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On some p–differential graded link homologies, II

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In a previous article, we constructed a link invariant categorifying the Jones polynomial at a $2p^{\text{th}}$ root of unity, where p is an odd prime. This categorification utilized an $N = 2$ specialization of a differential introduced by Cautis in an $\mathfrak{sl}_N$–link homology theory. Here we give a family of link homologies where the Cautis differential is specialized to a positive integer of the form $N = kp + 2$. When k is even, all these link homologies categorify the Jones polynomial evaluated at a $2p^{\text{th}}$ root of unity, but they are distinct link invariants.

57K18; 18G99

1 Introduction

Given any link L, Khovanov and Rozansky [5] constructed a triply graded link homology theory $\mathrm{HHH}(L)$ whose graded Euler characteristic is the HOMFLYPT polynomial of L using the theory of matrix factorizations. Khovanov reformulated this construction using categories of Soergel bimodules [3]. The connection between Soergel bimodules and link homology began with Rouquier's categorification of the braid group [14]. He also extended this categorification to a link homology [15].

Link homology theories are important examples of categorification. In 1994, Crane and Frenkel [2] introduced their categorification program with the purpose of constructing (3+1)–dimensional TQFTs by lifting the (2+1)–dimensional TQFTs coming from quantum groups. The (2+1)–dimensional TQFTs utilize quantum groups at roots of unity. Motivated by this goal, Khovanov [4] introduced the subject of hopfological algebra, which was further developed in [7]. The basic idea is to take a categorification of a quantum group (for a generic quantum parameter) or its representations, defined over a field of characteristic p and look for differentials ∂ such that $\partial^p = 0$. Searching for such p–differentials is equivalent to constructing an action of the Hopf algebra $H = \Bbbk[\partial]/(\partial^p)$ (hence the word "hopfological"). We refer the reader to [10] for a survey of some recent progress in this direction.

Cautis [1] defined an additional differential, depending upon a natural number N, on the chain groups for the triply graded theory, which produced a categorification of the quantum $\mathfrak{sl}_N$–link invariant (also known as the symmetric $\mathfrak{gl}_N$ homology). Independently, Robert and Wagner [13] and Queffelec, Rose and Sartori [12] constructed the same $\mathfrak{sl}_N$–link homology from different perspectives.

In a more recent work [6], Khovanov and Rozansky equipped the triply graded link homology with an action of the positive half of the Witt algebra. One of the Witt algebra generators (denoted by $L_1 = x^2 \frac{\partial}{\partial x}$) in [6] acts as a p–differential over a field of characteristic p on $\mathrm{HHH}(L)$. For degree reasons, this is the only Witt algebra generator that can play the role of a p–differential. In [11], we utilized this p–differential along with the Cautis differential for $N = 2$, to construct a categorification of the Jones polynomial evaluated at a $2p^{\text{th}}$ root of unity. The Cautis differential has the effect of applying $L_1 \wedge (\cdot)$ to $\mathrm{HHH}(L)$. A key property that facilitated the construction in [11] is that the two actions of L_1, as the p–differential and the Cautis differential, commute with each other.

In this work, we generalize the previous results by considering the Cautis differential for $N = kp + 2$ where p is an odd prime — this condition could be removed but was used in the braid group action in the prequel [11] — and k is a nonnegative integer. The essential reason that this generalization works is that, in characteristic p, the polynomial algebra generated by x^p lies in the center of the Witt algebra. Therefore the p–differential L_1 still commutes with $L_{kp+1} = x^{kp+2} \frac{\partial}{\partial x}$, the latter now serving as the Cautis differential. Thus, for each $N = kp + 2$ and braid β, we obtain a finite-dimensional object $p\mathrm{H}(\beta, kp + 2)$ which is well defined in the homotopy category of p–complexes. Our main result is the following.

Theorem 4.9 *Let L be a link presented as the closure of a braid β and p be an odd prime. The object $p\mathrm{H}(\beta, kp+2)$ is a finite-dimensional framed link invariant. When $k \in 2\mathbb{Z}$, its Euler characteristic is the Jones polynomial evaluated at a $2p^{th}$ root of unity.*

Varying the Cautis differential categorifies $\mathfrak{sl}_N$–link invariants for different ranks. But when q is a $2p^{\text{th}}$ root of unity, and k is even, $q^{kp+2} = q^2$ so the $\mathfrak{sl}_{kp+2}$–link invariant is just the Jones polynomial. While this is true on the decategorified level, we show in Section 5 that on the level of homology, the invariant for the Hopf link depends upon k. Thus we obtain a family of distinct link homologies categorifying the Jones polynomial at $2p^{\text{th}}$ roots of unity.

