Diffeomorphisms of odd-dimensional discs,
glued into a manifold

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Let $\mu_M : B\mathrm{Diff}_\partial(D^{2n+1}) \to B\mathrm{Diff}(M)$, for a compact $(2n+1)$–dimensional smooth manifold M, be the map defined by extending diffeomorphisms on an embedded disc by the identity. By a classical result of Farrell and Hsiang, the rational homotopy groups and the rational homology of $B\mathrm{Diff}_\partial(D^{2n+1})$ are known in the concordance stable range. We prove two results on the behaviour of the map μ_M in the concordance stable range. Firstly, it is *injective* on rational homotopy groups, and secondly, it is *trivial* on rational homology if M contains sufficiently many embedded copies of $S^n \times S^{n+1} \setminus \mathrm{int}(D^{2n+1})$. We also show that μ_M is generally not injective on homotopy groups outside the stable range.

The homotopical statement is probably not new and follows from the theory of smooth torsion invariants. The noninjectivity outside the stable range is based on recent work by Krannich and Randal-Williams. The homological statement relies on work by Botvinnik and Perlmutter on diffeomorphisms of odd-dimensional manifolds.

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1 Introduction

For a smooth compact manifold M with boundary, we denote by $\mathrm{Diff}(M)$ the topological group of diffeomorphisms of M, and by $\mathrm{Diff}_\partial(M) \subset \mathrm{Diff}(M)$ the subgroup of those diffeomorphisms which agree with the identity near ∂M. A celebrated classical result by Farrell and Hsiang [10] states that

$$\textbf{(1.1)} \qquad \pi_k(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } k \equiv 0 \pmod 4, \\ 0 & \text{if } k \not\equiv 0 \pmod 4, \end{cases}$$

in a range of degrees which was originally given by $k < \frac{1}{3}n$, but (1.1) holds more generally if $k \leq \phi^{\mathbb{Q}}(D^{2n})$, where $\phi^{\mathbb{Q}}(D^{2n})$ is the *rational concordance stable range for D^{2n}*, which we briefly recall.

For a compact smooth manifold M, let

$$C(M) := \mathrm{Diff}(M \times [0,1], M \times \{0\} \cup \partial M \times [0,1])$$

be the concordance diffeomorphism group of M, and let $\sigma_+ \colon C(M) \to C(M \times [0,1])$ be the (positive) suspension map defined in eg Igusa [16, Section 6.2]. Define $\phi(M)$ to be the largest integer k such that the maps $\sigma_+ \colon C(M \times [0,1]^m) \to C(M \times [0,1]^{m+1})$, for $m \geq 0$, are all k–connected. Similarly, define $\phi^{\mathbb{Q}}(M) \geq \phi(M)$ using rational connectivity instead of connectivity (this makes sense if $\dim(M) \geq 6$ as $\pi_0(C(M))$ is abelian in that case, by Hatcher and Wagoner [14, Lemma 1.1]).

Igusa's stability theorem [15, page 6] states that

(1.2) $$\phi(M^d) \geq \min\left(\tfrac{1}{2}(d-7), \tfrac{1}{3}(d-4)\right).$$

Recent work by Krannich and Randal-Williams [21] gives the optimal range in which (1.1) holds. Corollary B of [21] shows that

(1.3) $$\phi^{\mathbb{Q}}(D^d) = d - 4 \quad \text{if } d \geq 10,$$

and hence (1.1) holds if $k \leq 2n-4$, provided that $n \geq 5$. Theorem A of [21] improves this to $k \leq 2n-3$, again for $n \geq 5$. These results slightly exceed Krannich [20, Corollary B].

For an arbitrary smooth compact and nonempty manifold M of dimension $2n+1$, choose an embedding $D^{2n+1} \to \mathrm{int}\, M$. Extending diffeomorphisms by the identity gives a gluing map

$$\mu_M^{\partial} \colon \mathrm{Diff}_{\partial}(D^{2n+1}) \to \mathrm{Diff}_{\partial}(M).$$

We may also consider the composition

$$\mu_M \colon \mathrm{Diff}_{\partial}(D^{2n+1}) \xrightarrow{\mu_M^{\partial}} \mathrm{Diff}_{\partial}(M) \to \mathrm{Diff}(M).$$

The purpose of this note is to study the effect of the maps $B\mu_M$ and $B\mu_M^{\partial}$ on rational homotopy and homology. The precise choice of the embedding does not play a role for this question as long as M is connected. This is because the homotopy class of $B\mu_M$ only depends on the isotopy class of the embedding. If M is connected and not orientable there is only one isotopy class of embeddings, and if M is connected and orientable there are two such isotopy classes, which differ by the reflection automorphism of the group $\mathrm{Diff}_{\partial}(D^{2n+1})$.

Theorem 1.4 (homotopical theorem) *For every $(2n+1)$–dimensional manifold M, the maps*

$$(\mu_M)_* \colon \pi_k(B\mathrm{Diff}_{\partial}(D^{2n+1})) \otimes \mathbb{Q} \to \pi_k(B\mathrm{Diff}(M)) \otimes \mathbb{Q}$$

and
$$(\mu_M^\partial)_* \colon \pi_k(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \to \pi_k(B\mathrm{Diff}_\partial(M)) \otimes \mathbb{Q}$$
are injective when $k \neq 1$ and $k \leq \phi^{\mathbb{Q}}(D^{2n})$.

