Algebraic & Geometric Topology

Volume 23 (2023)

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Algebraic & Geometric Topology **23**:5 (2023) 2157–2190
DOI: 10.2140/agt.2023.23.2157
Published: 25 July 2023

Modifications preserving hyperbolicity of link complements

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Given a link in a 3–manifold such that the complement is hyperbolic, we provide two modifications to the link, called the chain move and the switch move, that preserve hyperbolicity of the complement, with only a relatively small number of manifold-link pair exceptions, which are also classified. These modifications provide a substantial increase in the number of known hyperbolic links in the 3–sphere and other 3–manifolds.

57K10, 57K32

1 Introduction

Thurston proved that every knot in the 3–sphere S^3 is either a torus knot, a satellite knot or a hyperbolic knot; by which we mean that its complement in S^3 admits a complete hyperbolic metric. By the Mostow–Prasad rigidity theorem, the complement of a hyperbolic knot in S^3 has a unique hyperbolic metric, which must have finite volume; hence, a hyperbolic knot in S^3 has associated to it a well-defined set of hyperbolic invariants such as volume, cusp volume, cusp shape, etc. More generally, Thurston proved that a link in a closed, orientable 3–manifold has hyperbolic complement (necessarily of finite volume) if and only if the exterior of the link contains no properly embedded essential disks, spheres, tori or annuli — terms that are described in Definition 2.1.

One would like to be able to identify link complements that satisfy Thurston's criteria, and that therefore possess a hyperbolic metric. In [12], Menasco proved that every non-2–braid prime alternating link in S^3 is hyperbolic. In [2], Adams extended this result to augmented alternating links, where additional nonparallel trivial components wrapping around two adjacent strands in the alternating projection were added to the link. These additional components bound twice-punctured disks, which are totally geodesic in the hyperbolic structure of the complement. By Adams [1], the link complement can be

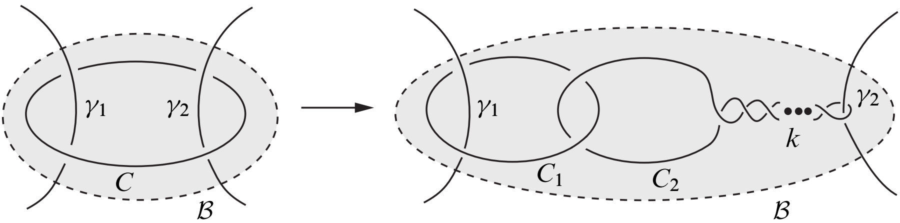

Figure 1: Replacing the left with the right preserves hyperbolicity of the complement.

cut open along such a twice-punctured disk, twisted a half-twist and reglued to obtain another hyperbolic link complement, with identical volume. This operation adds one crossing to the link projection. In many hyperbolic link complements, twice-punctured disks are particularly useful, because they are totally geodesic; see for instance the survey article by Purcell [15] and the references therein.

We consider two moves that one can perform on a link in a 3–manifold with hyperbolic complement. The first move we consider is called the *chain move*. Here, we start with a trivial component bounding a twice-punctured disk in a ball $\mathcal{B}$ as in Figure 1, and we replace the tangle on the left with the tangle on the right in Figure 1, where k is any integer. Assuming that the rest of the manifold outside $\mathcal{B}$ is not the complement of a rational tangle in a 3–ball (see Adams [4, Chapter 2] for this definition), the result is hyperbolic.

There are counterexamples to extending the result to the case where the manifold outside $\mathcal{B}$ is a rational tangle complement in a 3–ball, as demonstrated by the hyperbolic link in the 3–sphere appearing in Figure 2. When the chain move is applied with $k = 3$, the resultant 3–component link is 6^3_3 in Alexander–Rolfsen notation, which is not hyperbolic. However, in Lemma 3.4, we delineate explicitly the only possible exceptions.

The second move is called the *switch move*. Suppose we have a 3–manifold M and a link L in M with hyperbolic complement. Let α be an embedded arc that runs from L to L with interior that is isotopic to an embedded geodesic in the complement, as in Figure 3.

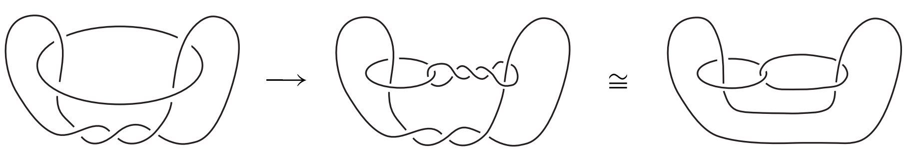

Figure 2: Applying the chain move to this hyperbolic link with $k = 3$ yields the nonhyperbolic link complement 6^3_3.

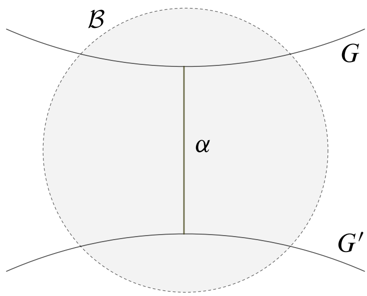

Figure 3: The trace of a geodesic α of $(M \setminus L, h)$ connects one or two components of L to one another, and a neighborhood $\mathcal{B}$ of α intersects L in two arcs.

Such a geodesic always exists since we could take one with minimal length outside fixed cusp boundaries. We consider the possibility that the arc runs from one component of L back to the same component or from one component to a second component. Let $\mathcal{B}$ be a neighborhood of α. Then $\mathcal{B}$ intersects L in two arcs, as in Figure 4, left. The switch move allows us to surger the link and add in a trivial component as in Figure 4, right, while preserving hyperbolicity.

Remark 1.1 The projection depicted in Figure 3 is not well defined, since if the two arcs are skew inside the ball, there are two different projections, depending on point of view. So in fact, for each such geodesic α, there are two switch moves possible. This is equivalent to cutting along the twice-punctured disk D bounded by C and twisting a half-twist in either direction before regluing. Once we prove that the switch move depicted in Figure 4 preserves hyperbolicity, the hyperbolicity of the half-twisted

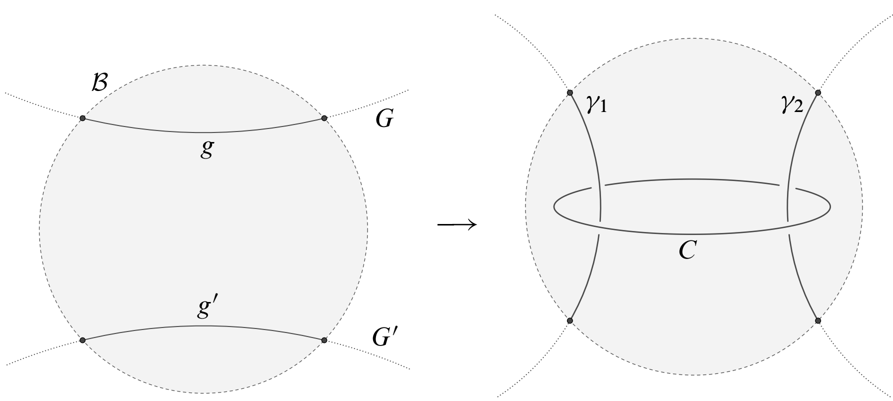

Figure 4: The switch move replaces the arcs g and g' by the tangle $\gamma_1 \cup \gamma_2 \cup C$.

version follows immediately from the previously mentioned results of [1], and the volumes of the resulting manifolds are identical. Further twists give link complements homeomorphic to the original or the half-twisted version.

These moves show that many additional link complements in 3–manifolds are hyperbolic. The authors [6] used the chain move and the switch move, together with the related *switch move gluing operation* described in Section 5, in the proof that for any given surface S of finite topology and negative Euler characteristic and any $H \in [0, 1)$, there exists a proper, totally umbilic embedding of S into some hyperbolic 3–manifold of finite volume with image surface having constant mean curvature H.

Moreover, Adams, Eisenberg, Greenberg, Kapoor, Liang, O'Connor, Pacheco-Tallaj and Wang [5] used the chain move in the proof that a virtual link obtained by taking a reduced classical prime alternating link projection and changing one crossing to be virtual yields a nonclassical virtual link.

We can also use the chain move and the switch move to obtain straightforward proofs of hyperbolicity of well-known classes of links.

Example 1.2 We can show that every chain link of five or more components, no matter how twisted, is hyperbolic. This was first proved by Oertel [14] (or see Neumann and Reid [13] for a proof using explicit hyperbolic structures for manifolds covered by these link complements).

Start with the alternating 4–chain, known to be hyperbolic by Menasco's work in [12]. Then apply the chain move repeatedly. This proves hyperbolicity of any chain link of five or more components with an arbitrary amount of twisting in the chain.

We note that the chain and switch moves apply more broadly than is apparent from Figures 1 and 4. In the case of the chain move, instead of specifying a hyperbolic link complement $M \setminus L$, we can start with a cusped hyperbolic 3–manifold M' containing a two-sided essential embedded thrice-punctured sphere S. Treating two of the boundary curves on the cusps as the meridional punctures of the disk in Figure 4 and the third as the longitudinal boundary of the disk, we can apply the chain move, removing the cusp that contains the longitude by doing a Dehn filling along a curve that crosses the longitude once and adding in the additional two components within a neighborhood of S. In the case that two of the boundaries of S are on the same cusp, they must play the role of the meridional punctures. (Note that if a two-sided thrice-punctured sphere has all three boundaries on the same cusp, no move is possible.)

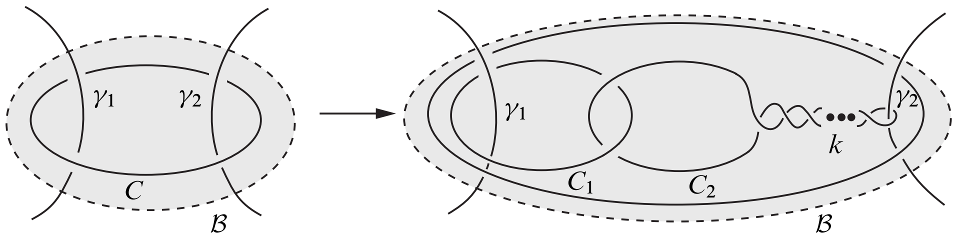

Figure 5: The augmented chain move.

In the case of the switch move, we can again begin with a cusped hyperbolic 3–manifold M'. For two cusps connected by an embedded geodesic, we can choose a nontrivial simple closed curve on each torus corresponding to each cusp. Then by Dehn filling along those curves we obtain a 3–manifold M for which M' is a link complement and the switch move applies.

The same procedure holds for a geodesic from a cusp back to the same cusp, and a specification of a nontrivial simple closed curve on the torus corresponding to the cusp, two copies of which play the role of the meridians around γ_1 and γ_2. Note that when applied to a link complement $M \setminus L$, but with a choice of curve other than meridians, the end result is not a new link complement in the same manifold.

Finally, we point out that there is a variant of the chain move called the *augmented chain move* as in Figure 5 wherein the two new components of the chain move are added in but the previous trivial component is not removed. We prove here that this move also preserves hyperbolicity.

To see this, we consider the link appearing in Figure 6, which is a twisted five-chain.

All five-chains are hyperbolic, as we just proved, so it has a hyperbolic complement. Now we apply the idea of a walnut as in [3]. We can cut the manifold M open along the twice-punctured disk E bounded by C, cut the 5–chain link complement open along the twice punctured disk bounding the bottom component in Figure 6 and then

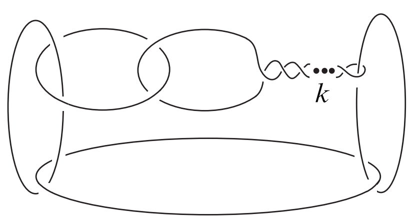

Figure 6: All 5–chains are hyperbolic.

glue copies of the twice-punctured disks to one another to insert the cut-open link complement into M. As in [1], since a twice-punctured disk is totally geodesic with a rigid unique hyperbolic structure, the gluings are isometries and the resulting manifold is hyperbolic with volume the sum of the volumes of the two manifolds.

