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Stable cohomology of the universal degree d hypersurface in $\mathbb{P}^n$

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We consider the universal hypersurface of degree d in $\mathbb{CP}^n$ and compute its stable cohomology (with respect to d). We describe the stable classes geometrically.

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1 Introduction

Let $U_{d,n}$ be the *parameter space* of smooth degree d hypersurfaces in $\mathbb{P}^n$. There is a natural inclusion $U_{d,n} \subseteq \mathbb{P}^{\binom{n+d}{d}} = \mathbb{P}(V_{d,n})$, where $V_{d,n}$ is the vector space of homogenous degree d complex polynomials in $n+1$ variables. Let

$$U_{d,n}^* := \{(f, p) \in U_{d,n} \times \mathbb{P}^n \mid f(p) = 0\}.$$

Let $\phi\colon U_{d,n}^* \to U_{d,n}$ be defined by $\phi(f, p) = f$. The map $\phi\colon U_{d,n}^* \to U_{d,n}$ is the *universal family* of smooth degree d hypersurfaces in $\mathbb{P}^n$; it satisfies the following property: given a family $\pi\colon E \to B$ of smooth degree d hypersurfaces in $\mathbb{P}^n$, there is a unique diagram

$$\begin{tikzcd} E \arrow[r, "\exists!"] \arrow[d] & U_{d,n}^* \arrow[d] \\ B \arrow[r, "\exists!"] & U_{d,n} \end{tikzcd}$$

In other words, any family of smooth degree d hypersurfaces is pulled back from this one. Our main result is as follows:

Theorem 1.1 *Let $d, n \geq 1$. Then there is an embedding of graded algebras,*

$$\phi\colon \mathrm{H}^*(\mathrm{PGL}_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n) \hookrightarrow \mathrm{H}^*(U_{d,n}^*; \mathbb{Q}),$$

where $|x| = 2$. Here $|\cdot|$ denotes the cohomological degree. Let $c_1(E)$ denote the first Chern class of the line bundle E.

(1) *The element* $\phi(x) = c_1(\mathscr{L})$, *where* $\mathscr{L}$ *is the fiberwise canonical bundle* (defined in Section 2).

(2) *Suppose* $d \geq 4n+1$. *Then* ϕ *is surjective in degree less than* $(d-1)/2$.

Let $X_{d,n} \subseteq V_{d,n}$ be the open subspace of polynomials defining a nonsingular hypersurface. The complement of $X_{d,n}$ in $V_{d,n}$ is known as the *discriminant hypersurface*; it is the zero locus of the classical discriminant polynomial. It is known to be highly singular.

A point of $X_{d,n}$ determines a projective hypersurface up to a scalar. There is a natural action of $\mathbb{C}^*$ on $X_{d,n}$ such that the quotient $X_{d,n}/\mathbb{C}^*$ is $U_{d,n}$. Let

$$X^*_{d,n} := \{(f, p) \mid f \in X_{d,n}, p \in \mathbb{P}^n, f(p) = 0\}.$$

There is a forgetful map $\pi\colon X^*_{d,n} \to X_{d,n}$ defined by $\pi(f, p) = f$. The fibres of π are

$$Z(f) := \pi^{-1}(f) = \{p \in \mathbb{P}^n \mid f(p) = 0\} \subseteq \mathbb{P}^n.$$

It is well known that the map π is a fibre bundle.

$X^*_{d,n}$ also has several interesting quotients. The action of GL_{n+1} on $X_{d,n}$ lifts to one on $X^*_{d,n}$. We obtain $U^*_{d,n} = X^*_{d,n}/\mathbb{C}^*$. The map $\pi\colon X^*_{d,n} \to X_{d,n}$ is $\mathbb{C}^*$–equivariant and descends to the map $\phi\colon U^*_{d,n} \to U_{d,n}$.

We define $M_{d,n} := U_{d,n}/\mathrm{PGL}_{n+1}(\mathbb{C})$, the *moduli space* of degree d smooth hypersurfaces in $\mathbb{P}^n$. We also define $M^*_{d,n} = X^*_{d,n}/\mathrm{GL}_{n+1}(\mathbb{C})$.

We can rewrite our result in terms of $X^*_{d,n}$ and $M^*_{d,n}$ as well. This is important to us as our proof will mostly involve understanding the space $X^*_{d,n}$. The space $M^*_{d,n}$ is important conceptually.

Theorem 1.2 *Let* $d, n \geq 1$.

(1) *There is an embedding of graded algebras*,

$$\psi\colon (\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n)) \hookrightarrow \mathrm{H}^*(X^*_{d,n}; \mathbb{Q}),$$

where $|x| = 2$.

(2) *There is an embedding of graded algebras*,

$$\varphi\colon \mathbb{Q}[x]/(x^n) \hookrightarrow \mathrm{H}^*(M^*_{d,n}; \mathbb{Q}),$$

where $|x| = 2$.

Suppose that $d \geq 4n+1$. *Then the maps* ψ *and* φ *are surjective in degree* $\leq (d-1)/2$.

Theorem 1.2 is equivalent to Theorem 1.1 after applying Theorem 2 of Peters and Steenbrink [6].

Nature of stable cohomology Throughout the course of the proof of Theorem 1.2 we also obtain the following description of the stable cohomology classes of $X^*_{d,n}$. The stable classes are tautological in the following sense: There is a line bundle $\mathscr{L}$ on $M^*_{d,n}$ defined by taking the canonical bundle fibrewise (we rigorously define $\mathscr{L}$ in Section 2). We will show that $c_1(\mathscr{L}), \dots, c_1(\mathscr{L})^{n-1}$ are nonzero in $H^*(M^*_{d,n};\mathbb{Q})$ and that stably the entire cohomology ring of $M^*_{d,n}$ is just the algebra generated by $c_1(\mathscr{L})$. By [6],

$$\mathrm{H}^*(X^*_{d,n};\mathbb{Q}) \cong \mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n};\mathbb{Q}).$$

In this way we have some qualitative understanding of the stable cohomology of $X^*_{d,n}$.

