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A uniqueness theorem for transitive Anosov flows obtained by gluing hyperbolic plugs

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In work with C Bonatti, we defined a general procedure to build new examples of Anosov flows in dimension 3. The procedure consists in gluing together some building blocks, called *hyperbolic plugs*, along their boundary in order to obtain a closed three-manifold endowed with a complete flow. The main theorem of that work states that (under some mild hypotheses) it is possible to choose the gluing maps so the resulting flow is Anosov. Here we show a uniqueness result for Anosov flows obtained by such a procedure. Roughly speaking, we show that the orbital equivalence class of these Anosov flows is insensitive to the precise choice of the gluing maps used in the construction. The proof relies on a coding procedure, which we find interesting for its own sake, and follows a strategy that was introduced by T Barbot in a particular case.

37D20; 57M99

1 Introduction

In a previous paper, written with C Bonatti [5], we have proved a result allowing one to construct transitive Anosov flows in dimension 3 by "gluing hyperbolic plugs along their boundaries". The purpose here is to study Anosov flows obtained by such a construction. We focus our attention on the diffeomorphisms that are used to glue together the boundaries of the hyperbolic plugs. We aim to understand what is the impact of the choice of these diffeomorphisms on the dynamics of the resulting Anosov flows. We will see that two gluing diffeomorphisms that are "strongly isotopic" yield some Anosov flows that are orbitally equivalent. In other words, in [5], we have proved the *existence* of Anosov flows constructed by a certain gluing procedure, and the goal here is to prove a *uniqueness result* for these Anosov flows.

In order to state some precise questions and results, we need to introduce some terminology. A *hyperbolic plug* is a pair (U, X), where U is a (not necessarily connected) compact three-dimensional manifold with boundary and X is a vector field on U, transverse to ∂U and such that the maximal invariant set $\Lambda_X := \bigcap_{t\in\mathbb{R}} X^t(U)$ is a saddle hyperbolic set for the flow (X^t). Given such a hyperbolic plug (U, X), we decompose ∂U as the disjoint union of an *entrance boundary* $\partial^{\text{in}} U$ (the union of the connected components of ∂U where the vector field X is pointing into U) and an *exit boundary* $\partial^{\text{out}} U$ (the union of the connected components of ∂U where the vector field X is pointing out of U). The stable lamination $W^s(\Lambda_X)$ of the maximal invariant set Λ_X intersects transversally the entrance boundary $\partial^{\text{in}} U$ and is disjoint from the exit boundary $\partial^{\text{out}} U$. Hence, $L^s_X := W^s(\Lambda_X) \cap \partial U$ is a one-dimensional lamination embedded in the surface $\partial^{\text{in}} U$. Similarly, $L^u_X := W^u(\Lambda_X) \cap \partial U$ is a one-dimensional lamination embedded in the surface $\partial^{\text{out}} U$. We call L^s_X and L^u_X the *entrance lamination* and the *exit lamination* of the hyperbolic plug (U, X). It can be proved that these laminations are quite simple:

(i) They contain only finitely many compact leaves.

(ii) Every half noncompact leaf is asymptotic to a compact leaf.

(iii) Each compact leaf may be oriented such that its holonomy is a contraction.

Hyperbolic plugs should be thought as the basic blocks of a building game, our goal being to build some Anosov flows by gluing a collection of such basic blocks together. From a formal viewpoint, a finite collection of hyperbolic plugs can always be viewed as a single nonconnected hyperbolic plug. For this reason, it is enough to consider a single hyperbolic plug (U, X) and a gluing diffeomorphism $\psi\colon \partial^{\text{out}} U \to \partial^{\text{in}} U$. For such (U, X) and ψ, the quotient space $M := U/\psi$ is a closed three-manifold, and the incomplete flow (X^t) on U induces a complete flow (Y^t) on M. The purpose of [5] was to describe some sufficient conditions on U, X and ψ for (Y^t) to be an Anosov flow. We will now explain these conditions.

We say that a one-dimensional lamination L is *filling a surface* S if every connected component C of $S \setminus L$ is "a strip whose width tends to 0 at both ends"; more precisely, C is simply connected, the accessible boundary of C consists of two distinct noncompact leaves ℓ^- and ℓ^+ of L, and these two leaves ℓ^- and ℓ^+ are asymptotic to each other at both ends. We say that two laminations L_1 and L_2 embedded in the same surface S are *strongly transverse* if they are transverse to each other and, moreover, every connected component C of $S \setminus (L_1 \cup L_2)$ is a topological disc whose boundary ∂C consists of exactly four arcs α_1, α_2, α'_1 and α'_2, where α_1 and α'_1 are arcs of leaves of the lamination L_1 and α_2 and α'_2 are arcs of leaves of the lamination L_2. We say that a hyperbolic plug (U, X) has *filling laminations* if the entrance lamination L^s_X is filling the surface $\partial^{\mathrm{in}} U$ and the exit lamination L^u_X is filling the surface $\partial^{\mathrm{out}} U$. Given a hyperbolic plug (U, X), we say that a gluing diffeomorphism $\psi\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$ is *strongly transverse* if the laminations L^s_X and $\psi_* L^u_X$ (both embedded in the surface $\partial^{\mathrm{in}} U$) are strongly transverse. If (U, X_1) and (U, X_2) are two hyperbolic plugs with the same underlying manifold U and $\psi_1, \psi_2\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$ are two gluing diffeomorphisms, we say that the triples (U, X_1, ψ_1) and (U, X_2, ψ_2) are *strongly isotopic* if one can find a continuous one-parameter family $\{(U, X_t, \psi_t)\}_{t\in[1,2]}$ such that (U, X_t) is a hyperbolic plug and $\psi_t\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$ is a strongly transverse gluing map for every t. The main technical result of [5] can be stated as follows:

Theorem 1.1 *Let (U, X_0) be a hyperbolic plug with filling laminations such that the maximal invariant set of (U, X_0) contains neither attractors nor repellers, and let $\psi_0\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$ be a strongly transverse gluing diffeomorphism. Then there exist a hyperbolic plug (U, X) with filling laminations and a strongly transverse gluing diffeomorphism $\psi\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$ such that (U, X_0, ψ_0) and (U, X, ψ) are strongly isotopic, and such that the vector field Y induced by X on the closed manifold $M := U/\psi$ is Anosov.*

The idea of building transitive Anosov flows by gluing hyperbolic plugs goes back to [7], where Bonatti and R Langevin consider a very simple hyperbolic plug (U, X) whose maximal invariant set is a single isolated periodic orbit and are able to find an explicit gluing diffeomorphism $\psi\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$ such that the vector field Y induced by X on the closed manifold $M := U/\psi$ generates a transitive Anosov flow. This example was later generalized by T Barbot, who defined a infinite family of transitive Anosov flows which he calls *BL flows*. These examples are obtained by considering the same very simple hyperbolic plug (U, X) as Bonatti and Langevin, but more general gluing diffeomorphisms.

Theorem 1.1 naturally raises the following question (see [5, Question 1.7]): *In the statement of Theorem 1.1, is the Anosov vector field Y well defined up to orbitally equivalence*? (Recall that two Anosov flows are said to be *orbitally equivalent* if there exists a homeomorphism between their phase space mapping the oriented orbits of the first flow to the oriented orbits of the second one.) One of the main purposes of the present paper is to provide a positive answer to this question. More precisely, we will prove the following:

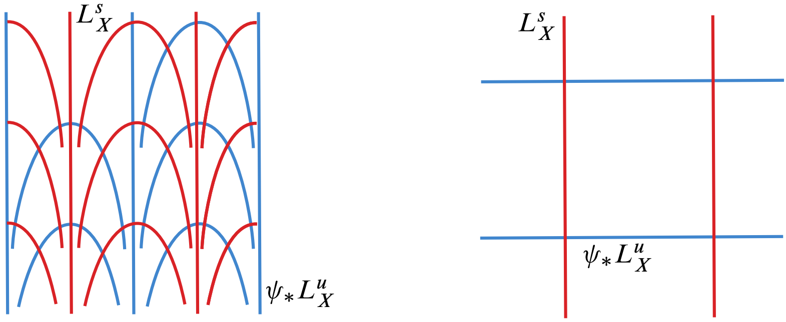

Figure 1: Two examples of strongly transverse gluing diffeomorphisms. On the left-hand side, the laminations are filling. The right-hand side corresponds to Bonatti and Langevin's example.

Theorem 1.2 *Let (U, X_1, ψ_1) and (U, X_2, ψ_2) be two hyperbolic plugs endowed with strongly transverse gluing diffeomorphisms. Let Y_1 and Y_2 be the vector fields induced by X_1 and X_2 on the closed manifolds $M_1 := U/\psi_1$ and $M_2 := U/\psi_2$. Suppose that:*

(0) *The manifolds U, M_1 and M_2 are orientable.*

(1) *Both Y_1 and Y_2 are transitive Anosov vector fields.*

(2) *The triples (U, X_1, ψ_1) and (U, X_2, ψ_2) are strongly isotopic.*

Then the flows (Y_1^t) and (Y_2^t) are orbitally equivalent.

Remark 1.3 In the statement of Theorem 1.2, we do not require that the hyperbolic plugs (U, X_1) and (U, X_2) have filling laminations. So Theorem 1.2 concerns a class of Anosov flows which is larger than the class of Anosov flows provided by Theorem 1.1. For example, Bonatti and Langevin's classical example and its generalizations by Barbot (BL flows) satisfy the hypotheses of Theorem 1.2.

Remark 1.4 On the other hand, we require the Anosov vector fields Y_1 and Y_2 to be transitive. The result is probably still true without this assumption. Nevertheless, at some point of our proof, we will need some leaves of the weak (un)stable foliations of Y_1 and Y_2 to be dense. This denseness is not true in general for nontransitive Anosov vector fields. Note that [5, Proposition 1.6] provides a sufficient condition for an Anosov vector field constructed by gluing some hyperbolic plugs to be transitive.

Remark 1.5 A possible application of Theorem 1.2 is to get some finiteness results. Suppose we are given a hyperbolic plug (U, X) and a diffeomorphism $\psi_0 : \partial^{\text{out}} U \to \partial^{\text{in}} U$. Consider the partition of the isotopy class of ψ_0 into strong isotopy classes. Although we did not write down a complete proof, it seems to us that this partition is finite.

Roughly speaking:

- The stable lamination $L^s_X = W^s(\Lambda_X) \cap \partial^{\mathrm{in}} U$ have finitely many compact leaves which cut $\partial^{\mathrm{in}} U$ in finitely many annuli $A^s_1, \dots, A^s_k$.
- The unstable lamination $L^u_X = W^u(\Lambda_X) \cap \partial^{\mathrm{out}} U$ have finitely many compact leaves which cut $\partial^{\mathrm{out}} U$ in finitely many annuli $A^u_1, \dots, A^u_\ell$.
- It seems that (except in a finite number of some very specific situations) the strong isotopy class of a gluing map ψ (isotopic to ψ_0) only depends on whether the annulus $\psi(A^u_i)$ intersects the annulus A^s_j for each (i, j) (which would of course imply that there are only finitely many possible strong isotopy classes for ψ_0.

Assume that the partition in strong isotopy classes is indeed finite. By Theorem 1.2, this means the following: up to orbital equivalence, there are only finitely many transitive Anosov flows that are built using the hyperbolic plug (U, X) and a gluing map $\psi\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$ isotopic to ψ_0. A further consequence should be that, if we consider some given hyperbolic plugs $(U_1, X_1), \dots, (U_n, X_n)$ such that $U_1, \dots, U_n$ are hyperbolic manifolds, and if we consider a manifold M, then, up to orbital equivalence, there should only finitely many transitive Anosov flows on M that are obtained by gluing $(U_1, X_1), \dots, (U_n, X_n)$.

An analog of Theorem 1.2 was proved by Barbot in the much more restrictive context of BL flows (see [2, Theorem B(2)]). Barbot's result can actually be considered as a particular case of Theorem 1.2: it corresponds to the case where the maximal invariant set of the hyperbolic plug (U_i, X_i) is a single isolated periodic orbit for $i = 1, 2$. Our proof of Theorem 1.2 roughly follows the same strategy as that of Barbot's result, but is far more intricate and requires some important new ingredients since we manipulate general hyperbolic plugs.

The proof is based on a coding procedure that we will describe now. Consider a hyperbolic plug (U, X) and a strongly transverse gluing diffeomorphism $\psi\colon \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U$. Let Y be the vector field induced by X on the closed manifold $M := U/\psi$, and assume that the flow (Y^t) is a transitive Anosov flow. The projection in M of ∂U is a closed surface transverse to the orbits of the Anosov flow (Y^t); we denote this surface by S. The projection in M of the entrance lamination of the plug (U, X) is a lamination in the surface S; we denote it by L^s. Consider the universal cover $\widetilde{M}$ of the manifold M and the lifts $(\widetilde{Y}^t)$, $\widetilde{S}$ and $\widetilde{L}^s$ of (Y^t), S and L^s. We will consider the (countable) alphabet $\mathcal{A}$ whose letters are the connected components of $\widetilde{S} \setminus \widetilde{L}^s$, and the symbolic space Σ whose elements are bi-infinite words on the alphabet $\mathcal{A}$. We will construct a coding map χ from (a dense subset of) the surface $\widetilde{S}$ to the symbolic space Σ, commuting with the natural actions of the fundamental group of M, and conjugating the Poincaré first return map of the flow $(\widetilde{Y}^t)$ on the surface $\widetilde{S}$ to the shift map on the symbolic space Σ. If Λ denotes the projection in M of the maximal invariant set of the plug (U, X), and $\widetilde{\Lambda}$ denotes the lift of Λ in $\widetilde{M}$, then the map χ is defined at every point of $\widetilde{S}$ which is neither in the stable nor in the unstable lamination of $\widetilde{\Lambda}$. This means that the dynamics of the flow (Y^t) can be decomposed into two parts: on the one hand, the orbits that converge towards to the maximal invariant set Λ in the past or in the future; on the other hand, the dynamics that is well described by the coding map χ.

Remark 1.6 Besides being the cornerstone of the proof of Theorem 1.2, this coding procedure is interesting for its own sake. Indeed, it allows one to understand the behaviour of the recurrent orbits of the Anosov flow (Y^t) that intersect the surface S (ie which do not correspond to recurrent orbits of the incomplete flow (X^t)). In a forthcoming paper [6], we will use this coding procedure to describe the free homotopy classes of theses orbits, and build new examples of transitive Anosov flows.

Let us now explain how this coding procedure yields a proof of Theorem 1.2. For $i = 1, 2$, we get a symbolic space Σ_i and a coding map χ_i with values in Σ_i. The strong isotopy between (U, X_1, ψ_1) and (U, X_2, ψ_2) implies that there is a natural map between the symbolic spaces Σ_1 and Σ_2. Together with the coding maps, this yields a conjugacy between the Poincaré return maps of the flows $(\widetilde{Y}_1^t)$ and $(\widetilde{Y}_2^t)$ on the surfaces $\widetilde{S}_1$ and $\widetilde{S}_2$. Unfortunately, this conjugacy is not well defined on the whole surfaces $\widetilde{S}_1$ and $\widetilde{S}_2$. So we need to extend it. In order to do that, we introduce some (partial) preorders on the leaf spaces of the lifts of the stable/unstable foliations of the Anosov flows (Y_1^t) and (Y_2^t), and prove that the conjugacy preserves these preorders. This is quite delicate since the coding maps χ_1 and χ_2 do not behave very well with respect to these preorders. Once the extension has been achieved, we obtain a homeomorphism between the orbits spaces of the flows $(\widetilde{Y}_1^t)$ and $(\widetilde{Y}_2^t)$ that is equivariant with respect to the actions of the fundamental groups of the manifolds M_1 and M_2. Using a classical result, this implies that the Anosov flows (Y_1^t) and (Y_2^t) are orbitally equivalent.

2 Coding procedure

In this section, we will consider a transitive Anosov flow obtained by gluing hyperbolic plugs. Our goal is to define a coding procedure for the orbits of this Anosov flow.

Actually, this coding procedure will only describe the behaviour of the orbits which do not remain in $\mathrm{int}(U)$ forever.

2.1 Setting

We consider a hyperbolic plug (U, X). Recall that this means that U is a (not necessarily connected)[1] compact three-dimensional manifold with boundary, and X is a vector field on U, transverse to ∂U, such that the maximal invariant set

$$\Lambda_X := \bigcap_{t \in \mathbb{R}} X^t(U)$$

is a saddle hyperbolic set for the flow of X. We decompose the boundary of U as

$$\partial U := \partial^{\mathrm{in}} U \sqcup \partial^{\mathrm{out}} U,$$

where $\partial^{\mathrm{in}} U$ (resp. $\partial^{\mathrm{out}} U$) is the union of the connected component of ∂U where X is pointing into (resp. out of) U. The stable manifold theorem implies that $W^s_X(\Lambda_X)$ and $W^u_X(\Lambda_X)$ are two-dimensional laminations transverse to ∂U. Moreover, $W^s_X(\Lambda_X)$ is obviously disjoint from $\partial^{\mathrm{out}} U$ and $W^u_X(\Lambda_X)$ is obviously disjoint from $\partial^{\mathrm{in}} U$. As a consequence,

$$L^s_X := W^s_X(\Lambda_X) \cap \partial U = W^s_X(\Lambda_X) \cap \partial^{\mathrm{in}} U,$$
$$L^u_X := W^u_X(\Lambda_X) \cap \partial U = W^u_X(\Lambda_X) \cap \partial^{\mathrm{out}} U$$

are one-dimensional laminations embedded in the surfaces $\partial^{\mathrm{in}} U$ and $\partial^{\mathrm{out}} U$, respectively. Note that L^s_X can be described as the set of points in $\partial^{\mathrm{in}} U$ whose forward (X^t)–orbit remains in U forever, ie does not intersect $\partial^{\mathrm{out}} U$. Similarly, L^u_X is the set of points in $\partial^{\mathrm{out}} U$ whose backward (X^t)–orbit remains in U forever, ie does not intersect $\partial^{\mathrm{in}} U$. These characterizations of L^s_X and L^u_X allow us to define a map

$$\theta_X : \partial^{\mathrm{in}} U \setminus L^s_X \to \partial^{\mathrm{out}} U \setminus L^u_X,$$

where $\theta_X(x)$ is the (unique) point of intersection the (X^t)–orbit of x with the surface $\partial^{\mathrm{out}} U$. Clearly, θ_X is a homeomorphism between $\partial^{\mathrm{in}} U \setminus L^s_X$ and $\partial^{\mathrm{out}} U \setminus L^u_X$. We call θ_X the *crossing map* of the plug (U, X).