Next, we explain the organization of the paper. First, we remark that it suffices to demonstrate our results when the ambient manifold is orientable. This property is proved by showing that the oriented cover of a related nonorientable link complement admits a hyperbolic metric and then one applies the Mostow–Prasad rigidity theorem to conclude that the associated order-two covering transformation is an isometry, which in turn implies that the hyperbolic metric on the oriented covering descends. In Section 2, we present some of the background material necessary to the proofs of our main results in the orientable setting. In Section 3, we prove the chain move theorem, stated there as Theorem 3.1. In Section 4, we prove the switch move theorem, Theorem 4.1. In Section 5 we prove the switch move gluing operation, Theorem 5.1, which allows us to glue together two diffeomorphic genus one boundary components from one or two hyperbolic 3–manifolds of finite volume and then operate to generate new hyperbolic 3–manifolds of finite volume.

Acknowledgements Thanks to the referee for very thorough and helpful comments, including a much shorter proof of Lemma 3.4 and a strengthened version of Theorem 5.1. Meeks and Ramos were partially supported by the CNPq Brazil, grant 400966/2014-0.

2 Preliminaries

In this section, we recall some definitions and results that are needed to understand hyperbolic 3–manifolds of finite volume and certain embedded surfaces in such ambient spaces. Our first goal is to understand the statement of Thurston's hyperbolization theorem in our setting. Before stating this result, we first explain some of the definitions and notations we use. Throughout this discussion, P will denote a connected, orientable, compact 3–manifold with nonempty boundary ∂P consisting of tori and $\operatorname{int}(P)$ will denote the interior of P. Moreover, a *surface* Σ *in* P means a properly embedded surface $\Sigma \subset P$, ie Σ is embedded in P with $\partial\Sigma = \Sigma \cap \partial P$.

Definition 2.1 (1) Given a surface Σ in P, a *compression disk for* Σ is a disk $E \subset P$ with $\partial E = E \cap \Sigma$ such that ∂E is homotopically nontrivial in Σ. If Σ does not admit any compression disk, we say Σ is *incompressible*.

(2) Given a surface Σ in P, a *boundary-compression disk for* Σ is a disk $E \subset P$ with $\partial E = E \cap (\Sigma \cup \partial P)$ such that $\partial E = \alpha \cup \beta$, where α and β are arcs intersecting only in their endpoints such that $\alpha = E \cap \Sigma$ and $\beta = E \cap \partial P$ and α does not cut a disk from Σ. If Σ does not admit any boundary-compression disk, we say Σ is *boundary-incompressible*.

(3) A torus T in P is *boundary parallel* if T is isotopic to a boundary component of P.

(4) An annulus A in P is *boundary parallel* if A is isotopic, relative to ∂A, to an annulus $A' \subset \partial P$.

(5) A sphere S in P is *essential* if S does not bound a ball in P.

(6) A disk E in P is *essential* if ∂E is homotopically nontrivial in ∂P.

(7) A torus T is *essential* in P if T is incompressible and not boundary parallel.

(8) An annulus A is *essential* in P if A is incompressible, boundary-incompressible and not boundary parallel.

Using the above definitions, Thurston's hyperbolization theorem implies that a connected, orientable, noncompact 3–manifold N admits a hyperbolic metric of finite volume if and only if N is diffeomorphic to $\mathrm{int}(P)$ as above and there are no essential spheres, disks, tori or annuli properly embedded in P. In this case, we shall say that N is *hyperbolic*. When a link L in a 3–manifold M has hyperbolic complement, we will say either $M \setminus L$ is hyperbolic, or L is hyperbolic.

A useful fact is that if α is an arc with endpoints in a link L in a 3–manifold M such that α corresponds to a geodesic in the hyperbolic link complement $M \setminus L$, then α cannot be homotoped through $M \setminus L$ into L while fixing its endpoints on L. This follows from the fact any such geodesic will lift to geodesics connecting distinct horospheres in the universal cover $\mathbb{H}^3$, whereas an arc that is homotopic into L will lift to arcs, each of which connects one and the same horosphere.

In the case that a manifold M has no essential disks, we say it is *boundary-irreducible*. In the case that a manifold M has no essential spheres, we say it is *irreducible*. Note that if M has only toroidal boundaries and it is not a solid torus, which is the situation we will consider, irreducibility implies boundary-irreducibility. This is because if there exists an essential disk D with boundary in a torus $T \subset \partial M$, then $\partial N(D \cup T)$ is a sphere which must bound a ball to the non-D side, implying M is a solid torus. Here and elsewhere, $N(G)$ denotes a regular neighborhood of a set $G \subset M$.

Given an annulus A properly embedded in an irreducible manifold M with toroidal boundary, we note that if A is boundary-compressible, it is boundary-parallel. This follows because we can surger the annulus along the boundary-compressing disk to obtain a properly embedded disk D, with trivial boundary on ∂M. Then ∂D bounds a disk D' in ∂M, and $D \cup D'$ is a sphere bounding a ball. This allows us to isotope A relative ∂A into ∂M.

Finally, we remark that if S is a two-sided, incompressible surface properly embedded in an irreducible manifold M with toroidal boundary, then either S is boundary-incompressible or S is a boundary parallel annulus; see for instance [11, Lemma 1.10].

3 The chain move theorem

Let L be a hyperbolic link in a 3–manifold M and let $\mathcal{B} \subset M$ be a ball in M that intersects L as in Figure 1, left. In this section we prove the chain move theorem, as stated by Theorem 3.1 below. The proof breaks up into two cases depending on whether or not the pair $(M \setminus \mathcal{B}, L \setminus \mathcal{B})$ is a rational tangle in a 3–ball; see [4, Chapter 2] for this definition and for the representation of a rational tangle by a sequence of integers.

Theorem 3.1 (chain move theorem) *Let L be a link in a 3–manifold M such that the link complement $M \setminus L$ admits a complete hyperbolic metric of finite volume. Suppose that there is a sphere $\mathcal{S}$ in M bounding a ball $\mathcal{B}$ that intersects L as in Figure 1, left. Let L' be the resulting link obtained by replacing $L \cap \mathcal{B}$ by the components as appear in Figure 1, right. Then if $(M \setminus \mathcal{B}, L \setminus (\mathcal{B} \cap L))$ is not any of the rational tangles $-k$, $-(k+1)$, or $-2-k$ in a 3–ball, then $M \setminus L'$ admits a complete hyperbolic metric of finite volume.*

In Figure 7, top, we see the new link components that are inserted into the ball $\mathcal{B}$. In Figures 7, bottom, we see, for any fixed integer k, the three cases of rational tangles in the exterior 3–ball that do not yield a hyperbolic link complement.

Remark 3.2 The crossings around the single trivial component need not be nonalternating for Theorem 3.1 to apply. If the crossings alternate (as shown in Figure 8, left), we could add a crossing to γ_2 and work in a subball as in Figure 8, right, so that the crossings are those shown in Figure 1, left.

Remark 3.3 Repeated application of the chain move theorem allows us to create a hyperbolic link complement with an arbitrarily long chain of trivial components and

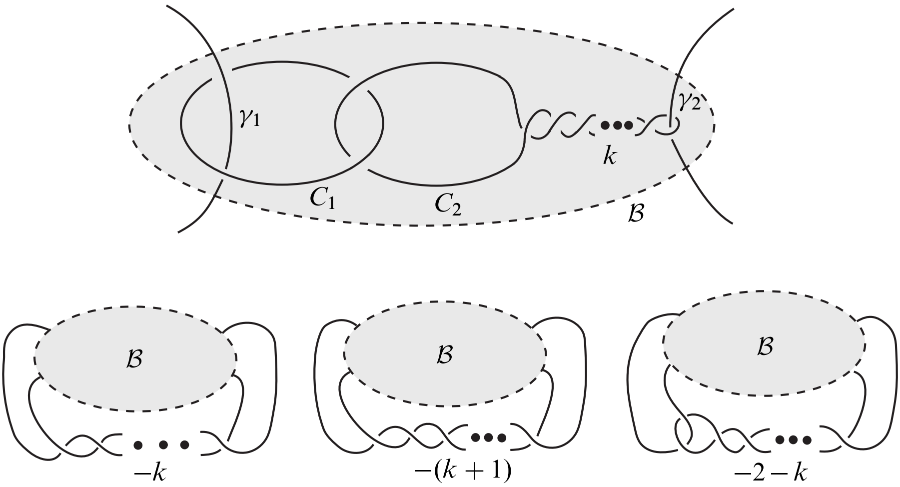

Figure 7: The link components we are inserting in $\mathcal{B}$ and the three rational tangles in an exterior ball that do not generate a hyperbolic link complement.

with any amount of twist. Moreover, if the original exterior tangle is assumed not to be rational, the subsequent exterior tangles to which we apply the move cannot be rational either, so all resulting link complements are hyperbolic. In fact, even if the initial exterior tangle is rational, if our first application of the move results in a hyperbolic link complement, all repeated applications will also be hyperbolic.

We set the stage for the proof of Theorem 3.1 with the following lemma.

Lemma 3.4 *Let L be a link in the* 3*–sphere such that the tangle $R = L \setminus \mathcal{B}$ is a rational tangle and the tangle $L \cap \mathcal{B}$ is the tangle T_k appearing in Figure 1, right, for some*

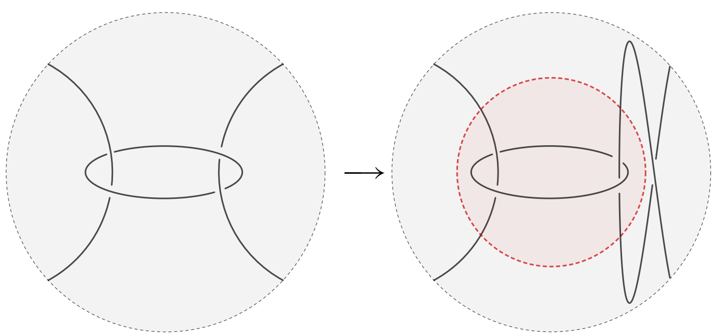

Figure 8: Using an isotopy within $\mathcal{B}$ to obtain a subball where the chain move theorem applies.

integer k. If R is none of the rational tangles ∞, $-k$, $-(k+1)$ or $-2-k$, the link complement is hyperbolic.

Proof We represent rational tangles by a fraction $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$. We also use the notation $K(p_1/q_1, \dots, p_n/q_n)$ to denote the Montesinos link created by the tangles $p_1/q_1, \dots, p_n/q_n$. For more details, see [14] or [16].

Note that if L is as stated in Lemma 3.4, then it is a Montesinos link of either three or four components. Furthermore, after untwisting the k half-twists into R, the rational tangles in $\mathcal{B}$ are $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$ and R is also a rational tangle. Thus, there exists $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ such that L is equivalent to $K\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{p}{q}\right)$. If $\frac{p}{q} = \infty$, then L is not prime (and not hyperbolic).

Next, we use the classification of all the nonhyperbolic Montesinos links given by work of Bonahon and Siebenmann [8] (or see [9] for a different proof) and Oertel [14, Corollary 5] to analyze the possibilities for $\frac{p}{q} \in \mathbb{Q}$ for which L is not hyperbolic. In [8], the Montesinos "torus links" are determined, all of which are nonhyperbolic. These include torus links in the usual sense but additionally allowing for the inclusion of the core curves of the solid tori to either side of the defining torus. In [14], the nonhyperbolic Montesinos links that are not "torus links" are determined. See [16] (Theorem 4.1 and the following paragraph) for a complete list of the nonhyperbolic Montesinos links.

If L has three components, then γ_1 and γ_2 are in the same component $C_3 \subset L$. But the only nonhyperbolic Montesinos links of three components are $L = K\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{m}\right)$, for $m \in 2\mathbb{N}$, $L = K\left(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, or their mirror images. Since $\operatorname{lk}(C_1, C_2) = \pm 1$, $\operatorname{lk}(C_2, C_3) = \pm 1$ and $\operatorname{lk}(C_3, C_1) = \pm 1$, the only possibility is $L = K\left(-\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}\right)$. In this situation, note that $K\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{p}{q}\right)$ is equivalent to $K\left(-\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}\right)$ if and only if $\frac{p}{q} = 0$ or $\frac{p}{q} = -1$.

In the case when L is a nonhyperbolic 4–component Montesinos link, the only possibility is $L = K\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$. But suppose it has another description as a 4–tangle Montesinos link. Let $L' = K\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{r}{s}\right)$, and suppose $L = L'$. Then $s = 2$ by consideration of the Seifert invariants of the double branched cover. Then r is odd, and L' is a chain link. But L is nonhyperbolic, while L' is hyperbolic by [13] unless $r = -1$ (corresponding to chain link $C(4, -2)$ in their notation).

