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We provide finite presentations for stated skein algebras and deduce that those algebras are Koszul and that they are isomorphic to the quantum moduli algebras appearing in lattice gauge field theory, generalizing previous results of Bullock, Frohman, Kania-Bartoszynska and Faitg.

57R56; 57K31

1 Introduction

Stated skein algebras and lattice gauge field theory A *punctured surface* is a pair $\mathbf{\Sigma} = (\Sigma, \mathcal{P})$, where Σ is a compact oriented surface and $\mathcal{P}$ is a (possibly empty) finite subset of Σ which nontrivially intersects each boundary component. We write $\Sigma_{\mathcal{P}} := \Sigma \setminus \mathcal{P}$. The set $\partial\Sigma \setminus \mathcal{P}$ consists of a disjoint union of open arcs, which we call *boundary arcs*.

Warning In this paper, the punctured surface $\mathbf{\Sigma}$ will be called open if the surface Σ has nonempty boundary and closed otherwise. This convention differs from the traditional one, where some authors refer to an open surface as a punctured surface $\mathbf{\Sigma} = (\Sigma, \mathcal{P})$ with Σ closed and $\mathcal{P} \neq \varnothing$ (in which case $\Sigma_{\mathcal{P}}$ is not closed).

The *Kauffman-bracket skein algebras* were introduced by Bullock and Turaev as a tool to study the SU(2) Witten–Reshetikhin–Turaev topological quantum field theories [45; 51]. They are associative unitary algebras $\mathcal{S}_\omega(\mathbf{\Sigma})$ indexed by a closed punctured surface $\mathbf{\Sigma}$ and an invertible element $\omega \in \mathbb{k}^\times$ in some commutative unital ring $\mathbb{k}$. Bonahon and Wong [12] and Lê [40] generalized the notion of Kauffman-bracket skein algebras to open punctured surfaces, where in addition to closed curves the algebras are generated by arcs whose endpoints are endowed with a sign, $\pm$ (a state). The motivation for the introduction of these so-called *stated skein algebras* is their good behavior for the operation of gluing two boundary arcs together. This property permitted the authors of [12] to define an embedding of the skein algebra into a quantum torus, named the quantum trace, and offers new tools to study the representation theory of skein algebras.

Except for genus 0 and 1 surfaces (see Bullock and Przytycki [21]), no finite presentation for the Kauffman-bracket skein algebras is known, though a conjecture in that direction was formulated in Santharoubane [46, Conjecture 1.2]. However, it is well known that they are finitely generated; see Abdiel and Frohman [1], Bullock [18], Frohman and Kania-Bartoszynska [30] and Santharoubane [46]. The corresponding problem for stated skein algebras of open punctured surfaces is easier. Finite presentations of stated skein algebras were given for a disc with two punctures on its boundary (for the bigon) and for the disc with three punctures on its boundary (for the triangle) in [40], for the disc with two punctures on its boundary and one inner puncture in Korinman [35] and for any connected punctured surface having exactly one boundary component, one puncture on the boundary and possibly inner punctures in Faitg [27].

Our first purpose is to provide explicit finite presentations for stated skein algebras of an arbitrary connected open punctured surface $\mathbf{\Sigma}$. Let us briefly sketch their construction; we refer to Section 2.2 for details.

The finite presentations we will define depend on the choice of a finite presentation $\mathbb{P}$ of some groupoid $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$. In brief, for each boundary arc a of $\mathbf{\Sigma}$, choose a point $v_a \in a$ and let $\mathbb{V}$ be the set of such points. The groupoid $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ is the full subcategory of the fundamental groupoid of $\Sigma_{\mathcal{P}}$ whose set of objects is $\mathbb{V}$. A finite presentation $\mathbb{P} = (\mathbb{G}, \mathbb{RL})$ for $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ will consist in a finite set $\mathbb{G}$ of generating paths relating points of $\mathbb{V}$ and a finite set $\mathbb{RL}$ of relations among those paths which satisfy some axioms (see Section 2.2 for details). For instance for the triangle $\mathbb{T}$ (the disc with three punctures on its boundary), the groupoid $\Pi_1(\mathbb{T}, \mathbb{V})$ admits the presentation with generators $\mathbb{G} = \{\alpha, \beta, \gamma\}$, drawn in Figure 1, and the unique relation $\alpha\beta\gamma = 1$.

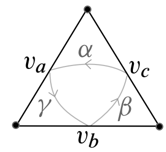

Figure 1: The triangle and some paths.

A path $\alpha \in \mathbb{G}$ can be seen as an arc in $\Sigma_{\mathcal{P}}$ and, after choosing some states $\varepsilon, \varepsilon' \in \{-, +\}$ for its endpoints, we get an element $\alpha_{\varepsilon\varepsilon'} \in \mathcal{S}_\omega(\mathbf{\Sigma})$ in the stated skein algebra. We denote by $\mathcal{A}^{\mathbb{G}} \subset \mathcal{S}_\omega(\mathbf{\Sigma})$ the (finite) set of such elements. It was proved in Korinman [38] that $\mathcal{A}^{\mathbb{G}}$ generates $\mathcal{S}_\omega(\mathbf{\Sigma})$ and its elements will be the generators of our presentations.

Concerning the relations, first for each $\alpha \in \mathbb{G}$, one has a *q–determinant* relation between the elements $\alpha_{\varepsilon\varepsilon'}$. For each pair $(\alpha, \beta) \in \mathbb{G}^2$ we will associate a finite set of *arc exchange relations* permitting us to express an element of the form $\alpha_{\varepsilon\varepsilon'}\beta_{\mu\mu'} \in \mathcal{S}_\omega(\mathbf{\Sigma})$ as a linear combination of elements of the form $\beta_{ab}\alpha_{cd}$. Finally, to each relation $R \in \mathbb{RL}$ in the finite presentation $\mathbb{P}$, we will associate a finite set of so-called *trivial loop relations*.

Theorem 1.1 *Let $\mathbf{\Sigma}$ be a connected open punctured surface and $\mathbb{P}$ a finite presentation of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$. Then the stated skein algebra $\mathcal{S}_\omega(\mathbf{\Sigma})$ is presented by the set of generators $\mathcal{A}^{\mathbb{G}}$ and by the q–determinant, arc exchange and trivial loop relations.*

For every open punctured surface, we can choose a finite presentation $\mathbb{P}$ of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ such that the set of relations is empty (for instance for the triangle of Figure 1, one might choose the presentation with generators $\mathbb{G} = \{\alpha, \beta\}$ and no relations). In this case, the presentation of $\mathcal{S}_\omega(\mathbf{\Sigma})$ is quadratic inhomogeneous and, by using the diamond lemma, we prove:

Theorem 1.2 *For $\mathbf{\Sigma}$ a connected open punctured surface, the quadratic inhomogeneous algebra $\mathcal{S}_\omega(\mathbf{\Sigma})$ is Koszul and admits a Poincaré–Birkhoff–Witt (PBW) basis.*

Theorem 1.2 implies that $\mathcal{S}_\omega(\mathbf{\Sigma})$ has an explicit minimal projective resolution (the so-called Koszul resolution), which permits us to effectively compute its cohomology (see Loday and Vallette [42] for details).

Let (Γ, c) be a ciliated graph, that is a finite graph with the data for each vertex of a linear ordering of its adjacent half-edges. Inspired by Fock and Rosly's original work in [29] on the Poisson structure of character varieties, Alekseev, Grosse and

Schomerus [2; 3; 5] and Buffenoir and Roche [15; 16] independently defined the so-called *quantum moduli algebras* $\mathcal{L}_\omega(\Gamma, c)$, which are combinatorial quantizations of relative character varieties (see Section 4.2 for details). Those algebras arise with some right comodule map $\Delta^{\mathcal{G}} : \mathcal{L}_\omega(\Gamma, c) \to \mathcal{L}_\omega(\Gamma, c) \otimes \mathcal{O}_q[\mathcal{G}]$, where $\mathcal{O}_q[\mathcal{G}] = \mathcal{O}_q[\mathrm{SL}_2]^{\otimes \mathring{V}(\Gamma)}$ is the so-called quantum gauge group Hopf algebra and $q := \omega^{-4}$. The subalgebra $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma) \subset \mathcal{L}_\omega(\Gamma, c)$ of coinvariant vectors plays an important role in combinatorial quantization. More precisely, as reviewed in Section 4.1, we associate to each ciliated graph (Γ, c) two punctured surfaces: an open one $\mathbf{\Sigma}^0(\Gamma, c)$ and a closed one $\mathbf{\Sigma}(\Gamma)$, such that the algebras $\mathcal{L}_\omega(\Gamma, c)$ and $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$ are quantizations of the $\mathrm{SL}_2(\mathbb{C})$ (relative) character varieties of $\mathbf{\Sigma}^0(\Gamma, c)$ and $\mathbf{\Sigma}(\Gamma)$, respectively, with their Fock–Rosly Poisson structures. We deduce from Theorem 1.1:

Theorem 1.3 *There exist isomorphisms of algebras* $\mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma, c)) \cong \mathcal{L}_\omega(\Gamma, c)$ *and* $\mathcal{S}_\omega(\mathbf{\Sigma}(\Gamma)) \cong \mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$.

Theorem 1.3 is not surprising and was already proved in some cases. First it is well known that (stated) skein algebras also induce deformation quantizations of (relative) character varieties: it follows from the work in Bullock [17], Przytycki and Sikora [44] and Turaev [50] for closed punctured surfaces and is proved in Korinman and Quesney [39, Theorem 1.3] and Costantino and Lê [26, Theorem 8.12] for open punctured surfaces. So Theorem 1.3 was expected; for instance its statement was conjectured in [26]. Next the skein origin of the defining relations of quantum moduli algebra was discovered by Bullock, Frohman and Kania-Bartoszynska in [19] where the authors already proved that $\mathcal{S}_\omega(\mathbf{\Sigma}(\Gamma))$ and $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$ are isomorphic in the particular case where $\Bbbk = \mathbb{C}[\![\hbar]\!]$ and $q := \omega^{-4} = \exp \hbar$. However, their proof does not extend to arbitrary ring (see item (vi) of Section 5). Finally, in the special case where (Γ, c) is the so-called daisy graph (it has only one vertex, so $\mathbf{\Sigma}^0(\Gamma, c)$ has exactly one boundary component with one puncture on it), Theorem 1.3 was proved by Faitg in [27] in the case where ω is not a root of unity. A detailed comparison between Faitg's isomorphism and ours is made in Section 4.4. Faitg's result can also be derived indirectly from the works in Ben-Zvi, Brochier and Jordan [9] and Gunningham, Jordan and Safronov [31], as detailed in Section 4.4. As pointed out to us by the anonymous referee, there is an important difference between our definition of quantum moduli algebras and the original one. In the original approaches, the algebra $\mathcal{L}_\omega(\Gamma, c)$ is seen as a $U_q\mathrm{sl}_2^{\otimes n}$–module, where n is the number of external vertices of Γ, and $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$ is then defined as the subalgebra of invariant vectors for this action. Here, $\mathcal{L}_\omega(\Gamma, c)$ is rather seen as an $\mathcal{O}_q[\mathrm{SL}_2]^{\otimes n}$–comodule and $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$ is defined as the subalgebra of coinvariant vectors instead. When q is generic both definitions coincide, however when q is a root of unity they differ in general (see Section 5 for details). In particular, the isomorphism in Theorem 1.3 holds with our definition of quantum moduli algebra and might fail with the original one, at roots of unity.

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2 Finite presentations for stated skein algebras

2.1 Definitions and first properties of stated skein algebras

Definition 2.1 A *punctured surface* is a pair $\mathbf{\Sigma} = (\Sigma, \mathcal{P})$ where Σ is a compact oriented surface and $\mathcal{P}$ is a finite subset of Σ which nontrivially intersects each boundary component. A *boundary arc* is a connected component of $\partial\Sigma \setminus \mathcal{P}$. We write $\Sigma_{\mathcal{P}} := \Sigma \setminus \mathcal{P}$.

Definition of stated skein algebras Before precisely stating the definition of stated skein algebras, let us sketch it informally. Given a punctured surface $\mathbf{\Sigma}$ and an invertible element $\omega \in \mathbb{k}^{\times}$ in some commutative unital ring $\mathbb{k}$, the stated skein algebra $\mathcal{S}_\omega(\mathbf{\Sigma})$ is the quotient of the $\mathbb{k}$–module freely spanned by isotopy classes of stated tangles in $\Sigma_{\mathcal{P}} \times (0,1)$ by some local skein relations. Figure 2, left, illustrates such a stated tangle: each point of $\partial T \subset \partial\Sigma_{\mathcal{P}}$ is equipped with a sign $+$ or $-$ (the state). Here the stated tangle is the union of three stated arcs and one closed curve. In order to work with two-dimensional pictures, we will consider the projection of tangles in $\Sigma_{\mathcal{P}}$ as in Figure 2, right; such a projection will be referred to as a diagram.

A *tangle* in $\Sigma_{\mathcal{P}} \times (0,1)$ is a compact framed, properly embedded one-dimensional manifold $T \subset \Sigma_{\mathcal{P}} \times (0,1)$ such that for every point of $\partial T \subset \partial\Sigma_{\mathcal{P}} \times (0,1)$ the framing is parallel to the $(0,1)$ factor and points in the direction of 1. Here, by framing, we refer to a thickening of T to an oriented surface. The *height* of $(v,h) \in \Sigma_{\mathcal{P}} \times (0,1)$ is h. If b is a boundary arc and T a tangle, we impose that no two points in $\partial_b T := \partial T \cap b \times (0,1)$ have the same heights, hence the set $\partial_b T$ is totally ordered by the heights. Two tangles are isotopic if they are isotopic through the class of tangles that preserve the boundary height orders. By convention, the empty set is a tangle only isotopic to itself.

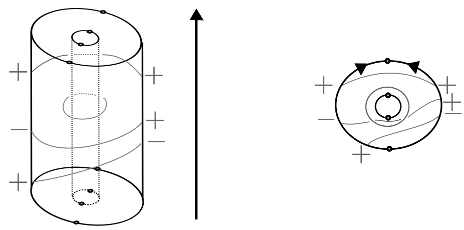
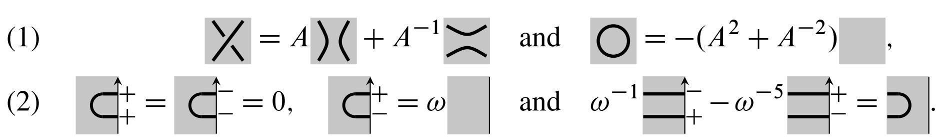


Figure 2: A stated tangle (left) and its associated diagram (right). The arrows represent the height orders.

Let $\pi\colon \Sigma_{\mathcal{P}} \times (0,1) \to \Sigma_{\mathcal{P}}$ be the projection with $\pi(v,h) = v$. A tangle T is in *generic position* if, for each of its points, the framing is parallel to the $(0,1)$ factor, points in the direction of 1 and is such that $\pi|_T\colon T \to \Sigma_{\mathcal{P}}$ is an immersion with at most transversal double points in the interior of $\Sigma_{\mathcal{P}}$. Every tangle is isotopic to a tangle in generic position. A *diagram* is the image $D = \pi(T)$ of a tangle in generic position, together with the over/undercrossing information at each double point. An isotopy class of diagram D together with a total order of $\partial_b D := \partial D \cap b$ for each boundary arc b uniquely define an isotopy class of a tangle. When choosing an orientation $\mathfrak{o}(b)$ of a boundary arc b and a diagram D, the set $\partial_b D$ receives a natural order by setting that the points are increasing when going in the direction of $\mathfrak{o}(b)$. We will represent tangles by drawing a diagram and an orientation (an arrow) for each boundary arc, as in Figure 2. When a boundary arc b is oriented we assume that the order of the heights of the points of $\partial_b D$ coincides with the order induced by the orientation of the boundary arc. A *state* of a tangle is a map $s\colon \partial T \to \{-,+\}$. A pair (T,s) is called a *stated tangle*. We define a *stated diagram* (D,s) in a similar manner.

Let $\omega \in \mathbb{k}^\times$ be an invertible element and write $A := \omega^{-2}$.

Definition 2.2 [40] The *stated skein algebra* $\mathcal{S}_\omega(\mathbf{\Sigma})$ is the free $\mathbb{k}$–module generated by isotopy classes of stated tangles in $\Sigma_{\mathcal{P}} \times (0,1)$ modulo the relations (1) and (2):



The product of two classes of stated tangles $[T_1,s_1]$ and $[T_2,s_2]$ is defined by isotoping T_1 and T_2 in $\Sigma_{\mathcal{P}} \times \left(\frac{1}{2},1\right)$ and $\Sigma_{\mathcal{P}} \times \left(0,\frac{1}{2}\right)$, respectively, and then setting $[T_1,s_1]\cdot[T_2,s_2]$ equal to $[T_1 \cup T_2, s_1 \cup s_2]$. Figure 3 illustrates this product.

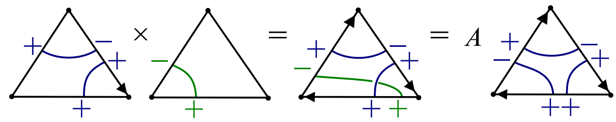


Figure 3: An illustration of the product in stated skein algebras.

For a closed punctured surface, $\mathcal{S}_\omega(\mathbf{\Sigma})$ coincides with the classical (Turaev's) Kauffman-bracket skein algebra.

Reflexion anti-involution Suppose $\Bbbk = \mathbb{Z}[\omega^{\pm 1}]$ and consider the $\mathbb{Z}$–linear involution $x \mapsto x^*$ on $\Bbbk$ sending ω to ω^{-1}. Let $r\colon \Sigma_\mathcal{P} \times (0,1) \xrightarrow{\cong} \Sigma_\mathcal{P}$ be the homeomorphism defined by $r(x,t) = (x, 1-t)$. Define an antilinear map $\theta\colon \mathcal{S}_\omega(\mathbf{\Sigma}) \xrightarrow{\cong} \mathcal{S}_\omega(\mathbf{\Sigma})$ by

$$\theta\Big(\sum_i x_i[T_i, s_i]\Big) := \sum_i x_i^*[r(T_i), s_i \circ r].$$

Proposition 2.3 [40, Proposition 2.7] *The map θ is an antimorphism of algebras*, *ie* $\theta(xy) = \theta(y)\theta(x)$.

Bases for stated skein algebras A closed component of a diagram D is trivial if it bounds an embedded disc in $\Sigma_\mathcal{P}$. An open component of D is trivial if it can be isotoped, relatively to its boundary, inside some boundary arc. A diagram is *simple* if it has neither double point nor trivial component. By convention, the empty set is a simple diagram. Let $\mathfrak{o}$ denote an arbitrary orientation of the boundary arcs of $\mathbf{\Sigma}$. For each boundary arc b we denote by $<_\mathfrak{o}$ the induced total order on $\partial_b D$. A state $s\colon \partial D \to \{-,+\}$ is $\mathfrak{o}$*–increasing* if, for any boundary arc b and any two points $x, y \in \partial_b D$, then $x <_\mathfrak{o} y$ implies $s(x) < s(y)$, with the convention $- < +$.

Definition 2.4 We denote by $\mathcal{B}^\mathfrak{o} \subset \mathcal{S}_\omega(\mathbf{\Sigma})$ the set of classes of stated diagrams (D, s) such that D is simple and s is $\mathfrak{o}$–increasing.

Theorem 2.5 [40, Theorem 2.11] *The set $\mathcal{B}^\mathfrak{o}$ is a basis of $\mathcal{S}_\omega(\mathbf{\Sigma})$.*

Remark 2.6 The basis $\mathcal{B}^\mathfrak{o}$ is independent of the choice of the ground ring $\Bbbk$ and of $\omega \in \Bbbk^\times$. This fact has the following useful consequence: Let $\Bbbk := \mathbb{Z}[\omega^{\pm 1}]$ and $\Bbbk'$ be any other commutative unital ring with an invertible element $\omega' \in \Bbbk'^\times$. There is a unique morphism of rings $\mu\colon \Bbbk \to \Bbbk'$ sending ω to ω' and the two $\Bbbk'$ algebras $\mathcal{S}_\omega(\mathbf{\Sigma}) \otimes_\Bbbk \Bbbk'$ and $\mathcal{S}_{\omega'}(\mathbf{\Sigma})$ are canonically isomorphic through the isomorphism preserving the basis $\mathcal{B}^\mathfrak{o}$. This fact permits us to prove formulas in $\Bbbk$ using the reflexion anti-involution θ and then apply them to any ring $\Bbbk'$ by changing the coefficients.

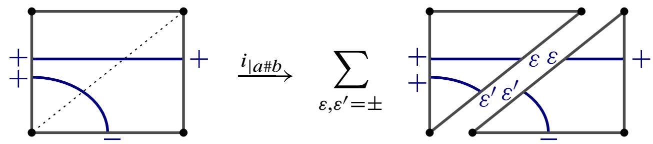


Figure 4: An illustration of the gluing map $i_{|a\#b}$.

Gluing maps Let a and b be two distinct boundary arcs of $\mathbf{\Sigma}$ and let $\mathbf{\Sigma}_{|a\#b}$ be the punctured surface obtained from $\mathbf{\Sigma}$ by gluing a and b. Denote by $\pi : \Sigma_{\mathcal{P}} \to (\Sigma_{|a\#b})_{\mathcal{P}_{|a\#b}}$ the projection and $c := \pi(a) = \pi(b)$. Let (T_0, s_0) be a stated framed tangle of ${\Sigma_{|a\#b}}_{\mathcal{P}_{|a\#b}} \times (0,1)$ transverse to $c \times (0,1)$ and such that the heights of the points of $T_0 \cap c \times (0,1)$ are pairwise distinct and the framing of the points of $T_0 \cap c \times (0,1)$ is vertical. Let $T \subset \Sigma_{\mathcal{P}} \times (0,1)$ be the framed tangle obtained by cutting T_0 along c. Any two states $s_a : \partial_a T \to \{-,+\}$ and $s_b : \partial_b T \to \{-,+\}$ give rise to a state (s_a, s, s_b) on T. Both the sets $\partial_a T$ and $\partial_b T$ are in canonical bijection with the set $T_0 \cap c$ by the map π. Hence the two sets of states s_a and s_b are both in canonical bijection with the set $\mathrm{St}(c) := \{s : c \cap T_0 \to \{-,+\}\}$.

