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Discrete real specializations of sesquilinear representations of the braid groups

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Using Salem numbers, this paper gives real specializations of sesquilinear representations of the braid groups that make the images discrete groups. This method is applied to the Burau, Jones and Lawrence–Krammer representations, and some details on the commensurability of the target groups are given.

20C08, 57M99

1 Introduction

Representations of the braid groups have attracted attention because of their wide variety of applications from discrete geometry to quantum computing. This paper takes the point of view that one should ask structural questions about the image of a braid group representation, in particular whether the image is discrete for specializations of the parameter. Venkataramana in [15] also followed this pursuit for discrete specializations of the Burau representation but with a different approach toward arithmeticity.

Since the Jones representations are used in modeling quantum computations, much work has been done to understand specializations at roots of unity, as explored by Funar and Kohno in [7], Freedman, Larsen and Wang in [6], and many others. However, there seems to be a lack of exploration of the *real* specializations of these representations. This paper takes a more general approach to find discrete real specializations of any sesquilinear group representation, and show how this can be applied to representations of the braid groups. The main theorem follows.

Theorem 1.1 *Let $\rho_t : G \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}])$ be a group representation with parameter t. Suppose there exists a matrix J_t such that*:

(1) *For all M in the image of ρ_t, $M^* J_t M = J_t$, where $M^*(t) = M^{\intercal}(1/t)$.*

(2) $J_t = (J_{1/t})^{\mathsf{T}}$.

(3) $J_t \in \mathrm{GL}_m(\mathbb{Q}(t))$, *where no entry of* J_t *has denominator with* 1 *as a root.*

(4) J_t *is positive definite for* t *in a neighborhood* η *of* 1 *in* $\mathbb{C}$.

Then there exist infinitely many Salem numbers s *such that the specialization representation* ρ_s *at* $t = s$ *is discrete.*

Further applying a classification theorem of hermitian forms from Scharlau [13] proves the following commensurability result of the target groups.

Corollary 1.2 *For* $\rho_t : G \to \mathrm{SL}_{2m+1}(\mathbb{Z}[t, t^{-t}])$ *as in Theorem 1.1, there exist infinitely many Salem numbers* s *such that for infinitely many integers* n *and* k *the specializations* ρ_{s^k} *at* $t = s^k$ *and* ρ_{s^n} *at* $t = s^n$ *map into commensurable lattices.*

Squier showed in [14] that the reduced Burau representation is sesquilinear and satisfies the criteria for Theorem 1.1. Example 3.2 gives explicit Salem numbers such that specializing the reduced Burau representation to these numbers is discrete. Using the Burau representation as motivation, Section 2 introduces the main tools of discrete generalized unitary groups and Salem numbers. In Section 3 we apply Theorem 1.1 to the Jones and Lawrence–Krammer representations of the braid group, and we suspect it also applies to all of the BMW representations. Lastly, Section 4 discusses the lattice structure and commensurability of the target groups for the Salem number specializations.

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2 Discrete representations using Salem numbers

2.1 Motivation from Squier and the Burau representation

The (reduced) Burau representation $\rho_{n,t} : B_{n+1} \to \mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ is an irreducible representation of the braid group. These representations depend on n, where $n+1$

is the number of braid strands, and are parametrized by a variable t. Squier showed in [14] that there is a nondegenerate n–dimensional matrix $J_{n,t}$ satisfying the equation

$$M^* J_{n,t} M = J_{n,t}, \tag{2-1}$$

for all M in the image of $\rho_{n,t}$. Here M^* is the transpose of M after replacing t with $1/t$ in the entries of M, $M^*(t) := M(1/t)^{\mathsf{T}}$. $J_{n,t}$ is sesquilinear with respect to $*$, $J_{n,t}^* = J_{n,t}$, and letting $t = x^2$, $J_{n,t}$ is given by the tridiagonal matrix

$$J_{n,t} = \begin{bmatrix} x+x^{-1} & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & & -1 \\ & & -1 & x+x^{-1} \end{bmatrix}.$$

If t is a unit complex number, (2-1) agrees with the usual unitary relation $(\overline{M})^{\mathsf{T}} M = \mathrm{Id}$. Representations that satisfy (2-1) are called *sesquilinear*, and are said to map into a *generalized unitary group*. This terminology will be made precise in the next section.

These generalized unitary groups are the key to finding discrete specializations. The method described here is to show that carefully chosen specializations of the parameter t make the entire generalized unitary group discrete, thus making the image of the representation discrete.

2.2 Unitary groups

In general, unitary groups are matrix groups which respect a form, or inner product. These notions heavily rely on the ring of coefficients and an involution of that ring. Let R be a ring and ϕ an order 2 automorphism of R. For a matrix M defined over R, let $M^* = (M^\phi)^{\mathsf{T}}$, where M^ϕ is the matrix obtained by applying ϕ to the entries of M. For the Burau representation in Section 2.1, ϕ is the map given by $t \mapsto 1/t$, and $R = \mathbb{Z}[t, t^{-1}]$.

Definition 2.1 For a matrix J such that $J^* = J$, the *generalized unitary group* is

$$U_m(J, \phi, R) := \{M \in \mathrm{GL}_m(R) \mid M^* J M = J\}.$$

Here, J is called a *sesquilinear form* with respect to ϕ. For example, in this notation the familiar unitary group U_m can be written as $U_m(\mathrm{Id}, -, \mathbb{C})$, where "–" is complex conjugation. A representation is called *sesquilinear* if its image is contained in a generalized unitary group.