Both the statement of Theorem 1.2 and our proof of it are heavily influenced by [8], in which Tommasi proves analogous theorems for $X_{d,n}$. Our techniques and approach are also similar to that of Das in [3], where he proves

$$\mathrm{H}^*(X^*_{3,3};\mathbb{Q}) \cong \mathrm{H}^*(\mathrm{GL}_3(\mathbb{C});\mathbb{Q}) \otimes \mathbb{Q}[x]/x^3$$

with $|x| = 2$. We would also like to mention the paper by Tommasi [7] where $\mathrm{H}^*(X_{2,4};\mathbb{Q})$ is computed.

In some sense, this paper shows that in a stable range, something similar to Das's theorem is true for marked hypersurfaces in general.

Some motivation and historical comments At this point we'd like to make some remarks on historical motivations for computing and understanding stable cohomology of moduli spaces of algebraic varieties.

The cohomology of moduli spaces are often interesting because they provide us with invariants for families of varieties. However, in many interesting cases the entire cohomology ring of the moduli space may be difficult to understand and compute. An example of such a phenomenon is the moduli space of curves of genus g, $\mathcal{M}_g$. In this setting, $H^*(\mathcal{M}_g;\mathbb{Q})$ is a huge ring which is not fully understood. However, the spaces $\mathcal{M}_g$ are known to satisfy homological stability and the stable cohomology ring can be explicitly described. For a survey, see Cohen [2].

Another motivation for computing the stable cohomology of moduli spaces has to do with arithmetic statistics. Let X be an algebraic variety over $\mathbb{Z}$. Often one would like to compute $\#X(\mathbb{F}_p)$ by studying the eigenvalues of Frob_p on $H^*_{\mathrm{et}}(X;\mathbb{Q}_l)$. There are often comparison theorems which relate the étale cohomology with the singular cohomology of $X(\mathbb{C})$ and computations of $H^*(X(\mathbb{C});\mathbb{Q})$ can often imply bounds on $\#(X(\mathbb{F}_p))$. For an introduction to this topic, see for instance Sections 1 and 2 of Church, Ellenberg and Farb [1].

Method of proof One could attempt to prove Theorem 1.2 by applying the Serre spectral sequence to the fibration $\pi\colon X^*_{d,n}\to X_{d,n}$. To successfully do this however, one would need to understand the groups $\mathrm{H}^p(X_{d,n};\mathrm{H}^q(Z(f);\mathbb{Q}))$. While we do a priori understand what the groups $\mathrm{H}^p(X_{d,n};\mathbb{Q})$ are (this is the main theorem of [8]), this is not sufficient for us to understand what the groups $\mathrm{H}^p(X_{d,n};\mathrm{H}^q(Z(f);\mathbb{Q}))$ are, since $\mathrm{H}^q(Z(f);\mathbb{Q})$ is a *nontrivial* local coefficient system. Instead we use an idea of Das and compute $\mathrm{H}^*(X^p_{d,n};\mathbb{Q})$, where $X^p_{d,n}:=\{f\in X_{d,n}\mid f(p)=0\}$ to avoid any computations with nontrivial coefficient systems. After we have proved Theorem 1.2 we can use it to deduce what these twisted cohomology groups are.

Corollary 1.3 *Let $d,n>0$. Suppose $d\ge 4n+1$ and $k<(d-1)/2$. Then*

$$\mathrm{H}^k(X_{d,n};\mathrm{H}^{n-1}(Z(f);\mathbb{Q}))=\begin{cases}\mathrm{H}^k(X_{d,n};\mathbb{Q}) & \text{if } n \text{ is odd},\\ 0 & \text{if } n \text{ is even}.\end{cases}$$

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2 A lower bound on $\mathrm{H}^k(X^*_{d,n})$

We begin by noting that there is an embedding of algebras

$$\mathrm{H}^k(\mathrm{GL}_{n+1}(\mathbb{C}))\otimes\mathbb{Q}[x]/(x^n)\hookrightarrow\mathrm{H}^k(X^*_d)$$

in the stable range. More precisely, we have the following:

Proposition 2.1 *Let $n\ge 0$ and $d>n+1$. There is a natural embedding of algebras*,

$$i\colon H^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q})\otimes\mathbb{Q}[x]/(x^n)\hookrightarrow H^*(X^*_{d,n};\mathbb{Q}),$$

where $|x|=2$.

Proof We first define the *fiberwise canonical bundle* $\mathscr{L}$ over $M^*_{d,n}$ as

$$\mathscr{L}=\{(f,p,v)\mid (f,p)\in M^*_d,\ v\in\wedge^{n-1}T^*_p(Z(f))\}.$$

We can pull back $\mathscr{L}$ to a bundle on $X^*_{d,n}$, which we will also denote by $\mathscr{L}$. By the same argument as in Theorem 1 of [6],

$$\mathrm{H}^*(X^*_{d,n};\mathbb{Q}) \cong \mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n}(\mathbb{C});\mathbb{Q}).$$

Let $f \in X_{d,n}$. Let $i\colon \mathrm{GL}_{n+1}(\mathbb{C}) \to X_{d,n}$ be the orbit map defined by $i(g) = g \cdot f$. More precisely, Theorem 1 of [6] states that the natural map

$$\pi^*\colon \mathrm{H}^*(M^*_{d,n};\mathbb{Q}) \to \mathrm{H}^*(X^*_{d,n};\mathbb{Q})$$

makes $\mathrm{H}^*(X^*_{d,n};\mathbb{Q})$ a free $\mathrm{H}^*(M^*_{d,n};\mathbb{Q})$–module with a basis given by some set $\{\alpha_i\}$ such that the pullbacks $\{i^*(\alpha_i)\}$ give a basis of $\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q})$. But since $\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q})$ is a free graded commutative algebra, this forces $H^*(X^*_{d,n};\mathbb{Q})$ to be isomorphic to $\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n}(\mathbb{C});\mathbb{Q})$ as an algebra.

