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HYUNGRYUL BAIK
EIKO KIN
HYUNSHIK SHIN
CHENXI WU

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Asymptotic translation lengths and normal generation for pseudo-Anosov monodromies of fibered 3–manifolds

HYUNGRYUL BAIK
EIKO KIN
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Let M be a hyperbolic fibered 3–manifold. We study properties of sequences $(S_{\alpha_n}, \psi_{\alpha_n})$ of fibers and monodromies for primitive integral classes in the fibered cone of M. The main object is the asymptotic translation length $\ell_{\mathcal{C}}(\psi_{\alpha_n})$ of the pseudo-Anosov monodromy ψ_{α_n} on the curve complex. We first show that there exists a constant $C > 0$ depending only on the fibered cone such that for any primitive integral class (S, ψ) in the fibered cone, $\ell_{\mathcal{C}}(\psi)$ is bounded from above by $C/|\chi(S)|$. We also obtain a moral connection between $\ell_{\mathcal{C}}(\psi)$ and the normal generating property of ψ in the mapping class group on S. We show that for all but finitely many primitive integral classes (S, ψ) in an arbitrary 2–dimensional slice of the fibered cone, ψ normally generates the mapping class group on S. In the second half of the paper, we study if it is possible to obtain a continuous extension of normalized asymptotic translation lengths on the curve complex as a function on the fibered face. An analogous question for normalized entropy has been answered affirmatively by Fried and the question for normalized asymptotic translation length on the arc complex in the fully punctured case has been answered negatively by Strenner. We show that such an extension in the case of the curve complex does not exist in general by explicit computation for sequences in the fibered cone of the magic manifold.

30F60, 37E30; 32G15, 37B40

1 Introduction

Let M be a hyperbolic fibered 3–manifold. Thurston introduced the so-called Thurston norm on the first cohomology group of M, and showed that the unit norm ball is a finite sided polyhedron. Let F be a top-dimensional face of this polyhedron and consider a primitive integral class contained in the open cone $\mathcal{C} = \mathcal{C}_F$ over F. Thurston showed

that if this cohomology class corresponds to a fibration of M over the circle S^1, then all primitive integral classes in $\mathscr{C}$ correspond to fibrations of M over S^1. In such a case, we call F a *fibered face* and the open cone $\mathscr{C}$ a *fibered cone*.

For each primitive integral class $\alpha \in \mathscr{C}$, let (S_α, ψ_α) be the pair of corresponding fiber and its monodromy. Since M is hyperbolic, the monodromy ψ_α is pseudo-Anosov by Thurston's hyperbolization theorem; see, for example Farb and Margalit [8, Theorem 13.4]. We will study the asymptotic translation length of ψ_α on the curve complex of the surface S_α and the normal generators of mapping class groups $\mathrm{Mod}(S_\alpha)$.

Let G be a group acting isometrically on a metric space (X, d_X). For $h \in G$, the *asymptotic translation length* (or *stable length*) of h is defined by

$$\ell_X(h) = \liminf_{n\to\infty} \frac{d_X(x, h^n x)}{n},$$

where x is a point in X. It is not hard to see that $\ell_X(h)$ is independent of the choice of x.

For a surface S, let $\mathcal{T}(S)$ be the Teichmüller space of S and let $\mathcal{C}(S)$ be the curve complex of S. Since ψ_α acts by an isometry on both $\mathcal{T}(S_\alpha)$ and $\mathcal{C}(S_\alpha)$, one can consider the asymptotic translation lengths of ψ_α on $\mathcal{T}(S_\alpha)$ and on $\mathcal{C}(S_\alpha)$, denoted by $\ell_{\mathcal{T}}(\psi_\alpha)$ and $\ell_{\mathcal{C}}(\psi_\alpha)$ respectively.

There has been a lot of work on $\ell_{\mathcal{T}}(\psi_\alpha)$ for primitive integral classes α in the fibered cone; see Fathi, Laudenbach and Poénaru [9], Fried [10; 11], Long and Oertel [22], Matsumoto [26], and McMullen [27]. In the case of $\ell_{\mathcal{C}}(\psi_\alpha)$, there has also been some progress in the literature; see Aougab and Taylor [1], Bowditch [5], Farb, Leininger and Margalit [7], Gadre, Hironaka, Kent and Leininger [12], Gadre and Tsai [13], Masur and Minsky [24], Valdivia [34; 35], and the authors [2; 3; 16].

The following is a general upper bound of $\ell_{\mathcal{C}}(\psi_\alpha)$ in the fibered cone in terms of the Euler characteristic $\chi(S_\alpha)$ of S_α.

Theorem 1.1 [3] *Let F be a fibered face of a closed hyperbolic fibered 3–manifold M and K be a compact subset of* $\mathrm{int}(F)$, *the interior of F. Then there exists a constant C depending on K such that for any sequence $(S_{\alpha_n}, \psi_{\alpha_n})$ of primitive integral classes which is contained in the intersection between the cone over K and a $(d+1)$–dimensional rational subspace of $H^1(M)$,*

$$\ell_{\mathcal{C}}(\psi_{\alpha_n}) \leq \frac{C}{|\chi(S_{\alpha_n})|^{1+1/d}}.$$

Here a $(d+1)$–dimensional rational subspace of $H^1(M)$ means a subspace of $H^1(M)$ which admits a basis $v_1, \dots, v_{d+1} \in H^1(M;\mathbb{Q})$. We note that in [3] the above theorem was stated in the case of closed hyperbolic fibered 3–manifolds, but almost the same proof can be adapted to the case of compact hyperbolic fibered 3–manifolds possibly with boundary; see Remark 2.5.

Two additional questions naturally arise from Theorem 1.1. First, what can we say if the sequence is not contained in the cone over any compact subset of the fibered face F? For instance, given a sequence that has a subsequence converging projectively to the boundary ∂F, can we determine the upper bound of the asymptotic translation length of the pseudo-Anosov monodromies? We answer the first question in the following theorem.

Theorem 3.1 *Let F be a fibered face of a compact hyperbolic fibered 3–manifold, possibly with boundary. Then there exists a constant C depending on F such that for any primitive integral class $(S,\psi) \in \mathscr{C}_F$,*

$$\ell_C(\psi) \leq \frac{C}{|\chi(S)|}.$$

We make a couple of remarks regarding Theorem 3.1. We first note that a version of Theorem 3.1 was obtained by Schleimer in [30]. Even though he used different language, [30, Theorem 4.4] can be reinterpreted to give a statement of the form of Theorem 3.1 when the manifold is closed. We give an alternative argument which works for the nonclosed case as well.

Secondly we remark that the upper bound in Theorem 3.1 is optimal. In Lemma 4.12, we give an explicit sequence $(S_{\alpha_n}, \psi_{\alpha_n})$ converging projectively to a point in ∂F such that the asymptotic translation length of the corresponding pseudo-Anosov monodromy is comparable to $1/|\chi(S_{\alpha_n})|$. That is, there exists a constant C such that

$$\frac{1}{C}\frac{1}{|\chi(S_{\alpha_n})|} \leq \ell_C(\psi_{\alpha_n}) \leq \frac{C}{|\chi(S_{\alpha_n})|}.$$

In general, for real-valued functions $A(x)$ and $B(x)$, we say that $A(x)$ is *comparable* to $B(x)$ if there exists a constant C independent of x such that $1/C \leq A(x)/B(x) \leq C$, and we denote it by $A(x) \asymp B(x)$.

The second question is whether the upper bound in Theorem 1.1 is sharp. It is noted in [3] that the bound is optimal for $d = 1$. We show that it is also optimal when $d = 2$ by constructing an example coming from the magic manifold N, which is the exterior of some 3–component link in the 3–sphere S^3.

Theorem 4.13 *Let F be a fibered face of the magic manifold. Then there exist two points, $\mathfrak{b}_0 \in \partial F$ and $\mathfrak{c}_0 \in \mathrm{int}(F)$, which satisfy the following*:

(1) *For any $r \in \mathbb{Q} \cap [1,2)$, there exists a sequence $(S_{\alpha_n}, \psi_{\alpha_n})$ of primitive integral classes in $\mathscr{C}_F$ converging projectively to $\mathfrak{b}_0$ as $n \to \infty$ such that*

$$\ell_{\mathcal{C}}(\psi_{\alpha_n}) \asymp \frac{1}{|\chi(S_{\alpha_n})|^r}.$$

(2) *For any $r \in \mathbb{Q} \cap \left[\frac{3}{2}, 2\right]$, there exists a sequence $(S_{\alpha_n}, \psi_{\alpha_n})$ of primitive integral classes in $\mathscr{C}_F$ converging projectively to $\mathfrak{c}_0$ as $n \to \infty$ such that*

$$\ell_{\mathcal{C}}(\psi_{\alpha_n}) \asymp \frac{1}{|\chi(S_{\alpha_n})|^r}.$$

In particular, the upper bound in Theorem 1.1 is optimal when $d = 2$.

As an immediate corollary of Theorem 4.13, we conclude that there is no normalization of the asymptotic translation length function defined on the rational classes of the fibered face, which continuously extends to the whole fibered face. More precisely, we have the following.

Corollary 4.15 *Let F be a fibered face of the magic manifold N. For $\alpha \in F \cap H^1(N;\mathbb{Q})$, let $(S_{\widetilde{\alpha}}, \psi_{\widetilde{\alpha}})$ be the fiber and pseudo-Anosov monodromy corresponding to the primitive integral class $\widetilde{\alpha}$ lying on the ray of α passing through the origin. Then there is no normalization of the asymptotic translation length function*

$$F \cap H^1(N;\mathbb{Q}) \to \mathbb{R}_{\geq 0}, \quad \alpha \mapsto \ell_{\mathcal{C}}(\psi_{\widetilde{\alpha}}),$$

in terms of the Euler characteristic $\chi(S_{\widetilde{\alpha}})$ which admits a continuous extension on F.

For the arc complex, Strenner defined in [31] the normalized asymptotic translation length function μ_d for each integer $d \geq 1$ on the rational classes of a fibered face with the fully punctured condition. Strenner proved in the same paper that the functions μ_d for $d \geq 2$ are typically nowhere continuous. His result and Corollary 4.15 stand in contrast to Fried's result [10]. See also Matsumoto [26] and McMullen [27]. They proved that the normalized entropy function of pseudo-Anosov monodromies has a continuous extension on the fibered face, which is strictly convex.

Now we turn our attention to normal generation of mapping class groups. Let $S = S_{g,n}$ be an orientable surface of genus g with n punctures, possibly $n = 0$. We denote $S_{g,0}$ by S_g. We say that an element h of a group G *normally generates* G if the normal

closure of h is equal to G. For a given primitive class (S_α, ψ_α) in the fibered cone $\mathcal{C}$, when does ψ_α normally generate $\mathrm{Mod}(S_\alpha)$? Normal generation in the mapping class group has been studied by many authors. For instance, D Long [21] asked if there exist pseudo-Anosov normal generators. Later Ivanov asked in [14] what properties are satisfied by the pseudo-Anosov normal generators. A work of Lanier and Margalit [20] (partially) answered the questions of Long and Ivanov. In particular, they showed that for a pseudo-Anosov element $f \in \mathrm{Mod}(S_g)$, if the stretch factor $\lambda(f)$ is smaller than $\sqrt{2}$, then f normally generates $\mathrm{Mod}(S_g)$. The normal closure of random elements was studied as well, for instance by Maher and Tiozzo [23]. They showed that with asymptotic probability 1, the normal closure of a random element is free. This in particular implies that random elements are not normal generators.

This connects to our brief discussion of asymptotic translation length, since the logarithm of the stretch factor, $\log \lambda(f)$, is equal to $\ell_{\mathcal{T}(f)}$. In other words, if a pseudo-Anosov element of $\mathrm{Mod}(S)$ is contained in some proper normal subgroup, then its asymptotic translation length on the Teichmüller space cannot be too small. It is natural to ask an analogous statement for the curve complexes, ie if a pseudo-Anosov element of $\mathrm{Mod}(S)$ is contained in some proper normal subgroup, then its asymptotic translation length on the curve complex cannot be too small in some sense. The following question was raised by Dan Margalit (via personal communication).