2.2.1 Creating discrete unitary groups The Burau representation can be written as $\rho_{n,t}\colon B_{n+1} \to U_n(J_{n,t}, \phi, \mathbb{Z}[t, t^{-1}])$. With the goal of parameter specialization in mind, the relevant choice for the coefficient ring is a number ring. Discreteness of the unitary group is a delicate relationship between the form J and the algebraic structure of the number ring. More precisely, let L be a totally real algebraic field extension of $\mathbb{Q}$ and let K be a degree 2 field extension of L, with $L, K \subseteq \mathbb{C}$. Let ϕ be the order 2 generator of $\mathrm{Gal}(K/L)$, and let $\mathcal{O}_K$ and $\mathcal{O}_L$ denote the rings of integers of K and L, respectively:

$$\begin{tikzcd} \mathcal{O}_K \arrow[loop above, "\phi"] \arrow[d, dash] & K \arrow[loop above, "\phi"] \arrow[d, dash, "2"] & K^\sigma \subseteq \mathbb{C} \arrow[loop above, "\phi^\sigma"] \\ \mathcal{O}_L & L & L^\sigma \subseteq \mathbb{R} \end{tikzcd}$$

Let σ be a complex place of K, which in this setting is a field homomorphism $\sigma\colon K \to \mathbb{C}$ different from ϕ and the identity map. We write $X^\sigma = \sigma(X)$ for any X in K. The algebraic structure is passed along by σ, meaning $\mathcal{O}_{K^\sigma} = (\mathcal{O}_K)^\sigma$ is the ring of integers for K^σ and $\phi^\sigma = \sigma\phi\sigma^{-1}$ is an involution on K^σ.

Let J be a matrix over $\mathcal{O}_K$ that is sesquilinear with respect to ϕ. Then J^σ is sesquilinear with respect to ϕ^σ. So, in particular,

$$U_m(J^\sigma, \phi^\sigma, \mathcal{O}_{K^\sigma}) = \{M \in \mathrm{GL}_m(\mathcal{O}_{K^\sigma}) \mid (M^{\phi^\sigma})^\intercal J^\sigma M = J^\sigma\}.$$

Since σ is a homomorphism, we can see that $(U_m(J, \phi, \mathcal{O}_K))^\sigma = U_m(J^\sigma, \phi^\sigma, \mathcal{O}_{K^\sigma})$ by applying σ to the equation $J = M^* J M$.

The following results outline compatibility requirements between J and $\mathcal{O}_K$, which show that $U_m(J, \phi, \mathcal{O}_K)$ is a discrete subgroup of $\mathrm{GL}_m(\mathbb{R})$ under the standard euclidean topology.

Proposition 2.2 *$U_m(J^\sigma, \phi^\sigma, \mathcal{O}_{K^\sigma})$ is a bounded group when J^σ is positive definite and ϕ^σ is complex conjugation.*

Proof Because J^σ is positive definite, by Sylvester's law of inertia and the Gram–Schmidt process, there exists a matrix $Q \in \mathrm{GL}_m(\mathbb{C})$ such that $J^\sigma = Q^* \,\mathrm{Id}\, Q$. This implies that $QU_m(J^\sigma, \phi^\sigma, \mathcal{O}_{K^\sigma})Q^{-1} \subseteq U_m(\mathrm{Id}, \phi^\sigma, \mathbb{C})$, which is a subgroup of the compact group U_m. □

Theorem 2.3 *$U_m(J, \phi, \mathcal{O}_K)$ is discrete if, for every complex place σ of K, J^σ is positive definite and ϕ^σ is complex conjugacy.*

Proof Assume that $\{M_n\}$ converges to the identity in $U_m(J, \phi, \mathcal{O}_K)$. To show $\{M_n\}$ is eventually constant, we will show that for n large there are only finitely many possibilities for the entries $(M_n)_{ij}$.

By assumption, for each σ the group $U_m(J^\sigma, \phi^\sigma, O_{K^\sigma})$ is bounded by Proposition 2.2. Also, for every M_n, $M_n^\sigma \in U_m(J^\sigma, \phi^\sigma, O_{K^\sigma})$. Thus, there exists a B such that for large n, for all i, j, and for all σ, we have that $|(M_n^\sigma)_{ij}| < B$.

For every $M \in U_m(J, \phi, \mathcal{O}_K)$, the equation $M^* J M = J$ can be rearranged to $JMJ^{-1} = ((M^\phi)^\intercal)^{-1}$, showing that M and $((M^\phi)^\intercal)^{-1}$ are simultaneously conjugate. Thus $\{M_n^\phi\}$ also converges to the identity. Convergent sequences are bounded, so for large enough n, $|(M_n)_{ij}| < B$ and $|(M_n)_{ij}^\phi| < B$ for every ij–entry.

L is a totally real degree 2 subfield of K, and ϕ generates $\mathrm{Gal}(K/L)$. So K has one nonidentity real embedding ϕ, and all other embeddings are complex. Thus we have shown above that for large n there is a uniform bound B for each entry $(M_n)_{ij}$ and each Galois conjugate of $(M_n)_{ij}$. There are only finitely many algebraic integers α such that $\deg(\alpha) \le \deg(K/\mathbb{Q})$, and with the property that α and all of the Galois conjugates of α have absolute values bounded above by B. So there are only finitely many possible entries for $(M_n)_{ij}$, which implies the sequence $\{M_n\}$ is eventually constant. □

Corollary 2.4 *If $\rho \colon G \to U_m(J, \phi, \mathcal{O}_K)$ is a representation of a group G such that for every nonidentity place σ of K, J^σ is positive definite and ϕ^σ is complex conjugacy, then ρ is a discrete representation.*

At first glance, the requirements for Corollary 2.4 seem very specific and perhaps it is doubtful that any such a representation could exist. However, as described in Section 2.1, Squier showed that the Burau representation maps into a generalized unitary group over $\mathbb{Z}[t, t^{-1}]$, so the next task is to find values of t such that the form and coefficient ring satisfy the specific hypothesis of Corollary 2.4.

2.3 Salem numbers

Salem numbers are the key ingredient to the application of Corollary 2.4, which requires a real algebraic number field with tight control and understanding of each of its complex embeddings.

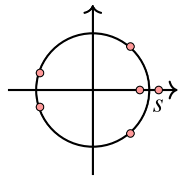


Figure 1: A schematic picture of an order 6 Salem number.

Definition 2.5 A Salem number s is a real algebraic unit greater than 1, with one real Galois conjugate $1/s$, and all of the complex Galois conjugates of s have absolute value equal to 1.