If we restrict $\mathscr{L}$ to a particular hypersurface Z, the bundle $\mathscr{L}|_Z = \mathscr{O}_Z(d-n-1)$. The Chern class of $\mathscr{L}|_Z$ satisfies the equality

$$c_1(\mathscr{O}_Z(d-n-1)) = (d-n-1)c_1(\mathscr{O}_Z(1)) = d(d-n-1)\omega_Z,$$

where ω_Z is the Kähler class of the variety Z. This implies that for $d > n+1$, the classes $c_1(\mathscr{L})|_Z, \dots, c_1^{n-1}(\mathscr{L})|_Z$ are nonzero since $\omega_Z, \dots, \omega_Z^{n-1}$ are nonzero. Now taking $x = c_1(\mathscr{L})$, this implies that $\mathrm{H}^*(M;\mathbb{Q})$ contains a subalgebra isomorphic to $\mathbb{Q}[x]/x^n$. □

3 The space X_d^p and the Vassiliev method

Given a space X, the n^{th} *ordered configuration space* of X, denoted by $\mathrm{PConf}_n X$, is

$$\mathrm{PConf}_n X := \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

There is a natural action of the symmetric group on n letters S_n on X by permuting the coordinates. The quotient $\mathrm{PConf}_n X/S_n$ is called the n^{th} *unordered configuration space* and denoted by $\mathrm{UConf}_n X$. In order to understand $X_{d,n}$ we will first look at the cohomology of a related space. For a fixed point $p \in \mathbb{P}^n$, we set

$$X_d^p = \{f \in X_{d,n} \mid f(p) = 0\}.$$

Then

$$X_d^p \subseteq V_d^p = \{f \in V_d \mid f(p) = 0\}.$$

The space V_d^p is a vector space. The complement of X_d^p in V_d^p will be called $\Sigma_{d,p}$. We will compute its Borel–Moore homology and use Alexander duality to compute $H^*(X_d^p)$.

Let $p \in \mathbb{P}^n$. By definition, p is a one-dimensional subspace $p \subseteq \mathbb{C}^{n+1}$. Choose a complementary subspace $W \subseteq \mathbb{C}^{n+1}$ (it is not unique, but we will fix a particular one). We define $G_p := \mathrm{GL}(W)$.

Let $x_1, \dots, x_n$ be local coordinates in a neighbourhood U containing p. Pick a local trivialization s of the line bundle $\mathbb{O}(d)$ in U. There is an induced map

$$f^* \colon T_0^*(\mathbb{O}(d)_p) \to T_p^*(\mathbb{P}^n).$$

Let us use our local coordinates to identify $T_0^*(\mathbb{O}(d)_p)$ with $\mathbb{C}$ and $T_p^*(\mathbb{P}^n)$ with $\mathbb{C}^n$.

Suppose $f \in X_d^p$. Then the map f^* is nonzero because f has a regular zero locus. This defines a map

$$\pi \colon X_d^p \to T_p^*(\mathbb{P}^n) - \{0\} \cong \mathbb{C}^n - \{0\}$$

given by $\pi(f) = f^*(1)$.

Proposition 3.1 *The map $\pi \colon X_d^p \to \mathbb{C}^n - \{0\}$ is a fibration.*

Proof The group G_p acts on $\mathbb{P}^n$ fixing p. Therefore it acts on both X_d^p and $\mathbb{C}^n - \{0\}$. The map π is equivariant with respect to these actions. The map π is therefore the pullback of a map π' from X_d^p/G_p to $\mathbb{C}^n - \{0\}/G_p$. But $\mathbb{C}^n - \{0\}/G_p$ is a point, and since π' is surjective it is a fibration. Since pullbacks of fibrations are fibrations, π is a fibration. □

Let $X_v := \pi^{-1}(v)$ and let

$$V_v := \{f \in V_d \mid f^*(1) = v\}.$$

Clearly, $X_v \subseteq V_v$. Let $\Sigma_v := V_v - X_v$. We will try to understand the Borel–Moore homology of Σ_v.

To accomplish this, the Vassiliev method [10] will be applied. The Vassiliev method to compute Borel–Moore homology involves stratifying a space and using the associated spectral sequence to compute its Borel–Moore homology. The space Σ_v will be stratified based on the points at which a section f is singular. The techniques used are very similar to that in [8] which contains many of the technical details.

We denote the k–simplex with vertex set $\{a_0, \dots, a_k\}$ by $\Delta_{\{a_0,\dots,a_k\}}$. We denote a k–simplex by Δ_k and an open k–simplex by Δ_k°

We will now construct a cubical space $\mathcal{C}$ which will be involved in understanding Σ_v. Let $N = (d-1)/2$. Let I be a subset of $\{1, \dots, N-1\}$. For $k < N$, let

$$\mathcal{C}_I := \{(f, p) \mid f \in \Sigma_v,\ p\colon I \to \mathbb{P}^n,\ p(I) \subseteq \text{singular zeroes of } f\}.$$

We define $\Sigma_v^{\geq N} = \{f \in \Sigma_v \mid f \text{ has at least } N \text{ singular zeroes}\}$. We define

$$\mathcal{C}_{I \cup \{N\}} := \{(f, p) \mid f \in \Sigma_v,\ p\colon I \to \mathbb{P}^n,\ p(I) \subseteq \text{singular zeroes of } f,\ f \in \overline{\Sigma}^{\geq N}\}.$$

If $I \subseteq J$ then we have a natural map from $\mathcal{C}_J \to \mathcal{C}_I$ defined by restricting p. This gives $\mathcal{C}$ the structure of a cubical space over the set $\{1, \dots, N\}$. We can take the geometric realization of $\mathcal{C}$, denoted by $|\mathcal{C}|$. Then there is a map $\rho\colon |\mathcal{C}| \to \Sigma_v$, induced by the forgetful maps $\mathcal{C}_I \to \Sigma_v$.

