AGT

Algebraic & Geometric Topology

Volume 23 (2023)

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Algebraic & Geometric Topology **23**:5 (2023) 2309–2327
DOI: 10.2140/agt.2023.23.2309
Published: 25 July 2023

Homotopy types of gauge groups over Riemann surfaces

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Let G be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$. We study the homotopy types of gauge groups of principal G–bundles over Riemann surfaces. This can be applied to an explicit computation of the homotopy groups of the moduli spaces of stable vector bundles over Riemann surfaces.

57S05; 55Q15

1 Introduction

Let G be a compact connected Lie group, and let P be a principal G–bundle over a finite complex X. The *gauge group* of P is defined to be the topological group of G–equivariant self-maps of P which fix X. There may be infinitely many distinct principal G–bundles over X. For example, there are infinitely many bundles when X is an orientable 4–manifold. Each bundle has a gauge group, so there may be potentially infinitely many gauge groups. However, Crabb and Sutherland [6] showed that these gauge groups have only finitely many homotopy types. Subsequently, the precise number of homotopy types of gauge groups for specific G and X has been intensely studied. The study began with simply connected Lie groups by Cutler [7], Hamanaka, Hasui, Kishimoto, Kono, So, Theriault and Tsutaya [10; 12; 15; 16; 18; 20; 30; 31], and recently, nonsimply connected cases are also studied by Hasui, Kamiyama, Kishimoto, Kono, Membrillo-Solis, Sato, Theriault and Tsukuda [11; 14; 17] and Rea [26].

In this paper, we study the homotopy types of gauge groups of principal G–bundles over a compact connected Riemann surface, where $\pi_1(G) \cong \mathbb{Z}$. This includes an important case: gauge groups of principal $\mathrm{U}(n)$–bundles over a Riemann surface, whose topology was first studied by Atiyah and Bott [2]. To state the results, we introduce a numerical invariant of G. Suppose $\pi_1(G) \cong \mathbb{Z}$. Then as in Mimura and Toda [24, Corollary 5.1, Chapter II], there is a compact connected simply connected Lie group H and a subgroup C of the center of $S^1 \times H$ such that

$$G \cong (S^1 \times H)/C. \tag{1-1}$$

In other words, G is locally isomorphic to $S^1 \times H$. Note that H is uniquely determined by G, but C is not. For example, if $G = S^1 \times H$, then C can be any finite subgroup of $S^1 \times 1 \subset S^1 \times H$. We define

$$s(G) = |p_2(C)|,$$

where $p_2\colon S^1 \times H \to H$ is the projection. By Theorem 1.4 below, we can see that $s(G)$ is independent of the choice of C.

Example 1.1 Since $\mathrm{U}(n)$ is the quotient of $S^1 \times \mathrm{SU}(n)$ by the diagonal central subgroup isomorphic to $\mathbb{Z}/n$, we have $s(\mathrm{U}(n)) = n$.

Let X be a compact connected Riemann surface. Then there is a one-to-one correspondence between principal G–bundles over X and $\pi_2(BG) \cong \mathbb{Z}$. Let $\mathcal{G}_k(X,G)$ denote the gauge group of a principal G–bundle over X corresponding to $k \in \mathbb{Z}$. Now we state our results.

Theorem 1.2 *Let G be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$, and let X be a compact connected Riemann surface. If $(k, s(G)) = (l, s(G))$, then $\mathcal{G}_k(X,G)$ and $\mathcal{G}_l(X,G)$ are homotopy equivalent after localizing at any prime or zero.*

We remark that the p–localization of a disconnected space will mean the disjoint union of the p–localization of path-connected components. For a prime p, Theriault [29] gave a p–local homotopy decomposition of $\mathcal{G}_k(X, \mathrm{U}(p))$, which implies the converse implication of Theorem 1.2 holds for $G = \mathrm{U}(p)$. We will prove the converse implication of Theorem 1.2 holds for other Lie groups.

Theorem 1.3 *Let G be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$, and let X be a compact connected Riemann surface. If G is locally isomorphic to $S^1 \times \mathrm{SU}(n)^r$ or $S^1 \times \mathrm{SU}(4n-2)^s \times Sp(2n-1)^t$, then the following statements are equivalent*:

1. $(k, s(G)) = (l, s(G))$.
2. *$\mathcal{G}_k(X,G)$ and $\mathcal{G}_l(X,G)$ are homotopy equivalent after localizing at any prime or zero.*

Note that since $\mathrm{U}(n) = (S^1 \times \mathrm{SU}(n))/(\mathbb{Z}/n)$ as in Example 1.1, Theorem 1.3 applies to the case $G = \mathrm{U}(n)$.

The homotopy type of a gauge group $\mathcal{G}_k(X, G)$ is closely related with a Samelson product in G, as we will see in Section 2. In our context, the Samelson product of a generator of $\pi_1(G) \cong \mathbb{Z}$ and the identity map of G is of particular importance. We will prove the following theorem, which is of independent interest.

Theorem 1.4 *Let G be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$, and let ϵ denote a generator of $\pi_1(G)$. Then the Samelson product $\langle \epsilon, 1_G \rangle$ in G is of order $s(G)$.*

Now we consider an application. Gauge groups over a Riemann surface are closely related to the moduli spaces of stable vector bundles over a Riemann surface as follows. Let X be a Riemann surface of genus g, and let $M(n,k)$ denote the moduli space of stable vector bundles over X of rank n and degree k. Daskalopoulos and Uhlenbeck [8] showed that there is an isomorphism

$$\pi_i(M(n,k)) \cong \pi_{i-1}(\mathcal{G}_k(X, \mathrm{U}(n)))$$

for $2 < i \leq 2(g-1)(n-1) - 2$ and $(n,k) \neq (2,2)$. There is a polystable Higgs bundle analog due to Bradlow, García-Prada and Gothen [5]. We can compute the homotopy groups of these moduli spaces in a range through the following homotopy decomposition.

Theorem 1.5 *Let G be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$, and let X be a compact connected Riemann surface of genus g. If $s(G)$ divides k, then*

$$\mathcal{G}_k(X, G) \simeq G \times (\Omega G)^{2g} \times \Omega^2 G.$$

Moreover, the above homotopy equivalence also holds after localizing at p whenever p does not divide $s(G)$.

The paper is structured as follows. Section 2 recalls a connection between gauge groups and Samelson products, and then proves Theorems 1.2 and 1.5 by assuming Theorem 1.4 holds. Section 3 shows some general results on Samelson products in a Lie group, which will be used for a practical computation. Sections 4 and 5 compute the Samelson products in G when H is simple. Finally, Section 6 collects all results so far together to prove Theorems 1.3 and 1.4.

Acknowledgements

The authors are grateful to Jérôme Scherer and the referee for useful comments. The authors were partly supported by JSPS KAKENHI grant numbers 17K05263 (Kameko), 17K05248 and 19K03473 (Kishimoto) and 21J10117 (Takeda).