Definition 2.7 Let $i_{|a\#b} : \mathcal{S}_\omega(\mathbf{\Sigma}_{|a\#b}) \to \mathcal{S}_\omega(\mathbf{\Sigma})$ be the linear map given, for any (T_0, s_0) as above, by

$$i_{|a\#b}([T_0, s_0]) := \sum_{s \in \mathrm{St}(c)} [T, (s, s_0, s)].$$

Theorem 2.8 [40, Theorem 3.1] *The linear map $i_{|a\#b} : \mathcal{S}_\omega(\mathbf{\Sigma}_{|a\#b}) \to \mathcal{S}_\omega(\mathbf{\Sigma})$ is an injective morphism of algebras. Moreover the gluing operation is coassociative in the sense that if a, b, c and d are four distinct boundary arcs, then $i_{|a\#b} \circ i_{|c\#d} = i_{|c\#d} \circ i_{|a\#b}$.*

Relation with $U_q\mathrm{sl}_2$ and $\mathcal{O}_q[\mathrm{SL}_2]$ Recall that $A = \omega^{-2}$ and write $q := A^2$. The stated skein algebra has deep relations with the quantum enveloping algebra $U_q\mathrm{sl}_2$ and the quantum group $\mathcal{O}_q(\mathrm{SL}_2)$, explored in [26; 27; 32; 39; 40], that we briefly reproduce here for later use by using the notation of [22; 33; 47]. Suppose that q is generic (not a root of unity) and let $\rho : U_q\mathrm{sl}_2 \to \mathrm{End}(V)$ be the standard representation of $U_q\mathrm{sl}_2$, where V is two-dimensional with basis (v_+, v_-) and

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

When q is a generic parameter, $U_q\mathrm{sl}_2$ has the structure of topological half-ribbon Hopf algebra in the sense of [47], that is, it admits an R–matrix

$$R = q^{\frac{1}{2}(H \otimes H)} \exp_q((q - q^{-1}) E \otimes F) \in \widetilde{U_q\mathrm{sl}_2^{\otimes 2}}$$

(see [22] for details) and a half-ribbon element $\Omega \in \widetilde{U_q \mathrm{sl}_2}$ (defined by Kirillov and Reshetikhin in [34], where Ω is denoted by w^{-1}) such that $\Delta(\Omega) = (\Omega \otimes \Omega) R$ and such that $(U_q \mathrm{sl}_2, R, \Omega^{-2})$ is a topological ribbon Hopf algebra. Note that the ribbon element $v := \Omega^{-2}$ is not the usual one (see [47; 49] for details) but the Kauffman-bracket one (the one for which $\mathrm{qdim}(V) = -q - q^{-1}$ instead of $q + q^{-1}$).

In the standard basis (v_+, v_-) of V, the matrix $C = \mathrm{Mat}_{(v_+, v_-)}(\Omega^{-1})$ is written

$$C = \begin{pmatrix} C_+^+ & C_-^+ \\ C_+^- & C_-^- \end{pmatrix} := \begin{pmatrix} 0 & \omega \\ -\omega^5 & 0 \end{pmatrix}.$$

Therefore

$$C^{-1} = -A^3 C = \begin{pmatrix} 0 & -\omega^{-5} \\ \omega^{-1} & 0 \end{pmatrix}.$$

Define the operators $\tau, q^{\frac{1}{2}(H \otimes H)} \in \mathrm{End}(V^{\otimes 2})$ by

$$\tau(v_i \otimes v_j) := v_j \otimes v_i \quad \text{and} \quad q^{\frac{1}{2}(H \otimes H)}(v_i \otimes v_j) = A^{ij} v_i \otimes v_j$$

for $i, j \in \{+, -\}$ (we identified $-$ with -1 and $+$ with $+1$). The braiding associated to the R–matrix is

$$\mathscr{R} = c_{V,V} := \tau \circ q^{\frac{1}{2}(H \otimes H)} \circ \exp_q((q - q^{-1}) \rho(E) \otimes \rho(F)) \\ = \tau \circ q^{\frac{1}{2}(H \otimes H)} \circ (\mathbb{1}_2 + (q - q^{-1}) \rho(E) \otimes \rho(F)).$$

In the basis $(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-)$, it is written

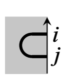
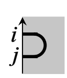
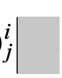
$$\mathscr{R} = \begin{pmatrix} \mathscr{R}_{++}^{++} & \mathscr{R}_{+-}^{++} & \mathscr{R}_{-+}^{++} & \mathscr{R}_{--}^{++} \\ \mathscr{R}_{++}^{+-} & \mathscr{R}_{+-}^{+-} & \mathscr{R}_{-+}^{+-} & \mathscr{R}_{--}^{+-} \\ \mathscr{R}_{++}^{-+} & \mathscr{R}_{+-}^{-+} & \mathscr{R}_{-+}^{-+} & \mathscr{R}_{--}^{-+} \\ \mathscr{R}_{++}^{--} & \mathscr{R}_{+-}^{--} & \mathscr{R}_{-+}^{--} & \mathscr{R}_{--}^{--} \end{pmatrix} := \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & 0 & A^{-1} & 0 \\ 0 & A^{-1} & A - A^{-3} & 0 \\ 0 & 0 & 0 & A \end{pmatrix},$$

so

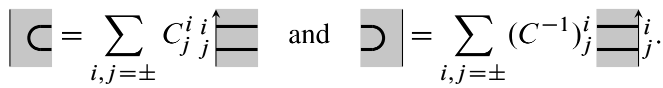
$$\mathscr{R}^{-1} = \begin{pmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & A^{-1} - A^3 & A & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & A^{-1} \end{pmatrix}.$$

We now list three families of skein relations, which are straightforward consequences of the definition, work regardless whether q is generic or a root of unity, and will be used later. Let $i, j \in \{-, +\}$.

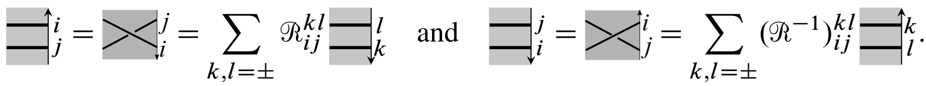
- The *trivial arc relations*, which are given by

(3)  $= C_j^i$  and  $= (C^{-1})_j^i$  .

- The *cutting arc relations*, which are given by

(4) 

- The *height exchange relations*, which are given by

(5) 

We refer to [40] for proofs.

The algebra $\mathcal{O}_q[\mathrm{SL}_2]$ is the algebra presented by generators $x_{\varepsilon\varepsilon'}, \varepsilon, \varepsilon' \in \{-,+\}$ and relations

$$
\begin{aligned}
x_{++}x_{+-} &= q^{-1}x_{+-}x_{++}, & x_{++}x_{-+} &= q^{-1}x_{-+}x_{++},\\
x_{--}x_{+-} &= qx_{+-}x_{--}, & x_{--}x_{-+} &= qx_{-+}x_{--},\\
x_{++}x_{--} &= 1+q^{-1}x_{+-}x_{-+}, & x_{--}x_{++} &= 1+qx_{+-}x_{-+},\\
x_{-+}x_{+-} &= x_{+-}x_{-+},
\end{aligned}
$$

It has a Hopf algebra structure characterized by

$$
\begin{pmatrix} \Delta(x_{++}) & \Delta(x_{+-}) \\ \Delta(x_{-+}) & \Delta(x_{--}) \end{pmatrix} = \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix} \otimes \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix},
$$

$$
\begin{pmatrix} \epsilon(x_{++}) & \epsilon(x_{+-}) \\ \epsilon(x_{-+}) & \epsilon(x_{--}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

$$
\begin{pmatrix} S(x_{++}) & S(x_{+-}) \\ S(x_{-+}) & S(x_{--}) \end{pmatrix} = \begin{pmatrix} x_{--} & -qx_{+-} \\ -q^{-1}x_{-+} & x_{++} \end{pmatrix}.
$$

When $q \in \mathbb{C}^*$ is generic (not a root of unity), $\mathcal{O}_q[\mathrm{SL}_2]$ is the subalgebra of the restricted dual of $U_q\mathrm{sl}_2$ generated by the matrix elements of the integrable modules; see [14; 22]. The *bigon* $\mathbb{B}$ is the punctured surface made of a disc with two punctures on its boundary. It has two boundary arcs a and b and is generated by the stated arcs $\alpha_{\varepsilon\varepsilon}, \varepsilon, \varepsilon' = \pm$ made of an arc α linking a to b with state ε on $\alpha \cap a$ and ε' on $\alpha \cap b$. Consider a disjoint union $\mathbb{B} \sqcup \mathbb{B}$ of two bigons; by gluing together the boundary arc b_1 of the first bigon with the boundary arc a_2 of the second, one obtains a morphism $\Delta := i_{|b_1 \# a_2} : \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{S}_\omega(\mathbb{B})^{\otimes 2}$ which endows $\mathcal{S}_\omega(\mathbb{B})$ with the structure of Hopf algebra where Δ is the coproduct.

Theorem 2.9 [26; 39; 40] *There is an isomorphism $\varphi: \mathcal{O}_q[\mathrm{SL}_2] \cong \mathcal{S}_\omega(\mathbb{B})$ of Hopf algebras sending the generator $x_{\varepsilon\varepsilon'} \in \mathcal{O}_q[\mathrm{SL}_2]$ to the element $\alpha_{\varepsilon\varepsilon'} \in \mathcal{S}_\omega(\mathbb{B})$.*

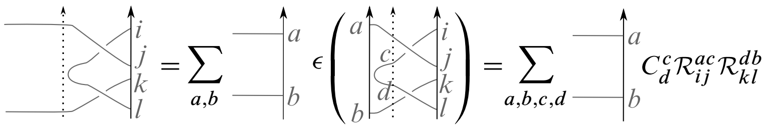


Figure 5: An example of the boundary skein relation.

More precisely, the fact that φ is an isomorphism of algebras is proved in [40] and the fact that it preserves the coproduct was noticed independently in [26; 39]. Throughout, we will (abusively) identify the Hopf algebras $\mathcal{O}_q[\mathrm{SL}_2]$ and $\mathcal{S}_\omega(\mathbb{B})$ using φ. Note that the definition of φ depends on an indexing by a and b of the boundary arcs of $\mathbb{B}$.

Now consider a punctured surface $\mathbf{\Sigma}$ and a boundary arc c. By gluing a bigon $\mathbb{B}$ along $\mathbf{\Sigma}$ while gluing b with c, one obtains a punctured surface isomorphic to $\mathbf{\Sigma}$, hence a map $\Delta^L_c := i_{|b\#c} : \mathcal{S}_\omega(\mathbf{\Sigma}) \to \mathcal{O}_q[\mathrm{SL}_2] \otimes \mathcal{S}_\omega(\mathbf{\Sigma})$ which endows $\mathcal{S}_\omega(\mathbf{\Sigma})$ with the structure of left $\mathcal{O}_q[\mathrm{SL}_2]$ comodule. Similarly, gluing c with a induces a right comodule morphism $\Delta^R_c := i_{|c\#a} : \mathcal{S}_\omega(\mathbf{\Sigma}) \to \mathcal{S}_\omega(\mathbf{\Sigma}) \otimes \mathcal{O}_q[\mathrm{SL}_2]$. The following theorem characterizes the image of the gluing map and was proved independently in [26; 39].

Theorem 2.10 [26, Theorem 4.7; 39, Theorem 1.1] *Let $\mathbf{\Sigma}$ be a punctured surface, and a and b two boundary arcs. The sequence*

$$0 \to \mathcal{S}_\omega(\mathbf{\Sigma}_{|a\#b}) \xrightarrow{i_{|a\#b}} \mathcal{S}_\omega(\mathbf{\Sigma}) \xrightarrow{\Delta^L_a - \sigma\circ\Delta^R_b} \mathcal{O}_q[\mathrm{SL}_2] \otimes \mathcal{S}_\omega(\mathbf{\Sigma})$$

is exact, where $\sigma(x \otimes y) := y \otimes x$.

An easy but very important consequence of the fact that Δ^L_a and Δ^R_a are comodule maps are the *boundary skein relations*

(6) $$(\epsilon \otimes \mathrm{id}) \circ \Delta^L_a = \mathrm{id} \quad \text{and} \quad (\mathrm{id} \otimes \epsilon) \circ \Delta^R_a = \mathrm{id}.$$

The image through the counit ϵ of a stated diagram in $\mathbb{B}$ can be computed using

(7)

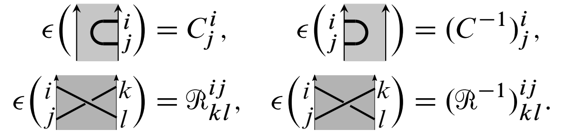


Figure 5 illustrates an instance of boundary skein relation (6). Here we draw a dotted arrow to illustrate where we cut the bigon. Note that all the trivial arc (3), cutting arc (4) and height exchange (5) relations are particular cases of (6).

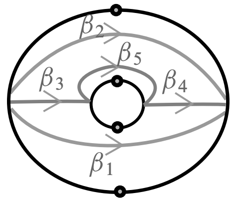


Figure 6: A punctured surface and a set of generators for its small fundamental groupoid.

2.2 The small fundamental groupoid and its finite presentations

In this section we fix a punctured surface $\mathbf{\Sigma} = (\Sigma, \mathcal{P})$ such that Σ is connected and has nonempty boundary. For each boundary arc a of $\mathbf{\Sigma}$, fix a point $v_a \in a$ and denote by $\mathbb{V}$ the set $\{v_a\}_a$.

Definition 2.11 The *small fundamental groupoid* $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ is the full subcategory of the fundamental groupoid $\Pi_1(\Sigma_{\mathcal{P}})$ generated by $\mathbb{V}$.

Said differently, $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ is the small groupoid whose set of objects is $\mathbb{V}$ and such that a morphism (called a path) $\alpha\colon v_1 \to v_2$ is a homotopy class of continuous maps $\varphi_\alpha\colon [0,1] \to \Sigma_{\mathcal{P}}$ with $\varphi_\alpha(0) = v_1$ and $\varphi_\alpha(1) = v_2$. The map φ_α will be referred to as a *geometric representative* of α. The composition is the concatenation of paths. For a path $\alpha\colon v_1 \to v_2$ we write $s(\alpha) = v_1$ (the source point) and $t(\alpha) = v_2$ (the target point), and $\alpha^{-1}\colon v_2 \to v_1$ is the path with opposite orientation ($\varphi_{\alpha^{-1}}(t) = \varphi_\alpha(1-t)$).

We will define the notion of *finite presentation* $\mathbb{P}$ of the groupoid $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ and attach to each such $\mathbb{P}$ a finite presentation of $\mathcal{S}_\omega(\mathbf{\Sigma})$. In order to get some intuition, consider the punctured surface in Figure 6: it is an annulus with two punctures per boundary component, so it has four boundary arcs. The figure shows some paths $\beta_1, \dots, \beta_5$ and we will say that $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ is finitely presented by the set of generators $\{\beta_1, \dots, \beta_5\}$ together with the relation $\beta_2^{-1}\beta_4\beta_5\beta_3 = 1$. We will deduce that $\mathcal{S}_\omega(\mathbf{\Sigma})$ is generated by the stated arcs $(\beta_i)_{\varepsilon\varepsilon'}$ and that the relation $\beta_2^{-1}\beta_4\beta_5\beta_3 = 1$ induces a relation among them. Alternatively, the fundamental groupoid of the same punctured surface has a presentation with the smaller set of generators $\{\beta_1, \dots, \beta_4\}$ and no relation. The induced finite presentation of $\mathcal{S}_\omega(\mathbf{\Sigma})$ will be simpler.

Definition 2.12 (i) A *set of generators* for $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ is a set $\mathbb{G}$ consisting of paths in $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ such that any path $\alpha \in \Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ decomposes as $\alpha = \alpha_1^{\varepsilon_1} \cdots \alpha_n^{\varepsilon_n}$ with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{G}$. We also require that each path $\alpha \in \mathbb{G}$ is the homotopy class of

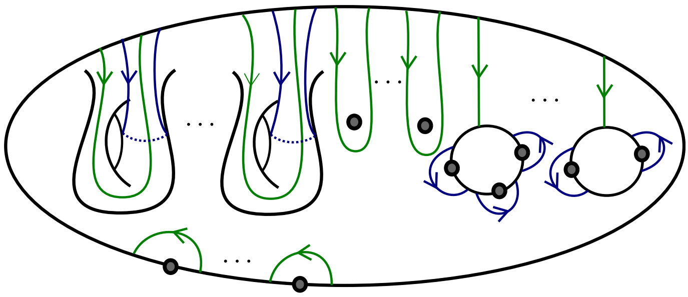


Figure 7: The geometric representatives of a set of generators for $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$.

some embedding $\varphi_\alpha : [0,1] \to \Sigma_{\mathcal{P}}$ such that the images of the φ_α do not intersect outside $\mathbb{V}$ and possibly intersect transversally at $\mathbb{V}$. The *generating graph* is the oriented ribbon graph $\Gamma \subset \Sigma_{\mathcal{P}}$ whose set of vertices is $\mathbb{V}$ and edges are the images of the φ_α. We will always assume implicitly that the geometric representatives φ_α are part of the data defining a set of generators. Moreover, when $\alpha \in \mathbb{G}$ is a path such that $s(\alpha) = t(\alpha)$ (ie α is a loop) we add the additional datum of a "height order" for its endpoints, that is we specify whether $h(s(\alpha)) < h(t(\alpha))$ or $h(t(\alpha)) < h(s(\alpha))$.

(ii) For a path $\alpha : v_1 \to v_2$ and $\varepsilon, \varepsilon' \in \{-,+\}$, we denote by $\alpha_{\varepsilon\varepsilon'} \in \mathcal{S}_\omega(\mathbf{\Sigma})$ the class of the stated arc (α, σ), where the state σ is given by $\sigma(v_1) = \varepsilon$ and $\sigma(v_2) = \varepsilon'$. When both endpoints lie in the same boundary arc (when $s(\alpha) = t(\alpha)$) we use the chosen height order to specify which endpoint lies on the top. Set

$$\mathcal{A}^{\mathbb{G}} := \{\alpha_{\varepsilon\varepsilon'} \mid \alpha \in \mathbb{G} \text{ and } \varepsilon, \varepsilon' \in \{-,+\}\} \subset \mathcal{S}_\omega(\mathbf{\Sigma}).$$

Example 2.13 For any connected open punctured surface $\mathbf{\Sigma}$, the groupoid $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ admits a finite set of generators depicted in Figure 7 and defined as follows. Denote by $a_0, \dots, a_n$ the boundary arcs, by $\partial_0, \dots, \partial_r$ the boundary components of Σ with $a_0 \subset \partial_0$, and write $v_i := a_i \cap \mathbb{V}$. Let $\overline{\Sigma}$ be the surface obtained from Σ by gluing a disc along each boundary component ∂_i for $1 \le i \le r$, and choose $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ some paths in $\pi_1(\Sigma_{\mathcal{P}}, v_0)$ (which equals $\mathrm{End}_{\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})}(v_0)$) such that their images in $\overline{\Sigma}$ generate the free group $\pi_1(\overline{\Sigma}, v_0)$ (said differently, the α_i and β_i are longitudes and meridians of Σ). For each inner puncture p choose a peripheral curve $\gamma_p \in \pi_1(\Sigma_{\mathcal{P}}, v_0)$ encircling p once and for each boundary puncture p_∂ between two boundary arcs a_i and a_j, consider the path $\alpha_{p_\partial} : v_i \to v_j$ represented by the corner arc in p_∂. Finally, for each boundary component ∂_j, with $1 \le j \le r$, containing a boundary arc $a_{k_j} \subset \partial_j$, choose a path $\delta_{\partial_j} : v_0 \to v_{k_j}$. The set

$$\mathbb{G}' := \{\alpha_i, \beta_i, \alpha_p, \delta_{\partial_j} \mid 1 \le i \le g,\ p \in \mathcal{P} \text{ and } 1 \le j \le r\}$$

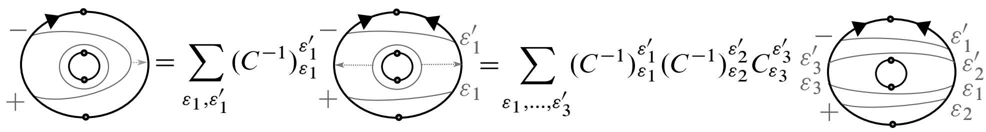


Figure 8: How an application of the cutting arc relations permits us to express any simple stated diagram in terms of the elements of $\mathcal{A}^{\mathbb{G}}$. Here $\mathbb{G} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is the set of generators of Figure 6. We draw dotted arrows to exhibit where we perform the cutting arc relations.

is a generating set for $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ and Figure 7 represents a set of geometric representatives for $\mathbb{G}'$. Moreover each of its generators which is not one of the δ_{∂_j} can be expressed as a composition of the other ones (we will soon say that there is a relation among those generators), therefore a set $\mathbb{G}$ obtained from $\mathbb{G}'$ by removing one of the element of the form α_i, β_i or γ_p is still a generating set for $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$. The height orders can be chosen arbitrarily. Note that $\mathbb{G}$ has cardinality $2g - 2 + s + n_\partial$, where g is the genus of Σ, $s := |\mathcal{P}|$ is the number of punctures and $n_\partial := |\pi_0(\partial\Sigma)|$ is the number of boundary components.

In the particular case where Σ has exactly one boundary component with one puncture on it (and possibly inner punctures), the generating graph of $\mathbb{G}$ is called the *daisy graph* (see Figure 9). The daisy graph was first considered in [4] in the context of classical lattice gauge field theory and in [5; 8; 27; 28] in the quantum case.

Proposition 2.14 [38, Proposition 3.4] *If $\mathbb{G}$ is a set of generators of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$, then the set $\mathcal{A}^{\mathbb{G}}$ generates $\mathcal{S}_\omega(\mathbf{\Sigma})$ as an algebra.*

The proof of Proposition 2.14 is an easy consequence of the cutting arc relations illustrated in Figure 8.