$|\mathcal{C}|$ is topologized in a nonstandard way so as to make ρ proper. We topologize it as follows: in [8], a space $|\mathscr{X}|$ is constructed with a map $\rho\colon |\mathscr{X}| \to \Sigma$. Here, $\Sigma = V_d - X_d$. The topology on $|\mathscr{X}|$ is chosen carefully so as to make ρ proper. The construction of $|\mathscr{X}|$ as a set identical to that of $|\mathcal{C}|$ except we replace Σ_v with Σ. There is a natural inclusion $|\mathcal{C}| \to |\mathscr{X}|$. We give $|\mathcal{C}|$ the subspace topology along this map.

Proposition 3.2 *The map $\rho\colon |\mathcal{C}| \to \Sigma_v$ is a proper homotopy equivalence.*

Proof This proof is nearly identical to that of Lemma 15 in [8]. The properness of $\rho\colon |\mathcal{C}| \to \Sigma_v$ follows from the properness of $\rho\colon |\mathscr{X}| \to \Sigma$ and the properties of the subspace topology. In our setting, having contractible fibres implies that the map ρ is a homotopy equivalence; this follows by combining Theorems 1.1 and 1.2 of [5]. We will now prove that the fibres are contractible. If $f \notin \overline{\Sigma}_v^{\geq N}$, let $\{p_1, \dots, p_k\}$ be the singular zeroes of f. In this case the fibre $\rho^{-1}(f)$ is a simplex with vertices given by the images of the points $(f, x_i) \in \mathcal{C}_{\{1\}} \times \Delta_{\{1\}}$. Now suppose $f \in \overline{\Sigma}_v^{\geq N}$. In this case the fibre $\rho^{-1}(f)$ is a cone over the point $f \in \mathcal{C}_N \times \Delta_{\{N\}}$. □

Now as in any geometric realization, $|\mathcal{C}|$ is filtered by

$$F_n = \operatorname{im}\Biggl(\coprod_{|I| \leq n} \mathcal{C}_I \times \Delta_k\Biggr).$$

The F_n form an increasing filtration of $|\mathcal{C}|$, ie $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq F_{n+1} \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} F_n = |\mathcal{C}|$.

Proposition 3.3 *Let $d, n \geq 1$ and $N = (d-1)/2$. For $k < N$, the space $F_k - F_{k-1}$ is a Δ_k°–bundle, over a vector bundle B_k over $\mathrm{UConf}_k(\mathbb{P}^n - p)$.*

Proof The space $F_k - F_{k-1}$ consists of the interiors of k simplices, labelled by $\{f, p_0, \dots, p_k\}$. Let

$$B_k = \{(f, \{p_0, \dots, p_k\}) \in \Sigma_v \times \mathrm{UConf}_k(\mathbb{P}^n - p) \mid p_i \text{ are singular zeroes of } f\}.$$

We have a map $\phi\colon F_k - F_{k-1} \to B_k$, defined by

$$\phi((f, \{p_0, \dots, p_k\}), s_0, \dots, s_k) = (f, \{p_0, \dots, p_k\}).$$

The map ϕ expresses $F_k - F_{k-1}$ as a fibre bundle over B_k with Δ_k° fibres, ie we have a diagram

$$\begin{tikzcd} \Delta_k^\circ \arrow[r] & F_k - F_{k-1} \arrow[d] \\ & B_k \end{tikzcd}$$

We have a map $B_k \to \mathrm{UConf}_k(\mathbb{P}^n - p)$ defined by $\{f, p_0, \dots, p_k\} \mapsto \{p_0, \dots, p_k\}$. This is a vector bundle by Lemma 3.2 in [9]. $\square$

We have a one-dimensional local coefficient system denoted by $\pm\mathbb{Q}$ on $\mathrm{UConf}_k(\mathbb{P}^n - p)$ defined in the following way: Let S_k be the symmetric group on k letters. We have a homomorphism $\pi_1 \mathrm{UConf}_k(\mathbb{P}^n - p) \to S_k$ associated to the covering space $\mathrm{PConf}_k(\mathbb{P}^n - p) \to \mathrm{UConf}_k(\mathbb{P}^n - p)$. Compose this homomorphism with the sign representation $S_k \to \pm 1 = \mathrm{GL}_1(\mathbb{Q})$ to obtain our local system.

Proposition 3.4 *Let $d, n \geq 1$ and $e_d = \dim_{\mathbb{C}}(V_v)$. For $k < (d-1)/2$,*

$$\overline{H}_*(F_k - F_{k-1}) \cong H_{*-(k+2e_d-2(n+1)(k+1))}(\mathrm{UConf}_k(\mathbb{P}^n - p), \pm\mathbb{Q}).$$

Proof By Proposition 3.3 the space $F_k - F_{k-1}$ is a bundle over $\mathrm{UConf}_k(\mathbb{P}^n - p)$. This fact implies that

$$\overline{H}_*(F_k - F_{k-1}) \cong H_{*-(k+2e_d-2(n+1)(k+1))}(\mathrm{UConf}_k(\mathbb{P}^n - p), \mathbb{Q}(\sigma)).$$

Here $\mathbb{Q}(\sigma)$ is the local system obtained by the action of $\pi_1(\mathrm{UConf}_k(\mathbb{P}^n - p))$ on the fibres $\overline{H}_k(\Delta_k^\circ)$, where in this case Δ_k° is the open k–simplex corresponding to the fibres of the map $F_k - F_{k-1} \to B_k$. But one observes that the action of $\pi_1(\mathrm{UConf}_k(\mathbb{P}^n - p))$ on this open simplex is by permutation of the vertices, which implies $\mathbb{Q}(\sigma) = \pm\mathbb{Q}$. $\square$