2 Gauge groups and Samelson products

This section recalls a connection between gauge groups and Samelson products, and then Theorems 1.2 and 1.5 are proved by assuming Theorem 1.4 holds. First, we recall a connection between gauge groups and mapping spaces. Let G be a topological group, and let P be a principal G–bundle over a base X, which is classified by a map $\alpha\colon X \to BG$. Recall that the gauge group of P, denoted by $\mathcal{G}(P)$, is the topological group of G–equivariant self-maps of P which fix X. Gottlieb [9] proved that there is a natural homotopy equivalence

$$B\mathcal{G}(P) \simeq \mathrm{map}(X, BG; \alpha),$$

where $\mathrm{map}(A, B; f)$ denotes the path component of the space of maps $\mathrm{map}(A, B)$ containing a map $f\colon A \to B$. Then evaluating at the basepoint of X yields a homotopy fibration

$$\mathrm{map}_*(X, BG; \alpha) \to B\mathcal{G}(P) \to BG, \tag{2-1}$$

where $\mathrm{map}_*(X, BG; \alpha)$ is the subspace of $\mathrm{map}(X, BG; \alpha)$ consisting of basepoint preserving maps. So the gauge group $\mathcal{G}(P)$ is homotopy equivalent to the homotopy fiber of the connecting map

$$\partial_\alpha\colon G \to \mathrm{map}_*(X, BG; \alpha)$$

of the above homotopy fibration.

Next, we assume $X = S^n$ for $n \geq 1$ and describe the connecting map ∂_α. Clearly, there is a homotopy equivalence $\mathrm{map}_*(S^n, BG; \alpha) \simeq \Omega_0^{n-1} G$, where $\Omega_0^{n-1} G$ denotes the path component of $\Omega^{n-1} G$ containing the constant map. Then by adjointing, the connecting map ∂_α corresponds to a map

$$d_\alpha\colon S^{n-1} \wedge G \to G.$$

The original definition of Whitehead products in [32] and adjointness of Whitehead products and Samelson products prove the following.

Lemma 2.1 *The map d_α is the Samelson product $\langle \bar{\alpha}, 1_G \rangle$ in G, where $\bar{\alpha}\colon S^{n-1} \to G$ is the adjoint of $\alpha\colon S^n \to BG$.*

The following lemma due to Theriault [27] shows how to identify the homotopy type of a gauge group $\mathcal{G}(P)$ from the order of a Samelson product $\langle \bar{\alpha}, 1_G \rangle$.

Lemma 2.2 *Suppose that a map $f\colon X \to Y$ into an H-space Y is of order $n < \infty$. Then $(n,k) = (n,l)$ implies $F_{k(p)} \simeq F_{l(p)}$ for any prime p, where F_k denotes the homotopy fiber of a map $k \circ f\colon X \to Y$.*

Finally, we recall a homotopy decomposition of a gauge group. Theriault [28] showed a homotopy decomposition of a gauge group over principal $\mathrm{U}(n)$–bundle over a Riemann surface. We can easily see that his proof works in verbatim for any compact connected Lie group G with $\pi_1(G) \cong \mathbb{Z}$. Then we get:

Proposition 2.3 *Let G be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$, and let X be a compact connected Riemann surface of genus g. Then there is a homotopy equivalence*

$$\mathcal{G}_k(X,G) \simeq (\Omega G)^{2g} \times \mathcal{G}_k(S^2,G).$$

Now we prove Theorems 1.2 and 1.5 by assuming Theorem 1.4 holds.

Proof of Theorem 1.2 Combine Lemmas 2.1 and 2.2, Proposition 2.3 and Theorem 1.4. □

Proof of Theorem 1.5 By Lemma 2.1 and Theorem 1.4, if k is divisible by $s(G)$, then $\mathcal{G}_k(S^2,G)$ is homotopy equivalent to the homotopy fiber of the constant map $G \to \Omega_0 G$. So since $\pi_2(G) = 0$, $\mathcal{G}_k(S^2,G) \simeq G \times \Omega^2 G$. Thus by Proposition 2.3, the proof is done. □

3 Samelson products in Lie groups

This section shows some criteria for computing Samelson products in a Lie group. For the rest of the paper, we will use the following notation:

- Let G be a compact connected Lie group with $\pi_1(G) \cong \mathbb{Z}$.
- Let ϵ_G denote a generator of $\pi_1(G) \cong \mathbb{Z}$.
- Let H and C be as in the decomposition (1-1).
- Let $j_H\colon \Sigma H \to BH$ denote the natural map.
- Let $p_G\colon S^1 \times H \to G$ denote the quotient map.
- Let $p_1\colon S^1 \times H \to S^1$ and $p_2\colon S^1 \times H \to H$ denote projections.
- Let $K = H/p_2(C)$.
- Let $q_G\colon G \to K$ and $\bar{q}_K\colon H \to K$ denote the quotient maps.

We will abbreviate ϵ_G, j_H, p_G, q_G and $\bar{q}_K$ to ϵ, j, p, q and $\bar{q}$, respectively, if G, H and K are clear from the context. First, we show two properties of the group C.

Lemma 3.1 *The abelian group $p_2(C)$ is cyclic.*

Proof There is a fibration

$$S^1 \to G \xrightarrow{q} K \tag{3-1}$$

and so by the homotopy exact sequence, we can see that $\pi_1(K) \cong p_2(C)$ is a quotient of $\pi_1(G) \cong \mathbb{Z}$. Then $p_2(C)$ is a cyclic group, as stated. □

Lemma 3.2 *We may choose a group C such that $|p_1(C)| = s(G)$.*

Proof Note that $p_2(C)$ is a cyclic group by (3-1). We prove that the inequality $|p_1(C)| \geq s(G)$ always holds. If $|p_1(C)| < s(G)$, then $C_1 = |p_1(C)|C$ is a nontrivial subgroup of the center of $1 \times H \subset S^1 \times H$. In particular, there is a covering

$$C/C_1 \to (S^1 \times H)/C_1 \to G.$$

Then $\pi_1(G) \cong \mathbb{Z}$ includes a nontrivial finite abelian group C_1, which is a contradiction. Thus $|p_1(C)| \geq s(G)$.

Suppose that $|p_1(C)| > s(G)$. Then $C_2 = s(G)C$ is a finite subgroup of $S^1 \times 1 \subset S^1 \times H$. Then $(S^1 \times H)/C_2 \cong S^1 \times H$, implying

$$G \cong (S^1 \times H)/C \cong ((S^1 \times H)/C_2)/(C/C_2) \cong (S^1 \times H)/(C/C_2).$$

Note that C is a subgroup of $p_1(C) \times p_2(C)$ generated by (g_1, g_2), where g_i is a generator of a cyclic group $p_i(C)$ for $i = 1, 2$. Then $C_2 = s(G)p_1(C) \times 0$, and so C/C_2 is identified with the diagonal subgroup of

$$(p_1(C)/s(G)p_1(C)) \times p_2(C) \cong \mathbb{Z}/s(G) \times Z/s(G).$$

Thus $|p_1(C/C_2)| = s(G)$, finishing the proof. □

By Lemma 3.1, $\pi_1(K) \cong p_2(C)$ is a cyclic group of order $s(G)$. For the rest of this section, we will also use the following notation:

- Let $\bar{\epsilon}_K$ denote a generator of $\pi_1(K)$.

