_0) - 2r \geq f((n+2)r) - 2r \geq (n+2)r - 2r = nr$$

and

$$i_0 + 2r \geq (n+4)r \geq nr.$$

Since α is the unique path outside the ρr -neighborhood of Y connecting x and y ,

$$\sigma_\rho^n(r) \leq d_{\rho r}(x, y) = f(i_0) - 2r \leq f(i_0).$$

Since $i_0 \leq (n+2)r + 1 \leq (n+3)r$ and f is nondecreasing, $f(i_0) \leq f((n+3)r)$. Thus, $\sigma_\rho^n(r) \leq f((n+3)r)$, which implies that σ_ρ^n is dominated by f .

Now we prove that for all $n \geq 3$ and $\rho \in (0, 1]$, σ_ρ^n dominates f by showing that $\sigma_\rho^n(r) \geq f(r) - 2r$ for each $r > r_0$. Let u and v be an arbitrary points in $\partial N_r(Y)$ such that $d(u, v) \geq nr$ and there is a path outside the r -neighborhood of Y connecting u and v . Therefore, u and v must lie in some segment J_{i_1} . We can assume that $d(u, x_{i_1-1}) = d(v, x_{i_1}) = r$. Therefore,

$$i_1 \geq d(x_{i_1-1}, x_{i_1}) \geq d(u, v) - 2r \geq nr - 2r \geq r.$$

This implies that $f(i_1) \geq f(r)$ since f is nondecreasing. Since the subpath β of J_{i_1} connecting u and v is the unique path outside the ρr -neighborhood of Y connecting these points,

$$d_{\rho r}(u, v) = f(i_1) - 2r \geq f(r) - 2r.$$

Therefore, $\sigma_\rho^n(r) \geq f(r) - 2r$ which implies that σ_ρ^n dominates f . Thus, the lower relative divergence $\text{div}(X, Y)$ is equivalent to f . \square

References

- [1] **C Abbott, J Behrstock, M G Durham**, *Largest acylindrical actions and stability in hierarchically hyperbolic groups*, *Trans. Amer. Math. Soc. Ser. B* 8 (2021) 66–104 MR Zbl
- [2] **I Agol**, *The virtual Haken conjecture*, *Doc. Math.* 18 (2013) 1045–1087 MR Zbl
- [3] **Y Antolín, M Mj, A Sisto, S J Taylor**, *Intersection properties of stable subgroups and bounded cohomology*, *Indiana Univ. Math. J.* 68 (2019) 179–199 MR Zbl
- [4] **T Aougab, M G Durham, S J Taylor**, *Pulling back stability with applications to $\text{Out}(F_n)$ and relatively hyperbolic groups*, *J. Lond. Math. Soc.* 96 (2017) 565–583 MR Zbl
- [5] **G N Arzhantseva**, *On quasiconvex subgroups of word hyperbolic groups*, *Geom. Dedicata* 87 (2001) 191–208 MR Zbl

- [6] **G N Arzhantseva, C H Cashen, D Gruber, D Hume**, *Characterizations of Morse quasi-geodesics via superlinear divergence and sublinear contraction*, Doc. Math. 22 (2017) 1193–1224 MR Zbl
- [7] **J Behrstock**, *A counterexample to questions about boundaries, stability, and commensurability*, from “Beyond hyperbolicity” (M Hagen, R Webb, H Wilton, editors), London Math. Soc. Lecture Note Ser. 454, Cambridge Univ. Press (2019) 151–159 MR Zbl
- [8] **J Behrstock, M F Hagen, A Sisto**, *Asymptotic dimension and small-cancellation for hierarchically hyperbolic spaces and groups*, Proc. Lond. Math. Soc. 114 (2017) 890–926 MR Zbl
- [9] **J Behrstock, M F Hagen, A Sisto**, *Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups*, Geom. Topol. 21 (2017) 1731–1804 MR Zbl
- [10] **J Behrstock, M Hagen, A Sisto**, *Hierarchically hyperbolic spaces II: Combination theorems and the distance formula*, Pacific J. Math. 299 (2019) 257–338 MR Zbl
- [11] **J Behrstock, M F Hagen, A Sisto**, *Quasiflats in hierarchically hyperbolic spaces*, Duke Math. J. 170 (2021) 909–996 MR Zbl
- [12] **M Bestvina, K Bromberg, K Fujiwara**, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publ. Math. Inst. Hautes Études Sci. 122 (2015) 1–64 MR Zbl
- [13] **B H Bowditch**, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. 22 (2012) art. id. 1250016 MR Zbl
- [14] **B H Bowditch**, *Coarse median spaces and groups*, Pacific J. Math. 261 (2013) 53–93 MR Zbl
- [15] **B H Bowditch**, *Uniform hyperbolicity of the curve graphs*, Pacific J. Math. 269 (2014) 269–280 MR Zbl
- [16] **B H Bowditch**, *Convex hulls in coarse median spaces*, preprint (2018) Available at <http://homepages.warwick.ac.uk/~masgak/papers/hulls-cms.pdf>
- [17] **J F Brock**, *The Weil–Petersson metric and volumes of 3–dimensional hyperbolic convex cores*, J. Amer. Math. Soc. 16 (2003) 495–535 MR Zbl
- [18] **C H Cashen**, *Morse subsets of CAT(0) spaces are strongly contracting*, Geom. Dedicata 204 (2020) 311–314 MR Zbl
- [19] **R Charney, H Sultan**, *Contracting boundaries of CAT(0) spaces*, J. Topol. 8 (2015) 93–117 MR Zbl
- [20] **F Dahmani, V Guirardel, D Osin**, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, Mem. Amer. Math. Soc. 1156, Amer. Math. Soc., Providence, RI (2017) MR Zbl
- [21] **C Druţu, M Kapovich**, *Geometric group theory*, American Mathematical Society Colloquium Publications 63, Amer. Math. Soc., Providence, RI (2018) MR Zbl
- [22] **C Druţu, S Mozes, M Sapir**, *Divergence in lattices in semisimple Lie groups and graphs of groups*, Trans. Amer. Math. Soc. 362 (2010) 2451–2505 MR Zbl