Remark 1.5 Note that $\pi_1(B\mathrm{Diff}_\partial(D^{2n+1})) = \pi_0(\mathrm{Diff}_\partial(D^{2n+1}))$ is always a finite group; this is trivial when $n = 0$ and follows from Cerf [7] for $n = 1$. For $n \geq 2$, Cerf [8, corollaire 2] and the h-cobordism theorem identify $\pi_0(\mathrm{Diff}_\partial(D^{2n+1}))$ with the group of homotopy $(2n+2)$–spheres, which is finite by Kervaire and Milnor [19].

Remark 1.6 By [21, Corollary B], Theorem 1.4 holds for $k \leq 2n - 4$ if $n \geq 5$. Theorem 1.4 is also true for $k = 2n-3$ and $n \geq 5$, since $\pi_{2n-3}(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} = 0$ for such n by [21, Theorem A].

Theorem 1.4 could have been proven with little effort in Badzioch, Dorabiała, Klein and Williams [1] at latest. It was in fact known by experts and we learnt the statement from Mauricio Bustamante. The proof is given here for the sake of completeness and to contrast it with our main result (Theorem 1.7 below), which seemingly goes the opposite direction.

Our result concerns the effect of μ_M in rational homology. Since $B\mathrm{Diff}_\partial(D^{2n+1})$ is a connected E_{2n+1}–space (and hence a homotopy commutative H-space), we have that $H_*(B\mathrm{Diff}_\partial(D^{2n+1}); \mathbb{Q})$ (with the Pontryagin product) is the free graded-commutative algebra generated by $\pi_*(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q}$. Therefore, in the concordance stable range, $H_*(B\mathrm{Diff}_\partial(D^{2n+1}); \mathbb{Q})$ is a polynomial algebra with one generator in each dimension divisible by 4.

Let
$$U_g^n := \#^g(S^n \times S^{n+1})$$
be the connected sum of g copies of $S^n \times S^{n+1}$, and let
$$U_{g,1}^n := U_g^n \setminus \mathrm{int}\, D^{2n+1}$$
be U_g^n with the interior of a disc removed.

Theorem 1.7 (homological theorem) *Let M be a connected manifold of dimension $2n + 1 \geq 9$ and suppose that M contains an embedded copy of $U_{g,1}^n$. Then the maps*
$$(\mu_M)_* \colon \widetilde{H}_k(B\mathrm{Diff}_\partial(D^{2n+1}); \mathbb{Q}) \to \widetilde{H}_k(B\mathrm{Diff}(M); \mathbb{Q})$$
and
$$(\mu_M^\partial)_* \colon \widetilde{H}_k(B\mathrm{Diff}_\partial(D^{2n+1}); \mathbb{Q}) \to \widetilde{H}_k(B\mathrm{Diff}_\partial(M); \mathbb{Q})$$
are trivial if $k \leq \phi^{\mathbb{Q}}(D^{2n}) + 1$ and $k \leq \frac{1}{2}(g - 4)$.

Finally, using the recent work [21], we can show that the range for the validity of Theorem 1.4 given in Remark 1.6 is optimal.

Theorem 1.8 *For even $n \geq 6$, there is a closed $(2n+1)$–dimensional smooth manifold M such that the kernel of*

$$(\mu_M)_* \colon \pi_{2n-2}(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \to \pi_{2n-2}(B\mathrm{Diff}(M)) \otimes \mathbb{Q}$$

is nonzero.

Acknowledgements

2 Proof of the homotopical theorem

The proof of Theorem 1.4 relies on higher torsion invariants as axiomatized by Igusa [18], and we review some background beforehand. Let $K(\mathbb{Z})$ be the algebraic K–theory spectrum of $\mathbb{Z}$ and let

$$u \colon Q(S^0) \to \Omega^\infty K(\mathbb{Z})$$

be the unit map on infinite loop spaces.

Let M be a finite CW complex of dimension d, and let $\pi \colon E \to B$ be a fibration with fibres homotopy equivalent to M. Let

(2.1) $$\rho_k(\pi) \colon B \to B\mathrm{GL}(H_k(M;\mathbb{Z}))$$

be the map induced by the monodromy action of the fundamental group on the homology of the fibre. The $\mathbb{Z}$–module $H_k(M;\mathbb{Z})$ is finitely generated, and hence there is a canonical map

$$\iota \colon B\mathrm{GL}(H_k(M;\mathbb{Z})) \to \Omega^\infty K(\mathbb{Z}).$$

Such a map exists even if the homology groups are not free, essentially because $\mathbb{Z}$ is a regular ring (each finitely generated $\mathbb{Z}$–module has a finite length resolution by projective finitely generated $\mathbb{Z}$–modules). See the discussion leading up to [9, Proposition 6.7] for details.

The map ι hits the component of $[H_k(M;\mathbb{Z})] \in K_0(\mathbb{Z}) = \pi_0(\Omega^\infty K(\mathbb{Z})) \cong \mathbb{Z}$. The algebraic K–theory Euler characteristic of the fibration π is the alternating sum

$$\chi(\pi) := \sum_{k=0}^{d} (-1)^k \iota \circ \rho_k(\pi) \colon B \to \Omega^\infty K(\mathbb{Z}),$$

where $d = \dim(M)$ and we have used the H–space structure on $\Omega^\infty K(\mathbb{Z})$ to form the sum. Of course, $\chi(\pi)$ hits the component indexed by $\chi(M) \in \mathbb{Z} = K_0(\mathbb{Z})$.