Hence, there are only four possibilities for $\frac{p}{q}$ which make L nonhyperbolic, namely ∞, 0, -1 and $-\frac{1}{2}$. After compensating for the k twists being moved into R, these correspond exactly to the four tangles in the statement of the lemma. $\square$

Proof of Theorem 3.1 Let $X = M \setminus L$ and, for $i = 1, 2$, let Γ_i be the connected component of L containing the arc γ_i (note that possibly $\Gamma_1 = \Gamma_2$). First, we assume that M is orientable. We let L' be the link formed by the replacement stated in Theorem 3.1. As stated in the introduction of this section, we will assume that $(M \setminus \mathcal{B}, L \setminus (\mathcal{B} \cap L))$ is not a rational tangle in a 3–ball as this special case has been dealt with by Lemma 3.4.

Note that we do not include the rational tangle ∞ as a tangle to exclude in the statement of Theorem 3.1 since, in the case of this tangle, the original link L is not prime and hence X is not hyperbolic. We prove Theorem 3.1 when M is orientable by showing that the resulting link complement $Y = M \setminus L'$ does not admit essential disks, spheres, tori or annuli. In order to do so, we first prove the following.

Claim 3.5 *The four-punctured sphere $\mathcal{Q} = \mathcal{S} \setminus L$ is incompressible and boundary-incompressible in X and also in Y.*

Proof We prove that if $\mathcal{Q}$ is compressible in X or in Y, then $(M \setminus \mathcal{B}, L \setminus (\mathcal{B} \cap L))$ is a rational tangle in a 3–ball. We first prove this property in X. Let γ be a nontrivial simple closed curve in $\mathcal{Q}$ and assume that there is a compact disk $E \subset X$ with $\partial E = \gamma = E \cap \mathcal{Q}$. Then each of the two disks E_1 and E_2 bounded by γ in $\mathcal{S}$ must contain exactly two of the punctures of $\mathcal{Q}$, otherwise we could attach a one-punctured disk in $\mathcal{Q}$ to E to find an essential disk in X, contradicting its hyperbolicity.

If E were contained in $\mathcal{B}$, then $E \cup E_1$ and $E \cup E_2$ are two spheres in $\mathcal{B}$, each punctured twice by L. Since both punctures in each sphere cannot come from distinct arcs in $L \cap \mathcal{B}$, E separates $\mathcal{B}$ into two balls B_1 and B_2, where $\gamma_1 \subset B_1$ and $\gamma_2 \subset B_2$, and it then follows that C cannot link γ_1 and γ_2 simultaneously, a contradiction.

Next assume that $E \cap \operatorname{int}(\mathcal{B}) = \varnothing$. Let $A_i = E \cup E_i \setminus L$ for $i = 1, 2$. Then each A_i is an annulus in X. Since each A_i is incompressible and X is hyperbolic, A_i is boundary parallel. Therefore, the closure of A_i in M bounds a closed ball $B_i \subset M \setminus \operatorname{int}(\mathcal{B})$ with $\partial B_i = E_i \cup E$ and such that $B_i \cap L$ is an unknotted arc in B_i. Hence, we can isotope $L \cap B_i$ through B_i to the surface $\mathcal{S}$. Then, after the isotopy, $\partial N(\mathcal{B})$ is a sphere in X. Since X is hyperbolic, $\partial N(\mathcal{B})$ bounds a ball which is disjoint from $\mathcal{B}$, and this is a contradiction unless $M = \mathbb{S}^3$.

If $M = \mathbb{S}^3$, then the fact $L \cap B_i$ can be isotoped through B_i to the surface $\mathcal{S}$ implies $L \setminus (\mathcal{B} \cap L)$ can be isotoped to be two disjoint embedded arcs on $\mathcal{S}$. Hence,

$$(M \setminus \mathcal{B}, L \setminus (\mathcal{B} \cap L))$$

is a rational tangle determined by γ, and up to isotopy, E is the only compression disk for $\mathcal{Q}$ in X.

Note that the above argument implies that $\mathcal{Q}$ is also incompressible in Y, as we next explain. If $E \subset Y$ was a compressing disk with $\partial E = E \cap \mathcal{Q}$, then E necessarily is contained in $\mathcal{B}$, otherwise $E \subset Y \setminus \mathcal{B}$ and $Y \setminus \mathcal{B} = X \setminus \mathcal{B}$ would give a compression disk for $\mathcal{Q}$ in X. Once again, $E \subset \mathcal{B}$ gives that E separates $\mathcal{B}$ into two balls, B_1 and B_2, such that $\gamma_1 \subset B_1$ and $\gamma_2 \subset B_2$. Then, since C_1 links γ_1, $C_1 \subset B_1$. And since C_2 links γ_2, $C_2 \subset B_2$. But then E separates C_1 from C_2 in $\mathcal{B}$, a contradiction to the fact they are linked in $\mathcal{B}$.

To prove boundary-incompressibility of $\mathcal{Q}$ in either X or Y, suppose E is a boundary-compression disk such that $\partial E = \alpha \cup \beta$ with $\alpha = E \cap \mathcal{Q}$. If α connects two distinct punctures of $\mathcal{Q}$ and $N(E)$ is a small neighborhood of E in M, then $\partial N(E) \setminus (\partial N(E) \cap \mathcal{B})$ is a compression disk for $\mathcal{Q}$, a contradiction.

If both endpoints of α are at the same puncture, then, since the interior of β is disjoint from $\mathcal{Q}$, β together with an arc in $\mathcal{Q}$ bound a disk $\widetilde{E}$ in $\partial N(L)$. Thus, $E \cup \widetilde{E}$ is a compression disk for $\mathcal{Q}$, a contradiction. $\square$

Claim 3.6 *Y does not admit any essential spheres or essential disks.*

Proof We argue by contradiction and first suppose that there is an essential sphere S in Y. If S intersects $\mathcal{Q}$, then, by incompressibility of $\mathcal{Q}$, we can exchange disks on S for disks on $\mathcal{Q}$ in order to obtain an essential sphere S' in Y that does not intersect $\mathcal{Q}$. If $S' \subset Y \setminus \mathcal{B}$, then $S' \subset X$, which implies S' is the boundary of a ball $B \subset X$. In this case, B must be disjoint from $\mathcal{B}$, since $C \subset \mathcal{B}$; hence, $B \subset Y$ which contradicts that S' is essential in Y. Thus, we may assume that S' is contained in $\mathcal{B}$, and so it bounds a subball B of $\mathcal{B}$. If B intersects $C_1 \cup C_2$, then, by the linking properties of these circles, $C_1 \cup C_2$ must be contained in B. As γ_1 links C_1 in $\mathcal{B}$, we arrive at a contradiction because the endpoints of γ_1 lie outside of B. This contradiction implies that $L' \cap \mathcal{B}$ is disjoint from B, which in turn implies that $B \subset Y$, contradicting that S' is essential in Y.

Suppose now that there is an essential disk D with boundary in $\partial N(L')$. Then there is a component J of L' such that $\partial D \subset \partial N(J)$, and we let $S = \partial N(D \cup N(J))$. It then follows that S is an essential sphere, as it splits J from the other components of L', contradicting the nonexistence of such spheres. $\square$

For the next arguments in the proof, let $D_1, D_2 \subset Y \cap \mathcal{B}$ denote two twice-punctured disks bounded respectively by $C_1, C_2 \subset L'$ and let $\overline{D}_i$ denote the closure of D_i in M; thus each $\overline{D}_i$ is a disk in $\mathcal{B}$. We prove the following.

Claim 3.7 *The twice punctured disks D_1 and D_2 are incompressible and boundary-incompressible in Y.*

Proof Suppose there were a disk $E \subset Y$, $\operatorname{int}(E) \cap D_i = \varnothing$ with nontrivial boundary in D_i. Since $\mathcal{Q}$ is incompressible and we may assume general position, any component in $E \cap \mathcal{Q}$ is a simple closed curve that is trivial both in E and in $\mathcal{Q}$. Choose an innermost curve $\alpha \subset E \cap \mathcal{Q}$ in the sense that the interior of the disk $E' \subset E$ bounded by α does not intersect $\mathcal{Q}$ and let E'' be the disk bounded by α in $\mathcal{Q}$. Then $E' \cup E''$ is a sphere that is either in the hyperbolic manifold X or in $Y \cap \mathcal{B}$. In either case, $E' \cup E''$ bounds a ball in Y that can be used to isotope E' to E'' and further to remove α from the intersection $E \cap \mathcal{Q}$. After repeating this disk replacement argument a finite number of times, we may assume that $E \subset \mathcal{B}$.

Let E' be the disk in $\overline{D}_i$ bounded by ∂E. Then $E \cup E'$ is a sphere in $\mathcal{B}$ which is punctured only once by at least one of the components in $L' \cap \mathcal{B}$, a contradiction that shows that D_1 and D_2 are incompressible in Y.

To finish the proof of Claim 3.7, we note that D_i is 2–sided and incompressible, and Y is irreducible by Claim 3.6. Thus, as explained in the end of Section 2, D_i is boundary-incompressible. □

Claim 3.8 *Y does not admit essential annuli.*

Proof Arguing by contradiction, assume there exists an essential annulus A in $M \setminus N(L')$. Let α_1 and α_2 denote the two boundary components for A. Then there are components J_1 and J_2 of L' such that $\alpha_1 \subset \partial N(J_1)$ and $\alpha_2 \subset \partial N(J_2)$. After an isotopy of A we will assume without loss of generality that both α_1 and α_2 are taut in the respective tori $\partial N(J_1)$ and $\partial N(J_2)$, in the sense that, in the product structure generated by respective meridional curves in $\partial N(J_i)$, each α_i is transverse to all meridians and also to all longitudes, unless α_i is one of them.

We next rule out the various possibilities for A, starting with the assumption that A does not intersect $D_1 \cup D_2$.

In this case, we may use the fact that $\partial N(D_1 \cup D_2) \setminus N(\gamma_1 \cup \gamma_2)$ is isotopic to $\mathcal{Q}$ to isotope A in $M \setminus N(L')$ to lie outside of $\mathcal{B}$. Thus, A is an annulus in X, and the fact that X is hyperbolic implies that A is either compressible or boundary parallel in X. If A is compressible in X, then we may use the fact that $\mathcal{Q}$ is incompressible in Y and a disk replacement argument to show that A is compressible in Y, a contradiction.

Next, we treat the case when A is boundary parallel in $M \setminus N(L)$. In this case, A defines a product region $W \subset M \setminus N(L)$ through which A is parallel to a subannulus in $\partial N(L)$. Since C lies outside of W, separation properties imply that $\mathcal{B}$ is disjoint from W; hence, $W \subset M \setminus N(L')$ from where it follows that A is boundary parallel in Y, a contradiction.

Now suppose that A intersects $D_1 \cup D_2$ and assume that A has the fewest number of intersection components in $A \cap (D_1 \cup D_2)$ for an essential annulus in Y. Note that for $i = 1, 2$, the intersection curves which may appear in $A \cap D_i$ are either simple closed curves or arcs with endpoints in ∂A.

We next eliminate the possibility that $A \cap D_i$ contains a simple closed curve. Since D_i is incompressible, by minimality of intersection curves, any simple closed curve in the intersection $A \cap D_i$ is nontrivial in A. Note that if $A \cap D_i$ contains a simple closed curve that circles one puncture, we may take an innermost such curve and use the once-punctured disk on D_i that it bounds to surger A to obtain two annuli, each with fewer intersection curves and at least one of them must be essential. So we may assume that all simple closed curves in $A \cap D_i$ circle both punctures of D_i. But then, the outermost of such intersection curves bounds an annulus that again allows us to surger A to obtain an essential annulus with fewer intersection curves. Hence, all curves in $A \cap D_i$ are arcs with endpoints in ∂A.

Next, we show that there are no arcs in $A \cap D_i$ that have endpoints on the same boundary component of A. Assume that α is such an arc and let E_1 be the disk defined by α in A. We assume that α is innermost in the sense that the interior of E_1 is disjoint from D_i. Since Claim 3.7 implies that D_i is boundary-incompressible, it follows that α must cut a disk E_2 from D_i. Then $E = E_1 \cup E_2$ is a disk with boundary $\partial E \subset \partial N(J)$. Since Y does not admit essential disks, it follows that ∂E is trivial in $\partial N(J)$, and we may use the fact that all spheres in Y bound balls to isotope A so that E_1 moves past E_2, thus eliminating the intersection curve α and contradicting minimality of the number of intersection components.