We now define the notion of relations for a generating set $\mathbb{G}$. Let $\mathcal{F}(\mathbb{G})$ denote the free semigroup generated by the elements of $\mathbb{G}$ and let $\mathrm{Rel}_{\mathbb{G}}$ denote the subset of $\mathcal{F}(\mathbb{G})$ of elements of the form $R = \beta_1 \star \cdots \star \beta_n$ such that $s(\beta_i) = t(\beta_{i+1})$ and such that the path $\beta_1 \dots \beta_n$ is trivial. We write $R^{-1} := \beta_n^{-1} \star \cdots \star \beta_1^{-1}$. A relation $R = \beta_1 \star \cdots \star \beta_n \in \mathrm{Rel}_{\mathbb{G}}$ is called *simple* if the β_i admit as representatives embedded curves whose concatenation forms a contractible simple closed curve γ in $\Sigma_{\mathcal{P}}$ whose orientation coincides with the orientation of the disc bounded by γ. Note that "being simple" depends on the choice of geometric representatives of the generators.

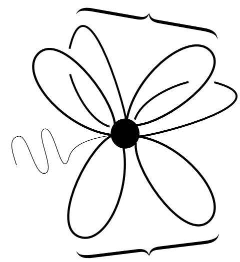


Figure 9: A daisy graph.

Definition 2.15 A finite subset $\mathbb{RL} \subset \mathrm{Rel}_{\mathbb{G}}$ is called a *finite set of relations* if its elements are simple and every word $R \in \mathrm{Rel}_{\mathbb{G}}$ can be decomposed as

$$R = \beta \star R_1^{\varepsilon_1} \star \cdots \star R_m^{\varepsilon_m} \star \beta^{-1},$$

where $R_i \in \mathbb{RL}$, $\varepsilon_i \in \{\pm 1\}$ and $\beta = \beta_1 \star \cdots \star \beta_n \in \mathcal{F}(\mathbb{G})$ is such that $s(\beta_i) = t(\beta_{i+1})$. The pair $\mathbb{P} := (\mathbb{G}, \mathbb{RL})$ is called a *finite presentation* of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$.

As illustrated in the introduction, the small fundamental groupoid of the triangle $\mathbb{T}$ admits the finite presentation with generating set $\mathbb{G} = \{\alpha, \beta, \gamma\}$ and unique relation $\mathbb{RL} = \{\alpha \star \beta \star \gamma\}$.

For a general connected open punctured surface $\mathbf{\Sigma}$, the set $\mathbb{G}$ of Example 2.13 is the generating set of a presentation of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ with no relations.

2.3 Relations among the generators of the stated skein algebras

We fix a connected open punctured surface $\mathbf{\Sigma}$, a finite presentation $\mathbb{P} = (\mathbb{G}, \mathbb{RL})$ of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$, and look for relations in $\mathcal{S}_\omega(\mathbf{\Sigma})$ among the elements of $\mathcal{A}^{\mathbb{G}}$.

Definition 2.16 An *oriented arc* β is a nonclosed connected simple diagram of $\Sigma_{\mathcal{P}}$ together with an orientation plus a possible height order of its endpoints in the case where they both lie in the same boundary arc. We will denote by $s(\beta)$ and $t(\beta)$ its endpoints so that β is oriented from $s(\beta)$ towards $t(\beta)$. For $\varepsilon, \varepsilon' \in \{-, +\}$, we denote by $\beta_{\varepsilon\varepsilon'} \in \mathcal{S}_\omega(\mathbf{\Sigma})$ the class of the stated diagram (β, σ) where $\sigma(s(\beta)) = \varepsilon$ and $\sigma(t(\beta)) = \varepsilon'$.

Note that to each oriented arc one can associate a path in $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ by first isotoping its endpoints to $\mathbb{V}$ and then taking its homotopy class. However a path in $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ can be associated to several distinct oriented arcs, so an oriented arc contains more information than a path in the small fundamental groupoid.

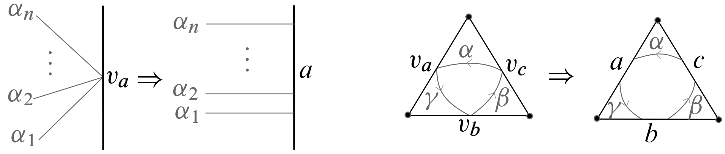


Figure 10: Left: an illustration of the local isotopy we perform to turn the set of edges of a (ribbon) presenting graph into a set of pairwise nonintersecting oriented arcs. Right: an example in the case of the triangle.

We want to see the elements of $\mathbb{G}$ as pairwise nonintersecting oriented arcs as illustrated in Figure 10. Recall that by Definition 2.12, any path $\alpha \in \mathbb{G}$ is endowed with a geometric representative φ_α whose image is an oriented arc $\underline{\alpha} \subset \Sigma_{\mathcal{P}}$ such that the $\underline{\alpha}$ pairwise do not intersect outside of $\mathbb{V}$ and they intersect transversally in $\mathbb{V}$. So each point $v_a \in \mathbb{V}$ is endowed with a total order $<_{v_a}$ on the set of its adjacent arcs (so the presenting graph has a ciliated ribbon graph structure).

The orientation of $\Sigma_{\mathcal{P}}$ induces an orientation of its boundary arcs, which, in turn, induces a total order $<_a$ on each boundary arc a, where $v_1 <_a v_2$ if a is oriented from v_1 towards v_2. After isotoping the $\underline{\alpha}$ in a small neighborhood of each v_a in such a way that the vertex order $<_{v_a}$ matches with the boundary arc order $<_a$ as illustrated in Figure 10, we get a family of pairwise nonintersecting oriented arcs representing the elements of $\mathbb{G}$.

Convention 2.17 From now on we consider the elements of $\mathbb{G}$ as pairwise nonintersecting oriented arcs.

Definition 2.18 Let α be an oriented arc, set $v_1 := s(\alpha)$ and $v_2 := t(\alpha)$ and denote by u and v the boundary arcs containing v_1 and v_2, respectively. The arc α is

- *of type a* if $u \neq v$,
- *of type b* if $u = v$, $h(v_1) < h(v_2)$ and $v_2 <_u v_1$,
- *of type c* if $u = v$, $h(v_2) < h(v_1)$ and $v_1 <_u v_2$,
- *of type d* if $u = v$, $h(v_1) < h(v_2)$ and $v_1 <_u v_2$,
- *of type e* if $u = v$, $h(v_2) < h(v_1)$ and $v_2 <_u v_1$.

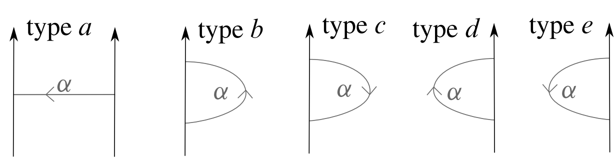


Figure 11: An illustration of the five types of oriented arcs.

Here $h(v)$ represents the height of v (h is the second projection $\Sigma_{\mathcal{P}} \times (0,1) \to (0,1)$). Figure 11 illustrates the five types of oriented arcs.

Notation 2.19 (i) For α an oriented arc, write $M(\alpha) := \begin{pmatrix} \alpha_{++} & \alpha_{+-} \\ \alpha_{-+} & \alpha_{--} \end{pmatrix}$, the 2×2 matrix with coefficients in $\mathcal{S}_\omega(\mathbf{\Sigma})$. The relations among the generators of $\mathcal{S}_\omega(\mathbf{\Sigma})$ that we will soon define are much more elegant when written using the matrix

$$N(\alpha) := \begin{cases} M(\alpha) & \text{if } \alpha \text{ is of type } a, \\ M(\alpha)C & \text{if } \alpha \text{ is of type } b, \\ M(\alpha)^t C & \text{if } \alpha \text{ is of type } c, \\ C^{-1}M(\alpha) & \text{if } \alpha \text{ is of type } d, \\ {}^tC^{-1}M(\alpha) & \text{if } \alpha \text{ is of type } e, \end{cases}$$

where ${}^tM$ denotes the transpose of M.

(ii) Let $M_{a,b}(R)$ be the ring of $a \times b$ matrices with coefficients in some ring R (here R will be $\mathcal{S}_\omega(\mathbf{\Sigma})$). The *Kronecker product* $\odot\colon M_{a,b}(R) \otimes M_{c,d}(R) \to M_{ac,bd}(R)$ is defined by $(A \odot B)^{i,k}_{j,l} = A^i_j B^k_l$. For instance,

$$M(\alpha) \odot M(\beta) = \begin{pmatrix} \alpha_{++}\beta_{++} & \alpha_{++}\beta_{+-} & \alpha_{+-}\beta_{++} & \alpha_{+-}\beta_{+-} \\ \alpha_{++}\beta_{-+} & \alpha_{++}\beta_{--} & \alpha_{+-}\beta_{-+} & \alpha_{+-}\beta_{--} \\ \alpha_{-+}\beta_{++} & \alpha_{-+}\beta_{+-} & \alpha_{--}\beta_{++} & \alpha_{--}\beta_{+-} \\ \alpha_{-+}\beta_{-+} & \alpha_{-+}\beta_{--} & \alpha_{--}\beta_{-+} & \alpha_{--}\beta_{--} \end{pmatrix}.$$

(iii) By abuse of notation τ also denotes the matrix of the flip map $\tau\colon V^{\otimes 2} \to V^{\otimes 2}$ given by $v_i \otimes v_j \mapsto v_j \otimes v_i$:

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(iv) For a 4×4 matrix $X = (X^{ij}_{kl})_{i,j,k,l=\pm}$, we define the 2×2 matrices $\mathrm{tr}_L(X)$ and $\mathrm{tr}_R(X)$ by

$$\mathrm{tr}_L(X)^b_a := \sum_{i=\pm} X^{ib}_{ia} \quad \text{and} \quad \mathrm{tr}_R(X)^b_a := \sum_{i=\pm} X^{bi}_{ai}.$$

(v) For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we set $\det_q(M) := ad - q^{-1}bc$ and $\det_{q^2}(M) := ad - q^{-2}bc$.

Lemma 2.20 (orientation-reversing formulas) *Let α be an oriented arc and α^{-1} be the same arc with opposite orientation. Then one has*

$$M(\alpha^{-1}) = {}^tM(\alpha).$$

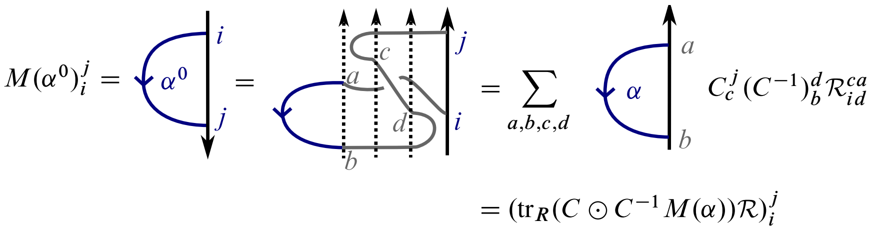


Figure 12: An illustration of the proof of (9) in the case where α is of type e.

Therefore,

$$N(\alpha^{-1}) = \begin{cases} {}^t N(\alpha) & \text{if } \alpha \text{ is of type } a, \\ {}^t C^{-1\,t} N(\alpha)^t C & \text{if } \alpha \text{ is of type } b \text{ or } d, \\ C^{-1\,t} N(\alpha) C & \text{if } \alpha \text{ is of type } c \text{ or } e. \end{cases} \tag{8}$$

Proof This is a straightforward consequence of the definitions. □

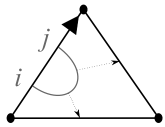
Lemma 2.21 (height-reversing formulas) *Let α be an oriented arc with both endpoints in the same boundary arc and let α^0 be the same arc with reversed height order for its endpoints. Then one has*

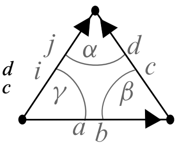
$$M(\alpha^0) = \begin{cases} \mathrm{tr}_R(\mathscr{R}^{-1}({}^t C^{-1} \odot M(\alpha)^t C)) & \text{if } \alpha \text{ is of type } b, \\ \mathrm{tr}_L(\mathscr{R}^{-1}(M(\alpha) C \odot C^{-1})) & \text{if } \alpha \text{ is of type } c, \\ \mathrm{tr}_L(({}^t C^{-1} M(\alpha) \odot {}^t C)\mathscr{R}) & \text{if } \alpha \text{ is of type } d, \\ \mathrm{tr}_R((C \odot C^{-1} M(\alpha))\mathscr{R}) & \text{if } \alpha \text{ is of type } e. \end{cases} \tag{9}$$

Proof Equation (9) is obtained by using the boundary skein relations (6). Figure 12 illustrates the proof in the case where α is of type e. The other cases are similar and left to the reader.

In Figure 12, we represent the curve α in blue to emphasize that, despite what the picture suggests, the curve can be arbitrarily complicated. Since the boundary arc relation only involves the intersection of α with a small neighborhood (a bigon) of the boundary arc (colored in gray), the exact structure of the blue part of the figure does not matter. □

Remark 2.22 Reversing the orientation of an arc exchanges type b with type c and type d with type e, whereas reversing the height order exchanges type b with type e and type c with type d. Therefore (8) and (9) permit us to switch between the types b, c, d and e; this will permit us to write the arc exchange and trivial loop relations in a simpler form by specifying the type of arc.

$(C^{-1})_i^j \overset{\text{trivial curve rel}}{=}$ 

$\overset{\text{cutting rel}}{=} \sum_{a,b,c,d=\pm} (C^{-1})_a^b (C^{-1})_c^d$ 

$$= \sum_{a,b,c,d=\pm} M(\alpha)_d^j (C^{-1})_c^d M(\beta)_b^c (C^{-1})_a^b M(\gamma)_i^a$$

Figure 13: An illustration of the proof of (10) in the case of the triangle.

Lemma 2.23 (trivial loop relations) *Let $R = \beta_k \star \cdots \star \beta_1$ be a simple relation. Suppose that all arcs β_i are either of type a or d. Then*

$$\mathbb{1}_2 = CM(\beta_k)C^{-1}M(\beta_{k-1})C^{-1}\cdots C^{-1}M(\beta_1). \tag{10}$$

Proof Equation (10) is a consequence of the trivial arc and cutting arc relations illustrated in Figure 13 in the case of the triangle with presentation whose generators are the arcs $\{\alpha, \beta, \gamma\}$ drawn in Figure 1 and where the relation is $\alpha \star \beta \star \gamma = 1$. Figure 13 shows the equality between the matrix coefficients of C^{-1} and $M(\alpha)C^{-1}M(\beta)C^{-1}M(\gamma)$.

Let us detail the proof in the general case. Since β_i is either of type a or d, it can be represented by a tangle $T(\beta_i)$ such that the height of the source endpoint of β_i (say v_i) is smaller than the height of its target endpoint (say w_i); said differently $h(v_i) < h(w_i)$. One can further choose the $T(\beta_i)$ so that $T(\beta_{i+1})$ lies on the top of $T(\beta_i)$ (so $h(v_1) < h(w_1) < h(v_2) < \cdots < h(w_k)$). Let T be the tangle made of the disjoint union of the $T(\beta_i)$. By the assumption that R is a simple relation, we can suppose that T is in generic position (in the sense of Section 2.1) and that its projection diagram is simple. Fix $i, j \in \{-,+\}$ and let α^0 be a trivial arc with endpoints $s(\alpha^0) = v_1$ and $t(\alpha^0) = w_k$ such that α^0 can be isotoped (relative to its boundary) to an arc inside $\partial\Sigma_{\mathcal{P}}$. One the one hand, the trivial arc relation (3) gives the equality $\alpha^0_{ij} = (C^{-1})_i^j$. On the other hand, the cutting arc relation (4) gives the equality

$$\begin{aligned}(C^{-1})_i^j = \alpha^0_{ij} &= \sum_{\substack{s\in \mathrm{St}(T)\\ s(v_1)=i\\ s(w_k)=j}} [T,s](C^{-1})^{s(v_2)}_{s(w_1)}(C^{-1})^{s(v_3)}_{s(w_2)}\cdots(C^{-1})^{v_k}_{s(w_{k-1})}\\ &= \sum_{\mu_1,\dots,\mu_{2k-2}=\pm} M(\beta_k)^j_{\mu_1}(C^{-1})^{\mu_1}_{\mu_2}M(\beta_{k-1})^{\mu_3}_{\mu_2}\cdots M(\beta_1)^{\mu_{2k-2}}_i\\ &= (M(\beta_k)C^{-1}M(\beta_{k-1})C^{-1}\cdots M(\beta_1))^j_i.\end{aligned}$$

□

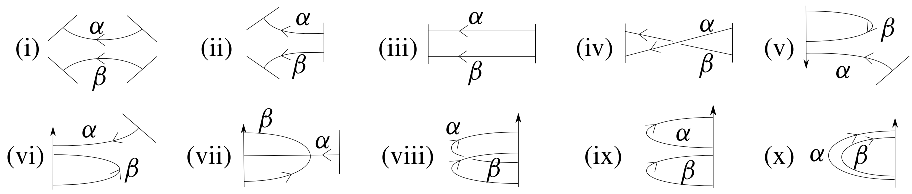


Figure 14: Ten configurations for two nonintersecting oriented arcs.

Let α and β be two nonintersecting oriented arcs. Denote by a, b, c and d the boundary arcs containing $s(\alpha)$, $t(\alpha)$, $s(\beta)$ and $t(\beta)$, respectively. Reversing the orientation and the height order of α or β if necessary, we have ten different possibilities illustrated in Figure 14. The proof of the following lemma is very similar to the computations made by Faitg in [27].

Lemma 2.24 (i) *If the elements of* $\{a,b,c,d\}$ *are pairwise distinct, one has*

$$N(\alpha) \odot N(\beta) = \tau(N(\beta) \odot N(\alpha))\tau. \tag{11}$$

(ii) *When* $a = c$, $\{a,b,d\}$ *has cardinality* 3 *and* $s(\beta) <_a s(\alpha)$, *one has*

$$N(\alpha) \odot N(\beta) = \tau(N(\beta) \odot N(\alpha))\mathscr{R}. \tag{12}$$

(iii) *When* $a = c \neq b = d$, $s(\beta) <_a s(\alpha)$ *and* $t(\alpha) <_b t(\beta)$, *one has*

$$N(\alpha) \odot N(\beta) = \mathscr{R}^{-1}(N(\beta) \odot N(\alpha))\mathscr{R}. \tag{13}$$

(iv) *When* $a = c \neq b = d$, $s(\beta) <_a s(\alpha)$ *and* $t(\beta) <_b t(\alpha)$, *one has*

$$N(\alpha) \odot N(\beta) = \mathscr{R}(N(\beta) \odot N(\alpha))\mathscr{R}. \tag{14}$$

(v) *When* $b = c = d \neq a$, $s(\beta) <_a t(\beta) <_a t(\alpha)$ *and* $h(s(\beta)) < h(t(\beta))$, *one has*

$$N(\alpha) \odot N(\beta) = \mathscr{R}^{-1}(N(\beta) \odot \mathbb{1}_2)\mathscr{R}(N(\alpha) \odot \mathbb{1}_2). \tag{15}$$

(vi) *When* $b = c = d \neq a$, $t(\alpha) <_a t(\beta) <_a s(\beta)$ *and* $h(s(\beta)) < h(t(\beta)) < h(t(\alpha))$, *one has*

$$N(\alpha) \odot N(\beta) = \mathscr{R}^{-1}(N(\beta) \odot \mathbb{1}_2)\mathscr{R}(N(\alpha) \odot \mathbb{1}_2). \tag{16}$$

(vii) *When* $b = c = d \neq a$, $t(\beta) <_a t(\alpha) <_a s(\beta)$ *and* $h(s(\beta)) < h(t(\alpha)) < h(t(\beta))$, *one has*

$$N(\alpha) \odot N(\beta) = \mathscr{R}(N(\beta) \odot \mathbb{1}_2)\mathscr{R}(N(\alpha) \odot \mathbb{1}_2). \tag{17}$$

(viii) *When $a=b=c=d$, $s(\beta)<_a s(\alpha)<_a t(\beta)<_a t(\alpha)$ and*

$$h(s(\beta))<h(s(\alpha))<h(t(\beta))<h(t(\alpha)),$$

one has

$$(\mathbb{1}_2\odot N(\alpha))\mathcal{R}^{-1}(\mathbb{1}_2\odot N(\beta))\mathcal{R}^{-1}=\mathcal{R}(\mathbb{1}_2\odot N(\beta))\mathcal{R}^{-1}(\mathbb{1}_2\odot N(\alpha)). \tag{18}$$

(ix) *When $a=b=c=d$, $s(\beta)<_a t(\beta)<_a s(\alpha)<_a t(\alpha)$ and*

$$h(s(\beta))<h(t(\beta))<h(s(\alpha))<h(t(\alpha)),$$

one has

$$\mathcal{R}^{-1}(\mathbb{1}_2\odot N(\alpha))\mathcal{R}(\mathbb{1}_2\odot N(\beta))=(\mathbb{1}_2\odot N(\beta))\mathcal{R}^{-1}(\mathbb{1}_2\odot N(\alpha))\mathcal{R}. \tag{19}$$

(x) *When $a=b=c=d$, $s(\alpha)<_a s(\beta)<_a t(\beta)<_a t(\alpha)$ and*

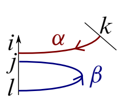
$$h(s(\alpha))<h(s(\beta))<h(t(\beta))<h(t(\alpha)),$$

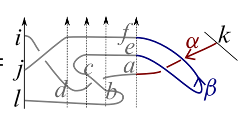
one has

$$(\mathbb{1}_2\odot N(\alpha))\mathcal{R}^{-1}(\mathbb{1}_2\odot N(\beta))\mathcal{R}=\mathcal{R}(\mathbb{1}_2\odot N(\beta))\mathcal{R}^{-1}(\mathbb{1}_2\odot N(\alpha)). \tag{20}$$

Proof Equation (11) says that in case (i) any α_{ij} commutes with any β_{kl}, which is obvious. Equations (12), (13) and (14) in cases (ii), (iii) and (iv) are straightforward consequences of the height exchange relation (5). All other cases will be derived using the boundary skein relations (6). As in the proof of Lemma 2.21, we will color the arcs α and β in red and blue to remind the reader that they might be much more complicated than they look in the picture: in the computations we perform while using the boundary skein relation we only care about the restriction of the diagrams (depicted in gray) in a small bigon in the neighborhood of the boundary arc a and not the actual shape of the blue and red parts.