We will abbreviate it by $\bar{\epsilon}$ if K is clear from the context.

Next, we show an upper bound and a lower bound for the order of $\langle \epsilon, 1_G \rangle$.

Lemma 3.3 *The order of $\langle \epsilon, 1_G \rangle$, hence $\langle \epsilon, p \rangle$, divides $s(G)$.*

Proof The proof of Lemma 3.1 implies $q \circ \epsilon = \bar{\epsilon}$. Then since q is a homomorphism, we get

$$q_*(s(G)\langle \epsilon, 1_G \rangle) = s(G)\langle q \circ \epsilon, q \rangle = \langle s(G)\bar{\epsilon}, q \rangle = 0.$$

So since there is a fibration (3-1), $s(G)\langle \epsilon, 1_G \rangle$ lifts to a map $S^1 \wedge G \to S^1$. Since $S^1 \wedge G$ is simply connected, this lift is trivial, and thus $s(G)\langle \epsilon, 1_G \rangle$ itself is trivial, completing the proof. □

Lemma 3.4 *The order of $\langle \bar{\epsilon}, \bar{q} \rangle$ divides the order of $\langle \epsilon, p \rangle$.*

Proof Let $i : H \to S^1 \times H$ denote the inclusion. By definition, $q \circ p \circ i = \bar{q}$, and the proof of Lemma 3.2 implies that $q \circ \epsilon = \bar{\epsilon}$. Then

$$(1 \wedge i)^* \circ q_*(\langle \epsilon, p \rangle) = q_*(\langle \epsilon, p \circ i \rangle) = \langle q \circ \epsilon, q \circ p \circ i \rangle = \langle \bar{\epsilon}, \bar{q} \rangle$$

and so the proof is done. □

Finally, we give a cohomological criterion for the Samelson product $\langle \bar{\epsilon}, \bar{q} \rangle$ being nontrivial. For an algebra A, let QA denote the module of indecomposables.

Lemma 3.5 *Suppose there are $x, y, z \in QH^*(BK; \mathbb{Z}/p)$ and a Steenrod operation θ satisfying the following conditions*:

(1) $|y| = 2$ *and* $QH^n(BK; \mathbb{Z}/p) = \langle z \rangle$ *for* $n > 2$.

(2) $\theta(x)$ *is decomposable and includes the term* $y \otimes z$.

(3) $(\bar{q} \circ j)^*(z)$ *is nontrivial and not included in any element of* $\theta(H^*(\Sigma H; \mathbb{Z}/p))$.

Then the Samelson product $\langle \bar{\epsilon}, \bar{q} \rangle$ is nontrivial.

Proof Suppose that $\langle \bar{\epsilon}, \bar{q} \rangle$ is trivial. Let $\hat{\epsilon} \colon S^2 \to BK$ and $\hat{q} \colon \Sigma H \to BK$ denote the adjoint of $\bar{\epsilon}$ and $\bar{q}$, respectively. Then by adjointness of Samelson products and Whitehead products, the Whitehead product $[\hat{\epsilon}, \hat{q}]$ is trivial, so that there is a homotopy commutative diagram

$$\begin{tikzcd} S^2 \vee \Sigma H \arrow[r, "\hat{\epsilon} \vee \hat{q}"] \arrow[d] & BK \arrow[d, equal] \\ S^2 \times \Sigma H \arrow[r, "\mu"] & BK \end{tikzcd}$$

Since BK is simply connected, $H^1(BK; \mathbb{Z}/p) = 0$ and $H^2(BK; \mathbb{Z}/p) = \langle y \rangle$. Then by the Hurewicz theorem and the first condition in the statement, we may assume $\hat{\epsilon}^*(y) = u$, where u is a generator of $H^2(S^2; \mathbb{Z}/p) \cong \mathbb{Z}/p$. Hence by the first and the second conditions, $\mu^*(\theta(x))$ includes the term $u \otimes \hat{q}^*(z)$. Since $\hat{q} = \bar{q} \circ j$, the third condition implies $u \otimes \hat{q}^*(z) \neq 0$. On the other hand, by the third condition, $\theta(\mu^*(x))$ cannot include the term $u \otimes \hat{q}^*(z)$. Thus since $\mu^*(\theta(x)) = \theta(\mu^*(x))$, we obtain a contradiction. Therefore $\langle \bar{\epsilon}, \bar{q} \rangle$ is nontrivial, completing the proof. $\square$

Recall that compact simply connected simple Lie groups with nontrivial center are

$$\mathrm{SU}(n), \qquad \mathrm{Sp}(n), \qquad \mathrm{Spin}(n) \quad (n \geq 7), \qquad E_6, \qquad E_7.$$

Then in the following two sections, we will compute the Samelson product $\langle \epsilon, p \rangle$ for H being one of the above Lie groups.

4 Classical case

This section determines the order of the Samelson product $\langle \epsilon, p \rangle$ for $H = \mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{Spin}(n)$.

4.1 The case $H = \mathrm{SU}(n)$

First we consider the case $H = \mathrm{SU}(n)$.

Proposition 4.1 *If $H = \mathrm{SU}(n)$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.*

Proof By Lemma 3.3, it suffices to show that the order of $\langle \epsilon, p \rangle$ is a nonzero multiple of $s(G)$. The center of $\mathrm{SU}(n)$ is isomorphic to $\mathbb{Z}/n$. Then since $\mathrm{U}(n) = S^1 \times_{\mathbb{Z}/n} \mathrm{SU}(n)$, it follows from Lemma 3.2 that there is a homomorphism $\rho\colon G \to \mathrm{U}(n)$ which is a $n/s(G)$ sheeted covering. Let α_{2i-1} denote a generator of $\pi_{2i-1}(\mathrm{U}(n)) \cong \mathbb{Z}$ for $i = 1, 2, \ldots, n$. Then

$$\rho_*(\epsilon) = \frac{n}{s(G)} \alpha_1.$$

On the other hand, it is shown in [4] that the order of $\langle \alpha_1, \alpha_{2n-1} \rangle$ is a nonzero multiple of n. Since $\rho_*\colon \pi_{2n-1}(G) \to \pi_{2n-1}(\mathrm{U}(n))$ is an isomorphism, there is an $\tilde{\alpha} \in \pi_{2n-1}(G)$ such that $\rho_*(\tilde{\alpha}) = \alpha_{2n-1}$. Then since