In particular, if A intersects D_i, both α_1 and α_2 must intersect D_i, and none of the intersection arcs on $A \cap D_i$ can cut a disk off D_i, as if they did, A would be boundary-compressible and hence boundary-parallel since Y is irreducible. Note that because there is at least one arc of intersection of A with a D_i, and such arc goes from α_1 to α_2, we have $\partial A \subset (\partial N(C_1) \cup \partial N(C_2) \cup \partial N(\Gamma_1) \cup \partial N(\Gamma_2))$. Moreover, since both α_1 and α_2 intersect $D_1 \cup D_2$ and we assume minimality of intersection components in $\partial A \cap (D_1 \cup D_2)$, no component of ∂A can be a meridian in $\partial N(\Gamma_1)$ or in $\partial N(\Gamma_2)$; hence any closed curve in $A \cap \mathcal{Q}$ must be trivial in A, and, consequently, trivial in $\mathcal{Q}$.

We next consider the case that $\partial A \subset \partial N(C_1) \cup \partial N(C_2)$. Then by incompressibility of $\mathcal{Q}$, we can isotope A to lie inside $\mathcal{B}$. Moreover, $C_1 \cup C_2$ is a Hopf link with complement in the 3–sphere that is a thickened torus $T \times [0,1]$. Thus, $\mathcal{B} \setminus (N(C_1) \cup N(C_2))$ is the complement of a ball B in $T \times [0,1]$, where we identify $\partial N(C_1)$ with $T \times \{0\}$ and $\partial N(C_2)$ with $T \times \{1\}$.

Assume that both boundary components of A are on $\partial N(C_1)$. Then A is an annulus in $(T \times [0,1]) \setminus B$ with both boundaries on $T \times \{0\}$. In particular, in $T \times [0,1]$, A is boundary-parallel through a solid torus V that A cuts from $T \times [0,1]$. Since ∂V is a closed surface in the interior of the three-ball $\mathcal{B}$, it defines a unique compact region disjoint from $\partial \mathcal{B} = \partial B$, from where it follows that B must be disjoint from V. But then both the arcs γ_1 and γ_2, which have endpoints on $\partial \mathcal{B}$, must also be disjoint from V, meaning that $V \subset Y$, and then A is boundary-parallel in Y, a contradiction. By symmetry, the same argument also proves that A cannot have both boundary components on $\partial N(C_2)$.

Next, suppose that one boundary of A is on $\partial N(C_1)$ and the other is on $\partial N(C_2)$. Then again, A can be seen as an annulus in $(T \times [0,1]) \setminus B$, but now its boundary is a pair of nontrivial parallel curves on $T \times \{0\}$ and $T \times \{1\}$. These curves are respectively realized as a (p,q)–curve[1] on $\partial N(C_1)$ and a (q,p)–curve on $\partial N(C_2)$. But there exist arcs $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ on $\mathcal{Q}$ such that the closed curve $\gamma_1 \cup \widetilde{\gamma}_1$ wraps meridionally around C_1 and the closed curve $\gamma_2 \cup \widetilde{\gamma}_2$ wraps meridionally around C_2, where in $T \times [0,1]$, a meridian of $\partial N(C_2)$ corresponds to a longitude of $N(C_1)$. Hence, when we add γ_1 and γ_2 to $T \times [0,1] \setminus B$, one wrapping meridionally around $T \times [0,1]$ and the other wrapping longitudinally, at least one will puncture A, a contradiction.

Thus, at least one boundary component of A, say α_1, must be on $\partial N(\Gamma_i)$, for some $i \in \{1,2\}$. As already explained, α_1 is not a meridian on $\partial N(\Gamma_i)$.

Next, assume that α_2 is on $\partial N(C_1)$ or $\partial N(C_2)$. Since $\mathcal{Q}$ is incompressible and Y is irreducible, after performing a disk replacement argument, we may assume that $A \cap \mathcal{Q}$ is

[1] For given relatively prime integers p and q, a (p,q)–curve is a torus knot that winds p times around the meridian of the torus and q times around its longitude.

a family of pairwise disjoint arcs, each with both endpoints in α_1. Let a be one of such arcs and assume that a cuts an innermost disk D from A, in the sense that $D \cap \mathcal{Q} = a$. If $D \subset \mathcal{B}$, then, if we let $b = \partial D \setminus a$, it follows that $b \subset (\partial N(\gamma_i)) \cap \mathcal{B}$ and our assumptions on α_1 being taut imply that b joins two distinct punctures of $\mathcal{Q}$. But then it follows that ∂D links C_i on $\mathcal{B}$, and D must be punctured by C_i, a contradiction. Hence, it follows that D is to the outside of $\mathcal{B}$. Once again, our assumptions on α_1 imply that a joins two distinct punctures of $\mathcal{Q}$, from where it follows that D is a boundary-compression disk for $\mathcal{Q}$, which contradicts Claim 3.5.

It remains to rule out the case where $\alpha_1 \cup \alpha_2 \subset \partial N(\Gamma_1) \cup \partial N(\Gamma_2)$. Let a be an arc of intersection $A \cap (D_1 \cup D_2)$. Then our previous arguments give that a joins α_1 and α_2 and that a cannot cut a disk off D_i. In particular, a must necessarily intersect the disk D_j for $j \neq i$ and that creates another arc $b \subset A \cap D_j$ which meets a transversely at a point p and joins α_1 and α_2. In particular, $\Gamma_1 = \Gamma_2$. The point p separates both arcs a and b, and that defines a unique disk $D \subset A$ with boundary given by one arc in a, one arc in b and one arc c in α_1. Note that $D \cap D_i \subset a \cup b$, since any arc in $A \cap D_i$ must join α_1 and α_2. Let E be a connected component of $D \setminus \mathcal{B}$ that contains a subarc of c in its boundary. Such component exists because the endpoints of a and b on D are on distinct disks, D_1 and D_2, and hence c cannot be contained in $\mathcal{B}$. Once again, the fact that α_1 is taut gives that $\partial E \cap \mathcal{Q}$ is an arc joining two distinct punctures of $\mathcal{Q}$. But then, E is a boundary-compression disk for $\mathcal{Q}$, a contradiction.

The cases treated above rule out the possibility that Y admits an essential annulus, thereby proving Claim 3.8. □

Claim 3.9 *Y does not admit essential tori.*

Proof We argue by contradiction and suppose that $T \subset Y$ is a torus which is incompressible and not boundary-parallel in Y. First, suppose that T does not intersect $D_1 \cup D_2$. Then we can isotope T in Y to assume that $T \cap \mathcal{B} = \varnothing$, and then $T \subset X$. Since X is hyperbolic, either T admits a compression disk in X or T is boundary parallel in X.

First assume that $E \subset X$ is a compression disk for $T \subset X \setminus \mathcal{B}$. Since $\mathcal{Q}$ is incompressible in X, after disk replacements, we may assume that E is disjoint from $\mathcal{Q}$. In particular, $E \subset X \setminus \mathcal{B} \subset Y$, which is a contradiction.

Next, suppose that T is parallel to the boundary of a neighborhood of one of the components J of L, and let $W \subset X$ be the related proper product region with boundary T.

We claim that $\mathcal{Q}$ must be disjoint from W. Otherwise, $\mathcal{Q} \subset W$ which would imply that $\mathcal{B} \setminus (\gamma_1 \cup \gamma_2 \cup C) \subset W$; this is a contradiction because W has only one end corresponding to a single component of L. Since $\mathcal{Q}$ separates X and is disjoint from W, we have $W \subset Y$, which means T is boundary parallel in Y. This proves that any essential torus in Y must intersect $D_1 \cup D_2$.

Let $T \subset Y$ be an essential torus that intersects D_i, for some $i \in \{1, 2\}$. Next, we prove that Y must contain an essential annulus, which contradicts Claim 3.8. After possibly replacing disks in T by disks in the incompressible surface D_i, we may assume that any component in $T \cap D_i$ is homotopically nontrivial in D_i; let $\gamma \subset T \cap D_i$ be one such component. First assume that γ encircles a single puncture in D_i and choose it to be an innermost such curve in $T \cap D_i$. Using the once-punctured disk bounded by γ in D_i to surger T, we obtain an essential annulus in Y, as claimed. Next, assume that γ encircles both punctures of D_i and that it is an outermost such curve on $T \cap D_i$. In this case, we may use the outer annulus on D_i to surger T in order to obtain an essential annulus in Y, thereby proving Claim 3.9. $\square$

Having proved that there are no essential disks, spheres, tori or annuli in Y, it follows that Y satisfies Thurston's conditions for hyperbolicity, proving Theorem 3.1 when M is orientable.

The case when M is nonorientable can be proved using the orientable case as we next explain. Suppose that M is nonorientable and that L, L' and $\mathcal{B}$ are as stated. Let $\Pi\colon \widehat{M} \to M$ be the oriented two-sheeted covering of M and let $\widehat{L} = \Pi^{-1}(L)$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ be the two connected components of $\Pi^{-1}(\mathcal{B})$. Then, $\widehat{L}$ is a hyperbolic link in $\widehat{M}$ and $\widehat{L} \setminus \mathcal{B}_1$ is not a rational tangle in a 3–ball, since $\widehat{L} \cap \mathcal{B}_2$ is diffeomorphic to $L \cap \mathcal{B}$. Then, we may use the chain move to modify $\widehat{L}$ in $\mathcal{B}_1$, replacing $\widehat{L} \cap \mathcal{B}_1$ by a tangle diffeomorphic to $L' \cap \mathcal{B}$, which creates a hyperbolic link $\widehat{L}'$ in $\widehat{M}$. Then, since $\widehat{L}' \cap \mathcal{B}_2 = \widehat{L} \cap \mathcal{B}_2$ and $\widehat{M} \setminus \widehat{L}'$ is hyperbolic, we can use the chain move in $\mathcal{B}_2$ to replace $\widehat{L}' \cap \mathcal{B}_2$ by a tangle diffeomorphic to $L' \cap \mathcal{B}$ and create another hyperbolic link $\widehat{L}''$ in $\widehat{M}$. Since we may do this second replacement in an equivariant manner with respect to the nontrivial covering transformation σ defined by Π, the restriction of Π to the hyperbolic manifold $\widehat{M} \setminus \widehat{L}''$ is the two-sheeted covering space of $M \setminus L'$. Since σ is an order-two diffeomorphism of $\widehat{M} \setminus \widehat{L}''$, the Mostow–Prasad rigidity theorem implies that we may consider σ to be an isometry of the hyperbolic metric of $\widehat{M} \setminus \widehat{L}''$. Hence, the hyperbolic metric of $\widehat{M} \setminus \widehat{L}''$ descends to $M \setminus L'$ via Π, which finishes the proof of Theorem 3.1. $\square$

4 The switch move theorem

Theorem 4.1 (switch move theorem) *Let L be a link in a 3–manifold M such that $M \setminus L$ admits a complete hyperbolic metric of finite volume. Let $\alpha \subset M$ be a compact arc which intersects L transversely in its two distinct endpoints, and such that $\operatorname{int}(\alpha)$ is a complete, properly embedded geodesic of $M \setminus L$. Let $\mathcal{B}$ be a closed ball in M containing α in its interior and such that $\mathcal{B} \cap L$ is composed of two arcs in L, as in Figure 3. Let L' be the resulting link in M obtained by replacing $L \cap \mathcal{B}$ by the components as appearing in Figure 4, right. Then $M \setminus L'$ admits a complete hyperbolic metric of finite volume.*

Proof We begin the proof by setting the notation. Let G and G' be the connected components of L containing the arcs g and g' respectively, as in Figure 4, left. Note that it can be the case $G = G'$. Let L' be the link formed by replacing $g \cup g'$ in $\mathcal{B}$ by $\gamma_1 \cup \gamma_2 \cup C$. For $i = 1, 2$, let Γ_i be the component of L' containing γ_i. Note that possibly $\Gamma_1 = \Gamma_2$ and let $\Gamma = \Gamma_1 \cup \Gamma_2$.