In order to create a closed manifold equipped with a transitive Anosov flow, we consider a diffeomorphism

$$\psi : \partial^{\mathrm{out}} U \to \partial^{\mathrm{in}} U.$$

[1] Hence, a finite collection of hyperbolic plugs can always be considered as a single, nonconnected, hyperbolic plug.

The quotient space

$$M := U/\psi$$

is a closed three-dimensional topological manifold. We denote by $\pi\colon U \to M$ the natural projection map. The topological manifold M can equipped with a differential structure (compatible with the differential structure of U) so that the vector field

$$Y := \pi_* X$$

is well defined (and as smooth as X). We adopt the following hypotheses:

(0) The manifolds U and M are orientable.

(1) The flow (Y^t) is a transitive Anosov flow on the manifold M.

(2) The diffeomorphism ψ is a strongly transverse gluing diffeomorphism.

Recall that (2) means that the laminations L^s_X and $\psi_*(L^u_X)$ are transverse in the surface $\partial^{\mathrm{in}} U$ and moreover that every connected component C of $\partial^{\mathrm{in}} U \setminus (L^s_X \cup \psi_*(L^u_X))$ is a topological disc whose boundary ∂C consists of exactly four arcs α^s, $\alpha^{s\prime}$, α^u and $\alpha^{u\prime}$, where α^s and $\alpha^{s\prime}$ are arcs of leaves of L^s_X and α^u and $\alpha^{u\prime}$ are arcs of leaves $\psi_*(L^u_X)$).

Remark 2.1 We insist on the fact that (2) implies that every connected components of $\partial^{\mathrm{in}} U \setminus (L^s_X \cup \psi_*(L^u_X))$ is a topological disc, even if some of the connected components of $\partial^{\mathrm{in}} U \setminus L^s_X$ and $\partial^{\mathrm{in}} U \setminus \psi_*(L^u_X)$ might be annuli (eg in Bonatti and Langevin's construction). Further properties which follow from (0)–(2) will be stated and proven in Section 2.2. Anyhow, recall that the second part of [5] as well as [7] or [2] provide many examples of hyperbolic plugs (U, X) and gluing maps ψ for which (0)–(2) are satisfied.

We define

$$S := \pi(\partial^{\mathrm{in}} U) = \pi(\partial^{\mathrm{out}} U), \quad \Lambda := \pi(\Lambda_X), \quad L^s := \pi_*(L^s_X), \quad L^u := \pi_*(L^u_X).$$

By construction, S is a closed surface, embedded in the manifold M, transverse to the vector field Y. The set Λ is the union of the orbits of (Y^t) that do not intersect the surface S. It is an invariant saddle hyperbolic set for the Anosov flow (Y^t). Our assumptions imply that L^s and L^u are two strongly transverse one-dimensional laminations in the surface S. The lamination L^s (resp. L^u) can be described as the set of points in S whose forward (resp. backward) (Y^t)–orbit does not intersect S.

Similarly, L^u is a strict subset of $W^u(\Lambda) \cap S$. The homeomorphism θ_X induces a homeomorphism

$$\theta = (\pi|_{\partial^{\mathrm{out}} U}) \circ \theta_X \circ (\pi|_{\partial^{\mathrm{in}} U})^{-1} \colon S \setminus L^s \to S \setminus L^u.$$

Note that θ is nothing but the Poincaré first return map of the orbits of the Anosov flow (Y^t) on the surface S.

Since (Y^t) is an Anosov flow, it comes with a stable foliation $\mathcal{F}^s$ and an unstable foliation $\mathcal{F}^u$. These are two-dimensional foliations, transverse to each other, and transverse to the surface S. Hence, they induce two transverse one-dimensional foliations

$$F^s := \mathcal{F}^s \cap S \quad \text{and} \quad F^u := \mathcal{F}^u \cap S$$

on the surface S. Clearly, L^s and L^u are sublaminations (ie union of leaves) of the foliations F^s and F^u, respectively.

In order to code the orbits of the Anosov flow (Y^t), we cannot work directly in the manifold M; we need to unfold the leaves of the laminations L^s and L^u by lifting them to the universal cover of M. We denote this universal cover by $p \colon \widetilde{M} \to M$, and we denote by

$$\widetilde{S}, \quad \widetilde{\Lambda}, \quad \widetilde{W}^s(\Lambda), \quad \widetilde{W}^u(\Lambda), \quad \widetilde{L}^s, \quad \widetilde{L}^u, \quad \widetilde{\mathcal{F}}^s, \quad \widetilde{\mathcal{F}}^u, \quad \widetilde{F}^s, \quad \widetilde{F}^u$$

the complete lifts of the surface S, the hyperbolic set Λ, the laminations $W^s(\Lambda)$, $W^u(\Lambda)$, L^s and L^u, and the foliations $\mathcal{F}^s$, $\mathcal{F}^u$, F^s and F^u. We insist that $\widetilde{S}$ is the *complete* lift of S; that is, $\widetilde{S} := p^{-1}(S)$. In particular, $\widetilde{S}$ has infinitely many connected components. By construction, $\widetilde{F}^s$ and $\widetilde{F}^u$ are two transverse one-dimensional foliations on the surface $\widetilde{S}$, and $\widetilde{L}^s$ and $\widetilde{L}^u$ are sublaminations of $\widetilde{F}^s$ and $\widetilde{F}^u$, respectively. We also lift the vector field Y to a vector field $\widetilde{Y}$ on M. Of course, $\widetilde{Y}$ is transverse to the surface $\widetilde{S}$, so we can consider the Poincaré return map

$$\widetilde{\theta} \colon \widetilde{S} \setminus \widetilde{L}^s \to \widetilde{S} \setminus \widetilde{L}^u$$

of the orbits of $(\widetilde{Y}^t)$ on the surface $\widetilde{S}$. Obviously, $\widetilde{\theta}$ is a lift of the map θ.

2.2 Connected components of $\widetilde{S} \setminus \widetilde{L}^s$

We next collect some information about the connected components of $\widetilde{S} \setminus \widetilde{L}^s$ and the action of the Poincaré map $\widetilde{\theta}$ on these connected components. This information will be used in Section 2.3. Let us start by the topology of the surface $\widetilde{S}$.

Proposition 2.2 *Every connected component of $\widetilde{S}$ is a properly embedded topological plane.*

Proof The surface S is transverse to the Anosov flow (Y^t). Hence, S is a collection of incompressible tori in M (see eg [8, Corollary 2.2]). □

This allows us to describe the topology of the leaves of the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$:

Proposition 2.3 *Every leaf of the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$ is a properly embedded topological line. A leaf of $\widetilde{F}^s$ and a leaf of $\widetilde{F}^u$ intersect in no more than one point.*

Proof The first assertion follows immediately from Proposition 2.2: it is a classical consequence of the Poincaré–Hopf theorem that the leaves of a foliation of a plane are properly embedded topological lines.

The second assertion is again a consequence Proposition 2.2, together with the transversality of the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$. To prove it, we argue by contradiction: Consider a leaf ℓ^s of $\widetilde{F}^s$ and a leaf ℓ^u of $\widetilde{F}^u$, and assume that ℓ^s and ℓ^u intersect at more than one point. Then one can find two arcs $\alpha^s \subset \ell^s$ and $\alpha^u \subset \ell^u$ which share the same endpoints and have disjoint interiors. The union $\alpha^s \cup \alpha^s$ is a simple closed curve in $\widetilde{S}$. Since every connected component of $\widetilde{S}$ is a topological plane, $\alpha^s \cup \alpha^s$ bounds a topological disc $C \subset \widetilde{S}$. Consider two copies of C, and glue them along α^s in order to obtain a new topological disc D. The boundary of D is the union of two copies of α^u, and hence is piecewise smooth. The foliation $\widetilde{F}^s$ provides a one-dimensional foliation on D, which is topologically transverse to boundary ∂D. This contradicts the Poincaré–Hopf theorem. □

The next three propositions below concern the action of the Poincaré map $\widetilde{\theta}$ on the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$ and the laminations $\widetilde{L}^s$ and $\widetilde{L}^u$. We recall that $\widetilde{L}^s$ and $\widetilde{L}^u$ are sublaminations (ie union of leaves) of the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$, respectively.

Proposition 2.4 *The Poincaré map $\widetilde{\theta}\colon \widetilde{S} - \widetilde{L}^s \to \widetilde{S} - \widetilde{L}^u$ preserves the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$.*

Remark 2.5 Proposition 2.4 states that the foliation $(\widetilde{F}^s)|_{\widetilde{S}-\widetilde{L}^s}$ is mapped by $\widetilde{\theta}$ to the foliation $(\widetilde{F}^s)|_{\widetilde{S}-\widetilde{L}^u}$. The leaves of $(\widetilde{F}^s)|_{\widetilde{S}-\widetilde{L}^s}$ are full leaves of the foliation $\widetilde{F}^s$. On the contrary, a leaf of the foliation $(\widetilde{F}^s)|_{\widetilde{S}-\widetilde{L}^u}$ is never a full leaf of $\widetilde{F}^s$ (because every leaf of $\widetilde{F}^s$ is "cut into infinitely many pieces" by the transverse lamination $\widetilde{L}^u$). As a consequence, $\widetilde{\theta}$ maps leaves of $\widetilde{F}^s$ to pieces of leaves of $\widetilde{F}^s$. Similarly, $\widetilde{\theta}$ maps pieces of leaves of $\widetilde{F}^u$ to full leaves of $\widetilde{F}^u$.

Proof of Proposition 2.4 Recall that $\widetilde{F}^s$ is defined as the intersection of the foliation $\widetilde{\mathcal{F}}^s$ with the transverse surface $\widetilde{S}$. The foliation $\widetilde{\mathcal{F}}^s$ is leafwise invariant under the flow $(\widetilde{Y}^t)$. As a consequence, $\widetilde{F}^s = \widetilde{\mathcal{F}}^s \cap \widetilde{S}$ is invariant under the Poincaré return map of $(\widetilde{Y}^t)$ on $\widetilde{S}$. □

Proposition 2.6 *For every $n \geq 0$, $\bigcup_{p=0}^{n} \widetilde{\theta}^{-p}(\widetilde{L}^s)$ is a closed sublamination of the foliation $\widetilde{F}^s$.*

Proof The foliation $\widetilde{F}^s$ is invariant under the Poincaré map $\widetilde{\theta}\colon \widetilde{S} - \widetilde{L}^s \to \widetilde{S} - \widetilde{L}^u$. Since $\widetilde{L}^s$ is a union of leaves of $\widetilde{F}^s$, it follows that $\widetilde{\theta}^{-1}(\widetilde{L}^s)$ is a union of leaves of $\widetilde{F}^s$. Moreover, since $\widetilde{L}^s$ is a closed subset of $\widetilde{S}$, its preimage $\widetilde{\theta}^{-1}(\widetilde{L}^s)$ must be a closed subset of $\widetilde{S} - \widetilde{L}^s$ (remember that $\widetilde{\theta}$ is well defined on $\widetilde{S} - \widetilde{L}^s$). Therefore $\bigcup_{p=0}^{1} \widetilde{\theta}^{-p}(\widetilde{L}^s)$ is a closed subset of $\widetilde{S}$. So $\bigcup_{p=0}^{1} \widetilde{\theta}^{-p}(\widetilde{L}^s)$ is a closed union of leaves of $\widetilde{F}^s$, ie a closed sublamination of $\widetilde{F}^s$. Repeating the same arguments, one proves by induction that $\bigcup_{p=0}^{n} \widetilde{\theta}^{-p}(\widetilde{L}^s)$ is a closed sublamination of $\widetilde{F}^s$ for every $n \geq 0$. □

Proposition 2.7
$$\bigcup_{p=0}^{\infty} \widetilde{\theta}^{-p}(\widetilde{L}^s) = \widetilde{W}^s(\Lambda) \cap \widetilde{S}.$$

Proof By definition, $W^s(\Lambda) \cap S$ is the set of all points $x \in S$ such that the forward orbit of x converges towards the set Λ, which is disjoint from S. As a consequence, for every point $x \in W^s(\Lambda) \cap S$, the forward orbit of x intersects the surface S only finitely many times, say $p(x)$ times. We have observed that L^s is the set of all points $y \in S$ such that the forward orbit of y does not intersect S and converges towards the set Λ (see Section 2.1). It follows that, for every $x \in W^s(\Lambda) \cap S$, the last intersection point $\theta^{p(x)}$ of the forward orbit of x with S is in L^s. This proves the inclusion $W^s(\Lambda) \cap S \subset \bigcup_{p=0}^{\infty} \theta^{-p}(L^s)$. The converse inclusion is straightforward. Hence, $\bigcup_{p=0}^{\infty} \theta^{-p}(L^s) = W^s(\Lambda) \cap S$. The equality $\bigcup_{p=0}^{\infty} \widetilde{\theta}^{-p}(\widetilde{L}^s) = \widetilde{W}^s(\Lambda) \cap \widetilde{S}$ follows by lifting to the universal cover. □

Of course, $\widetilde{W}^s(\Lambda) \cap \widetilde{S}$ and $\widetilde{W}^u(\Lambda) \cap \widetilde{S}$ are unions of leaves of the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$, respectively. But these sets are not closed. More precisely:

Proposition 2.8 *Both $\widetilde{W}^s(\Lambda) \cap \widetilde{S}$ and $\widetilde{S} - \widetilde{W}^s(\Lambda)$ are dense in $\widetilde{S}$.*

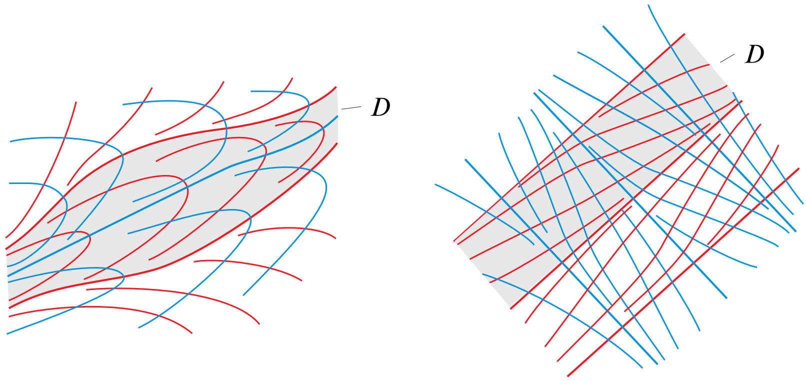

Figure 2: Left: a proper stable strip. Right: a trivially bifoliated proper stable strip.

Proof Recall that (Y^t) is a transitive Anosov flow on M. Hence, every leaf of the weak stable foliation $\mathcal{F}^s$ is dense in M. Since both $W^s(\Lambda)$ and $M \setminus W^s(\Lambda)$ are nonempty unions of leaves of the foliation $\mathcal{F}^s$, and since the leaves of $\mathcal{F}^s$ are transversal to the surface S, it follows that both $W^s(\Lambda) \cap S$ and $S \setminus W^s(\Lambda)$ are dense in S. Lifting to the universal cover, we obtain that $\widetilde{W}^s(\Lambda) \cap \widetilde{S}$ and $\widetilde{S} - \widetilde{W}^s(\Lambda)$ are dense in $\widetilde{S}$. $\square$

Of course, the analogs of Propositions 2.6, 2.7 and 2.8 for $\widetilde{L}^u$ and $W^u(\widetilde{\Lambda})$ hold ($\widetilde{\theta}^{-p}$ should be replaced by $\widetilde{\theta}^p$ in Propositions 2.6 and 2.7). We will now describe the topology of the connected components of $\widetilde{S} \setminus \widetilde{L}^s$. We first introduce some vocabulary.

Definition 2.9 We call a *proper stable strip* every topological open disc D of $\widetilde{S}$ whose boundary is the union of two leaves of the foliation $\widetilde{F}^s$.

If D is a proper stable strip, one can easily construct a homeomorphism h from the closure of D to $\mathbb{R} \times [-1, 1]$. We will need the following stronger notion:

Definition 2.10 We say that a proper stable strip D is *trivially bifoliated* if there exists a homeomorphism h from the closure of D to $\mathbb{R} \times [-1, 1]$ mapping the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$ to the horizontal and vertical foliations on $\mathbb{R} \times [-1, 1]$.

Of course, *proper unstable strips* and *trivially bifoliated proper unstable strips* can be defined similarly. The proposition below gives a fairly precise description of the positions of the connected components of $\widetilde{S} - \widetilde{L}^s$ with respect to the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$:

Proposition 2.11 *Every connected component of $\widetilde{S} - \widetilde{L}^s$ is a trivially bifoliated proper stable strip bounded by two leaves of the lamination $\widetilde{L}^s$.*

Proof Let D be a connected component of $\widetilde{S} - \widetilde{L}^s$. Denote by P the connected component of $\widetilde{S}$ containing D. Since P is a topological plane (Proposition 2.2), and since each leaf of $\widetilde{L}^s$ is a properly embedded topological line (Proposition 2.3) which separates P into two connected components, it follows that D is a topological disc. The boundary of D is a union of leaves of $\widetilde{L}^s$ (which we call *the boundary leaves of* D). We denote by $\overline{D}$ the closure of D.

Claim 1 *Let ℓ^u be a leaf of the foliation $\widetilde{F}^u$ intersecting $\overline{D}$, and α^u be a connected component of $\ell^u \cap \overline{D}$. Then α^u is an arc joining two different boundary leaves of D.*

Let R be a connected component of $D \setminus \widetilde{L}^u$ such that α^u is included in the closure $\overline{R}$ of R (actually R is unique, but we will not use this fact). Observe that R is a connected component of $\widetilde{S} - (\widetilde{L}^s \cup \widetilde{L}^u)$. Our assumptions (specifically the strong transversality of the gluing map ψ) imply that R is a relatively compact topological disc whose boundary ∂R is made of four arcs α^s_-, α^s_+, α^u_- and α^u_+, where α^s_- and α^s_- are disjoint and lie in some leaves of $\widetilde{L}^s$, and where α^u_- and α^u_+ are disjoint and lie in some leaves of $\widetilde{L}^u$. Loosely speaking, R is a rectangle with two sides α^s_- and α^s_+ in $\widetilde{L}^s$ and two sides α^u_- and α^u_+ in $\widetilde{L}^u$. Proposition 2.3 implies that ℓ^u intersects α^s_- and α^s_+ at no more than one point. Since ℓ^u is a proper line and $\overline{R}$ is a compact set, it follows that α^u must be an arc going from α^s_- to α^s_+. Using again Proposition 2.3, it also follows that α^s_- to α^s_+ cannot be in the same leaf of $\widetilde{F}^s$. The claim is proved.