$$\rho_*(\langle \epsilon, \tilde{\alpha} \rangle) = \langle \rho_*(\epsilon), \rho_*(\tilde{\alpha}) \rangle = \left\langle \frac{n}{s(G)} \alpha_1, \alpha_{2n-1} \right\rangle = \frac{n}{s(G)} \langle \alpha_1, \alpha_{2n-1} \rangle,$$

the order of $\rho_*(\langle \epsilon, \tilde{\alpha} \rangle)$ is a nonzero multiple of $s(G)$. Thus, since the map

$$\rho_*\colon \pi_{2n}(G) \to \pi_{2n}(\mathrm{U}(n))$$

is an isomorphism, the order of $\langle \epsilon, \tilde{\alpha} \rangle$ is a nonzero multiple of $s(G)$ too. Since

$$p_*\colon \pi_{2n-1}(S^1 \times \mathrm{SU}(n)) \to \pi_{2n-1}(G)$$

is an isomorphism, there is a $\beta \in \pi_{2n-1}(S^1 \times \mathrm{SU}(n))$ such that $p \circ \beta = \tilde{\alpha}$. Thus since $(1 \wedge \beta)^*(\langle \epsilon, p\rangle) = \langle \epsilon, \tilde{\alpha}\rangle$, the order of $\langle \epsilon, p\rangle$ is a nonzero multiple of $s(G)$, completing the proof. □

4.2 The case $H = \mathrm{Sp}(n)$

Next, we consider the case $H = \mathrm{Sp}(n)$. Recall that the center of $\mathrm{Sp}(n)$ is isomorphic to $\mathbb{Z}/2$, and the quotient of $\mathrm{Sp}(n)$ by its center is denoted by $\mathrm{PSp}(n)$. We apply Lemma 3.5 to the case $H = \mathrm{Sp}(n)$. To this end, we compute the mod 2 cohomology of $B\mathrm{PSp}(2n)$ in low dimensions.

Lemma 4.2 *Let $\Delta = \{\pm(1, \dots, 1) \in \mathrm{Sp}(2)^n\}$. Then for $* \le 7$,*

$$H^*(B(\mathrm{Sp}(2)^n/\Delta); \mathbb{Z}/2) = \mathbb{Z}/2[x_2, x_3, x_5] \otimes \bigotimes_{k=1}^{n} \mathbb{Z}/2[x_{4,k}], \quad \mathrm{Sq}^2 x_{4,k} = x_2 x_{4,k},$$

where $|x_i| = i$ and $|x_{4,k}| = 4$.

Proof Consider the Serre spectral sequence for a homotopy fibration

$$\mathbb{R}P^\infty \to B\mathrm{Sp}(2)^n \to B(\mathrm{Sp}(2)^n/\Delta).$$

Since $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[w]$ with $|w| = 1$,

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \Delta(w, \mathrm{Sq}^1 w, \mathrm{Sq}^2 \mathrm{Sq}^1 w)$$

for $* \le 7$, where $\Delta(a_1, \dots, a_k)$ denotes the simple system of generators in $a_1, \dots, a_k$. Clearly, $\tau(w) = x_2$ for a generator x_2 of $H^2(B(\mathrm{Sp}(2)^n/\Delta); \mathbb{Z}/2) \cong \mathbb{Z}/2$, where τ denotes the transgression. Then by [23, Corollary 6.9], $\mathrm{Sq}^1 w$ and $\mathrm{Sq}^2 \mathrm{Sq}^1 w$ are also transgressive, and so we get $H^*(B(\mathrm{Sp}(2)^n/\Delta); \mathbb{Z}/2)$ for $* \le 7$ as stated. It remains to show $\mathrm{Sq}^2 x_{4,k} = x_2 x_{4,k}$. Recall that

$$H^*(B\mathrm{SO}(n); \mathbb{Z}/2) = \mathbb{Z}/2[w_2, w_3, \dots, w_n],$$

$$\mathrm{Sq}^i w_j = \sum_{k=0}^{i} \binom{j+k-i-1}{k} w_{i-k} w_{j+k}, \tag{4-1}$$

where w_i is the i^{th} Stiefel–Whitney class. Then since $\mathrm{PSp}(2) \cong \mathrm{SO}(5)$,

$$H^*(B\mathrm{PSp}(2); \mathbb{Z}/2) = \mathbb{Z}/2[y_2, y_3, y_4, y_5], \quad \mathrm{Sq}^2 y_4 = y_2 y_4,$$

where $|y_i| = i$. Let $q_k \colon B(\mathrm{Sp}(2)^n/\Delta) \to B\mathrm{PSp}(2)$ denote the induced map of the k^{th} projection for $k = 1, 2, \dots, n$. Then $q_k^*(y_2) = x_2$ and $q_k^*(y_4) = x_{4,k}$. Thus we obtain $\mathrm{Sq}^2 x_{4,k} = x_2 x_{4,k}$, completing the proof. □

Proposition 4.3 *For $* \leq 7$,*

$$H^*(B\mathrm{PSp}(n);\mathbb{Z}/2) = \mathbb{Z}/2[x_2, x_3, x_4, x_5], \quad \mathrm{Sq}^2 x_4 = x_4 x_2, \quad |x_i| = i.$$

Proof We can compute the mod 2 cohomology of $B\mathrm{PSp}(2n)$ in the same way as in the proof of Lemma 4.2 by considering a homotopy fibration

$$\mathbb{R}P^\infty \to B\mathrm{Sp}(2n) \to B\mathrm{PSp}(2n).$$

Then it remains to show $\mathrm{Sq}^2 x_4 = x_4 x_2$. Let Δ be as in Lemma 4.2. Then there is an inclusion $i : \mathrm{Sp}(2)^n/\Delta \to \mathrm{PSp}(2n)$. Clearly, $i^*(x_2) = x_2$ and $i^*(x_4) = x_{4,1} + \cdots + x_{4,n}$. Then we obtain $\mathrm{Sq}^2 x_4 = x_4 x_2$ by Lemma 4.2. □

Now we prove:

Proposition 4.4 *If $H = \mathrm{Sp}(n)$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.*

Proof Since the center of $\mathrm{Sp}(n)$ is isomorphic to $\mathbb{Z}/2$, we only consider

$$G = S^1 \times_{\mathbb{Z}/2} \mathrm{Sp}(n).$$

In this case, $s(G) = 2$, so by Lemma 3.3, it suffices to show $\langle \epsilon, p \rangle$ is nontrivial. First, we consider the case $G = S^1 \times_{\mathbb{Z}/2} \mathrm{Sp}(2n-1)$. The natural inclusion

$$\mathrm{Sp}(2n-1) \to \mathrm{SU}(4n-2)$$

sends the center of $\mathrm{Sp}(2n-1)$ injectively into the center of $\mathrm{SU}(4n-2)$. Then we get a homomorphism $G \to S^1 \times_{\mathbb{Z}/2} \mathrm{SU}(4n-2)$ which is an isomorphism in π_1. It is well known that the induced map $\pi_{8n-5}(\mathrm{Sp}(2n-1)) \to \pi_{8n-5}(\mathrm{SU}(4n-2))$ is an isomorphism; hence so is $\pi_{8n-5}(G) \to \pi_{8n-5}(S^1 \times_{\mathbb{Z}/2} \mathrm{SU}(4n-2))$. Then the proof of Proposition 4.1 implies that the Samelson product $\langle \epsilon, p \rangle$ is nontrivial.