- [23] **M G Durham**, *The augmented marking complex of a surface*, J. Lond. Math. Soc. 94 (2016) 933–969 MR Zbl
- [24] **M G Durham, M F Hagen, A Sisto**, *Boundaries and automorphisms of hierarchically hyperbolic spaces*, Geom. Topol. 21 (2017) 3659–3758 MR Zbl
- [25] **M G Durham, S J Taylor**, *Convex cocompactness and stability in mapping class groups*, Algebr. Geom. Topol. 15 (2015) 2839–2859 MR Zbl
- [26] **A Eskin, H Masur, K Rafi**, *Large-scale rank of Teichmüller space*, Duke Math. J. 166 (2017) 1517–1572 MR Zbl
- [27] **B Farb**, *Some problems on mapping class groups and moduli space*, from “Problems on mapping class groups and related topics” (B Farb, editor), Proc. Sympos. Pure Math. 74, Amer. Math. Soc., Providence, RI (2006) 11–55 MR Zbl
- [28] **A Genevois**, *Hyperbolicities in CAT(0) cube complexes*, Enseign. Math. 65 (2019) 33–100 MR Zbl
- [29] **A Genevois**, *Quasi-isometrically rigid subgroups in right-angled Coxeter groups*, Algebr. Geom. Topol. 22 (2022) 657–708 MR Zbl
- [30] **R Gitik**, *Ping-pong on negatively curved groups*, J. Algebra 217 (1999) 65–72 MR Zbl
- [31] **M Gromov**, *Hyperbolic groups*, from “Essays in group theory” (S M Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR Zbl
- [32] **M Gromov**, *Asymptotic invariants of infinite groups*, from “Geometric group theory, II” (G A Niblo, M A Roller, editors), London Math. Soc. Lecture Note Ser. 182, Cambridge Univ. Press (1993) 1–295 MR Zbl
- [33] **D Groves, J F Manning**, *Hyperbolic groups acting improperly*, preprint (2018) arXiv 1808.02325
- [34] **T Haettel**, *Hyperbolic rigidity of higher rank lattices*, Ann. Sci. Éc. Norm. Supér. 53 (2020) 439–468 MR Zbl
- [35] **M F Hagen, H Petyt**, *Projection complexes and quasimedians maps*, Algebr. Geom. Topol. 22 (2022) 3277–3304 MR Zbl
- [36] **G C Hruska**, *Relative hyperbolicity and relative quasiconvexity for countable groups*, Algebr. Geom. Topol. 10 (2010) 1807–1856 MR Zbl
- [37] **A Karrer**, *Right-angled Coxeter groups with totally disconnected Morse boundaries*, preprint (2021) arXiv 2105.04029
- [38] **H Kim**, *Stable subgroups and Morse subgroups in mapping class groups*, Internat. J. Algebra Comput. 29 (2019) 893–903 MR Zbl
- [39] **T Koberda, J Mangahas, S J Taylor**, *The geometry of purely loxodromic subgroups of right-angled Artin groups*, Trans. Amer. Math. Soc. 369 (2017) 8179–8208 MR Zbl
- [40] **H A Masur, Y N Minsky**, *Geometry of the complex of curves, I: Hyperbolicity*, Invent. Math. 138 (1999) 103–149 MR Zbl
- [41] **H A Masur, Y N Minsky**, *Geometry of the complex of curves, II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000) 902–974 MR Zbl