For example, the largest real root of Lehmer's polynomial, called Lehmer's number,

$$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1,$$

is a Salem number. Trivial Salem numbers of degree 2 are solutions to s^2-ns+1 for $n \in \mathbb{N}$ and $n>2$. It is well known that there are infinitely many Salem numbers of arbitrarily large absolute value and degree. In particular, if s is a Salem number, then s^m is also a Salem number for every positive integer m. One geometric consequence of the property that powers of Salem numbers are Salem numbers is that by taking powers, one can control the spatial configuration of the complex Galois conjugates of a Salem number, as described in Lemma 2.6.

Lemma 2.6 *For any Salem number s, and for any interval containing* 1 *on the complex unit circle, there exist infinitely many integers m such that every complex Galois conjugate of s^m lies in the interval.*

Proof Let $e^{i\theta_1},\dots,e^{i\theta_k}$ be all the Galois conjugates of the Salem number s with positive imaginary part. Suppose that $\prod_{j=1}^{k}(e^{i\theta_j})^{m_j}=1$. Let ϕ be the automorphism of the Galois closure of s with the property that $\phi(e^{i\theta_1})=s$. Since ϕ must permute the

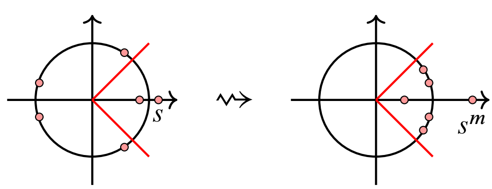


Figure 2: A schematic picture of Lemma 2.6.

Galois conjugates of s, for $j \neq 1$, $\phi(e^{i\theta_j})$ is again on the complex unit circle. Thus,

$$1=\phi\biggl(\prod_{j=1}^{k}(e^{i\theta_j})^{m_j}\biggr)=s^{m_1}\prod_{j=2}^{k}\phi(e^{i\theta_j})^{m_j}, \quad \text{which implies} \quad \prod_{j=2}^{k}\phi(e^{i\theta_j})^{m_j}=1/s^{m_1}.$$

Since each $\phi(e^{i\theta_j})$ is a unit complex number, it must be the case that each $m_j = 0$. This shows that the point $p = (e^{i\theta_1}, \dots, e^{i\theta_k})$ satisfies the criteria for Kronecker's theorem. In particular, the set $\overline{\{p^m \mid\in \mathbb{Z}\}}$ is dense in the torus T^k. □

Fixing an arbitrary Salem number s, let $K = \mathbb{Q}(s)$, $L = \mathbb{Q}(s + 1/s)$ and $\mathcal{O}_K$ be the ring of integers of K:

$$\begin{array}{c} \mathbb{Q}(s) = K \\ \Big|\,{\scriptstyle 2} \\ \mathbb{Q}(s+1/s) = L \\ \Big| \\ \mathbb{Q} \end{array}$$

Since s and $1/s$ are real and all other Galois conjugates of s are complex, K has exactly two real embeddings. For a complex embedding σ of K, $(s+1/s)^\sigma = 2\,\mathrm{Re}(s^\sigma)$, which is real. This shows that all embeddings of L are real, and that L is a totally real subfield of K. Since s is a root of $X^2 - (s + 1/s)X + 1$, K is degree 2 over L.

The Galois group of K/L is generated by ϕ, which maps s to $1/s$. (This exactly matches the involution $t \mapsto 1/t$ needed in the sesquilinear condition for the Burau representation.) On the complex unit circle, inversion is the same as complex conjugation. So for the complex embeddings σ of K, ϕ^σ is complex conjugacy. Notice for a sesquilinear matrix J_t over $\mathcal{O}_K$ with a parameter t, specializing $t = s$ leaves J_s^σ hermitian.

Theorem 1.1 *Let $\rho_t : G \to \mathrm{GL}_m(\mathbb{Z}[t, t^{-1}])$ be a representation of a group G. Suppose there exists a matrix J_t such that:*

(1) $M^* J_t M = J_t$ *for all* M *in the image of* ρ_t.

(2) $J_t = (J_{1/t})^{\intercal}$.

(3) $J_t \in \mathrm{GL}_m(\mathbb{Q}(t))$, *where no entry of* J_t *has denominator with* 1 *as a root.*

(4) J_t *is positive definite for* t *in a neighborhood* η *of* 1 *in* $\mathbb{C}$.

Then there exist infinitely many Salem numbers s such that the specialization ρ_s at $t = s$ is discrete.

Proof The neighborhood η can be chosen so that no entry of J_t has a denominator with a root in η. By Lemma 2.6, there are infinitely many Salem numbers with the property that all the complex Galois conjugates lie in η. Let s be one such Salem number. Specializing t to s gives $\rho_s\colon G \to U_m(J_s, \phi, O_{\mathbb{Q}(s)})$, where ϕ is the usual map given by $s \mapsto 1/s$.

Let σ be a complex place of $\mathbb{Q}(s)$ which is given by $s \mapsto z$ for z a complex Galois conjugate of s. Then $J_s^\sigma = J_z$, and since $z \in \eta$, J_z is positive definite. By Corollary 2.4, the specialization ρ_s at $t = s$ is discrete. $\square$

Remark 2.7 If the representations in Theorem 1.1 all have determinant 1, then the image is more than just discrete, and in fact is a subgroup of a lattice. See Section 4 for more details.

3 Applications to braid group representations

3.1 The Burau representation

Proposition 3.1 *There are infinitely many Salem numbers s such that the Burau representation specialized to $t = s$ is discrete.*

Proof The specialization of $\rho_{n,1}$ at $t = 1$ collapses to an irreducible representation of the symmetric group. As a representation of a finite group, $\rho_{n,1}$ fixes a positive definite form which is unique up to scaling, by Lemma 3.7, which is proved later. At $t = 1$, $J_{n,1}$ is positive definite, and the signature of $J_{n,t}$ can only change at zeroes of its determinant.

An inductive computation shows that $\det(J_{n,t}) = (t^{2n+2} - 1)/(t^n(t^2 - 1))$ for $t \neq 1$, and the zeroes of $\det(J_{n,t})$ occur at $(n+1)^{\text{th}}$ roots of unity. Thus, $J_{n,t}$ remains positive definite for unit complex values of t with argument less than $2\pi/(n + 1)$. This shows the Burau representation satisfies the criteria of Theorem 1.1. $\square$

Example 3.2 The Burau representation $\rho_{4,t}$ of B_4 is discrete when specializing t to the following Salem numbers:

- Lehmer's number raised to the powers 16, 32 and 47,
- the largest real root of $1-x^4-x^5-x^6+x^{10}$ raised to the powers 17, 23, and 43.