Question 1.2 For a subgroup H of $\mathrm{Mod}(S_g)$, set

$$L_{\mathcal{C}}(H) = \min\{\ell_{\mathcal{C}}(f) \mid f \text{ is pseudo-Anosov and } f \in H\}.$$

Is there a constant $C > 0$ such that

$$L_{\mathcal{C}}(H) \geq \frac{C}{g}$$

for any $g \geq 2$ and for any proper normal subgroup H of $\mathrm{Mod}(S_g)$?

As a partial evidence toward this question, it is shown by Baik and Shin [2] that

$$L_{\mathcal{C}}(\mathcal{I}_g) \asymp \frac{1}{g},$$

where $\mathcal{I}_g$ is the Torelli group, ie the proper normal subgroup of $\mathrm{Mod}(S_g)$ whose action on the first homology is trivial. In fact, by [2, Theorem 3.2],

$$L_{\mathcal{C}}(\mathcal{I}_g) \geq \frac{1}{96(g-1)}$$

for all $g \geq 2$.

Combining with Theorem 3.1, we propose the following conjecture regarding the normal generators of mapping class groups contained in the fibered cone which was originally asked as a question by Dan Margalit (via personal communication).

Conjecture 1.3 Let F be a fibered face of a closed hyperbolic fibered 3–manifold M. Then for all but finitely many primitive classes $(S_\alpha, \psi_\alpha) \in \mathscr{C}_F$, ψ_α normally generates $\mathrm{Mod}(S_\alpha)$.

We give a partial answer when primitive integral classes are contained in a 2–dimensional rational subspace of $H^1(M)$. See also Remark 3.7.

Theorem 3.4 *Let F be a fibered face of a closed hyperbolic fibered 3–manifold M, and let L be a 2–dimensional rational subspace of $H^1(M)$. Then for all but finitely many primitive integral classes (S, ψ) in $\mathscr{C}_F \cap L$, ψ normally generates $\mathrm{Mod}(S)$. In particular, if the rank of $H^1(M)$ equals 2, then Conjecture 1.3 is true.*

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2 Arithmetic sequences in the fibered cone

For a hyperbolic 3–manifold M, possibly with boundary ∂M, Thurston [32] defined a norm $\|\cdot\|$ on $H_2(M, \partial M; \mathbb{R})$. It turns out the unit norm ball B_M with respect to the Thurston norm is a finite-sided polyhedron. Let F be a top-dimensional face of B_M. We consider an open cone $\mathscr{C} = \mathscr{C}_F$ over F. Thurston showed that if M is a fibered 3–manifold, then either all integral points in $\mathscr{C}$ are fibered or none of them are fibered. (When an integral cohomology class corresponds to a fibration of M over S^1, we say the integral point is fibered.) In the former case, we call $\mathscr{C}$ a *fibered cone*. We denote by $\overline{\mathscr{C}}$ the closure of the fibered cone $\mathscr{C}$.

By abuse of notation, the first cohomology classes are treated as their dual second homology classes throughout this paper without explicitly mentioning it. Furthermore, we will write a primitive integral class $\alpha \in H^1(M)$ as a pair (S, ψ) when S and ψ are the fiber and the monodromy for the fibration over S^1 corresponding to α.

In this section, we will show a key property of infinite arithmetic sequences in a fibered cone for the proof of Theorem 3.4. Here by an arithmetic sequence we mean a sequence $(\alpha + n\beta)_{n\in\mathbb{Z}_{\geq 0}}$ where α (resp. β) is a primitive integral class in a fibered cone $\mathcal{C}$ (resp. the closure $\overline{\mathcal{C}}$ of the fibered cone $\mathcal{C}$). We first need to find some criterion for a given element of the mapping class group to be a normal generator. In [20], the so-called *well-suited curve criterion* is introduced. Roughly speaking, this criterion says that if there is a simple closed curve c such that the configuration of $c \cup f(c)$ is simple enough, then f is a normal generator for the mapping class group.

Here we state one special case that we need and show its proof for the sake of completeness. For more general statements, see [20, Sections 2, 7 and 9]. For a closed curve c in the surface S_g without specified orientation, $[c]$ means the homology class in $H_1(S_g)$ with arbitrary orientation.

Lemma 2.1 [20, Lemma 2.3] *Let $f \in \mathrm{Mod}(S_g)$ for $g \geq 3$. Suppose that there is a nonseparating curve c in S_g such that c and $f(c)$ are disjoint and*

$$\pm[c] \neq [f(c)] \in H_1(S_g).$$

Then the normal closure of f is $\mathrm{Mod}(S_g)$.

Proof Let f and c be as in the statement of the lemma. Then one can find nonseparating curves a, b, d, x and y which satisfy the following conditions.

- a, b, c and d bound a subsurface S of S_g which is homeomorphic to a 4–punctured sphere.
- Each of the triple of curves (a, b, x), (b, d, y) and $(b, c, f(c))$ bounds a pair of pants contained in S.
- No two of the curves a, b, c, d, x, y and $f(c)$ are homologous.

To see the existence of such curves, start with Figure 1, left, which is the surface of genus 0 with four boundary components (ie, a 4–punctured sphere) labeled A, B, C and D. Glue a pair of pants along the boundary components labeled A and B, and glue another pair of pants along the boundary components labeled C and D. Then we get a

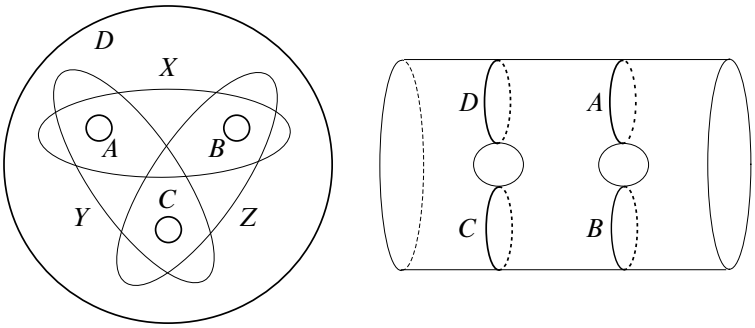

Figure 1: Left: a 4–punctured sphere. Right: a genus 2 surface with two boundary components.

surface of genus 2 with two boundary components (Figure 1, right). Along the two boundary components, we glue in another surface of genus $k \geq 0$ with two boundary components. The resulting surface is a closed surface of genus $3+k$. We take k so that $3+k=g$ which is the genus of our given surface S_g. This is our model surface, and we let Σ denote the model surface. If we set $a=A$, $b=B$, $c=C$, $d=D$, $x=X$, $y=Y$ and $f(c)=Z$, then the above conditions are satisfied by construction.

By the classification of the compact orientable surfaces, for any two pairs of disjoint nonhomologous simple closed curves on the surface, there exists a homeomorphism which maps one pair to the other. (This is a special case of the so-called change of coordinates principle. See for instance [8].) Hence, there exists a homeomorphism Φ from Σ to S_g such that $\Phi(C)=c$ and $\Phi(Z)=f(c)$. Now set $a=\Phi(A)$, $b=\Phi(B)$, $d=\Phi(D)$, $x=\Phi(X)$ and $y=\Phi(Y)$. Then we get the desired set of curves a, b, d, x and y which satisfy all the conditions together with c and $f(c)$.

For any curve γ on S_g, let T_γ be the left-handed Dehn twist about γ. Then by the lantern relation, we have $T_aT_bT_cT_d = T_{f(c)}T_xT_y$. Using the commutativity of the Dehn twists about disjoint curves, one can rewrite the lantern relation as

$$T_d = T_c^{-1}T_{f(c)}T_a^{-1}T_xT_b^{-1}T_y.$$

Note that $T_c^{-1}T_{f(c)} = T_c^{-1}(fT_cf^{-1}) = (T_c^{-1}fT_c)f^{-1}$ which is contained in the normal closure of f.

As before, by the change of coordinates principle, there exists an orientation-preserving homeomorphism h of S_g such that $h(c)=a$ and $h(f(c))=x$. Then

$$T_a^{-1}T_x = T_{h(c)}^{-1}T_{h(f(c))} = h^{-1}T_c^{-1}T_{f(c)}h,$$

ie it is just a conjugate of $T_c^{-1}T_{f(c)}$. Hence $T_a^{-1}T_x$ is in the normal closure of f. Similarly, $T_b^{-1}T_y$ is also contained in the normal closure of f.

This shows that T_d lies in the normal closure of f. From the fact that there exists only one mapping class group orbit of nonseparating simple closed curves and the Dehn twists about nonseparating simple closed curves generate the mapping class group, we can now conclude that the entire mapping class group $\mathrm{Mod}(S_g)$ is contained in the normal closure of f. □

Now we prove the key proposition on the sequences in the fibered cone.

Proposition 2.2 *Let $\mathscr{C}$ be a fibered cone for a closed hyperbolic fibered 3–manifold M. Let $\alpha \in \mathscr{C}$ and $\beta \in \overline{\mathscr{C}}$ be integral classes. Then there is some integer $n_0 > 0$ depending on α and β which satisfies the following. If $(S, \psi) = \alpha + n\beta \in \mathscr{C}$ is a primitive integral class for $n \geq n_0$, then there is an essential simple closed curve c on S such that $c, \psi(c), \dots, \psi^{n-1}(c)$ are disjoint, and $\pm[c] \neq [\psi(c)]$ in $H_1(S)$.*

Proof Let n be a positive integer such that $\alpha + n\beta$ is a primitive integral class. Let S_α and S_β be embedded surfaces in M which represent α and β respectively. Note that their orientations are assigned, and each connected component of those surfaces has genus at least 2, since M is a closed hyperbolic 3–manifold. In what follows, we explain how to choose these representatives more explicitly.

For any primitive integral class in $\mathscr{C}$, one obtains a suspension flow $\mathcal{F}$ of the monodromy. Fried showed that when M is a closed hyperbolic fibered 3–manifold, the flow $\mathcal{F}$ is an invariant of $\mathscr{C}$ in the following sense: if one considers the suspension flows from two primitive integral classes in $\mathscr{C}$, then they are the same flow up to reparametrization and conjugation by homeomorphisms on M. Moreover Fried showed that if an embedded surface S in M is a fiber for a primitive integral class in $\mathscr{C}$, then S can be made transverse to $\mathcal{F}$, and the first return map along the flow $\mathcal{F}$ represents the monodromy; see [11] and [9, Theorem 14.11 and Lemma 14.12].

Surely S_α can be made transverse to $\mathcal{F}$, since $\alpha \in \mathscr{C}$. If $\beta \in \mathscr{C}$, then the same holds for S_β. However if $\beta \in \partial\mathscr{C} = \overline{\mathscr{C}} \setminus \mathscr{C}$, then this may or may not be possible for representatives of β. The transverse surface theorems by Mosher [28] and Landry [19] including the case of compact hyperbolic 3–manifolds tells us that, for any integral class $\beta \in \overline{\mathscr{C}}$, there exists a flow $\widehat{\mathcal{F}}$, which is semiconjugate to $\mathcal{F}$, such that a representative S_β of β is transverse to $\widehat{\mathcal{F}}$. Here $\widehat{\mathcal{F}}$ is obtained from $\mathcal{F}$ by using the dynamic blowup of some

(possibly empty) singular periodic orbits of $\mathcal{F}$. The flow $\widehat{\mathcal{F}}$ is called a dynamic blowup of $\mathcal{F}$ for $\beta \in \overline{\mathscr{C}}$. (The dynamic blowups of $\mathcal{F}$ may not be unique.) For more details of the dynamic blowup of singular orbits, see [28, pages 8–9] and [19, Section 3.1].

We now explain some relevant properties of $\widehat{\mathcal{F}}$ which are needed in the proof of Proposition 2.2. The new flow $\widehat{\mathcal{F}}$ is obtained from $\mathcal{F}$ by replacing the singular orbits of $\mathcal{F}$ by a set of annuli such that flow lines in the interior of each annulus spiral toward boundary components of the annulus. Moreover $S_\alpha \cap \mathcal{A}$ is a union of embedded trees in S_α, where $\mathcal{A}$ is the collection of annuli created during the finitely many blowups of singular orbits. When $\beta \in \mathscr{C}$, it is shown in the transverse surface theorem that $\widehat{\mathcal{F}}$ is obtained by dynamically blowing up $\mathcal{F}$ along an empty collection of periodic orbits, and hence $\widehat{\mathcal{F}}$ is the same as $\mathcal{F}$. Now S_β is transverse to $\widehat{\mathcal{F}}$. From the construction of $\widehat{\mathcal{F}}$, we may suppose that S_α is still transverse to $\widehat{\mathcal{F}}$.