- [42] **H Masur, S Schleimer**, *The geometry of the disk complex*, J. Amer. Math. Soc. 26 (2013) 1–62 MR Zbl
- [43] **H T Nguyen, H C Tran**, *On the coarse geometry of certain right-angled Coxeter groups*, Algebr. Geom. Topol. 19 (2019) 3075–3118 MR Zbl
- [44] **A Y Ol’shanskii, D V Osin, M V Sapir**, *Lacunary hyperbolic groups*, Geom. Topol. 13 (2009) 2051–2140 MR Zbl
- [45] **D Osin**, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016) 851–888 MR Zbl
- [46] **N Petrosyan**, *Decomposing groups by codimension-1 subgroups*, Proc. Amer. Math. Soc. 150 (2022) 4587–4601 MR Zbl
- [47] **K Rafi**, *A combinatorial model for the Teichmüller metric*, Geom. Funct. Anal. 17 (2007) 936–959 MR Zbl
- [48] **J Russell**, *From hierarchical to relative hyperbolicity*, Int. Math. Res. Not. 2022 (2022) 575–624 MR Zbl
- [49] **A Sisto**, *On metric relative hyperbolicity*, preprint (2012) arXiv 1210.8081
- [50] **A Sisto**, *Quasi-convexity of hyperbolically embedded subgroups*, Math. Z. 283 (2016) 649–658 MR Zbl
- [51] **D Spriano**, *Hyperbolic HHS, II: Graphs of hierarchically hyperbolic groups*, preprint (2018) arXiv 1801.01850
- [52] **H Sultan**, *Hyperbolic quasi-geodesics in CAT(0) spaces*, Geom. Dedicata 169 (2014) 209–224 MR Zbl
- [53] **H C Tran**, *Relative divergence of finitely generated groups*, Algebr. Geom. Topol. 15 (2015) 1717–1769 MR Zbl
- [54] **H C Tran**, *On strongly quasiconvex subgroups*, Geom. Topol. 23 (2019) 1173–1235 MR Zbl
- [55] **D T Wise**, *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics 117, Amer. Math. Soc., Providence, RI (2012) MR Zbl

Math Department, Rice University

Houston, TX, United States

Mathematical Institute, University of Oxford

Oxford, United Kingdom

Department of Mathematics, The University of Oklahoma

Norman, OK, United States

`jacob.russell@rice.edu`, `davide.spriano@chch.ox.ac.uk`,
`hungtran280687@gmail.com`

Received: 16 September 2020

Revised: 17 June 2021

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Norihiko Minami	Nagoya Institute of Technology nori@nitech.ac.jp
Tara E. Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Indira Chatterji	CNRS & Université Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Thomas Nikolaus	University of Münster nikolaus@uni-muenster.de
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris 13 bobol@math.univ-paris13.fr
Corneli Druţu	University of Oxford cornelia.drutu@maths.ox.ac.uk	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Zoltán Szabó	Princeton University szabo@math.princeton.edu
David Futер	Temple University dfuter@temple.edu	Ulrike Tillmann	Oxford University tillmann@maths.ox.ac.uk
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Nathalie Wahl	University of Copenhagen wahl@math.ku.dk
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Chris Wendl	Humboldt-Universität zu Berlin wendl@math.hu-berlin.de
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu	Daniel T. Wise	McGill University, Canada daniel.wise@mcgill.ca
Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr		


See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2023 is US \$650/year for the electronic version, and \$940/year (+ \$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2023 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23

Issue 3 (pages 963–1462)

2023

- Projective naturality in Heegaard Floer homology 963
MICHAEL GARTNER
- Geometrically bounding 3–manifolds, volume and Betti numbers 1055
JIMING MA and FANGTING ZHENG
- Constrained knots in lens spaces 1097
FAN YE
- Convexity in hierarchically hyperbolic spaces 1167
JACOB RUSSELL, DAVIDE SPRIANO and HUNG CONG TRAN
- Finite presentations for stated skein algebras and lattice gauge field theory 1249
JULIEN KORINMAN
- On the functoriality of \mathfrak{sl}_2 tangle homology 1303
ANNA BELIAKOVA, MATTHEW HOGANCAMP,
KRZYSZTOF K PUTYRA and STEPHAN M WEHRLI
- Asymptotic translation lengths and normal generation for pseudo-Anosov monodromies of fibered 3–manifolds 1363
HYUNGRYUL BAIK, EIKO KIN, HYUNSHIK SHIN and
CHENXI WU
- Geometric triangulations and highly twisted links 1399
SOPHIE L HAM and JESSICA S PURCELL