In a parallel direction [8], we show that the root of unity categorification of [11] can be extended to the colored case. Combining the approach of [8] with the current work, one can construct certain colored $\mathfrak{sl}_N$–link homologies, which we plan to explore.

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2 Background

In this section, we recall some background material from [11]. We assume the reader has some familiarity with the constructions in [11].

2.1 p–DG algebras and their relative homotopy categories

Let $\Bbbk$ be a field of characteristic $p > 2$. For any graded or ungraded algebra B over $\Bbbk$, denote by d_0 the zero superdifferential ($d_0^2 = 0$) and by ∂_0 the zero p–differential ($\partial_0^p = 0$) on B, while letting B sit in homological degree zero. When B is graded, the homological grading is independent of the internal grading of B. We will usually refer to the internal grading as the *q–degree* in what follows.

We will let $\mathcal{C}(B, d_0)$ and $\mathcal{C}(B, \partial_0)$ stand for the homotopy categories and p–homotopy categories of B respectively. For more details on hopfological algebra of p–homotopy categories, see [4; 7].

For a graded module M over a graded algebra B, we let $M\{n\}$ denote the module M, where the internal grading has been shifted up by n. When convenient, we sometimes call this shifted module $q^n M$.

We will need the following functor introduced in [11, Section 2.1] which is called the *p–extension functor*. Let B be a $\Bbbk$–algebra. Given a chain complex of B–modules, we repeat every term sitting in odd homological degrees $(p-1)$ times while keeping even degree terms unchanged. More explicitly, for a given complex

$$\cdots \xrightarrow{d_{2k+2}} M_{2k+1} \xrightarrow{d_{2k+1}} M_{2k} \xrightarrow{d_{2k}} M_{2k-1} \xrightarrow{d_{2k-1}} M_{2k-2} \xrightarrow{d_{2k-2}} \cdots,$$

the p–extended complex looks like

$$\cdots \xrightarrow{d_{2k+2}} M_{2k+1} = \cdots = M_{2k+1} \xrightarrow{d_{2k+1}} M_{2k} \xrightarrow{d_{2k}} M_{2k-1} = \cdots = M_{2k-1} \xrightarrow{d_{2k-1}} M_{2k-2} \xrightarrow{d_{2k-2}} \cdots$$

Similarly, for chain maps of B–modules, the odd degree maps are repeated $p-1$ times while the even ones are kept unchanged. In [9, Proposition 2.3], it is shown that this construction leads to an exact functor between homotopy categories

$$\mathcal{P}\colon \mathcal{C}(B,d_0) \to \mathcal{C}(B,\partial_0). \tag{2-1}$$

The exactness of $\mathcal{P}$ means that it commutes with homological shifts, denoted by $[\pm 1]_d$ and $[\pm 1]_\partial$, respectively, on $\mathcal{C}(B,d_0)$ and $\mathcal{C}(B,\partial_0)$, and preserves the class of distinguished triangles.

Suppose (A,∂_A) is a p–DG algebra, ie a graded algebra equipped with a differential ∂_A of degree two, satisfying

$$\partial_A^p(a) \equiv 0, \quad \partial_A(ab) = \partial_A(a)b + a\partial_A(b) \tag{2-2}$$

for all $a, b \in A$. In other words, A is an algebra object in the module category of the graded Hopf algebra $H_q = \Bbbk[\partial_q]/(\partial_q^p)$, where the primitive degree-two generator $\partial_q \in H_q$ acts on A by the differential ∂_A. Below we will usually take B to be a certain smash product algebra associated with (A,∂_A), which we next recall.

Given a p–DG algebra A, we may form the *smash product algebra $A \# H_q$* in this case. As a $\Bbbk$–vector space, $A \# H_q$ is isomorphic to $A \otimes H_q$. The multiplication on the smash product, given in pure tensor elements, is determined by

$$(a \otimes \partial_q)(b \otimes \partial_q) = ab \otimes \partial_q^2 + a\partial_A(b) \otimes \partial_q. \tag{2-3}$$

Notice that, by construction, $A \otimes 1$ and $1 \otimes H_q$ sit in $A \# H_q$ as subalgebras.