For any positive integer n, we can consider n parallel copies of S_β, say $S_1, \dots, S_n$ such that the S_i are very close to each other. Whenever we are in this situation, the n copies of S_i are labeled so that for $1 \le i < n$, S_i gets mapped to S_{i+1} by the flow $\widehat{\mathcal{F}}$ before touching any other S_j. Note that n is not fixed.

We now describe the surgery, ie cut and paste, on $S_\alpha, S_1, \dots, S_n$ along the intersection locus to get a surface S which represents $\alpha + n\beta$. Along each component of the intersection between S_α and each copy of S_β, we cut those surfaces. Locally there are four sheets of surfaces, two from S_α and two from the copy of S_β. Glue one sheet from S_α to one sheet from S_β so that the orientations on those sheets match up. One can do the same for the other two remaining sheets. The resulting surface S represents $\alpha + n\beta$. Clearly S is transverse to $\widehat{\mathcal{F}}$. We note that this is a standard operation. For instance, it is the same as the oriented sum in [6].

The transversality of S to $\widehat{\mathcal{F}}$ implies two things. First of all, this means S is transverse to $\mathcal{A}$. Since the original flow $\mathcal{F}$ is obtained from $\widehat{\mathcal{F}}$ by collapsing the annuli in $\mathcal{A}$ to singular orbits of $\mathcal{F}$, S is transverse to $\mathcal{F}$ after the collapsing. Second, the intersection $S \cap \mathcal{A}$ is a collection of trees on S by transversality together with the construction of $\mathcal{A}$ in the dynamic blowup. Now let $\widehat{\Psi}$ and Ψ be the first return maps on S for $\widehat{\mathcal{F}}$ and $\mathcal{F}$, respectively. Since $\widehat{\Psi}$ and Ψ differ only on the trees and each tree is contractible, $\widehat{\Psi}$ and Ψ are clearly homotopic to each other. Therefore $\widehat{\Psi}$ represents the monodromy $\psi = [\Psi]$ for $\alpha + n\beta$.

Note that because all S_i are parallel copies of S_β, any curve or region on S_β gives rise to a curve or region on each of the S_i that are parallel to it. Hence, in what follows,

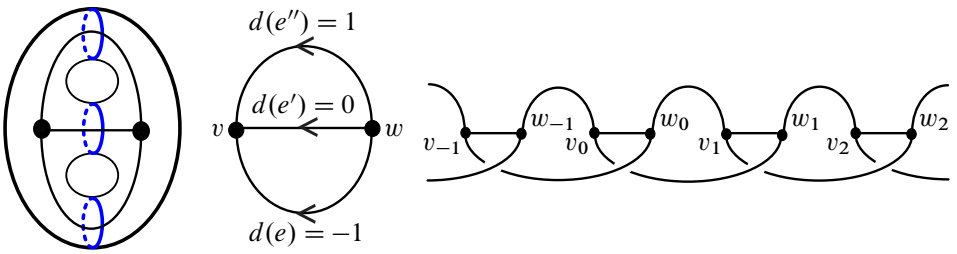

Figure 2: Left: a multicurve C together with its 3–regular graph G on $S_\beta \simeq$ closed surface of genus 2. Middle: an example of a cochain d on G: for three edges from w to v, their values are -1, 0 and 1 respectively. Right: a $\mathbb{Z}$–fold cover G' corresponding to d of the middle diagram.

whenever we specify any multicurve on S_β we implicitly specify multicurves on all of the S_i which are parallel to each other.

Let C be a multicurve on S_β such that all the connected components of $S_\beta \setminus C$ have genus 0 with three ends (Figure 2, left). Furthermore, we assume that every intersecting curve between S_α and S_β is parallel to one of the curves in C. Such a multicurve C always exists. To construct one, group the intersecting curves between S_α and S_β into parallel families, choose one in each parallel family and use them to form a multicurve C'. Now, if some connected component of $S_\beta \setminus C'$ has genus greater than 0, or has more than three ends, then we can add an extra curve to C' to break it into components of lower complexity, and repeat this process until all the connected components of $S_\beta \setminus C'$ have genus 0 with three ends.

Now we make use of the graph theoretic lemma below.

Lemma 2.3 *Let G be a 3–regular finite graph. Let d be an integer valued cellular cochain on G whose value on each edge is bounded above by $k \geq 0$, and let G' be the $\mathbb{Z}$–fold cover constructed from d (ie the vertices of G' are $\mathbb{Z}$–copies of the vertices of G and each edge e in G from w to v is lifted to edges from the j^{th} lift of w to the $(j+d(e))^{\text{th}}$ lift of v; see Figure 2, middle and right). Then there is some R depending only on k and the number of edges $|E(G)|$ of G such that G' has a simple loop γ' of length no more than $2R$.*

Proof Suppose there are no such loops of length less than $2R$ in G' for any R. Then the R–neighborhood (ie neighborhood with radius R assigning each edge length 1) of any vertex v_0 in G' must be a tree whose vertices have valence 1 or 3. Hence it contains $3 \times (2^R - 1)$ edges. However, such a neighborhood must contain at most

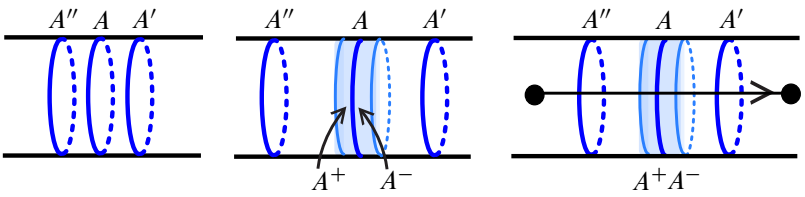

Figure 3: Left: parallel curves on S_β which are some components of the intersection between S_α and S_β. Middle: an annular neighborhood of A and the side of $A^\pm$. Right: for an edge e starting from the side of A^+, A contributes to $d(e)$ by $+1$.

$(2Rk+1)|E(G)|$ edges. (This is because in R steps, one can travel up at most Rk levels, ie Rk copies of the fundamental domain, or travel down at most Rk levels. Together with the original level, there are $(2Rk+1)$ levels in total that one might be able to pass through, and hence there are at most $(2Rk+1)|E(G)|$ edges in them.)

Since the exponential function grows faster than the linear one, one can set R sufficiently large to reach a contradiction. □

We continue the proof of Proposition 2.2. Note that the multicurve C above gives a pants decomposition of S_β. Let G be the 3–regular graph where each vertex corresponds to a pair of pants in the pants decomposition of S_β, and each edge corresponds to the component of the multicurve between two pairs of pants; see Figure 2, left. Now we define the cochain d on G which only depends on S_α and S_β as follows; see Figure 3.

Consider the surface S obtained from the cut and paste construction of S_α and n copies of S_β. If a curve A is one component of the intersection between S_α and S_β, we cut S_β along A (hence we cut each copy of S_β along a curve corresponding to A) which results in two boundary curves for each copy of S_β, say A^+ and A^-. The labeling A^+ and A^- are determined as follows: in the surface obtained from S_α and the copies of S_β via the cut and paste construction, an annular piece of S_α connecting the i^{th} copy of S_β to the $(i+1)^{\text{th}}$ copy of S_β is attached to the i^{th} copy of S_β along A^+ (the index of each copy of S_β is understood as an integer modulo n). We label the other boundary component A^-.

Now the labeling on each copy of S_β is well defined, and if one considers an annular neighborhood of A, then one can make sense of the statement that one side is the side of A^+ and the other side is the side of A^-.

Let us consider an edge e on G which intersects the curve A. If e is with the orientation so that it passes from the side of A^+ to that of A^-, then A contributes to $d(e)$ by $+1$,

and A contributes to $d(e^{-1})$ by -1, where e^{-1} is the same edge as e with the opposite orientation. The number $d(e)$ is obtained by summing up all the contributions of curves in $S_\alpha \cap S_\beta$ that the edge e passes through. Note that the cochain d does not depend on n but only on S_α and S_β, since we consider copies of S_β very close to each other, the intersection with S_α looks exactly the same in any copy of S_β.

Let k be the maximum of the values of d on all edges on G, and let R be the constant from Lemma 2.3. Now let n be any integer so that $n \geq 2Rk+2$, and consider the surface S obtained from S_α and n copies of S_β by a cut and paste construction. (In other words, here we will argue that the integer n_0 in Proposition 2.2 can be chosen as $2Rk+2$.) Let γ' be a simple loop in G' in Lemma 2.3. The fact that $|d(e)| \leq k$ implies that γ' passes through at most $2Rk+1$ consecutive fundamental domains of the deck group action on G'. The embedding of these $2Rk+1$ fundamental domains, together with one more, to $2Rk+2$ copies of S_β after the surgery, sends γ' to some simple loop γ on the surface S. (To do that, pick a point in each pant in S_i. Now pick a starting vertex v_0 on γ', and let γ start at the point associated to the corresponding pant in S_{Rk+2}. Now, we travel along γ', and for each edge, connect the points in the two pants associated with the two end points of the edge. The construction of G' and Lemma 2.3 imply that the resulting path γ will also be closed.)

Let $c \in C$ be a component of the multicurve on S_β and let c_i be the corresponding copies of c on the i^{th} copy S_i of S_β. Suppose that c is chosen such that c_l is crossed by γ once for some l, and that γ does not cross the lowest copy S_1; see Figure 4. One can choose such a c since the length of γ' is no more than $2R$. Note that all c_i survive under surgery because they do not cross the intersections between S_i and S_α. Furthermore, except for the top c_n, their images under the first return map are $\psi(c_i) = c_{i+1}$. By the construction of S, it follows that $c_1, \psi(c_1) = c_2, \ldots, \psi^{n-1}(c_1) = c_n$ are disjoint. For the proof of Proposition 2.2, we only need to show that $[c_2] \pm [c_1]$ is not homologous to 0. (This also implies that c_1 on S is essential.) To do so, one only needs to show that

$$(\psi_*^{l-2} + \psi_*^{l-3} + \cdots + \mathrm{id}_*)([c_2] - [c_1]) = [c_l] - [c_1]$$

and

$$(\psi_*^{l-2} - \psi_*^{l-3} + \cdots + (-1)^{l-2}\mathrm{id}_*)([c_2] + [c_1]) = [c_l] + (-1)^{l-2}[c_1]$$

are not 0. Since γ passes through c_l and it does not pass through c_1, the simple closed curves c_l and c_1 do not bound a subsurface. Therefore $[c_l] \neq \pm[c_1]$. This completes the proof of Proposition 2.2. □

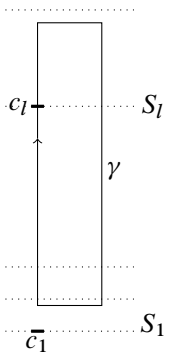

Figure 4: The horizontal line segments (with dots) represent the copies $S_1, S_2, \dots$ of S_β, and the curve with arrow represents the loop γ which passes through S_l but not the lowest copy S_1 of S_β.

We now consider a compact hyperbolic fibered 3–manifold M. In order to obtain an estimate for the asymptotic translation length of monodromies from the arithmetic sequences in the fibered cone for M, we show the following variant of Proposition 2.2.

Proposition 2.4 *Let $\mathscr{C}$ be a fibered cone for a compact hyperbolic fibered 3–manifold M possibly with boundary, let $\alpha \in \mathscr{C}$ and $\beta \in \overline{\mathscr{C}}$ be integral classes, and suppose $(S, \psi) = \alpha + n\beta \in \mathscr{C}$ is a primitive integral class for an integer $n \geq 2$. Then there is an essential simple closed curve c on S or essential arc on S such that $c, \psi(c), \dots, \psi^{n-1}(c)$ are disjoint. In particular,*

$$\ell_C(\psi) \leq \frac{2}{n-1}.$$

Proof Let $\mathcal{F}$ be the suspension flow for the fibered cone $\mathscr{C}$. In [19, Appendix A], Landry generalized Fried's theory on the fibered cone (for closed hyperbolic fibered 3–manifolds) to the case of compact hyperbolic fibered 3–manifolds M possibly with boundary. In particular $\mathcal{F}$ is an invariant of $\mathscr{C}$ as well. Then we use the transverse surface theorem [19; 28] for compact hyperbolic fibered 3–manifolds M again. Let $\widehat{\mathcal{F}}$ be a dynamic blowup of $\mathcal{F}$ for $\beta \in \overline{\mathscr{C}}$. We can take representatives S_α and S_β of α and β respectively so that S_α and S_β are transverse and they intersect the new flow $\widehat{\mathcal{F}}$ transversely. We may assume that S_α and S_β intersect minimally, ie the number of components of the intersection between S_α and S_β is minimal among all representatives of α and β. The surface obtained from S_α and S_β by a cut and paste construction is a fiber of the fibration associated with $\alpha + \beta \in \mathscr{C}$. This implies that S_α and S_β are minimal representatives of α and β.