As with any filtered space, we have a spectral sequence with

$$E_1^{p,q} = \overline{H}_{p+q}(F_p - F_{p-1}; \mathbb{Q})$$

converging to $\overline{H}_*(Y; \mathbb{Q})$. Now for $p < N$, by Proposition 3.4,

$$E_1^{p,q} = \overline{H}_{q-(2e_d-2(n+1)(p+1))}(\mathrm{UConf}_p(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

We would like to claim that $E_1^{N,q}$ doesn't matter in the stable range. To be more precise, we have the following:

Lemma 3.5 *Let $d, n \geq 1$, let $N = (d-1)/2$, and let $k > 2e_d - N$. Then*

$$\overline{H}_k(|\mathcal{C}| - F_N; \mathbb{Q}) \cong \overline{H}_k(|\mathcal{C}|; \mathbb{Q}).$$

Proof We first will try to bound the $\overline{H}_*(F_N; \mathbb{Q})$ and then use the long exact sequence of the pair. F_N is the union of locally closed subspaces

$$\phi_k = \{(f, x_1, \dots, x_k), p \mid f \in \Sigma^{\geq N}, x_i \text{ are singular zeroes of } f, p \in \Delta_k\}.$$

We have a surjection $\pi : \phi_k \to \mathrm{UConf}_k(\mathbb{P}^n - p)$. This map π is in fact a fibre bundle with fibres $\Delta^k \times \mathbb{C}^{e_d - N(n+1)}$. The space $\mathrm{UConf}_k(\mathbb{P}^n - p)$ is kn–dimensional. Therefore,

$$\overline{H}_*(\phi_k; \mathbb{Q}) = 0 \quad \text{if } * > 2(e_d - (n+1)N) + kn < 2e_d - N.$$

This implies that for all k, $\overline{H}_*(\phi_k; \mathbb{Q}) = 0$ if $* > 2e_d - N$. This implies $\overline{H}_*(F_N; \mathbb{Q}) = 0$ if $* > 2e_d - N$. By the long exact sequence in Borel–Moore homology associated to the pair $F_N \hookrightarrow Y$, $\overline{H}_k(Y - F_N; \mathbb{Q}) \cong \overline{H}_k(Y; \mathbb{Q})$ for $k > 2e_d - N$. □

4 Interlude

In [8], Tommasi proves the following result:

Theorem 4.1 [8] *Let $d, n \geq 1$, let $f \in X_{d,n}$, and let $\psi : \mathrm{GL}_{n+1}(\mathbb{C}) \to X_{d,n}$ be the orbit map defined by $\psi(g) = g \cdot f$. Then $\psi^* : H^k(X_{d,n}, \mathbb{Q}) \to H^k(\mathrm{GL}_{n+1}(\mathbb{C}), \mathbb{Q})$ is an isomorphism for $k < (d+1)/2$.*

In this section we shall look at the proof of Theorem 4.1 in [8] and use it to prove an identity used later on in this paper. One of the ingredients in the proof of Theorem 4.1 is a Vassiliev spectral sequence. We introduce a new convention, by letting h denote the dimension of H. We also define $\mathrm{Gr}(p, n)$ to be the Grassmannian of p–planes in $\mathbb{C}^n$. In what follows we shall need a few basic facts about $H_*(\mathrm{Gr}(p,n);\mathbb{Q})$ and Schubert symbols. Let

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = \mathbb{C}^n$$

be a complete flag. Given $U \in \mathrm{Gr}(p,n)$, we can associate to it a sequence of numbers, $a_i = \dim U \cap E_i$. These a_i satisfy the conditions

$$0 \le a_{i+1} - a_i \le 1, a_0 = 0 \quad \text{and} \quad a_n = p.$$

Such sequences are called *Schubert symbols*. Let $\boldsymbol{a} = (a_0, \dots, a_n)$. We call $\boldsymbol{a}$ a Schubert symbol if $0 \le a_{i+1} - a_i \le 1$, $a_0 = 0$ and $a_n = p$. Associated to each Schubert symbol $\boldsymbol{a}$ we have a subvariety $W_{\boldsymbol{a}} \subseteq \mathrm{Gr}(p, \mathbb{C}^n)$ defined as

$$W_{\boldsymbol{a}} := \overline{\{U \subseteq \mathbb{C}^n \mid \dim(U \cap \mathbb{C}^i) = a_i\}}.$$

The main result we will be using is the following.

Theorem 4.2 *Let $\boldsymbol{a}$ be a Schubert symbol. The classes $[W_{\boldsymbol{a}}] \in H_*(\mathrm{Gr}(p,n);\mathbb{Q})$ form a basis.*

For a proof of Theorem 4.2 see page 1071 of [4].

Proposition 4.3 *Let n be a positive integer. Then*

$$\sum_{k,p} h_k(\mathrm{Gr}(p,\mathbb{C}^n);\mathbb{Q}) = 2^n.$$

Proof By Theorem 4.2,

$$\begin{aligned}\sum_{k,p} h_k(\mathrm{Gr}(p,\mathbb{C}^n);\mathbb{Q}) &= \sum_p \#\{(a_0,\dots,a_n) \mid 0 \le a_{i+1}-a_i \le 1, a_0 = 0, a_n = p\} \\ &= \#\{(a_0,\dots,a_n) \mid 0 \le a_{i+1}-a_i \le 1, a_0 = 0\} \\ &= \#\{(b_1,\dots,b_n) \in \{0,1\}\}.\end{aligned}$$

The last equality follows because if we are given a sequence of a_i, we can uniquely obtain a sequence of b_i, by letting $b_i = a_i - a_{i-1}$. □

Our main aim of this section is to prove the following technical result.

Theorem 4.4 *The Vassiliev spectral sequence in* [8] *degenerates in the stable range*: *if* $p < (d+1)/2$ and $q > 0$, *then* $E_1^{p,q} \cong E_{p,q}^{\infty}$.