Claim 2 *D has exactly two boundary leaves.*

In order to prove this claim, we endow the foliation $\widetilde{F}^u$ with an orientation (this is possible since $\widetilde{F}^u$ is a foliation on a collection of topological planes). For every $x \in \overline{D}$, we denote by $\ell^u(x)$ the leaf of the foliation $\widetilde{F}^u$ passing through x, and denote by $\alpha^u(x)$ the connected component of $\ell^u_x \cap \overline{D}$ containing x. Note that $\ell^u(x)$ and $\alpha^u(x)$ are oriented by the orientation of $\widetilde{F}^u$. By Claim 1, $\alpha^u(x)$ is an arc whose endpoints lie on two boundary leaves $\ell^s_-(x)$ and $\ell^s_+(x)$ of D. By transversality of the foliations $\widetilde{F}^u$ and $\widetilde{F}^s$, the maps $x \mapsto \ell^s_-(x)$ and $x \mapsto \ell^s_+(x)$ are locally constant. Since $\overline{D}$ is connected, these maps are constant. In other words, one can find two boundary leaves ℓ^s_- and ℓ^s_+ of D such that $\alpha^u(x)$ is an arc from ℓ^s_- to ℓ^s_+ for every $x \in \overline{D}$. It follows that ℓ^s_- and ℓ^s_+ are the only accessible boundary leaves of D: otherwise, one can consider another boundary leaf ℓ^s, take a point $x \in \ell^s$, and get a contradiction since one end of α^u_x is on ℓ^s. As a further consequence, the accessible boundary of D is closed (recall that ℓ^s_- and ℓ^s_+ are properly embedded lines), and therefore coincides with the boundary of D. We finally conclude that ℓ^s_- and ℓ^s_+ are the only boundary leaves of D, and Claim 2 is proved.

Claims 1 and 2 already imply that D is a proper stable strip bounded by two leaves ℓ^s_- and ℓ^s_+ of $\widetilde{L}^s$. We are left to prove that D is trivially bifoliated. Recall that $\widetilde{S}$ is a topological plane (Proposition 2.2), and that ℓ^s_- and ℓ^s_+ are properly embedded topological lines (Proposition 2.3). By easy planar topology, it follows that there exists a homeomorphism h from $\overline{D}$ to $\mathbb{R}\times[-1,1]$ mapping ℓ^s_- and ℓ^s_+ to $\mathbb{R}\times\{-1\}$ and $\mathbb{R}\times\{1\}$, respectively. Claim 1 implies that $h_*(\widetilde{F}^u_{\overline{D}})$ is a foliation of $\mathbb{R}\times[-1,1]$ by arcs going from $\mathbb{R}\times\{-1\}$ and $\mathbb{R}\times\{1\}$. One can easily construct a self-homeomorphism h' of $\mathbb{R}\times[-1,1]$ mapping this foliation on the vertical foliation of $\mathbb{R}\times[-1,1]$. Up to replacing h by $h'\circ h$, we will assume that h maps $\widetilde{F}^u_{\overline{D}}$ on the vertical foliation of $\mathbb{R}\times[-1,1]$. Now we consider a leaf ℓ^s of the foliation $\widetilde{F}^s$ included in $\overline{D}$. According to Proposition 2.3, ℓ^s intersects each leaf of $\widetilde{F}^u$ at no more than one point. Hence, $h(\ell^s)$ intersects each vertical segment in $\mathbb{R}\times[-1,1]$ at no more than one point. Let E be the set of $t\in\mathbb{R}$ such that $h(\ell^s)$ intersects the vertical segment $\{t\}\times[-1,1]$. Since ℓ^s is a proper topological line transversal to $\widetilde{F}^u$, the set E_t must be open and closed in $\mathbb{R}$. Therefore, $h(\ell^s)$ intersects every vertical segment in $\mathbb{R}\times[-1,1]$ at exactly one point. In other words, the leaves of $h_*(\widetilde{F}^s_{\overline{D}})$ are graphs over the first coordinate in $\mathbb{R}\times[-1,1]$. One can easily modify the homeomorphism h so that $h_*(\widetilde{F}^s_{\overline{D}})$ is the horizontal foliation of $\mathbb{R}\times[-1,1]$. Hence, D is a trivially bifoliated proper stable strip. □

Of course, the unstable analog of Proposition 2.11 holds true: every connected component of $\widetilde{S}-\widetilde{L}^u$ is a trivially bifoliated proper unstable strip bounded by two leaves of the lamination $\widetilde{L}^u$. On the other hand, $\widetilde{\theta}$ maps connected components of $\widetilde{S}-\widetilde{L}^s$ to connected component of $\widetilde{S}-\widetilde{L}^u$. So, we obtain:

Corollary 2.12 *If D is a connected component of $\widetilde{S}-\widetilde{L}^s$, then $\widetilde{\theta}(D)$ is a trivially bifoliated proper unstable strip, disjoint from $\widetilde{L}^u$, bounded by two leaves of the lamination $\widetilde{L}^u$.*

The following proposition describes the action of $\widetilde{\theta}$ on the connected components of $\widetilde{S}-\widetilde{L}^s$:

Proposition 2.13 *Let D be a connected component of $\widetilde{S}-\widetilde{L}$, and D' be any trivially bifoliated proper stable strip. Assume that $D\cap\widetilde{\theta}^{-1}(D')$ is nonempty. Then $D\cap\widetilde{\theta}^{-1}(D')$ is a trivially bifoliated proper stable substrip of D.*

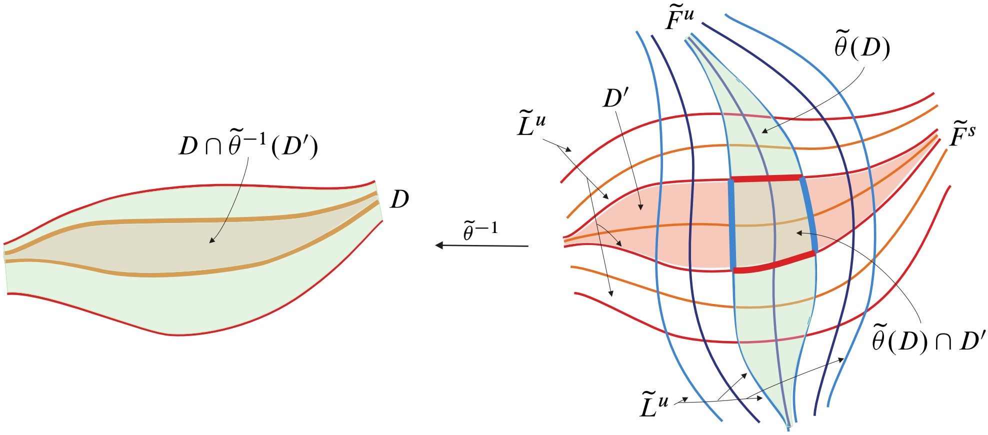

Figure 3: The proof of Proposition 2.13.

Proof We call a *trivially bifoliated rectangle* every topological open disc $R \subset \widetilde{S}$ such that there exists a homeomorphism from the closure of R to $[-1,1]^2$ mapping the restrictions of $\widetilde{F}^s$ and $\widetilde{F}^u$ to the horizontal and vertical foliations of $[-1,1]^2$. In particular, the boundary of such a trivially bifoliated rectangle is made of two stable sides and two unstable sides.

According to Corollary 2.12, $\widetilde{\theta}(D)$ is a trivially bifoliated proper unstable strip, disjoint from $\widetilde{L}^u$, bounded by two leaves of $\widetilde{L}^u$. By assumption, D' is a trivially bifoliated proper stable strip. It easily follows that $\widetilde{\theta}(D) \cap D'$ is a trivially bifoliated rectangle, disjoint from $\widetilde{L}^u$, whose unstable sides are in $\widetilde{L}^u$ (see Figure 3). Observe that the interiors of two stable sides of $\widetilde{\theta}(D) \cap D'$ are full leaves of $\widetilde{F}^s|_{\widetilde{S}-\widetilde{L}^u}$. Hence:

($\star$) $\theta(D) \cap D'$ is a connected subset of $\theta(D)$ and the boundary of $\theta(D) \cap D'$ in $\theta(D)$ is made of two disjoint leaves of $\widetilde{F}^s|_{\widetilde{S}-\widetilde{L}^u}$.

Now recall that $\widetilde{\theta}^{-1}$ is a homeomorphism from $\widetilde{S} - \widetilde{L}^u$ to $\widetilde{S} - \widetilde{L}^s$, mapping leaves of $\widetilde{F}^s|_{\widetilde{S}-\widetilde{L}^u}$ to full leaves of $\widetilde{F}^s$ (see Proposition 2.4 and Remark 2.5). Also observe that $D \cap \widetilde{\theta}^{-1}(D')$ is a subset of D. As a consequence, property ($\star$) implies:

($\star'$) $D \cap \widetilde{\theta}^{-1}(D')$ is a connected subset of D, and the boundary of $D \cap \widetilde{\theta}^{-1}(D')$ is made of two disjoint leaves of $\widetilde{F}^s$.

Since D is a trivially foliated proper stable strip D, Property ($\star'$) clearly implies that $D \cap \widetilde{\theta}^{-1}(D')$ is a trivially bifoliated proper stable substrip of D. See Figure 3. □

2.3 The coding procedure

In this section, we will use the connected components of $\widetilde{S}\setminus\widetilde{L}^s$ to describe the itinerary of the orbits the flow $(\widetilde{Y}^t)$ that do not belong to $\widetilde{W}^s(\Lambda)\cup\widetilde{W}^u(\Lambda)$. We consider the alphabet

$$\mathcal{A} := \{\text{connected components of } \widetilde{S}\setminus\widetilde{L}^s\},$$

and the symbolic spaces

$$\begin{aligned}\Sigma^s &= \{\bar{D}^s = (D_p)_{p\geq 0} \mid D_p\in\mathcal{A} \text{ and } \widetilde{\theta}(D_p)\cap D_{p+1}\neq\varnothing \text{ for every } p\},\\ \Sigma^u &= \{\bar{D}^u = (D_p)_{p< 0} \mid D_p\in\mathcal{A} \text{ and } \widetilde{\theta}(D_p)\cap D_{p+1}\neq\varnothing \text{ for every } p\},\\ \Sigma &= \{\bar{D} = (D_p)_{p\in\mathbb{Z}} \mid D_p\in\mathcal{A} \text{ and } \widetilde{\theta}(D_p)\cap D_{p+1}\neq\varnothing \text{ for every } p\}.\end{aligned}$$

In order to define the coding maps, we need to introduce some leaf spaces. We will denote by f^s the leaf space of the foliation $\widetilde{F}^s$ (equipped with the quotient topology). We will denote by $f^{s,\infty}$ the subset of f^s made of the leaves that are not in $\widetilde{W}^s(\Lambda)$. Similarly, we denote by f^u the leaf space of $\widetilde{F}^u$, and by $f^{u,\infty}$ the subset fo f^u made of the leaves that are not in $\widetilde{W}^u(\Lambda)$. Finally, we denote by $\widetilde{S}^\infty$ the set of points in $\widetilde{S}$ that are neither in $\widetilde{W}^s(\Lambda)$ nor in $\widetilde{W}^u(\Lambda)$. That is,

$$\begin{aligned}f^{s,\infty} &= \{\text{leaves of } \widetilde{F}^s \text{ that are not in } \widetilde{W}^s(\Lambda)\},\\ f^{u,\infty} &= \{\text{leaves of } \widetilde{F}^u \text{ that are not in } \widetilde{W}^u(\Lambda)\},\\ \widetilde{S}^\infty &= \widetilde{S}-(\widetilde{W}^s(\Lambda)\cup\widetilde{W}^u(\Lambda)).\end{aligned}$$

By Proposition 2.7, if $\ell^s\in f^{s,\infty}$, then $\widetilde{\theta}^p(\ell^s)$ is included in a connected component of $\widetilde{S}-\widetilde{L}^s$ for every $p\geq 0$. Similarly, if $\ell^u\in f^{u,\infty}$, then $\widetilde{\theta}^p(\ell^u)$ is included in a connected component of $\widetilde{S}-\widetilde{L}^u$ for every $p\leq 0$. Since $\widetilde{\theta}^{-1}$ maps homeomorphically $\widetilde{S}-\widetilde{L}^u$ to $\widetilde{S}-\widetilde{L}^s$, we deduce that, if $\ell^u\in f^{u,\infty}$, then $\widetilde{\theta}^p(\ell^u)$ is included in a connected component of $\widetilde{S}-\widetilde{L}^s$ for every $p<0$. As a further consequence, if x is a point of $\widetilde{S}^\infty$, then $\widetilde{\theta}^p(x)$ is in a connected component of $\widetilde{S}-\widetilde{L}^s$ for every $p\in\mathbb{Z}$. This shows that the following coding maps are well defined:

$$\begin{aligned}&\chi^s\colon f^{s,\infty}\to\Sigma^s, && \ell^s\mapsto\bar{D}^s=(D_p)_{p\geq 0}, && \text{where } \widetilde{\theta}^p(\ell^s)\subset D_p \text{ for every } p\geq 0;\\ &\chi^u\colon f^{u,\infty}\to\Sigma^u, && \ell^u\mapsto\bar{D}^u=(D_p)_{p<0}, && \text{where } \widetilde{\theta}^p(\ell^u)\subset D_p \text{ for every } p<0;\\ &\chi\colon \widetilde{S}^\infty\to\Sigma, && x\mapsto\bar{D}=(D_p)_{p\in\mathbb{Z}}, && \text{where } \widetilde{\theta}^p(x)\in D_p \text{ for every } p\in\mathbb{Z}.\end{aligned}$$

The following proposition is an important ingredient of the proof of Theorem 1.2:

Proposition 2.14 *The maps χ^s, χ^u and χ are bijective.*

Lemma 2.15 (1) *For every $\overline{D}^s = (D_p)_{p\geq 0} \in \Sigma^s$, the set $\bigcap_{p\geq 0} \widetilde{\theta}^{-p}(D_p)$ is a stable leaf $\ell^s \in f^{s,\infty}$.*

(2) *For every $\overline{D}^u = (D_p)_{p<0} \in \Sigma^u$, the set $\bigcap_{p<0} \widetilde{\theta}^{-p}(D_p)$ is an unstable leaf $\ell^u \in f^{u,\infty}$.*

(3) *For every $\overline{D} = (D_p)_{p\in\mathbb{Z}} \in \Sigma$, the set $\bigcap_{p\in\mathbb{Z}} \widetilde{\theta}^{-p}(D_p)$ is a single point $x \in \widetilde{S}^{\infty}$.*

Remark 2.16 Lemma 2.15 is completely false if we replace the connected components of $\widetilde{S} \setminus \widetilde{L}^s$ by the connected components of $S \setminus L^s$ (and $\widetilde{\theta}$ by θ). For example, if $(D_p)_{p\geq 0}$ is a sequence of connected components of $S \setminus L^s$, then $\bigcap_{p\geq 0} \theta^{-p}(D_p)$, if not empty, will be the union of uncountably many leaves of the foliation F^s. This is the reason why we need to work in the universal cover of M.

Proof of Lemma 2.15 Let us prove the first item. Consider a sequence $\overline{D}^s = (D_p)_{p\geq 0} \in \Sigma^s$. By Proposition 2.11, D_0 is a trivially bifoliated proper stable strip. Proposition 2.13 and a straightforward induction imply that, for every $n \in \mathbb{N}$, the set $\bigcap_{p=0}^{n} \widetilde{\theta}^{-p}(D_p)$ is a substrip of D_0. So $\big(\bigcap_{p=0}^{n} \widetilde{\theta}^{-p}(D_p)\big)_{n\geq 0}$ is a decreasing sequence of substrips of the trivially bifoliated proper stable strip D_0. It easily follows that $\bigcap_{p\geq 0} \widetilde{\theta}^{-p}(D_p)$ is a substrip of D_0. In particular, $\bigcap_{p\geq 0} \widetilde{\theta}^{-p}(D_p)$ is a connected union of leaves of $\widetilde{F}^s$. On the other hand, since $D_0, D_1, \dots$ are connected components of $\widetilde{S} - \widetilde{L}^s$, the set $\bigcap_{\geq 0} \widetilde{\theta}^{-p}(D_p)$ is disjoint from $\bigcup_{p\geq 0} \widetilde{\theta}^{-p}(\widetilde{L}^s) = \widetilde{W}^s(\Lambda) \cap \widetilde{S}$ (see Proposition 2.7). But $\widetilde{W}^s(\Lambda) \cap \widetilde{S}$ is dense in $\widetilde{S}$ (Proposition 2.8). It follows that $\bigcap_{p\geq 0} \widetilde{\theta}^{-p}(D_p)$ must be a single leaf of $\widetilde{F}^s$. This completes the proof of (1).