Equations (15) and (16) in cases (v) and (vi) are proved in a very similar way; we detail the proof of (16) and leave (17) to the reader. In case (vi), one has

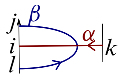
$$(M(\alpha)\odot M(\beta))^{ij}_{kl}=\alpha_{ki}\beta_{lj}=$$


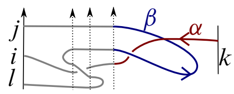
$$=$$


$$=\sum_{a,b,c,d,e,f=\pm}(\mathcal{R}^{-1})^{ij}_{fd}M(\beta)^f_e C^e_c\mathcal{R}^{cd}_{ab}M(\alpha)^a_k(C^{-1})^b_l$$
$$=\big(\mathcal{R}^{-1}(M(\beta)C\odot\mathbb{1}_2)\mathcal{R}(M(\alpha)\odot C^{-1})\big)^{ij}_{kl}.$$

To handle cases (vii)–(x), we introduce the 4×4 matrix $V=(V^{ij}_{kl})_{i,j,k,l\in\{-,+\}}$, where $V^{ij}_{kl}=[\alpha\cup\beta,\sigma_{ijkl}]\in\mathcal{S}_\omega(\mathbf{\Sigma})$ is the class of the simple diagram $\alpha\cup\beta$ with state σ_{ijkl} sending $t(\alpha)$, $t(\beta)$, $s(\alpha)$ and $s(\beta)$ to i, j, k and l, respectively. Here the height order of the points of $\partial(\alpha\cup\beta)$ is given by the boundary arc orientation drawn in Figure 14. The trick is to compute V in two different ways and then equate the two obtained formulas.

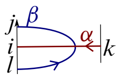
In case (vii), on the one hand, we first prove $V=\tau(M(\beta)C\odot\mathbb{1}_2)\mathscr{R}(M(\alpha)\odot C^{-1})$:

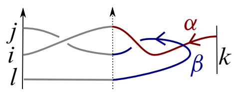
$V^{ij}_{kl}=$ 

$=$ 

$=((M(\beta)C\odot\mathbb{1}_2)\mathscr{R}(M(\alpha)\odot C^{-1}))^{ji}_{kl}.$

On the other hand, we prove $V=\tau\mathscr{R}^{-1}(M(\alpha)\odot M(\beta))$:

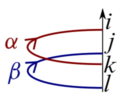
$V^{ij}_{kl}=$ 

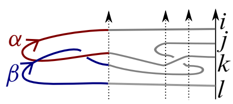
$=$ 

$=(\mathscr{R}^{-1}(M(\alpha)\odot M(\beta)))^{ji}_{kl}.$

So we get the equality $\mathscr{R}^{-1}(M(\alpha)\odot M(\beta))=(M(\beta)C\odot\mathbb{1}_2)\mathscr{R}(M(\alpha)\odot C^{-1})$ (which equals τV) and (17) follows.

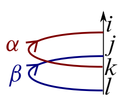
In case (viii), on the one hand, we first prove $V=\tau(C\odot M(\alpha))\mathscr{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\beta))$:

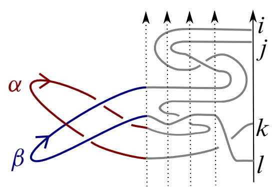
$V^{ij}_{kl}=$ 

$=$ 

$=((C\odot M(\alpha))\mathscr{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\beta)))^{ji}_{kl}.$

On the other hand, we prove $V=\tau(C\odot C)\mathscr{R}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathscr{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha))\mathscr{R}$:

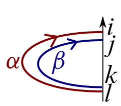
$V^{ij}_{kl}=$ 

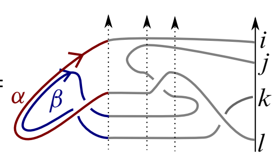
$=$ 

$$=((C\odot C)\mathscr{R}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathscr{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha))\mathscr{R})^{ji}_{kl}.$$

Equation (18) follows by equating the two obtained expressions for V.

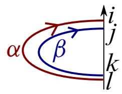
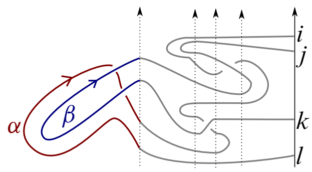
In case (x), on the one hand, we first prove $V=(C\odot M(\alpha))\mathscr{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathscr{R}$:

$V^{ij}_{kl}=$ 

$=$ 

$=((C\odot M(\alpha))\mathscr{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathscr{R})^{ij}_{kl}.$

On the other hand, we prove $V=(C\odot C)\mathcal{R}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathcal{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha))$:

$V^{ij}_{kl}=$  $=$ 

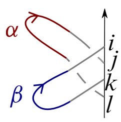
$$=((C\odot C)\mathcal{R}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathcal{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha)))^{ij}_{kl}.$$

Therefore, we obtain the following equality that will be used in the proof of Lemma 2.25:

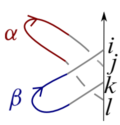
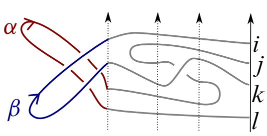
$$\begin{aligned}V&=(C\odot M(\alpha))\mathcal{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathcal{R}\\&=(C\odot C)\mathcal{R}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathcal{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha)).\end{aligned}\tag{21}$$

Equation (20) follows.

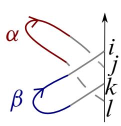
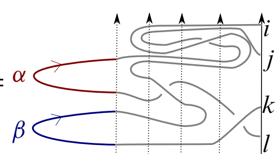
In (ix) we slightly change strategy. Define the 4×4 matrix $W=(W^{ij}_{kl})_{i,j,k,l\in\{-,+\}}$ by

$W^{ij}_{kl}:=$ 

We first prove $W=(C\odot M(\beta))\mathcal{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha))$:

$W^{ij}_{kl}=$  $=$  $=((C\odot M(\beta))\mathcal{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha)))^{ij}_{kl}.$

Next, we prove $W=(C\odot C)\mathcal{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha))\mathcal{R}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathcal{R}^{-1}$:

$W^{ij}_{kl}=$  $=$ 

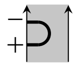
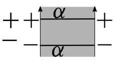
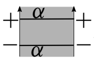
$$=((C\odot C)\mathcal{R}^{-1}(\mathbb{1}_2\odot C^{-1}M(\alpha))\mathcal{R}(\mathbb{1}_2\odot C^{-1}M(\beta))\mathcal{R}^{-1})^{ij}_{kl}.$$

Equation (19) follows by equating the two obtained expressions for W. □

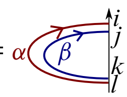
Lemma 2.25 (q–determinant relations) *Let α be an oriented arc. Then*

$$\det_q(N(\alpha))=1 \text{ if } \alpha \text{ is of type } a, \text{ and } \det_{q^2}(N(\alpha))=1 \text{ otherwise.}\tag{22}$$

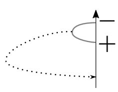
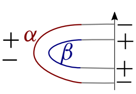
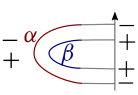
Proof First suppose that α is of type a. Applying the trivial arc and cutting arc relation, we obtain

$(C^{-1})_+^- =$  $= (C^{-1})_-^+$  $+ (C^{-1})_+^-$ ,

which is equivalent to the equation $\alpha_{++}\alpha_{--} - q^{-1}\alpha_{+-}\alpha_{-+} = 1$ as claimed. Next we suppose that α is of type d. Let β be an arc isotopic to and disjoint from α, placed as in Figure 14(x) (so $\beta_{ij} = \alpha_{ij}$). Consider the matrix $V = (V_{kl}^{ij})_{i,j,k,l \in \{-,+\}}$, where $V_{kl}^{ij} = [\alpha \cup \beta, \sigma_{ijkl}] \in \mathcal{S}_\omega(\mathbf{\Sigma})$ is the class of the simple diagram $\alpha \cup \beta$ with state σ_{ijkl} sending $t(\alpha)$, $t(\beta)$, $s(\alpha)$ and $s(\beta)$ to i, j, k and l respectively, like in the proof of Lemma 2.24, ie

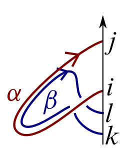
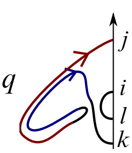
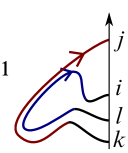
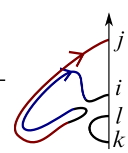
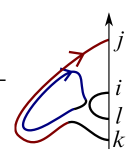
$V_{kl}^{ij} =$ 

Again, using the trivial arc and cutting arc relation, we obtain

(23) $C_+^- =$  $= (C^{-1})_-^+$  $+ (C^{-1})_+^-$ 

$$\iff A^5 V_{+-}^{-+} - A^3 V_{-+}^{-+} = 1.$$

To develop the elements V_{kl}^{ij} as linear combinations of the $\alpha_{ab}\alpha_{cd}$ we can either consider the matrix coefficients of the equality $V = (C \odot M(\alpha))\mathcal{R}^{-1}(\mathbb{1}_2 \odot C^{-1}M(\beta))\mathcal{R}$ proved in the proof of Lemma 2.24, or we can perform the skein computation

$\alpha_{ij}\alpha_{kl} =$  $= q$  $+ q^{-1}$  $+$  $+$ 

$$= qC_k^j C_l^i + C_i^j C_k^l + q^{-1} V_{lk}^{ji} + C_l^i (C^{-1})_+^- V_{+k}^{j-} + C_l^i (C^{-1})_-^+ V_{-k}^{j+},$$

from which we deduce the equalities

$$V_{+-}^{-+} = q\alpha_{+-}\alpha_{-+} + A^{-1}, \quad V_{-+}^{+-} = q\alpha_{-+}\alpha_{+-} + A^{-1}, \quad V_{-+}^{-+} = \alpha_{--}\alpha_{++} - A^{-3}.$$

Now, using the skein relation (2), we find

$$V_{+-}^{-+} = qV_{+-}^{+-} + A^{-1} = V_{-+}^{+-},$$

so $V_{+-}^{-+} = V_{-+}^{+-}$, which implies that $\alpha_{+-}\alpha_{-+} = \alpha_{-+}\alpha_{+-}$.

Next, replacing the elements V_+^{-+} and V_{-+}^{-+} in (23) by their expressions in terms of the $\alpha_{ij}\alpha_{kl}$, we find

$$\alpha_{--}\alpha_{++} - q^2\alpha_{+-}\alpha_{-+} = A. \tag{24}$$

Using $\alpha_{+-}\alpha_{-+}=\alpha_{-+}\alpha_{+-}$ we obtain the desired equality:

$$\det_{q^2}(N(\alpha))=\det_{q^2}\begin{pmatrix}-\omega^{-5}\alpha_{-+} & -\omega^{-5}\alpha_{--}\\ \omega^{-1}\alpha_{++} & \omega^{-1}\alpha_{+-}\end{pmatrix}=-A^3\alpha_{-+}\alpha_{+-}+A^{-1}\alpha_{--}\alpha_{++}=1.$$

Now, if α is of type e, then α^{-1} is of type d. A simple computation shows that if $M=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ is such that $ad=da$ then $\det_{q^2}(M)=\det_{q^2}(C^{-1\,t}MC)$, so we deduce the q–determinant formula for α of type e from the fact that it holds for α^{-1}, from the orientation-reversing formula in Lemma 2.20 and from $\alpha_{+-}\alpha_{-+}=\alpha_{-+}\alpha_{+-}$.

Suppose that α is of type c and choose $\Bbbk=\mathbb{Z}[\omega^{\pm 1}]$. Recall from Section 2.1 the reflexion anti-involution θ. The image $\theta(\alpha)$ is of type d, so applying θ to (24),

$$\alpha_{++}\alpha_{--}-q^{-2}\alpha_{-+}\alpha_{+-}=A^{-1}. \tag{25}$$

By Remark 2.6, since (25) holds for $\Bbbk=\mathbb{Z}[\omega^{\pm 1}]$, it also holds for any other ring. Also using θ, we find that $\alpha_{+-}\alpha_{-+}=\alpha_{-+}\alpha_{+-}$ and the equation $\det_{q^2}(N(\alpha))=1$ follows. Finally, when α is of type b, we deduce the q–determinant relation from the fact that it holds for α^{-1} (of type c), from the orientation-reversing formulas of Lemma 2.20 and from the identity $\alpha_{+-}\alpha_{-+}=\alpha_{-+}\alpha_{+-}$. □

Definition 2.26 Let $\mathbb{P}=(\mathbb{G},\mathbb{RL})$ be a finite presentation of $\Pi_1(\Sigma_{\mathcal{P}},\mathbb{V})$. The set $\mathcal{A}^{\mathbb{G}}$ generates $\mathcal{S}_\omega(\mathbf{\Sigma})$ by Proposition 2.14, and we have found three families of relations:

- (i) For each $\alpha\in\mathbb{G}$ we have either the relation $\det_q(N(\alpha))=1$ or $\det_{q^2}(N(\alpha))=1$ by (22) in Lemma 2.25; we call these the *q–determinant relations*.
- (ii) For each $R\in\mathbb{RL}$, we have four relations obtained by considering the matrix coefficients in (10) in Lemma 2.23; we call these *trivial loop relations*.
- (iii) For each pair (α,β) of elements in $\mathbb{G}$, we have 16 relations obtained by considering the matrix coefficients in one of (11)–(20) of Lemma 2.24 after having possibly replaced α or β by α^{-1} or β^{-1}, if necessary, and using the inversion formula (8); we call these *arc exchange relations*.

3 Proof of Theorems 1.1 and 1.2

Definition 3.1 Let $\mathcal{L}_\omega(\mathbb{P})$ be the algebra generated by the elements of $\mathbb{G}$ modulo the q–determinant, trivial loops and arc exchange relations, and write $\Psi\colon\mathcal{L}_\omega(\mathbb{P})\to\mathcal{S}_\omega(\mathbf{\Sigma})$ the obvious algebra morphism.

By Proposition 2.14, Ψ is surjective and we need to show that Ψ is injective to prove Theorem 1.1. We cut the proof of Theorem 1.1 in three steps: In step 1, we show that it is sufficient to consider the case where $\mathbb{P}$ has no relations (as in Example 2.13); in this particular case, the finite presentation defining $\mathcal{L}_\omega(\mathbb{P})$ is inhomogeneous quadratic and we will use the diamond lemma to extract PBW bases of $\mathcal{L}_\omega(\mathbb{P})$ and to prove it is Koszul. In step 2 we extract the rewritten rules and their leading terms from the q–determinant and arc exchange relations, and exhibit the associated spanning family $\underline{\mathcal{B}}^{\mathbb{G}} \subset \mathcal{L}_\omega(\mathbb{P})$. Finally in step 3, we show that the image by Ψ of $\underline{\mathcal{B}}^{\mathbb{G}}$ is a basis; this will prove both the injectivity of Ψ and the fact that $\underline{\mathcal{B}}^{\mathbb{G}}$ is a Poincaré–Birkhoff–Witt basis, and conclude the proofs of Theorems 1.1 and 1.2.

3.1 Step 1: reduction to the case where $\mathbb{P}$ has no relations

Let Γ be the presenting graph of $\mathbb{P}$ and consider its fundamental groupoid $\Pi_1(\Gamma)$: the objects of $\Pi_1(\Gamma)$ are the vertices of Γ (ie the set $\mathbb{V}$) and the morphisms are compositions $\alpha_k^{\varepsilon_k} \cdots \alpha_1^{\varepsilon_1}$ where $\alpha_i \in \mathbb{G}$. The inclusion $\Gamma \subset \Sigma_{\mathcal{P}}$ induces a functor $F\colon \Pi_1(\Gamma) \to \Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$, which is the identity on the objects. The fact that $\mathbb{G}$ is a set of generators implies that F is full and $\mathbb{P}$ has no relations if and only if F is faithful. Fix $v_0 \in \mathbb{V}$. For a relation $R \in \mathbb{RL}$ of the form $R = \beta_k \star \cdots \star \beta_1$, the *basepoint* of R is $s(\beta_1) = t(\beta_k)$. By inspecting the trivial loop relation (10), we see that changing a relation R by a relation $\beta \star R \star \beta^{-1}$ does not change the algebra $\mathcal{L}_\omega(\mathbb{P})$. Since $\Sigma_{\mathcal{P}}$ is assumed to be connected, we can suppose that all relations in $\mathbb{RL}$ have the same basepoint v_0, so each relation $R = \beta_k \star \cdots \star \beta_1$ induces an element $[R] = \beta_k \cdots \beta_1 \in \pi_1(\Gamma, v_0)$. The functor F induces a surjective group morphism $F_{v_0}\colon \pi_1(\Gamma, v_0) \to \pi_1(\Sigma_{\mathcal{P}}, v_0)$ and the fact that $\mathbb{RL}$ is a set of relations implies that $\{[R] \mid R \in \mathbb{RL}\}$ generates $\ker(F_{v_0})$. Since $\pi_1(\Gamma, v_0)$ is a free group, so is $\ker(F_{v_0})$. Let $R_1, \dots, R_m \in \mathbb{RL}$ be such that $\{[R_1], \dots, [R_m]\}$ is a minimal set of generators for the free group $\ker(F_{v_0})$. For each R_i, choose an element $\beta_i \in \mathbb{G}$ such that either β_i or β_i^{-1} appears in the expression of R_i and such that the set $\mathbb{G}'$ obtained from $\mathbb{G}$ by removing the β_i is a generating set. So if Γ' is the presenting graph of $\mathbb{G}'$, the morphism $F'_{v_0}\colon \pi_1(\Gamma', v_0) \to \pi_1(\Sigma_{\mathcal{P}}, v_0)$ is injective, thus the functor $F'\colon \Pi_1(\Gamma') \to \Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ is faithful and $\mathbb{P}' := (\mathbb{G}', \varnothing)$ is a finite presentation of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ with no relations.

The inclusion $\mathbb{G}' \subset \mathbb{G}$ induces an algebra morphism $\tilde{\varphi}\colon \mathcal{T}[\mathbb{G}'] \hookrightarrow \mathcal{T}[\mathbb{G}]$ on the free tensor algebras generated by $\mathbb{G}'$ and $\mathbb{G}$, respectively, and $\tilde{\varphi}$ sends q–determinant and arc exchange relations to q–determinant and arc exchange relations, so it induces an algebra morphism

$$\varphi\colon \mathcal{L}_\omega(\mathbb{P}') \to \mathcal{L}_\omega(\mathbb{P}).$$

Lemma 3.2 *The morphism φ is an isomorphism.*

Proof To prove the surjectivity we need to show that, for each removed path $\beta_i \in \mathbb{G}\setminus\mathbb{G}'$, the stated arcs $(\beta_i)_{\varepsilon\varepsilon'}$ can be expressed as a polynomial in the stated arcs $(\alpha^{\pm 1})_{\mu\mu'}$ for $\alpha \in \mathbb{G}'$. This follows from the trivial loop relation (10) associated to the relation $R_i \in \mathbb{RL}$ containing $\beta_i^{\pm 1}$. Injectivity of φ is a straightforward consequence of the definition. □

3.2 Step 2: Poincaré–Birkhoff–Witt bases and Koszulness

Convention 3.3 In the rest of the section, we suppose that $\mathbb{P} = (\mathbb{G}, \varnothing)$ is a presentation with no relations and that every arc in $\mathbb{G}$ is either of type a, c or d.

Note that the convention on the type of the generators is not restrictive but purely conventional since we can always replace a generator α by α^{-1} without changing the set $\mathcal{A}^{\mathbb{G}}$ of generators of $\mathcal{S}_\omega(\mathbf{\Sigma})$.

Since $\mathbb{P}$ has no relations, the defining presentation of $\mathcal{L}_\omega(\mathbb{P})$ contains only q–determinant and arc exchange relations. All these relations are quadratic (inhomogeneous) in the generators $\mathcal{A}^{\mathbb{G}}$ and we want to apply the diamond lemma to prove that $\mathcal{L}_\omega(\mathbb{P})$ is Koszul.

Reminder of the diamond lemma for PBW bases Following the exposition in Section 4 of [42], we briefly recall the statement of the diamond lemma for PBW bases: Let V be a free finite rank $\Bbbk$–module, denote by $T(V) := \oplus_{n\geq 0} V^{\otimes n}$ the tensor algebra and fix $R \subset V^{\otimes 2}$ a finite subset. The quotient algebra $\mathcal{A} := T(V)/(R)$ is called a *quadratic algebra*. Let $\{v_i\}_{i\in I}$ be a totally ordered basis of V and write $I = \{1,\dots,k\}$ so that $v_i < v_{i+1}$. Then the set $J := \bigsqcup_{n\geq 0} I^n$ (where $I^0 = \{0\}$) is totally ordered by the lexicographic order and the set of elements $v_{\boldsymbol{i}} = v_{i_1}\cdots v_{i_n}$, for $\boldsymbol{i} = (i_1,\dots,i_k)$, forms a basis of $T(V)$. We suppose that the elements $r \in R$ (named relators) have the form

$$r = v_i v_j - \sum_{(k,l)<(i,j)} \lambda_{kl}^{ij} v_k v_l.$$

The term $v_i v_j$ is called the *leading term* of r. We assume that two distinct relators have distinct leading terms. Define the family

(26) $\qquad \mathcal{B} := \{v_{i_1}\cdots v_{i_n} \mid v_{i_k} v_{i_{k+1}} \text{ is not a leading term for all } 1 \leq k \leq n-1\},$

and denote by $\mathcal{B}^{(3)} \subset \mathcal{B}$ the subset of elements of length 3 (of the form $v_{i_1} v_{i_2} v_{i_3}$). Obviously the set $\mathcal{B}$ spans $\mathcal{A}$.