Equivalently, *for* $k < (d+1)/2$,

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\mathbb{Q}) = h_k(\mathrm{GL}_{n+1};\mathbb{Q}). \tag{1}$$

Remark 4.5 The statements are equivalent because the group $\mathrm{H}^k(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q})$ is a subquotient of

$$\bigoplus \mathrm{H}_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\mathbb{Q}).$$

Proof We already know that

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) \geq h_k(\mathrm{GL}_{n+1};\mathbb{Q})$$

because the left hand side of (1) are the appropriate terms in a spectral sequence converging to the right hand side of (1).

It suffices to prove that

$$\sum_k \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) = \sum_k h_k(\mathrm{GL}_{n+1};\mathbb{Q}) = 2^{n+1}.$$

Lemma 2 in [10] states that

$$h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n),\pm\mathbb{Q}) = h_{2(p+1)(n+1)-p-k-1-p(p-1)}(\mathrm{Gr}(p,\mathbb{C}^{n+1});\mathbb{Q}).$$

Therefore

$$\sum_k \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) = \sum_k \sum_p h_k(\mathrm{Gr}(p,\mathbb{C}^{n+1});\mathbb{Q}).$$

By Proposition 4.3, this is equal to 2^{n+1}. □

5 Computation

We would like to know what the groups $\overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n - p);\pm\mathbb{Q})$ are. First note that in [10] Vassiliev proves that:

Proposition 5.1 [10] *Let* $k, n > 0$. *Then*

$$H_*(\mathrm{UConf}_k(\mathbb{P}^n);\pm\mathbb{Q}) \cong H_{*-(k)(k-1)}(\mathrm{Gr}_k(\mathbb{C}^{n+1});\mathbb{Q}).$$

Also note that in light of Theorem 4.2 the homology of Grassmannians is well understood in terms of Schubert cells.

Consider the long exact sequence in Borel–Moore homology associated to

$$\mathrm{UConf}_{k+1}(\mathbb{P}^n - p) \subseteq \mathrm{UConf}_{k+1}(\mathbb{P}^n) \hookleftarrow \mathrm{UConf}_k(\mathbb{P}^n - p).$$

The last inclusion is defined by the map $\phi\colon \mathrm{UConf}_k(\mathbb{P}^n - p) \to \mathrm{UConf}_{k+1}(\mathbb{P}^n)$, where $\phi(\{x_1, \dots, x_n\}) = \{x_1, \dots, x_n, p\}$.

We consider the long exact sequence in Borel–Moore homology associated to the pair $(\mathrm{UConf}_{k+1}(\mathbb{P}^n), \mathrm{UConf}_{k+1}(\mathbb{P}^n - p))$. Here $\mathrm{UConf}_{k+1}(\mathbb{P}^n - p)$ is an open subset of $\mathrm{UConf}_{k+1}(\mathbb{P}^n)$ with complement homeomorphic to $\mathrm{UConf}_k(\mathbb{P}^n - p)$. A segment of this exact sequence is

$$\overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \to \overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \to \overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n - p); \pm\mathbb{Q}) \tag{2}$$

Proposition 5.2 *Let $k, n > 0$. Then there is a canonical decomposition*

$$\overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \cong \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \oplus \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}),$$

due to the fact that (2) splits.

Proof Lemma 2 of [10] implies that (2) decomposes into split short exact sequences,

$$\overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \cong \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \oplus \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}). \qquad \square$$

Remark 5.3 In fact $H_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q})$ has a basis given by Schubert symbols with $a_1 = 0$.

Proposition 5.4 *If the Vassiliev spectral sequence has no nonzero differentials and $k < (d-1)/2$, then $H^k(X_v) \cong H^k(G_p)$ as vector spaces.*

Proof Now in our spectral sequence we had

$$E_1^{p,q} = \overline{H}_{q-(2e_d - 2(p+1)(n+1))}(\mathrm{UConf}_{p+1}(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

First collect all terms in the main diagonal, ie

$$V := \bigoplus_{p+q=l} \overline{H}_{q-(2D_n-2(p+1)(n+1))}(\mathrm{UConf}_{p+1}(\mathbb{P}^n - p); \pm\mathbb{Q})$$

It will suffice to prove that

$$\dim V = \sum_{p \le 2D_n - k} h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h^k(\mathrm{GL}_n; \mathbb{Q}). \tag{3}$$

Theorem 4.4 implies

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = h_k(\mathrm{GL}_{n+1}; \mathbb{Q}). \tag{4}$$

Proposition 5.1 implies

$$h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = 0 \quad \text{if } p > n.$$

So as long as $n < 2(D_n + n + 1) - k$,

$$\sum_{p \le 2(D_n+n+1)-k} h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}).$$

But the condition $n < 2(D_n + n + 1) - k$ is equivalent to $k < 2(D_n + 1) + n$, which is true if $k < N$. We have another equality from Proposition 5.2,

$$h_k(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) + h_k(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h_k(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}).$$

Plugging this into (4),

$$\begin{aligned} h^k(\mathrm{GL}_{n+1}; \mathbb{Q}) &= \sum h_{2(p+1)(n+1)-p-k}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) \\ &= \sum h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) \\ &\qquad + h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}). \end{aligned}$$

We have the identity

$$h^k(\mathrm{GL}_n; \mathbb{Q}) + h^{k-(2n+1)}(\mathrm{GL}_n; \mathbb{Q}) = h^k(\mathrm{GL}_{n+1}; \mathbb{Q}).$$

This implies

$$\begin{aligned} & h^k(\mathrm{GL}_n; \mathbb{Q}) + h^{k-(2n+1)}(\mathrm{GL}_n; \mathbb{Q}) \\ &\quad = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) \\ &\qquad\qquad + h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}). \end{aligned} \tag{5}$$

Now we will try to prove (3) by induction on k. For $k=0$, (3) is trivial. By induction,

$$h^{k-(2n+1)}(\mathrm{GL}_n;\mathbb{Q}) = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}).$$

Putting this into (5), we obtain

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h^k(\mathrm{GL}_n;\mathbb{Q}). \qquad \square$$

Now we can look at the Serre spectral sequence associated to the fibration

$$X_v \hookrightarrow X_p \to \mathbb{C}^n - 0.$$

We observe that if there are no nonzero differentials, then

$$H^*(X_p;\mathbb{Q}) \cong H^*(X_v;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/e_{2n-1}^2.$$

This is because the Serre spectral sequence degenerates and since $\mathbb{Q}[e_{2n-1}]/e_{2n-1}^2$ is a free graded commutative algebra the ring structure of the total space is forced to be the tensor product.