Do surgery (as in the proof of Proposition 2.2) at the intersection locus of S_α and n copies of S_β to obtain a surface S representing $\alpha + n\beta$. We now find the desired essential simple closed curve on S or an essential arc c on S. Let c be one of the intersection curves or arcs between S_α and S_β, and let S_1 be the lowest copy of S_β. The fact that c is essential on S_α and on S_β follows from the fact that the intersection between S_α and S_β is minimal; see [32] or [6, Lemma 5.8]. It is not hard to see from the cut and paste construction that c is also essential on S.

From the choice of c, it follows that c and $\psi^{n-1}(c)$ are disjoint. They are distinct in the arc and curve complex $\mathcal{AC}(S)$, since ψ is pseudo-Anosov. Thus the distance between c and $\psi^{n-1}(c)$ in $\mathcal{AC}(S)$ equals 1. This implies that $(n-1)\ell_{\mathcal{AC}}(\psi) = \ell_{\mathcal{AC}}(\psi^{n-1}) \leq 1$ — cf [16, Lemma 2.1] — where $\ell_{\mathcal{AC}}(\psi)$ is the asymptotic translation length of ψ on $\mathcal{AC}(S)$. It is known that the inclusion map $\mathcal{C}(S) \to \mathcal{AC}(S)$ is 2–bilipschitz; see, for instance, [25, Lemma 2.2] or [18]. In particular, this tells us that

$$\ell_{\mathcal{C}}(\psi) \leq 2\ell_{\mathcal{AC}}(\psi).$$

Thus we have $\ell_{\mathcal{C}}(\psi) \leq 2\ell_{\mathcal{AC}}(\psi) \leq 2/(n-1)$. □

Remark 2.5 In [3], Theorem 1.1 was proved in the case of closed hyperbolic fibered 3–manifolds. We note that almost the same proof can be adapted to the case of compact hyperbolic fibered 3–manifold. In fact, one only needs to modify the last paragraph (after Lemma 8) in the proof of [3, Theorem 5] to allow γ and γ' to be either an essential simple closed curve or an essential simple arc. Then one obtains the same conclusion of Theorem 1.1 by the fact that inclusion map $\mathcal{C}(S) \to \mathcal{AC}(S)$ is 2–bilipschitz as in the proof of Proposition 2.4 in this paper.

3 Applications of arithmetic sequences

3.1 Asymptotic translation lengths in fibered cones

In this section, we show the following estimate for the asymptotic translation lengths in the curve complexes.

Theorem 3.1 *Let F be a fibered face of a compact hyperbolic fibered 3–manifold possibly with boundary. Then there exists a constant C depending on F such that for any primitive integral class $(S, \psi) \in \mathcal{C}_F$,*

$$\ell_{\mathcal{C}}(\psi) \leq \frac{C}{|\chi(S)|}.$$

To prove this theorem, we need the following lemma about rational cones. Here a rational cone in Euclidean space $\mathbb{R}^m$ is the set of the points of the form

$$\{\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{R}^m \mid A\boldsymbol{x}^t \geq \mathbf{0}\}$$

for some $k \times m$ matrix A with integer entries (where $\boldsymbol{x}^t$ is the transpose of $\boldsymbol{x}$.) We further assume that this set has nonempty interior.

Lemma 3.2 *Let P be a rational cone in $\mathbb{R}^m$, and let $\mathrm{int}(P)$ be its interior. Then there exist two nonempty finite sets $\Omega_0 \subset \mathrm{int}(P) \cap \mathbb{Z}^m$ and $\Omega \subset P \cap \mathbb{Z}^m$ such that*

$$\mathrm{int}(P) \cap \mathbb{Z}^m = \left\{ a + \sum_{b \in \Omega} k_b b \;\middle|\; a \in \Omega_0, k_b \in \mathbb{Z}, k_b \geq 0 \right\}.$$

Proof It is a classical result — see [33, Proposition 3.4] — that $P \cap \mathbb{Z}^m$ is a finitely generated monoid. Let Ω be a finite set of generators of $P \cap \mathbb{Z}^m$, and let

$$\Omega_0 = \left\{ \sum_{b \in W} b \;\middle|\; W \subset \Omega, W \not\subset F \text{ for all faces } F \text{ of } \partial P \right\}.$$

Here a face of ∂P is a polytope of dimension $m-1$ which is the intersection of ∂P with a $(m-1)$–dimensional subspace of $\mathbb{R}^m$. Note that W can possibly contain only a single point in $\mathrm{int}(P)$. Clearly Ω_0 is a finite set with at most $2^{|\Omega|}$ elements.

Note that a linear combination of elements in Ω with nonnegative coefficients lie on a face of ∂P if and only if all the coefficients for those generators that are not on this face are 0. In other words, if $\sum_{b \in \Omega} k_b b$ is in $\mathrm{int}(P)$ and k_b are all nonnegative, then the set $\{b \in \Omega \mid k_b \geq 1\}$ must not be contained in any face of ∂P. Hence

$$\mathrm{int}(P) \cap \mathbb{Z}^m = \left\{ a + \sum_{b \in \Omega} k_b b \;\middle|\; a \in \Omega_0, k_b \in \mathbb{Z}, k_b \geq 0 \right\}$$

and in particular $\Omega_0 \subset \mathrm{int}(P) \cap \mathbb{Z}^m$ as we desire. □

Here is an example of the two finite sets Ω_0 and Ω for a rational cone in $\mathbb{R}^2$.

Example 3.3 Let us consider the following rational cone in $\mathbb{R}^2$.

$$P = \left\{ \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

One can take $\Omega = \{b_1 = (1,0), b_2 = (1,1), b_3 = (2,3)\}$ as a set of generators of $P \cap \mathbb{Z}^2$. There are two faces of ∂P. One is $\{(x,0) \mid x \geq 0\}$ which contains $\{b_1\}$ as a

subset, and the other is $\{(x, \frac{3}{2}x) \mid x \geq 0\}$ which contains $\{b_3\}$ as a subset. One sees that Ω_0 consists of five elements, b_2, $b_1 + b_2 = (2, 1)$, $b_1 + b_3 = (3, 3)$, $b_2 + b_3 = (3, 4)$ and $b_1 + b_2 + b_3 = (4, 4)$.

Proof of Theorem 3.1 For a fibered cone $\mathscr{C}$, the closure $\overline{\mathscr{C}}$ is a rational cone in $H^1(M)$, because the unit Thurston norm ball is a polytope whose vertices are rational points [32]. By Lemma 3.2, if an integral class δ is in $\mathscr{C}$, then it can always be written of the form $\delta = a + \sum_{b\in\Omega} k_b b$, where $a \in \Omega_0$ and k_b is a nonnegative integer. If S is a norm-minimizing surface of δ, then we have $\|\delta\| = |\chi(S)|$ and it is bounded above by

$$\max\Big(1, \max_{b\in\Omega}(k_b)\Big)\bigg(\|a\| + \sum_{b\in\Omega}\|b\|\bigg).$$

Hence, when $|\chi(S)| > \max_{a\in\Omega_0}\|a\| + \sum_{b\in\Omega}\|b\|$,

$$|\chi(S)| \leq \max_{b\in\Omega}(k_b)\bigg(\|a\| + \sum_{b\in\Omega}\|b\|\bigg).$$

Therefore,

$$\max_{b\in\Omega}(k_b) \geq \frac{|\chi(S)|}{\|a\| + \sum_{b\in\Omega}\|b\|} \geq \frac{|\chi(S)|}{\max_{a\in\Omega_0}\|a\| + \sum_{b\in\Omega}\|b\|}.$$

Let b_m be a b in Ω that maximizes k_b. We set $\alpha = a + \sum_{b\in\Omega, b\neq b_m} k_b b$, $\beta = b_m$ and $n = k_{b_m}$. We have $\alpha \in \mathscr{C}$ and $\beta \in \overline{\mathscr{C}}$. Then δ is given by $\delta = \alpha + n\beta$ with

$$n \geq \frac{|\chi(S)|}{\max_{a\in\Omega_0}\|a\| + \sum_{b\in\Omega}\|b\|}.$$

Note that the denominator in the right hand side only depends on the fibered cone. Now, when $\|\delta\| = |\chi(S)| > \max_{a\in\Omega_0}\|a\| + \sum_{b\in\Omega}\|b\|$, the conclusion of the theorem follows directly from Proposition 2.4. The remaining case $\|\delta\| \leq \max_{a\in\Omega_0}\|a\| + \sum_{b\in\Omega}\|b\|$ consisting of finitely many primitive integral classes δ; hence the theorem is proved. □

3.2 Normal generation in the fibered cone

In this section we prove the following theorem as a partial answer to Conjecture 1.3.

Theorem 3.4 *Let F be a fibered face of a closed hyperbolic fibered 3–manifold M, and let L be a 2–dimensional rational subspace of $H^1(M)$. Then for all but finitely many primitive integral classes (S, ψ) in $\mathscr{C}_F \cap L$, ψ normally generates* $\mathrm{Mod}(S)$. *In particular, if the rank of $H^1(M)$ equals 2, then Conjecture 1.3 is true.*

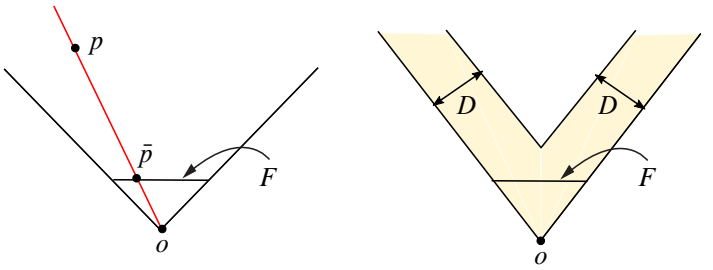

Figure 5: Left: the fibered face F in the fibered cone $\mathcal{C}$ (p and $\bar{p}$ lie on the same ray in $\mathcal{C}_F$ passing through the origin). Right: the subset $\mathcal{N}_D \subset \mathcal{C}$.

For the proof of Theorem 3.4, we first prove the following result.

Theorem 3.5 *Let $\mathcal{C}$ be a fibered cone of a closed hyperbolic fibered 3–manifold M. Then there exists some $x \in \mathcal{C}$ such that for each primitive integral class $(S, \psi) \in x + \mathcal{C}$, ψ normally generates* $\mathrm{Mod}(S)$, *where $x + \mathcal{C} = \{x + v \mid v \in \mathcal{C}\}$.*

Proof Let d be any Euclidean metric on $H^1(M)$. Let F be the fibered face corresponding to $\mathcal{C}$. For every point $p \in \mathcal{C}$, let $\bar{p}$ be the intersection of F with the ray starting from the origin and passing p (Figure 5, left). By [27, Corollary 5.4], we have a real analytic, strictly concave and degree 1 homogeneous function $y = 1/\log K(\cdot)$ defined on $\mathcal{C}$, such that the stretch factor $\lambda(p)$ for $p \in \mathcal{C}$ is equal to $K(p)$ and $y(p) = 1/\log K(p) \to 0$ as $p \to \partial F$. The concavity implies that there must be some $k > 0$ (independent of the choice of $\bar{p}$) such that

$$\frac{1}{\log(K(\bar{p}))} \geq k \cdot d(\bar{p}, \partial\mathcal{C}).$$

A way to see the existence of k is as follows. Concavity of y implies that there is some point $p_0 \in F$ where $y(p_0) > 0$. Then, for any point $\bar{p} \in F$, consider the line segment from p_0 to the boundary of F passing through $\bar{p}$. Then concavity of y means that on this line segment, y is bounded from below by the linear function L which takes value 0 at one end and $y(p_0)$ at another end. Hence it has a slope $s = s(\bar{p})$ that depends on $\bar{p}$ and $s = s(\bar{p})$ is continuous on $\bar{p}$. On the other hand, the function $d(\cdot, \partial\mathcal{C})$, restricted to this line segment, is piecewise linear, and hence it is also bounded from above by a linear function L' taking value 0 at the end on ∂F. We choose such linear function L' with the smallest slope $s' = s'(\bar{p})$. Then $s' = s'(\bar{p})$ is continuous on $\bar{p}$. Now k can be chosen as any number below the ratio s/s' between these two slopes. As both slopes depend continuously on $\bar{p}$, and F has compact closure, we can choose a universal k that works on the whole face F.