The fibration π has an associated transfer map [3]

$$\mathrm{trf}_\pi \colon \Sigma^\infty B_+ \to \Sigma^\infty E_+$$

on the level of suspension spectra; we mostly consider its adjoint, also written

$$\mathrm{trf}_\pi \colon B \to Q(E_+).$$

The Dwyer–Weiss–Williams index theorem [9, Corollary 8.12] implies that if π is a *smooth* fibre bundle, the diagram

(2.2)
$$
\begin{array}{ccc}
B & \xrightarrow{\ \mathrm{trf}_\pi\ } & Q(E_+) \\
 & {\scriptstyle \chi(\pi)} \searrow & \big\downarrow {\scriptstyle u \circ c} \\
 & & \Omega^\infty K(\mathbb{Z})
\end{array}
$$

commutes, up to a preferred homotopy (here $u \circ c$ is the composition of the unit map with the collapse map $c \colon Q(E_+) \to Q(S^0)$).

Remark 2.3 Actually, Theorem 8.5 of [9] proves a stronger version involving the algebraic K–theory $A(E)$ of the space E and a fibrewise version thereof. Raptis and Steimle gave a substantially simpler proof of the homotopy-commutativity of (2.2) in [24]; they also showed [9, Theorem 8.5] for smooth bundles in [25].

The diagram (2.2) can be used to define secondary invariants under additional hypotheses on the bundle π; we follow the approach of [2] here, with some modifications. The extra hypothesis to be made is that the monodromy action of $\pi_1(B)$ on $H_k(M;\mathbb{Z})$ is unipotent for all k (in [2] the authors consider homology with coefficients in a field,

but the construction generalizes to regular rings such as $\mathbb{Z}$; see [2, Remark 6.11]). Moreover, we assume as in [2] that the base space B is a compact manifold, possibly with boundary.

Under these assumptions, the map $\chi(\pi)$ comes with a preferred homotopy to the constant map to the point

$$\chi(M) := \sum_{k=0}^{d} (-1)^k [H_k(M;\mathbb{Z})] \in \Omega^\infty K(\mathbb{Z});$$

see [2, Theorem 6.7]. Combining this homotopy with the preferred homotopy from (2.2) yields a map

$$(2.4) \qquad T(\pi): B \to \mathrm{hofib}_{\chi(M)}(u) \simeq \mathrm{hofib}_0(u),$$

where we use the infinite loop space structures to identify the homotopy fibres. Using Borel's computation [5] of $\pi_*(\Omega^\infty K(\mathbb{Z})) \otimes \mathbb{R}$, we define characteristic classes of unipotent smooth bundles as follows. Note that

$$(2.5) \qquad H^*(\mathrm{hofib}_0(u); \mathbb{R}) = \mathbb{R}[a_4, a_8, \dots]$$

for certain generators a_{4k} of degree $4k$ (the transgression of a_k is the Borel class in $H^{4k+1}(\Omega^\infty K(\mathbb{Z}); \mathbb{R})$). Following [2, Section 7] (but using the notation of [1]), define

$$(2.6) \qquad t_{4k}^s(\pi) := T(\pi)^* a_{4k} \in H^{4k}(B; \mathbb{R}).$$

It is convenient for us to replace the coefficient field by $\mathbb{Q}$, which can be done as follows. First, the Borel class comes from a spectrum cohomology class $b_k \in H^{4k+1}(K(\mathbb{Z}); \mathbb{R})$. Second, Borel showed that $H^{4k+1}(K(\mathbb{Z}); \mathbb{R})$ is 1–dimensional, and so there is $\alpha_k \in \mathbb{R}^\times$ such that $\alpha_k b_k$ lies in $H^*(K(\mathbb{Z}); \mathbb{Q})$. We now define

$$(2.7) \qquad \bar{t}_{4k}^s(\pi) = \bar{t}_{4k}^s(E) := \alpha_k t_{4k}^s(\pi) \in H^{4k}(B; \mathbb{Q}).$$

The construction of (2.4) is given in [2] only for compact manifold bases; the definition of (2.7) can be generalized to arbitrary base spaces as follows. For an arbitrary unipotent bundle $E \to B$, we define $\bar{t}_{4k}^s(E) \in H^{4k}(B; \mathbb{Q})$ as the class corresponding to the homomorphism

$$(2.8) \qquad \Omega_{4k}^{\mathrm{fr}}(B) \to \mathbb{Q}; \quad [X, f] \mapsto \langle \bar{t}_{4k}^s(f^*E); [X] \rangle$$

from the framed bordism group of B under the isomorphism

$$H^{4k}(B; \mathbb{Q}) \cong \mathrm{Hom}(\Omega_{4k}^{\mathrm{fr}}(B); \mathbb{Q})$$

(we need [2, Proposition 7.3] to show that (2.8) is well defined).

Now let $\mathrm{Torr}(M) \subset \mathrm{Diff}(M)$ be the *Torelli diffeomorphism group*, ie the subgroup of those diffeomorphisms which act as the identity on $H_*(M; \mathbb{Z})$ and $H_*(M, \partial M; \mathbb{Z})$. The universal M–bundle over $B\mathrm{Torr}(M)$ is clearly unipotent, and combining the classes $\bar{t}^s_{4k}$, we obtain a map

$$(2.9) \qquad \tau_M : B\mathrm{Torr}(M) \to \prod_{k \geq 1} K(\mathbb{Q}, 4k).$$

Observing that $\mathrm{Torr}(D^{2n+1}) = \mathrm{Diff}^+(D^{2n+1})$ is the group of orientation-preserving diffeomorphisms, we obtain in particular

$$(2.10) \quad \tau_{D^{2n+1}} : B\mathrm{Diff}_\partial(D^{2n+1}) \to B\mathrm{Diff}^+(D^{2n+1})$$
$$= B\mathrm{Torr}(D^{2n+1}) \to \prod_{k \geq 1} K(\mathbb{Q}, 4k).$$

Farrell and Hsiang's theorem may be restated as follows.