Theorem 3.4 (diamond lemma for PBW bases, Bergman [10]; see also Loday and Vallette [42, Theorem 4.3.10]) *If $\mathcal{B}^{(3)}$ is free, then $\mathcal{B}$ is a (Poincaré–Birkhoff–Witt) basis and $\mathcal{A}$ is Koszul.*

The arc exchange relations defining $\mathcal{L}_\omega(\mathbb{P})$ are quadratic, however the q–determinant relations are not (because of the 1 in $\det_q(N(\alpha)) = 1$), so $\mathcal{L}_\omega(\mathbb{P})$ is not quadratic but rather inhomogeneous quadratic. An *inhomogeneous quadratic algebra* is an algebra of the form $\mathcal{A} := T(V)/(R)$, where $R \subset V^{\otimes 2} \oplus V \oplus \Bbbk \subset T(V)$. We further assume

$$(ql_1) \qquad R \cap V = \{0\}$$

and

$$(ql_2) \qquad (R \otimes V + V \otimes R) \cap V^{\otimes 2} \subset R \cap V^{\otimes 2}.$$

The hypothesis (ql_2) says that one cannot create new relations by adding an element to R, so it is not restrictive. Like before, we fix an ordered basis $\{v_i\}_{i \in I}$ of V and suppose that the relators of R have the form

$$(27) \qquad r = v_i v_j - \sum_{(k,l)<(i,j)} \lambda^{ij}_{kl} v_k v_l - c_{i,j},$$

where $c_{i,j}$ are some scalars and we suppose that two distinct relators have distinct leading terms. The associated quadratic algebra $q\mathcal{A}$ is the algebra with same generators v_i but where the relators have been changed by replacing the scalars $c_{i,j}$ by 0. Let $q\mathcal{B} \subset q\mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}$ be the two generating families defined by (26).

Theorem 3.5 [42, Theorem 4.3.18] *Suppose that $q\mathcal{B}^{(3)} \subset q\mathcal{A}$ is free. Then both $q\mathcal{B}$ and $\mathcal{B}$ are (PBW) bases of $q\mathcal{A}$ and $\mathcal{A}$, respectively, and both $q\mathcal{A}$ and $\mathcal{A}$ are Koszul.*

There exists a linear surjective morphism $\varphi: q\mathcal{A} \to \mathcal{A}$ sending the generating family $q\mathcal{B}$ to $\mathcal{B}$; see [42, Section 4.2.9]. So, if $\mathcal{B}$ is a basis of $\mathcal{A}$, then $q\mathcal{B}$ is free, therefore Theorem 3.5 implies that $\mathcal{A}$ is Koszul. Therefore:

Theorem 3.6 *If $\mathcal{B}$ is a basis of $\mathcal{A}$, then $\mathcal{A}$ is Koszul.*

The relators of the stated skein presentations and PBW bases For $\alpha \in \mathbb{G}$, define $\mathcal{B}(\alpha)$ as

$$\{(\alpha_{++})^a(\alpha_{+-})^b(\alpha_{--})^c \mid a,b,c \geq 0\} \cup \{(\alpha_{++})^a(\alpha_{-+})^b(\alpha_{--})^c \mid a,b,c \geq 0\} \subset \mathcal{L}_\omega(\mathbb{P}).$$

Fix a total order $<$ on the set $\mathbb{G}$ of generators and index its elements as $\mathbb{G} = \{\alpha_1, \dots, \alpha_n\}$, where $\alpha_i < \alpha_{i+1}$. Let

$$\underline{\mathcal{B}}^{\mathbb{G}} := \{m_1 m_2 \cdots m_n \mid m_i \in \mathcal{B}(\alpha_i)\} \subset \mathcal{L}_\omega(\mathbb{P}).$$

We want to apply Theorem 3.6 to prove that $\mathcal{L}_\omega(\mathbb{P})$ is Koszul. By definition, $\mathcal{L}_\omega(\mathbb{P})$ is an inhomogeneous quadratic algebra with generators $\mathcal{A}^{\mathbb{G}} = \{\alpha_{ij} \mid \alpha \in \mathbb{G} \text{ and } i,j = \pm\}$ and whose relations are the arc exchange and q–determinant relations.

We first define a total order $\prec$ on $\mathcal{A}^{\mathbb{G}}$ by imposing that $\alpha_{ab} \prec \beta_{cd}$ if $\alpha < \beta$ and that $\alpha_{++} \prec \alpha_{+-} \prec \alpha_{-+} \prec \alpha_{--}$.

The goal of this subsection is to rewrite the q–determinant and arc exchange relations so that they define a set of relators of the form (27) whose leading terms are pairwise distinct, satisfying (ql_1) and (ql_2) and such that the set of leading terms is

(28) $$\mathrm{LT} := \{\alpha_{ab}\beta_{cd} \mid \text{ either } \alpha > \beta, \text{ or } \alpha = \beta \text{ and either } a < c \text{ or } b < d\}.$$

The set $\underline{\mathcal{B}}^{\mathbb{G}}$ is the generating set defined by (26) with this set of leading terms (ie $\underline{\mathcal{B}}^{\mathbb{G}}$ is the set of elements $v_1 \cdots v_n$ where $v_i \in \mathcal{A}^{\mathbb{G}}$ and $v_i v_{i+1}$ is not in LT). At this stage, it will become clear that $\underline{\mathcal{B}}^{\mathbb{G}}$ spans $\mathcal{L}_\omega(\mathbb{P})$. Once we perform this task, we will prove in step 3 that $\underline{\mathcal{B}}^{\mathbb{G}}$ is free by showing that its image through $\Psi\colon \mathcal{L}_\omega(\mathbb{P}) \to \mathcal{S}_\omega(\mathbf{\Sigma})$ is a basis of $\mathcal{S}_\omega(\mathbf{\Sigma})$. This will imply that Ψ is an isomorphism (proving Theorem 1.1) and Theorem 3.6 will imply that $\mathcal{L}_\omega(\mathbb{P})$ is Koszul (proving Theorem 1.2).

Consider two distinct generators $\alpha, \beta \in \mathbb{G}$ such that $\alpha > \beta$. For each $a, b, c, d \in \{\pm\}$, we have an arc exchange relation of the form

$$\alpha_{ab}\beta_{cd} = \sum_{ijkl=\pm} c^{i,j,k,l}_{a,b,c,d}\beta_{ij}\alpha_{kl},$$

where $c^{i,j,k,l}_{a,b,c,d}$ are some scalars. We associate the relator

$$r = \alpha_{ab}\beta_{cd} - \sum_{ijkl=\pm} c^{i,j,k,l}_{a,b,c,d}\beta_{ij}\alpha_{kl},$$

whose leading term is $\alpha_{ab}\beta_{cd}$ (because $\alpha > \beta$ implies that $\alpha_{ab}\beta_{cd} \succ \beta_{ij}\alpha_{kl}$) and denote by $R_{\alpha,\beta}$ the set (of cardinality 16) of such relators.

Now suppose that $\alpha \in \mathbb{G}$ is of type a. The set of relations between the generators α_{ij} is given by

$$M(\alpha) \odot M(\alpha) = \mathscr{R}^{-1}(M(\alpha) \odot M(\alpha))\mathscr{R} \quad \text{and} \quad \det_q(M(\alpha)) = 1.$$

Note that in this case, the subalgebra of $\mathcal{L}_\omega(\mathbb{P})$ generated by the α_{ij} is isomorphic to $\mathcal{O}_q[\mathrm{SL}_2] \cong \mathcal{S}_\omega(\mathbb{B})$. We rewrite those relations:

(Ra) $$\begin{aligned} &\alpha_{+-}\alpha_{++} = q\alpha_{++}\alpha_{+-}, && \alpha_{-+}\alpha_{++} = q\alpha_{++}\alpha_{-+},\\ &\alpha_{--}\alpha_{+-} = q\alpha_{+-}\alpha_{--}, && \alpha_{--}\alpha_{-+} = q\alpha_{-+}\alpha_{--},\\ &\alpha_{+-}\alpha_{-+} = q\alpha_{++}\alpha_{--} - q, && \alpha_{-+}\alpha_{+-} = q\alpha_{++}\alpha_{--} - q,\\ &\alpha_{--}\alpha_{++} = q^2\alpha_{++}\alpha_{--} + 1 - q^2. \end{aligned}$$

The associated set of relators R_α is defined by assigning, to each of the seven equalities of the form $x=y$ in the system (Ra), the relator $r := x-y$ with leading term x. Note that the set of leading terms of the elements of R_α is the set of elements $\alpha_{ab}\alpha_{cd}$ such that either $a<c$ or $b<d$.

Now suppose that $\alpha \in \mathbb{G}$ is of type d. The set of relations between the generators α_{ij} are given by

$$(\mathbb{1}_2 \odot N(\alpha))\mathcal{R}^{-1}(\mathbb{1}_2 \odot N(\alpha))\mathcal{R} = \mathcal{R}(\mathbb{1}_2 \odot N(\alpha))\mathcal{R}^{-1}(\mathbb{1}_2 \odot N(\alpha)), \quad \det_{q^2}(N(\alpha)) = 1,$$

where $N(\alpha) = C^{-1}M(\alpha)$. These relations generate the same ideal as the set of relations

$$\text{(Rd)}\quad \begin{aligned} &\alpha_{-+}\alpha_{++} = \alpha_{++}\alpha_{-+} + (q-q^{-1})q^2\alpha_{+-}\alpha_{--}, \quad \alpha_{+-}\alpha_{++} = q^2\alpha_{++}\alpha_{+-},\\ &\alpha_{--}\alpha_{-+} = \alpha_{-+}\alpha_{--} + (q-q^{-1})q^2\alpha_{+-}\alpha_{--}, \quad \alpha_{--}\alpha_{+-} = q^2\alpha_{+-}\alpha_{--},\\ &\alpha_{+-}\alpha_{-+} = \alpha_{++}\alpha_{--} - (q-q^{-1})^2\alpha_{+-}^2 - A,\\ &\alpha_{-+}\alpha_{+-} = \alpha_{++}\alpha_{--} - (q-q^{-1})^2\alpha_{+-}^2 - A,\\ &\alpha_{--}\alpha_{++} = q^2\alpha_{++}\alpha_{--} - q^2(q-q^{-1})^2\alpha_{+-}^2 + A(1-q^2). \end{aligned}$$

As before, we denote by R_α the set of relators obtained from system (Rd) by assigning, to each of the seven equalities of the form $x=y$ in the system (Rd), the relator $r := x-y$ with leading term x. Again, the set of leading terms of the elements of R_α is the set of elements $\alpha_{ab}\alpha_{cd}$ such that either $a<c$ or $b<d$.

For $\alpha \in \mathbb{G}$ of type c, the set of relations between the elements α_{ij} can be obtained from the system (Rd) using the reflection anti-involution. Rearranging the terms, we get the system of relations

$$\text{(Rc)}\quad \begin{aligned} &\alpha_{-+}\alpha_{++} = \alpha_{++}\alpha_{-+} + (q-q^{-1})\alpha_{+-}\alpha_{--}, \quad \alpha_{+-}\alpha_{++} = q^2\alpha_{++}\alpha_{+-},\\ &\alpha_{--}\alpha_{-+} = \alpha_{-+}\alpha_{--} + (q-q^{-1})\alpha_{+-}\alpha_{--}, \quad \alpha_{--}\alpha_{+-} = q^2\alpha_{+-}\alpha_{--},\\ &\alpha_{+-}\alpha_{-+} = q^2\alpha_{++}\alpha_{--} - A^3, \qquad \alpha_{-+}\alpha_{+-} = q^2\alpha_{++}\alpha_{--} - A^3,\\ &\alpha_{--}\alpha_{++} = q^2\alpha_{++}\alpha_{--} + (q-q^{-1})^2\alpha_{+-}^2 + A^{-1}(1-q^2). \end{aligned}$$

Like previously, we denote by R_α the associated set of relators and note that the set of leading terms is the set of elements $\alpha_{ab}\alpha_{cd}$ such that either $a<c$ or $b<d$.

Let V be the free $\mathbb{k}$–module with basis $\mathcal{A}^{\mathbb{G}}$ and $R \subset \mathbb{k} \oplus V^{\otimes 2} \subset T(V)$ be the union of the sets of relators $R_{\alpha,\beta}$ and R_α, where $\alpha, \beta \in \mathbb{G}$ and $\alpha > \beta$. Then $\mathcal{L}_\omega(\mathbb{P}) = T(V)/(R)$, the leading terms of R are pairwise distinct and they form the set LT of (28), and the hypotheses (ql_1) and (ql_2) are obviously satisfied. Therefore, if we prove that $\underline{\mathcal{B}}^{\mathbb{G}}$ is a basis of $\mathcal{L}_\omega(\mathbb{P})$ then Theorem 3.6 would imply that $\mathcal{L}_\omega(\mathbb{P})$ is Koszul.

3.3 Step 3: injectivity of Ψ

Denote by $\mathcal{B}^{\mathbb{G}} \subset \mathcal{S}_\omega(\mathbb{P})$ the image of $\underline{\mathcal{B}}^{\mathbb{G}}$ under $\Psi\colon \mathcal{L}_\omega(\mathbb{P}) \to \mathcal{S}_\omega(\mathbf{\Sigma})$.

Theorem 3.7 *The set $\mathcal{B}^{\mathbb{G}}$ is a basis of $\mathcal{S}_\omega(\mathbf{\Sigma})$.*

Corollary 3.8 (i) *The morphism $\Psi\colon \mathcal{L}_\omega(\mathbb{P}) \to \mathcal{S}_\omega(\mathbf{\Sigma})$ is an isomorphism.*

(ii) *The family $\mathcal{B}^{\mathbb{G}}$ is a PBW basis and $\mathcal{S}_\omega(\mathbf{\Sigma})$ is Koszul.*

The fact that $\mathcal{B}^{\mathbb{G}}$ linearly spans $\mathcal{S}_\omega(\mathbf{\Sigma})$ follows from the surjectivity of Ψ (so follows from Proposition 2.14), however we will reprove this fact. The proof of Theorem 3.7 is divided into two steps. First we introduce another family $\mathcal{B}^{\mathbb{G}}_+ \subset \mathcal{S}_\omega(\mathbf{\Sigma})$ and prove that $\mathcal{B}^{\mathbb{G}}_+$ is free by relating it to the basis $\mathcal{B}$. Next we use a filtration of $\mathcal{S}_\omega(\mathbf{\Sigma})$ to deduce that $\mathcal{B}^{\mathbb{G}}$ is free from the fact that $\mathcal{B}^{\mathbb{G}}_+$ is free.

For $\alpha \in \mathbb{G}$ and $n \geq 0$, we denote by $\alpha^{\langle n \rangle}$ the simple diagram made of n pairwise nonintersecting copies of α. For $\boldsymbol{n} \in \mathbb{N}^{\mathbb{G}}$, we denote by $D(\boldsymbol{n})$ the simple diagram $\bigsqcup_{\alpha \in \mathbb{G}} \alpha^{\langle n(\alpha) \rangle}$. Denote by v and w the two endpoints of α, and by a and b the (not necessarily distinct) boundary arcs containing v and w, respectively. Write $v_1, \dots, v_n$ and $w_1, \dots, w_n$ the endpoints of $\alpha^{\langle n \rangle}$ so that $v_i <_a v_{i+1}$ and $w_i <_b w_{i+1}$ (so v_i and w_i are not necessarily the boundary points of the same component of $\alpha^{\langle n \rangle}$). A state $s \in \mathrm{St}(D(\boldsymbol{n}))$ is *positive* if for all $\alpha \in \mathbb{G}$ and for all $i \leq j$ one has $s(v_i) \leq s(v_j)$ and $s(w_i) \leq s(w_j)$; we let $\mathrm{St}^+(D(\boldsymbol{n}))$ denote the set of positive states.

Definition 3.9 We denote by $\mathcal{B}^{\mathbb{G}}_+ \subset \mathcal{S}_\omega(\mathbf{\Sigma})$ the set of classes $[D(\boldsymbol{n}), s]$ for $\boldsymbol{n} \in \mathbb{N}^{\mathbb{G}}$ and $s \in \mathrm{St}^+(D(\boldsymbol{n}))$.

Proposition 3.10 *The family $\mathcal{B}^{\mathbb{G}}_+$ is a basis of $\mathcal{S}_\omega(\mathbf{\Sigma})$.*

The fact that $\mathcal{B}^{\mathbb{G}}_+$ is free will follow from this elementary lemma, which basically says that an upper triangular matrix with invertible diagonal elements is invertible:

Lemma 3.11 *Let V be a free $\Bbbk$–module, $\mathcal{B}$ a basis of V equipped with a partial order $\leq$, and $\mathcal{B}' \subset V$ a family such that there exist two maps $m\colon \mathcal{B}' \to \mathcal{B}$ and $c\colon \mathcal{B}' \to \Bbbk^\times$ such that*

(i) *m is injective, and*

(ii) *every element $b' \in \mathcal{B}'$ decomposes as*

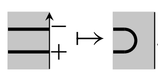
$$b' = c(b')m(b') + \sum_{b > m(b')} \alpha_{b,b'} b.$$

Then $\mathcal{B}'$ is free.

Proof Consider a vanishing linear combination $\sum_{b' \in \mathcal{B}'} x_{b'} b' = 0$, where $x_{b'} \in \Bbbk$. Set $S := \{m(b') \mid x_{b'} \neq 0\}$. For contradiction, suppose that $S \neq \varnothing$ and choose b_0 a minimum for S. Let $b'_0 \in \mathcal{B}'$ be the unique element such that $m(b'_0) = b_0$. Then the equality $\sum_{b' \in \mathcal{B}'} x_{b'} b' = 0$ together with the decomposition hypothesis imply that $c(b'_0) x_{b'_0} = 0$. Since $c(b'_0) \in \Bbbk^\times$ is invertible by hypothesis, $x_{b'_0} = 0$, so we have a contradiction. □

Notation 3.12 (i) Let (D, s) be a stated diagram and a a be boundary arc. We denote by $d_a([D, s]) \in \mathbb{N}$ the number of pairs (v, w) in $\partial_a D$ such that $v <_a w$ and $(s(v), s(w)) = (+, -)$; recall that the orientation of $\Sigma_{\mathcal{P}}$ induces an orientation of a which in turn induces the order $<_a$. We also write $d([D, s]) = \sum_a d_a([D, s])$. Note that $[D, s] \in \mathcal{B}$ if and only if $d([D, s]) = 0$.

(ii) Let $\mathcal{D}$ denote the set of stated diagrams (D, s) with D simple, so both $\mathcal{B}$ and $\mathcal{B}^{\mathbb{G}}_{+}$ are subsets of $\mathcal{D}$. Define a binary operation $\mapsto_{\mathfrak{o}}$ on $\mathcal{D}$ as follows. If (D, s) contains a pair (v, w) in $\partial_a D$ of consecutive points for the height ordering (there is no $z \in \partial_a D$ such that $v <_a z <_a w$) with $v <_a w$ and such that $(s(v), s(w)) = (+, -)$, let (D', s') be the stated diagram obtained by joining v and w to a single point and then pushing it to the interior of Σ, that is (D', s') is obtained from (D, s) by the local move



Let (D'', s'') be obtained from (D', s') by removing the possible trivial component if any. In this case, we write $(D, s) \mapsto_{\mathfrak{o}} (D'', s'')$. Since $d([D'', s'']) < d([D, s])$, the relation $\mapsto_{\mathfrak{o}}$ is terminal, with $\mathcal{B}$ as the set of terminal objects. Define a partial order $\leq_{\mathfrak{o}}$ by setting $(D, s) >_{\mathfrak{o}} (D', s')$ if there exists a sequence $(D, s) \mapsto_{\mathfrak{o}} (D_2, s_2) \mapsto_{\mathfrak{o}} \cdots \mapsto_{\mathfrak{o}} (D', s')$. Clearly, $\leq_{\mathfrak{o}}$ is filtrant, ie if $(D_1, s_2) \leq_{\mathfrak{o}} (D, s)$ and $(D_2, s_2) \leq_{\mathfrak{o}} (D, s)$ there exists (D_0, s_0) such that $(D_0, s_0) \leq_{\mathfrak{o}} (D_i, s_i)$ for $i = 1, 2$.

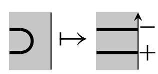
(iii) Let α be an oriented arc. Since $\mathbb{G}$ is a generating set, the associated path in $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ decomposes as $\alpha = \beta_1^{\varepsilon_1} \cdots \beta_k^{\varepsilon_k}$ and, since $(\mathbb{G}, \varnothing)$ is a presentation with no relation, this decomposition is unique. We denote by $\mathrm{lw}(\alpha) := k$ its length. For $(D, s) \in \mathcal{D}$, where $D = \alpha_1 \cup \cdots \cup \alpha_n$ with α_i connected, we set

$$l(D, s) := \sum_{i=1}^{n} (\mathrm{lw}(\alpha_i) - 1).$$

Note that $\mathcal{B}^{\mathbb{G}}_{+}$ is the subset of elements $(D, s) \in \mathcal{D}$ such that $l(D, s) = 0$.

Figure 15: An element $b \in \mathcal{B}_+^{\mathbb{G}}$ (left) and its associated element $m(b) \in \mathcal{B}$ (right). Here $\mathbb{G} = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ is the set of generators of Figure 6.

(iv) We define a binary operation $\mapsto_{\mathbb{B}}$ on $\mathcal{D}$ as follows. Let $(D, s) \in \mathcal{D}$ and α a connected component of D with $\mathrm{lw}(\alpha) > 1$. Choose a decomposition $\alpha = \alpha_1\alpha_2$ where $\mathrm{lw}(\alpha_i) < \mathrm{lw}(\alpha)$, set $D' := (D \setminus \alpha) \cup \alpha_1 \cup \alpha_2$ and fix the height orders and the state s' such that (D', s') is obtained from (D, s) by the local move



In this case, we write $(D, s) \mapsto_{\mathbb{G}} (D', s')$. Since $l(D', s') < l(D, s)$, the relation $\mapsto_{\mathbb{G}}$ is terminal with $\mathcal{B}_+^{\mathfrak{o}}$ as the set of terminal objects. Define a partial order $\leq_{\mathbb{G}}$ on $\mathcal{D}$ by setting $(D, s) >_{\mathbb{G}} (D', s')$ if there exists a sequence $(D, s) \mapsto_{\mathbb{G}} (D_2, s_2) \mapsto_{\mathbb{G}} \cdots \mapsto_{\mathbb{G}} (D', s')$. It follows from the unicity of the decomposition of a path in $\mathbb{G}$ (so from the fact that $(\mathbb{G}, \varnothing)$ is a presentation with no relation) that $\leq_{\mathbb{G}}$ is filtrant.

(v) Let $m\colon \mathcal{B}_+^{\mathbb{G}} \to \mathcal{B}$ be the map sending a class $[D, s]$ to the class of the unique minimum for $\leq_{\mathfrak{o}}$ of the successors of (D, s) (the existence and unicity are guaranteed by the fact that $\leq_{\mathfrak{o}}$ is terminal and filtrant). Similarly, let $m'\colon \mathcal{B} \to \mathcal{B}_+^{\mathbb{G}}$ be the map sending a class $[D, s]$ to the class of the unique minimum for $\leq_{\mathbb{G}}$ of the set of successors of (D, s); see Figure 15 for an example.