For later use, let us record a family of *balanced* H_q–modules

$$V_i := \Big(\overset{-i}{\Bbbk} = \overset{-i+2}{\Bbbk} = \cdots = \overset{i-2}{\Bbbk} = \overset{i}{\Bbbk} \Big) \tag{2-4}$$

for each q–degree i in $\{0, \dots, p-1\}$. Here the module sits in a single homological degree, while the labels on top indicate the various q–degrees that the module lives in. As graded modules over H_q, we have $V_i \cong q^{-i} H_q/(\partial_q^{i+1})$.

We will also need a relative version of certain homotopy categories that play an essential role in [11]. There is an exact forgetful functor between the usual homotopy categories of chain complexes of graded $A \# H_q$–modules

$$\mathcal{F}_d : \mathcal{C}(A \# H_q, d_0) \to \mathcal{C}(A, d_0).$$

An object $K_\bullet$ in $\mathcal{C}(A \# H_q, d_0)$ lies inside the kernel of the functor if and only if, when forgetting the H_q–module structure on each term of $K_\bullet$, the complex of graded A–modules $\mathcal{F}_d(K_\bullet)$ is nullhomotopic. The nullhomotopy map on $\mathcal{F}_d(K_\bullet)$, though, is not required to intertwine H_q–actions.

Likewise, there is an exact forgetful functor

$$\mathcal{F}_\partial : \mathcal{C}(A \# H_q, \partial_0) \to \mathcal{C}(A, \partial_0).$$

Similarly, an object $K_\bullet$ in $\mathcal{C}(A \# H_q, \partial_0)$ lies inside the kernel of the functor if and only if, when forgetting the H_q–module structure on each term of $K_\bullet$, the p–complex of A–modules $\mathcal{F}(K_\bullet)$ is nullhomotopic. The nullhomotopy map on $\mathcal{F}(K_\bullet)$, though, is not required to intertwine H_q–actions.

Definition 2.1 Given a p–DG algebra (A, ∂_A), the *relative homotopy category* is the Verdier quotient

$$\mathcal{C}^{\partial_q}(A, d_0) := \frac{\mathcal{C}(A \# H_q, d_0)}{\mathrm{Ker}(\mathcal{F}_d)}.$$

Likewise, the *relative p–homotopy category* is the Verdier quotient

$$\mathcal{C}^{\partial_q}(A, \partial_0) := \frac{\mathcal{C}(A \# H_q, \partial_0)}{\mathrm{Ker}(\mathcal{F}_\partial)}.$$

The superscripts in the definitions are to remind the reader of the H_q–module structures on the objects.

The categories $\mathcal{C}^{\partial_q}(A, d_0)$ and $\mathcal{C}^{\partial_q}(A, \partial_0)$ are triangulated. By construction, there are factorizations of the forgetful functors

$$\begin{array}{ccc} \mathcal{C}(A \# H_q, d_0) & \xrightarrow{\mathcal{F}_d} & \mathcal{C}(A, d_0) \\ & \searrow \quad \nearrow & \\ & \mathcal{C}^{\partial_q}(A, d_0) & \end{array} \qquad \begin{array}{ccc} \mathcal{C}(A \# H_q, \partial_0) & \xrightarrow{\mathcal{F}_\partial} & \mathcal{C}(A, \partial_0) \\ & \searrow \quad \nearrow & \\ & \mathcal{C}^{\partial_q}(A, \partial_0) & \end{array}$$

Proposition 2.2 [11, Proposition 2.13] *The p–extension functor*

$$\mathcal{P}\colon \mathcal{C}(A \# H_q, d_0) \to \mathcal{C}(A \# H_q, \partial_0)$$

descends to an exact functor, still denoted by $\mathcal{P}$, between the relative homotopy categories,

$$\mathcal{P}\colon \mathcal{C}^{\partial_q}(A, d_0) \to \mathcal{C}^{\partial_q}(A, \partial_0).$$

2.2 p–DG bimodules over the polynomial algebra

The polynomial algebra $R_n = \Bbbk[x_1, \dots, x_n]$ has a natural graded algebra structure by setting the degree of each x_i to be two. We can equip R_n with a p–DG algebra structure, where the generator $\partial_q \in H_q$ acts as a derivation determined by $\partial_q(x_i) = x_i^2$ for $i = 1, \dots, n$. As before, the internal grading on R_n will be referred to as the q–degree. When n is clear from the context, we will abbreviate R_n by just R.