Next, we consider $G = S^1 \times_{\mathbb{Z}/2} \mathrm{Sp}(2n)$. We apply Lemma 3.5 to $K = \mathrm{PSp}(2n)$ by setting $x = z = x_4$, $y = x_2$ and $\theta = \mathrm{Sq}^2$. By Proposition 4.3, the first and the second conditions of Lemma 3.5 are satisfied. The proof of Proposition 4.3 implies $\bar{q}^*(x_4)$ is nontrivial, where $H^4(B\mathrm{Sp}(2n);\mathbb{Z}/2) \cong QH^4(B\mathrm{Sp}(2n);\mathbb{Z}/2) \cong \mathbb{Z}/2$. Since the map

$$j^* \colon QH^4(B\mathrm{Sp}(2n);\mathbb{Z}/2) \to \Sigma QH^3(\mathrm{Sp}(2n);\mathbb{Z}/2)$$

is an isomorphism, we have $(\bar{q} \circ j)^*(x_4) \neq 0$. Moreover, for degree reasons, $(\bar{q} \circ j)^*(x_4)$ is not included in any element of $\theta(H^*(\Sigma\mathrm{Sp}(2n);\mathbb{Z}/2))$. Then the third condition of Lemma 3.5 is also satisfied. Thus $\langle \bar{\epsilon}, \bar{q} \rangle$ is nontrivial, and so by Lemma 3.4, $\langle \epsilon, p \rangle$ is nontrivial too. □

4.3 The case $H = \mathrm{Spin}(n)$

Finally, we consider the case $H = \mathrm{Spin}(n)$. We show some properties of the mod 2 cohomology of $B\mathrm{Spin}(n)$ that we are going to use. Recall that the mod 2 cohomology of $B\mathrm{SO}(n)$ is given as in (4-1).

Lemma 4.5 (1) *The mod 2 cohomology of $B\mathrm{Spin}(n)$ is given by*

$$H^*(B\mathrm{Spin}(n);\mathbb{Z}/2)=\mathbb{Z}/2[u_2,u_3,\dots,u_n,z]/(u_2,\mathrm{Sq}^{2^k}\mathrm{Sq}^{2^{k-1}}\cdots\mathrm{Sq}^1 u_2\mid k\geq 0),$$

where $\bar{q}^*_{\mathrm{SO}(n)}(w_j)=u_j$, $|z|=2^h$ for some $h>0$ and $\mathrm{Sq}^i u_j$ is computed by replacing w_j with u_j in (4-1).

(2) *For $2\leq i\leq n$ with $i\neq 2^k+1$, $j^*_{\mathrm{Spin}(n)}(u_i)\neq 0$.*

Proof Item (1) is a result of Quillen [25]. We prove statement (2). It is well known that $(j')^*(w_i)\neq 0$ for $i=2,3,\dots,n$, where $j'\colon \Sigma\mathrm{SO}(n)\to B\mathrm{SO}(n)$ is the natural map. On the other hand, it is shown in [13] that $(\Sigma\bar{q}_{\mathrm{SO}(n)})^*\circ(j')^*(w_i)\neq 0$. Then for $2\leq i\leq n$ with $i\neq 2^k+1$,

$$0\neq(\Sigma\bar{q}_{\mathrm{SO}(n)})^*\circ(j')^*(w_i)=j^*\circ\bar{q}_{\mathrm{SO}(n)}(w_i)=j^*(u_i). \qquad \square$$

The following lemma is easily deduced from the formula (4-1).

Lemma 4.6 *In $H^*(B\mathrm{SO}(n);\mathbb{Z}/2)$, we have:*

1. *If $n\equiv 0,1 \bmod 4$, then $\mathrm{Sq}^2 w_i$ for $i=n-3,n-1$ are decomposable and $\mathrm{Sq}^2 w_{n-1}$ includes the term w_2w_{n-1}.*
2. *If $n\equiv 2 \bmod 8$, then $\mathrm{Sq}^5 w_i$ for $i=n-4,n-9$ are decomposable and $\mathrm{Sq}^5 w_{n-4}$ includes the term w_2w_{n-1}.*
3. *If $n\equiv 6 \bmod 8$, then $\mathrm{Sq}^3 w_i$ for $i=n-2,n-4$ are decomposable and $\mathrm{Sq}^3 w_{n-2}$ includes the term w_2w_{n-1}.*
4. *If $n\equiv 3 \bmod 4$, then $\mathrm{Sq}^2 w_i$ for $i=n-2,n$ are decomposable and $\mathrm{Sq}^2 w_n$ includes the term w_2w_n.*

Let C_n denote the center of $\mathrm{Spin}(n)$. Then we have:

1. $C_{2n+1}\cong\mathbb{Z}/2$ and $\mathrm{Spin}(2n+1)/C_{2n+1}\cong\mathrm{SO}(2n+1)$.
2. $C_{4n+2}\cong\mathbb{Z}/4$ and $\mathrm{Spin}(4n+2)/(\mathbb{Z}/2)\cong\mathrm{SO}(4n+2)$.
3. $C_{4n}\cong\mathbb{Z}/2\times\mathbb{Z}/2$, $\mathrm{Spin}(4n)/(\mathbb{Z}/2\times 1)\cong\mathrm{SO}(4n)$ and $\mathrm{Spin}(4n)/(1\times\mathbb{Z}/2)\cong Ss(4n)$.