We split the proof into two cases, depending on whether or not $(M \setminus \mathcal{B}, L \setminus (\mathcal{B} \cap L))$ is a rational tangle in a 3–ball.

Claim 4.2 *If $(M \setminus \mathcal{B}, L \setminus (\mathcal{B} \cap L))$ is a rational tangle in a 3–ball, then $M \setminus L'$ is hyperbolic.*

Proof A rational tangle in a 3–ball always has a projection that is alternating; see for instance [10]. Then L is a rational, alternating link in S^3 that is prime, since $M \setminus L$ is hyperbolic. By [12, Corollary 2], a rational, alternating link in S^3 that is prime is hyperbolic if and only if it is nontrivial and not a 2–braid. After forming L', we consider the link L'' obtained from L' by doing a half-twist on the twice-punctured disk bounded by C to add a crossing so that $L'' \setminus C$ has an alternating projection, as in Figure 9. Then L'' is in an augmented alternating link projection obtained from a prime, nonsplit reduced alternating projection. If $L'' \setminus C$ is neither trivial nor a 2–braid, L'' is hyperbolic by [2]. However, if $L'' \setminus C$ is trivial, then L is a 2–braid and hence it does not satisfy the hypothesis that $M \setminus L$ is hyperbolic. And if $L'' \setminus C$ is a 2–braid, then L is a trivial knot, again not satisfying the same hypothesis. So L'' is a hyperbolic link in S^3. But by [1, Theorem 4.1], L'' is hyperbolic if and only if L' is hyperbolic. □

Remark 4.3 If $M \setminus L$ is hyperbolic and $(M \setminus \mathcal{B}, L \setminus (\mathcal{B} \cap L))$ is a rational tangle in a 3–ball, then L is either a rational link or a rational knot in S^3 which is hyperbolic.

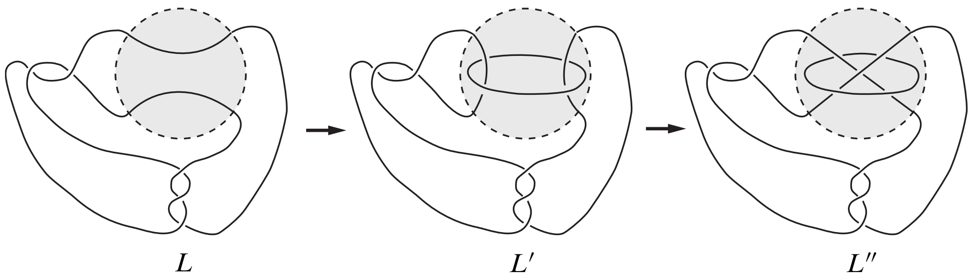

Figure 9: Creating L, L' and L''.

In this case, there is always an arc α as depicted in Figure 3 which is isotopic to a geodesic and hence the switch move can be applied. This follows because α can be chosen to be part of the fixed point set of an involution of the complement, which is realized by an isometry, and fixed-point sets of isometries must be geodesics (see [7] for the details).

From now on, we assume that $(M \setminus \mathcal{B}, L \setminus (\mathcal{B} \cap L))$ is not a rational tangle in a 3–ball. As in the proof of the chain move theorem (Theorem 3.1), we first assume that M is orientable. We also let $X = M \setminus L$ and $Y = M \setminus L'$ and we will prove that Y is hyperbolic by showing that there are no essential disks, spheres, tori or annuli in Y. Once again, we let $\mathcal{S} = \partial\mathcal{B}$, $\mathcal{Q} = \mathcal{S} \setminus L = \mathcal{S} \setminus L'$ and notice that the same arguments used to prove Claim 3.5 can be used to prove that $\mathcal{Q}$ is incompressible and boundary-incompressible in Y; the details are left to the reader.

Claim 4.4 *Y does not admit essential spheres or essential disks.*

Proof We first show that there are no essential spheres in Y. Suppose that $S \subset Y$ is a sphere and first assume that $S \cap \mathcal{B} = \varnothing$. Then $S \subset X$, and, since there are no essential spheres in X, it follows that S bounds a ball $B \subset X$. Since L intersects $\mathcal{B}$, this gives that $B \cap \mathcal{B} = \varnothing$, hence $B \subset Y$, proving that S is not essential in Y.

Next, we treat the case where S intersects $\mathcal{B}$. We can take S to have the least number of intersection curves in $S \cap \mathcal{Q}$ over all essential spheres. If S were contained in $\mathcal{B}$, it bounds a ball in $\mathcal{B}$ which is also a ball in $Y = M \setminus L'$, since $S \cap L' = \varnothing$. Next, we assume that $S \cap \mathcal{Q} \neq \varnothing$. Then there exists a disk $E \subset S$ with $\partial E = E \cap \mathcal{Q}$. After a standard disk replacement argument using that $\mathcal{Q}$ is incompressible and that there are no essential spheres that do not intersect $\mathcal{Q}$, we isotope S to lower the number of components in $S \cap \mathcal{Q}$, which proves that there are no essential spheres in Y.

To prove that there are no essential disks in Y, we argue by contradiction and assume that E is such a disk with boundary on a regular neighborhood of a component J of L'. Then $S = \partial N(E \cup N(J))$ is an essential sphere in Y, as it splits J from the other components of L', a contradiction. □

Let D be the interior of a twice-punctured disk in $\mathcal{B} \setminus L'$ bounded by C and let $\overline{D}$ be its closure in M.

Claim 4.5 *D is incompressible and boundary-incompressible in Y.*

Proof Using the facts that $\mathcal{Q}$ is incompressible in Y, X is hyperbolic and $Y \setminus \mathcal{B} = X \setminus \mathcal{B}$, we may use a disk replacement argument to assume that any compression disk for D is contained in $\mathcal{B} \setminus L'$. Arguing by contradiction, assume that $E \subset \mathcal{B} \setminus L'$ is a disk with $\partial E = E \cap D$, and that ∂E is nontrivial in D. Let $E_1 \subset \overline{D}$ be the subdisk bounded by ∂E in $\overline{D}$. Let $S = E_1 \cup E$. Then, S is a two-sphere in the ball $\mathcal{B}$ which is punctured only once by at least one of the arcs γ_1 or γ_2, which is impossible.

In order to prove that D is boundary-incompressible, we proceed as in the proof of Claim 3.7 and just observe that D is two-sided, incompressible, properly embedded in the irreducible manifold Y. □

Using that both D and $\mathcal{Q}$ are incompressible and boundary incompressible, we next proceed with the proof of Theorem 4.1.

Claim 4.6 *There are no essential annuli in $M \setminus N(L')$.*

Proof Suppose that A is an essential annulus in $M \setminus N(L')$. Our next arguments rule out the several distinct possibilities for A, which are separated into cases.

Case 1 *Assume that $A \cap \mathcal{B} = \varnothing$.*

In this case, $A \subset M \setminus N(L)$ and it must either compress or be boundary-parallel in $M \setminus N(L)$. First, let us assume that $E \subset M \setminus N(L)$ is a compression disk to A with boundary β. Then β separates A into two subannuli A_1 and A_2, and $E \cup A_1$ and $E \cup A_2$ give rise to two essential disks in X, which contradicts its hyperbolicity.

Hence, we may assume that A is boundary-parallel in $M \setminus N(L)$. Then, there is a component J of L and an annulus $A' \subset \partial N(J)$ such that $\partial A' = \partial A$ and $A \cup A'$ bounds a solid torus W in $M \setminus N(L)$, through which A is parallel to A'. If $\mathcal{B} \cap W = \varnothing$, then A is boundary parallel in Y, a contradiction. Hence, we may assume that $\mathcal{B} \cap W \neq \varnothing$. Since $A \cap \mathcal{B} = \varnothing$ and $L \cap W = \varnothing$, then A' must intersect $\mathcal{B}$ and J must be either G or G', which could be the same component. Suppose first that G and G' are distinct. Then if λ is an arc in $\mathcal{B} \setminus N(L)$ with an endpoint in $\partial N(G)$ and the other in $\partial N(G')$, at least one endpoint of λ is not in W. Since $\operatorname{int}(\lambda)$ cannot intersect A, it follows that $G' \subset W$, a contradiction.

Suppose now G and G' are the same component J. Since $A \cap \mathcal{B} = \varnothing$, ∂A is a pair of meridians on $\partial N(J)$. Then, there is a ball B' in $N(J)$ bounded by A' and two meridional disks in $N(J) \setminus \mathcal{B}$ bounded by ∂A. Then $W' = W \cup B'$ is a ball in M, and $J \cap W'$ is an unknotted properly embedded arc within it. Since $\mathcal{B} \cap W \neq \varnothing$, we have $\mathcal{B} \cap W' \neq \varnothing$. But then, the fact that $\partial W' \cap \mathcal{B} = \varnothing$ implies that $\mathcal{B} \subset W'$. Hence α can be homotoped into $\partial N(J)$, contradicting the fact it is a geodesic with endpoints on L.

Case 2 *Assume that $A \subset \mathcal{B}$.*

Let α_1 and α_2 denote the two components of ∂A. First, we assume that $\alpha_1 \subset \partial N(\Gamma)$ and $\alpha_2 \subset \partial N(C)$. Since $A \subset \mathcal{B} \setminus N(L')$, α_1 is either a meridian of $\partial N(\gamma_1)$ or a meridian of $\partial N(\gamma_2)$, and the symmetry between γ_1 and γ_2 allows us to assume $\alpha_1 \subset \partial N(\gamma_1)$. Take a meridional disk E_1 in $N(\gamma_1) \cap \mathcal{B}$ with $\partial E_1 = \alpha_1$. Then $E = A \cup E_1$ is a disk in $\mathcal{B} \setminus N(C)$ with $\partial E = \alpha_2 \subset \partial N(C)$. Hence, α_2 is a longitude of $\partial N(C)$. In particular, α_2 links γ_2 in $\mathcal{B}$, and hence γ_2 must puncture E, which is a contradiction. This contradiction shows that if $A \subset \mathcal{B} \setminus N(L')$ is an essential annulus, then α_1 and α_2 are either both parallel curves on $\partial N(C)$ or both meridians on $\partial N(\Gamma)$.

Assume that A is an essential annulus in $M \setminus N(L')$ such that $A \subset \mathcal{B}$ and α_1 and α_2 are meridians on $\partial N(\Gamma)$. Let E_1 and E_2 be two meridional disks in $N(\Gamma)$ with respective boundaries α_1 and α_2. Then $A \cup E_1 \cup E_2$ is a sphere in $\mathcal{B}$ that bounds a ball $B \subset \mathcal{B}$, which is either punctured once by each γ_1 and γ_2, which is not possible, or twice by one of them, say γ_1. Since A is not boundary parallel, $C \subset B$. However, since C links both γ_1 and γ_2, γ_2 must be contained in B, which is a contradiction.

Still assuming that $A \subset \mathcal{B}$, it remains to obtain a contradiction when both α_1 and α_2 are (p,q)–curves on $\partial N(C)$. In this case, $\mathcal{B} \setminus (N(\gamma_1) \cup N(C))$ is diffeomorphic to $T \times [0,1]$, where $T = \mathbb{S}^1 \times \mathbb{S}^1$ is a torus, and we identify $\partial N(C)$ with $T \times \{0\}$. Since any annulus in $T \times [0,1]$ with boundary in $T \times \{0\}$ is parallel to an annulus in $T \times \{0\}$, it follows that A is parallel to an annulus $A' \subset \partial N(C)$ with $\partial A' = \alpha_1 \cup \alpha_2$, in the sense that there is a solid torus region $W \subset T \times [0,1]$ with $\partial W = A \cup A'$. Since $N(\gamma_2) \subset T \times [0,1]$ and does not intersect ∂W, the fact that the endpoints of γ_2 lie in $T \times \{1\}$ implies that $N(\gamma_2)$ is disjoint from W. Therefore, A is boundary parallel in $\mathcal{B} \setminus N(L')$, contradicting the assumption that A was essential.

Having proved Claim 4.6 in Cases 1 and 2, from now on, we assume that A intersects $\mathcal{Q}$. We also assume that A minimizes the number of intersection curves of an essential annulus of $M \setminus N(L')$ with $\mathcal{Q}$. In particular, since $\mathcal{Q}$ is incompressible, the connected components of $A \setminus \mathcal{Q}$ are either annuli or disks whose boundary intersect ∂A.