The differential is invariant under the permutation action of the symmetric group S_n on the indices of the variables. Therefore let the subalgebra of polynomials symmetric in variables x_i and x_{i+1} with its inherited H_q–module structure be denoted by

$$R_n^i = \Bbbk[x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \dots, x_n].$$

More generally, given a (Young) subgroup $G \subset S_n$, the invariant subalgebra R_n^G inherits an H_q–algebra structure from R_n (and is thus a p–DG algebra). In particular, we will also use the H_q–subalgebra $R_n^{i,i+1} := R_n^{S_3}$, where S_3 is identified with the subgroup generated by permuting the indices i, $i+1$ and $i+2$.

The (R, R)–bimodule $B_i = R \otimes_{R^i} R$ has the structure of an H_q–module (and is thus a p–DG bimodule) where the differential acts via the Leibniz rule: for any $h \otimes g \in R \otimes_{R^i} R$,

$$\partial_q(h \otimes g) = \partial_q(h) \otimes g + h \otimes \partial_q(g).$$

With respect to $\otimes_R$, the monoidal category of (R, R)–bimodules generated by the B_i has an H_q–module structure, where the ∂_q action is given by the Leibniz rule. We denote this category by $(R, R)\#H_q$–mod.

Let $f = \sum_{i=1}^n a_i x_i \in \mathbb{F}_p[x_1, \dots, x_n] \subset R$ be a linear function. We twist the H_q–action on the bimodule B_i to obtain a bimodule B_i^f defined as follows. As an (R, R)–bimodule, it is the same as B_i but the action of H_q is twisted by defining

$$\partial_q(1 \otimes 1) = (1 \otimes 1) f. \tag{2-5a}$$

Similarly, we define ${}^f B_i$ where now

$$\partial_q(1 \otimes 1) = f(1 \otimes 1). \tag{2-5b}$$

For R_n as a bimodule over itself, it is clear that ${}^f R_n \cong R_n^f$ as p–DG bimodules. It follows that there are p^n ways to put an H_q–module structure on a rank-one free module over R_n. Each such H_q–module is quasi-isomorphic to a finite-dimensional p–complex. Choose numbers $b_i \in \{2, \dots, p, p+1\}$ such that $b_i \equiv a_i \pmod{p}$, $i = 1, \dots, n$, and define the H_q–ideal of R,

$$I = (x_1^{p+1-b_1}, \dots, x_n^{p+1-b_n}). \tag{2-6}$$

Then the natural quotient map

$$\pi \colon R^f \twoheadrightarrow R^f / (I \cdot R^f) \tag{2-7}$$

is readily seen to be a quasi-isomorphism. The right hand side of (2-7) computes the *slash homology* (see [11, Section 2.1] for more details), denoted by $\mathrm{H}^{/}_{\bullet}$, of R^f.

Lemma 2.3 [11, Lemma 3.1] *For each $f = \sum_i a_i x_i$, the rank-one p–DG module R^f has slash homology*

$$\mathrm{H}^{/}_{\bullet}(R^f) \cong \bigotimes_{i=1}^{n} V_{p-a_i}\{p - a_i\}.$$

In particular, the slash homology is finite-dimensional, and vanishes if any a_i of $f = \sum_i a_i x_i$ is equal to one.

Corollary 2.4 [11, Corollary 3.2] *Let M be a p–DG module over R which is equipped with a finite filtration, whose subquotients are isomorphic to R^f for various f. Then M has finite-dimensional slash homology.*

2.3 Relative p–Hochschild homology

In [11, Section 2.3], we introduced an absolute version of the p–Hochschild (co)homology functor. In what follows, we will instead need a relative version of p–Hochshild homology for a p–DG algebra, which we recall now. An important reason for introducing the relative homotopy category is that the relative p–Hochschild homology functor descends to this category.

Let (A, ∂_A) be a p–DG algebra. Equip A with the zero differential d_0 and zero p–differential ∂_0, and denote the resulting trivial (p–)DG algebras by (A_0, d_0) and (A_0, ∂_0) respectively. Likewise, for a (p–)DG bimodule M over A, we temporarily denote by M_0 the A–bimodule equipped with zero (p–)differentials.