Proposition 5.5 *Let $d>0$ and $p \in \mathbb{P}^n$. Then*

$$H^*(X_{d,p};\mathbb{Q}) \cong H^*(G_p;\mathbb{Q}) \otimes A,$$

where A is $H^*(X_d^p/G_p;\mathbb{Q})$.

Proof This follows immediately from Theorem 2 in [6]. $\square$

We will also need the following fact, which is a special case of Lemma 2.6 in [3].

Proposition 5.6 *Let $d>0$, let $k<(d-1)/2$, and let $U_d^* = X_d^*/\mathbb{C}^*$. Then*

$$H^*(X_d^*;\mathbb{Q}) \cong H^*(U_d^*;\mathbb{Q}) \otimes \mathbb{Q}[e_1]/(e_1^2),$$

where $|e_1| = 1$.

Proposition 5.6 implies if there are no nonzero differentials in both our Vassiliev spectral sequence and in the Serre spectral sequence associated to the fibration $X_{d,n}^p \to \mathbb{C}^n - 0$ then

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(G_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2)$$

for $* < (d-1)/2$. In case there are nonzero differentials in either spectral sequence, then $H^*(U_{d,p};\mathbb{Q}) \cong H^*(G_p;\mathbb{Q})$ for $* < (d-1)/2$.

6 Comparing fibre bundles

In this section we finish the proof of Theorem 1.2.

Proof of Theorem 1.2 We compare three related fibre bundles and their associated spectral sequences. This is similar to the proof of Theorem 1.1 in [3].

Let $\mathrm{PG}_p := \mathrm{Stab}_{\mathrm{PGL}(n+1)}(p)$:

$$\begin{tikzcd}
\mathrm{PG}_p \arrow[r] \arrow[dr] & U_{d,p} \arrow[r] \arrow[dr] & U_d \arrow[dr] & \\
 & \mathrm{PGL}_{n+1}(\mathbb{C}) \arrow[r] \arrow[d] & U_d^* \arrow[r] \arrow[d] & U_d \times \mathbb{P}^n \arrow[d] \\
 & \mathbb{P}^n \arrow[r, equal] & \mathbb{P}^n \arrow[r, equal] & \mathbb{P}^n
\end{tikzcd} \tag{6}$$

We denote the exterior algebra on generators $a_1, \dots, a_n$ by

$$\Lambda\langle a_1, \dots, a_n\rangle.$$

By Proposition 5.4 and [6, Theorem 1], there are two possibilities for $H^*(U_{d,p};\mathbb{Q})$: either

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(\mathrm{PG}_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2) \cong \Lambda\langle u_1, u_3, \dots, u_{2n-1}, e_{2n-1}\rangle$$

or

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(\mathrm{PG}_p;\mathbb{Q}) = \Lambda\langle u_1, u_3, \dots, u_{2n-1}\rangle.$$

Suppose for the sake of contradiction $H^*(U_{d,p}) = \Lambda\langle u_3, \dots, u_{2n-1}\rangle$ for $* < (d-1)/2$. In this case, $H^*(U_{d,p};\mathbb{Q}) \cong H^*(\mathrm{PG}_p;\mathbb{Q})$ for $* < (d-1)/2$. Then since the homology of the base and the fibres are isomorphic, $H^*(U_d^*;\mathbb{Q}) \cong H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q})$ for $* < (d-1)/2$. However, by Proposition 2.1,

$$H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathbb{Q}[x]/x^n) \subseteq H^*(U_d^*;\mathbb{Q}).$$

But $H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q})$ does not contain a subalgebra isomorphic to

$$H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathbb{Q}[x]/x^n).$$

This is a contradiction. So we must be in the case where

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(\mathrm{PG}_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2).$$

Consider the Serre spectral sequence associated to the fibration $U_d^* \to \mathbb{P}^n$. Its E_2 page has terms

$$E_2^{p,q} = H^p(\mathbb{P}^n, H^q(U_d^p;\mathbb{Q})) \cong H^p(\mathbb{P}^n;\mathbb{Q}) \otimes H^q(U_d^p;\mathbb{Q}).$$

Now

$$H^q(U_d^p;\mathbb{Q}) \cong H^q(\mathrm{PG}_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2).$$

Consider the trivial fibre bundle $U_d \times \mathbb{P}^n \to \mathbb{P}^n$. There is a natural inclusion of fibre bundles as shown in (6). This induces a map of spectral sequences between the associated Serre spectral sequences.

Note that any class $\alpha \in H^q(U_d^p;\mathbb{Q})$ that lies in the image of $H^q(U_d;\mathbb{Q})$ is mapped to zero under any differential thanks to the fact that all differentials are zero in the spectral sequence associated to a trivial fibration. The only possible nonzero differential in the E_2 page of the Serre spectral sequence associated to the fibration $U_d^* \to \mathbb{P}^n$ is $d(e_{2n-1})$.

Suppose for contradiction that $d(e_{2n-1}) = 0$. This implies that

$$H^k(U_d^*;\mathbb{Q}) \cong (H^*(U_{d,p};\mathbb{Q}) \otimes H^*(\mathbb{P}^n;\mathbb{Q}))_k = (H^*(\mathrm{PG}_p;\mathbb{Q}) \otimes H^*(\mathbb{P}^n,\mathbb{Q}))_k$$

for $k < (d-1)/2$.