Proof of Proposition 3.10 We will apply Lemma 3.11 to the map $m\colon \mathcal{B} \to \mathcal{B}_+^{\mathbb{G}}$, where we equip $\mathcal{B}$ with the partial order $\prec$ where $[D, s] \prec [D', s']$ if $|\partial D| < |\partial D'|$.

The map *m* is injective By definition, if $(D, s) \mapsto_{\mathbb{G}} (D', s')$ then $(D', s') \mapsto_{\mathfrak{o}} (D, s)$ (the converse if false in general). Therefore, for $[D, s] \in \mathcal{B}$, given a sequence

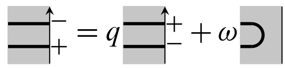
$$(D, s) \mapsto_{\mathbb{G}} (D_2, s_2) \mapsto_{\mathbb{G}} \cdots \mapsto_{\mathbb{G}} (D_n, s_n) \mapsto_{\mathbb{G}} m(D, s)$$

one has a sequence

$$m(D, s) \mapsto_{\mathfrak{o}} (D_n, s_n) \mapsto_{\mathfrak{o}} \cdots \mapsto_{\mathfrak{o}} (D_2, s_2) \mapsto_{\mathfrak{o}} D.$$

This implies that $m'(m(D)) = D$ so $m' \circ m = \mathrm{id}$ and m is injective.

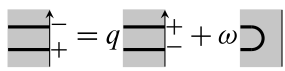
$\mathcal{B}^{\mathbb{G}}_{+}$ is upper triangular Suppose that $(D,s)\mapsto_{\mathfrak{o}}(D',s')$. The skein relation



shows that $[D,s]=\omega[D',s']+q[D'',s'']$, where $|\partial D'|<|\partial D''|$. So for $[D,s]\in\mathcal{B}^{\mathbb{G}}_{+}$ with $m([D,s])=[D_0,s_0]$ and given $(D,s)\mapsto_{\mathfrak{o}}(D_2,s_2)\mapsto_{\mathfrak{o}}\cdots\mapsto_{\mathfrak{o}}(D_n,s_n)\mapsto_{\mathfrak{o}}(D_0,s_0)$, we have

$$[D,s]=\omega^n m(D,s)+\text{higher terms},$$

where "higher terms" is a linear combination of elements (D',s') with $|\partial D'|>|\partial D_0|$. Since $\mathcal{B}$ is free, Lemma 3.11 implies that $\mathcal{B}^{\mathbb{G}}_{+}$ is free. To prove that it spans $\mathcal{S}_\omega(\mathbf{\Sigma})$ we note that if $(D,s)\mapsto_{\mathbb{G}}(D',s')$, the same skein relation



implies that

$$[D,s]=\omega^{-1}[D',s']-\omega^{-5}[D'',s'']$$

for another element $(D'',s'')\in\mathcal{D}$ such that $l(D',s')<l(D,s)$ and $l(D'',s'')<l(D,s)$. We then prove that any element of $\mathcal{B}$ is a linear combination of elements of $\mathcal{B}_{\mathbb{G}}^{+}$ by induction on $l(D,s)$. $\square$

We now want to deduce that $\mathcal{B}^{\mathbb{G}}$ is a basis from the fact that $\mathcal{B}^{\mathbb{G}}_{+}$ is a basis. The argument is based on the use of an algebra filtration of $\mathcal{S}_\omega(\mathbf{\Sigma})$ that we now introduce:

Definition 3.13 For $\boldsymbol{n}\in\mathbb{N}^{\mathbb{G}}$, we let $|\boldsymbol{n}|:=\sum_{\alpha\in\mathbb{G}}\boldsymbol{n}(\alpha)$. For a class $[D(\boldsymbol{n}),s]$, we set $\|[D(\boldsymbol{n}),s]\|:=(|\boldsymbol{n}|,-d([D(\boldsymbol{n}),s]))\in\mathbb{N}\times\mathbb{Z}$. Denote by $<$ the lexicographic order on $\mathbb{N}\times\mathbb{Z}$, ie $(k_1,k_2)<(k_1',k_2')$ if either $k_1<k_1'$, or $k_1=k_1'$ and $k_2<k_2'$. Finally, to $\mathbb{k}=(k_1,k_2)\in\mathbb{N}\times\mathbb{Z}$ we associate the submodule

$$\mathcal{F}_{\mathbb{k}}:=\operatorname{Span}\big([D(\boldsymbol{n}),s]:\|[D(\boldsymbol{n}),s]\|\le\mathbb{k}\big).$$

In order to prove that $\{\mathcal{F}_{\mathbb{k}}\}$ forms an algebra filtration, the following elementary observation will be quite useful:

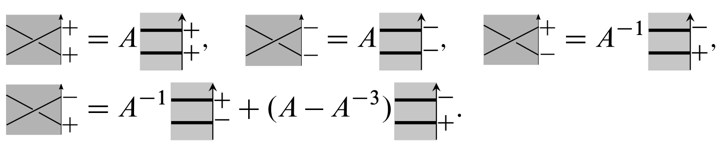
Lemma 3.14 *Let T and T' be two tangles in $\Sigma_{\mathcal{P}}\times(0,1)$ which are isotopic through an isotopy that does not preserves the height orders. Let $s\in\mathrm{St}(T)$ and $s'\in\mathrm{St}(T')$ be two states such that for a boundary arc a, if $\partial_aT=\{v_1,\dots,v_n\}$ and $\partial_aT'=\{w_1,\dots,w_n\}$*

are ordered so that $h(v_i) < h(v_{i+1})$ *and* $h(w_i) < h(w_{i+1})$, *then* $s(v_i) = s'(w_i)$ *for all* $i \in \{1, \dots, n\}$. *Then one has*

$$[T,s] = \omega^n [T',s'] + \sum_{\sigma \in \mathrm{St}(T'),\, d([T',\sigma]) < d([T',s'])} x_\sigma [T',\sigma], \tag{29}$$

where $n \in \mathbb{Z}$, $x_\sigma \in \Bbbk$ *and the sum in the right-hand side is over states* σ *of* T' *such that* $d([T',\sigma]) < d([T',s'])$.

Proof We say that a tangle T_i is obtained from a tangle T_{i+1} by an elementary height exchange if there exists a boundary arc a and two consecutive points v and w in $\partial_a T_i$ with $h(v) < h(w)$ ("consecutive" means that there does not exist any $p \in \partial_a T_i$ such that $h(v) < h(p) < h(w)$) such that T_{i+1} is the tangle obtained from T_i by exchanging the heights of v and w. Since T and T' are isotopic, through an isotopy that does not preserve the height orders, we can obtain T' from T by a finite sequence $T = T_1 \mapsto T_2 \mapsto \cdots \mapsto T_n = T'$ of elementary height exchanges. It is clear that if one has a development (29) when the pair (T, T') is equal to a pair (T_i, T_{i+1}) and a pair (T_{i+1}, T_{i+2}), then it holds for the pair (T_i, T_{i+2}). So by induction on the size n of the finite sequence, it is sufficient to prove the lemma in the particular case where T and T' differ by an elementary height exchange. In this case, (29) follows from the height exchange relations



□

Notation 3.15 Let $b \in \mathcal{B}^{\mathbb{G}}$, so by definition $b = b_{\alpha_1} \cdots b_{\alpha_n}$, where $b_{\alpha_i} \in \mathcal{B}(\alpha_i)$. That is, one has either $b_{\alpha_i} = \alpha_{++}^{a_i} \alpha_{+-}^{b_i} \alpha_{--}^{c_i}$ or $b_{\alpha_i} = \alpha_{++}^{a_i} \alpha_{-+}^{b_i} \alpha_{--}^{c_i}$ for some $a_i, b_i, c_i \ge 0$. Let $\boldsymbol{n} \in \mathbb{N}^{\mathbb{G}}$ be defined by $\boldsymbol{n}(\alpha_i) := a_i + b_i + c_i$. Let $T(\boldsymbol{n})$ be the tangle underlying $D(\boldsymbol{n})$. Let (T, s) be a stated tangle (unique up to isotopy) such that $b = [T, s]$, so that $T(\boldsymbol{n})$ is obtained from T by an isotopy that does not necessarily preserve the height order. Finally we define the element $b^+ := [T(\boldsymbol{n}), s^+] \in \mathcal{B}_+^{\mathbb{G}}$, where $s^+ \in \mathrm{St}^+(T(\boldsymbol{n}))$ is the unique state such that (T, s) and $(T(\boldsymbol{n}), s^+)$ satisfy the assumptions of (29). Note that the induced map $(\cdot)^+ : \mathcal{B}^{\mathbb{G}} \to \mathcal{B}_+^{\mathbb{G}}$, sending b to b^+, is a bijection.

Lemma 3.16 (i) *For* $\Bbbk, \Bbbk' \in \mathbb{N} \times \mathbb{Z}$, *one has* $\mathcal{F}_{\Bbbk} \cdot \mathcal{F}_{\Bbbk'} \subset \mathcal{F}_{\Bbbk + \Bbbk'}$.

(ii) *For* $b \in \mathcal{B}^{\mathbb{G}}$, *one has*

$$b = \omega^n b^+ + \textit{lower terms}, \tag{30}$$

where $n \in \mathbb{Z}$ and "lower terms" is a linear combination of basis elements $b_i^+ \in \mathcal{B}_+^{\mathbb{G}}$ such that $\|b_i^+\| < \|b^+\|$.

Note that the second assertion of Lemma 3.16 implies that $\mathcal{B}^{\mathbb{G}}$ spans $\mathcal{S}_\omega(\mathbf{\Sigma})$, so reproves Proposition 2.14.

Proof (i) Let $x := [T(\boldsymbol{n}), s]$ and $y := [T(\boldsymbol{n}'), s']$, and denote by $(T(\boldsymbol{n}) \cup T(\boldsymbol{n}'), s \cup s')$ the stated tangle obtained by stacking $(T(\boldsymbol{n}), s)$ on top of $(T(\boldsymbol{n}'), s')$, so that

$$x \cdot y = [T(\boldsymbol{n}) \cup T(\boldsymbol{n}'), s \cup s'].$$

The tangles $T(\boldsymbol{n}) \cup T(\boldsymbol{n}')$ and $T(\boldsymbol{n} + \boldsymbol{n}')$ differ by an isotopy that does not necessarily preserve the height orders, so Lemma 3.14 implies that $x \cdot y$ is a linear combination of elements of the form $[D(\boldsymbol{n} + \boldsymbol{n}'), \sigma]$ such that $\|[D(\boldsymbol{n} + \boldsymbol{n}'), \sigma]\| \leq \|x\| + \|y\|$. This proves the first assertion.

(ii) Using Notation 3.15, we apply Lemma 3.14 to $b = [T, s]$ and $b^+ = [T(\boldsymbol{n}), s^+]$, and (30) is just a rewriting of (29). □

Proof of Theorem 3.7 Both Proposition 2.14 and the second assertion of Lemma 3.16 imply that $\mathcal{B}^{\mathbb{G}}$ generates $\mathcal{S}_\omega(\mathbf{\Sigma})$. To prove that $\mathcal{B}^{\mathbb{G}}$ is free, we apply Lemma 3.11 to the injective map $(\cdot)^+ : \mathcal{B}^{\mathbb{G}} \to \mathcal{B}_+^{\mathbb{G}}$ where we equip $\mathcal{B}_+^{\mathbb{G}}$ with the ordering $[D, s] < [D', s']$ if $\|[D, s]\| > \|[D', s']\|$. The hypotheses of Lemma 3.11 are satisfied by virtue of Proposition 3.10 and Lemma 3.16, so $\mathcal{B}$ is free. □

4 Lattice gauge field theory

4.1 Ciliated graphs and quantum gauge group coaction

Since the pioneering work of Fock and Rosly [29], constructions in lattice gauge field theory are based on ciliated graphs. As we now explain, to a ciliated graph (Γ, c) one can associate a punctured surface $\mathbf{\Sigma}^0$ together with a finite presentation $\mathbb{P}$ of its associated groupoid.

Definition 4.1 (i) A *ribbon graph* Γ is a finite graph together with the data, for each vertex, of a cyclic ordering of its adjacent half-edges. An *orientation* for a ribbon graph is the choice of an orientation for each of its edges.

(ii) A *ciliated ribbon graph* (Γ, c) is a ribbon graph Γ together with a lift, for each vertex, of the cyclic ordering of the adjacent half-edges to a linear ordering. If the half-edges adjacent to a vertex have the cyclic ordering $e_1 < e_2 < \cdots < e_n < e_1$ that we lift to the linear ordering $e_1 < e_2 < \cdots < e_n$, we draw a *cilium* between e_n and e_1.

(iii) We associate surfaces to ribbon graphs as follows.

(a) Place a disc D_v on top of each vertex v and a band B_e on top of each edge e, then glue the discs to the bands using the cyclic ordering. We thus get a surface $S(\Gamma)$ named the *fattening of* Γ.

(b) The *closed punctured surface* $\mathbf{\Sigma}(\Gamma) = (\Sigma(\Gamma), \mathcal{P})$ *associated to* Γ is the closed punctured surface obtained from $S(\Gamma)$ by gluing a disc to each boundary component and placing a puncture inside each added disc. So $S(\Sigma)$ deformation retracts to $\Sigma_{\mathcal{P}}(\Gamma)$.

(c) The *open punctured surface* $\mathbf{\Sigma}^0(\Gamma, c) = (\Sigma^0(\Gamma, c), \mathcal{P}^0)$ *associated to* (Γ, c) is obtained from $S(\Gamma)$ by first pushing each vertex v to the boundary of $S(\Gamma)$ in the direction of the associated cilium. Said differently, if the ordered half-edges adjacent to v are $e_1 < e_2 < \cdots < e_n$, we push v in the boundary of D_v so that it lies between the band B_{e_n} and the band B_{e_1}. Next place a puncture p_v next to v (in the counterclockwise direction) on the same boundary component as v. Finally, to each boundary component of $S(\Gamma)$ which does not contain any puncture p_v, glue a disc and place a puncture inside the disc. In the so-obtained punctured surface $\mathbf{\Sigma}^0(\Gamma, c)$, each boundary arc contains exactly one vertex v of Γ, so we denote by a_v the boundary arc containing v. Suppose that Γ is oriented. Then the oriented edges of Γ form a set $\mathbb{G}$ of generators of $\Pi_1(\Sigma^0_{\mathcal{P}}, \mathbb{V})$ such that $\mathbb{P}(\Gamma, c) := (\mathbb{G}, \varnothing)$ is a finite presentation without relations.

(iv) For v_1 and v_2 two distinct vertices of (Γ, c), the ciliated graph $(\Gamma_{v_1 \# v_2}, c_{v_1 \# v_2})$ is obtained by gluing the vertices v_1 and v_2 to a vertex v in such a way that if $e_1 < \cdots < e_n$ and $f_1 < \cdots < f_m$ are the ordered half-edges adjacent to v_1 and v_2, respectively, then the linear order of the half-edges adjacent to v is $e_1 < \cdots < e_n < f_1 < \cdots < f_m$. Note that $c_{v_1 \# v_2} \neq c_{v_2 \# v_1}$.

Figure 16 illustrates two examples having the same ribbon graph but different ciliated structures: the punctured surface $\mathbf{\Sigma}^0(\Gamma, c)$ is a disc with two inner punctures and two boundary punctures whereas $\mathbf{\Sigma}^0(\Gamma, c')$ is an annulus with one puncture per boundary component and one inner puncture.

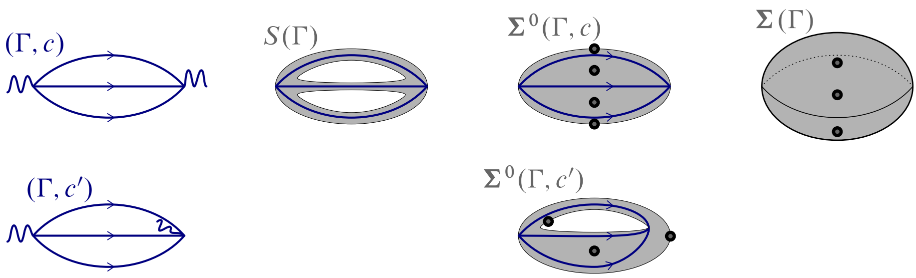


Figure 16: Top, from left to right: a ciliated graph (Γ, c), its fattening $S(\Gamma)$, its open punctured surface $\mathbf{\Sigma}^0(\Gamma, c)$ and its closed punctured surface $\mathbf{\Sigma}(\Gamma)$. Bottom: the same ribbon graph with a different ciliated structure c' (left) and the associated open punctured surface $\mathbf{\Sigma}^0(\Gamma, c')$ (right).

Remark 4.2 In [26] Costantino and Lê made the following important remark: the punctured surface $\mathbf{\Sigma}^0(\Gamma_{v_1\#v_2}, c_{v_1\#v_2})$ is obtained from $\mathbf{\Sigma}^0(\Gamma, c) \sqcup \mathbb{T}$ by gluing the boundary arcs a_{v_1} and a_{v_2} to two faces of the triangle $\mathbb{T}$. In particular, when $\Gamma = \Gamma_1 \sqcup \Gamma_2$ with $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$, this property, together with Theorem 2.10, permitted the authors of [26] to prove that $\mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma_{v_1\#v_2}, c_{v_1\#v_2}))$ is the cobraided tensor product of $\mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma_1, c_1))$ with $\mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma_2, c_2))$. The same gluing property was first discovered by Alekseev, Grosse and Schomerus in [2; 3] for the quantum moduli spaces (see [37] for a survey on the classical and quantum versions of the fusion operation).

For an oriented ciliated graph (Γ, c), we denote by $V(\Gamma)$ its set of vertices and $\mathcal{E}(\Gamma)$ its set of (oriented) edges. Like in the previous section, we see the elements of $\mathcal{E}(\Gamma)$ as oriented arcs. Denote by $\mathbb{D}_0$ the punctured surface made of a disc with a single puncture on its boundary. The closed punctured surface $\mathbf{\Sigma}(\Gamma)$ is obtained from the open one $\mathbf{\Sigma}^0(\Gamma, c)$ by gluing a copy $\mathbb{D}_0$ along each boundary arc a_v. Therefore, writing $\widehat{\mathbb{D}} := \bigsqcup_{v \in V(\Gamma)} \mathbb{D}_0$, by Theorem 2.10 one has the exact sequence

$$(31) \quad 0 \to \mathcal{S}_\omega(\mathbf{\Sigma}(\Gamma)) \xrightarrow{i} \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma, c) \sqcup \widehat{\mathbb{D}}) \xrightarrow{\Delta^R - \sigma \circ \Delta^L} \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma, c) \sqcup \widehat{\mathbb{D}}) \otimes \mathcal{O}_q[\mathrm{SL}_2]^{\otimes V(\Gamma)},$$

where i represents the gluing map.

Using the isomorphism $\mathcal{S}_\omega(\mathbb{D}_0) \cong \mathbb{k}$ sending the class of the empty stated tangle to the neutral element $1 \in \mathbb{k}$, we define an isomorphism

$$\kappa : \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma, c) \sqcup \widehat{\mathbb{D}}) \cong \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma, c)) \otimes \bigotimes_{v \in V(\Gamma)} \mathcal{S}_\omega(\mathbb{D}_0) \cong \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma, c)).$$

Denote by $\iota\colon \mathcal{S}_\omega(\mathbf{\Sigma}(\Gamma)) \hookrightarrow \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c))$ the injective morphism $\iota := \kappa\circ i$. Also denote by $\Delta^{\mathcal{G}}\colon \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c)) \to \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c))\otimes \mathcal{O}_q[\mathrm{SL}_2]^{\otimes V(\Gamma)}$ the (unique) morphism making the following diagram commute:

$$\begin{array}{ccc} \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c)\sqcup\widehat{\mathbb{D}}) & \xrightarrow{\Delta^R} & \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c)\sqcup\widehat{\mathbb{D}})\otimes\mathcal{O}_q[\mathrm{SL}_2]^{\otimes V(\Gamma)} \\ \cong\big\downarrow\kappa & & \cong\big\downarrow\kappa\otimes\mathrm{id} \\ \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c)) & \xrightarrow{\Delta^{\mathcal{G}}} & \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c))\otimes\mathcal{O}_q[\mathrm{SL}_2]^{\otimes V(\Gamma)} \end{array}$$

Definition 4.3 The *quantum gauge group* is the Hopf algebra $\mathcal{O}_q[\mathcal{G}] := \mathcal{O}_q[\mathrm{SL}_2]^{\otimes V(\Gamma)}$. The (right) Hopf-comodule map $\Delta^{\mathcal{G}}\colon \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c)) \to \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c))\otimes\mathcal{O}_q[\mathcal{G}]$ is called the *quantum gauge group coaction*.

Note that, by definition, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c)\sqcup\widehat{\mathbb{D}}) & \xrightarrow{\sigma\circ\Delta^L} & \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c)\sqcup\widehat{\mathbb{D}})\otimes\mathcal{O}_q[\mathcal{G}] \\ \cong\big\downarrow\kappa & & \cong\big\downarrow\kappa\otimes\mathrm{id} \\ \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c)) & \xrightarrow{\mathrm{id}\otimes\epsilon} & \mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c))\otimes\mathcal{O}_q[\mathcal{G}] \end{array}$$

Therefore the exactness of (31) implies that we have the exact sequence

$$(32)\qquad 0\to\mathcal{S}_\omega(\mathbf{\Sigma}(\Gamma))\xrightarrow{\iota}\mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c))\xrightarrow{\Delta^{\mathcal{G}}-\mathrm{id}\otimes\epsilon}\mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c))\otimes\mathcal{O}_q[\mathcal{G}].$$

Said differently, $\iota(\mathcal{S}_\omega(\mathbf{\Sigma}(\Gamma)))$ is the subalgebra of $\mathcal{S}_\omega(\mathbf{\Sigma}^0(\Gamma,c))$ of coinvariant vectors for the quantum gauge group coaction.