The usual Hochschild homology of M_0 over (A_0, d_0) in this case carries a natural H_q–action, since the H_q–action commutes with all differentials in the usual simplicial bar complex for A_0.

Definition 2.5 The *relative Hochschild homology* of a p–DG bimodule (M, ∂_M) over (A, ∂_A) is the usual Hochschild homology of M_0 over (A_0, d_0) equipped with the induced H_q–action from ∂_M and ∂_A, and denoted by

$$\mathrm{HH}_\bullet^{\partial_q}(M) := \mathrm{HH}_\bullet(A_0, M_0).$$

Replacing the usual simplicial bar complex by Mayer's p–simplicial bar complex (see [11, Definition 2.10]), we make the next definition (see [11, Section 2.3] for details). Mayer's p–simplicial bar complex is obtained by removing the alternating signs in the usual simplicial bar complex of an algebra. In turn this results in a p–complex bimodule resolution of an algebra.

Definition 2.6 The *relative p–Hochschild homology* of M is the p–complex

$$p\mathrm{HH}_\bullet^{\partial_q}(M) := \mathrm{H}_\bullet^{/}(A_0 \otimes^{\boldsymbol{L}}_{A_0 \otimes A_0^{\mathrm{op}}} M_0) = \mathrm{H}_\bullet^{/}(\boldsymbol{p}(A_0) \otimes_{A_0 \otimes A_0^{\mathrm{op}}} M_0),$$

where the notation $\otimes^{\boldsymbol{L}}$ is the derived tensor functor. Here, the usual simplicial bar resolution of M_0 over A_0 is replaced by *Mayer's p–simplicial bar complex* $\boldsymbol{p}(A_0)$.

Similar to the usual Hochschild homology, the relative p–Hochschild homology is also covariant functor: if $f : M \to N$ is a morphism of p–DG bimodules over A, it induces

$$p\mathrm{HH}_\bullet^{\partial_q}(f) := \mathrm{H}_\bullet^{/}(\mathrm{Id}_{A_0} \otimes f) \colon \mathrm{H}_\bullet^{/}(A_0 \otimes^{\boldsymbol{L}}_{A_0 \otimes A_0^{\mathrm{op}}} M_0) \to \mathrm{H}_\bullet^{/}(A_0 \otimes^{\boldsymbol{L}}_{A_0 \otimes A_0^{\mathrm{op}}} N_0).$$

Proposition 2.7 [11, Proposition 2.20] *The relative p–Hochschild homology descends to a functor defined on the relative homotopy category $\mathcal{C}^{\partial_q}(A, \partial_0)$ of p–DG bimodules over A.*

We also have the trace-like property for relative p–Hochschild homology.

Proposition 2.8 [11, Proposition 2.21] *Given two p–DG bimodules M and N over A, there is an isomorphism of p–complexes of H_q–modules*

$$p\mathrm{HH}_\bullet^{\partial_q}(M \otimes_A^{\boldsymbol{L}} N) \cong p\mathrm{HH}_\bullet^{\partial_q}(N \otimes_A^{\boldsymbol{L}} M).$$

We next recall a technical tool that allows us to use a simpler bimodule resolution to compute the relative Hochschild homology than the usual simplicial bar resolution.

Theorem 2.9 [11, Theorem 2.22] *Let M be a p–DG bimodule over A. Suppose $f\colon Q_\bullet \to M$ is a p–complex resolution of M over (A_0, ∂_0) which is H_q–equivariant, and each term of $Q_\bullet$ is projective as an $A_0 \otimes A_0^{\mathrm{op}}$–module. Then f induces an isomorphism of H_q–modules*

$$\mathrm{H}_\bullet^{/}(A_0 \otimes_{A_0 \otimes A_0^{\mathrm{op}}} Q_\bullet) \cong p\mathrm{HH}_\bullet^{\partial_q}(M).$$

2.4 Elementary braiding complexes

Here and below, for ease of notation, we will abbreviate $t^n = [n]_d$ for homological shifts, where $n \in \mathbb{Z}$. Recall that in [11], we show that there are $(R, R)\#H_q$–module homomorphisms

(i) $rb_i\colon R \to q^{-2} B_i^{-(x_i + x_{i+1})}$, where $1 \mapsto (x_{i+1} \otimes 1 - 1 \otimes x_i)$;

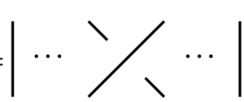
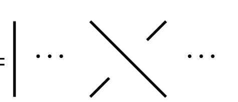
(ii) $br_i\colon B_i \to R$, where $1 \otimes 1 \mapsto 1$.