Case 3 *Assume that there is an intersection arc in $A \cap \mathcal{Q}$ that cuts a disk from A.*

Let a be an intersection arc in $A \cap \mathcal{Q}$ that cuts a disk E from A. Then both endpoints of a are on the same boundary component of A and $E \cap \mathcal{Q} \subset \partial E$. Because $\mathcal{Q}$ is boundary-incompressible, it must be the case that a cuts a disk E_1 from $\mathcal{Q}$. Then $E_2 = E \cup E_1$ is a disk properly embedded in $M \setminus N(L')$. Since there are no essential disks in $M \setminus N(L')$, then ∂E_2 bounds a disk E_3 in $\partial N(L')$. Then $E_2 \cup E_3$ is a sphere that bounds a ball in $M \setminus N(L')$, through which E can be isotoped to E_1, and just beyond to eliminate a from $A \cap \mathcal{Q}$, contradicting that we assumed a minimal number of intersection components.

Thus, we now know that there are only two possibilities for the intersection curves in $A \cap \mathcal{Q}$. Either they are all parallel nontrivial closed curves on A or they are all arcs with endpoints on distinct boundary components of A.

Case 4 *Assume that $\partial A \cap \mathcal{Q} = \varnothing$, with $A \cap \mathcal{Q} \neq \varnothing$.*

In this case, there are no arcs in $A \cap \mathcal{Q}$. Since A and $\mathcal{Q}$ are incompressible in $M \setminus L'$, the minimality condition on the number of curves in $A \cap \mathcal{Q}$ implies that any curve in $A \cap \mathcal{Q}$ is nontrivial on both A and on $\mathcal{Q}$.

Next, we prove that any curve in $A \cap \mathcal{Q}$ must encircle two of the punctures of $\mathcal{Q}$. Arguing by contradiction, assume that a is a simple closed curve in $A \cap \mathcal{Q}$ and assume that a bounds a once-punctured disk E in $\mathcal{Q}$. Without loss of generality, we may assume that E is innermost in the sense that $E \cap A = a$. Using E to surger A, we obtain two annuli in $M \setminus L'$, where at least one is still essential, and, after a small isotopy, with a lesser number of intersection components with $\mathcal{Q}$, which is a contradiction. Thus, any curve in $A \cap \mathcal{Q}$ encircles two of the punctures of $\mathcal{Q}$ and all intersection curves must be parallel on $\mathcal{Q}$, separating one pair of punctures from the other pair.

Still under the assumption that $\partial A \cap \mathcal{Q} = \varnothing$ and $A \cap \mathcal{Q} \neq \varnothing$, we next rule out the case where at least one boundary component of A, say α_1, lies in $\partial N(C)$. In this case, let A_1 be the connected component of $A \cap \mathcal{B}$ containing α_1 and let $a = \partial A_1 \setminus \alpha_1$ denote the other boundary component of the annulus A_1. Let E be one of the two disks defined by a in $\mathcal{S}$. Then $A_1 \cup E$ is a disk in $\mathcal{B} \setminus N(C)$ which has nontrivial boundary in $\partial N(C)$; hence, α_1 is a longitude. After an isotopy on A_1, we may assume that $\alpha_1 \cap D = \varnothing$, and thus $\partial A_1 \cap D = \varnothing$. Since D is incompressible, we may isotope A_1 in $\mathcal{B} \setminus L'$ to assume that $A_1 \cap D$ does not contain any trivial curves. Moreover, if $\beta \subset A_1 \cap D$ is a nontrivial simple closed curve both in D and in A_1, then β cannot encircle one puncture in D, since this would generate a sphere in $\mathcal{B}$ punctured three times by L', a contradiction. Hence, any curve in $A_1 \cap D$ encircles both punctures of D; this gives rise to solid tori regions in $\mathcal{B} \setminus N(L')$ that can be used to further isotope A_1 in $\mathcal{B} \setminus L'$ to assume that $A_1 \cap D = \varnothing$. In particular, after capping α_1 with a longitudinal disk in $\mathcal{B} \setminus (N(C) \cup A_1)$, it follows that a is the boundary of a disk in $\mathcal{B} \setminus L$.

Since any other curve in $A \cap \mathcal{Q}$ must be parallel to a, $A \cap \mathcal{Q}$ is a family $\{a_1, a_2, \dots, a_n\}$ of pairwise disjoint simple closed curves, all parallel to each other both in $\mathcal{Q}$ and in A. In particular, for each $i \in \{1, 2, \dots, n\}$, a_i generates $\pi_1(A)$ and bounds a disk $E_i \subset \mathcal{B} \setminus L$, punctured once by the arc α. Note that $n \geq 2$, since otherwise $\alpha_2 \subset \partial N(J)$, where J is a component of L and then capping A with a disk in $\mathcal{B} \setminus L$ bounded by α_1 would yield an essential disk in X. This implies that there exists a subannulus $A_2 \subset A \setminus \mathcal{B}$ with boundary $\partial A_2 \subset \mathcal{Q}$. Let us assume that $\partial A_2 = a_1 \cup a_2$. Then (after possibly isotoping the disks E_1 and E_2 in $\mathcal{B} \setminus L$ so they become disjoint) $S = A_2 \cup E_1 \cup E_2$ is a sphere in X, which bounds a ball $B \subset X$. Let $V = B \setminus \mathcal{B}$, then V is a solid torus in $X \setminus \mathcal{B} = Y \setminus \mathcal{B}$ and we may use V to isotope A in Y to reduce the number of intersection components in $A \cap \mathcal{Q}$, a contradiction.

At this point in the proof of Case 4 of Claim 4.6, it remains to rule out the case where no boundary component of A is on $\partial N(C)$. Then $\partial A \cap \mathcal{B} = \varnothing$, since otherwise a boundary component of A would be a meridian in $\partial N(\Gamma)$ and we could isotope A to reduce the number of intersection components in $A \cap \mathcal{Q}$. Next, we show that, after an isotopy, $A \cap D = \varnothing$. Indeed, since D and A are both incompressible, after a disk replacement argument we may assume that any curve in $A \cap D$ is a simple closed curve that generates $\pi_1(A)$ and either encircles one or two of the punctures of D. If there is a curve $a \subset A \cap D$, we may assume that either a encircles one puncture of D and is innermost or that a encircles the two punctures of D and is outermost. In either case, we can surger A to obtain two annuli in Y, where at least one is still essential in Y and

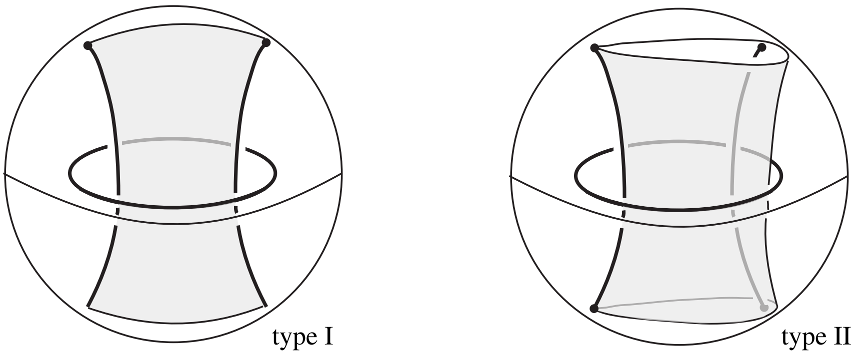

Figure 10: Possibilities for E, a connected component of $A \cap \mathcal{B}$ when all intersections of $A \cap \mathcal{Q}$ are arcs.

with less intersection components with $\mathcal{Q}$, a contradiction that proves that $A \cap D = \varnothing$. Next, using $\mathcal{Q} \times [0, 1]$ as a coordinate system for $\mathcal{B} \setminus N(D \cup L')$, we can isotope A in Y to make A disjoint from $\mathcal{B}$. Since we already showed that there are no essential annuli in Y disjoint from $\mathcal{B}$, this is a contradiction.

Case 5 *Assume that each intersection curve in $A \cap \mathcal{Q}$ is an arc with endpoints on distinct boundary components of A.*

The arcs in $A \cap \mathcal{Q}$ cut A into a collection of disks. Because $\mathcal{S}$ separates M, there must be an even number of such arcs and hence such disks, and the disks must alternate between lying inside and outside $\mathcal{B}$.

There are no such arcs that cut a disk from $\mathcal{Q}$. Indeed, if there were such a disk, by choosing an innermost one, we could surger A along this disk to obtain a disk Δ with boundary in $\partial N(L')$. Since there are no essential disks in $M \setminus N(L')$, $\partial\Delta$ must bound a disk Δ' on $\partial N(L')$. Then $\Delta \cup \Delta'$ is a sphere bounding a ball in $M \setminus N(L')$. Thus we can isotope Δ to Δ' through the ball, and hence isotope A to an annulus in $\partial N(L')$, contradicting the fact that A is not boundary parallel in $M \setminus N(L')$.

Let E be a connected component of $A \cap \mathcal{B}$, which necessarily is a disk. Next, we show that there are two possibilities for E up to isotopy and switching the roles of γ_1 and γ_2. These two possibilities are depicted in Figure 10.

Let $\partial E = \beta_1 \cup \mu_1 \cup \beta_2 \cup \mu_2$ where β_1 and β_2 lie in $\mathcal{Q}$ and μ_1 and μ_2 lie in $\partial N(L')$. Note that each of μ_1 and μ_2 must begin and end at distinct components of $\partial N(\Gamma) \cap \mathcal{Q}$, since otherwise we could lower the number of intersection curves of A with $\mathcal{Q}$.

For the arguments that follow, we set coordinates and consider

$$\mathcal{B} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\},$$

D a horizontal disk in $\{z = 0\}$ and the two arcs γ_1 and γ_2 parallel to the z–axis. Let A' be the annular connected component of $(\mathcal{B} \setminus N(C)) \cap \{z = 0\}$. Then A' is annulus with one boundary component in $\mathcal{Q}$ and the other boundary component a longitude on $\partial N(C)$.

We assume that we have isotoped E in $\mathcal{B} \setminus N(L')$ to minimize the number of intersection curves in $E \cap A'$, and next we prove that $E \cap A' = \varnothing$. First, we claim that $E \cap A'$ does not contain any arc. Indeed, if there were an arc $\phi \subset E \cap A'$, since $\partial E \cap \partial N(C) = \varnothing$, ϕ would cut a disk H_1 from A' and a disk H_2 from E. Let $H_3 = H_1 \cup H_2$. If ϕ has both endpoints in the same β_i, then $\partial H_3 \subset \mathcal{Q}$, which, by incompressibility of $\mathcal{Q}$, implies that ∂H_3 is a trivial curve bounding a disk $H_4 \subset \mathcal{Q}$. Then $H_3 \cup H_4$ is a sphere bounding a ball, through which we can isotope H_1 through H_2, and lower the number of intersection curves in $E \cap A'$, a contradiction.

If ϕ has one endpoint in β_1 and the other in β_2, then ∂H_3 consists of one arc in $\mathcal{Q}$ and one taut arc on $\partial N(\gamma_i)$. Then we can use H_3 to isotope γ_i to $\mathcal{Q}$, a contradiction to the fact that C links γ_i in $\mathcal{B}$. So $E \cap A'$ can only contain simple closed curves.

If $\phi \subset E \cap A'$ is a simple closed curve, then there is a disk $E' \subset E$ with $\partial E' = \phi$. In particular, ϕ is nontrivial in A', since otherwise we could use a disk replacement argument to isotope E removing ϕ from $E \cap A'$. Since ϕ is isotopic to C through A', we could obtain a disk in Y with nontrivial boundary in $\partial N(C)$, a contradiction. Thus, $A' \cap E = \varnothing$.

If μ_1 and μ_2 lie on $\partial N(\gamma_1)$ and $\partial N(\gamma_2)$ respectively, then by an isotopy on

$$\mathcal{Q} \cup \partial N(\gamma_1 \cup \gamma_2),$$

we can assume that μ_1 and μ_2 are vertical arcs that do not wind around $\partial N(\gamma_1)$ or $\partial N(\gamma_2)$. Then because β_1 and β_2 cannot cross the equator $\partial A' \cap \mathcal{Q}$, after possibly reindexing, β_1 connects the top two punctures of $\mathcal{Q}$ and β_2 connects the bottom two punctures. Since ∂E must be trivial as an element of the fundamental group of the handlebody $\mathcal{B} \setminus N(\gamma_1 \cup \gamma_2)$, there can be no twisting around the punctures, and E must appear as in Figure 10, left.