Proposition 4.7 *If $H=\mathrm{Spin}(n)$ and $K=\mathrm{SO}(n)$, then $\langle\epsilon,p\rangle$ is of order $s(G)$.*

Proof We only give a proof for n odd because the case n even is quite similarly proved. We apply Lemma 3.5 by setting $x = z = w_{n-1}$, $y = w_2$ and $\theta = \mathrm{Sq}^2$. By Lemma 4.6, the first and the second conditions of Lemma 3.5 are satisfied. By Lemmas 4.5 and 4.6, $(\bar{q} \circ j)^*(w_{n-1})$ is nontrivial and not included in any element of $\mathrm{Sq}^2(H^*(\Sigma\mathrm{Spin}(n); \mathbb{Z}/2))$. Then the third condition of Lemma 3.5 is also satisfied, so $\langle \bar{\epsilon}, \bar{q} \rangle \neq 0$. Thus, since $s(G) = 2$, Lemmas 3.3 and 3.4 complete the proof. □

Let $\mathrm{PO}(n) = \mathrm{Spin}(n)/C_n$. Then we have:

Corollary 4.8 *If $H = \mathrm{Spin}(4n+2)$ and $K = \mathrm{PO}(4n+2)$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.*

Proof Let $\bar{\rho}\colon \mathrm{SO}(4n+2) \to \mathrm{PO}(4n+2)$ denote the projection. Then $\bar{\rho}_*(\bar{\epsilon}_{\mathrm{SO}(4n+2)}) = 2\bar{\epsilon}_{\mathrm{PO}(4n+2)}$. Since $S^1 \wedge \mathrm{Spin}(4n+2)$ is simply connected, the map

$$\bar{\rho}_*\colon [S^1 \wedge \mathrm{Spin}(4n+2), \mathrm{SO}(4n+2)] \to [S^1 \wedge \mathrm{Spin}(4n+2), \mathrm{PO}(4n+2)]$$

is an isomorphism. By definition, $\bar{q}_{\mathrm{PO}(4n+2)} = \bar{\rho} \circ \bar{q}_{\mathrm{SO}(4n+2)}$. So by Proposition 4.7,

$$2\langle \bar{\epsilon}_{\mathrm{PO}(4n+2)}, \bar{q}_{\mathrm{PO}(4n+2)} \rangle = \bar{\rho}_*(\langle \bar{\epsilon}_{\mathrm{SO}(4n+2)}, \bar{q}_{\mathrm{SO}(4n+2)} \rangle) \neq 0.$$

Then by Lemma 3.3, the order of $\langle \bar{\epsilon}_{\mathrm{PO}(4n+2)}, \bar{q}_{\mathrm{PO}(4n+2)} \rangle$ is a nonzero multiple of $s(G) = 4$. Thus the proof is complete by Lemmas 3.3 and 3.4. □

Let Δ denote the diagonal subgroup of $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Proposition 4.9 *If $H = \mathrm{Spin}(4n)$ and $p_2(C) = 1 \times \mathbb{Z}/2, \Delta$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.*

Proof By triality of $\mathrm{Spin}(8)$, the case $H = \mathrm{Spin}(8)$ is proved by Proposition 4.7. Then we assume $n > 2$. The mod 2 cohomology of $\mathrm{PO}(4n)$ was determined by Baum and Browder [3] such that

$$H^*(\mathrm{PO}(4n); \mathbb{Z}/2) = \mathbb{Z}/2[v]/(v^{2^r}) \otimes \Delta(u_1, \dots, \hat{u}_{2^r-1}, \dots, u_{n-1}), \quad \bar{\rho}^*(u_i) = w_i,$$

where $4n = 2^r(2m+1)$, $|v| = 1$ and $|u_i| = i$. The elements v and u_1 correspond respectively to generators of subgroups $1 \times \mathbb{Z}/2$ and $\mathbb{Z}/2 \times 1$ of $C_{4n} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. The Hopf algebra structure of $H^*(\mathrm{PO}(4n); \mathbb{Z}/2)$ was also determined such that

$$\bar{\phi}(v) = 0 \quad \text{and} \quad \bar{\phi}(u_i) = \sum_{j=1}^{i-1} \binom{i}{j} u_j \otimes v^{i-j},$$

where $\bar{\phi}$ is the reduced diagonal map. Let $\gamma\colon \mathrm{PO}(4n)^2 \to \mathrm{PO}(4n)$ denote the commutator map. Since $\bar{\epsilon}(v) \neq 0$, it suffices to show $\gamma^*(x)$ includes the term $v \otimes y$

such that $\rho^*(y) \neq 0$, where $\rho\colon \mathrm{Spin}(4n) \to \mathrm{PO}(4n)$ denotes the projection. Let $\mu\colon \mathrm{PO}(4n)^2 \to \mathrm{PO}(4n)$ and $\Delta\colon \mathrm{PO}(4n) \to \mathrm{PO}(4n)^2$ denote the multiplication and the diagonal map, respectively. Let $\iota\colon \mathrm{PO}(4n) \to \mathrm{PO}(4n)$ be a map given by $\iota(x) = x^{-1}$, and let $T\colon \mathrm{PO}(4n)^2 \to \mathrm{PO}(4n)^2$ be the switching map. Then

$$\gamma = \mu \circ (\mu \times \mu) \circ (1 \times 1 \times \iota \times \iota) \circ (1 \times T \times 1) \circ (\Delta \times \Delta).$$

Let $I_k = \widetilde{H}^*(\mathrm{PO}(n)^k; \mathbb{Z}/2)$. Now we compute $\gamma^*(u_i)$:

$$\begin{aligned}
u_i &\xmapsto{\mu^*} u_i \otimes 1 + 1 \otimes u_i + i u_{i-1} \otimes v \bmod I_2^3 \\
&\xmapsto{(\mu\times\mu)^*} i(u_{i-1}\otimes v\otimes 1\otimes 1 + 1\otimes 1\otimes u_{i-1}\otimes v + u_{i-1}\otimes 1\otimes 1\otimes v + 1\otimes u_{i-1}\otimes v\otimes 1) \\
&\qquad\qquad \bmod I_1\otimes 1\otimes I_1\otimes 1 + 1\otimes I_1\otimes 1\otimes I_1 + I_4^3 \\
&\xmapsto{(1\times 1\times\iota\times\iota)^*} i(u_{i-1}\otimes v\otimes 1\otimes 1 + 1\otimes 1\otimes u_{i-1}\otimes v + u_{i-1}\otimes 1\otimes 1\otimes v + 1\otimes u_{i-1}\otimes v\otimes 1) \\
&\qquad\qquad \bmod I_1\otimes 1\otimes I_1\otimes 1 + 1\otimes I_1\otimes 1\otimes I_1 + I_4^3 \\
&\xmapsto{(1\times T\times 1)^*} i(u_{i-1}\otimes 1\otimes v\otimes 1 + 1\otimes u_{i-1}\otimes 1\otimes v + u_{i-1}\otimes 1\otimes 1\otimes v + 1\otimes v\otimes u_{i-1}\otimes 1) \\
&\qquad\qquad \bmod I_1\otimes I_1\otimes 1\otimes 1 + 1\otimes 1\otimes I_1\otimes I_1 + I_4^3 \\
&\xmapsto{(\Delta\times\Delta)^*} i(u_{i-1}\otimes v + v\otimes u_{i-1}) \bmod I_1\otimes 1 + 1\otimes I_1 + I_2^3.
\end{aligned}$$

Then for n odd, $\gamma^*(u_7)$ includes the term $v \otimes u_6$, where $\rho^*(u_6) \neq 0$ by Lemma 4.5, and for n even, $\gamma^*(u_{11})$ includes the term $v \otimes u_{10}$, where $\rho^*(u_{10}) \neq 0$ by Lemma 4.5. Thus the Samelson product $\langle \bar{\epsilon}, \bar{q} \rangle$ is nontrivial, completing the proof by Lemmas 3.3 and 3.4 because $s(G) = 2$. □

5 Exceptional case

First, we consider the case $H = E_6$.