Notation 4.4 For $x\in\mathcal{O}_q[\mathrm{SL}_2]$ and $v_0\in V(\Gamma)$ the element of the form $\bigotimes_v y_v$, where $y_v = 1$ for $v\neq v_0$ and $y_{v_0} = x$, is denoted by $x^{(v_0)}\in\mathcal{O}_q[\mathcal{G}] = \mathcal{O}_q[\mathrm{SL}_2]^{\otimes V(\Gamma)}$.

Let α be an arc of type either a or d and write v_1 and v_2 for the elements of $\mathbb{V}$ corresponding to the boundary arcs containing $s(\alpha)$ and $t(\alpha)$, respectively. The quantum gauge group coaction is characterized by the following formula illustrated in Figure 17:

$$(33)\qquad \Delta^{\mathcal{G}}(\alpha_{ij}) = \sum_{a,b=\pm}\alpha_{ab}\otimes x_{jb}^{(v_2)}x_{ia}^{(v_1)}.$$

In order to prepare the comparison between stated skein algebras at $\omega=+1$ and relative character varieties in the next subsection, let us derive from Theorem 1.1 an alternative

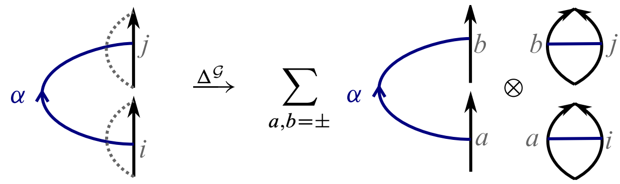


Figure 17: An illustration of (33).

presentation of $\mathcal{S}_\omega(\mathbf{\Sigma})$. During the rest of the section, we fix a finite presentation $\mathbb{P} = (\mathbb{G}, \mathbb{RL})$ of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ such that every arc of $\mathbb{G}$ is either of type a or d.

When comparing skein algebras with character varieties, there is a well-known sign issue which requires some attention. When $\mathbf{\Sigma}$ is closed, the skein algebra $\mathcal{S}_{+1}(\mathbf{\Sigma})$ is generated by the classes of closed curves γ whereas the algebra $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ of regular functions of the character variety is generated by curve functions τ_γ, sending a class $[\rho]$ of representation $\rho\colon \pi_1(\Sigma_{\mathcal{P}}) \to \mathrm{SL}_2(\mathbb{C})$ to $\tau_\gamma([\rho]) := \mathrm{tr}(\rho(\gamma))$. However there is no isomorphism $\mathcal{S}_{+1}(\mathbf{\Sigma}) \cong \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ sending γ to τ_γ. Instead, we fix a spin structure on $\Sigma_{\mathcal{P}}$ with associated Johnson quadratic form $\Omega\colon \mathrm{H}_1(\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ and define $w(\gamma) := 1 + \Omega([\gamma])$. Then it follows from [7; 17; 44] that we have an isomorphism $\mathcal{S}_{+1}(\mathbf{\Sigma}) \cong \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ sending γ to $(-1)^{w(\gamma)}\tau_\gamma$. A similar sign issue appears when dealing with stated skein algebras and relative character varieties; this was studied in [39] to which we refer for further details (see also [26; 48] for an elegant interpretation of this sign issue in term of *twisted character variety*).

In short, the authors defined in [39] the notion of *relative spin structure* to which one can associate a map $w\colon \mathbb{G} \to \mathbb{Z}/2\mathbb{Z}$ having the property that for any simple relation $R = \beta_k \star \cdots \star \beta_1$, one has $\sum_{i=1}^k w(\beta_i) = 1$. We will call a map $w\colon \mathbb{G} \to \mathbb{Z}/2\mathbb{Z}$ satisfying this property a *spin function*.

Notation 4.5 Let w be a spin function. For $\alpha \in \mathbb{G}$, we denote by $U(\alpha)$ the 2×2 matrix with coefficients in $\mathcal{S}_\omega(\mathbf{\Sigma})$ defined by

$$U(\alpha) := \begin{cases} (-1)^{w(\alpha)}\omega C^{-1}M(\alpha) & \text{if } \alpha \text{ is of type } a, \\ (-1)^{w(\alpha)}C^{-1}M(\alpha) = (-1)^{w(\alpha)}N(\alpha) & \text{if } \alpha \text{ is of type } d. \end{cases} \tag{34}$$

Proposition 4.6 (i) *The stated skein algebra $\mathcal{S}_\omega(\mathbf{\Sigma})$ admits the alternative presentation with generators the elements $U(\alpha)_i^j$ and with $\alpha \in \mathbb{G}$ and $i, j = \pm$, together with the following relations*:

- *The q–determinant relations* $\det_q(U(\alpha)) = 1$ *when α is of type a, and* $\det_{q^2}(U(\alpha)) = 1$ *when α is of type d.*

- *For* $R = \beta_k \star \cdots \star \beta_1 \in \mathbb{RL}$ *a relation where* l *generators* β_i *are of type* a*, the trivial loop relation*

$$U(\beta_k)\cdots U(\beta_1) = A^3\omega^l. \tag{35}$$

- *For each pair of generators in* $\mathbb{G}$*, the arc exchange relations obtained from the relations in Lemma 2.24 by replacing* $N(\alpha)$ *by* $U(\alpha)$ *if* α *is of type* d *or by* $CN(\alpha)$ *if* α *is of type* a.

(ii) *The quantum gauge group coaction is characterized by the formula*

$$\Delta^{\mathcal{G}}(U(\alpha)_i^j) = \sum_{a,b=\pm} U(\alpha)_a^b \otimes S(x_{bj})^{(v_2)} x_{ia}^{(v_1)}, \tag{36}$$

where we use the same notation as in (33).

Proof It is clear from (34) that the matrix elements $U(\alpha)_i^j$ generate the same algebra as the elements $M(\alpha)_i^j = \alpha_{ij}$, so they generate $\mathcal{S}_\omega(\mathbf{\Sigma})$. We need to check that the q–determinant, trivial loop and arc exchange relations for the elements α_{ij} are equivalent to the relations of the proposition for the elements $U(\alpha)_i^j$. When $\alpha \in \mathbb{G}$ is of type d, clearly the relation $\det_{q^2}(N(\alpha)) = 1$ is equivalent to the relation $\det_{q^2}(U(\alpha)) = 1$. When $\alpha \in \mathbb{G}$ is of type a, the equivalence

$$\det_q(M(\alpha)) = 1 \iff \det_q(U(\alpha)) = 1$$

follows from a straightforward computation (and is the reason for the ω in the expression $U(\alpha) = (-1)^{w(\alpha)}\omega C^{-1} M(\alpha)$). The equivalence between (10) and (35) is straightforward (and is responsible for the introduction of the spin function and for the $(-1)^{w(\alpha)}$ factor in the definition of $U(\alpha)$). The fact that the arc exchange relations are equivalent to the same relations with $N(\alpha)$ replaced by $U(\alpha)$ or $CU(\alpha)$ depending whether α is of type d or a follows from the definition of $U(\alpha)$ and the fact that the arc exchange relations are homogeneous.

It remains to derive the formula (36) from (33). This is done by direct computation, left to the reader, using the fact that for the two 2×2 matrices

$$X = \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix} \quad \text{and} \quad S(X) = \begin{pmatrix} S(x_{++}) & S(x_{+-}) \\ S(x_{-+}) & S(x_{--}) \end{pmatrix}$$

with coefficients in $\mathcal{O}_q[\mathrm{SL}_2]$, one has $S(X) = C^{-1}\,{}^tXC$. Figure 18 illustrates (36). In Figure 18, we use a special convention: we have drawn stated diagrams that go "outside" of $\Sigma_{\mathcal{P}}$ in some small bigon neighborhoods of the boundary arcs. It must be

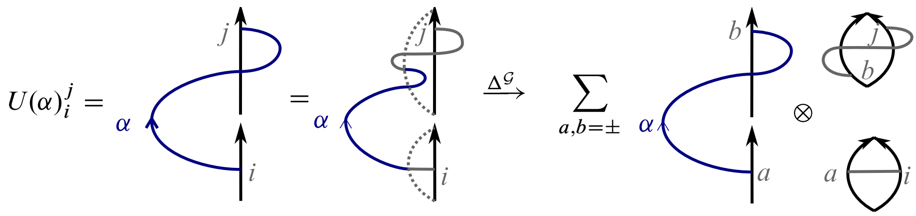


Figure 18: An illustration of (36).

understood that we need to apply a boundary skein relation in those neighborhoods. This convention permits us to draw the matrix coefficients $(C^{-1}M(\alpha))_i^j$. Note also that in Figure 18 we drop the scalar factor $(-1)^{w(\alpha)}$. □

4.2 Relative character varieties

Since the quantum moduli algebras are deformation quantizations of the (relative) character varieties studied by Fock and Rosly in [29], we briefly recall their construction and refer to [6] for a detailed survey.

Let $\mathbf{\Sigma}$ be a punctured surface and $\mathbb{V} \subset \Sigma_{\mathcal{P}}$ be a finite subset which intersects each boundary arc exactly once and each connected component of Σ at least once. Denote by $\mathring{V} := \mathbb{V} \cap \mathring{\Sigma}_{\mathcal{P}}$ its (possibly empty) subset of inner points and let $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ be the full subcategory of $\Pi_1(\Sigma_{\mathcal{P}})$ generated by $\mathbb{V}$. The representation space $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})$ is the set of functors $\rho\colon \Pi_1(\Sigma, \mathbb{V}) \to \mathrm{SL}_2(\mathbb{C})$. The discrete gauge group is $\mathcal{G}_{\mathbb{V}} := \mathrm{SL}_2(\mathbb{C})^{\mathring{V}}$ and it acts on $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})$ by

$$(\rho\cdot g)(\alpha) := g(t(\alpha))^{-1}\rho(\alpha)g(s(\alpha)) \quad \text{for } \rho \in \mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V}),\ g \in \mathcal{G}_{\mathbb{V}},\ \alpha \in \Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V}).$$

We claim that $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})$ can be given the structure of affine variety in such a way that the action of the reducible algebraic group $\mathcal{G}_{\mathbb{V}}$ is algebraic, so we can define the GIT quotient

$$\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}) := \mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})/\!\!/\mathcal{G}_{\mathbb{V}},$$

which we call the *relative character variety*. To prove this, consider a finite presentation $\mathbb{P} = (\mathbb{G}, \mathbb{RL})$ of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ and write $\mathbb{G} = (\alpha_1, \dots, \alpha_n)$ and $\mathbb{RL} = (R_1, \dots, R_m)$. Consider the regular map $\mathcal{R}\colon \mathrm{SL}_2(\mathbb{C})^{\mathbb{G}} \to \mathrm{SL}_2(\mathbb{C})^{\mathbb{RL}}$ written $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_m)$, where the coordinate $\mathcal{R}_i$ associated to a relation $R_i = \alpha_{i_1}^{\varepsilon_1} \star \cdots \star \alpha_{i_k}^{\varepsilon_k}$ is the polynomial function

$$\mathcal{R}_i(g_1, \dots, g_n) = g_{i_1}^{\varepsilon_1} \cdots g_{i_k}^{\varepsilon_k}.$$

Clearly one has $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V}) = \mathcal{R}^{-1}(\mathbb{1}_2, \dots, \mathbb{1}_2)$, where $\mathbb{1}_2$ is the identity matrix, so $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})$ is a subvariety of $\mathrm{SL}_2(\mathbb{C})^{\mathbb{G}}$.

Note that the algebra $\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})]$ of regular functions lies in the exact sequence

$$(37)\qquad \mathbb{C}[\mathrm{SL}_2(\mathbb{C})]^{\otimes \mathbb{RL}} \xrightarrow{\mathcal{R}^* - \eta^{\otimes \mathbb{G}} \circ \epsilon^{\otimes \mathbb{RL}}} \mathbb{C}[\mathrm{SL}_2(\mathbb{C})]^{\otimes \mathbb{G}} \to \mathbb{C}[\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})] \to 0,$$

where η and ϵ are the unit and counit of $\mathbb{C}[\mathrm{SL}_2]$. So we have turned $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})$ into an affine variety. Now the discrete gauge group action is induced by the Hopf comodule map $\Delta^{\mathcal{G}} \colon \mathbb{C}[\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})] \to \mathbb{C}[\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}, \mathbb{V})] \otimes \mathbb{C}[\mathcal{G}_{\mathbb{V}}]$, which is the restriction of the right comodule map $\widetilde{\Delta}_{\mathcal{G}} \colon \mathbb{C}[\mathrm{SL}_2(\mathbb{C})]^{\otimes \mathbb{G}} \to \mathbb{C}[\mathrm{SL}_2(\mathbb{C})]^{\otimes \mathbb{G}} \otimes \mathbb{C}[\mathrm{SL}_2(\mathbb{C})]^{\otimes \mathring{V}}$ defined by

$$\widetilde{\Delta}^{\mathcal{G}}(x^{(\alpha)}) = \sum x''^{(\alpha)} \otimes S(x''')^{(v_2)} x'^{(v_1)} \quad \text{for } x \in \mathcal{O}_q[\mathrm{SL}_2] \text{ and } \alpha \colon v_1 \to v_2 \in \mathbb{G},$$

using Sweedler's notation $\Delta^{(2)}(x) = \sum x' \otimes x'' \otimes x'''$. In particular, when $x = x_{ij}$ with $i, j \in \{-, +\}$, the formula gives

$$(38)\qquad \Delta^{\mathcal{G}}(x_{ij}^{(\alpha)}) = \sum_{a,b=\pm} x_{ab}^{(\alpha)} \otimes S(x_{bj})^{(v_2)} x_{ia}^{(v_1)}.$$

Note the analogy between (38) and (36).

Finally, the algebra of regular functions of the relative character variety is defined as the set of coinvariant vectors for this coaction, that is by the exact sequence

$$(39)\qquad 0 \to \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})] \to \mathbb{C}[\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma})] \xrightarrow{\Delta^{\mathcal{G}} - \mathrm{id} \otimes \epsilon} \mathbb{C}[\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma})] \otimes \mathbb{C}[\mathcal{G}_{\mathbb{V}}].$$

The relative character variety $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ does not depend (up to unique isomorphism) on the choice of the triple $(\mathbb{V}, \mathbb{G}, \mathbb{RL})$ used to define it, but only on $\mathbf{\Sigma}$; we refer to [36] for a proof. Note that in the particular case where $\mathbb{V} \subset \partial \Sigma_{\mathcal{P}}$, the gauge group is trivial so $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}) = \mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma})$. Moreover, if the presentation $\mathbb{P}$ does not have any relations, then $\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}) = \mathrm{SL}_2(\mathbb{C})^{\mathbb{G}}$. As we saw in Example 2.13, such a presentation $\mathbb{P}$ always exists when $\mathbf{\Sigma}$ is a connected punctured surface with nontrivial boundary, therefore in that case one has

$$\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}) = \mathrm{SL}_2(\mathbb{C})^{\mathbb{G}}.$$

Now consider an oriented ciliated graph (Γ, c) and consider the associated finite presentation $(\mathbb{V}, \mathbb{G}, \mathbb{RL})$ of the groupoid $\Pi_1(\Sigma^0_{\mathcal{P}}(\Gamma, c), \mathbb{V})$ associated to the open punctured surface defined in the previous subsection. The same triple $(\mathbb{V}, \mathbb{G}, \mathbb{RL})$ also gives a finite presentation of $\Pi_1(\Sigma_{\mathcal{P}}(\Gamma), \mathbb{V})$ associated to the *closed* punctured surface, where this time all elements of $\mathbb{V}$ are inner vertices of $\Sigma_{\mathcal{P}}(\Gamma)$. Therefore one has

$$\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}^0(\Gamma, c)) = \mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}(\Gamma)) = \mathrm{SL}_2(\mathbb{C})^{\mathcal{E}(\Gamma)},$$

where as before $\mathcal{E}(\Gamma)$ denotes the set of edges of Γ. So the exact sequence (39) can be rewritten as

$$0 \to \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\boldsymbol{\Sigma}(\Gamma))] \to \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\boldsymbol{\Sigma}^0(\Gamma,c))] \xrightarrow{\Delta^{\mathcal{G}}-\mathrm{id}\otimes\epsilon} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\boldsymbol{\Sigma}^0(\Gamma,c))] \otimes \mathbb{C}[\mathcal{G}_{\mathbb{V}}].$$

Note the analogy with the exact sequence (32). The main achievement of Fock and Rosly in [29] is the construction of Poisson structures on $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\boldsymbol{\Sigma}^0(\Gamma,c))] = \mathbb{C}[\mathrm{SL}_2]^{\otimes \mathcal{E}(\Gamma)}$ and $\mathbb{C}[\mathcal{G}_{\mathbb{V}}] = \mathbb{C}[\mathrm{SL}_2]^{\otimes \mathring{V}(\Gamma)}$ such that the coaction $\Delta^{\mathcal{G}}$ is a Poisson morphism. Therefore, using the above exact sequence, the affine variety $\mathcal{X}_{\mathrm{SL}_2}(\boldsymbol{\Sigma}(\Gamma))$ receives a (quotient) Poisson structure. A good plan then is to show that this Poisson structure only depends on the surface $\Sigma_{\mathcal{P}}(\Gamma)$ and not on (Γ, c). This strategy permitted the authors of [29] to extend the Atiyah–Bott–Goldman Poisson structure from unpunctured closed surfaces to closed general punctured surfaces (see also [36] for a general treatment in the language of punctured surfaces rather than ciliated graphs and using groupoid cohomology, and for a Goldman type formula for the Poisson bracket).

Let us conclude this subsection with the following observation. It is well known that the (stated) skein algebra $\mathcal{S}_{+1}(\boldsymbol{\Sigma})$ is isomorphic (though noncanonically) to the algebra $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\boldsymbol{\Sigma})]$ of regular functions of the (relative) character variety. For closed punctured surfaces this was shown by Bullock [17] under the assumption that $\mathcal{S}_{+1}(\boldsymbol{\Sigma})$ is reduced; this assumption was proved in [44] (see also [23] for an alternative proof). For open punctured surfaces this was proved independently in [39, Theorem 1.3] and [26, Theorem 8.12] using triangulations of surfaces. Let us note that Theorem 1.1 gives a straightforward alternative proof of this result with the additional assumption that $\mathcal{P} \neq \varnothing$.

Theorem 4.7 [17; 26; 39; 44] *The algebras $\mathcal{S}_{+1}(\boldsymbol{\Sigma})$ (where $\Bbbk = \mathbb{C}$) and $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\boldsymbol{\Sigma})]$ are isomorphic.*

Proof First suppose that $\boldsymbol{\Sigma}$ is an open connected punctured surface, let $\mathbb{V}$ be such that each of its vertices are on the boundary (so the representation and relative character varieties are the same), let $\mathbb{P} = (\mathbb{G}, \mathbb{RL})$ be a finite presentation of $\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V})$ whose generators are either of type a or d and fix a spin function w. By (37), the algebra $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\boldsymbol{\Sigma})]$ is presented by the generators $x_{ij}^{(\alpha)}$ for $\alpha \in \mathbb{G}$ and $i, j \in \{-, +\}$, with

- the exchange relations $x_{ij}^{(\alpha)} x_{kl}^{(\beta)} = x_{kl}^{(\beta)} x_{ij}^{(\alpha)}$ for all $\alpha, \beta \in \mathbb{G}$ and $i, j \in \{-, +\}$,
- the determinant relations $\det(X(\alpha)) = 1$ for all $\alpha \in \mathbb{G}$,
- the trivial loop relations $X(\beta_k) \cdots X(\beta_1) = \mathbb{1}_2$ for $R = \beta_k \star \cdots \star \beta_1 \in \mathbb{RL}$,

where we set

$$X(\alpha) := \begin{pmatrix} x^{(\alpha)}_{++} & x^{(\alpha)}_{+-} \\ x^{(\alpha)}_{-+} & x^{(\alpha)}_{--} \end{pmatrix}.$$

By comparing this presentation of $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ with the presentation of $\mathcal{S}_\omega(\mathbf{\Sigma})$ obtained in Proposition 4.6 by setting $\omega = +1$, we see that one has an isomorphism of algebras $\Theta\colon \mathcal{S}_{+1}(\mathbf{\Sigma}) \xrightarrow{\cong} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ sending $U(\alpha)$ to $X(\alpha)$; note that $\mathscr{R} = \tau$ when $\omega = +1$, so all arc exchange relations become $U(\alpha) \odot U(\beta) = \tau U(\alpha) \odot U(\beta)\tau$ giving relations $\alpha_{ij}\beta_{kl} = \beta_{kl}\alpha_{ij}$. Moreover, by comparing (38) and (36), we see that Θ is equivariant for the gauge group coactions.

Now suppose that $\mathbf{\Sigma}$ is closed and connected with $\mathcal{P} \neq \varnothing$, and let (Γ, c) be a ciliated fat graph such that $\mathbf{\Sigma}(\Gamma) = \mathbf{\Sigma}$. By the preceding case, one has an equivariant isomorphism $\Theta\colon \mathcal{S}_{+1}(\mathbf{\Sigma}^0(\Gamma, c)) \xrightarrow{\cong} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}^0(\Gamma, c))]$, so one has a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{S}_{+1}(\mathbf{\Sigma}(\Gamma))] & \longrightarrow & \mathcal{S}_{+1}(\mathbf{\Sigma}^0(\Gamma, c))] & \xrightarrow{\Delta^{\mathcal{G}} - \mathrm{id} \otimes \epsilon} & \mathcal{S}_{+1}(\mathbf{\Sigma}^0(\Gamma, c))] \otimes \mathbb{C}[\mathcal{G}_{\mathbb{V}}] \\
& \cong \Big\downarrow \exists! & & \cong \Big\downarrow \Theta & & \cong \Big\downarrow \Theta \otimes \mathrm{id} \\
0 \to & \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}(\Gamma))] & \to & \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}^0(\Gamma, c))] & \xrightarrow{\Delta^{\mathcal{G}} - \mathrm{id} \otimes \epsilon} & \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}^0(\Gamma, c))] \otimes \mathbb{C}[\mathcal{G}_{\mathbb{V}}]
\end{array}$$

Since both lines are exact there exists an isomorphism $\mathcal{S}_{+1}(\mathbf{\Sigma}(\Gamma)) \xrightarrow{\cong} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}(\Gamma))]$ obtained by restriction of Θ. □

4.3 Combinatorial quantizations of (relative) character varieties

The work of Fock and Rosly suggests a natural way of quantizing character varieties. The following problem was raised and solved independently by Alekseev, Grosse and Schomerus [2; 3] and Buffenoir and Roche [15] (see also [20] for a survey):

Problem 4.8 Associate to each oriented ciliated graph (Γ, c) an (associative unital) algebra $\mathcal{L}_\omega(\Gamma, c)$ over the ring $\Bbbk := \mathbb{C}[\omega^{\pm 1}]$ satisfying:

(A1) As a $\Bbbk$–module, $\mathcal{L}_\omega(\Gamma, c)$ is just the (free) module

$$\mathbb{C}[\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}^0(\Gamma, c))] \otimes_{\mathbb{C}} \Bbbk \cong \mathbb{C}[\mathrm{SL}_2]^{\otimes \mathcal{E}(\Gamma)} \otimes_{\mathbb{C}} \Bbbk.$$

(A2) As before, write $\mathcal{O}_q[\mathcal{G}] := \mathcal{O}_q[\mathrm{SL}_2]^{\otimes V(\Gamma)}$. The linear map

$$\Delta^{\mathcal{G}}\colon \mathcal{L}_\omega(\Gamma, c) \to \mathcal{L}_\omega(\Gamma, c) \otimes \mathcal{O}_q[\mathcal{G}]$$

defined by the formulas

$$\Delta^{\mathcal{G}}(x^{(\alpha)}_{ij}) = \sum_{a,b=\pm} x^{(\alpha)}_{ab} \otimes S(x_{bj})^{(v_2)} x^{(v_1)}_{ia}$$

is a Hopf-comodule map. In particular, it is a morphism of algebras.