If μ_1 and μ_2 both lie on $\partial N(\gamma_1)$, then β_1 and β_2 are loops on $\mathcal{Q}$ based at a puncture and restricted to the upper and lower hemisphere. Since no arcs in $A \cap \mathcal{Q}$ can cut disks off $\mathcal{Q}$, each β_1 and β_2 circle a puncture in $\mathcal{Q}$. Hence, E must appear as in Figure 10, right. A similar case occurs when μ_1 and μ_2 both lie on $\partial N(\gamma_2)$.

This argument allows us to introduce the following language. If $\beta \subset A \cap \mathcal{Q}$ is any arc, then there is a unique disk $E \subset A \cap \mathcal{B}$ with $\beta \subset \partial E$. If E is a type I disk (where type I and type II are defined as in Figure 10), we shall say that β is a type I arc. Otherwise, we will say that β is a type II arc.

Next, we show that all intersection arcs in $A \cap \mathcal{Q}$ are of the same type. If $\Gamma_1 \neq \Gamma_2$, then, if there exists a type I disk, the two boundaries of A are on different components and only type I disks can occur. If there is not a type I disk, then all disks are type II. On the other hand, if $\Gamma_1 = \Gamma_2$, both components of ∂A intersect $\partial N(\gamma_1)$ and $\partial N(\gamma_2)$ the same equal number of times. In this case, the existence of a type II disk E_1 with boundary intersecting $\partial N(\gamma_1)$ in two arcs, implies that there exists a type II disk E_2 intersecting $\partial N(\gamma_2)$. But E_1 and E_2 would then intersect, a contradiction.

Assume that all arcs in $A \cap \mathcal{Q}$ are of type I and let E be a connected component of $A \setminus \mathcal{B}$. Then, when we switch from L' to L, E can be extended to a disk properly embedded in $M \setminus N(L)$. Thus, there is a trivial component in L, a contradiction to its hyperbolicity.

Our next argument eliminates the last case when all intersections of $A \cap \mathcal{Q}$ are of type II, and $\Gamma_1 \neq \Gamma_2$, since we cannot mix the two types of type II intersections. Until the end of the proof we will assume that $\partial A \subset \partial N(\Gamma_1)$. Let $E \subset A$ be a connected component of $A \setminus \mathcal{B}$, and we label $\partial E = \beta_1 \cup \mu_1 \cup \beta_2 \cup \mu_2$, where β_1 and β_2 lie in $\mathcal{Q}$ and μ_1 and μ_2 lie in $\partial N(\Gamma_1)$. Then μ_1 and μ_2 define two disks, Δ and $\widetilde{\Delta}$, in the annulus $\partial N(\Gamma_1) \setminus \mathcal{B}$. We assume that the disk Δ is the one that makes $\widetilde{A} = E \cup \Delta$ an annulus in $Y \setminus \mathcal{B}$ with both boundary components in $\mathcal{Q}$ parallel to the punctures that come from Γ_2.

After capping $\widetilde{A}$ with the two once-punctured disks bounded by $\partial \widetilde{A}$ in $\mathcal{Q}$, we create an incompressible annulus $\widehat{A}$ in $Y \setminus \mathcal{B}$ which also lives and is incompressible in $X \setminus \mathcal{B}$. Since X is hyperbolic, it follows that $\widehat{A}$ must be boundary-parallel to Γ_2. But this implies that μ_1 is parallel in $X \setminus \mathcal{B}$ to the arc $j_2 = \Gamma_2 \setminus \mathcal{B}$, and there exists a disk $E' \subset X \setminus \mathcal{B}$ with $\partial E' = \mu_1 \cup \nu_1 \cup j_2 \cup \nu_2$, where $\nu_1 \cup \nu_2 = E' \cap \mathcal{Q}$ are two arcs joining the respective two upper punctures and the two lower punctures of $\mathcal{Q}$ which avoid the equator of $\mathcal{Q}$. It then follows that $\nu_1 \cup g$ and $\nu_2 \cup g'$ bound two respective disks in $\mathcal{B} \setminus L$, and the union of those disks with E' is an essential disk in $M \setminus L$, contradicting hyperbolicity of X and finishing the proof of Claim 4.6. □

Claim 4.7 *Y does not admit essential tori.*

Proof We argue by contradiction and suppose that T is a torus which is incompressible and not boundary-parallel in Y. First, suppose that $T \cap D = \varnothing$. Then, after an isotopy in Y, we may assume that $T \cap \mathcal{B} = \varnothing$. Hence, $T \subset X$ and, since X is hyperbolic, T is either compressible or boundary parallel in X. If T is boundary parallel, since both G and G' intersect $\mathcal{B}$, T must be parallel to a component J of L which lives in L', contradicting that T is essential in Y.

Next, we treat the case where T is compressible in X; let $E \subset X$ be a compression disk for T and assume that E has the least number of intersection curves with $\mathcal{Q}$ among compression disks for T. Since T is incompressible in Y, E intersects $\mathcal{B} \cap L'$ and the arc α, which is a complete geodesic in the hyperbolic metric of X. Let $\overline{N}(E) \subset X$ be a closed neighborhood of E with coordinates $E \times [0,1]$ and such that $(\partial E \times [0,1]) \subset T$. Since α is transverse to E, we may choose such a coordinate system on $\overline{N}(E)$ in such a way that, for each $t \in [0,1]$, each component of $\alpha \cap \overline{N}(E)$ intersects $E \times \{t\}$ transversely in a single point.

Let $S = (T \setminus (\partial E \times [0,1])) \cup (E \times \{0\}) \cup (E \times \{1\})$. Then S is a sphere in X and $T \setminus S = \partial E \times (0,1)$. Since X is hyperbolic, S separates and must bound a closed ball $B \subset X$. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the arcs in $\alpha \cap \overline{N}(E)$. We claim that each α_i is contained in B. This follows because the endpoints of α are in L, $L \cap B = \varnothing$ and $T \cap \mathcal{B} = \varnothing$. In particular, $\overline{N}(E) \subset B$.

Let $W = B \setminus \overline{N}(E)$. Then $\partial W = T$ and W is a knot exterior in B bounded by T (in fact, we think of W as obtained from B by removing a potentially knotted hole; see Figure 11). Since $T \cap L' = \varnothing$ and L' intersects $\overline{N}(E)$, we have $W \subset Y$. Our next argument is to show that $\overline{N}(E)$ is unknotted in B; thus W is a solid torus bounded by T, which contradicts the essentiality of T in Y.

Let $\Pi \colon \mathbb{H}^3 \to X$ be the Riemannian universal covering map of X. By appropriately choosing a neighborhood $N(L)$, it follows that $\Pi^{-1}(\partial N(L))$ is a collection of horospheres in $\mathbb{H}^3$. Moreover, $\Pi^{-1}(\alpha)$ is a collection of geodesics connecting these horospheres. On the other hand, B lifts to a collection of balls, one of which is a ball $\widetilde{B}$, containing a lift of W, denoted by $\widetilde{W}$.

In order for T to be incompressible in Y, a lift of α, which we denote by $\tilde{\alpha}$, must pass through the hole $\widetilde{B} \setminus \widetilde{W}$ in $\widetilde{B}$. Since $\tilde{\alpha}$ is a geodesic in $\mathbb{H}^3$, it follows that $\tilde{\alpha}$ is unknotted, which implies that $\widetilde{W}$ is a solid torus. Since W is homeomorphic to $\widetilde{W}$, this gives a contradiction, as previously explained.

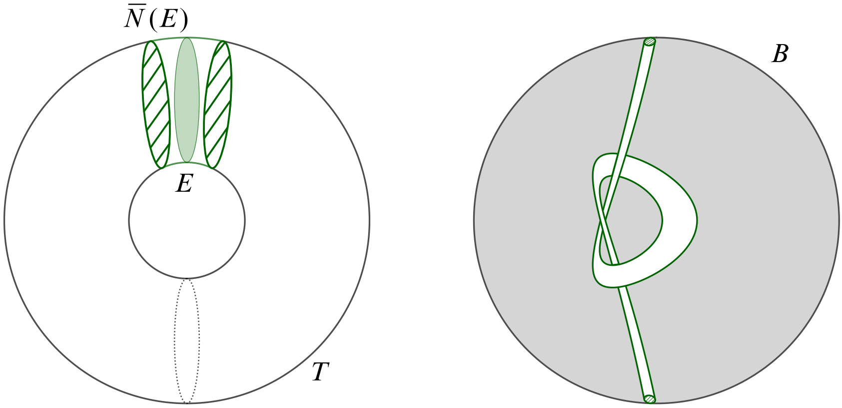

Figure 11: Left: the compressing disk E for the torus T and the neighborhood $\overline{N}(E)$. Right: the sphere S bounding the ball B and the (possibly knotted) region $\overline{N}(E) \subset B$, which defines the highlighted knot exterior $W = B \setminus \overline{N}(E)$.

It remains to prove that there are no essential tori in Y which intersect D. Arguing by contradiction, assume that T is such a torus. Since D is incompressible in Y, a disk replacement argument allows us to further assume that any curve in $T \cap D$ is nontrivial both in T and in D. Let β be a curve in $T \cap D$. If β encircles one puncture of D, take an innermost curve in such intersection and use the one-punctured disk it bounds in D to surger T and obtain an essential annulus in Y with boundary in $\partial N(\Gamma_i)$. If β encircles both punctures of D, take an outermost curve on $T \cap D$ and use the outer annulus on D to surger T and obtain an essential annulus with boundary in $\partial N(C)$. Since Claim 4.6 gives that Y does not admit essential annuli, this proves Claim 4.7. □

Thus, having proved that Y satisfies Thurston's hyperbolicity conditions, Theorem 4.1 follows when M is orientable.

Next, we assume that M is nonorientable and that L, L', α and $\mathcal{B}$ are as before. Let $\Pi\colon \widehat{M} \to M$ be the two-sheeted oriented covering map of M. Then $\widehat{L} = \Pi^{-1}(L)$ is a hyperbolic link in $\widehat{M}$. We also let $\widehat{L}'' = \Pi^{-1}(L')$, $\mathcal{B}_1$ and $\mathcal{B}_2$ be the connected components of $\Pi^{-1}(\mathcal{B})$ and α_1 and α_2 be the connected components of $\Pi^{-1}(\alpha)$. We claim that $\widehat{Y} = \widehat{M} \setminus \widehat{L}''$ is also hyperbolic. Note that, as explained in the proof of the nonorientable setting for the chain move, the fact that $\widehat{Y}$ is hyperbolic implies that $Y = M \setminus L'$ is hyperbolic.

Since α is a complete geodesic in the hyperbolic metric of $M \setminus L$, both α_1 and α_2 are complete geodesics in $\widehat{M} \setminus \widehat{L}$. In particular, since $\widehat{M}$ is orientable, the switch move allows us to replace $\widehat{L} \cap \mathcal{B}_1$ by a tangle diffeomorphic to $L' \cap \mathcal{B}$ to create a new hyperbolic link $\widehat{L}'$ in $\widehat{M}$. Note that $\widehat{L}''$ may be obtained from $\widehat{L}'$ by replacing the tangle $\widehat{L}' \cap \mathcal{B}_2 = \widehat{L} \cap \mathcal{B}_2$ by a tangle diffeomorphic to $L \cap \mathcal{B}$.

Since it might be the case that α_2 is not isotopic to a geodesic in the hyperbolic metric of $\widehat{X} = \widehat{M} \setminus \widehat{L}'$, one cannot directly apply the switch move a second time. However, most of the arguments in its proof can be repeated without change for this setting. We next guide the reader over the steps in the proof that need some adaptation.

First, the arguments in the proof of the orientable case for the switch move can be used to prove that the four-punctured sphere $\mathcal{Q}_1 = \partial\mathcal{B}_1 \setminus \widehat{L}''$ is incompressible and boundary-incompressible in $\widehat{X}$ and in $\widehat{Y} = \widehat{M} \setminus \widehat{L}''$, that $\mathcal{Q}_2 = \partial\mathcal{B}_2 \setminus \widehat{L}''$ is incompressible and boundary-incompressible in $\widehat{Y}$ and that $\widehat{Y}$ does not admit any essential disks and essential spheres.