3.2 The Jones representations

The *Hecke algebra* (of type A_n), denoted by $H_n(q)$, is the complex algebra generated by invertible elements $g_1, \dots, g_{n-1}$ with relations

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} && \text{for all } i < n, \\ g_i g_j &= g_j g_i && \text{for } |i-j| > 1, \\ g_i^2 &= (1-q) g_i + q && \text{for all } i < n. \end{aligned} \tag{3-1}$$

Here, q is viewed as a complex parameter. $H_n(q)$ is a quotient of $\mathbb{C}[B_n]$ by the relation (3-1). The *Jones representations* of B_n are defined by precomposing a representation of $H_n(q)$ by the quotient map from $\mathbb{C}[B_n]$. The Jones representations have matrix entries in $\mathbb{Z}[t, t^{-1}]$.

3.2.1 The Jones representations are sesquilinear The Hecke algebras have a natural automorphism, denoted here by ϕ, which sends q to $1/q$. Taking q to be a unit complex number, this automorphism becomes complex conjugacy. The Jones representations are known to be sesquilinear with respect to ϕ for various complex specializations of q and with many different types of proofs, as in [2; 3; 9; 17]. To be overtly clear that all of the criteria of Theorem 1.1 are satisfied by the Jones representations, we provide a simple proof of sesquilinearity here that is very similar to [3, Proposition 3.7] by Brunat and Marin.

The irreducible representations of $H_n(q)$ are parametrized by the Young diagrams. (For a more detailed discussion of Young diagrams see [18], and for a construction of the Jones representations see [16].) Every Young diagram contains sub-Young diagrams, obtained by removing boxes in a way that retains the weakly decreasing row length condition. If λ is a Young diagram with n boxes, then we will call the sub-Young diagrams found by removing one box from λ the $(n-1)$–*subdiagrams of* λ. A Young diagram is completely determined by any two of its $(n-1)$–subdiagrams. These $(n-1)$–subdiagrams also determine representations of the Hecke algebras in a powerful way. The following theorem, originally due to Curtis, Iwahori and Kilmoyer in [5] and popularized by Jones in [8], states concretely the relationship between Young diagrams and the representations of the Hecke algebras.

Theorem 3.3 *Up to equivalence, the finite-dimensional irreducible representations of $H_n(q)$, for generic q, are in one to one correspondence with the Young diagrams of n boxes. Moreover, if ρ is a representation corresponding to Young diagram λ, then ρ*

restricted to $H_{n-1}(q)$ is equivalent to the representation $\bigoplus_{i=1}^{k} \rho_{\lambda_i}$, where $\lambda_1, \dots, \lambda_k$ are all of the $(n-1)$–subdiagrams of λ and each ρ_{λ_i} is an irreducible representation of $H_{n-1}(q)$ corresponding to λ_i.

Here equivalence means the existence of an intertwining isomorphism, made precise by the following definition.

Definition 3.4 The representations $\varphi: G \to \mathrm{GL}(V)$ and $\psi: G \to \mathrm{GL}(W)$ are *equivalent* if there exists a linear isomorphism $T: V \to W$ such that $T\varphi(g)(v) = \psi(g)T(v)$ for all $g \in G$ and $v \in V$, or that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi(g)"] \arrow[d, "T"] & V \arrow[d, "T"] \\ W \arrow[r, "\psi(g)"] & W \end{tikzcd}$$

Choosing bases for V and W, the equivalence T gives the matrix equation

$$[T][\varphi(g)][T]^{-1} = [\psi(g)].$$

At the level of matrices, representations are equivalent exactly when they are simultaneously conjugate. In the context of Theorem 3.3, the restriction of ρ to $H_{n-1}(q)$ is equivalent to the representation $\bigoplus_{i=1}^{k} \rho_{\lambda_i}$, which means there is a change of basis such that the restriction of ρ is block diagonal. These restriction rules are combinatorially depicted in the Young lattice of Young diagrams; see [18].

A representation is sesquilinear if there exists an invertible matrix J such that for every M in the image of the representation the following equation is satisfied:

$$M^* J M = J. \tag{3-2}$$

Rearranging this equation, we see that $M = J^{-1}((M^{\phi})^{\mathsf{T}})^{-1} J$, showing that M and $((M^{\phi})^{\mathsf{T}})^{-1}$ are simultaneously conjugate and the conjugating matrix J is the sesquilinear form. Changing views slightly, consider the following definition.

Definition 3.5 For $\varphi: G \to \mathrm{GL}(V)$ a complex linear representation, $\tilde{\varphi}: G \to \mathrm{GL}(V^*)$ is called the *ϕ–twisted contragredient representation* of φ and is given by

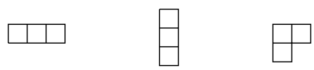
$$\tilde{\varphi}(g) f(v) = f(\varphi(g^{-1})^{\phi} v)$$

for every $g \in G$, $v \in V$ and $f \in V^*$.

If a basis for V is chosen, then as matrices, $[\tilde{\varphi}(g)] = ([\varphi(g)^{\phi}]^{\mathsf{T}})^{-1}$. So another way to view a sesquilinear representation is one that is *equivalent to its ϕ–twisted contragredient*.

Lemma 3.6 *Every finite-dimensional irreducible representation of the Hecke algebra is equivalent to its ϕ–twisted contragredient representation, when q is a generic complex number.*

Proof We can establish this result for $n = 3$. There are three nonequivalent irreducible representations of $H_3(q)$ corresponding to the following Young diagrams:



Up to equivalence, the first two representations are 1–dimensional, given by $g_i \mapsto q$ and $g_i \mapsto -1$, and are in fact equal to their ϕ–twisted contragredient representations. The third representation is known to be the Burau representation for B_3. As described earlier, Squier showed that the Burau representations are sesquilinear and are therefore equivalent to their ϕ–twisted contragredient.