Item (2) follows from exactly the same arguments as (1). In order to prove the last item, we consider a sequence $\overline{D} = (D_p)_{p\in\mathbb{Z}}$ in Σ. According to (1)–(2), $\bigcap_{p\geq 0} \widetilde{\theta}^{-p}(D_p)$ is a leaf ℓ^s of the foliation $\widetilde{F}^s$ and $\bigcap_{p<0} \widetilde{\theta}^{-p}(D_p)$ is a leaf ℓ^u of the foliation $\widetilde{F}^u$. Since $\overline{D} = (D_p)_{p\in\mathbb{Z}}$ is in Σ, the intersection $D_0 \cap \widetilde{\theta}^{(}D_{-1})$ is not empty. Since D_0 is a trivially bifoliated proper stable strip (Proposition 2.11) and $\widetilde{\theta}^{(}D_{-1})$ is a trivially bifoliated proper unstable strip (Corollary 2.12), every leaf of $\widetilde{F}^s$ in D_0 intersects every leaf of $\widetilde{F}^u$ in $\widetilde{\theta}(D_{-1})$ at exactly one point. In particular, $\bigcap_{p\in\mathbb{Z}} \widetilde{\theta}^{-p}(D_p) = \ell^s \cap \ell^u$ is made of exactly one point x. Since the leaves ℓ^s and ℓ^u are disjoint from $\widetilde{W}^s(\Lambda)$ and $\widetilde{W}^u(\Lambda)$, respectively, the point x must be in $\widetilde{S}^{\infty}$. □

Proof of Proposition 2.14 Lemma 2.15 allows us to define some inverse maps for χ^s, χ^u and χ. Therefore, χ^s, χ^u and χ are bijective. □

Deck transformation preserve the surface $\widetilde{S}$, the foliations $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$, and the laminations $W^s(\widetilde{\Lambda})$ and $W^u(\widetilde{\Lambda})$. This induces some natural actions of $\pi_1(M)$ on the set $\widetilde{S}^\infty$, on the leaf spaces $f^{s,\infty}$ and $f^{u,\infty}$, on the alphabet $\mathcal{A}$, and therefore on the symbolic spaces Σ, Σ^s and Σ^u. From the definition of the coding maps, one easily checks that:

Proposition 2.17 *The coding maps χ, χ^s and χ^u commute with the actions of the fundamental group of M on $\widetilde{S}^\infty$ f^s, f^u, Σ Σ^s and Σ^u.*

The definition of the coding maps also implies that:

Proposition 2.18 *The coding map χ (resp. χ^s and χ^u) conjugates the action of the Poincaré first return map $\widetilde{\theta}$ on $\widetilde{S}^\infty$ (resp. f^s and f^u) to the left shift on the symbolic space Σ (resp. Σ^s and Σ^u).*

Given an integer $n \geq 0$ and some connected components $D_0^0, \dots, D_n^0$ of $\widetilde{S} - \widetilde{L}^s$, we define the cylinder

$$[D_0^0 \dots D_n^0]^s := \{(D_p)_{p\geq 0} \in \Sigma^s \mid D_p = D_p^0 \text{ for } 0 \leq p \leq n\}.$$

Similarly, given $n < 0$ and some connected components $D_n^0, \dots, D_{-1}^0$ of $\widetilde{S} - \widetilde{L}^s$, we define the cylinder

$$[D_n^0 \dots D_{-1}^0]^u := \{(D_p)_{p<0} \in \Sigma^u \mid D_p = D_p^0 \text{ for } n \leq p \leq -1\}.$$

The following proposition will be used in the next subsection:

Proposition 2.19 (1) *For $n \geq 0$ and $D_0, \dots, D_n \in \mathcal{A}$, the set*

$$(\chi^s)^{-1}([D_0 D_1 \dots D_n]^s) = \bigcap_{0\leq p\leq n} \widetilde{\theta}^{-p}(D_p)$$

is either empty or a substrip of the trivially foliated proper stable strip D_0 bounded by two leaves of $\widetilde{\theta}^{-n}(\widetilde{L}^s)$.

(2) *For $n < 0$ and $D_n, \dots, D_{-1} \in \mathcal{A}$, the set*

$$(\chi^u)^{-1}([D_n D_{n+1} \dots D_{-1}]) = \bigcap_{-n\leq p\leq -1} \widetilde{\theta}^{-p+1}(D_p)$$

is a substrip of the trivially foliated proper unstable strip $\widetilde{\theta}(D_{-1})$ bounded by two leaves of $\widetilde{\theta}^{K-1}(\widetilde{L}^u)$.

Proof This follows from the arguments of the proof of Lemma 2.15. □

2.4 Partial orders on the leaf spaces and the symbolic spaces

We will now describe a partial preorder on the leaf space f^s. The preservation of this partial preorder will be a fundamental ingredient of our proof of Theorem 1.2 in Section 3.

Let us start by choosing some orientations. First of all, we choose an orientation of the hyperbolic plug U. The orientation of U, together with the vector field X, provides an orientation of ∂U: if ω is a 3–form defining the orientation on U, then the 2–form $i_X U$ defines the orientation on ∂U. The orientation of U induces an orientation of the manifold $M = U/\psi$ (we have assumed that the manifold M is orientable, which is equivalent to assuming that the gluing map ψ preserves the orientation of ∂U), and the orientation of ∂U induces an orientation of the surface $S = \pi(\partial^{\mathrm{in}} U) = \pi(\partial^{\mathrm{out}} U)$. The orientations of M and S induce some orientations on $\widetilde{M}$ and $\widetilde{S}$. Now, since every connected component of $\widetilde{S}$ is a topological plane, the foliation $\widetilde{F}^s$ is orientable. We fix an orientation of $\widetilde{F}^s$. This automatically induces an orientation of the foliation $\widetilde{F}^u$ as follows: the orientation of $\widetilde{F}^u$ is chosen so that, if Z^s and Z^u are vector fields tangent to $\widetilde{F}^s$ and $\widetilde{F}^u$, respectively, and pointing in the direction of the orientation of the leaves, then the frame field (Z^s, Z^u) is positively oriented with respect to the orientation of $\widetilde{S}$.

Remarks 2.20 (1) By construction, the orientations of the manifold $\widetilde{M}$ and the surface $\widetilde{S}$ are related as follows: if ω is a 3–form defining the orientation on $\widetilde{M}$, then the 2–form $i_{\widetilde{Y}} \widetilde{M}$ defines the orientation on $\widetilde{S}$. As a consequence, the Poincaré return map $\widetilde{\theta}$ of the orbits of $\widetilde{Y}$ on $\widetilde{S}$ preserves the orientation of $\widetilde{S}$.

(2) Consequently, for any connected component D of $\widetilde{S} - \widetilde{L}^s$, if the Poincaré map $\widetilde{\theta}|_D$ preserves (resp. reverses) the orientation of the foliation $\widetilde{F}^s$, then it also preserves (resp. reverses) the orientation of the foliation $\widetilde{F}^u$.

Let ℓ be a leaf of the foliation $\widetilde{F}^s$, contained in a connected component $\widetilde{S}_\ell$ of $\widetilde{S}$. Recall that $\widetilde{S}_\ell$ is a topological plane, and ℓ is a properly embedded line in $\widetilde{S}_\ell$. As a consequence, $\widetilde{S}_\ell \setminus \ell$ has two connected components.

Definition 2.21 We denote by $L(\ell)$ and $R(\ell)$ the two connected components of $\widetilde{S} \setminus \ell$ so that the oriented leaves of $\widetilde{F}^u$ crossing ℓ go from $L(\ell)$ towards $R(\ell)$. The points of $L(\ell)$ are said to be *on the left* of ℓ; the points of $R(\ell)$ are said to be *on the right* of ℓ.

Now we can define a preorder on the leaf space f^s.

Definition 2.22 (preorder on f^s) Given two leaves $\ell \neq \ell'$ of the foliation $\widetilde{F}^s$, we write $\ell \prec \ell'$ if there exists an arc of a leaf of $\widetilde{F}^u$ with endpoints $a \in \ell$ and $a' \in \ell'$ such that the orientation of $\widetilde{F}^u$ goes from a towards a'.

Proposition 2.23 *$\prec$ is a preorder on f^s: the relations $\ell \prec \ell'$ and $\ell' \prec \ell$ are incompatible.*

Proof The relation $\ell \prec \ell'$ implies that the leaf ℓ' is on the right of ℓ; that is, $\ell' \subset R(\ell)$. Similarly, the relation $\ell \prec \ell'$ implies $\ell' \subset L(\ell)$. The proposition follows since $L(\ell) \cap R(\ell) = \varnothing$. □

The proposition below is very easy to prove, but fundamental (it will be used in a crucial way to extend some conjugating maps in the next section, see Corollary 3.12):

Proposition 2.24 *$\prec$ is a local total order on f^s: for every leaf ℓ_0 of $\widetilde{F}^s$, there exists a neighbourhood $\mathcal{V}_0$ of ℓ_0 in f^s such that any two different leaves $\ell, \ell' \in \mathcal{V}_0$ are comparable* (*ie satisfy either* $\ell \prec \ell'$ *or* $\ell' \prec \ell$).

Proof Consider a leaf ℓ_0 of $\widetilde{F}^s$ and a leaf ℓ^u of $\widetilde{F}^u$ such that $\ell^u \cap \ell_0 \neq \varnothing$. By transversality of the foliations $\widetilde{F}^s$ and $\widetilde{F}^u$, there exists a neighbourhood $\mathcal{V}_0$ of ℓ_0 in f^s such that ℓ^u crosses every leaf in $\mathcal{V}_0$. As a consequence, any two different leaves $\ell, \ell' \in \mathcal{V}_0$ are comparable for the preorder $\prec$. □

The proposition below shows that the preorder $\prec$ is "compatible" with the connected components decomposition of $\widetilde{S} - \widetilde{L}^s$:

Proposition 2.25 *Given two different elements D and D' of $\mathcal{A}$, the following are equivalent*:

(1) *There exist some leaves $\ell_0, \ell_0' \in f^s$ such that $\ell_0 \subset D$, $\ell_0' \subset D'$ and $\ell_0 \prec \ell_0'$.*
(2) *All leaves $\ell, \ell' \in f^s$ such that $\ell \subset D$ and $\ell' \subset D'$ satisfy $\ell \prec \ell'$.*

Proof Assume that (1) is satisfied. Since $\ell_0 \prec \ell_0'$, there must be a leaf ℓ^u of the foliation $\widetilde{F}^u$ intersecting both ℓ_0 and ℓ_0'. Proposition 2.11 implies that $\alpha := \ell^u \cap D$ and $\alpha' := \ell^u \cap D'$ are two disjoint arcs in the leaf ℓ^u. Consider some leaves ℓ and ℓ' of $\widetilde{F}^s$ contained in D and D', respectively. Again Proposition 2.11 implies that ℓ intersects ℓ^u at some point $a_\ell \in \alpha$ and ℓ' intersects ℓ^u at some point $a_{\ell'} \in \alpha'$. Since $\ell_0 \preceq \ell_0'$, the orientation of ℓ^u goes from α towards α', and hence from a_ℓ towards $a_{\ell'}$. This shows that $\ell \prec \ell'$. □

Definition 2.26 (preorder on $\mathcal{A}$) Given two different elements D and D' of $\mathcal{A}$, we write $D \prec D'$ if there exist some leaves $\ell_0, \ell'_0 \in f^s$ such that $\ell_0 \subset D$, $\ell'_0 \subset D'$ and $\ell_0 \prec \ell'_0$.

Definition 2.27 (preorder on Σ^s) The partial preorder $\prec$ on $\mathcal{A}$ induces a lexicographic partial preorder on $\Sigma^s \subset \mathcal{A}^{\mathbb{N}}$, which will also be denoted by $\prec$: for $\overline{D} = (D_p)_{p\geq 0}$ and $\overline{D'} = (D'_p)_{p\geq 0}$ in Σ^s, we write $\overline{D} \prec \overline{D'}$ if and only if there exists $p_0 \geq 0$ such that $D_p = D'_p$ for $p \in \{0, \dots, p_0 - 1\}$ and $D_{p_0} \prec D'_{p_0}$.

We have defined a preorder on the leaf space f^s (Definition 2.22) and a preorder on the symbolic space Σ^s (Definition 2.27). It is natural to wonder whether the coding map $\chi^s : f^{s,\infty} \to \Sigma^s$ is compatible with these preorders or not. For pedagogical reasons, we first consider the simple situation where the two-dimensional foliation $\mathcal{F}^u$ is orientable:

Proposition 2.28 *Assume that the unstable foliation $\mathcal{F}^u$ is orientable. Then the coding map $\chi^s : f^{s,\infty} \to \Sigma^s$ preserves the preorders*, ie *for $\ell, \ell' \in f^{s,\infty}$, $\ell \prec \ell'$ if and only if $\chi^s(\ell) \prec \chi^s(\ell')$.*

Proof Since the two-dimensional foliation $\mathcal{F}^u$ is orientable, its lift $\widetilde{\mathcal{F}}^u$ is also orientable. Recall that the vector field $\widetilde{Y}$ is tangent to the leaves of the foliation $\widetilde{\mathcal{F}}^u$. So the orientability of the two-dimensional foliation $\widetilde{\mathcal{F}}^u$ implies that the return map $\widetilde{\theta}$ of the orbits of the vector field $\widetilde{Y}$ on the surface $\widetilde{S}$ preserves the orientation of the one-dimensional foliation $\widetilde{F}^u = \widetilde{\mathcal{F}}^u \cap \widetilde{S}$.

Consider two leaves $\ell, \ell' \in f^{s,\infty}$ such that $\ell \prec \ell'$. Let $\chi^s(\ell) = (D_p)_{p\geq 0}$ and $\chi^s(\ell) = (D'_p)_{p\geq 0}$. Recall that this means that

$$\ell = \bigcap_{p\geq 0} \widetilde{\theta}^{-p}(D_p) \quad \text{and} \quad \ell' = \bigcap_{p\geq 0} \widetilde{\theta}^{-p}(D'_p).$$

Consider the integer $p_0 = \min\{p \geq 0 \mid D_p \neq D'_p\}$ and the set

$$\widehat{D} := \bigcap_{p=0}^{p_0-1} \widetilde{\theta}^{-p}(D_p).$$

Both the leaves ℓ and ℓ' are included in $\widehat{D}$, and, according to Proposition 2.19, $\widehat{D}$ is a trivially bifoliated proper stable strip. So we can consider an arc α^u of a leaf ℓ^u of the foliation $\widetilde{F}^u$ such that α^u is included in the trivially bifoliated proper stable strip $\widehat{D}$ and the ends a and a' of α^u are on ℓ and ℓ', respectively. Since $\ell \prec \ell'$, the orientation of $\widetilde{F}^u$ goes from a towards a'. Now observe that $\widehat{D}$ is a connected component of $\widetilde{S} - \bigcup_{p=0}^{p_0-1} \widetilde{\theta}^p(\widetilde{L}^s)$. As a consequence, the map $\widetilde{\theta}^{p_0}$ is well defined on $\widetilde{D}$. In particular, we can consider $\beta^u := \widetilde{\theta}^{p_0}(\alpha^u)$. Observe that β^u is an arc of a leaf of the foliation $\widetilde{F}^u$. Its ends $b := \widetilde{\theta}^{p_0}(a)$ and $b' := \widetilde{\theta}^{p_0}(a')$ are respectively in $\widetilde{\theta}^{p_0}(\ell) \subset D_{p_0}$ and $\widetilde{\theta}^{p_0}(\ell') \subset D'_{p_0}$. Since the return map $\widetilde{\theta}^{p_0}$ preserves the orientation of the foliation $\widetilde{F}^u$, the orientation of $\widetilde{F}^u$ goes from b towards b'. It follows that $\widetilde{\theta}^{p_0}(\ell) \prec \widetilde{\theta}^{p_0}(\ell')$ and therefore $D_{p_0} \prec D'_{p_0}$. As a further consequence,

$$\chi^s(\ell) = (D_0, D_1, \dots, D_{p_0-1}, D_{p_0}, \dots) \prec (D_0, D_1, \dots, D_{p_0-1}, D'_{p_0}, \dots) = \chi^s(\ell').$$

This completes the proof of the implication $\ell \prec \ell' \Longrightarrow \chi^s(\ell) \prec \chi^s(\ell')$. The converse implication follows from the very same arguments in reversed order. □

In general, the relationship between the order on the leaf space f^s and the symbolic space Σ^s is more complicated:

Proposition 2.29 *Let ℓ and ℓ' be two different elements of $f^{s,\infty}$. Let $(D_p)_{p\geq 0} := \chi^s(\ell)$ and $(D'_p)_{p\geq 0} := \chi^s(\ell')$. Let p_0 be the smallest integer p such that $D_p \neq D'_p$.*

(1) *If the map $\widetilde{\theta}^{p_0}|_{\bigcap_{p=0}^{p_0-1} \widetilde{\theta}^{-p}(D_p)}$ preserves the orientation of the foliation $\widetilde{F}^u$, then*
$$\ell \prec \ell' \iff D_{p_0} \prec D'_{p_0} \iff \chi^s(\ell) \prec \chi^s(\ell').$$

(2) *If the map $\widetilde{\theta}^{p_0}|_{\bigcap_{p=0}^{p_0-1} \widetilde{\theta}^{-p}(D_p)}$ reverses the orientation of the foliation $\widetilde{F}^u$, then*
$$\ell \prec \ell' \iff D'_{p_0} \prec D_{p_0} \iff \chi^s(\ell') \prec \chi^s(\ell).$$

Proof The arguments are exactly the same as in the proof of Proposition 2.28. □

3 Topological equivalence of Anosov flows

We will now prove Theorem 1.2 with the help of the coding procedure implemented in Section 2.

3.1 A simplification

We begin by explaining why it is enough to prove Theorem 1.2 in the particular case where the vector fields X_1 and X_2 coincide.

Let (U, X_1, ψ_1) and (U, X_2, ψ_2) be two triples satisfying the hypotheses of Theorem 1.2. In particular, (U, X_1, ψ_1) and (U, X_2, ψ_2) are strongly isotopic. This means that there exists a continuous one-parameter family $\{(U, X_t, \psi_t)\}_{t\in[1,2]}$ such that (U, X_t) is a hyperbolic plug and $\psi_t\colon \partial^{\text{out}}U \to \partial^{\text{in}}U$ is a strongly transverse gluing map for every t. By standard hyperbolic theory, hyperbolic plugs are structurally stable. Hence, this means that we can find a continuous family $(h_t)_{t\in[1,2]}$ of self-homeomorphisms of U such that $h_1 = \text{Id}$ and h_t induces an orbital equivalence between X_1 and X_t. For $t \in [1, 2]$, define

$$\widehat{\psi}_t := (h_t|_{\partial^{\text{in}}U})^{-1} \circ \psi_t \circ (h_t|_{\partial^{\text{out}}U})$$

and observe that $\widehat{\psi}_1 = \psi_1$. For sake of clarity, let $X := X_1$. Then:

- The triples $(U, X, \widehat{\psi}_1)$ and $(U, X, \widehat{\psi}_2)$ are strongly isotopic; the strong isotopy is given by the continuous path $\{(U, X, \widehat{\psi}_t)\}_{t\in[1,2]}$.
- For $t \in [1, 2]$, the flow induced by the vector field X on the manifold $\widehat{M}_t := U/\widehat{\psi}_t$ is orbitally equivalent to the flow induced by the vector field X_t on the manifold $M_t := U/\psi_t$; the orbital equivalence is induced by the homeomorphism h_t.