(Inv) The subalgebra $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma) \subset \mathcal{L}_\omega(\Gamma, c)$ defined by the exact sequence

$$0 \to \mathcal{L}_\omega^{\mathrm{inv}}(\Gamma) \to \mathcal{L}_\omega(\Gamma, c) \xrightarrow{\Delta^{\mathcal{G}} - \mathrm{id} \otimes \epsilon} \mathcal{L}_\omega(\Gamma, c) \otimes \mathcal{O}_q[\mathcal{G}]$$

only depends (up to canonical isomorphism) on the (homeomorphism class of) surface $S(\Gamma)$.

(Q) Let $\mathbb{k}_\hbar := \mathbb{C}[\![\hbar]\!]$ and write $\omega_\hbar := \exp(-(i\pi)/(2\hbar)) \in \mathbb{k}_\hbar$ so that $\mu\colon \mathbb{k} \to \mathbb{k}_\hbar$ defined by $\mu(\omega) := \omega_\hbar$ is a ring morphism. Then the $\mathbb{k}_\hbar$ algebra $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma) \otimes_\mu \mathbb{k}_\hbar$ is a deformation quantization of the Poisson algebra $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}(\Gamma))]$ equipped with its Fock–Rosly Poisson structure.

Theorem 4.9 (Alekseev, Grosse and Schomerus [2; 3; 5], Buffenoir and Roche [15; 16]) *Problem 4.8 admits the solution $\mathcal{L}_\omega(\Gamma, c) := \mathcal{L}_\omega(\mathbf{\Sigma}^0(\Gamma, c))$, where the $\mathbb{k}$–module isomorphism $\mathcal{L}_\omega(\Gamma, c) \cong \mathbb{C}[\mathcal{R}_{\mathrm{SL}_2}(\mathbf{\Sigma}^0(\Gamma, c))] \otimes_{\mathbb{C}} \mathbb{k}$ is given by sending $U(\alpha)$ to $X(\alpha)$.*

The algebras $\mathcal{L}_\omega(\Gamma, c)$ are the so-called *quantum moduli algebras* and Theorem 1.3 is an obvious consequence of Theorem 1.1.

More precisely, the ciliated graphs considered in [15; 16] are those whose underlying graph is the 1–skeleton of some combinatorial triangulation of a Riemann surface. By combinatorial we mean that each edge has two distinct endpoints, so every arc is of type a and the only arc exchange relations among distinct arcs are in configurations (i) or (ii) (in the notation of Lemma 2.24). In [2; 3; 5] general ciliated graphs are considered, though in [3; 5] special attention is given to the quantum moduli algebras of the daisy graphs defined in Example 2.13 (they are called *standard graphs* in [3; 5]) and are further studied and related to stated skein algebras in [27]. In those daisy graphs, the arcs are of type d and the more complicated arc exchange relations in configurations (viii), (ix) and (x) appear under the name braid relations; see [3, Definition 12].

Note that, except for the study of the Poisson structure (which could have been easily done), we reproved Theorem 4.9. In [43], Meusburger and Wise proved that the solution of Problem 4.8 is unique, provided that we add some natural axioms for the operation of gluing graphs together. Actually the authors of [43] consider quantum moduli algebras associated to finite-dimensional ribbon algebras, whereas here we consider the infinite-dimensional one $U_q\mathrm{sl}_2$, but their proof extends word-for-word to our context.

4.4 Comparison with previous works

Let $\mathbf{\Sigma}^0$ be a connected punctured surface with one boundary component, one puncture on its boundary and possibly some inner punctures. Let (Γ, c) be its daisy graph and $\mathbb{P} = (\mathbb{G}, \varnothing)$ be the associated finite presentation as defined in Example 2.13 (so $\boldsymbol{\Sigma}^0 = \boldsymbol{\Sigma}^0(\Gamma, c)$). In this case, since the presentation has no relations, one can consider the spin function w sending every generator to $0 \in \mathbb{Z}/2\mathbb{Z}$. Since every generator $\alpha \in \mathbb{G}$ is of type d, the isomorphism $\Psi\colon \mathcal{S}_\omega(\boldsymbol{\Sigma}^0) \xrightarrow{\cong} \mathcal{L}_\omega(\Gamma, c)$ sends $U(\alpha) = C^{-1} M(\alpha)$ to $X(\alpha)$. By precomposing with the reflection anti-involution θ, one obtains an isomorphism

$$\Psi'\colon \mathcal{S}_{\omega^{-1}}(\boldsymbol{\Sigma}^0)^{op} \xrightarrow{\cong} \mathcal{L}_\omega(\Gamma, c),$$

which corresponds to Faitg's isomorphism in [27]. Let us stress that our notation is quite different from that in [27]; in particular:

- The letter q in [27] is what we denoted by A (so our q corresponds to q^2 in [27]).
- The letter $\mathcal{R}$ in [27] is related to our $\mathscr{R}$ by $\mathscr{R} = \tau \circ \mathcal{R}$.
- Faitg actually considered $\mathcal{S}_{\omega^{-1}}(\boldsymbol{\Sigma}^0)^{op}$, the opposite of the stated skein algebra.

As Faitg, Jordan and Safronov kindly explained to the author, the existence of an isomorphism $\Psi\colon \mathcal{S}_\omega(\boldsymbol{\Sigma}^0) \xrightarrow{\cong} \mathcal{L}_\omega(\Gamma, c)$ could have been derived from [9; 31] as we now briefly explain using the notation in [31] to which we refer for further details. Set $\Bbbk = \mathbb{C}[\omega^{\pm 1}]$ and fix a structure of a Riemann surface Σ. To any $\Bbbk$–ribbon category $\mathcal{A}$, one can associate a skein category $\mathrm{SkCat}_{\mathcal{A}}(\Sigma)$ whose objects are oriented embeddings of finitely many disjoint discs $\mathbb{D} \to \Sigma$ colored by objects in $\mathcal{A}$ and whose morphisms are framed $\mathcal{A}$–colored ribbon graphs in $\Sigma \times [0, 1]$ considered up to skein relations; see [24, Section 4.2] for a precise definition. We denote by $\mathbb{1} \in \mathrm{SkCat}_{\mathcal{A}}(\Sigma^0)$ the empty set. Let Σ^0 be obtained from a connected closed oriented surface Σ by removing an open disc. Fixing an arbitrary disc embedding $\mathbb{D} \to \Sigma^0$ gives a functor $\mathcal{P}\colon \mathcal{A} \to \mathrm{SkCat}_{\mathcal{A}}(\Sigma^0)$ in an obvious way. Let $\widehat{\mathcal{A}} := \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Vect})$ be the free cocompletion of $\mathcal{A}$ (which inherits a monoidal structure from $\mathcal{A}$). The *internal skein algebra* is defined as the coend

$$\mathrm{SkCat}^{\mathrm{int}}_{\mathcal{A}}(\Sigma^0) := \int^{x \in \mathcal{A}} \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^0)}(\mathcal{P}(x), \mathbb{1}) \otimes x \in \widehat{\mathcal{A}}.$$

The functor $\mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^0)}(\mathcal{P}(\cdot), \mathbb{1})\colon \mathcal{A}^{op} \to \mathrm{Vect}$ has a natural lax monoidal structure, given by stacking ribbon graphs on top of each other, which endows $\mathrm{SkCat}^{\mathrm{int}}_{\mathcal{A}}(\Sigma^0)$ with the structure of an algebra object in $\widehat{\mathcal{A}}$. If $\mathcal{A}$ is Tannakian, that is if it is equipped with a fully faithful monoidal functor $\mathrm{for}\colon \mathcal{A} \to \mathrm{Vect}$, then

$$\mathcal{S}_{\mathcal{A}}(\Sigma^0) := \mathrm{for}(\mathrm{SkCat}^{\mathrm{int}}_{\mathcal{A}}(\Sigma^0)) = \int^{x \in \mathcal{A}} \mathrm{Hom}_{\mathrm{SkCat}_{\mathcal{A}}(\Sigma^0)}(\mathcal{P}(x), \mathbb{1}) \otimes \mathrm{for}(x) \in \mathrm{Vect}$$

is a unital associative algebra that we might call the *stated skein algebra* associated to $\mathcal{A}$ and Σ^0. Let us consider two Tannakian ribbon categories: the (Cauchy closure of the) Temperley–Lieb category TL and the category of finite-dimensional $U_q\mathrm{sl}_2$ left modules $\mathrm{Rep}_q^{\mathrm{fd}}(\mathrm{SL}_2)$ (recall that q is generic here). The Tannakian structure $\mathrm{forget}\colon \mathrm{Rep}_q^{\mathrm{fd}}(\mathrm{SL}_2) \to \mathrm{Vect}$ is just the forgetful functor. It is well known that one has a monoidal braided equivalence of categories (which does not preserve the pivotal structure) $G\colon \mathrm{TL} \to \mathrm{Rep}_q^{\mathrm{fd}}(\mathrm{SL}_2)$ sending the one strand ribbon $[1] \in \mathrm{TL}$ to the fundamental representations V of Section 2.1 with basis $\{v_+, v_-\}$, thus we get a Tannakian structure $\mathrm{forget} \circ G\colon \mathrm{TL} \to \mathrm{Vect}$.

On the one hand, there is a natural algebra morphism

$$\Psi_1\colon \mathcal{S}_\omega(\mathbf{\Sigma}^0) \to \mathcal{S}_{TL}(\Sigma^0)$$

sending the class $[T,s]$ of a stated tangle, where ∂T has n elements, to the class of $T \otimes v_s \in \mathrm{Hom}_{\mathrm{SkCat}_{\mathrm{TL}}(\Sigma^0)}(\mathcal{P}([1]^{\otimes n}), \mathbb{1}) \otimes V^{\otimes n}$, where $v_s \in V^{\otimes n}$ is obtained from the state s by identifying the signs $+$ and $-$ with the basis vectors v_+ and v_- of V. As noted in [31, Remark 2.21] and fully explored in [32], a detailed comparison of the definitions shows that Ψ_1 is an isomorphism.

On the other hand, thanks to Cooke's excision theorem in [24] and as proved in [31, Proposition 2.19], the internal skein algebra $\mathrm{SkCat}_{\mathcal{A}}^{\mathrm{int}}(\Sigma^0)$ is isomorphic to the so-called moduli algebra $\mathcal{A}_{\Sigma^0} = \underline{\mathrm{End}}(\mathbb{1}) \in \widehat{A}$ introduced in [9, Definition 5.3]. The authors of [9, Theorem 5.14] defined an explicit isomorphism $[\mathrm{Rep}_q^{\mathrm{fd}}(\mathrm{SL}_2)]_{\Sigma^0} \cong \mathcal{L}_\omega(\Gamma)$, so by composing the two isomorphisms, one get an isomorphism

$$\Psi_2\colon \mathcal{S}_{TL}(\Sigma^0) \xrightarrow{\cong} \mathcal{L}_\omega(\Gamma).$$

Putting Ψ_1 and Ψ_2 together, we get an alternative construction of Faitg's isomorphism.

Remark 4.10 The above construction generalizes the notion of a stated skein algebra $\mathcal{S}_{\mathcal{C}}(\Sigma^0)$ to an arbitrary Tannakian ribbon category $\mathcal{C}$ (how to replace $\mathbf{\Sigma}^0$ with an arbitrary punctured surface is obvious), and [9, Theorem 5.14] seems to permit us to give explicit finite presentations for $\mathcal{S}_{\mathcal{C}}(\Sigma^0)$. A detailed study of these generalized stated skein algebras will appear in a separate publication [25].

5 Concluding remarks

We conclude the paper by making some remarks concerning the usefulness of relating stated skein algebras and quantum moduli spaces (Theorem 1.3). We can see the stated skein algebras as defined by a huge set of generators (all stated tangles) and a huge set of relations (isotopy and skein relations) whereas the quantum moduli algebra is defined by a finite subset of generators and by a finite subset of relations. Both presentations have their own advantages.

(i) The fact that the quantum moduli algebra $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$ only depends, up to canonical isomorphism, on the thickened surface $S(\Gamma)$ (or equivalently $\mathbf{\Sigma}(\Gamma)$) is usually proved by defining elementary moves on graphs that preserve the thickened surface and showing that those elementary moves induce isomorphisms on the algebras. This strategy was pioneered by Fock and Rosly in the classical case of relative character varieties [29] and later carried on in [3; 16] for quantum moduli algebras (see also [43] for very detailed study). Thanks to the isomorphism $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma) \cong \mathcal{S}_\omega(\mathbf{\Sigma}(\Gamma))$ (and the fact that stated skein algebras depend on surfaces rather than graphs), this fact is also an immediate consequence of Theorem 1.3. Also, the image of a closed curve γ through the reverse isomorphism $\Psi^{-1}\colon \mathbf{\Sigma}(\Gamma) \to \mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$ is usually called its *holonomy* $\mathrm{Hol}(\gamma)$ or *Wilson loop operators*, and the expression of this holonomy in terms of generators as well as the proof of some composition properties is the subject of long and technical computations in [2; 3; 15; 16; 28; 43], whereas they become easy in the skein algebra setting.

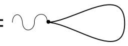
(ii) Since the quantum moduli algebra $\mathcal{L}_\omega(\Gamma, c)$ is quadratic homogeneous, we might have tried to prove that it is Koszul (proving that $\underline{\mathcal{B}}^{\mathbb{G}}$ is free) without the help of the stated skein algebra. The standard technique to prove that the family $\mathcal{B}$ of (26) is a PBW basis consists in examining the set of critical monomials of the form $v_i v_j v_k$ (we use the notation of Section 3.2) where both $v_i v_j$ and $v_j v_k$ are leading terms. To such a critical monomial we associate a finite graph (which might have the shape of a diamond) and the diamond lemma implies that if each of these graphs is confluent (has a terminal object) then $\mathcal{B}$ is a basis, so the quadratic algebra is Koszul; see [42, Section 4] for details. In our case, due to the huge amount of different kinds of relations in our presentation, this strategy would require us to verify the confluence of 6578 different graphs! This is way too much to be handled by hand. It is thanks to the fact that stated skein algebras have a lot of relations and generators that Lê was able to successfully use the diamond lemma in [40] to prove that $\mathcal{B}$ is basis, and our proof that $\underline{\mathcal{B}}^{\mathbb{G}}$ is a basis is directly derived from this fact. So proving the Koszulness of $\mathcal{L}_\omega(\Gamma, c)$ without the help of stated skein algebras could have been a very difficult problem.

(iii) Even if we could find PBW bases for the algebras $\mathcal{L}_\omega(\Gamma, c)$ without the help of skein algebras, finding bases for $\mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$ would be extremely difficult, since it is only defined as a kernel and no presentation is known. However, skein algebras $\mathcal{S}_\omega(\mathbf{\Sigma}(\Gamma)) \cong \mathcal{L}_\omega^{\mathrm{inv}}(\Gamma)$ have well-known bases (of multicurves).

(iv) As we saw in Section 4.2, the fact that $\mathcal{L}_{+1}(\mathbf{\Sigma}, \mathbb{P})$ is isomorphic to the algebra of regular functions of the (relative) character variety $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$ is very easy to prove, whereas relating the (stated) skein algebra $\mathcal{S}_{+1}(\mathbf{\Sigma})$ to $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ is not so obvious (see [17; 44] for closed surfaces and [26; 39] for open ones).

(v) In [13] Bonahon and Wong proved that the Kauffman-bracket skein algebra $\mathcal{S}_{+1}(\mathbf{\Sigma})$, with deforming parameter $+1$, embeds into the center of the skein algebra $\mathcal{S}_{\zeta}(\mathbf{\Sigma})$ with deforming parameter ζ a root of unity of odd order (see also [41] for an alternative proof). This result was generalized in [39] to stated skein algebras as well (see also [11] for generalizations). In [8], Baseilhac and Roche showed that the construction of this so-called Chebyshev–Frobenius morphism is much easier in the context of quantum moduli algebras (that is, using the finite presentations of Theorem 1.1). Even though their study only concerns genus 0 surfaces, their proofs seem to generalize easily to general surfaces, providing simpler proofs for the results in [13; 39].

(vi) Bullock, Frohman and Kania-Bartoszynska already proved in [19, Theorem 10] that $\mathcal{L}_{\omega}^{\mathrm{inv}}(\Gamma)$ and $\mathcal{S}_{\omega}(\mathbf{\Sigma}(\Gamma))$ are isomorphic when $\Bbbk = \mathbb{C}[\![\hbar]\!]$ and $\omega = -\exp\left(-\frac{1}{4}\hbar\right)$. Their proof consists of defining an algebra morphism $\Psi\colon \mathcal{L}_{\omega}^{\mathrm{inv}}(\Gamma) \to \mathcal{S}_{\omega}(\mathbf{\Sigma}(\Gamma))$ (by techniques similar to what we did in Section 2.2), and noting that under the $(\mathrm{mod}\,\hbar)$ identifications $\mathcal{L}_{\omega}^{\mathrm{inv}}(\Gamma)/(\hbar) \cong \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ and $\mathcal{S}_{\omega}(\mathbf{\Sigma}(\Gamma))/(\hbar) \cong \mathcal{S}_{-1}(\mathbf{\Sigma}(\Gamma))$, the morphism Ψ reduces modulo $\hbar$ to Bullock's isomorphism $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})] \cong \mathcal{S}_{-1}(\mathbf{\Sigma}(\Gamma))$. So the fact that the reduction of Ψ modulo $\hbar$ is an isomorphism implies that Ψ is an isomorphism. This proof does not seem (at least to the author) to generalize to prove the identification $\mathcal{L}_{\omega}^{\mathrm{inv}}(\Gamma) \cong \mathcal{S}_{\omega}(\mathbf{\Sigma}(\Gamma))$ for more general rings (such as $\Bbbk = \mathbb{C}$ and ω a root of unity), whereas our Theorem 1.3 works in full generality. A second reason why the approach in [19] does not work at roots of unity is described in (vii).

(vii) The following important remark was kindly explained to us by the anonymous referee, whom the author warmly thanks. In traditional papers in lattice gauge field theory (like [3; 8]) the algebras $\mathcal{L}_{\omega}(\Gamma, c)$ are seen as $U_q\mathrm{sl}_2^{\otimes n}$–modules instead of $\mathcal{O}_q[\mathrm{SL}_2]^{\otimes n}$–comodules (here n is the number of external vertices of Γ, ie the number of boundary arcs of $\mathbf{\Sigma}(\Gamma, c)$) and $\mathcal{L}_{\omega}^{\mathrm{inv}}(\Gamma)$ is then defined as the algebra of $U_q\mathrm{sl}_2^{\otimes n}$–invariant vectors instead of $\mathcal{O}_q[\mathrm{SL}_2]^{\otimes n}$–coinvariant vectors. When q is generic, there is a perfect pairing between the two Hopf algebras $U_q\mathrm{sl}_2$ and $\mathcal{O}_q[\mathrm{SL}_2]$ so that both definitions coincide. However, at roots of unity, the induced morphism $\mathcal{O}_q[\mathrm{SL}_2] \to U_q\mathrm{sl}_2^{\circ}$ is no longer injective nor surjective. As a consequence, the two definitions of $\mathcal{L}_{\omega}^{\mathrm{inv}}(\Gamma)$ do not coincide anymore and Theorem 1.3 only holds for the definition used in the present paper. For instance, consider the case where $(\Gamma, c) =$  so that $\boldsymbol{m}_1 := \boldsymbol{\Sigma}(\Gamma, c)$ is a once-punctured monogon, that is, a disc with one inner puncture and one boundary puncture. In this case $\mathcal{S}_\omega(\boldsymbol{m}_1) \cong \mathcal{L}_\omega(\Gamma, c)$ is Majid's braided quantum group; see [26; 8]. On the one hand, when $q := \omega^{-4}$ is a root of unity of odd order, Baseilhac and Roche have proved [8, page 41] that the subalgebra of $U_q\mathrm{sl}_2$–invariant vectors coincides with the center of $\mathcal{S}_\omega(\boldsymbol{m}_1)$ (denoted by $\mathcal{L}^{\varepsilon}_{0,1}$ in [8]). This center is generated by the peripheral curve γ_p encircling the inner puncture p together with the image of the Chebyshev–Frobenius morphism. On the other hand, the $\mathcal{O}_q[\mathrm{SL}_2]$–coinvariant vectors form the algebra $\mathbb{C}[\gamma_p]$ generated by the peripheral curve, isomorphic to the skein algebra of a punctured disc $\boldsymbol{\Sigma}^0(\Gamma, c)$ as expected. Therefore the subalgebra of $U_q\mathrm{sl}_2$–invariant vectors is bigger than the algebra of $\mathcal{O}_q[\mathrm{SL}_2]$–coinvariant vectors and Theorem 1.3 would fail with the original definition of $\mathcal{L}^{\mathrm{inv}}_\omega(\Gamma)$ at roots of unity.

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