Inductively moving forward, let $\rho\colon H_n(q) \to \mathrm{GL}(V)$ be a finite-dimensional irreducible representation and $\tilde{\rho}$ be the ϕ–twisted contragredient representation of ρ. Up to equivalence, ρ corresponds to a Young diagram λ. To show that ρ and $\tilde{\rho}$ are equivalent, it suffices to show that both representations correspond to the same λ. A Young diagram is completely characterized by its list of $(n-1)$–subdiagrams, which correspond to the restriction of the representation to $H_{n-1}(q)$. So it is enough to show that the restrictions of ρ and $\tilde{\rho}$ correspond to the same list of $(n-1)$–subdiagrams.

Writing $\rho| = \rho|_{H_{n-1}(q)}$, by Theorem 3.3 there is an equivalence T such that

$$T\rho|(h)T^{-1} = \bigoplus_{i=1}^{k} \rho_{\lambda_i}(h) \quad \text{for every } h \in H_{n-1}(q),$$

where each λ_i is an $(n-1)$–subdiagram of λ, k is the number of $(n-1)$–subdiagrams of λ, and ρ_{λ_i} is an irreducible representation of $H_{n-1}(q)$ corresponding to λ_i. Choosing a basis for V, the matrix for $[T\rho \mid (h)T^{-1}]$ is block diagonal. Taking the ϕ–twisted contragredient of a block diagonal matrix preserves the block decomposition, which gives

$$([T^{\phi}]^{\mathsf{T}})^{-1}[\tilde{\rho} \mid (h)][T^{\phi}]^{\mathsf{T}} = \bigoplus_{i=1}^{k} [\tilde{\rho}_{\lambda_i}(h)] \quad \text{for every } h \in H_{n-1}(q).$$

This equation shows that $\tilde{\rho}|$ is equivalent to $\bigoplus \tilde{\rho}_{\lambda_i}$. Since each ρ_{λ_i} is an irreducible representation of $H_{n-1}(q)$, we can inductively assume that ρ_{λ_i} is equivalent to $\tilde{\rho}_{\lambda_i}$, for all $i \leq k$. Therefore, ρ_{λ_i} and $\tilde{\rho}_{\lambda_i}$ correspond to the same Young diagram λ_i. Thus the restrictions of ρ and $\tilde{\rho}$ correspond to the same list of $(n-1)$–subdiagrams. $\square$

Lemma 3.7 *If an absolutely irreducible matrix representation has an invertible matrix J satisfying $M^* J M = J$ for all M in the representation, then J is unique up to scaling.*

Proof Suppose there were two such matrices J_1 and J_2. Then (3-2) gives, for all matrices M in the representation,

$$J_1 M J_1^{-1} = ((M^{\phi})^{\mathsf{T}})^{-1} = J_2 M J_2^{-1} \implies (J_1^{-1} J_2)^{-1} M (J_1^{-1} J_2) = M.$$

This shows that $J_1^{-1} J_2$ is in the centralizer of the entire irreducible representation. Schur's lemma gives that $J_1^{-1} J_2 = \alpha \cdot \mathrm{Id}$ for some scalar α, and finally $J_2 = \alpha J_1$. $\square$

Proposition 3.8 *If ρ is an irreducible Jones representation of B_n and q is a generic unit complex number close to 1, then there exists a nondegenerate, positive definite, sesquilinear matrix J with entries in $\mathbb{Q}(q)$ such that for all M in the image of ρ, $(M^{\phi})^{\mathsf{T}} J M = J$.*

Proof Let ρ be a finite-dimensional irreducible representation of $H_n(q)$ over V. By Lemma 3.6, ρ is equivalent to its ϕ–twisted contragredient representation $\tilde{\rho}$ by an equivalence T. Choose a basis for V and its dual basis for V^*, and let $\mathcal{T}$ be the matrix for T with respect to these bases. We will use this matrix $\mathcal{T}$ to find the desired matrix J. Let the superscript $*$ denote the ϕ–twisted transpose of a matrix to ease computation. For all $g \in H_n(q)$, we get the matrix equations

$$\begin{aligned} \mathcal{T}[\rho(g)]\mathcal{T}^{-1} = [\tilde{\rho}(g)] = ([\rho(g)]^{-1})^* &\implies (\mathcal{T}^{-1})^*[\rho(g)]^*\mathcal{T}^* = [\rho(g)]^{-1} \\ &\implies \mathcal{T}^*[\rho(g)](\mathcal{T}^*)^{-1} = ([\rho(g)]^{-1})^*. \end{aligned} \tag{3-3}$$

This shows that $\mathcal{T}$ and $\mathcal{T}^*$ are two possible forms for ρ. By Lemma 3.7, $\mathcal{T} = \alpha \mathcal{T}^*$ for some $\alpha \in \mathbb{C}$. Applying $*$ again gives $\mathcal{T} = \alpha\alpha^* \mathcal{T}$ and $\alpha\alpha^* = 1$.

Define $J = \beta\mathcal{T} + \beta^*\mathcal{T}^* = (\alpha\beta + \beta^*)\mathcal{T}^*$ where β is as follows. (Here β is needed to ensure that J is invertible.) If $\alpha \neq -1$, let $\beta = 1$, which gives that $\det J = \det((\alpha+1)\mathcal{T})$, which is nonzero. If $\alpha = -1$, let $\beta \in \mathbb{C}$ be such that $\beta^* \neq \beta$. Then

$$\det J = \det[(\alpha\beta + \beta^*)\mathcal{T}^*] = \det[(-\beta + \beta^*)\mathcal{T}]$$

is nonzero. So, in both cases, J is invertible.