Furthermore, the degree 1 homogeneity implies that

$$\frac{1}{\log(K(p))} = \frac{d(0,p)}{d(0,\bar{p})} \cdot \frac{1}{\log(K(\bar{p}))}.$$

For $D > 0$, we consider the set $\mathcal{N}_D$ (Figure 5, right),

$$\mathcal{N}_D = \{p \in \mathcal{C} \mid d(p, \partial\mathcal{C}) \leq D\}.$$

From the above computation, the stretch factor for $p \in \mathcal{C} \setminus \mathcal{N}_D$ satisfies

$$\begin{aligned}\lambda(p) = e^{\log K(p)} &= (e^{\log K(\bar{p})})^{d(0,\bar{p})/d(0,p)} \\ &\leq (e^{1/(kd(\bar{p},\partial\mathcal{C}))})^{d(0,\bar{p})/d(0,p)} = e^{1/(kd(p,\partial\mathcal{C}))} \leq e^{1/(kD)}.\end{aligned}$$

Hence as long as D is sufficiently large, $\lambda(p)$ can be made to be as close to 1 as needed. In particular it is smaller than $\sqrt{2}$ when D is large enough. This together with [20, Theorem 1.2] shows that for some D, all primitive integral classes in $\mathcal{C} \setminus \mathcal{N}_D$ are normal generators. The theorem now follows by picking an arbitrary $x \in \mathcal{C} \setminus \mathcal{N}_D$, due to the fact that the boundary of $\mathcal{N}_D$ must be parallel to that of $\partial\mathcal{C}$ itself; see Figure 5, right. □

The next result follows immediately from Lemma 2.1 and Proposition 2.2.

Theorem 3.6 *Let $\mathcal{C}$ be a fibered cone of a closed hyperbolic fibered 3–manifold. Suppose that $(S_{\alpha_n}, \psi_{\alpha_n})$ is a sequence of primitive integral classes in $\mathcal{C}$ such that $\alpha_n = v + nw$, where $v \in \mathcal{C}$ and $w \in \overline{\mathcal{C}}$ are fixed integral classes. Then ψ_{α_n} normally generates* $\mathrm{Mod}(S_{\alpha_n})$ *for sufficiently large n.*

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4 Let L be a 2–dimensional rational subspace of $H^1(M)$ satisfying the assumption of Theorem 3.4. Theorem 3.5 says that there is some $x \in \mathcal{C}$ such that all primitive integral classes (S, ψ) in $x + \mathcal{C}$ normally generate $\mathrm{Mod}(S)$. In particular this holds for all primitive integral classes in $(x + \mathcal{C}) \cap L$. Because L is of dimension 2, the integral classes in $(\mathcal{C} \setminus (x + \mathcal{C})) \cap L$ are the union of finitely many sequences of the form $(v + nw)_{n \in \mathbb{N}}$, where $v \in \mathcal{C}$ and $w \in \overline{\mathcal{C}}$. Thus by Theorem 3.6, for all but finitely many primitive integral classes (S, ψ) in $(\mathcal{C} \setminus (x + \mathcal{C})) \cap L$, ψ normally generates $\mathrm{Mod}(S)$. □

Remark 3.7 Our approach to Theorem 3.4 does not work when the dimension of the rational subspace L of $H^1(M)$ is more than 2. This is because in this case, the intersection $(\mathcal{C} \setminus (x + \mathcal{C})) \cap L$ no longer consists of finitely many sequences of primitive integral classes of the form $v + nw$, where $v \in \mathcal{C}$ and $w \in \overline{\mathcal{C}}$.

4 Sequences in the fibered cone of the magic manifold

Let $\mathcal{C}_3$ be the 3–chain link in S^3 as in Figure 6, left. The magic manifold N is the exterior of $\mathcal{C}_3$ (hence ∂N consists of three boundary tori), and it is a hyperbolic and fibered 3–manifold. We give some background on invariant train tracks in Section 4.1 and we discuss the fibered cone of N in Section 4.2. We compute the upper and lower bounds of the asymptotic translation length of particular sequences in the fibered cone of N in Sections 4.3 and 4.4. Then we prove Theorem 4.13 in Section 4.5.

4.1 Invariant train tracks for pseudo-Anosov maps

For definitions and basic results on train tracks, see [4; 8; 29]. Let $\psi : S \to S$ be a pseudo-Anosov homeomorphism defined on a surface S possibly with boundary/punctures. When S is a punctured surface, we say that ψ is *fully punctured* if the set of singularities of the unstable foliation for ψ is contained in the set of punctures of S.

Let τ be an invariant train track for ψ. Then $\psi : S \to S$ induces a map on τ to itself which takes switches (vertices) to themselves. Such a map is called the *train track map*. By abuse of notation, we denote the train track map on τ also by $\psi : \tau \to \tau$. Following [4, Section 3.3], we say that a branch e of τ is *real* if there exists an integer $m \geq 1$ such that $\psi^m(e)$ passes through all branches of τ. Otherwise we say that e is *infinitesimal*. The train track map $\psi : \tau \to \tau$ induces a finite digraph Γ by taking a vertex for each real branch of τ, and then adding m_{ij} directed edges from the j^{th} real branch e_j to the i^{th} real branch e_i, where $m_{i,j}$ is the number of times that the image $\psi(e_j)$ under the train track map ψ passes through e_i in either direction.

For the lower bound of $\ell_{\mathcal{C}}(\psi)$, we recall a result of Gadre and Tsai. The following statement is a consequence of [13, Lemma 5.2] together with the proof of [13, Theorem 5.1].

Proposition 4.1 *Let $\psi \in \mathrm{Mod}(S_{g,n})$ be a pseudo-Anosov element and let τ be an invariant train track for ψ. Suppose that r is a positive integer such that for any real branch e of τ, $\psi^r(e)$ passes through every real branch. If we set $h = r + 24|\chi(S_{g,n})| - 8n$, then $\psi^h(e)$ passes through every branch of τ (including infinitesimal branches). Moreover, if we set*

$$w = h + 6|\chi(S_{g,n})| - 2n = r + 30|\chi(S_{g,n})| - 10n \leq r + 30|\chi(S_{g,n})|,$$

then we have

$$\ell_{\mathcal{C}}(\psi) \geq \frac{1}{w} \geq \frac{1}{r + 30|\chi(S_{g,n})|}.$$

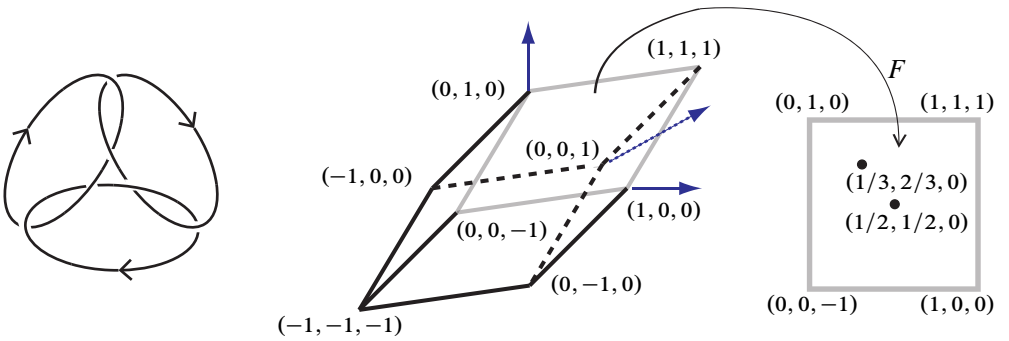

Figure 6: Left: the 3–chain link $\mathcal{C}_3$. Right: the Thurston norm ball of N and fibered face F.

4.2 Fibered cones of the magic manifold

We consider coordinates of integral classes in fibered cones of N. We assign orientations of the three components of $\mathcal{C}_3$ as in Figure 6, left. Let S_α, S_β and S_γ be the oriented 2–punctured disks bounded by these components of $\mathcal{C}_3$. We set $\alpha = [S_\alpha]$, $\beta = [S_\beta]$ and $\gamma = [S_\gamma]$ in $H_2(N, \partial N; \mathbb{Z}) \simeq H^1(N; \mathbb{Z})$. Then α, β and γ form a basis of $H_2(N, \partial N; \mathbb{Z})$. We denote by (x, y, z) the class $x\alpha + y\beta + z\gamma$. The Thurston norm ball B_N is the parallelepiped with vertices $\pm\alpha = \pm(1, 0, 0)$, $\pm\beta = \pm(0, 1, 0)$, $\pm\gamma = \pm(0, 0, 1)$ and $\pm(\alpha + \beta + \gamma) = \pm(1, 1, 1)$; see Figure 6, right.

A symmetry of $\mathcal{C}_3$ tells us that every top-dimensional face of B_N is a fibered face. Moreover all fibered faces of N are permuted transitively by homeomorphisms of N. Hence they have the same topological types in their fibers and the same dynamics of their monodromies. To study monodromies of fibrations on N, it suffices to pick a particular fibered face, say F with vertices $(1, 0, 0)$, $(1, 1, 1)$, $(0, 1, 0)$ and $(0, 0, -1)$; see Figure 6, right. For a primitive integral class $(S, \psi) \in \mathcal{C}_F$, the monodromy ψ is pseudo-Anosov defined on S with boundary components, since $\partial N \neq \varnothing$. Each connected component of ∂S is a simple closed curve which lies on one of the boundary tori of N. By abusing notation, we often regard boundary components of S as punctures of S by crushing each boundary component to a puncture. Hence we think of ψ as a pseudo-Anosov map defined on the punctured surface S. Such ambiguity does not matter for our purpose since the computation of the asymptotic translation lengths of the pseudo-Anosov monodromies on the curve complex will not be affected. Under this convention, one sees that for any primitive integral class $(S, \psi) \in \mathcal{C}_F$, the pseudo-Anosov monodromy ψ is fully punctured; see for example [15].

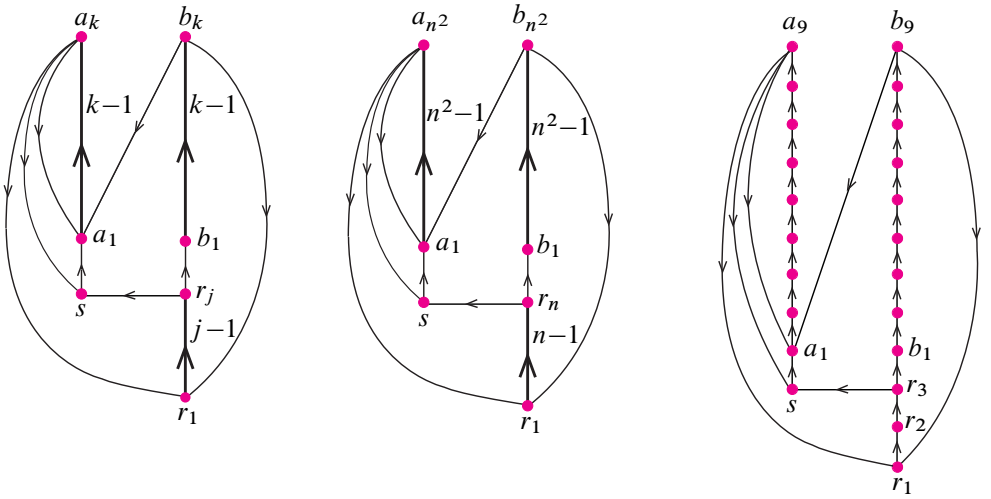

Figure 7: Digraphs $\Gamma_{(1,j,k)_+}$, left, $\Gamma_{(1,n,n^2)_+}$, middle, and $\Gamma_{(1,3,9)_+}$, right.