Theorem 2.11 *The map $\tau_{D^{2n+1}}$ induces an isomorphism on rational homotopy groups in degrees at most $\phi^{\mathbb{Q}}(D^{2n})$.*

Proof The map $\tau_{D^{2n+1}}$ factors through

$$B\mathrm{Diff}_\partial(D^{2n+1}) \to BC(D^{2n}) \to B\mathrm{Diff}^+(D^{2n+1}).$$

Consider the diagram

$$(2.12) \qquad \begin{array}{ccc} BC(D^{2n}) & \longrightarrow & B\mathrm{Diff}^+(D^{2n+1}) \\ \downarrow{\scriptstyle \sigma^{om}_+} & & \downarrow{\scriptstyle \times\, \mathrm{id}_{D^m}} \\ BC(D^{2n+m}) & \longrightarrow & B\mathrm{Diff}^+(D^{2n+m+1}) \end{array}$$

The left vertical map is a composition of the suspension map and the right vertical map is given by taking products with D^m and the identification $D^{2n+m+1} = D^{2n+1} \times D^m$. The square commutes up to homotopy by the definition of the suspension map σ_+. A special case of [1, Theorem 7.1] states that $\bar{t}^s_{4k}(E \times D^m) = \bar{t}^s_{4k}(E)$ for each unipotent bundle $E \to B$. It follows that $\tau_{D^{2n+1}} : B\mathrm{Diff}_\partial(D^{2n+1}) \to \prod_{k \geq 1} K(\mathbb{Q}, 4k)$ factors as

$$(2.13) \quad B\mathrm{Diff}_\partial(D^{2n+1}) \to BC(D^{2n}) \to BC(D^{2n})$$
$$:= \mathrm{hocolim}_m BC(D^{2n+m}) \to \prod_{m \geq 1} K(\mathbb{Q}, 4k).$$

All three maps in (2.13) induce isomorphisms on rational homotopy up to degree $\phi^{\mathbb{Q}}(D^{2n})$. This is true for the second map by definition.

The third map is a rational equivalence. Since the Whitehead group of $\pi_1(D^{2n})$ is trivial, $BC(D^{2n})$ is the stable h–cobordism space $\mathcal{H}(D^{2n})$. The stable h–cobordism theorem [27] states an equivalence $\mathcal{H}(D^{2n}) \simeq \mathrm{hofib}(Q(S^0) \to A(*))$, and the linearization map from $A(*)$ to $\Omega^\infty K(\mathbb{Z})$ induces a rational equivalence

$$\mathrm{hofib}(Q(S^0) \to A(*)) \to \mathrm{hofib}(Q(S^0) \to \Omega^\infty K(\mathbb{Z})).$$

Together with [5], this shows that the rational homotopy groups of $BC(D^{2n})$ and of $\prod_{m \geq 1} K(\mathbb{Q}, 4k)$ are abstractly isomorphic and at most 1–dimensional.

To conclude that the third map in (2.13) is a rational isomorphism, it is therefore enough to prove that the induced map on rational homotopy groups is nontrivial whenever its target is nonzero, and this amounts to proving that for each $k \geq 1$, there is an m and an element in $\pi_{4k}(BC(D^{2n+m}))$ such that $\bar{t}^s_{D^{2n+m+1}}$ is nontrivial on that element. This was done by Igusa in [16, Theorem 6.4.2], but with the *higher Franz–Reidemeister torsion* classes $t^{\mathrm{FR}}_{4k} \in H^{4k}(BC(D^{2n+m}); \mathbb{R})$ in place of $\bar{t}^s_{4k}$. These were constructed using ideas from Morse theory in [16, Section 5.7.2] (for bundles with structure group $\mathrm{Torr}(M)$) and in [17, Section 2.11] for unipotent bundles. The main theorem of [1] shows that there is a universal constant $\lambda_{4k} \in \mathbb{R}^\times$ such that $\bar{t}^s_{4k}(\pi) = \lambda_{4k} t^{\mathrm{IK}}_{4k}(\pi)$ for all unipotent bundles over compact manifold bases, and so the third map in (2.13) is a rational equivalence.

It is shown in [16, Section 6.5] that the first map in (2.13) induces an isomorphism on rational homotopy groups up to degree $\phi^\mathbb{Q}(D^{2n})$. In loc. cit., the result is stated in terms of (1.2), so we give a few more details here. Consider the fibre sequence

$$\mathrm{Diff}_\partial(D^{2n+1}) \to C(D^{2n}) \to \mathrm{Diff}_\partial(D^{2n}).$$

The two maps are compatible with the following involutions on the spaces: the group inversion on $\mathrm{Diff}_\partial(D^{2n})$, an involution defined at the beginning of [16, Section 6.5] on $C(D^{2n})$, and the involution

(2.14) $I : \mathrm{Diff}_\partial(D^{2n+1}) \to \mathrm{Diff}_\partial(D^{2n+1})$

given by conjugation with the reflection map

$$r(x_1, \ldots, x_{2n+1}) := (x_1, \ldots, x_{2n}, -x_{2n+1}).$$

The rational homotopy sequence of the fibration splits into negative and positive eigenspaces of these involutions. An Eckmann–Hilton argument proves that

$$\pi^+_*(\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \to \pi^+_*(C(D^{2n})) \otimes \mathbb{Q}$$

is an isomorphism in all degrees. On the other hand, $I_* = \mathrm{id}$ on $\pi_k(\_) \otimes \mathbb{Q}$ for $k \leq \phi^{\mathbb{Q}}(D^{2n})$; Corollary 6.5.3 of [16] states this when k is in the range given by (1.2), but the proof clearly works for $k \leq \phi^{\mathbb{Q}}(D^{2n})$. Hence, for $k \leq \phi^{\mathbb{Q}}(D^{2n}) + 1$,

$$\pi_k(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} = \pi_k^+(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \cong \pi_k^+(BC(D^{2n})) \otimes \mathbb{Q}.$$