To prove that $\widehat{Y}$ does not admit any essential annuli, the arguments in Claim 4.6 apply to show that if A is an essential annulus in $\widehat{Y}$, then both boundary components of A are meridians in a component $\widehat{G}'$ of $\widehat{L}'$ that intersects $\mathcal{B}_2$ and that we may isotope A in $\widehat{Y}$ to assume that $A \cap \mathcal{B}_2 = \varnothing$. Since $\widehat{X} \setminus \mathcal{B}_2 = \widehat{Y} \setminus \mathcal{B}_2$, A is an incompressible annulus in $\widehat{X}$, and A must be boundary-parallel in $\widehat{M} \setminus N(\widehat{L}')$. In particular, after an isotopy in $\widehat{Y}$ that does not change the property $A \cap \mathcal{B}_2 = \varnothing$, we may assume that $\partial A \cap \mathcal{B}_1 = \varnothing$ and that if A intersects $\mathcal{B}_1$, then each connected component of $A \cap \mathcal{B}_1$ is an annulus parallel to one of the two arcs in the tangle $\widehat{L}' \cap \mathcal{B}_1$.

If $A \cap \mathcal{B}_1 = \varnothing$, A is an incompressible annulus in the hyperbolic manifold $\widehat{M} \setminus \widehat{L}$, and the same arguments in the proof of Claim 4.6 apply to show that the neighborhood through which A is parallel to an annulus A' in $\partial N(\widehat{L})$ can be capped off by meridional disks to define a ball W' in $\widehat{M}$ that contains both $\mathcal{B}_1$ and $\mathcal{B}_2$ and may be used to homotope the arcs α_1 and α_2 to $\partial N(\widehat{L})$, a contradiction with the fact that both α_1 and α_2 are geodesics in the hyperbolic metric of $\widehat{M} \setminus \widehat{L}$.

Hence, there must exist A_0 a connected component of $A \cap \mathcal{B}_1$. We assume that A_0 is innermost in the sense that no other component of $A \cap \mathcal{B}_1$ lies in the ball region defined by A_0 in $\mathcal{B}_1$. Then each boundary component of A_0 is a curve in $\mathcal{Q}_1$ that encircles one puncture, defining a once-punctured disk in $\mathcal{Q}_1$. Using these two once-punctured disks to surger A gives three incompressible annuli in $\widehat{M} \setminus N(\widehat{L}')$, all disjoint from $\mathcal{B}_2$. One of them lies in $\mathcal{B}_1$ and at least one of the other two must be essential in $\widehat{Y}$. By induction on the number of components in $A \cap \mathcal{B}_1$, this argument yields an essential annulus $\widehat{A}$ in $\widehat{Y}$, with both boundary components being meridians, and that is disjoint both from $\mathcal{B}_1$ and from $\mathcal{B}_2$. As already shown, this is a contradiction that proves that $\widehat{Y}$ does not admit any essential annuli.

The proof that $\widehat{Y}$ does not admit any essential tori uses the arguments in Claim 4.7. Among all possible essential tori, the only case that still needs an adaptation is when V is an essential torus in $\widehat{Y}$ that can be isotoped to be disjoint from $\mathcal{B}_2$. Let D_1 be a twice punctured disk in $\mathcal{B}_1 \setminus \widehat{L}'$ bounded by the trivial component $\widehat{C}_1$ of $\widehat{L}' \cap \mathcal{B}_1$. Then D_1 is incompressible and we may isotope V in $\widehat{Y}$ to assume that there are no trivial curves in $V \cap D_1$. Hence, $V \cap D_1 = \varnothing$, since the existence of a nontrivial curve in $V \cap D_1$ allows us to surger V to produce an essential annulus in $\widehat{Y}$, which we already proved that cannot exist. In particular, V can also be isotoped in $\widehat{Y}$ to be disjoint from $\mathcal{B}_1$, and then V is a torus in the hyperbolic manifold $\widehat{M} \setminus \widehat{L}$. Since V cannot be boundary parallel in $\widehat{M} \setminus \widehat{L}$, there exists a compressing disk E for V in $\widehat{M} \setminus \widehat{L}$, and the fact that V is incompressible in $\widehat{Y}$ implies that E must necessarily intersect the arcs α_1 and α_2, which are geodesics in the hyperbolic metric of $\widehat{M} \setminus \widehat{L}$. Now, the same arguments in the proof of Claim 4.7 apply to show that V bounds a unknotted solid region W in $\widehat{Y}$, contradicting the fact that V is essential in $\widehat{Y}$. This argument finishes the proof that $\widehat{Y}$ satisfies Thurston's hyperbolicity conditions and, as already explained, proves the switch move theorem for the nonorientable case. □

5 The switch move gluing operation

We describe in Theorem 5.1 below a method to obtain new hyperbolic 3–manifolds of finite volume from previously given ones; this method uses a variant of the switch move (Theorem 4.1). Before stating this result, we set the notation.

Let P be a compact 3–manifold with nonempty genus one boundaries and let L be a link in P. We allow for P to consist of one connected manifold or two connected manifolds. Let $M = P \setminus L$ and assume that $\operatorname{int}(M)$ admits a complete hyperbolic metric of finite volume. Let T_1 and T_2 be two distinct, diffeomorphic components of ∂P and, for $i \in \{1, 2\}$, let α_i be a complete geodesic in the hyperbolic metric of $\operatorname{int}(M)$, with one endpoint in T_i and the other in a component J_i of L. Note that J_1 can equal J_2. Let $\Phi\colon T_1 \to T_2$ be a gluing diffeomorphism that maps the endpoint of α_1 in T_1 to the endpoint of α_2 in T_2. Let $P' = P/\Phi$ be the manifold obtained from P by identifying T_1 and T_2 to a genus one surface T using Φ. Note that P' is compact (possibly with empty boundary, if $\partial P = T_1 \cup T_2$), connected and that $L \subset P'$. Let $X = P' \setminus L$.

Theorem 5.1 (switch move gluing operation) *With the above notation, let α be the concatenation of α_1 and α_2^{-1} in P'. Let $\mathcal{B}$ be a ball neighborhood of α in P' that intersects L in two arcs $g \subset J_1$ and $g' \subset J_2$ and intersects T in a disk Δ. Let L' be the resulting link obtained in P' by replacing $g \cup g'$ by the tangle $\gamma_1 \cup \gamma_2 \cup C$ as in Figure 4, right. Then the manifold $Y = P' \setminus L'$ is hyperbolic.*

After choosing ϕ as above, as in the case of the switch move, the operation described above may yield two distinct hyperbolic 3–manifolds depending on the projection of the strands g and g'; see Remark 1.1.

Proof of Theorem 5.1 We first prove the theorem in the case when P is orientable. In this circumstance, the setting in Theorem 5.1 is the same as in the switch move theorem (Theorem 4.1), with the exception that X is no longer hyperbolic. However, X is close to being hyperbolic in the following sense:

Claim 5.2 *X does not admit any essential spheres, essential disks and essential annuli. Moreover, any essential torus in X is isotopic to T.*

Proof Suppose there were an essential disk E in X. Since $\mathrm{int}(M)$ is hyperbolic, it follows that E must intersect T. But because T is incompressible and ∂E is disjoint from T, we may replace subdisks in E by disks in T to obtain an essential disk in M, a contradiction. The same argument shows that an essential sphere in X would generate an essential sphere in M, also a contradiction.

Because $\mathrm{int}(M)$ is hyperbolic, an essential torus in X that does not intersect T must be parallel to T, and hence isotopic to T. To prove that T is the only possible essential torus up to isotopy, we argue by contradiction. Suppose T' is an essential torus in X that is not isotopic to T and has the fewest number of intersection curves with T. Then any curve in $T' \cap T$ is nontrivial both in T and in T'. It follows that there is a component of $T' \setminus T$ that is an essential annulus in $\mathrm{int}(M)$, a contradiction. Analogously, we may show that X does not admit any essential annuli, and this proves Claim 5.2. □

Having proved Claim 5.2, we observe that $L \setminus \mathcal{B}$ is not a rational tangle in a 3–ball as X contains an essential torus that intersects the 3–ball in an essential punctured torus, which cannot exist in a rational tangle complement. Then we note that the arguments in the proof of the switch move theorem apply directly to show that Y does not admit any essential spheres or essential disks and that the four-punctured sphere $\mathcal{Q} = \partial\mathcal{B} \setminus L = \partial\mathcal{B} \setminus L'$ and the twice-punctured disk D bounded by C on T are incompressible and boundary-incompressible in Y.

To prove that Y does not admit any essential annuli, the proof of Claim 4.6 applies directly. Hence, to prove Theorem 5.1 when M is orientable, it remains to show that Y does not admit any essential tori.

We argue by contradiction and assume that V is an essential torus in $M \setminus L'$ that has the least number of intersection components with T among all essential tori in Y. Then, after assuming general position, $T \cap V$ is a finite collection of pairwise disjoint simple closed curves. Let γ be one of such intersection components. If $\gamma \subset D$, then it does not bound a disk in $T \setminus L'$ and either encircles one or two of the punctures of D. Then we can choose a component γ' in $V \cap D$ (if γ encircles one puncture, we choose γ' as an innermost curve, otherwise we choose γ' as an outermost curve) and surger V (along a punctured disk in the first case and an annulus in the second case) to obtain an essential annulus in Y, a contradiction. It then follows that $V \cap D = \varnothing$, and then V can be isotoped through Y to be disjoint from $\mathcal{B}$, without increasing the number of intersection components in $V \cap T$. Hence, V is a torus that is contained in X.

In X, V is not isotopic to T, since $V \subset M \setminus L'$ and any torus isotopic to T is punctured by L'. We claim that $V \cap T = \varnothing$. Argue by contradiction and assume that there exists a curve γ in $V \cap T$. Then γ does not intersect D and there are two possibilities: either γ is a nontrivial curve in T or γ, together with C, bounds an annulus in $T \setminus D$. In the latter case, we may use this annulus to surger V and obtain an essential annulus in Y. Since Y does not admit essential annuli, γ is nontrivial in T. Then there is a component of $V \setminus T$ that is an essential annulus in X, which cannot occur, proving that $V \cap T = \varnothing$.

Thus, V is a torus in the hyperbolic manifold $\operatorname{int}(M)$. Note that V cannot be boundary parallel in M, since this either contradicts its essentiality in Y or the fact that it is not isotopic to T. Hence, it must be the case that V is compressible in M. Let $E \subset M$ be a compression disk for V. Then, since V is incompressible in Y, the geodesic α_1 must intersect E. Now, the same arguments used in the proof of Claim 4.7 for the case when T was an essential torus in Y, disjoint from $\mathcal{B}$ and compressible in X apply to obtain that V is compressible in Y, a contradiction that proves Theorem 5.1 when M is orientable.

Next, we sketch the arguments that prove Theorem 5.1 when M is nonorientable, using the notation already introduced. Let $\widehat{X}$ be the oriented double cover of X and let $\Pi\colon \widehat{X} \to X$ be the covering map. Let $\widehat{T} = \Pi^{-1}(T)$ in $\widehat{X}$. Then $\widehat{T}$ consists of one or two tori.

In the case of two tori V_1 and V_2, $\Pi^{-1}(\alpha)$ consists of two arcs μ_1 and μ_2 each of which intersects one of the two tori. Since $\widehat{X}$ is orientable, we may apply Theorem 5.1 twice first for V_1 and μ_1 and then, in an equivariant manner to the first switch move with respect to the covering translation, for V_2 and μ_2, to obtain a hyperbolic link complement. Then, by Mostow–Prasad rigidity, the covering translation can be realized as an isometry, proving that the switch move gluing operation on the original nonorientable manifold M yields a hyperbolic manifold.

In the case that $\widehat{T}$ consists of one torus, both copies μ_1 and μ_2 intersect $\widehat{T}$. In this situation, we may apply Theorem 5.1 on μ_1, obtaining a hyperbolic link complement where Theorem 4.1 can be performed, in an equivariant manner with respect to the covering translation, on a neighborhood of μ_2 and again the result is hyperbolic. Realizing the covering translation as an isometry allows us to prove that the switch move gluing operation on the nonorientable manifold M yields a hyperbolic manifold Y. □

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Received: 30 December 2020 Revised: 3 February 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY
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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

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mathematical sciences publishers
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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23 Issue 5 (pages 1935–2414) 2023