Next note that J is sesquilinear, that is, $J^* = (\beta\mathcal{T} + \beta^*\mathcal{T}^*)^* = \beta^*\mathcal{T}^* + \beta\mathcal{T} = J$. If M is a matrix in the image of ρ, rearranging the right-hand equation of (3-3) gives $M^*\mathcal{T}^*M = \mathcal{T}$. So inserting J gives

$$M^*JM = M^*(\alpha\beta + \beta^*)\mathcal{T}^*M = (\alpha\beta + \beta^*)M^*\mathcal{T}^*M = (\alpha\beta + \beta^*)\mathcal{T} = J.$$

To show that the entries of J are in $\mathbb{Q}(q)$ we will proceed by induction on n, as in the proof of Lemma 3.6. As a base case with $n = 3$, Squier's form for the Burau representation has entries in $\mathbb{Q}(q)$. Let ρ be an irreducible Jones representation of B_n with $\rho|_{B_{n-1}} = \bigoplus_{i=1}^k \rho_i$, where each ρ_i is an irreducible Jones representation of B_{n-1}. We can inductively assume each ρ_i is sesquilinear with form J_i whose coefficients are in $\mathbb{Q}(q)$. Thus, there exist some scalars α_i such that $J = [\alpha_i J_i]$, the block diagonal matrix, and J is the sesquilinear form for ρ.

It remains to show that J is positive definite. Taking $q = 1$, ρ is an irreducible representation of the symmetric group Σ_n. As a linear representation of a finite group, V admits an inner product that is invariant under the action of Σ_n, given by a positive definite nondegenerate matrix $\hat{J}$. Lemma 3.7 guarantees that $\hat{J}$ is unique up to scaling. Since $J|_{q=1}$ is also a form for this representation, it must be that $\hat{J}$ is a multiple of $J|_{q=1}$, which gives that J is positive definite for $q = 1$. Since J is Hermitian for unit complex q it has real eigenvalues, and continuity of the determinant map finally gives that either J or $-J$ is positive definite for q close to 1. □

Corollary 3.9 *For each irreducible Jones representation, there are infinitely many Salem numbers s such that specializing $q = s^m$, for some m, is a discrete representation.*

3.3 The Lawrence–Krammer and BMW representations

The BMW algebras are a 2–parameter family of algebras, denoted by $C_n(l, m)$, with $n - 1$ generators and parameters l and m. The BMW representations of the braid group come from representations of the BMW algebras [1; 12]. Similar to how the Burau representation is one irreducible summand of the Jones representations, Zinno proved in [19] that the Lawrence–Krammer representation is one summand of the BMW representations. To make sense of the $*$ operation, the relevant involution for the BMW algebra is given by $l \mapsto 1/l$, $m \mapsto m$ and $\alpha \mapsto 1/\alpha$, where $m = \alpha + 1/\alpha$. Budney proved that the Lawrence–Krammer representation is sesquilinear [4], and Brunat and Marin give a more general proof that all the BMW representations are sesquilinear [3]; see also [10]. It is also known that the sesquilinear forms $J_{l,m}$ are positive definite for a neighborhood of $(1,1)$ in the unit complex sphere in $\mathbb{C}^2$ [17]; see [2, Theorem 1.2] for a concise restatement. It is suspected that the forms $J_{l,m}$ have coefficients in $\mathbb{Q}(m,l)$, and this is known to be true for the Lawrence–Krammer representation.

Corollary 3.10 *For the Lawrence–Krammer representation there are infinitely many Salem numbers s such that specializing $l = s^{k_1}$ and $\alpha = s^{k_2}$, for some k_1 and k_2, is a discrete representation.*

Example 3.11 Let ρ be the Lawrence–Krammer representation of B_4 given on page 272 of [1]. Taking the Salem number $S = 1/2 + 1/\sqrt{2} + 1/(2\sqrt{-1+2\sqrt{2}})$, specializing $\alpha = S^{15}$ and $l = S^3$ makes ρ a discrete representation.

4 Commensurability

Ideally, we would like to find *real* specializations so that the Jones representations have images that are not just discrete, but are arithmetic groups or lattices in $\mathrm{GL}_n(\mathbb{R})$. A first step in this direction is to further study the unitary groups coming from Salem number specializations, and consider when the images are subsets of lattices. Specializing to two different powers of the same Salem number can give *commensurable* unitary groups, but the defining sesquilinear forms might be very different.

Recall the notation of K, L, $\mathcal{O}_K$ and ϕ from Section 2.3. In general, fixing a number ring $\mathcal{O}_K$ and dimension m, the group $U_m(J,\phi,\mathcal{O}_K)$ is determined by the form J. Notice that $U_m(J,\phi,\mathcal{O}_K) = U_m(\lambda J,\phi,\mathcal{O}_K)$ for every $\lambda \in L$, and that the form J is not completely unique. This motivates that following definition.

Definition 4.1 Matrices J and H are equivalent over K if $Q^*JQ = \lambda H$ for some $Q \in \mathrm{GL}_m(K)$ and $\lambda \in \mathrm{Fix}(\phi)$.

It would be nice if equivalent forms gave rise to *equal* unitary groups, but this is not true in general. However, in the careful scenario that the unitary group is a lattice in $\mathrm{SL}_m(\mathbb{R})$, changing the form by equivalence yields "the same" lattice, up to commensurability in the following sense.

Definition 4.2 Two groups G_1 and G_2 are *commensurable* if there are finite-index subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that H_1 is isomorphic to H_2.

Definition 4.3 A *lattice* in a semisimple Lie group is a discrete subgroup with finite covolume.

For our purposes, we will take $\mathrm{SL}_m(\mathbb{R})$ or $\mathrm{PSL}_m(\mathbb{R})$ as the semisimple Lie group.

Proposition 4.4 *Assume that* $\mathrm{SU}_m(J_1, \phi, \mathcal{O}_K)$ *and* $\mathrm{SU}_m(J_2, \phi, \mathcal{O}_K)$ *are lattices in* $\mathrm{SL}_m(\mathbb{R})$*. If* J_1 *and* J_2 *are equivalent over* K*, then* $\mathrm{SU}_m(J_1, \phi, \mathcal{O}_K)$ *is commensurable to* $\mathrm{SU}_m(J_2, \phi, \mathcal{O}_K)$

Proof Let $\lambda J_1 = Q^* J_2 Q$ for some $Q \in \mathrm{GL}_m(K)$ and $\lambda \in \mathrm{Fix}(\phi)$. For notational clarity, write $\mathrm{SU}(J_i, \mathcal{O}_K) = \mathrm{SU}_m(J_i, \phi, \mathcal{O}_K)$.