The open face $\mathrm{int}(F)$ is written by

$$\mathrm{int}(F) = \{(x, y, z) \mid x + y - z = 1, x > 0, y > 0, x > z, y > z\}.$$

This implies that $(x, y, z) \in \mathscr{C}_F$ if and only if $x > 0$, $y > 0$, $x > z$ and $y > z$. The next lemma tells us the topological type of the corresponding fiber $S_{(x,y,z)}$.

Lemma 4.2 [17] *For a primitive integral class $(x, y, z) \in \mathscr{C}_F$, let $|\partial S_{(x,y,z)}|$ denote the number of the boundary components of $S_{(x,y,z)}$. The Thurston norm*

$$\|(x, y, z)\| = |\chi(S_{(x,y,z)})|$$

equals $x + y - z$, and $|\partial S_{(x,y,z)}|$ is given by

$$|\partial S_{(x,y,z)}| = \gcd(x, y + z) + \gcd(y, z + x) + \gcd(z, x + y).$$

More precisely, each term in the right-hand side expresses the number of boundary components of $S_{(x,y,z)}$ which lie on one of the boundary tori of N.

We introduce another coordinate, $(i, j, k)_+$. For $i, j, k \geq 0$, define

$$(i, j, k)_+ = i(1, 1, 1) + j(0, 1, 0) + k(1, 1, 0) = (i + k, i + j + k, i).$$

Note that $(1, 1, 0) \in \mathscr{C}_F$, but $(0, 1, 0) \notin \mathscr{C}_F$ and $(1, 1, 1) \notin \mathscr{C}_F$ (in fact the two classes lie on ∂F); see Figure 6, right. We denote by $\overline{(i, j, k)}_+$, the class with the Thurston norm 1 which is projectively equal to $(i, j, k)_+$.

If i, j and k are integers with $i \geq 0$, $j \geq 0$ and $k > 0$, then $(i, j, k)_+ \in \mathscr{C}_F$. If $(i, j, k)_+$ is a primitive integral class in $\mathscr{C}_F$, then we let $(S_{(i,j,k)_+}, \psi_{(i,j,k)_+})$ be the pair of the fiber and its monodromy. In [15, Section 3], the second author constructs an invariant train track $\tau = \tau_{(i,j,k)_+}$ and the digraph $\Gamma = \Gamma_{(i,j,k)_+}$ of the train track map $\psi = \psi_{(i,j,k)_+} : \tau \to \tau$ for each primitive integral class $(i, j, k)_+ \in \mathscr{C}_F$. Figure 7, left, illustrates $\Gamma = \Gamma_{(1,j,k)_+}$ when $i = 1$, $j > 0$ and $k > 0$; see also [15, Figure 22(4)]. The vertices in the left column of Γ are denoted by $s, a_1, \dots, a_k$ from bottom to top; vertices in the right column of Γ are denoted by $r_1, \dots, r_j, b_1, \dots, b_k$ from bottom to top. (Recall that each vertex of Γ corresponds to a real branch of τ.) The numbers $j-1$ and $k-1$ near the "thick" edges of Γ indicate their lengths of paths. For instance, the edge $r_1 \xrightarrow{j-1} r_j$ from r_1 to r_j indicates the edge path $r_1 \to \cdots \to r_{j-1} \to r_j$. See Figure 7, right, for the concrete example. When $j = 1$ or $k = 1$, the corresponding "thick" edges collapse; see Figure 11.

4.3 Computing the lower bounds

For fixed positive integers p and q, we consider the sequence

$$(1, n^p, n^q)_+ = (1 + n^q, 1 + n^p + n^q, 1) \in \mathscr{C}_F$$

for a varying positive integer n. The integral class $(1, n^p, n^q)_+$ is primitive, since $\gcd(1, n^p, n^q) = 1$. From the formula of the Thurston norm in Lemma 4.2, it is immediate to see the following lemma. See also Figure 6, right.

Lemma 4.3 *Let $\overline{(1, n^p, n^q)_+}$ be the projective class of $(1, n^p, n^q)_+$.*

(1) *If $p = q$, then $\overline{(1, n^p, n^q)_+} \to \left(\frac{1}{3}, \frac{2}{3}, 0\right) \in \operatorname{int}(F)$ as $n \to \infty$.*

(2) *If $p < q$, then $\overline{(1, n^p, n^q)_+} \to \left(\frac{1}{2}, \frac{1}{2}, 0\right) \in \operatorname{int}(F)$ as $n \to \infty$.*

(3) *If $p > q$, then $\overline{(1, n^p, n^q)_+} \to (0, 1, 0) \in \partial F$ as $n \to \infty$.*

Here we consider the following three cases: $q < p < 2q$, $p < q \leq 2p$ and $2p \leq q$. We define

$$k = k_{p,q} = \begin{cases} n^q(2n^q + 1) & \text{if } q < p < 2q, \\ n^q(2n^p + 1) & \text{if } p < q \leq 2p, \\ n^q(2n^{q-p} + 1) & \text{if } 2p \leq q. \end{cases}$$

Proposition 4.4 *For any two vertices v and w of $\Gamma = \Gamma_{(1,n^p,n^q)_+}$, there exists an edge path from v to w of length $k + 2n^p + 3n^q$.*

In other words, if we set $k' = k_{p,q} + 2n^p + 3n^q$, then for any real branch v of τ, $\psi^{k'}(v)$ passes through every real branch. For the proof of Proposition 4.4, we need some lemmas. Recall that s is the bottom vertex in the left column of Γ. Let v_0 be the top vertex a_{n^q} in the left column of Γ; see Figure 9.

Lemma 4.5 *For any vertex v in the left column of Γ, there exists an edge path from s to v of length k.*

Proof We have an edge path $s \to a_1 \xrightarrow{n_q-1} a_{n^q} = v_0$ from s to v_0 of length n^q. For the proof of the lemma, it suffices to show that for any vertex v in the left column of Γ, there exists an edge path from v_0 to v of length $k - n^q$. Then the desired path can be obtained from the concatenation of the two paths, the path from s to v_0 and the path from v_0 to v. Equivalently, we show that for any $i = 0, \dots, n^q$, there exists a cycle based at v_0 of length $k - n^q + i$.

It is easy to find two cycles based at v_0 in Γ of lengths n^q and $n^q + 1$; see Figure 7, left. We have another cycle based at v_0 in Γ of length $n^p + n^q + 1$,

$$v_0 = a_{n^q} \to r_1 \xrightarrow{n_p-1} r_{n^p} \to s \to a_1 \xrightarrow{n_q-1} a_{n^q} = v_0$$

We show that repeated use of these three cycles is enough to produce the cycles we desire. Suppose $q < p < 2q$. Then $k - n^q = 2n^{2q}$. We now show that for any $i = 0, \dots, n^q$, there exist nonnegative integers a, b and c such that

$$an^q + b(n^q + 1) + c(n^q + n^p + 1) = 2n^{2q} + i.$$

This is done by setting $c = 0$, $b = i$ and $a = 2n^q - i$.

Suppose $p < q \leq 2p$. Then $k - n^q = 2n^{p+q}$. We claim that for any $i = 0, \dots, n^q$, there exist nonnegative integers a, b, and c such that

$$an^q + b(n^q + 1) + c(n^q + n^p + 1) = 2n^{p+q} + i.$$

This can be done by setting

$$c = \left\lfloor \frac{i}{n^p + 1} \right\rfloor, \quad b = i - (n^p + 1)\left\lfloor \frac{i}{n^p + 1} \right\rfloor, \quad a = 2n^p - b - c,$$

where $\lfloor \cdot \rfloor$ is the floor function. Here b and c are nonnegative integers by definition, and b is the remainder of i divided by $n^p + 1$. Hence b must be no larger than n^p. On the other hand $c \leq n^{q-p}$, because $i \leq n^q < n^{q-p}(n^p + 1)$. Thus $b + c \leq n^p + n^{q-p} \leq 2n^p$, which implies that a is nonnegative.

Lastly, suppose $2p \le q$. Then $k - n^q = 2n^{2q-p}$. We claim that for any $i = 0, \dots, n^q$, there exist nonnegative integers a, b, and c such that

$$an^q + b(n^q + 1) + c(n^q + n^p + 1) = 2n^{2q-p} + i.$$

This can be done by setting

$$c = \left\lfloor \frac{i}{n^p + 1} \right\rfloor, \quad b = i - (n^p + 1)\left\lfloor \frac{i}{n^p + 1} \right\rfloor, \quad a = 2n^{q-p} - b - c.$$

Here b and c are nonnegative integers by definition, and b is the remainder of i divided by $n^p + 1$. Hence b must be no larger than n^p. On the other hand $c \le n^{q-p}$, because $i \le n^q < n^{q-p}(n^p + 1)$. Thus $b + c \le n^p + n^{q-p} \le 2n^{q-p}$, which says that a is nonnegative. This finishes the proof. □

Lemma 4.6 *For any vertex v in the left column of Γ and for any $m \ge 0$, there exists an edge path from s to v of length $k + m$.*

Proof Let v be any vertex in the left column of Γ. For any $m \ge 0$, one can find a vertex v' in the left column of Γ such that there is an edge path from v' to v of length m. (To see this, use the above cycles based at v_0 of lengths n^q and $n^q + 1$.) Lemma 4.5 tells us that there exists an edge path from s to v' of length k. The concatenation of these edge paths is a desired edge path of length $k + m$. □

Lemma 4.7 *For any vertex v in the right column of Γ and for any $m \ge 0$, there exists an edge path from s to v of length $k + n^p + n^q + m$.*

Proof Let v be an arbitrary vertex in the right column of Γ. Then there exists an edge path from v_0 to v of length ℓ with $1 \le \ell \le n^p + n^q$. To see this, use the path

$$v_0 = a_{n^q} \to r_1 \xrightarrow{n_p - 1} r_{n^q} \to b_1 \xrightarrow{n_q - 1} b_{n^q}$$

from v_0 to b_{n^q}. On the other hand, Lemma 4.6 tells us that there exists an edge path from s to v_0 of length $k + (n^p + n^q - \ell) + m$. Here $(n^p + n^q - \ell) + m$ plays the role of m in Lemma 4.6. Concatenating these two paths, one obtains an edge path from s to v of length $k + n^p + n^q + m$. □

By Lemmas 4.6 and 4.7, we immediately have the following lemma.

Lemma 4.8 *For any vertex v of Γ and for any $m \ge 0$, there exists an edge path from s to v of length $k + n^p + n^q + m$.*

We are now ready to prove Proposition 4.4.

Proof of Proposition 4.4 Note that for any vertex v, there exists an edge path from v to s of length $0 \le \ell \le n^p + 2n^q$. To see this, one can use the edge path of length $n^p + 2n^q$ passing through all vertices of Γ,

$$r_1 \xrightarrow{n_p - 1} r_{n^q} \to b_1 \xrightarrow{n_q - 1} b_{n^q} \to a_1 \xrightarrow{n_q - 1} a_{n^q} \to s.$$

By Lemma 4.8 there exists an edge path from s to any vertex w of length exactly $k + (2n^p + 3n^q - \ell)$, since $2n^p + 3n^q - \ell \ge n^p + n^q$. The concatenation of the two paths has length $k + 2n^p + 3n^q$. □

Now we are ready to compute the lower bounds. For real-valued functions $A(x)$ and $B(x)$, we write $A(x) \gtrsim B(x)$ if there is a constant $C > 0$ independent of x such that $A(x) \ge C \cdot B(x)$.

Theorem 4.9 *The sequence $(1, n^p, n^q)_+$ in $\mathcal{C}_F$ satisfies*

$$\ell_{\mathcal{C}}(\psi_{(1,n^p,n^q)_+}) \gtrsim \begin{cases} 1/n^{2q} & \text{if } q < p < 2q, \\ 1/n^{p+q} & \text{if } p < q \le 2p, \\ 1/n^{2q-p} & \text{if } 2p \le q. \end{cases}$$

Proof By Lemma 4.2, it is not hard to see that

$$(k_{p,q} + 2n^p + 3n^q) + 30|\chi(S_{(1,n^p,n^q)_+})| \asymp \begin{cases} n^{2q} & \text{if } q < p < 2q, \\ n^{p+q} & \text{if } p < q \le 2p, \\ n^{2q-p} & \text{if } 2p \le q. \end{cases}$$

Then the desired claim follows from Propositions 4.1 and 4.4. □

4.4 Computing the upper bounds

To prove Theorem 4.13, we will also compute the upper bound of the asymptotic translation length of $\psi_{(1,n^p,n^q)_+}$.