Thus we have complexes of $(R, R)\#H_q$–modules

$$T_i := \big(tB_i \xrightarrow{br_i} R\big), \quad T_i' := \big(R \xrightarrow{rb_i} q^{-2} t^{-1} B_i^{-(x_i + x_{i+1})}\big). \tag{2-8}$$

In the coming sections we will, for presentation reasons, often omit the various shifts built into the definitions of T_i and T_i'.

We associate respectively to the left and right crossings σ_i and σ_i' between the i^{th} and $(i+1)^{\text{st}}$ strands in (2-9) the chain complexes of $(R, R)\#H_q$–bimodules T_i and T_i',

(2-9) $\sigma_i :=$  $\qquad \sigma_i' :=$ 

More generally, if $\beta \in \mathrm{Br}_n$ is a braid group element written as a product in the elementary generators $\sigma_{i_i}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$, where $\epsilon_i \in \{\varnothing, \prime\}$, we assign the chain complex of $(R, R)\#H_q$–bimodules

$$T_\beta := T_{i_1}^{\epsilon_1} \otimes_R \cdots \otimes_R T_{i_k}^{\epsilon_k}. \tag{2-10}$$

The complex is well defined in the relative homotopy category thanks to the following result.

Theorem 2.10 *The complexes of T_i and T_i' are mutually inverse complexes in the relative homotopy category $\mathcal{C}^{\partial_q}(R, R, d_0)$. They satisfy the braid relations*

- $T_i T_j \cong T_j T_i$ *if* $|i - j| > 1$,
- $T_i T_{i+1} T_i \cong T_{i+1} T_i T_{i+1}$ *for all* $i = 1, \ldots, n-1$.

Consequently, given any braid group element $\beta \in \mathrm{Br}_n$, the chain complex of T_β associated to it is a well-defined element of the relative homotopy category $\mathcal{C}^{\partial_q}(R, R, d_0)$.

Proof This is proven in [11, Section 3]. □

3 Specialized HOMFLYPT theories

3.1 HOMFLYPT homologies

In this section we categorify the HOMFLYPT polynomial of any link using analogous arguments from [1], [13] and [15] adapted to the p–DG setting.

For the first construction, we will allow complexes of Soergel bimodules to sit in half-integer degrees in the Hochschild (a) and the homological, sometimes called the topological, (t) degrees when considering the usual complexes of vector spaces.

We modify the elementary braiding complexes of (2-8) to be

$$T_i := (at)^{-\frac{1}{2}} q^{-2}\big(tB_i \xrightarrow{br_i} R\big), \quad T_i' := (at)^{\frac{1}{2}} q^2\big(R \xrightarrow{rb_i} q^{-2}t^{-1} B_i^{-(x_i+x_{i+1})}\big). \tag{3-1}$$

Here we have extended the degree shift convention for q–degrees (see the beginning of Section 2) to a– and t–degrees.

Let $\beta \in \mathrm{Br}_n$ be a braid group element in n strands. By Theorem 2.10, there is a chain complex of $(R_n, R_n)\#H_q$–bimodules T_β, well defined up to homotopy, associated with β. Then set

$$T_\beta = \big(\cdots \xrightarrow{d_0} T_\beta^{i+1} \xrightarrow{d_0} T_\beta^{i} \xrightarrow{d_0} T_\beta^{i-1} \xrightarrow{d_0} \cdots\big). \tag{3-2}$$

Definition 3.1 The *untwisted H_q–HOMFLYPT homology* of β is the object

$$\widehat{\mathrm{HHH}}^{\partial_q}(\beta) := a^{-\frac{n}{2}} t^{\frac{n}{2}} \mathrm{H}_\bullet\big(\cdots \to \mathrm{HH}_\bullet^{\partial_q}(T_\beta^{i+1}) \xrightarrow{d_t} \mathrm{HH}_\bullet^{\partial_q}(T_\beta^{i}) \xrightarrow{d_t} \mathrm{HH}_\bullet^{\partial_q}(T_\beta^{i-1}) \to \cdots\big)$$

in the category of triply graded H_q–modules, where $d_t := \mathrm{HH}_\bullet^{\partial_q}(d_0)$ is the induced map of d_0 on relative Hochschild homology. Here, the relative Hochschild homology is defined in Definition 2.5, and $\mathrm{H}_\bullet$ means the usual homology of a chain complex.