Let $p(t)$ be the Poincaré polynomial of U_d^*. We already know that

$$H^*(U_d^*;\mathbb{Q}) \cong H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes H^*(U_d^*/\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}).$$

So $(1+t^3)\cdots(1+t^{2n+1}) \mid p(t)$. On the other hand, if $de_{2n-1} = 0$ then

$$p(t) = (1+t^3)\cdots(1+t^{2n-1})(1+t^2+t^4+\cdots+t^{2n}) \mod t^{(d-1)/2}.$$

If $d \geq 4n+1$, then this implies that $(1+t^{2n+1}) \nmid p(t)$. This is a contradiction.

So we must have a differential killing the class in $H^{2n}(\mathbb{P}^n, H^0(U_{d,p}));\mathbb{Q})$. The differential must come from e_{2n-1}; ie $d(e_{2n-1}) = ax^n$ for some $a \in \mathbb{Q}^*$. This (along with multiplicativity of differentials) determines all differentials and implies (1). By Proposition 5.6, (1) implies (2). By Theorem 1 of [6],

$$H^*(X_{d,n}^*;\mathbb{Q}) \cong H^*(M_{d,n}^*;\mathbb{Q}) \otimes (H^*(\mathrm{GL}_{n+1})(\mathbb{C});\mathbb{Q}).$$

In light of this, (2) implies (3). □

Having finished the proof of Theorem 1.2 we can prove Corollary 1.3.

Proof of Corollary 1.3 Consider the fibration

$$\begin{tikzcd} Z(f) \arrow[r] & X_d^* \arrow[d] \\ & X_d \end{tikzcd}$$

and its associated Serre spectral sequence whose E_2 page is of the form

$$H^p(X_d; H^q(Z(f);\mathbb{Q})) \Rightarrow H^*(X_d^*;\mathbb{Q}).$$

By Theorem 4.1 for $* < (d+1)/2$,

$$H^*(X_d;\mathbb{Q}) \cong H^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q}).$$

By Theorem 1.2, we know that the classes in the E_2 page corresponding to the group $H^p(\mathrm{GL}_{n+1}(\mathbb{C}); c_1(\mathscr{L})^q)$ survive until the E^∞ page, and in the stable range all other terms are killed by differentials.

Now suppose n is even. Then the only other terms in the spectral sequence are of the form $H^p(X_d; H^{n-1}(Z(f);\mathbb{Q}))$. However it is not possible for any such term to be in the image or in the preimage of a nonzero differential. This is because all other terms survive, so any possible nonzero differential must be from $H^{p_1}(X_d; H^{n-1}(Z(f);\mathbb{Q}))$ to $H^{p_2}(X_d; H^{n-1}(Z(f);\mathbb{Q}))$ for some choice of p_1 and p_2. However no differential is of bidegree $(p_2-p_1, 0)$. This implies that

$$H^p(X_d; H^{n-1}(Z(f);\mathbb{Q})) \cong 0.$$

A similar argument shows that if n is odd, $H^p(X_d; H^{n-1}(Z(f);\mathbb{Q})) \cong H^p(X_d;\mathbb{Q})$. Essentially the only difference between the even case and the odd case is that in the odd case we have a class $c_1(\mathscr{L})^{(n-1)/2} \in H^{n-1}(Z(f);\mathbb{Q})$. Let $A = \mathbb{Q}$–span$(c_1(\mathscr{L})^{(n-1)/2})$ By Theorem 1.2, we know that $H^p(X_{d,n}; A)$ survives until the E^∞ page. An argument similar to that in the even case shows that

$$H^p(X_d; H^{n-1}(Z(f);\mathbb{Q})) \cong H^p(X_d; A). \qquad \square$$

References

[1] **T Church**, **J S Ellenberg**, **B Farb**, *Representation stability in cohomology and asymptotics for families of varieties over finite fields*, from "Algebraic topology: applications and new directions", Contemp. Math. 620, Amer. Math. Soc., Providence, RI (2014) 1–54 MR Zbl

[2] **R L Cohen**, *Stability phenomena in the topology of moduli spaces*, from "Geometry of Riemann surfaces and their moduli spaces" (L Ji, S A Wolpert, S-T Yau, editors), Surv. Differ. Geom. 14, International, Somerville, MA (2009) 23–56 MR Zbl

[3] **R Das**, *Cohomology of the universal smooth cubic surface*, Q. J. Math. 72 (2021) 795–815 MR Zbl

[4] **S L Kleiman**, **D Laksov**, *Schubert calculus*, Amer. Math. Monthly 79 (1972) 1061–1082 MR Zbl

[5] **R C Lacher**, *Cell-like mappings, I*, Pacific J. Math. 30 (1969) 717–731 MR Zbl

[6] **C A M Peters**, **J H M Steenbrink**, *Degeneration of the Leray spectral sequence for certain geometric quotients*, Mosc. Math. J. 3 (2003) 1085–1095 MR Zbl

[7] **O Tommasi**, *Rational cohomology of the moduli space of genus 4 curves*, Compos. Math. 141 (2005) 359–384 MR Zbl

[8] **O Tommasi**, *Stable cohomology of spaces of non-singular hypersurfaces*, Adv. Math. 265 (2014) 428–440 MR Zbl

[9] **R Vakil**, **M M Wood**, *Discriminants in the Grothendieck ring*, Duke Math. J. 164 (2015) 1139–1185 MR Zbl

[10] **V A Vassiliev**, *How to calculate the homology of spaces of nonsingular algebraic projective hypersurfaces*, Tr. Mat. Inst. Steklova 225 (1999) 132–152 MR Zbl In Russian; translated in Proc. Steklov Inst. Math. 225 (1999) 121–140

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