Finally, $\pi_k^-(BC(D^{2n})) \otimes \mathbb{Q} = 0$ for $k \leq \phi^{\mathbb{Q}}(D^{2n})$. To see this, observe that the stabilization map $BC(D^d)) \to BC(D^{d+1})$ switches the eigenspaces of the involutions by [16, Lemma 6.5.1]. Hence it is enough to check that $\pi_k^-(BC(D^{2n})) \otimes \mathbb{Q} = 0$ for very large n, and this follows from Theorem 6.4.2 and Lemma 6.5.4 of [16], using that the third map in (2.13) is a rational equivalence. □

Proof of Theorem 1.4 for closed M We first consider the case where M is closed. The cohomology classes $\bar{t}_{4k}^s$ have the following additivity property: for unipotent bundles $\pi_j \colon E_j \to B$ with common (vertical) boundary bundle $E_{01} = \partial E_0 = \partial E_1$, the boundary bundle $\pi_{01} \colon E_{01} \to B$ and the glued bundle $\pi \colon E = E_0 \cup_{\partial E_j} E_1 \to B$ are also unipotent, and

$$(2.15) \qquad \bar{t}_{4k}^s(\pi) + \bar{t}_{4k}^s(\pi_{01}) = \bar{t}_{4k}^s(\pi_1) + \bar{t}_{4k}^s(\pi_2) \in H^{4k}(B; \mathbb{Q}).$$

This is proven for B a compact manifold in [1, Corollary 5.2], the case of a general base follows by using framed bordism as in the construction of $\bar{t}_{4k}^s$ for general base spaces.

Next, $\mu_M \colon B\mathrm{Diff}_\partial(D^{2n+1}) \to B\mathrm{Diff}(M)$ lifts to $\tilde{\mu}_M \colon B\mathrm{Diff}_\partial(D^{2n+1}) \to B\mathrm{Torr}(M)$, and (2.15) shows that

$$(2.16) \qquad\qquad\qquad \tau_M \circ \tilde{\mu}_M \sim \tau_{D^{2n+1}}.$$

By Theorem 2.11, it follows that $\tilde{\mu}_M$ is injective on $\pi_k(\_) \otimes \mathbb{Q}$ when $k \leq \phi^{\mathbb{Q}}(D^{2n})$. Because $\mathrm{Torr}(M) \subset \mathrm{Diff}(M)$ is a union of path components,

$$p_* \colon \pi_k(B\mathrm{Torr}(M)) \to \pi_k(B\mathrm{Diff}(M))$$

is injective when $k = 1$ and an isomorphism when $k \geq 2$, and $\mu_M = p \circ \tilde{\mu}_M$ is injective on rational homotopy groups up to degree $\phi^{\mathbb{Q}}(D^{2n})$. □

We have used that M is closed in order to apply (2.15), which in the quoted source is only covered for closed M. The case of a general M reduces to the closed case by "doubling". Let M be a manifold with boundary, let $A \subset \partial M$ be a compact codimension-0 submanifold, and form $M \cup_{\partial M - \mathrm{int}\, A} M$. This is a manifold with

boundary $A \cup_{\partial A} A$. Let $\mathrm{Diff}_A(M)$ be the group of diffeomorphisms which fix A pointwise. There is a doubling map

$$d_A \colon B\mathrm{Diff}_A(M) \to B\mathrm{Diff}_\partial(M \cup_{\partial M - \mathrm{int}\, A} M)$$

given by extending a diffeomorphism with its reflection. The diagram

$$
\text{(2.17)} \quad
\begin{array}{ccc}
B\mathrm{Diff}_\partial(D^{2n+1}) & \xrightarrow{\ \ \mu_M\ \ } & B\mathrm{Diff}_A(M) \\
\downarrow{\scriptstyle d_{D^{2n}}} & & \downarrow{\scriptstyle d_A} \\
B\mathrm{Diff}_\partial(D^{2n+1} \cup_{D^{2n}} D^{2n+1}) & \xrightarrow{\ \mu_{M \cup_{\partial M - \mathrm{int}\, A} M}\ } & B\mathrm{Diff}_\partial(M \cup_{\partial M - \mathrm{int}\, A} M)
\end{array}
$$

commutes up to homotopy, where $D^{2n} \subset \partial D^{2n+1}$ denotes a half disc in the boundary.

Lemma 2.18 *The doubling map induces an isomorphism*

$$(d_{D^{2n}})_* \colon \pi_k(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \to \pi_k(B\mathrm{Diff}_\partial(D^{2n+1} \cup_{D^{2n}} D^{2n+1})) \otimes \mathbb{Q}$$

when $k \leq \phi^{\mathbb{Q}}(D^{2n}) + 1$.

Proof By the Eckmann–Hilton argument, the effect of d on rational homotopy groups is the map

$$1 + (BI)_* \colon \pi_*(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \to \pi_*(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q}$$

(here I is the involution (2.14), and we identify $D^{2n+1} \cup_{D^{2n}} D^{2n+1} = D^{2n+1}$). The lemma now follows from the fact that $BI_* = \mathrm{id}$ on $\pi_k(\_) \otimes \mathbb{Q}$ for $k \leq \phi^{\mathbb{Q}}(D^{2n}) + 1$ (see the proof of Theorem 2.11 above for more details). $\qquad \square$

Remark 2.19 The bound given in Lemma 2.18 is optimal: [21, Corollary 8.4] shows that the involution acts nontrivially on $\pi_{2n-2}(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q}$ when $n \geq 5$, while $\phi^{\mathbb{Q}}(D^{2n}) + 1 = 2n - 3$ for such n.