Proposition 5.1 *If $H = E_6$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.*

Proof Since the center of E_6 is isomorphic to $\mathbb{Z}/3$, we only need to consider the case $G = S^1 \times_{\mathbb{Z}/3} E_6$. The mod 3 cohomology of $\mathrm{Ad}(E_6)$, which is the quotient of E_6 by its center, was determined by Kono [19] as

$$H^*(\mathrm{Ad}(E_6); \mathbb{Z}/3) = \mathbb{Z}/3[x_2, x_8]/(x_2^9, x_8^3) \otimes \Lambda(x_1, x_3, x_7, x_9, x_{11}, x_{16})$$

such that

$$\bar{\phi}(x_9) = x_8 \otimes x_1 + x_2 \otimes x_7 - x_2^3 \otimes x_3 + x_2^4 \otimes x_1 \quad \text{and} \quad \bar{q}^*(x_8) \neq 0,$$

where $|x_i| = i$. Then by the same computation as in the proof of Proposition 4.9, we can see that $\langle \bar{\epsilon}, \bar{q} \rangle$ is nontrivial. Thus by Lemmas 3.3 and 3.4, $\langle \epsilon, 1_G \rangle$ is of order $s(G) = 3$. □

Next, we consider the case $H = E_7$. Because the center of E_7 is isomorphic to $\mathbb{Z}/2$, we only need to consider the case $G = S^1 \times_{\mathbb{Z}/2} E_7$. The Hopf algebra structure of $H^*(\mathrm{Ad}(E_7); \mathbb{Z}/2)$ was determined by Ishitoya, Kono and Toda [13], from which we can see that the same computation as $\mathrm{Ad}(E_6)$ does not apply to $\mathrm{Ad}(E_7)$. So we apply Lemma 3.5. Kono and Mimura [21] showed that the mod 2 cohomology of $B\mathrm{Ad}(E_7)$ is generated by elements x_i for $i \in \{2, 3, 6, 7, 10, 11, 18, 19, 34, 35, 64, 66, 67, 96, 112\}$, where $|x_i| = i$. We determine $\mathrm{Sq}^2 x_6$.

Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Elements of the spin group $\mathrm{Spin}(n)$ are expressed by using $e_1, e_2, \dots, e_n$. See [1, Chapter 3]. Recall from [1, Proposition 4.2] that there are two representations

$$\Delta^+_{2n}, \Delta^-_{2n} \colon \mathrm{Spin}(2n) \to \mathrm{SU}(2^{n-1})$$

such that Δ^+_n has weights $\frac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_n)$ with even numbers of minus signs and Δ^-_n has weights $\frac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_n)$ with odd numbers of minus signs.

Proposition 5.2 *There is a natural isomorphism*

$$\mathrm{Spin}(4) \cong \mathrm{Ker}\, \Delta^+_4 \times \mathrm{Ker}\, \Delta^-_4.$$

Proof There is a product decomposition $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ such that

$$\Delta^\pm_4 \colon \mathrm{Spin}(4) \to \mathrm{SU}(2)$$

are identified with projections $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(2)$. □

As in [1, Theorem 6.1], there is a homomorphism

$$\theta \colon \mathrm{Spin}(16) \to E_8$$

whose kernel is $\{1, e_1 e_2 \cdots e_{16}\}$. Let $\mu \colon \mathrm{Spin}(4) \times \mathrm{Spin}(12) \to \mathrm{Spin}(16)$ denote the homomorphism covering the inclusion

$$\mathrm{SO}(4) \times \mathrm{SO}(12) \to \mathrm{SO}(16), \quad (A, B) \mapsto \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

Define $\bar{\mu} = \theta \circ \mu \colon \mathrm{Spin}(4) \times \mathrm{Spin}(12) \to E_8$. Then

$$\mathrm{Ker}\, \bar{\mu} = \{(1,1), (-1,-1), (e_1e_2e_3e_4, e_5e_6 \cdots e_{16}), (-e_1e_2e_3e_4, -e_5e_6 \cdots e_{16})\}.$$

Recall from [1, Chapter 8] that E_7 is defined as the centralizer of $\bar{\mu}(\mathrm{Ker}\,\Delta_4^+ \times 1)$ in E_8. Then, by Proposition 5.2, there is a homomorphism

$$\hat{\mu}\colon \mathrm{Ker}\,\Delta_4^- \times \mathrm{Spin}(12) \to E_7.$$

Since $-e_1e_2e_3e_4 \in \mathrm{Ker}\,\Delta_4^+$, $\bar{\mu}(-e_1e_2e_3e_4, 1)$ commutes with every element of E_7 in E_8. Moreover, $\bar{\mu}(-e_1e_2e_3e_4, 1) = \bar{\mu}(e_1e_2e_3e_4, -1) = \hat{\mu}(e_1e_2e_3e_4, -1)$, which belongs to E_7 and is not the unit of E_7. Then we obtain:

Proposition 5.3 *The center of E_7 is $\{1, \hat{\mu}(e_1e_2e_3e_4, -1)\}$.*

Let $L = (\mathrm{Ker}\,\Delta_4^- \times \mathrm{Spin}(12))/\{(1,1), (e_1e_2e_3e_4, -1)\}$. Then by Proposition 5.3, there is a map

$$\rho\colon L \to \mathrm{Ad}(E_7),$$

which is an isomorphism in the second mod 2 cohomology.

Lemma 5.4 *In $H^*(B\mathrm{Ad}(E_7); \mathbb{Z}/2)$, $\mathrm{Sq}^2 x_6$ is decomposable and includes the term x_2x_6.*

Proof By [21; 22], $(\bar{\mu} \circ (1 \times \bar{q}))^*(x_6)$ includes the term $1 \otimes u_6$, where u_i is as in Lemma 4.5. Note that the composition

$$\mathrm{Spin}(12) \to \mathrm{Ker}\,\Delta_4^- \times \mathrm{Spin}(12) \to L \xrightarrow{q_2} \mathrm{SO}(12)$$

is the natural projection, where q_2 is the second projection. Then by degree reasons,

$$\rho^*(x_6) + a\rho^*(x_2)^3 + b\rho^*(x_3)^2 = q_2^*(w_6)$$

for some $a, b \in \mathbb{Z}/2$. On the other hand, $q_2^*\colon H^2(B\mathrm{SO}(12); \mathbb{Z}/2) \to H^2(BL; \mathbb{Z}/2)$ is an isomorphism, implying $\rho^*(x_2) = q_2^*(w_2)$. Then since $\mathrm{Sq}^2 w_6 = w_2w_6$ by (4-1) and $\mathrm{Sq}^2 x_6$ is decomposable by degree reasons, $\mathrm{Sq}^2 x_6$ is decomposable and includes the term x_2x_6, as stated. □