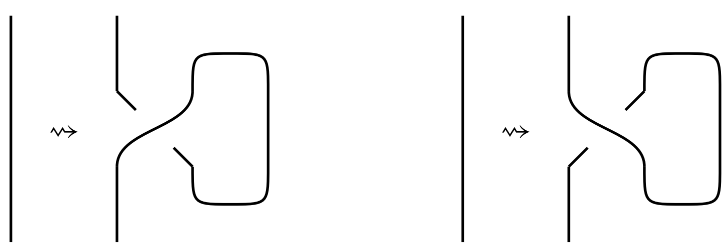
By construction, the space $\widehat{\mathrm{HHH}}^{\partial_q}(\beta)$ is triply graded by topological (t) degree, Hochschild (a) degree as well as quantum (q) degree. When necessary to emphasize each graded piece of the space, we will write $\widehat{\mathrm{HHH}}^{\partial_q}_{i,j,k}(\beta)$ to denote the homogeneous component concentrated in t–degree i, a–degree j and q–degree k.

The following theorem is a particular case of the main result of [6], where we have only kept track of the degree two p–nilpotent differential — which is denoted by L_1 in [6] — in finite characteristic p. The detailed verification given in Section 3.2, however, uses the main ideas of [15] and differs from that of [6]. This proof serves as the model for the other link homology theories in this paper.

Theorem 3.2 *The untwisted H_q–HOMFLYPT homology of β depends only on the braid closure of β as a framed link in $\mathbb{R}^3$.*

As a convention for the framing number of braid closure, if a strand for a component of link is altered as in the left of (3-3), then we say that the framing of the component is increased by 1 (with respect to the blackboard framing). If a strand for a component of link is altered as in the right of (3-3), then we say that the framing of the component is decreased by 1.

(3-3)



Denote by $\mathtt{f}_i(L)$ the framing number of the i^{th} component of a link L. Then, under the Reidemeister moves of (3-3), $\mathtt{f}_i(L)$ is increased or decreased by one when changing from the corresponding left local picture to the right local picture.

We next seek to define a triply graded analogue with a–, t– and q–degrees in the homotopy category of p–complexes. Let us first discuss what degrees of freedom we have in the constructions.

First, we may adapt (3-1) into

$$pT_i := a^u t^v q^w [n]^a_\partial [m]^t_\partial \big(B_i[1]^t_\partial \xrightarrow{br_i} R\big), \tag{3-4a}$$

$$pT_i' := a^{-u} t^{-v} q^{-w} [-n]^a_\partial [-m]^t_\partial \big(R \xrightarrow{rb_i} q^{-2} B_i^{-x_i - x_{i+1}} [-1]^t_\partial\big). \tag{3-4b}$$

Here, the superscripts in homological shifts indicate in which of the three gradings they are occurring. See the discussion around (2-1) for the meaning of the subscripts in the notation. We let $u, v, w, m, n \in \mathbb{Z}$ denote possible grading shifts to be determined, which will be made into the simplest possible form at the end of the next subsection.

Definition 3.3 Let $\beta \in \mathrm{Br}_n$ be a braid group element written as a product in the elementary generators $\sigma_{i_i}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$, where $\epsilon_i \in \{\varnothing, \prime\}$. We assign to β the p–chain complex of $(R_n, R_n)\#H_q$–bimodules

$$pT_\beta := pT_{i_1}^{\epsilon_1} \otimes_R \cdots \otimes_R pT_{i_k}^{\epsilon_k}. \tag{3-5}$$

We will denote the boundary maps in the p–complex pT_β by ∂_0 ($\partial_0^p = 0$), in contrast to the usual, also called the topological, differential d_0 satisfying $d_0^2 = 0$.