Theorem 4.10 *For any fixed positive integers p and q with $q < p < 2q$, the sequence $(1, n^p, n^q)_+$ of primitive integral classes in $\mathcal{C}_F$ converges projectively to $(0, 1, 0) \in \partial F$ as $n \to \infty$, and*

$$\ell_{\mathcal{C}}(\psi_{(1,n^p,n^q)_+}) \le \frac{4}{n^{2q}}.$$

The first half of Theorem 4.10 follows from Lemma 4.3(3). For the rest of the proof, we first introduce the dual arcs of real branches of train tracks. Consider an invariant train track τ for the monodromy ψ defined on the fiber S of a fibration on N. If we

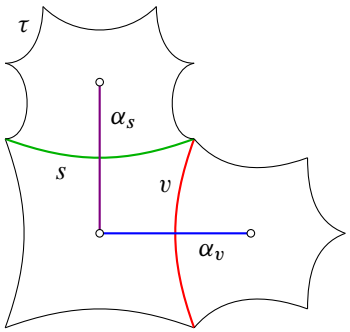

Figure 8: Cell decomposition, branches, and dual arcs.

think of the surface S with boundary as the punctured surface which is again denoted by S abusing notation, each component of the complement $S \setminus \tau$ of the train track is a once-punctured ideal polygon, because ψ is fully punctured. Consider the cell decomposition of S corresponding to τ. That is, 0–cells are switches of τ, 1–cells are branches of τ, and 2–cells are ideal polygons of $S \setminus \tau$.

Given a real branch v, the *dual arc* α_v of v is defined to be the edge of the dual cell complex that connects the punctures in two polygons (possibly the same polygon) sharing the real branch v; see Figure 8.

Notice that the dual arc α_v is an essential arc. In order to see this, consider a rectangle associated with the real branch v, contained in a Markov partition for a pseudo-Anosov homeomorphism which represents ψ. Then v corresponds to leaves of the unstable foliation and the dual arc α_v corresponds to leaves of the stable foliation in this rectangle. If the dual arc is not essential, then this implies that the real branch v cannot support a positive transverse measure, which is a contradiction to a property of pseudo-Anosov homeomorphisms.

Readers may notice that the dual arc associated to a real branch is a general notion for fully punctured pseudo-Anosov homeomorphisms. More precisely, if τ is an invariant train track for a fully punctured pseudo-Anosov ψ, then for a real branch v of τ, one can define the dual arc α_v which is essential.

Proof of Theorem 4.10 Let $(S, \psi) = (S_{(1,n^p,n^q)_+}, \psi_{(1,n^p,n^q)_+})$ be the pair of the fiber and its monodromy for $(1, n^p, n^q)_+$. Let Γ be the digraph of the train track τ for $(1, n^p, n^q)_+$, and let $\psi_*\colon V(\Gamma) \to V(\Gamma)$ be the induced map, where $V(\Gamma)$ is the set of vertices of Γ. The map ψ_* can be read off Figure 9.

Here is the outline of the proof. We will compute the upper bound of the asymptotic translation length $\ell_{\mathcal{AC}}(\psi)$ of ψ on the arc and curve complex $\mathcal{AC}(S)$. Since $\mathcal{C}(S)$ and $\mathcal{AC}(S)$ are quasi-isometric, this gives an upper bound on $C(S)$. We show that there are distinct vertices t and v in Γ, ie distinct real branches t and v of τ, such that $\psi_*^{n^{2q}}(t)$ does not contain v. Using this fact, we also show that there are disjoint arcs β_t and α_v in $\mathcal{AC}(S)$ such that $\psi^{n^{2q}}(\beta_t)$ and α_v are disjoint. This implies that the distance in $\mathcal{AC}(S)$ satisfies $d_{\mathcal{AC}}(\beta_t, \psi^{n^{2q}}(\beta_t)) \leq 2$, and we deduce that $\ell_{\mathcal{AC}}(\psi) \leq 2/n^{2q}$.

Step 1 *$\mathcal{C}(S)$ and $\mathcal{AC}(S)$ are quasi-isometric.*

Proof Just recall that the inclusion map $\mathcal{C}(S) \to \mathcal{AC}(S)$ is 2–bilipschitz. □

Hence for the proof of Theorem 4.10, it is enough to show that the asymptotic translation length ψ on $\mathcal{AC}(S)$ satisfies

$$\ell_{\mathcal{AC}}(\psi) \leq \frac{2}{n^{2q}}.$$

Step 2 *Let t be the vertex b_{n^q} of Γ. Then $\psi_*^{n^{2q}}(t)$ doesn't contain all vertices in Γ.*

Proof We will show that there is a vertex v that is not contained in $\psi_*^{n^{2q}}(t)$. Consider the partition $\{A, B, R_1, R_2, \ldots, R_{n^{p-q}}\}$ of vertices a_i, b_i, and r_i of Γ, where each partition element consists of n^q vertices as in Figure 9. Under the iteration of the $(n^q)^{\text{th}}$ power $\psi_*^{n^q}$ of ψ_*, one can see that

$$\begin{aligned}
\psi_*^{n^q}(t) &= \{a_{n^q}, r_{n^q}\},\\
\psi_*^{2n^q}(t) &= \{a_{n^q}, a_{n^q-1}, r_{n^q}, r_{2n^q}\},\\
\psi_*^{3n^q}(t) &= \{a_{n^q}, a_{n^q-1}, a_{n^q-2}, r_{n^q}, r_{n^q-1}, r_{2n^q}, r_{3n^q}\},\\
&\vdots
\end{aligned}$$

and that the number of vertices in each partition element contained in $\psi_*^{j\cdot n^q}(t)$ is increasing by at most one as j increases. Hence one can see that there are vertices in each R_k $(k = 1, \ldots, n^{p-q})$ that are not contained in $\psi_*^{n^{2q}}(t)$. More precisely, consider $R_1 = \{r_1, r_2, \ldots, r_{n^q}\}$. One can check that for vertices in R_1, the image $\psi_*^{j\cdot n^q}(t)$ contains only

$$\{r_{n^q}, r_{n^q-1}, \ldots, r_{n^q-j+2}\} \subset R_1$$

for $2 \leq j \leq n^q$. Therefore $\psi_*^{n^{2q}}(t)$ does not contain r_1, and we may choose v to be r_1. □

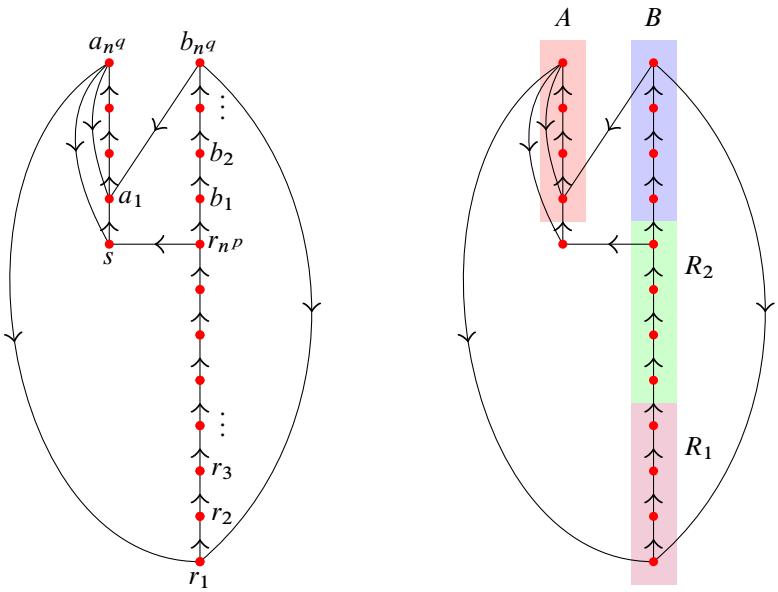

Figure 9: Left: digraph $\Gamma_{(1,n^p,n^q)_+}$. Right: digraph $\Gamma_{(1,2^3,2^2)_+}$ with partition $\{A, B, R_1, R_2\}$.

Step 3 *There are distinct arcs α_v and β_t in $\mathcal{AC}(S)$ such that $\psi^{n^{2q}}(\beta_t)$ and α_v are disjoint.*

Before proving Step 3, we first discuss some properties of the primitive integral class $(1, j, k)_+$ with $j > 0$ and $k > 0$. Recall that $r_1, \dots, r_j, b_1, \dots, b_k$ are vertices of $\Gamma = \Gamma_{(1,j,k)_+}$ which lie on the right column of Γ (Figure 7, left). There is a single ideal polygon $P = P_{(1,j,k)_+}$ containing a single puncture c_P of the fiber $S = S_{(1,j,k)_+}$ such that the two endpoints of each real branch b_i $(i = 1, \dots, k)$ are switches (of τ) in the boundary ∂P of P; see Figure 10. From the construction of τ in [15], it follows that ∂P consists of periodic branches, ie infinitesimal branches, and $\psi = \psi_{(1,j,k)_+}$ maps c_P to itself (and hence the ideal polygon P is preserved by ψ). To see $\psi(c_P) = c_P$, we consider the fiber $S = S_{(i,j,k)_+}$ with boundary. (So we now think of the above c_P as a boundary component of S.) By using Lemma 4.2 for the primitive integral class $(1, j, k)_+$, we see that there is a boundary torus T of N such that c_P is the only boundary component of S which lies on T. This implies c_P is preserved by ψ.

For the real branch r_i $(i = 1, \dots, j)$, consider its dual arc α_{r_i}. Let c_{r_i} and c'_{r_i} be boundary components in ∂S which are connected by α_{r_i}. (Possibly $c_{r_i} = c'_{r_i}$.) Then there is another boundary torus T' of N on which the both c_{r_i} and c'_{r_i} lie.

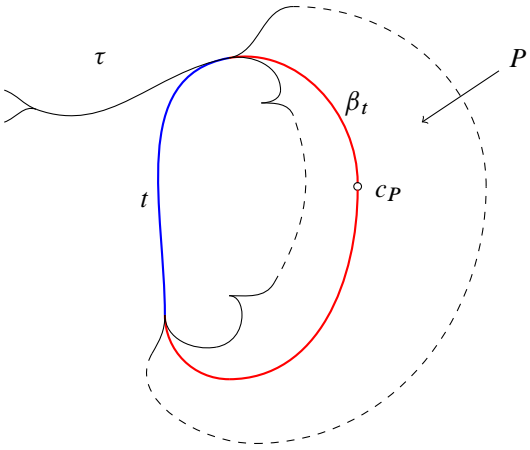

Figure 10: Part of the train track τ. The ideal polygon P, real branch $t = b_{n^q}$, and arc β_t based at c_P.

Proof of Step 3 Consider the primitive integral class $(1, n^p, n^q)_+$ in question. The two endpoints of the real branch $t = b_{n^q}$ are switches (of τ) in ∂P. Join c_P and each endpoint of the real branch t by an arc and then we obtain an arc β_t in S; see Figure 10. Since t is a real branch, one sees that the arc β_t is essential. Since ψ maps c_P to itself, $\psi^\ell(\beta_t)$ is an essential arc based at the same c_P for each $\ell > 0$. Moreover $\psi^\ell(\beta_t)$ is not homotopic to β_t for each $\ell > 0$, since ψ is pseudo-Anosov. Let us consider the dual arc α_v of $v = r_1$. Recall that c_v and c_v' which are connected by α_v lie on a boundary torus T' of N, yet c_P lies on the different boundary torus T of N. The arc β_t has end points at c_P, and hence β_t is not homotopic to α_v.