We are ready to prove:

Proposition 5.5 *If $H = E_7$, then $\langle \epsilon, p \rangle$ is of order $s(G)$.*

Proof As mentioned above, we only need to consider $G = S^1 \times_{\mathbb{Z}/2} E_7$. We apply Lemma 3.5 by setting $x = z = x_6$, $y = x_2$ and $\theta = \mathrm{Sq}^2$. By Lemma 5.4, the first and second conditions of Lemma 3.5 are satisfied. As in [22], $\bar{q}^*(x_6)$ is a generator of $H^6(BE_7; \mathbb{Z}/2)$ such that $(\bar{q} \circ j)^*(x_6)$ is nontrivial. Then by degree reasons, the third condition of Lemma 3.5 is also satisfied, implying $\langle \bar{\epsilon}, \bar{q} \rangle$ is nontrivial. Since $s(G) = 2$, the proof is complete by Lemmas 3.3 and 3.4. □

6 Proofs of Theorems 1.3 and 1.4

This section proves Theorems 1.3 and 1.4. First, we prove Theorem 1.4.

Proof of Theorem 1.4 Suppose $H \cong H_1 \times \cdots \times H_k$, where each H_i is a simple Lie group. Let $r_i \colon S^1 \times H \to S^1 \times H_i$ be the projection, and let $G_i = (S^1 \times H_i)/(r_i(C))$ for $i = 1, 2, \dots, k$. By definition, $s(G)$ is the least common multiple of $s(G_1), \dots, s(G_k)$. Let $\bar{r}_i \colon G \to G_i$ and $\iota_i \colon S^1 \times H_i \to S^1 \times H$ denote the projection and the inclusion, respectively. Then $\bar{r}_i \circ \epsilon_G = \epsilon_{G_i}$ and $\bar{r}_i \circ p_G \circ \iota_i = p_{G_i}$, so

$$(1 \wedge \iota_i)^* \circ (\bar{r}_i)_*(\langle \epsilon_G, p_G \rangle) = \langle \bar{r}_i \circ \epsilon_G, \bar{r}_i \circ p_G \circ \iota_i \rangle = \langle \epsilon_{G_i}, p_{G_i} \rangle.$$

Thus the order of $\langle \epsilon_G, p_G \rangle$ is a nonzero multiple of the order of $\langle \epsilon_{G_i}, p_{G_i} \rangle$. So by Propositions 4.1, 4.4, 4.7, 5.1 and 5.5, the order of $\langle \epsilon_G, p_G \rangle$ is a nonzero multiple of $s(G_i)$ for $i = 1, 2, \dots, k$; hence so is $\langle \epsilon_G, 1_G \rangle$. Therefore, by Lemma 3.3, the proof is complete. □

Next, we prove Theorem 1.3.

Proof of Theorem 1.3 First, we prove the case $H = \mathrm{SU}(n)^r$. The implication (1) $\Longrightarrow$ (2) follows from Theorem 1.2. We prove the implication (2) $\Longrightarrow$ (1). Let $\partial_k \colon G \to \mathrm{map}_*(S^2, BG; k) \simeq \Omega_0 G$ be as in Section 2, and let $q_i \colon H \to \mathrm{SU}(n)$ be the projection onto the i^{th} $\mathrm{SU}(n)$. Then by Lemma 2.1, the proof of Proposition 4.1 implies that the image of the map

$$(\partial_k)_* \colon \pi_{2n-1}(G) \to \pi_{2n-1}(\Omega_0 G)$$

is isomorphic to $\prod_{i=1}^r \mathbb{Z}/\frac{n!}{(k, |q_i(C)|)}$, where $\pi_{2n-1}(\Omega_0 G) \cong (\mathbb{Z}/n!)^r$. By (2-1), there is an exact sequence

$$0 \to \prod_{i=1}^r \mathbb{Z}\Big/\frac{n!}{(k, |q_i(C)|)} \to \pi_{2n-1}(B\mathcal{G}_k(S^2, G)) \to \pi_{2n-1}(BG) \cong \pi_{2n-1}(B\mathrm{SU}(n)^r) = 0.$$

Then since $\pi_{2n-1}(B\mathcal{G}_k(S^2, G)) \cong \pi_{2n-2}(\mathcal{G}_k(S^2, G))$,

$$\pi_{2n-2}(\mathcal{G}_k(S^2, G)) \cong \prod_{i=1}^r \mathbb{Z}/(k, |q_i(C)|).$$

So if $\mathcal{G}_k(X, G) \simeq \mathcal{G}_l(X, G)$, then $\pi_{2n-2}(\mathcal{G}_k(S^2, G)) \simeq \pi_{2n-2}(\mathcal{G}_l(S^2, G))$, implying

$$\big(k, |q_1(C)|\big) \cdots \big(k, |q_r(C)|\big) = \big(l, |q_1(C)|\big) \cdots \big(l, |q_r(C)|\big).$$

As in the proof of Theorem 1.4, $s(G)$ is the least common multiple of

$$|q_1(C)|, \dots, |q_r(C)|.$$

Then it is easy to see that the above equality implies $(k, s(G)) = (l, s(G))$.

Next, we prove the case $H = \mathrm{SU}(4n-2)^s \times \mathrm{Sp}(2n-1)^t$. Note that

$$\pi_{8n-4}(\mathrm{Sp}(2n-1)) \cong \mathbb{Z}/2.$$

Then similarly to the above case, the proofs of Propositions 4.1 and 4.4 imply that the image of the map

$$(\partial_k)_* \colon \pi_{8n-5}(G) \to \pi_{8n-5}(\Omega_0 G)$$

is isomorphic to

$$\prod_{i=1}^{s} \mathbb{Z}\Big/\frac{(4n-2)!}{\bigl(k, |q_i(C)|\bigr)} \times \prod_{i=1}^{t} \mathbb{Z}\Big/\frac{2}{(k, q_i(C))}.$$

So we also get an exact sequence

$$\begin{aligned} 0 \to \prod_{i=1}^{s} \mathbb{Z}\Big/\frac{(4n-2)!}{\bigl(k, |q_i(C)|\bigr)} \times \prod_{i=1}^{t} \mathbb{Z}\Big/\frac{2}{\bigl(k, |q_i(C)|\bigr)} &\to \pi_{8n-5}(B\mathcal{G}_k(S^2, G)) \\ &\to \pi_{2n-1}(BG) \cong \pi_{8n-5}(B\mathrm{SU}(4n-2)^s \times B\mathrm{Sp}(2n-1)^t) = 0. \end{aligned}$$

Thus, by arguing as above, we obtain $(k, s(G)) = (l, s(G))$ whenever $\mathcal{G}_k(X, G) \simeq \mathcal{G}_l(X, G)$. Therefore, the proof is complete. □

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Received: 3 August 2021 Revised: 6 February 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23 Issue 5 (pages 1935–2414) 2023