Since scalar multiplication commutes with matrix multiplication, $M^* J M = J$ if and only if $M^* \lambda J M = \lambda J$. So scaling the form preserves the unitary group, and without loss of generality we may assume $\lambda = 1$.

It is clear that $M^* J M = J$ if and only if $(Q^* M^* Q^{*-1})(Q^* J Q)(Q^{-1} M Q) = Q^* J Q$, which seems like it implies that $\mathrm{SU}(Q^* J_1 Q, \mathcal{O}_K) = Q^{-1}\,\mathrm{SU}(J_1, \mathcal{O}_K) Q$. However, since Q has coefficients in K, $Q^{-1} M Q$ may not have coefficients in $\mathcal{O}_K$, so we can only conclude that $Q^{-1}\,\mathrm{SU}(J, \mathcal{O}_K) Q \subseteq \mathrm{SU}(Q^* J Q, K)$. To avoid this, we need to pass to a finite-index subgroup.

Since K is the ring of fractions of $\mathcal{O}_K$, there exists $\gamma \in \mathcal{O}_K$ such that $\gamma Q \in M_m(\mathcal{O}_K)$. As a ring of integers of an algebraic extension, $\mathcal{O}_K$ is a Dedekind domain and every quotient is finite. So $\mathcal{O}_K / \langle \gamma^2 \rangle$ is finite and $\mathrm{SU}(J_1, \mathcal{O}_K / \langle \gamma^2 \rangle)$ is finite. The kernel N of the quotient map $\mathrm{SU}(J_1, \mathcal{O}_K) \to \mathrm{SU}(J_1, \mathcal{O}_K / \langle \gamma^2 \rangle)$ has finite index in $\mathrm{SU}(J_1, \mathcal{O}_K)$.

Any element B in the kernel has the form $B = \mathrm{Id} + \gamma^2 A$ for some matrix $A \in M_m(\mathcal{O}_K)$. Substituting $Q^* J_2 Q$ for J_1 in the equation $B^* J_1 B = J_1$ gives that $Q B Q^{-1}$ fixes the form J_2. Because Q has coefficients over K, $Q B Q^{-1}$ has coefficients in K and not necessarily in $\mathcal{O}_K$. However, since $Q B Q^{-1} = \mathrm{Id} + (\gamma Q) A (\gamma Q^{-1})$, and both A and γQ are integral, $Q B Q^{-1}$ is also integral. Thus $Q B Q^{-1} \in \mathrm{SU}(J_2, \mathcal{O}_K)$.

Since $\mathrm{SU}(J_1, \mathcal{O}_K)$ is a lattice and N is a finite-index subgroup, N is also a lattice in $\mathrm{SL}(\mathbb{R})$ with finite covolume. Thus $Q N Q^{-1}$ has finite covolume in $\mathrm{SL}(\mathbb{R})$ and is therefore a lattice. So $Q N Q^{-1}$ is a sublattice of $\mathrm{SU}(J_2, \mathcal{O}_K)$ and must have finite index by Margulis's theorem for lattices.

This shows that N is a finite-index subgroup of $\mathrm{SU}(J_1, \mathcal{O}_K)$ and $Q N Q^{-1}$ has finite index in $\mathrm{SU}(J_2, \mathcal{O}_K)$. $\square$

So how does this lattice information apply to the Jones representations? Firstly, after rescaling and reparametrization the Jones representations can be made to have determinant ± 1, allowing the image to land in $\mathrm{PSU}(J,\phi,\mathcal{O}_K)$ instead of just $U(J,\phi,\mathcal{O}_K)$. Secondly, an arithmetic group theory result of Harish and Chandra that is formalized in our setting in Chapter 6 of Morris [11], states that $\mathrm{SU}_m(J,\phi,\mathcal{O}_K)$ is a lattice in $\mathrm{SL}_m(\mathbb{R})$ under the exact Salem number circumstances as required by Theorem 1.1. So Corollary 3.9 can be restated using this new vocabulary.

Corollary 4.5 *For each irreducible Jones representation, after a change of parameter, there are infinitely many Salem numbers s such that specializing q to a power of s maps into a lattice in $\mathrm{PSL}_m(\mathbb{R})$.*

Proof Let ρ_q be an irreducible Jones representation of dimension m. The images of the braid generators under ρ_q have determinant $\pm q^k$ for some $k \in \mathbb{N}$. After a change of variable $q = y^m$ and scaling the generators by $1/y^{m-k}$, this adjusted representation $\tilde{\rho}_y$ maps into $\mathrm{PSU}_m(J^y, \mathbb{Z}[y^{\pm 1}])$.

The subgroup B_n^{even} of even braids (the preimage of the alternating group under the standard projection to S_n) is a noncentral normal subgroup of B_n of finite index. The restriction $\tilde{\rho}_y|$ maps B_n^{even} into $\mathrm{SU}_m(J_y, \mathbb{Z}[y^{\pm 1}])$, and by Theorem 1.1 there exist infinitely many Salem numbers s such that the specialization $\rho_s|$ at $y = s$ is discrete. Further, by the results in Chapter 6 of [11] described above, these specializations make $\mathrm{SU}_m(J_s,\mathcal{O}_K)$ lattices in $\mathrm{SL}_m(\mathbb{R})$. Finite-index arguments imply $\mathrm{PSU}_m(J_s,\mathcal{O}_K)$ is a lattice in $\mathrm{PSL}_m(\mathbb{R})$. □

Since the goal is to obtain commensurable lattices as images of our Jones representations, and it is more natural to think of lattices in $\mathrm{SL}_m(\mathbb{R})$ instead of in $\mathrm{PSL}_m(\mathbb{R})$, one may simply pass to the finite-index subgroup B_n^{even} and continue to think only about lattices in $\mathrm{SL}_m(\mathbb{R})$. To apply Proposition 4.4 requires equivalent defining forms. In general, it is difficult to determine when two forms are equivalent. The following theorem gives a complete classification of the sesquilinear forms in a very specific algebraic setting that applies to the Salem number field scenario.