Proof of Theorem 1.4 for general M To prove the statement for μ_M, use (2.17) with $A = \varnothing$ and apply Lemma 2.18. The statement for μ_M^∂ follows from that for μ_M in view of the definition of μ_M. $\qquad \square$

Remark 2.20 From the proof of Theorem 1.4 given above, one can also deduce a statement about $H_*(B\mathrm{Torr}(M); \mathbb{Q})$, namely that $\tilde{\mu}_M$ is injective on rational homology in the concordance stable range, at least when M is closed or orientable.

In the case where M is closed, this follows from (2.16) and Theorem 2.11. For manifolds with nonempty boundary, a bit more care is needed to check that doubling really gives a map $\mathrm{Torr}(M) \to \mathrm{Torr}(M \cup_{\partial M} M)$. For oriented M, the argument goes as follows.

Mapping both halves of $M \cup_{\partial M} M$ to M and excision in homology gives a $\mathrm{Diff}(M)$–equivariant isomorphism $H_*(M \cup_{\partial M} M; \mathbb{Z}) \cong H_*(M; \mathbb{Z}) \oplus H_*(M, \partial M; \mathbb{Z})$, from which it follows that the double of f also induces the identity on homology.

We leave it to the reader to figure out statements in cohomology or a variant for nonorientable M.

3 Proof of the homological theorem

We now turn to the proof of Theorem 1.7, which relies on work by Botvinnik and Perlmutter [6; 23]. To state their results, let

$$V_g^n := \natural^g (S^n \times D^{n+1})$$

be the boundary connected sum of g copies of $S^n \times D^{n+1}$, and let $D = D^{2n} \subset \partial V_g^n$ be a disk in the boundary of V_g^n. Note that $V_0^n = D^{2n+1}$. There is a stabilization map

(3.1) $$B\mathrm{Diff}_D(V_g^n) \to B\mathrm{Diff}_D(V_{g+1}^n),$$

given by taking the boundary connected sum with $S^n \times D^{n+1}$ at D and extending diffeomorphisms by the identity. Perlmutter [23, Theorem 1.1] proved that the map (3.1) induces an isomorphism in homology in degrees $* \leq \frac{1}{2}(g-4)$ provided that $n \geq 4$. Botvinnik and Perlmutter [6] computed the homology of $B\mathrm{Diff}_D(V_g^n)$ in the stable range. Let

$$\theta_n \colon BO(2n+1)\langle n \rangle \to BO(2n+1)$$

be the n–connected cover of $BO(2n+1)$. Let $\pi \colon E \to B$ be a bundle with fibre V_g^n and structure group $\mathrm{Diff}_D(V_g^n)$. The vertical tangent bundle $T_v E$ admits a θ_n–structure, ie a bundle map $\ell \colon T_v E \to \theta_n^* \gamma_{2n+1}$ to the pullback of the universal bundle over $BO(2n+1)$. This θ_n–structure is unique up to contractible choice once the following condition is imposed. Inside E, there is a trivial D–subbundle $B \times D$. The restriction of the vertical tangent bundle $T_v E$ to $B \times D$ has a canonical trivialization, and one requires that ℓ is compatible with that trivialization (see [6, Proposition 6.16] for all this).

Let $\underline{\ell}\colon E \to BO(2n+1)\langle n\rangle$ be the map of spaces underlying ℓ. Let

$$\alpha_\pi\colon B \xrightarrow{\mathrm{trf}_\pi} Q(E_+) \xrightarrow{Q(\ell)} Q(BO(2n+1)\langle n\rangle_+)$$

be the composition of the transfer with the map induced by $\underline{\ell}$. In particular, we can apply this construction to the universal bundle over $B\mathrm{Diff}_D(V_g^n)$ and obtain a map

(3.2) $$\alpha_g\colon B\mathrm{Diff}_D(V_g^n) \to Q_{1+(-1)^n g}(BO(2n+1)\langle n\rangle_+).$$

The target is the path component indexed by

$$\chi(V_g^n) = 1 + (-1)^n g \in \mathbb{Z} = \pi_0\big(Q(BO(2n+1)\langle n\rangle_+)\big).$$

The following is essentially [6, Corollary B].

Theorem 3.3 (Botvinnik and Perlmutter) *Let $n \geq 4$. Then the map (3.2) induces an isomorphism in integral homology in degrees $* \leq \frac{1}{2}(g-4)$.*

Theorem 3.3 as stated above differs from the formulation given in [6] in so far as loc. cit. does not mention the transfer at all, so some remarks have to be made here. For a fibration $\theta\colon X \to BO(d)$, Genauer [13] introduced the cobordism category $\mathrm{Cob}_\theta^\partial$ of $(d-1)$–dimensional θ–manifolds with boundaries and their cobordisms (which have corners). He proved that there is a weak equivalence $B\mathrm{Cob}_\theta^\partial \simeq \Omega^{\infty-1}\Sigma^\infty X_+$, and the equivalence is given by a parametrized Pontryagin–Thom construction (this result is parallel to the well-known result [12] for the usual cobordism category). Given any bundle $\pi\colon E \to B$ of smooth compact d–manifolds with boundary equipped with a θ–structure ℓ on the vertical tangent bundle, one obtains a tautological map $B \to \Omega B\mathrm{Cob}_\theta^\partial$, and from the description of the transfer for smooth bundles, one sees that the composition of this tautological map with Genauer's equivalence agrees with the composition $B \xrightarrow{\mathrm{trf}_\pi} Q(E_+) \xrightarrow{Q(\ell_+)} Q(X_+)$. Using this observation, one derives Theorem 3.3 from the results of [6].