This shows that the hypotheses and the conclusion of Theorem 1.2 are satisfied for the triples (U, X_1, ψ_1) and (U, X_2, ψ_2) if and only if they are satisfied for the triples $(U, X, \widehat{\psi}_1)$ and $(U, X, \widehat{\psi}_2)$. This allows us to replace the vector fields X_1 and X_2 by a single vector field X in the proof of Theorem 1.2.

3.2 Setting

From now on, we consider a hyperbolic plug (U, X) endowed with two strongly transverse gluing diffeomorphisms $\psi_1, \psi_2\colon \partial^{\text{out}}U \to \partial^{\text{in}}U$. We denote by $\Lambda := \bigcap_{t\in\mathbb{R}} X^t(U)$ the maximal invariant set of the plug (U, X). For $i = 1, 2$, the quotient space $M_i := U/\psi_i$ is a closed three-dimensional manifold, and X induces a vector field Y_i on M_i. We assume that the hypotheses of Theorem 1.2 are satisfied; that is:

(0) The manifolds U, M_1 and M_2 are orientable.

(1) For $i = 1, 2$, the flow (Y_i^t) of the vector field Y_i is a transitive Anosov flow.

(2) The gluing maps ψ_1 and ψ_2 are strongly isotopic, ie there exists an isotopy $(\psi_s)_{s\in[1,2]}$ such that, for every s, the laminations L^s and $\psi_s(L^u_X)$ are strongly transverse.

In order to prove Theorem 1.2, we have to construct a homeomorphism $H\colon M_1 \to M_2$ mapping the oriented orbits of the Anosov flow (Y_1^t) to the orbits of the Anosov flow (Y_2^t). The construction will be divided into several steps.

3.3 Starting point of the construction: diffeomorphisms $\phi_{\mathrm{in}}, \phi_{\mathrm{out}}\colon S_1 \to S_2$

For $i = 1, 2$, we denote by π_i the projection of U on the closed three-dimensional manifold $M_i = U/\psi_i$. We denote by

$$S_i = \pi_i(\partial^{\mathrm{in}} U) = \pi_i(\partial^{\mathrm{out}} U)$$

the projection of the boundary of U. The surface S_i is endowed with the strongly transverse laminations

$$L_i^s := \pi_i(L_X^s) \quad \text{and} \quad L_i^u := \pi_i(L_X^u).$$

The maps $\pi_i|_{\partial^{\mathrm{in}} U}\colon \partial^{\mathrm{in}} U \to S_i$ and $\pi_i|_{\partial^{\mathrm{out}} U}\colon \partial^{\mathrm{out}} U \to S_i$ are invertible. This provides us with two diffeomorphisms

$$\phi_{\mathrm{in}} := \pi_2|_{\partial^{\mathrm{in}} U} \circ (\pi_1|_{\partial^{\mathrm{in}} U})^{-1}\colon S_1 \to S_2 \quad \text{and} \quad \phi_{\mathrm{out}} := \pi_2|_{\partial^{\mathrm{out}} U} \circ (\pi_1|_{\partial^{\mathrm{out}} U})^{-1}\colon S_1 \to S_2.$$

The diffeomorphisms ϕ_{in} and ϕ_{out} are the starting point of our construction. Observe that, at this step, we are very far from getting an orbital equivalence. Indeed, ϕ_{in} and ϕ_{out} are in no way compatible with the actions of the flows (Y_1^t) and (Y_2^t) (ie they do not conjugate the Poincaré return maps of (Y_1^t) and (Y_2^t) on the surfaces S_1 and S_2).

Nevertheless, the definitions of the diffeomorphisms ϕ_{in} and ϕ_{out} imply that

$$\phi_{\mathrm{in}}(L_1^s) = \pi_2|_{\partial^{\mathrm{in}} U} \circ (\pi_1|_{\partial^{\mathrm{in}} U})^{-1}(L_1^s) = \pi_2(L_X^s) = L_2^s.$$
$$\phi_{\mathrm{out}}(L_1^u) = \pi_2|_{\partial^{\mathrm{out}} U} \circ (\pi_1|_{\partial^{\mathrm{out}} U})^{-1}(L_1^u) = \pi_2(L_X^u) = L_2^u.$$

Remark 3.1 Be careful: in general, $\phi_{\mathrm{in}}(L_1^u) \neq L_2^u$ and $\phi_{\mathrm{out}}(L_1^s) \neq L_2^s$.

On the other hand, the strong isotopy connecting the gluing maps ψ_1 and ψ_2 can be used to construct an isotopy between the diffeomorphisms ϕ_{in} and ϕ_{out}:

Proposition 3.2 *There exists a continuous family $(\phi_t)_{t\in[0,1]}$ of diffeomorphisms from S_1 to S_2 such that $\phi_0 = \phi_{\mathrm{out}}$, such that $\phi_1 = \phi_{\mathrm{in}}$ and such that the laminations $\phi_t(L_1^u)$ and L_2^s are strongly transverse for every t.*

Proof By assumption, the gluing maps ψ_1 and ψ_2 are connected by a continuous path $(\psi_s)_{s\in[1,2]}$ of diffeomorphisms from $\partial^{\mathrm{out}} U$ to $\partial^{\mathrm{in}} U$ such that the laminations $\psi_s(L^u)$ and L^s are strongly transverse for every s. For $t \in [0, 1]$, we set

$$\phi_t := \pi_2|_{\partial^{\mathrm{out}} U} \circ \psi_2^{-1} \circ \psi_{2-t} \circ (\pi_1|_{\partial^{\mathrm{out}} U})^{-1}.$$

From this formula, we immediately get

$$\phi_0 = \pi_2|_{\partial^{\mathrm{out}} U} \circ (\pi_1|_{\partial^{\mathrm{out}} U})^{-1} = \phi_{\mathrm{out}}.$$

Plugging the equality $\pi_i|_{\partial^{\mathrm{in}}U} \circ \psi_i = \pi_i|_{\partial^{\mathrm{out}}U}$ into the definition of ϕ_1, we get

$$\phi_1 = \pi_2|_{\partial^{\mathrm{out}}U} \circ \psi_2^{-1} \circ \psi_1 \circ (\pi_1|_{\partial^{\mathrm{out}}U})^{-1} = \pi_2|_{\partial^{\mathrm{in}}U} \circ (\pi_1|_{\partial^{\mathrm{in}}U})^{-1} = \phi_{\mathrm{in}}.$$

We know that the laminations L^s_X and $\psi_{2-t}(L^u_X)$ are strongly transverse for every t. As a consequence, the laminations

$$\pi_2|_{\partial^{\mathrm{out}}U} \circ \psi_2^{-1}(L^s_X) = \pi_2|_{\partial^{\mathrm{in}}U}(L^s_X) = L^s_2$$

and

$$\pi_2|_{\partial^{\mathrm{out}}U} \circ \psi_2^{-1} \circ \psi_{2-t}(L^u_X) = \phi_t \circ \pi_1|_{\partial^{\mathrm{out}}U}(L^u_X) = \phi_t(L^u_1)$$

are strongly transverse for every t. □

It is important to observe that the diffeomorphism ϕ_{in} can be obtained as the restriction of a diffeomorphism from M_1 to M_2:

Proposition 3.3 *The diffeomorphism $\phi_{\mathrm{in}}\colon S_1 \to S_2$ is the restriction of a diffeomorphism $\Phi_{\mathrm{in}}\colon M_1 \to M_2$.*

Proof Once again, we use the existence of a continuous path $(\psi_s)_{s\in[1,2]}$ of diffeomorphisms from $\partial^{\mathrm{out}}U$ to $\partial^{\mathrm{in}}U$ connecting the gluing maps ψ_1 and ψ_2. We consider a collar neighbourhood V of $\partial^{\mathrm{out}}U$ in U, and a diffeomorphism $\xi\colon \partial^{\mathrm{out}}U \times [0,1] \to V$ of V such that $\xi(\partial^{\mathrm{out}}U \times \{0\}) = \partial^{\mathrm{out}}U$. We define a diffeomorphism $\overline{\Phi}_{\mathrm{in}}\colon U \to U$ by setting $\overline{\Phi}_{\mathrm{in}}(\xi(x,t)) := \psi_{2-t}^{-1} \circ \psi_1(x)$ for every $(x,t) \in \partial^{\mathrm{out}}U \times [0,1]$, and $\overline{\Phi}_{\mathrm{in}} = \mathrm{Id}$ on $U \setminus V$. By construction, this diffeomorphism satisfies

$$\overline{\Phi}_{\mathrm{in}} = \begin{cases} \mathrm{Id} & \text{on } \partial^{\mathrm{in}}U, \\ \psi_2^{-1} \circ \psi_1 & \text{on } \partial^{\mathrm{out}}U. \end{cases}$$

As a consequence, the relation $\pi_2 \circ \overline{\Phi}_{\mathrm{in}} = \overline{\Phi}_{\mathrm{in}} \circ \pi_1$ holds, and therefore $\overline{\Phi}_{\mathrm{in}}$ induces a diffeomorphism $\Phi_{\mathrm{in}}\colon M_1 \to M_2$. Since $\overline{\Phi}_{\mathrm{in}} = \mathrm{Id}$ on $\partial^{\mathrm{in}}U$, it follows that $\Phi_{\mathrm{in}}|_{S_1} = \pi_2|_{\partial^{\mathrm{in}}U} \circ (\pi_2|_{\partial^{\mathrm{in}}U})^{-1} = \phi_{\mathrm{in}}$, as desired. □

Now, we introduce the return maps on the surface S_1 and S_2. We first consider the crossing map of the plug (U,X)

$$\theta_X\colon \partial^{\mathrm{in}}U \setminus L^s \to \partial^{\mathrm{out}}U \setminus L^u.$$

By definition, $\theta_X(x)$ is the unique intersection point of the forward (X^t)–orbit of the point x with the surface $\partial^{\mathrm{out}}U$. For $i = 1,2$, the map θ_X induces a map

$$\theta_i := \pi_i|_{\partial^{\mathrm{out}}U} \circ \theta_X \circ (\pi_i|_{\partial^{\mathrm{in}}U})^{-1}\colon S_i \setminus L^s_i \to S_i \setminus L^u_i.$$

This map θ_i is just the Poincaré return map of the flow (Y^t_i) on the surface S_i.

Proposition 3.4 *The diffeomorphisms θ_1, θ_2, ϕ_{in} and ϕ_{out} are related by*

$$\theta_2 \circ \phi_{\mathrm{in}} = \phi_{\mathrm{out}} \circ \theta_1.$$

Proof This follows immediately from the formulas defining θ_1, θ_2, ϕ_{in} and ϕ_{out}. □

Now we lift all the objects to the universal covers of M_1 and M_2. We pick a point $x_1 \in M_1$ which will serve as the basepoint of the fundamental group of the manifold M_1. The point $x_2 := \Phi_{\mathrm{in}}(x_1)$ will be used as the basepoint of fundamental group of the manifold M_2. The diffeomorphism Φ_{in} provides us with an isomorphism $(\Phi_{\mathrm{in}})_*$ between the fundamental groups $\pi_1(M_1, x_1)$ and $\pi_1(M_2, x_2)$. For $i = 1, 2$, we denote by $p_i \colon \widetilde{M}_i \to M_i$ the universal cover of the manifold M_i. We denote by $\widetilde{Y}_i$ the lift of the vector field Y_i on $\widetilde{M}_i$. Observe that $\widetilde{Y}_i$ is equivariant under the action of $\pi_1(M_i, x_i)$: for $\gamma \in \pi_1(M_i, x_i)$, one has $\widetilde{Y}_i(\gamma\tilde{x}) = D_{\tilde{x}}\gamma.\widetilde{Y}_i(\tilde{x})$. We denote by $\widetilde{S}_i$ the complete lift of the surface S_i (ie $\widetilde{S}_i := p_i^{-1}(S_i)$).

We denote by $\widetilde{L}_i^s$ and $\widetilde{L}_i^u$ the complete lifts of the laminations L_i^s and L_i^u. We denote by

$$\tilde{\theta}_i \colon \widetilde{S}_i \setminus \widetilde{L}_i^s \to \widetilde{S}_i \setminus L_i^u$$

the first return map of the flow of the vector field $\widetilde{Y}_i$ on the surface $\widetilde{S}_i$. Clearly, $\tilde{\theta}_i$ is a lift of the map θ_i. Moreover, $\tilde{\theta}_i$ commutes with the deck transformations:

(1) $$\tilde{\theta}_i \circ \gamma = \gamma \circ \tilde{\theta}_i \quad \text{for every } \gamma \in \pi_1(M_i, x_i).$$

This commutation relation is an immediate consequence of the equivariance of $\widetilde{Y}_i$ (see above). Now we fix a lift $\widetilde{\Phi}_{\mathrm{in}} \colon \widetilde{M}_1 \to \widetilde{M}_2$ of the diffeomorphism Φ_{in} (note that, unlike what happens for θ_1 and θ_2, there is no canonical lift of Φ_{in}). Recall that the diffeomorphism Φ_{in} maps the surface S_1 to the surface S_2, and that the restriction of Φ_{in} to S_1 coincides with ϕ_{in}. As a consequence, the lift $\widetilde{\Phi}_{\mathrm{in}}$ maps the surface $\widetilde{S}_1$ to $\widetilde{S}_2$, and the restriction of $\widetilde{\Phi}_{\mathrm{in}}$ to $\widetilde{S}_1$ is a lift $\tilde{\phi}_{\mathrm{in}}$ of the diffeomorphism ϕ_{in}. By construction, this lift satisfies

(2) $$\tilde{\phi}_{\mathrm{in}} \circ \gamma = (\Phi_{\mathrm{in}})_*(\gamma) \circ \tilde{\phi}_{\mathrm{in}} \quad \text{for every } \gamma \in \pi_1(M_1, x_1).$$

Now recall that, according to Proposition 3.2, there exists a continuous arc $(\phi_t)_{t\in[0,1]}$ of diffeomorphisms from S_1 to S_2 such that $\phi_0 = \phi_{\mathrm{in}}$ and $\phi_1 = \phi_{\mathrm{out}}$, and such that the laminations $\phi_t(L_1^u)$ and L_2^s are strongly transverse for every t. We lift this isotopy, starting at the lift $\tilde{\phi}_{\mathrm{in}}$ of $\phi_{\mathrm{in}} = \phi_0$. This yields a continuous arc $(\tilde{\phi}_t)_{t\in[0,1]}$ of diffeomorphisms from $\widetilde{S}_1$ to $\widetilde{S}_2$ such that $\tilde{\phi}_0 = \tilde{\phi}_{\mathrm{in}}$ and such that the laminations $\tilde{\phi}_t(\widetilde{L}_1^u)$ and $\widetilde{L}_2^s$ are strongly transverse for every t. The diffeomorphism $\tilde{\phi}_{\mathrm{out}} := \tilde{\phi}_1$ is a lift of the diffeomorphism ϕ_{out}. By continuity, the relation (2) remains true if we replace $\widetilde{\phi}_{\text{in}} = \widetilde{\phi}_0$ by $\widetilde{\phi}_t$ for any $t \in [0,1]$. In particular, the diffeomorphism $\widetilde{\phi}_{\text{out}}$ satisfies

$$\widetilde{\phi}_{\text{out}} \circ \gamma = (\Phi_{\text{in}})_*(\gamma) \circ \widetilde{\phi}_{\text{out}} \quad \text{for every } \gamma \in \pi_1(M_1, x_1). \tag{3}$$

Proposition 3.5 *The diffeomorphisms $\widetilde{\theta}_1$, $\widetilde{\theta}_2$, $\widetilde{\phi}_{\text{in}}$ and $\widetilde{\phi}_{\text{out}}$ are related by*

$$\widetilde{\theta}_2 \circ \widetilde{\phi}_{\text{in}} = \widetilde{\phi}_{\text{out}} \circ \widetilde{\theta}_1.$$

Proof According to Proposition 3.4, the diffeomorphisms $\theta_2 \circ \phi_{\text{in}}$ and $\phi_{\text{out}} \circ \theta_1$ coincide. Hence, the diffeomorphisms $\widetilde{\theta}_2 \circ \widetilde{\phi}_{\text{in}}$ and $\widetilde{\phi}_{\text{out}} \circ \widetilde{\theta}_1$ are two lifts of the same diffeomorphism. It follows that there exists a deck transformation $\gamma_0 \in \pi_1(M_2, y_0)$ such that

$$\widetilde{\theta}_2 \circ \widetilde{\phi}_{\text{in}} = \gamma_0 \circ \widetilde{\phi}_{\text{out}} \circ \widetilde{\theta}_1.$$

Now consider a deck transformation $\gamma \in \pi_1(M_1, x_0)$. On the one hand, using (2) and (1), we get

$$\widetilde{\theta}_2 \circ \widetilde{\phi}_{\text{in}} \gamma = \widetilde{\theta}_2 \circ (\Phi_{\text{in}})_*(\gamma) \circ \widetilde{\phi}_{\text{in}} = (\Phi_{\text{in}})_*(\gamma) \circ \widetilde{\theta}_2 \circ \widetilde{\phi}_{\text{in}} = ((\Phi_{\text{in}})_*(\gamma) \cdot \gamma_0) \circ \widetilde{\phi}_{\text{out}} \circ \widetilde{\theta}_1.$$

On the other hand, using (1) and (3), we get

$$\widetilde{\theta}_2 \circ \widetilde{\phi}_{\text{in}} \circ \gamma = \gamma_0 \circ \widetilde{\phi}_{\text{out}} \circ \widetilde{\theta}_1 \circ \gamma = \gamma_0 \circ \widetilde{\phi}_{\text{out}} \circ \gamma \circ \widetilde{\theta}_1 = (\gamma_0 \cdot (\Phi_{\text{in}})_*(\gamma)) \circ \widetilde{\phi}_{\text{out}} \circ \widetilde{\theta}_1.$$

Hence,

$$(\Phi_{\text{in}})_*(\gamma) \cdot \gamma_0 = \gamma_0 \cdot (\Phi_{\text{in}})_*(\gamma).$$