Theorem 4.6 (Scharlau [13, Chapter 10]) *If L is a global field and $K = L(\sqrt{\delta})$, sesquilinear forms over K/L are classified by dimension, determinant class and the signatures for those orderings of L for which δ is negative.*

This classification relies on the *determinant class*, which is defined here. Recall for a Salem number s the following tower of fields:

$$\begin{array}{ccc} \mathbb{Q}(s) = K & K^\times & \\ \Big| \, 2 & \Big\downarrow \text{Norm} & \\ \mathbb{Q}(s+1/s) = L & L^\times & (L^\times)^2 \subseteq \mathrm{Norm}(K^\times) \\ \Big| & & \\ \mathbb{Q} & & \end{array}$$

The Galois group of K/L is generated by ϕ, which maps s to $1/s$. There is a multiplicative group homomorphism $\mathrm{Norm}\colon K^\times \to L^\times$ given by $\mathrm{Norm}(\alpha) = \alpha\alpha^\phi$, where $K^\times = K - \{0\}$. Notice for $\beta \in L$ we have $\mathrm{Norm}(\beta) = \beta\beta^\phi = \beta^2$, and so $(L^\times)^2 \subseteq \mathrm{Norm}(K)$.

Definition 4.7 The *determinant class* of a sesquilinear form H over K/L is the coset of $\det(H)$ in $K^\times/\mathrm{Norm}(K^\times)$:

$$[\det(H)] = \det(H)\,\mathrm{Norm}(K).$$

Taking $\delta = (s - 1/s)^2$, K can be rewritten as $K = L(\sqrt{\delta})$. Thus we can restate Scharlau's classification in the specific context of Salem numbers.

Theorem 4.8 (Scharlau restated) *Sesquilinear forms over K/L are classified by dimension, determinant class and the signatures for those orderings of L for which $(s-1/s)^2$ is negative.*

In odd dimensions it is very simple to show that *all* sesquilinear forms have the same determinant class, up to scaling. However, for even dimensions, the situation is very unclear.

Proposition 4.9 *For every odd-dimensional invertible sesquilinear matrices H and J over K, $[\det(H)] = [\det(\lambda J)]$ for $\lambda \in L$.*

Proof Let H and J be sesquilinear matrices over K of dimension $2k+1$. Since H and J are Hermitian, they are both diagonalizable with diagonal entries fixed by ϕ. So, the determinants of both H and J are elements in L. Let d_H and d_J denote the nonzero determinants of H and J. Thus,

$$d_H = \frac{d_H}{d_J} d_J \overset{\mathrm{mod}(L^\times)^2}{\equiv} \left(\frac{d_H}{d_J}\right)^{2k+1} d_J = \det\!\left(\frac{d_H}{d_J} J\right).$$

Since $(L^\times)^2 \subseteq \mathrm{Norm}(K)$, we have that H and λJ have the same determinant class for $\lambda = d_H/d_J \in L$. $\square$

As a result, to determine whether two forms of the same odd dimension are equivalent, it suffices to only check that they have the same signatures.

Theorem 4.10 *For J_t a sesquilinear form that is positive definite for t in a neighborhood η of 1, there are infinitely many Salem numbers s and integers n and m such that, in all odd dimensions, $\mathrm{SU}_{2k+1}(J_{s^n}, \phi, \mathcal{O}_K)$ and $\mathrm{SU}_{2k+1}(J_{s^m}, \phi, \mathcal{O}_K)$ are commensurable, discrete groups.*

Proof By Lemma 2.6 there are infinitely many Salem numbers s and integers n and m such that every complex Galois conjugate of s^m and s^n lies in η. Fix one such Salem number s, and K, L and δ as above.

By Theorem 4.6, sesquilinear forms are completely classified by dimension, determinant class, and the signatures for the places of L for which $(s-1/s)^2$ is negative. By Proposition 4.9, J_{s^n} and λJ_{s^m} have the same determinant class for λ in L, namely $\lambda = \det J_{s^n}/\det J_{s^m}$.

Let σ be a complex placement of L. Then $\sigma(s^m)$ is a complex Galois conjugates of s^m, and similarly for $\sigma(s^n)$ and s^n. Since n and m were chosen so that all of the complex Galois conjugates of s^m and s^n have arguments in η, $J_{\sigma(s^m)}$ and $J_{\sigma(s^n)}$ are positive definite. Moreover, $\det J_{s^n}/\det J_{s^m}$ and $\sigma(\det J_{s^n}/\det J_{s^m})$ are both positive, making $\lambda > 0$. So regardless of whether $\sigma((s-1/s)^2)$ is positive or negative, the forms $J_{\sigma(s^i)}$ have the same signature.

Therefore J_{s^n} is equivalent to λJ_{s^m}, and so $\mathrm{SU}(J_{s^n}, \phi, \mathcal{O}_K)$ is commensurable to $\mathrm{SU}(J_{s^m}, \phi, \mathcal{O}_K)$. The groups are discrete by Theorem 2.3. $\square$

Corollary 4.11 *Let $\rho_t\colon G \to \mathrm{SL}_{2k+1}(\mathbb{Z}[t, t^{-t}])$ be a group representation with a parameter t. Suppose there exists a matrix J_t such that:*

1. *For all M in the image of ρ_t, $M^* J_t M = J_t$, where $M^*(t) = M^\intercal(1/t)$.*
2. $J_t = (J_{1/t})^\intercal$.
3. *$J_t \in \mathrm{GL}_m(\mathbb{Q}(t))$, where no entry of J_t has denominator with 1 as a root.*
4. *J_t is positive definite for t in an neighborhood η of 1.*

Then there exist infinitely many Salem numbers s such that for infinitely many integers n and m the specializations ρ_{s^m} at $t = s^m$ and ρ_{s^n} at $t = s^n$ map into commensurable lattices of $\mathrm{SL}_{2k+1}(\mathbb{R})$.

Example 4.12 The reduced Burau representation of B_4 is 3–dimensional and, after the appropriate rescaling to have determinant 1, satisfies Corollary 4.11. So certain powers of the specializations in Example 3.2 map into commensurable lattices in $\mathrm{SL}_3(\mathbb{R})$.

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