Corollary 3.4 *If $n \geq 4$, the iterated stabilization map*

$$B\mathrm{Diff}_D(V_0^n) \to B\mathrm{Diff}_D(V_g^n)$$

induces the zero map on integral reduced homology in degrees $* \leq \frac{1}{2}(g-4)$.

Proof The transfer has an additivity property [4] which implies that

$$
\begin{array}{ccc}
B\mathrm{Diff}_D(V_0^n) & \longrightarrow & B\mathrm{Diff}_D(V_g^n) \\
\downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\alpha_g} \\
Q_1(BO(2n+1)\langle n\rangle_+) & \longrightarrow & Q_{1+(-1)^n g}(BO(2n+1)\langle n\rangle_+)
\end{array}
$$

commutes up to homotopy. The lower map takes the sum with a fixed point in $Q_{(-1)^n g}(BO(2n+1)\langle n\rangle_+)$ and is a weak equivalence. We shall show that the left vertical map is trivial in reduced homology (in all degrees); this will imply the claim by Theorem 3.3.

The map α_0 factors as

$$B\mathrm{Diff}_D(V_0^n) \xrightarrow{\mathrm{trf}} Q_1((E\mathrm{Diff}_D(V_0^n) \times_{\mathrm{Diff}_D(V_0^n)} V_0^n)_+) \xrightarrow{Q(l)} Q_1(BO(2n+1)\langle n\rangle_+).$$

The map $Q(\underline{l})$ is induced from the vertical tangent bundle of the universal bundle $E\mathrm{Diff}_D(V_0^n) \times_{\mathrm{Diff}_D(V_0^n)} V_0^n \to B\mathrm{Diff}_D(V_0^n)$, which is trivial by the following argument: The choice of a point in $D \subset \partial V_0^n$ determines a section s of the bundle, which is a homotopy equivalence as V_0^n is a disc, and the pullback of the vertical tangent bundle along s is trivial. Hence $Q(\underline{l})$ factors through $Q_1(S^0)$, which is rationally acyclic. Therefore α_0 induces the zero map on rational homology. This finishes the proof for rational homology, which is all we need for the proof of Theorem 1.7.

For the integral version, we use that the transfer is defined more generally for fibrations with finite CW fibres. It follows that there is a commutative diagram

$$
\begin{array}{ccc}
B\mathrm{Diff}_D(V_0^n) & \xrightarrow{\;\mathrm{trf}\;} & Q_1((E\mathrm{Diff}_D(V_0^n) \times_{\mathrm{Diff}_D(V_0^n)} V_0^n)_+) \\
\downarrow & & \downarrow \\
B\mathrm{Homeo}_D(V_0^n) & \xrightarrow{\;\mathrm{trf}\;} & Q_1((E\mathrm{Homeo}_D(V_0^n) \times_{\mathrm{Homeo}_D(V_0^n)} V_0^n)_+)
\end{array}
$$

Because the map

$$\underline{\ell} \colon E\mathrm{Diff}_D(V_0^n) \times_{\mathrm{Diff}_D(V_0^n)} V_0^n \to BO(2n+1)\langle n\rangle$$

is nullhomotopic as we just argued, it extends to a map

$$\underline{\ell}' \colon E\mathrm{Homeo}_D(V_0^n) \times_{\mathrm{Homeo}_D(V_0^n)} V_0^n \to BO(2n+1)\langle n\rangle.$$

Therefore α_0 factors up to homotopy through $B\mathrm{Homeo}_D(V_0^n)$, which is contractible by the Alexander trick. $\qquad\square$

Proof of Theorem 1.7 Since both μ_M and μ_M^∂ factor through

$$\mu_{U_{g,1}^n}^\partial \colon B\mathrm{Diff}_\partial(D^{2n+1}) \to B\mathrm{Diff}_\partial(U_{g,1}^n),$$

it suffices to show that this map induces the trivial map in rational homology in the indicated range of degrees. Note that

$$V_g^n \cup_{\partial V_g^n \setminus \mathrm{int}\, D} V_g^n = U_{g,1}^n.$$

Pick an embedding $f\colon D^{2n+1} \to V_0^n \subset V_g^n$ disjoint from the disc $D \subset V_0^n$ such that $f^{-1}(\partial V_0^n) = D^{2n}$ is a disc in ∂D^{2n+1}. This gives $\mu\colon B\mathrm{Diff}_\partial(D^{2n+1}) \to B\mathrm{Diff}_D(V_0^n)$, and by Corollary 3.4 composition with the stabilization map to $B\mathrm{Diff}_D(V_g^n)$ is trivial in integral homology in degrees $* \le \frac{1}{2}(g-4)$.