Since $(\Phi_{\text{in}})_*(\gamma)$ ranges over the whole fundamental group $\pi_1(M_2, y_0)$, it follows that γ_0 is in the centre of the fundamental group $\pi_1(M_2, y_0)$. If $\gamma_0 \neq \text{Id}$, this implies that $\pi_1(M_2, y_0)$ has a nontrivial centre. It follows that M_2 is a Seifert manifold (see eg [1, Theorem 2.5.5]). Then an easy generalization of a well-known theorem of É Ghys implies that, up to finite cover, the Anosov flow (X_2^t) must be topologically equivalent to the geodesic flow on the unit tangent bundle of a closed hyperbolic surface (see [9] or [3, théorème 3.1]). This is clearly impossible, since X_2 admits a transverse torus (any connected component of the surface S_2 is such a torus). As a consequence, γ_0 must be the identity, and the desired relation $\widetilde{\theta}_2 \circ \widetilde{\phi}_{\text{in}} = \widetilde{\phi}_{\text{in}} \circ \widetilde{\theta}_1$ is proved. □

3.4 Construction of maps $\Delta^s : f_1^{s,\infty} \to f_2^{s,\infty}$ and $\Delta^u : f_1^{u,\infty} \to f_2^{u,\infty}$

In Section 2, we have defined some symbolic spaces which allow us to code certain orbits of certain Anosov flows. Let us introduce these symbolic space in our particular setting. For $i = 1, 2$, we consider the alphabet

$$\mathcal{A}_i := \{\text{connected components of } \widetilde{S}_i \setminus \widetilde{L}_i^s\},$$

and the symbolic space

$$\Sigma_i := \{(D_p)_{p\in\mathbb{Z}} \mid D_p \in \mathcal{A}_i \text{ and } \widetilde{\theta}_i(D_p) \cap D_{p+1} \neq \varnothing \text{ for every } p\}.$$

In order to code stable and unstable leaves, we consider the subspaces Σ_i^s and Σ_i^u of Σ_i defined by

$$\Sigma_i^s := \{(D_p)_{p\geq 0} \mid D_p \in \mathcal{A}_i \text{ and } \widetilde{\theta}_i(D_p) \cap D_{p+1} \neq \varnothing \text{ for every } p\}$$

and

$$\Sigma_i^u := \{(D_p)_{p< 0} \mid D_p \in \mathcal{A}_i \text{ and } \widetilde{\theta}_i(D_p) \cap D_{p+1} \neq \varnothing \text{ for every } p\}.$$

Proposition 3.6 *Let D_1 and D_1' be two elements of $\mathcal{A}_1$. Let $D_2 := \widetilde{\phi}_{\text{in}}(D_1)$ and $D_2' := \widetilde{\phi}_{\text{in}}(D_1')$. Then $\widetilde{\theta}_1(D_1)$ intersects D_1' if and only if $\widetilde{\theta}_2(D_2)$ intersects D_2'.*

Proof We have the sequence of equivalences

$$\begin{aligned}
\widetilde{\theta}_1(D_1) \cap D_1' \neq \varnothing &\iff \widetilde{\phi}_{\text{in}}(\widetilde{\theta}_1(D_1)) \cap \widetilde{\phi}_{\text{in}}(D_1') \neq \varnothing \\
&\iff \widetilde{\phi}_{\text{out}}(\widetilde{\theta}_1(D_1)) \cap \widetilde{\phi}_{\text{in}}(D_1') \neq \varnothing \\
&\iff \widetilde{\theta}_2(\widetilde{\phi}_{\text{in}}(D_1)) \cap \widetilde{\phi}_{\text{in}}(D_1') \neq \varnothing \\
&\iff \widetilde{\theta}_2(D_2) \cap D_2' \neq \varnothing.
\end{aligned}$$

The first equivalence is straightforward. The last one is nothing but the definition of the connected components D_2 and D_2'. The third equivalence follows from Proposition 3.5. It remains to prove the second equivalence. For that purpose, observe that $\widetilde{\theta}_1(D_1)$ is a strip bounded by two leaves of $\widetilde{L}_1^u$, and $\widetilde{\phi}_{\text{in}}(D_1')$ is a strip bounded by two leaves of $\widetilde{L}_2^s$. Now recall that there exists an isotopy $(\widetilde{\phi}_t)_{t\in[0,1]}$ joining $\widetilde{\phi}_{\text{in}}$ to $\widetilde{\phi}_{\text{out}}$ such that the lamination $\widetilde{\phi}_t(\widetilde{L}_1^u)$ is strongly transverse to the lamination $\widetilde{L}_2^s$. It follows that $\widetilde{\phi}_{\text{out}}(\widetilde{\theta}_1(D_1))$ intersects $\widetilde{\phi}_{\text{in}}(D_1')$ if and only if $\widetilde{\phi}_{\text{in}}(\widetilde{\theta}_1(D_1))$ intersects $\widetilde{\phi}_{\text{in}}(D_1')$. □

Now we consider the map

$$(\widetilde{\phi}_{\text{in}})^{\otimes\mathbb{Z}} \colon \mathcal{A}_1^{\mathbb{Z}} \to \mathcal{A}_2^{\mathbb{Z}}, \quad (D_p)_{p\in\mathbb{Z}} \mapsto (\widetilde{\phi}_{\text{in}}(D_p))_{p\in\mathbb{Z}}.$$

As an immediate consequence of Proposition 3.6, we get:

Corollary 3.7 $(\widetilde{\phi}_{\text{in}})^{\otimes\mathbb{Z}} \colon \mathcal{A}_1^{\mathbb{Z}} \to \mathcal{A}_2^{\mathbb{Z}}$ *maps Σ_1 to Σ_2.*

Corollary 3.7 entails that $(\widetilde{\phi}_{\text{in}})^{\otimes \mathbb{Z}_{\geq 0}}$ maps Σ_1^s to Σ_2^s, and $(\widetilde{\phi}_{\text{in}})^{\otimes \mathbb{Z}_{<0}}$ maps Σ_1^u to Σ_2^u. Hence, the map $\widetilde{\phi}_{\text{in}}$ builds a bridge between the symbolic spaces associated to the vector field Y_1 and those associated to the vector field Y_2.

Let us recall the definition of the coding maps constructed in Section 2.3. For $i = 1, 2$, we denote by $\mathcal{F}_i^s$ and $\mathcal{F}_i^u$ the weak stable and the weak unstable foliations of the Anosov flow (Y_i^t) on the manifold M_i. These two-dimensional foliations induce two one-dimensional foliations F_i^s and F_i^u on the surface S_i. We denote by $\widetilde{F}_i^s$ and $\widetilde{F}_i^u$ the lifts of F_i^s and F_i^u on $\widetilde{S}_i$. We denote by f_i^s and f_i^u the leaf spaces of the foliations $\widetilde{F}_i^s$ and $\widetilde{F}_i^u$. We denote by $f_i^{s,\infty}$ the subset of f_i^s made of the leaves that are not in $\widetilde{W}^s(\Lambda_i)$ (recall that $\widetilde{W}^s(\Lambda_i)$ is a union of leaves of $\widetilde{\mathcal{F}}_i^s$ and therefore $\widetilde{W}^s(\Lambda_i) \cap \widetilde{S}_i$ is a union of leaves of F_i^s). Similarly, we denote by $f_i^{u,\infty}$ the subset of f_i^u made of the leaves that are not in $\widetilde{W}^u(\Lambda_i)$. The construction of Section 2.3 provides two bijective coding maps

$$\chi_i^s : \tilde{f}_i^{s,\infty} \to \Sigma_i^s, \quad \ell \mapsto (D_p)_{p \geq 0}, \quad \text{where } \widetilde{\theta}_i^p(\ell) \subset D_p \text{ for every } p \geq 0,$$

and

$$\chi_i^u : \tilde{f}_i^{u,\infty} \to \Sigma_i^u, \quad \ell \mapsto (D_p)_{p<0}, \quad \text{where } \widetilde{\theta}_i^p(\ell) \subset D_p \text{ for every } p < 0.$$

Hence, we obtain two natural bijective maps

$$\Delta^s := (\chi_2^s)^{-1} \circ (\widetilde{\phi}_{\text{in}})^{\otimes \mathbb{Z}_{\geq 0}} \circ \chi_1^s : \tilde{f}_1^{s,\infty} \to \tilde{f}_2^{s,\infty}$$

and

$$\Delta^u := (\chi_2^u)^{-1} \circ (\widetilde{\phi}_{\text{in}})^{\otimes \mathbb{Z}_{<0}} \circ \chi_1^u : \tilde{f}_1^{u,\infty} \to \tilde{f}_2^{u,\infty}.$$

3.5 Extension of the maps Δ^s and Δ^u

We wish to extend the map Δ^s in order to obtain a bijective map between the leaf spaces $\tilde{f}_1^s$ and $\tilde{f}_2^s$. Observe that Δ^s is already defined from a dense subset of $\tilde{f}_1^s$ onto a dense subset of $\tilde{f}_2^s$. We will prove that Δ^s preserves the orders on $\tilde{f}_1^s$ and $\tilde{f}_2^s$. Of course, these are only partial orders. Nevertheless, according to Proposition 2.24, every leaf of $\widetilde{F}_1^s$ (resp. $\widetilde{F}_2^s$) admits a neighbourhood in $\tilde{f}_1^s$ (resp. $\tilde{f}_2^s$) which is totally ordered. As a consequence, the preservation of the order will be sufficient to extend Δ^s.

Our first task is to write a precise definition of the partial orders on $\tilde{f}_1^s$ and $\tilde{f}_2^s$. First we choose an orientation of the lamination $L_X^u \subset \partial^{\text{out}} U$. Pushing this orientation by the maps π_1 and π_2, this defines some orientations of the laminations $L_1^u = (\pi_1)_*(L_X^u) \subset S_1$ and $L_2^u = (\pi_2)_*(L_X^u) \subset S_2$. Since L_i^u is a sublamination of the foliation F_i^u (and since L_i^u intersects every connected component of S_i), the orientations of the laminations L_1^u and L_2^u define some orientations of the foliations F_1^u and F_2^u. Finally, these orientations can be lifted, providing orientations of the lifted foliations $\widetilde{F}_1^u$ and $\widetilde{F}_2^u$. It is important to notice that our choice of orientations for $\widetilde{F}_1^u$ and $\widetilde{F}_2^u$ are not independent from each other. More precisely, the orientations are chosen so that $\phi_{\text{out}} = \pi_2|_{\partial^{\text{out}}U} \circ (\pi_2|_{\partial^{\text{out}}U})^{-1}$ maps the orientation of the lamination L_1^u to the orientation of the lamination L_2^u, and therefore:

(4) $\widetilde{\phi}_{\text{out}}$ maps the orientated lamination $\widetilde{L}_1^u$ to the orientated lamination $\widetilde{L}_2^u$.

As explained in Section 2.4, the orientation of the foliation $\widetilde{F}_i^u$ induces a partial order $\prec_i$ on the leaf space $\tilde{f}_i^s$ defined as follows: given two leaves $\ell_i, \ell_i' \in \tilde{f}_i^s$ satisfy $\ell_i \prec_i \ell_i'$ if there exists an arc segment of an oriented leaf of $\widetilde{F}_i^u$ going from a point of ℓ_i to a point of ℓ_i'. Proposition 2.23 proves that this indeed defines an order on $\tilde{f}_i^s$. Moreover, this order on $\tilde{f}_i^s$ induces a partial order on the alphabet $\mathcal{A}_i$: given two elements D_i and D_i' of $\mathcal{A}_i$, we write $D_i \prec_i D_i'$ if there exists a leaf $\widetilde{\alpha}_i$ of $\widetilde{F}_i^s$ included in D_i and a leaf $\widetilde{\alpha}_i'$ of $\widetilde{F}_i^s$ included in D_i' such that $\widetilde{\alpha}_i \prec_i \widetilde{\alpha}_i'$. Proposition 2.25 shows that we can replace "there exists" by "for every" in this definition. It follows that $\prec_i$ is indeed a partial order on $\mathcal{A}_i$. Now comes the technical result which will allow us to extend the map Δ^s:

Proposition 3.8 *The map* $\Delta^s : (f_1^{s,\infty}, \prec_1) \to (f_2^{s,\infty}, \prec_2)$ *is order-preserving.*

In order to prove Proposition 3.8, we need several intermediary results.

Lemma 3.9 *The map* $\widetilde{\phi}_{\text{in}} : (\mathcal{A}_1, \prec_1) \to (\mathcal{A}_2, \prec_2)$ *is order-preserving.*

Proof Consider two elements D_1 and D_1' of $\mathcal{A}_1$. Assume that $D_1 \prec_1 D_1'$. This means that there exists a leaf ℓ_1 of the oriented lamination $\widetilde{L}_1^u$ which crosses D_1 before crossing D_1'. As a consequence, if we endow $\widetilde{\phi}_{\text{in}}(\ell_1)$ with the image under $\widetilde{\phi}_{\text{in}}$ of the orientation of α_1, then $\widetilde{\phi}_{\text{in}}(\ell_1)$ crosses $\widetilde{\phi}_{\text{in}}(D_1)$ before crossing $\widetilde{\phi}_{\text{in}}(D_1')$. Now recall that:

- $\widetilde{\phi}_{\text{in}}(D_1)$ and $\widetilde{\phi}_{\text{in}}(D_1')$ are strips bounded by leaves of the lamination $\widetilde{\phi}_{\text{in}}(\widetilde{L}_1^s) = \widetilde{L}_2^s$.
- There exists an isotopy $(\widetilde{\phi}_t)$ joining $\widetilde{\phi}_{\text{in}}$ to $\widetilde{\phi}_{\text{out}}$ such that the lamination $\widetilde{\phi}_t(\widetilde{L}_1^u)$ is strongly transverse to the lamination $\widetilde{L}_2^s$ for every t.

We deduce that, if we endow $\widetilde{\phi}_{\text{out}}(\ell_1)$ with the image under $\widetilde{\phi}_{\text{out}}$ of the orientation of ℓ_1, then $\widetilde{\phi}_{\text{out}}(\ell_1)$ crosses $\widetilde{\phi}_{\text{in}}(D_1)$ before crossing $\widetilde{\phi}_{\text{in}}(D_1')$. According to (4), this means that there is a leaf of the oriented lamination $\widetilde{L}_1^u$ which crosses $\widetilde{\phi}_{\text{in}}(D_1)$ before crossing $\widetilde{\phi}_{\text{in}}(D_1')$. By definition of the partial order $\prec_2$, this means that $\widetilde{\phi}_{\text{in}}(D_1) \prec_2 \widetilde{\phi}_{\text{in}}(D_1')$. □

Lemma 3.10 *Let D_1 be a connected component of $\widetilde{S}_1 \setminus \widetilde{L}^s_1$. Set $D_2 := \widetilde{\phi}_{\mathrm{in}}(D_1)$. Then the following are equivalent*:

(1) *The map $\widetilde{\theta}_1$ restricted to the strip D_1 preserves the orientation of the foliation $\widetilde{F}^u_1$.*

(2) *The map $\widetilde{\theta}_2$ restricted to the strip D_2 preserves the orientation of the foliation $\widetilde{F}^u_2$.*

Proof The proof is a bit intricate, because we need to introduce no fewer than six leaves and compare their orientations. Recall that we have chosen some orientations for the foliations $\widetilde{F}^u_1$ and $\widetilde{F}^u_2$. In the sequel, we will also consider the foliations $(\widetilde{\phi}_{\mathrm{in}})_* \widetilde{F}^u_1$, $(\widetilde{\phi}_{\mathrm{out}})_* \widetilde{F}^u_1$ and $(\widetilde{\phi}_t)_* \widetilde{F}^u_1$; we endow them with the images under $\widetilde{\phi}_{\mathrm{in}}$, $\widetilde{\phi}_{\mathrm{out}}$ and $\widetilde{\phi}_t$ of the orientation of $\widetilde{F}^u_1$.

We pick a leaf ℓ_1 of the lamination $\widetilde{L}^u_1$ so that $\ell_1 \cap D_1 \neq \varnothing$ (such a leaf always exists since the laminations $\widetilde{L}^s_1$ and $\widetilde{L}^u_1$ are strongly transverse). Then we set

$$\ell_2 := \widetilde{\phi}_{\mathrm{out}}(\ell_1), \quad \widehat{\ell}_2 := \widetilde{\phi}_{\mathrm{in}}(\ell_1),$$
$$\ell'_1 := \widetilde{\theta}_1(\ell_1 \cap D_1), \quad \ell'_2 := \widetilde{\theta}_2(\ell_2 \cap D_2), \quad \widehat{\ell}'_2 := \widetilde{\theta}_2(\widehat{\ell}_2 \cap D_2).$$

Observe that

$$\widehat{\ell}'_2 = \widetilde{\theta}_2(\widetilde{\phi}_{\mathrm{in}}(\ell_1) \cap D_2) = \widetilde{\theta}_2 \circ \widetilde{\phi}_{\mathrm{in}}(\ell_1 \cap D_1) = \widetilde{\phi}_{\mathrm{out}} \circ \widetilde{\theta}_1(\ell_1 \cap D_1) = \widetilde{\phi}_{\mathrm{out}}(\ell'_1) \tag{5}$$

(the third equality follows from Proposition 3.5). Now recall that, for $i = 1, 2$, both $\widetilde{L}^u_i$ and $(\widetilde{\theta}_i)_*(\widetilde{L}^u_i \cap D^s_i)$ are sublaminations of the foliation $\widetilde{\mathcal{F}}^u_i$. Also recall that $\widetilde{\phi}_{\mathrm{out}}(\widetilde{L}^u_1) = \widetilde{L}^u_2$. This provides some natural orientations on ℓ_1, ℓ'_1, ℓ_2, ℓ'_2, $\widehat{\ell}_2$ and $\widehat{\ell}'_2$:

- ℓ_1 and ℓ'_1 are leaves of the foliation $\widetilde{F}^u_1$, and hence inherit the orientation of $\widetilde{F}^u_1$.
- ℓ_2 and ℓ'_2 are leaves of the foliation $\widetilde{F}^u_2$, and hence inherit the orientation of $\widetilde{F}^u_2$; we endow them with the orientation of this foliation.
- $\widehat{\ell}_2$ is a leaf of the foliation $(\widetilde{\phi}_{\mathrm{in}})_* \widetilde{F}^u_1$, and hence inherits the orientation of $(\widetilde{\phi}_{\mathrm{in}})_* \widetilde{F}^u_1$;
- $\widehat{\ell}'_2$ is a leaf of the foliation $(\widetilde{\phi}_{\mathrm{out}})_* \widetilde{F}^u_1$, and hence inherits the orientation of $(\widetilde{\phi}_{\mathrm{out}})_* \widetilde{F}^u_1$.