Now we prove that $\psi^{2q}(\beta_t)$ and α_v are disjoint. The ideal polygon P is preserved by ψ, and $\psi^{n^{2q}}(t)$ is carried by τ since τ is invariant under ψ. Moreover, since $\psi^{n^{2q}}(t)$ does not pass through v by the proof of Step 2, it follows that $\psi^{n^{2q}}(\beta_t)$ is disjoint from v, and hence also disjoint from its dual arc α_v. □

Step 4 $$\ell_{\mathcal{AC}}(\psi) \leq \frac{2}{n^{2q}}.$$

Proof Clearly β_t and α_v are disjoint. Since $\psi^{n^{2q}}(\beta_t)$ is an essential arc based at c_P, we have $\psi^{n^{2q}}(\beta_t) \neq \alpha_v$ in $\mathcal{AC}(S)$ by the same argument as in the proof of Step 3. This together with the fact that the geometric intersection number $i(\psi^{n^{2q}}(\beta_t), \alpha_v) = 0$

implies that β_t and $\psi^{n^{2q}}(\beta_t)$ are at most distance 2 in $\mathcal{AC}(S)$, ie $d_{\mathcal{AC}}(\beta_t, \psi^{n^{2q}}(\beta_t)) \leq 2$. By the definition of the asymptotic translation length, it follows that

$$\ell_{\mathcal{AC}}(\psi) \leq \frac{2}{n^{2q}}. \qquad \square$$

Thus we have finished the proof of Theorem 4.10. $\square$

Theorem 4.11 *For any fixed positive integers p and q with $2p \leq q$, the sequence $(1, n^p, n^q)_+$ of primitive integral classes in $\mathcal{C}_F$ converges projectively to $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ in* $\mathrm{int}(F)$ *as $n \to \infty$, and*

$$\ell_{\mathcal{C}}(\psi_{(1,n^p,n^q)_+}) \leq \frac{C}{n^{2q-p}},$$

where C is a constant independent of n.

Proof The first half of the claim follows from Lemma 4.3(2). For the rest of the proof, let $\psi = \psi_{(1,n^p,n^q)_+}$. Consider the digraph $\Gamma = \Gamma_{(1,n^p,n^q)_+}$ and the induced map $\psi_*\colon V(\Gamma) \to V(\Gamma)$. Let t be the vertex b_{n^q} of Γ. By using a similar argument as in Step 2 of the proof of Theorem 4.10, one can show that the set of vertices $\psi_*^{j\cdot n^q}(t)$ is contained in $V(\Gamma)\setminus R$ for $j = 1, \dots, \lfloor (n^q-1)/(n^p+1)\rfloor$, where $R = \{r_1, r_2, \dots, r_{n^p}\}$. In other words, each vertex in R is not contained in $\psi_*^{j\cdot n^q}(t)$ for such j. In particular, if we set $D = D(n) = \lfloor (n^q-1)/(n^p+1)\rfloor$, then r_1 is not contained in $\psi_*^{Dn^q}(t)$. Then we consider the two arcs β_t and α_v as in Step 3 of the proof of Theorem 4.10. By the same argument, it follows that β_t, α_t and $\psi^{Dn^q}(\beta_t)$ are distinct elements in $\mathcal{AC}(S)$. Moreover we have $i(\psi^{Dn^q}(\beta_t), \alpha_v) = 0$ and $i(\beta_t, \alpha_v) = 0$. Therefore β_t and $\psi^{Dn^q}(\beta_t)$ are at most distance 2 in $\mathcal{AC}(S)$, and we have $\ell_{\mathcal{AC}}(\psi) \leq 2/(Dn^q)$, which implies that $\ell_{\mathcal{C}}(\psi) \leq 4/(Dn^q)$. Since $Dn^q \asymp n^{2q-p}$, we have finished the proof. $\square$

4.5 The behaviors of asymptotic translation lengths

We prove the following lemma which implies that the upper bound of Theorem 3.1 is optimal.

Lemma 4.12 *The sequence $(1, n, 1)_+$ of primitive integral classes in $\mathcal{C}_F$ converges projectively to a point in ∂F as $n \to \infty$, and*

$$\ell_{\mathcal{C}}(\psi_{(1,n,1)_+}) \asymp \frac{1}{|\chi(S_{(1,n,1)_+})|}.$$

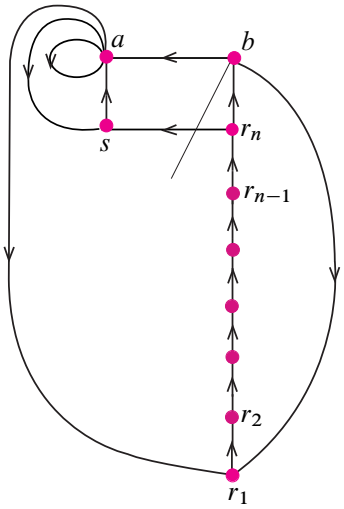

Figure 11: The digraph $\Gamma_{(1,n,1)_+}$.

Proof The first half of the claim follows from that fact that $\overline{(1,n,1)}_+ \to (0,1,0) \in \partial F$ as $n \to \infty$. Since $|\chi(S_{(1,n,1)_+})| = n+3$, it is enough to prove that $\ell_{\mathcal{C}}(\psi_{(1,n,1)_+}) \asymp 1/n$. By the digraph $\Gamma = \Gamma_{(1,n,1)_+}$ (see Figure 11) together with Proposition 4.1, it is not hard to see that $\ell_{\mathcal{C}}(\psi_{(1,n,1)_+}) \gtrsim 1/n$.

Now we compute the upper bound. Let $(S,\psi) = (S_{(1,n,1)_+}, \psi_{(1,n,1)_+})$ and let t be the vertex b of Γ. We have

$$\psi_*(t) = \{r_1\}, \psi_*^2(t) = \{r_2\}, \dots, \psi_*^n(t) = \{r_n\}.$$

In particular this implies that $\psi^n(t)$ does not pass through the real branch r_1 of $\tau = \tau_{(1,n,1)_+}$. We consider the essential arc β_t for t as in the proof of Theorem 4.10, and consider the dual arc α_{r_1} of r_1. By the same argument as in the proof of Theorem 4.10, one sees that the three arcs β_t, $\psi^n(\beta_t)$ and α_{r_1} are distinct elements in $\mathcal{AC}(S)$. Furthermore for the geometric intersection numbers between arcs, we have $i(\beta_t, \alpha_{r_1}) = 0$ and $i(\psi^n(\beta_t), \alpha_{r_1}) = 0$. Therefore β_t and $\psi^n(\beta_t)$ are at most distance 2 in $\mathcal{AC}(S)$, and we have $\ell_{\mathcal{AC}}(\psi) \le 2/n$, which gives the desired upper bound $\ell_{\mathcal{C}}(\psi) \le 4/n$. □

Now we are ready to prove the following theorem.

Theorem 4.13 *Let F be a fibered face of the magic manifold. Then there exist two points, $\mathfrak{b}_0 \in \partial F$ and $\mathfrak{c}_0 \in \mathrm{int}(F)$, which satisfy the following*:

(1) *For any $r \in \mathbb{Q} \cap [1,2)$, there exists a sequence $(S_{\alpha_n}, \psi_{\alpha_n})$ of primitive integral classes in $\mathscr{C}_F$ converging projectively to $\mathfrak{b}_0$ as $n \to \infty$ such that*

$$\ell_{\mathcal{C}}(\psi_{\alpha_n}) \asymp \frac{1}{|\chi(S_{\alpha_n})|^r}.$$

(2) *For any $r \in \mathbb{Q} \cap \left[\frac{3}{2}, 2\right]$, there exists a sequence $(S_{\alpha_n}, \psi_{\alpha_n})$ of primitive integral classes in $\mathscr{C}_F$ converging projectively to $\mathfrak{c}_0$ as $n \to \infty$ such that*

$$\ell_{\mathcal{C}}(\psi_{\alpha_n}) \asymp \frac{1}{|\chi(S_{\alpha_n})|^r}.$$

In particular, the upper bound in Theorem 1.1 is optimal when $d = 2$.

Proof Because of the symmetry of the Thurston norm ball B_N, it suffices to prove the theorem for the fibered face as we picked in Section 4.2. For (1), if $1 < r < 2$, let p and q be positive integers such that $r = 2q/p$ with $q < p < 2q$. By Lemma 4.3, the sequence$(1, n^p, n^q)_+$ converges projectively to $(0,1,0) \in \partial F$. By Theorems 4.9 and 4.10, we have $\ell_{\mathcal{C}}(\psi_{(1,n^p,n^q)_+}) \asymp 1/n^{2q}$. Since we have $\|(1, n^p, n^q)_+\| \asymp n^p$, it follows that

$$\ell_{\mathcal{C}}(\psi_{(1,n^p,n^q)_+}) \asymp \frac{1}{|\chi(S_{(1,n^p,n^q)_+})|^{2q/p}} = \frac{1}{|\chi(S_{(1,n^p,n^q)_+})|^r},$$

where $r = 2q/p \in (1,2)$. If $r = 1$, it follows from Lemma 4.12.

For (2), if $\frac{3}{2} \le r < 2$, let p and q be positive integers such that $r = 2 - p/q$ with $2p \le q$. By Lemma 4.3, the sequence $(1, n^p, n^q)_+$ converges projectively to $\left(\frac{1}{2}, \frac{1}{2}, 0\right) \in \operatorname{int}(F)$ as $n \to \infty$. By Theorems 4.9 and 4.11, we have $\ell_{\mathcal{C}}(\psi_{\alpha_n}) \asymp 1/n^{2q-p}$. Since we have $\|(1, n^p, n^q)_+\| \asymp n^q$, it follows that

$$\ell_{\mathcal{C}}(\psi_{(1,n^p,n^q)_+}) \asymp \frac{1}{|\chi(S_{(1,n^p,n^q)_+})|^{2-p/q}} = \frac{1}{|\chi(S_{(1,n^p,n^q)_+})|^r},$$

where $r = 2 - p/q \in \left[\frac{3}{2}, 2\right)$. For $r = 2$, one can choose a sequence of primitive integral classes contained in the intersection between the cone over some compact set $K \subset \operatorname{int}(F)$ and some 2–dimensional rational subspace of $H^1(M)$, eg the sequence $(1, n, n)_+$. Then the sequence satisfies the desired property from [3, Corollary 1].

Finally we consider the upper bound in Theorem 1.1 when $d = 2$. If $(p,q) = (1,2)$, then

$$\ell_{\mathcal{C}}(\psi_{(1,n,n^2)_+}) \asymp \frac{1}{|\chi(S_{(1,n,n^2)_+})|^{1+1/2}}.$$

Then Theorem 1.1 implies that the sequence $(1, n, n^2)_+$ of primitive integral classes can not be contained in any finite union of 2–dimensional rational subspaces of $H^1(N)$. The fibered cone $\mathcal{C}_F$ is a (2+1)–dimensional rational subspace of $H^1(N)$. Thus Theorem 1.1 is optimal when $d = 2$. □

In light of Theorem 4.13(1), we ask the following question.

Question 4.14 Let F be a fibered face of a compact hyperbolic fibered 3–manifold. Does there exist a sequence $(S_{\alpha_n}, \psi_{\alpha_n})$ of primitive integral classes in $\mathcal{C}_F$ converging projectively to ∂F as $n \to \infty$ such that $\ell_{\mathcal{C}}(\psi_{\alpha_n}) \asymp 1/|\chi(S_{\alpha_n})|^2$?

By Theorem 4.13, we immediately have the following corollary.

Corollary 4.15 *Let F be a fibered face of the magic manifold N. For $\alpha \in F \cap H^1(N;\mathbb{Q})$, let $(S_{\widetilde{\alpha}}, \psi_{\widetilde{\alpha}})$ be the fiber and pseudo-Anosov monodromy corresponding to the primitive integral class $\widetilde{\alpha}$ lying on the ray of α passing through the origin. Then there is no normalization of the asymptotic translation length function*

$$F \cap H^1(N;\mathbb{Q}) \to \mathbb{R}_{\geq 0}, \quad \alpha \mapsto \ell_{\mathcal{C}}(\psi_{\widetilde{\alpha}}),$$

in terms of the Euler characteristic $\chi(S_{\widetilde{\alpha}})$ which admits a continuous extension on F.

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Department of Mathematical Sciences, KAIST
Daejeon, South Korea
Center for Education in Liberal Arts and Sciences, Osaka University
Osaka, Japan
Reinsurance Group of America
Chesterfield, MO, United States
Department of Mathematics, University of Wisconsin at Madison
Madison, WI, United States
hrbaik@kaist.ac.kr, kin.eiko.celas@osaka-u.ac.jp, hshin153@gmail.com, cwu367@math.wisc.edu

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