Diagram (2.17) becomes

(3.5)
$$\begin{array}{ccc}
B\mathrm{Diff}_\partial(D^{2n+1}) & \longrightarrow & B\mathrm{Diff}_D(V_g^n) \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\
B\mathrm{Diff}_\partial(D^{2n+1} \cup_{D^{2n}} D^{2n+1}) & \xrightarrow{\ \mu\ } & B\mathrm{Diff}_\partial(U_{g,1}^n)
\end{array}$$

Lemma 2.18 shows that the left vertical in (3.5) induces an isomorphism in rational homotopy in degrees at most $\phi^{\mathbb{Q}}(D^{2n}) + 1$. The same is true in rational homology since both spaces are connected H–spaces and their rational homology is the free graded commutative algebra on the rational homotopy, and so the proof is complete. $\square$

4 Optimality of the range in the homotopical theorem

Proof of Theorem 1.8 The composition

$$B\mathrm{Diff}_\partial(D^{2n+1}) \xrightarrow{\mu_M} B\mathrm{Diff}(M) \to B\mathrm{Homeo}(M)$$

factors through $B\mathrm{Homeo}_\partial(D^{2n+1}) \simeq *$. Hence μ_M factors through the space

$$\mathrm{hofib}(B\mathrm{Diff}(M) \to B\mathrm{Homeo}(M)).$$

By [21, Theorem A], $\pi_{2n-2}(B\mathrm{Diff}_\partial(D^{2n+1})) \otimes \mathbb{Q} \ne 0$ if $n \ge 6$. Therefore, it is enough to find a closed $(2n+1)$–manifold M such that

$$\pi_{2n-2}(\mathrm{hofib}(B\mathrm{Diff}(M) \to B\mathrm{Homeo}(M))) \otimes \mathbb{Q} = 0.$$

Now, by smoothing theory [26], $\mathrm{hofib}(B\mathrm{Diff}(M) \to B\mathrm{Homeo}(M))$ is homotopy equivalent to a union of path components of the section space

$$\Gamma\left(M; \mathrm{Fr}(M) \times_{O(2n+1)} \frac{\mathrm{Top}(2n+1)}{O(2n+1)}\right).$$

We now prove that

(4.1) $\pi_{2n-2}\left(\Gamma\left(\mathbb{RP}^2 \times S^{2n-1}; \mathrm{Fr}(\mathbb{RP}^2 \times S^{2n-1}) \times_{O(2n+1)} \frac{\mathrm{Top}(2n+1)}{O(2n+1)}\right)\right) \otimes \mathbb{Q} = 0$

when n is even. Homotopy groups of section spaces can be computed by means of the Federer spectral sequence [11]; see [22, Section 5.2] for the variant we shall be using. Let $E \to B$ be a fibration over a finite-dimensional CW-complex with simply connected fibre F and a fixed section s. Then there is a spectral sequence

$$E^2_{p,q} = H^{-p}(B; \pi_q(F)) \Rightarrow \pi_{p+q}(\Gamma(B; E), s)$$

(the coefficient systems in the E_2–term are twisted). Hence $\pi_m(\Gamma(B; E), s)$ admits a finite filtration whose filtration quotients are subquotients of $H^{-p}(B; \pi_q(F))$ with $p + q = m$, and so in order to prove that $\pi_m(\Gamma(B; E), s) \otimes \mathbb{Q} = 0$, it suffices to show that $H^{-p}(B; \pi_q(F)) \otimes \mathbb{Q} = 0$ if $p + q = m$.

Because $\text{Top}(2n+1)/O(2n+1)$ is simply connected by [26, 5.0(4)–(5)], we can apply the Federer spectral in our case. Furthermore, by loc. cit., $\text{Top}(2n+1)/O(2n+1)$ is rationally $(2n+2)$–connected. So the only entries in the E_2–page which could potentially be rationally nonzero and contribute to (4.1) are

$$H^{2n+1-i}\left(\mathbb{RP}^2 \times S^{2n-1}; \pi_{4n-1-i}\left(\frac{\text{Top}(2n+1)}{O(2n+1)}\right) \otimes \mathbb{Q}\right)$$

for $0 \leq i \leq 2$. It remains to be shown that

(4.2) $$H^{2-i}\left(\mathbb{RP}^2; \pi_{4n-1-i}\left(\frac{\text{Top}(2n+1)}{O(2n+1)}\right) \otimes \mathbb{Q}\right) = 0$$

when $0 \leq i \leq 2$.

The fundamental group $\pi_1(\mathbb{RP}^2) = C_2$ acts on $\pi_{4n-1-i}(\text{Top}(2n+1)/O(2n+1))$ by conjugation with an isometry of determinant -1. There is an isomorphism

$$\pi_{4n-1-i}\left(\frac{\text{Top}(2n+1)}{O(2n+1)}\right) = \pi_{2n-2-i}(B\text{Diff}_\partial(D^{2n+1}))$$

coming from Morlet's theorem, [26, Section 3.4] which states that

$$B\text{Diff}_\partial(D^{2n+1}) \simeq \Omega_0^{2n+1}\left(\frac{\text{Top}(2n+1)}{O(2n+1)}\right).$$

By the discussion in [21, Section 8.2], we have that the action of the generator of C_2 on $\pi_{4n-1-i}(\text{Top}(2n+1)/O(2n+1))$ corresponds under this isomorphism to *minus* the involution $(BI)_*$ considered in the proof of Lemma 2.18 above. By [21, Corollary 8.4],

we therefore have, for *even n*, that

$$\pi_{4n-1}\left(\frac{\text{Top}(2n+1)}{O(2n+1)}\right) \otimes \mathbb{Q} = \mathbb{Q}^+,$$

$$\pi_{4n-2}\left(\frac{\text{Top}(2n+1)}{O(2n+1)}\right) \otimes \mathbb{Q} = 0,$$

$$\pi_{4n-3}\left(\frac{\text{Top}(2n+1)}{O(2n+1)}\right) \otimes \mathbb{Q} = \mathbb{Q}^-$$

as C_2–modules. Since $H^2(\mathbb{RP}^2; \mathbb{Q}^+) = H^0(\mathbb{RP}^2; \mathbb{Q}^-) = 0$, we obtain (4.2), which concludes the proof. □

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