By symmetry, it is enough to prove the implication (1) $\Longrightarrow$ (2). So we assume that the restriction of $\widetilde{\theta}_1$ to D^s_1 preserves the orientation of $\widetilde{F}^u_1$; in particular:

$$\widetilde{\theta}_1 \text{ maps the orientation of } \ell_1 \text{ to that of } \ell'_1. \tag{6}$$

According to (4):

$$\widetilde{\phi}_{\mathrm{out}} \text{ maps the orientation of } \ell_1 \text{ to that of } \ell_2. \tag{7}$$

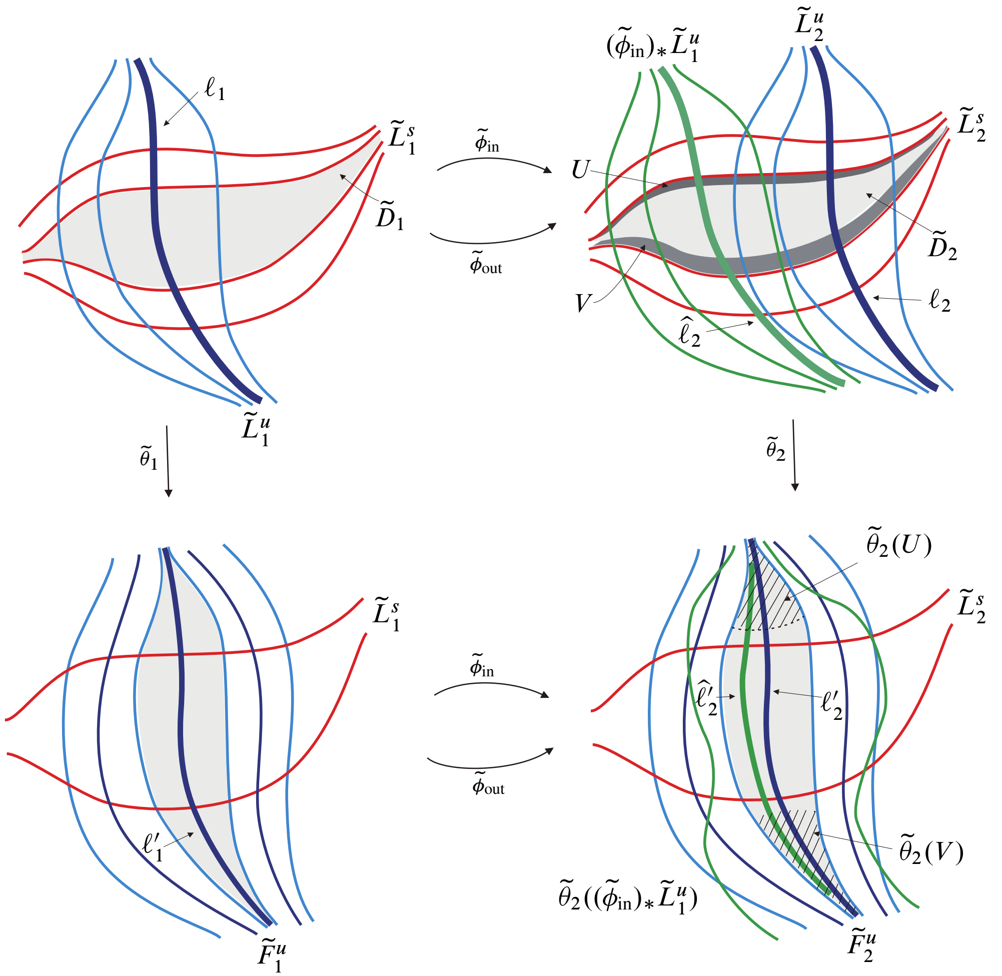

Figure 4: Proof of Lemma 3.10.

The orientations of ℓ_1, ℓ_2, $\widehat{\ell}_2$ and $\widehat{\ell}'_2$ are chosen in such a way that $\widetilde{\phi}_{\mathrm{in}}^{-1}$ maps the orientation of $\widehat{\ell}_2$ to that of ℓ_1, and $\widetilde{\phi}_{\mathrm{out}}$ maps the orientation of ℓ'_1 to that of $\widehat{\ell}'_2$. Putting this together with (6), we obtain that $\widetilde{\phi}_{\mathrm{out}} \circ \widetilde{\theta}_1 \circ \widetilde{\phi}_{\mathrm{in}}^{-1}$ maps the orientation of $\widehat{\ell}_2$ to that of $\widehat{\ell}'_2$. Using Proposition 3.5, we obtain:

$$\widetilde{\theta}_2 \text{ maps the orientation of } \widehat{\ell}_2 \text{ to that of } \widehat{\ell}'_2. \tag{8}$$

Our final goal is to prove that $\widetilde{\theta}_2$ maps the orientation of ℓ_2 to that of ℓ'_2. So, in view of (8), we need to compare the orientations of ℓ_2 and $\widehat{\ell}_2$ on the one hand, and the orientations ℓ'_2 and $\widehat{\ell}'_2$ on the other hand. We start with ℓ_2 and $\widehat{\ell}_2$.

Recall that D_2 is a strip in $\widetilde{S}_2$ bounded by two leaves of the stable lamination $\widetilde{L}^s_2$. We denote these two leaves by α and β in such a way that oriented unstable leaf ℓ_2 enters in D_2 by crossing α and exits D_2 by crossing β. According to (7), the orientation of $\ell_2 = (\widetilde{\phi}_{\mathrm{out}})_*\ell_1$ as a leaf of $\widetilde{L}^u_2 \subset \widetilde{F}^u_2$ coincides with the orientation as a leaf of $(\widetilde{\phi}_{\mathrm{out}})_*\widetilde{L}^u_1 \subset (\widetilde{\phi}_{\mathrm{out}})_*F^u_1$. Moreover, recall that there exists an isotopy $(\widetilde{\phi}_t)_{t\in[0,1]}$ joining $\widetilde{\phi}_0 = \widetilde{\phi}_{\mathrm{in}}$ to $\widetilde{\phi}_1 = \widetilde{\phi}_{\mathrm{out}}$ such that the lamination $\widetilde{\phi}_t(\widetilde{L}^u_1)$ is strongly transverse to the lamination $\widetilde{L}^u_2$ for every t. We deduce that $\widehat{\ell}_2 = (\widetilde{\phi}_{\mathrm{in}})_*(\ell_1)$ crosses D_2 in the same direction as $\ell_2 = (\widetilde{\phi}_{\mathrm{out}})_*\ell_1$. In other words:

(9) Both ℓ_2 and $\widehat{\ell}_2$ enter D_2 by crossing α and exit D_2 by crossing β.

Let U and V be some disjoint neighbourhoods of the stable leaves α and β in the strip D_2. Assertion (9) can be reformulated as follows:

(10) The arcs of oriented leaves $\ell_2 \cap D_2$ and $\widehat{\ell}_2 \cap D_2$ both go from U to V.

We are left to compare the orientations of ℓ'_2 and $\widehat{\ell}'_2$. First observe that $\widetilde{\theta}_2(D_2)$ is an open strip in $\widetilde{S}_2$, bounded by two leaves of the unstable lamination $\widetilde{L}^u_2 = (\widetilde{\phi}_{\mathrm{out}})_*\widetilde{L}^u_1$. The closure $\mathrm{Cl}(\widetilde{\theta}_2(D_2))$ of $\widetilde{\theta}_2(D_2)$ is the union of the open strip $\widetilde{\theta}_2(D_2)$ and its two boundary leaves. The boundary components of $\widetilde{\theta}_2(D_2)$ are leaves of both the foliations F^u_2 and $(\widetilde{\phi}_{\mathrm{out}})_*F^u_1$. Moreover, F^u_2 and $(\widetilde{\phi}_{\mathrm{out}})_*F^u_1$ induce two trivial oriented foliations on the closed strip $\mathrm{Cl}(\widetilde{\theta}_2(D_2))$. In particular, the leaves of F^u_2 and $(\widetilde{\phi}_{\mathrm{out}})_*F^u_1$ in $\mathrm{Cl}(\widetilde{\theta}_2(D_2))$ go from one end of $\mathrm{Cl}(\widetilde{\theta}_2(D_2))$ to the other end. In order to distinguish the two ends of the closed strip $\mathrm{Cl}(\widetilde{\theta}_2(D_2))$, we use the sets $\mathrm{Cl}(\widetilde{\theta}_2(U))$ and $\mathrm{Cl}(\widetilde{\theta}_2(V))$. These sets are disjoint neighbourhoods of the two ends of $\mathrm{Cl}(\widetilde{\theta}_2(D_2))$. So we just need to decide if the leaves go from $\mathrm{Cl}(\widetilde{\theta}_2(U))$ to $\mathrm{Cl}(\widetilde{\theta}_2(V))$, or the contrary. On the one hand, putting (8) and (10) together, we obtain that $\widehat{\ell}_2$ goes from $\mathrm{Cl}(\widetilde{\theta}_2(U))$ to $\mathrm{Cl}(\widetilde{\theta}_2(V))$. On the other hand, F^u_2 and $(\widetilde{\phi}_{\mathrm{out}})_*F^u_1$ are trivial oriented foliations on $\mathrm{Cl}(\widetilde{\theta}_2(D_2))$, and, according to (4), they induce the same orientation on the boundary leaves of D'_2. So we conclude that all the leaves of both the oriented foliations F^u_2 and $(\widetilde{\phi}_{\mathrm{out}})_*F^u_1$ go from $\mathrm{Cl}(\widetilde{\theta}_2(U))$ to $\mathrm{Cl}(\widetilde{\theta}_2(V))$. In particular:

(11) The oriented leaves ℓ'_2 and $\widehat{\ell}'_2$ go from $\widetilde{\theta}_2(U)$ to $\widetilde{\theta}_2(V)$.

From (10) and (11), we deduce that $\widetilde{\theta}_2|_{D_2}$ maps the orientation of ℓ_2 to that of ℓ'_2. By definition of the orientations of ℓ_2 and ℓ'_2, this means that the restriction of $\widetilde{\theta}_2$ to the strip D_2 preserves the orientation of the foliation $\widetilde{F}^u_2$. This completes the proof of the implication (1) $\Longrightarrow$ (2). $\square$

Corollary 3.11 *Let $D_{1,0}, \dots, D_{1,p_0-1}$ be connected components of $\widetilde{S}_1 \setminus \widetilde{L}_1^s$ such that $\bigcap_{p=0}^{p_0-1} \widetilde{\theta}_1^p(D_{1,p})$ is nonempty. For $p = 1, \dots, p_0 - 1$, let $D_{2,p} := \widetilde{\phi}_{\mathrm{in}}(D_{1,p})$. Then the following are equivalent:*

(1) *The map $\widetilde{\theta}_1^{p_0}$ restricted to $\bigcap_{p=0}^{p_0-1} \widetilde{\theta}_1^p(D_{1,p})$ preserves the orientation of the foliation $\widetilde{F}_1^u$.*
(2) *The map $\widetilde{\theta}_2^{p_0}$ restricted to $\bigcap_{p=0}^{p_0-1} \widetilde{\theta}_2^p(D_{2,p})$ preserves the orientation of the foliation $\widetilde{F}_2^u$.*

Proof For $i = 1, 2$, consider the set $J_i \subset \{0, \dots, p_0 - 1\}$ so that the restriction of $\widetilde{\theta}_i$ to $D_{i,p}$ preserves the orientation of $\widetilde{F}_i^u$. On the one hand, Lemma 3.10 implies that the sets J_1 and J_2 coincide. On the other hand, it is clear that the restriction of $\widetilde{\theta}_i$ to $\bigcap_{j=0}^{p_0-1} \widetilde{\theta}_i^p(D_{i,p})$ preserves the orientation of the leaves of $\widetilde{F}_i^u$ if and only if the cardinality of J_i is even. □

Proof of Proposition 3.8 We consider two leaves γ_1 and γ_1' in $f_1^{s,\infty}$, we define $\gamma_2 := \Delta^s(\gamma_1)$ and $\gamma_2' := \Delta^s(\gamma_1')$, and we assume that $\gamma_1 \prec_1 \gamma_1'$. We aim to prove $\gamma_2 \prec_2 \gamma_2'$. Let $\chi_1^s(\widetilde{\gamma}_1) = (D_{1,p})_{p\geq 0}$, $\chi_1^s(\widetilde{\gamma}_1') = (D'_{1,p})_{p\geq 0}$, $\chi_2^s(\widetilde{\gamma}_2) = (D_{2,p})_{p\geq 0}$ and $\chi_2^s(\widetilde{\gamma}_2') = (D'_{2,p})_{p\geq 0}$. By definition of the map χ_i^s, this means that, for $i = 1, 2$,

$$\widetilde{\gamma}_i = \bigcap_{p\geq 0} \widetilde{\theta}_i^{-p}(D_{i,p}) \quad \text{and} \quad \widetilde{\gamma}_i' = \bigcap_{p\geq 0} \widetilde{\theta}_i^{-p}(D'_{i,p}).$$

And, since $\widetilde{\gamma}_2 = \Delta^s(\widetilde{\gamma}_1)$ and $\widetilde{\gamma}_2' = \Delta^s(\widetilde{\gamma}_1')$, we have

$$D_{2,p} = \phi_{\mathrm{in}}(D_{1,p}) \quad \text{and} \quad D'_{2,p} = \phi_{\mathrm{in}}(D'_{1,p})$$

for every $p \geq 0$. We denote by p_0 the smallest integer p such that $D_{1,p} \neq D'_{1,p}$.

Let us consider the case where the map $\widetilde{\theta}_1^{p_0}$ restricted to $\bigcap_{p=0}^{p_0-1} \widetilde{\theta}_1^{-p}(D_{1,p})$ preserves the orientation of the foliation $\widetilde{F}_1^u$.

- Proposition 2.29 implies that $D_{1,p_0} \prec_1 D'_{1,p_0}$.
- Since $\phi_{\mathrm{in}}\colon \mathcal{A}_1 \to \mathcal{A}_2$ is order-preserving (Lemma 3.9), $D_{2,p_0} \prec_2 D'_{2,p_0}$.
- Corollary 3.11 implies that the map $\widetilde{\theta}_2^{p_0}$ restricted to $\bigcap_{p=0}^{p_0-1} \widetilde{\theta}_2^{-p}(D_{2,p})$ preserves the orientation of the foliation $\widetilde{F}_2^u$.
- Using again Proposition 2.29, we deduce from the two last items above that $\widetilde{\gamma}_2 \prec_2 \widetilde{\gamma}_2'$, as desired.

The case where the map $\widetilde{\theta}_1^{p_0}$ restricted to $\bigcap_{p=0}^{p_0-1} \widetilde{\theta}_1^{-p}(D_{1,p})$ reverses the orientation of the foliation $\widetilde{F}_1^u$ follows from the very same arguments. □

Corollary 3.12 *The map $\Delta^s\colon f_1^{s,\infty} \to f_2^{s,\infty}$ extends in a unique way to an order-preserving bijection $\Delta^s : f_1^s \to f_2^s$.*

Proof This is an immediate consequence of the following facts:

- $\Delta_s\colon f_1^{s,\infty} \to f_2^{s,\infty}$ is an order-preserving map (Proposition 3.8).
- For $i = 1, 2$, $f_i^{s,\infty}$ is a dense subset of the (nonseparated) one-dimensional manifold f_i^s (Proposition 2.8).
- For $i = 1, 2$, each leaf $\ell \in f_i^s$ has a neighbourhood U_ℓ in f_i^s such that the leaves in U_ℓ are totally ordered (Proposition 2.24). □

Of course, the stable and the unstable directions play some symmetric roles, so the same arguments as above allow one to prove the following analog of Corollary 3.12:

Corollary 3.13 *The map $\Delta^u\colon f_1^{u,\infty} \to f_2^{u,\infty}$ extends in a unique way to an order-preserving bijection $\widehat{\Delta}^u\colon f_1^u \to f_2^u$.*

3.6 Mating $\widehat{\Delta}^s$ and $\widehat{\Delta}^u$: construction of the map $\widehat{\Delta}$

Now, we will mate the maps $\widehat{\Delta}^s$ and $\widehat{\Delta}^u$ to obtain a $\widehat{\Delta}\colon \widetilde{S}_1 \to \widetilde{S}_2$. In view to that goal, we need the following lemma:

Lemma 3.14 *Consider a leaf ℓ_1^s of the stable foliation $\widetilde{\mathcal{F}}_1^s$ and a leaf ℓ_1^u of the unstable foliation $\widetilde{\mathcal{F}}_1^u$. Then ℓ_1^s intersects ℓ_1^u if and only if $\widehat{\Delta}^s(\ell_1^s)$ intersects $\widehat{\Delta}^u(\ell_1^u)$.*

Proof The case where the leaves ℓ_1^s and ℓ_1^u belong to $f_1^{s,\infty}$ and $f_1^{u,\infty}$ is a consequence of Proposition 3.6 (together with the definitions of the maps Δ^s, Δ^u and Δ): the leaves ℓ_1^s and ℓ_1^u intersect at x if and only if the leaves $\widehat{\Delta}^s(\ell_1^s) = \Delta^s(\ell_1^s)$ and $\widehat{\Delta}^u(\ell_1^u) = \Delta^u(\ell_1^u)$ intersect at $\Delta(x)$. The general case follows by density of $f_i^{s,\infty}$ and $f_i^{u,\infty}$ in $f_i^{s,\infty}$ and $f_i^{u,\infty}$. □

Now we define a map $\widehat{\Delta}\colon \widetilde{S}_1 \to \widetilde{S}_2$. Let $\tilde{x}$ be any point in $\widetilde{S}_1$. Denote by ℓ_1^s (resp. ℓ_1^u) the leaf of the stable foliation $\widetilde{\mathcal{F}}_1^s$ (resp. the unstable foliation $\widetilde{\mathcal{F}}_1^u$) passing through x. Recall that x is the unique intersection point of ℓ_1^s and ℓ_1^u. According to the preceding lemma, the stable leaf $\widehat{\Delta}^s(\ell_1^s)$ and the unstable leaf $\widehat{\Delta}^u(\ell_1^u)$ do intersect. According to Proposition 2.3, the intersection is a single point. We define $\widehat{\Delta}(\tilde{x})$ to be the unique intersection point of the leaves $\widehat{\Delta}^s(\ell_1^s)$ and $\widehat{\Delta}^u(\ell_1^u)$. In other words, $\widehat{\Delta}$ is defined by

$$\widehat{\Delta}(\ell_1^s \cap \ell_1^u) = \widehat{\Delta}^s(\ell_1^s) \cap \widehat{\Delta}^u(\ell_1^u). \tag{12}$$

By construction, the map $\widehat{\Delta}$ is bijective and maps the foliations $\widetilde{F}^s_1$ and $\widetilde{F}^u_1$ to the foliations $\widetilde{F}^s_2$ and $\widetilde{F}^u_2$, preserving the orders on the leaf spaces. Since the leaf spaces are locally totally ordered (Proposition 2.24), it follows that $\widehat{\Delta}$ is continuous. Hence, $\widehat{\Delta}$ is a homeomorphism.

Proposition 3.15 *The map $\widehat{\Delta}\colon \widetilde{S}_1 \to \widetilde{S}_2$ is equivariant with respect to the actions of the fundamental groups: for every γ of $\pi_1(M_1)$,*

$$\widehat{\Delta} \circ \gamma = (\widetilde{\Phi}_{\text{in}})_*(\gamma) \circ \widehat{\Delta}.$$

Proof This is a rather immediate consequence of the construction of $\widehat{\Delta}$. First recall that $\widehat{\Delta}$ is a continuous extension of the map $\Delta\colon \widetilde{S}^\infty_1 \to \widetilde{S}^\infty_2$ and recall that $\widetilde{S}^\infty_1$ and $\widetilde{S}^\infty_2$ are dense subsets of $\widetilde{S}_1$ and $\widetilde{S}_2$. As a consequence, it is enough to prove that Δ is equivariant with respect to the actions of the fundamental groups. Now recall that Δ is defined as the composition of three maps:

$$\Delta = (\chi_2)^{-1} \circ (\widetilde{\phi}_{\text{in}})^{\otimes\mathbb{Z}} \circ \chi_1.$$

But we know that:

- The map χ_i commutes with the action of the fundamental group $\pi_1(M_i)$ for $i = 1, 2$ (Proposition 2.17).
- The map $\widetilde{\phi}_{\text{in}}$ satisfies $\widetilde{\phi}_{\text{in}} \circ \gamma = (\widetilde{\Phi}_{\text{in}})_*(\gamma) \circ \widetilde{\phi}_{\text{in}}$ (equation (2)).

This shows that the map Δ satisfies the equivariance relation $\Delta \circ \gamma = (\widetilde{\Phi}_{\text{in}})_*(\gamma) \circ \Delta$. □

Proposition 3.16 *The map $\widehat{\Delta}\colon \widetilde{S}_1 \to \widetilde{S}_2$ conjugates the Poincaré maps $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$; that is,*

$$\widehat{\Delta} \circ \widetilde{\theta}_1 = \widetilde{\theta}_2 \circ \widehat{\Delta}.$$

Proof On the one hand, for $i = 1, 2$, the coding map χ^s_i conjugates the Poincaré map $\widetilde{\theta}_i$ on $\widetilde{S}_i$ to the shift map on the symbolic space Σ^s_i (Proposition 2.18). On the other hand, the map $(\widetilde{\phi}_{\text{in}})^{\otimes\mathbb{Z}_{\geq 0}}$ obviously conjugates the shift map on Σ^s_1 to the shift map on Σ^s_2. Hence, $\Delta^s = (\chi^s_2)^{-1} \circ (\widetilde{\phi}_{\text{in}})^{\otimes\mathbb{Z}_{\geq 0}} \circ \chi^s_1$ conjugates the action $\widetilde{\theta}_1$ on $f^{s,\infty}_1$ to the action of $\widetilde{\theta}_2$ on $f^{s,\infty}_2$. By density of $f^{s,\infty}_1$ in f^s_i, it follows that $\widehat{\Delta}^s$ conjugates the action $\widetilde{\theta}_1$ on f^s_1 to the action of $\widetilde{\theta}_2$ on f^s_2. Similarly, $\widehat{\Delta}^u$ conjugates the action $\widetilde{\theta}_1$ on f^u_1 to the action of $\widetilde{\theta}_2$ on f^u_2. Finally, since $\widetilde{\Delta}$ is defined by mating $\widehat{\Delta}^s$ and $\widehat{\Delta}^u$ (see (12)), this implies that $\widetilde{\Delta}$ conjugates $\widetilde{\theta}_1$ to $\widetilde{\theta}_2$. □

3.7 From the map $\widehat{\Delta}$ to the orbital equivalence

To conclude the proof of Theorem 1.2, we need to introduce the orbit spaces of the Anosov flows (Y_1^t) and (Y_2^t). The *orbit space* of (Y_i^t) is by definition the quotient of the manifold $\widetilde{M}_i$ by the action of the flow (Y_i^t). We denote it by O_i, and we denote by pr_i the natural projection of $\widetilde{M}_i$ on O_i. The action of the fundamental group $\pi_1(M_i)$ on $\widetilde{M}_i$ induces an action of this group on O_i. The two-dimensional foliations $\widetilde{\mathcal{F}}_i^s$ and $\widetilde{\mathcal{F}}_i^u$ are leafwise invariant under the flow (Y_i^t) and therefore can be projected in the orbit space O_i. They induce a pair (g_i^s, g_i^u) of transverse one-dimensional foliations on O_i.

The orbit space O_i by itself does not carry much information: indeed, O_i is always a separated manifold diffeomorphic to $\mathbb{R}^2$ (see [8, Proposition 2.1] or [2, Theorem 3.2]). The pair of transverse foliations (g_i^s, g_i^u) carries much more interesting information (see the work of Barbot and Fenley on the subject; good references are Barbot's habilitation memoir [3] and Barthelmé's lecture notes [4]). The action of $\pi_1(M_i)$ on O_i carries even richer dynamical information: actually, this action characterizes the flow (Y_i^t) up to topological equivalence (see Theorem 3.22 below).

Recall that Λ denotes the maximal invariant set of the initial hyperbolic plug (U, X), that Λ_i denotes the projection of Λ in the manifold $M_i = U/\psi_i$, and that $\widetilde{\Lambda}_i$ the complete lift of Λ_i in the universal cover $\widetilde{M}_i$. Now we denote by L_i the projection of the set $\widetilde{\Lambda}_i$ in O_i.

Lemma 3.17 *The projection $\mathrm{pr}_i(\widetilde{S}_i)$ of the surface $\widetilde{S}_i$ in the orbit space O_i is exactly the complement of the set L_i in O_i.*

Proof The set Λ is the union of the orbits of the vector field X which remain in U forever, ie which do not intersect ∂U. Hence, the set $\Lambda_i = \pi_i(\Lambda)$ is the union of the orbits of the vector field $Y_i = (\pi_i)_* X$ which do not intersect the surface $S_i = \pi_i(\partial U)$. As a further consequence, $\widetilde{\Lambda}_i$ is the union of the orbits of the vector field $\widetilde{Y}_i$ which do not intersect the surface $\widetilde{S}_i$. This means that the projection of $\widetilde{S}_i$ in the orbit space O_i is exactly the complement of the projection of the set $\widetilde{\Lambda}_i$. □

Proposition 3.16 can be rephrased as follows: two points $x, x' \in \widetilde{S}_1$ belong to the same orbit of the flow $(\widetilde{Y}_1^t)$ if and only if the points $\widehat{\Delta}(x)$ and $\widehat{\Delta}(x')$ belong to the same orbit of the flow $(\widetilde{Y}_2^t)$. As a consequence, the homeomorphism $\widehat{\Delta}\colon \widetilde{S}_1 \to \widetilde{S}_2$ induces a homeomorphism

$$\delta\colon \mathrm{pr}_1(\widetilde{S}_1) = O_1 \setminus L_1 \to \mathrm{pr}_2(\widetilde{S}_2) = O_2 \setminus L_2.$$

Since $\widehat{\Delta}$ is equivariant with respect to the actions of the fundamental groups (Proposition 3.15), the homeomorphism δ is also equivariant: for every $\gamma \in \pi_1(M_1)$,

$$\delta \circ \gamma = (\widetilde{\Phi}_{\text{in}})_*(\gamma) \circ \delta.$$

Our next step is to extend the map η to the whole orbit spaces.

Proposition 3.18 *The homeomorphism $\delta\colon O_1 \setminus L_1 \to O_2 \setminus L_2$ can be extended in a unique way to a homeomorphism $\overline{\delta}\colon O_1 \to O_2$ which is equivariant with respect to the actions of the fundamental groups of M_1 and M_2.*

We shall use the following general lemma of planar topology:

Lemma 3.19 *Let A and B be totally discontinuous subsets of $\mathbb{R}^2$ and $h\colon \mathbb{R}^2 \setminus A \to \mathbb{R}^2 \setminus B$. Assume that, for every compact subset K of $\mathbb{R}^2$, the set $h(K \setminus A)$ is relatively compact in $\mathbb{R}^2$. Then h can be extended to a homeomorphism of $\bar{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$.*

This lemma is easy and certainly well known to people working in planar topology, but we were not able to find it in the literature. We provide a proof for sake of completeness.

Proof We proceed to the definition of $\bar{h}$. Let x be a point in A. We pick a decreasing sequence $(X_n)_{n\geq 0}$ of compact connected subsets of $\mathbb{R}^2$ so that $X_n \neq \{x\}$ for every n and so that $\bigcap_n X_n = \{x\}$. For every $n \geq 0$, let Y_n be the closure in $\mathbb{R}^2$ of the set $h(X_n \setminus A)$. Our assumptions imply that $(Y_n)_{n\geq 0}$ is a decreasing sequence of nonempty compact connected subsets of $\mathbb{R}^2$. As a consequence, the intersection $\bigcap_n Y_n$ must be a nonempty compact connected subset of $\mathbb{R}^2$. Moreover, since $\bigcap_n X_n = \{x\} \subset A$, the intersection $\bigcap_n Y_n$ must be included in B. Since B is totally disconnected, it follows that $\bigcap_n Y_n$ must be a singleton $\{y\}$. Standard arguments show that the point y does not depend on the choice of the sequence (X_n). We set $\bar{h}(x) := y$. Repeating the same procedure for each point $x \in A$, we get an extension $\bar{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of h. The continuity of $\bar{h}$ follows easily from its definition.

Of course, the same procedure yields a continuous extension $\overline{h^{-1}}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the map $h^{-1}\colon \mathbb{R}^2 \setminus B \to \mathbb{R}^2 \setminus A$. Since $\mathbb{R}^2 \setminus A$ and $\mathbb{R}^2 \setminus B$ are dense in $\mathbb{R}^2$, the equalities $h \circ h^{-1} = \mathrm{Id}_{\mathbb{R}^2 \setminus B}$ and $h^{-1} \circ h = \mathrm{Id}_{\mathbb{R}^2 \setminus A}$ extend to $\bar{h} \circ \overline{h^{-1}} = \overline{h^{-1}} \circ \bar{h} = \mathrm{Id}_{\mathbb{R}^2}$. This shows that $\bar{h}$ is a homeomorphism. □

Lemma 3.20 *For $i = 1, 2$, the set L_i is totally discontinuous in $O_i \simeq \mathbb{R}^2$.*

Let us introduce some terminology that will be used in the proof of Lemma 3.20. By a *local section* of a vector field Z on a three-manifold P, we mean a compact surface with boundary embedded in P and transverse to Z. A (Z^t)–invariant set $\Omega \subset P$ is said to be *transversally totally discontinuous* if $\Omega \cap \Sigma$ is totally discontinuous for every local section Σ of Z.

Proof By our assumptions, the maximal invariant set Λ_X of the hyperbolic plug (U, X) contains neither attractors nor repellers. Since Λ_X is a hyperbolic set, it follows that Λ_X is transversally totally discontinuous. Hence, the projection Λ_i of Λ_X in the manifold M_i is also transversally totally discontinuous (recall that Λ_X sits in the interior of U and that the projection $p_i : U \to M_i$ is a homeomorphism in restriction to the interior of U). As a further consequence, the complete lift $\widetilde{\Lambda}_i$ of Λ_i in the universal cover $\widetilde{M}_i$ is also transversally totally discontinuous.

Now recall that $(\widetilde{M}_i, \widetilde{Y}_i)$ is topologically equivalent to $\mathbb{R}^3$ equipped with the trivial vertical unit vector field. As a consequence, for every point $x \in \widetilde{M}_i$, we can find a local section Σ of $\widetilde{Y}_i$ such that $x \in \Sigma$ and no orbit of $\widetilde{Y}_i$ intersects Σ twice. This implies that the restriction to Σ of the projection $\mathrm{pr}\colon \widetilde{M}_i \to \mathcal{O}_i$ is one-to-one, and hence a homeomorphism onto its image. Since $\widetilde{\Lambda}_i$ is transversally totally discontinuous, it follows that the set $L_i = \mathrm{pr}(\widetilde{\Lambda}_i)$ is totally discontinuous in $\mathcal{O}_i$. □

Lemma 3.21 *For every compact set $K \subset \mathcal{O}_1 \simeq \mathbb{R}^2$, the set $\eta(K \setminus L_1)$ has compact closure in $\mathcal{O}_2 \simeq \mathbb{R}^2$.*

Proof For $i = 1, 2$, the surface $\mathcal{O}_i \setminus L_i$ has infinitely many ends. One of them is the end of $\mathcal{O}_i \simeq \mathbb{R}^2$, which we denote by ∞_i. The other ends are in one-to-one correspondence with the points of L_i (since L_i is totally discontinuous). Proving Lemma 3.21 is equivalent to proving that the homeomorphism $\eta\colon \mathcal{O}_1 \setminus L_1 \to \mathcal{O}_2 \setminus L_2$ maps the end ∞_1 to the end ∞_2.

From the viewpoint of the topology of the surface $\mathcal{O}_i \setminus L_i$, nothing distinguishes ∞_i from the other ends. Hence, we need to introduce some dynamical invariants to prove that η necessarily maps ∞_1 to ∞_2.

For $i = 1, 2$, the foliation $\widetilde{\mathcal{F}}^s_i$ induces a one-dimensional foliation g^s_i on the space $\mathcal{O}_i$. We denote by $g^s_{i,0}$ the restriction of the foliation g^s_i to $\mathcal{O}_i \setminus L_i$. According to Lemma 3.17, $g^s_{i,0}$ can be obtained as the projection on $\mathcal{O}_i$ of the foliation $\widetilde{\mathcal{F}}^s_i \cap \widetilde{S}_i = \widetilde{F}^s_i$. As a consequence, η maps the foliation $g^s_{1,0}$ to the foliation $g^s_{2,0}$.

Since O_i is a plane, every leaf of the foliation g_i^s is a properly embedded line, going from ∞_i to ∞_i (recall that ∞_i is the unique end of O_i). The leaves of $g_i^s = (\mathrm{pr}_i)_*\widetilde{\mathcal{F}}_i^s$ that intersect $L_i = \mathrm{pr}_i(\Lambda_i)$ are the projections of the leaves of the lamination $W^s(\widetilde{\Lambda}_i)$. In particular, there exist leaves of g_i^s that do not intersect L_i. As a consequence, there exist leaves of $g_{i,0}^s$ going from ∞_i to ∞_i. On the other hand, if x is an end of $O_i \setminus L_i$ corresponding to a point of L_i, then there does not exist any leaf of $g_{i,0}^s$ going from x to x (because every leaf ℓ of $g_{i,0}^s$ is a connected component of $\widehat{\ell} \setminus L_i$, where $\widehat{\ell}$ is a line in O_i going from ∞_i to ∞_i). So the foliation $g_{i,0}^s$ allows us to distinguish ∞_i from the other ends of $O_i \setminus L_i$. Since η maps $g_{1,0}^s$ to $g_{2,0}^s$, it follows that η must map ∞_1 to ∞_2. Since ∞_i is the unique end of O_i, this exactly means that, for a compact set $K \subset O_1 \simeq \mathbb{R}^2$, the set $\eta(K \setminus L_1)$ has compact closure in $O_2 \simeq \mathbb{R}^2$. □

Proof of Proposition 3.18 Lemmas 3.20 and 3.21, together with the fact that O_1 and O_2 are homeomorphic to $\mathbb{R}^2$, show that we are exactly in the situation of Lemma 3.19. Applying this lemma, we get a homeomorphism $\overline{\delta}\colon O_1 \to O_2$ extending η. The equivariance of $\overline{\eta}$ follows from that of δ, by continuity and by density of $O_i \setminus L_i$ in O_i. □

We will now conclude the proof of Theorem 1.2 by using a result of Barbot.

Theorem 3.22 (see [2, Theorem 3.4] or [3, proposition 1.36 and corollaire 1.42]) *Two transitive Anosov flows are topologically equivalent if and only if there exists a homeomorphism between their orbit spaces which is equivariant with respect to the actions of the fundamental groups and which does not exchange the stable/unstable directions.*

Proof of Theorem 1.2 The theorem is an immediate consequence of Proposition 3.18 and Theorem 3.22. □

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References

[1] **M Aschenbrenner**, **S Friedl**, **H Wilton**, *3–Manifold groups*, Eur. Math. Soc., Zürich (2015) MR Zbl

[2] **T Barbot**, *Generalizations of the Bonatti–Langevin example of Anosov flow and their classification up to topological equivalence*, Comm. Anal. Geom. 6 (1998) 749–798 MR Zbl

[3] **T Barbot**, *De l'hyperbolique au globalement hyperbolique*, habilitation, Université Claude Bernard – Lyon I (2005) https://tel.archives-ouvertes.fr/tel-00011278

[4] **T Barthelmé**, *Anosov flows in* 3*–manifolds*, lecture notes (2017) https://sites.google.com/site/thomasbarthelme/research

[5] **F Béguin**, **C Bonatti**, **B Yu**, *Building Anosov flows on* 3*–manifolds*, Geom. Topol. 21 (2017) 1837–1930 MR Zbl

[6] **F Béguin**, **B Yu**, *Coding periodic orbits for Anosov flows obtained by gluing hyperbolic plugs and applications*, in preparation

[7] **C Bonatti**, **R Langevin**, *Un exemple de flot d'Anosov transitif transverse à un tore et non conjugué à une suspension*, Ergodic Theory Dynam. Systems 14 (1994) 633–643 MR Zbl

[8] **S R Fenley**, *Quasigeodesic Anosov flows and homotopic properties of flow lines*, J. Differential Geom. 41 (1995) 479–514 MR Zbl

[9] **E Ghys**, *Flots d'Anosov sur les* 3*–variétés fibrées en cercles*, Ergodic Theory Dynam. Systems 4 (1984) 67–80 MR Zbl

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