Definition 3.4 The *untwisted H_q–HOMFLYPT p–homology* of β is the object

$$p\widehat{\mathrm{HHH}}^{\partial_q}(\beta) := q^{f(n)} \mathrm{H}^{/}_{\bullet}\big(\cdots \to p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i+1}_\beta) \xrightarrow{\partial_t} p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i}_\beta) \xrightarrow{\partial_t} p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i-1}_\beta) \to \cdots\big)$$

in the homotopy category of bigraded H_q–modules, where $f(n)$ is a function on $\mathbb{N}$ which is determined below in (3-16). Here ∂_t stands for the induced map of the topological differentials on p–Hochschild homology groups $\partial_t := p\mathrm{HH}^{\partial_q}_{\bullet}(\partial_0)$.

In the definition of the H_q–HOMFLYPT p–homology, we have applied the p–extensions in both the topological and the Hochschild direction so that they can be collapsed into a single degree. The reason will become clearer later when categorifying certain $\mathfrak{sl}_N$ polynomials at prime roots of unity. Therefore, in contrast to $\widehat{\mathrm{HHH}}(\beta)$, $p\widehat{\mathrm{HHH}}(\beta)$ is only doubly graded, and we will adopt the notation $p\widehat{\mathrm{HHH}}_{i,j}(\beta)$ as above to stand for its homogeneous components in topological degree i and q–degree j. Further, the overall grading shift in the definition will be utilized in the invariance under the Markov II move below.

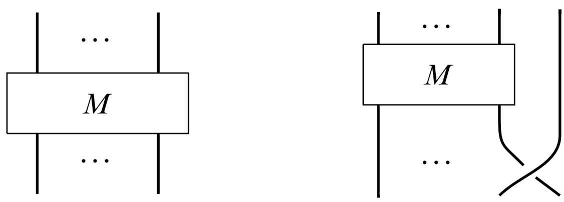
Theorem 3.5 *The untwisted H_q–HOMFLYPT p–homology of β depends only on the braid closure of β as a framed link in $\mathbb{R}^3$.*

The proof of Theorems 3.2 and 3.5 will occupy the next few subsections, after we introduce the H_q–equivariant (p–)Koszul resolutions.

3.2 Examining Markov II invariance

In this subsection, let us examine the invariance under the Markov II move for $p\widehat{\mathrm{HHH}}$.

In order to satisfy the second Markov move, one needs to show that for a Soergel bimodule M (or a complex of Soergel bimodules) over the polynomial p–DG algebra R_n, that the H_q–HOMFLYPT (p–)homologies of the bimodules (3-6) are isomorphic (up to grading shifts and twists in H_q actions),

(3-6) 

By definition, the one-variable p–extended Koszul complex is given by

$$pC_1 = q^2\Bbbk[x]^x \otimes \Bbbk[x]^x[1]^a_\partial \xrightarrow{x\otimes 1 - 1\otimes x} \Bbbk[x]\otimes\Bbbk[x]. \tag{3-7}$$

Set $pC_{n+1} := pC_1^{\otimes n+1}$. For the ease of notation, we will write pC_1' for the p–extended Koszul complex pC_1 in the variable x_{n+1}. Using the isomorphism of p–DG bimodules,

$$\begin{aligned} pC_{n+1} &\otimes_{(R_{n+1},R_{n+1})} ((M\otimes\Bbbk[x_{n+1}])\otimes_{R_{n+1}} pT_n) \\ &= (pC_n\otimes pC_1')\otimes_{(R_{n+1},R_{n+1})}((M\otimes\Bbbk[x_{n+1}])\otimes_{R_{n+1}} pT_n) \\ &\cong pC_n\otimes_{(R_n,R_n)}(M\otimes_{R_n}(pC_1'\otimes_{(\Bbbk[x_{n+1}],\Bbbk[x_{n+1}])} pT_n)), \end{aligned} \tag{3-8}$$

we are reduced to analyzing the p–homology of the "square" $pC_1'\otimes_{\Bbbk[x_{n+1}],\Bbbk[x_{n+1}]} pT_n$:

$$\begin{array}{ccc} a^u t^v q^w ({}^{x_{n+1}}B_i^{x_{n+1}})[k+1]^a_\partial[m+1]^t_\partial & \longrightarrow & a^u t^v q^w R^{2x_{n+1}}[k+1]^a_\partial[m]^t_\partial \\ \downarrow & & \downarrow \\ a^u t^v q^w B_n[k]^a_\partial[m+1]^t_\partial & \longrightarrow & a^u t^v q^w R[k]^a_\partial[m]^t_\partial \end{array} \tag{3-9a}$$

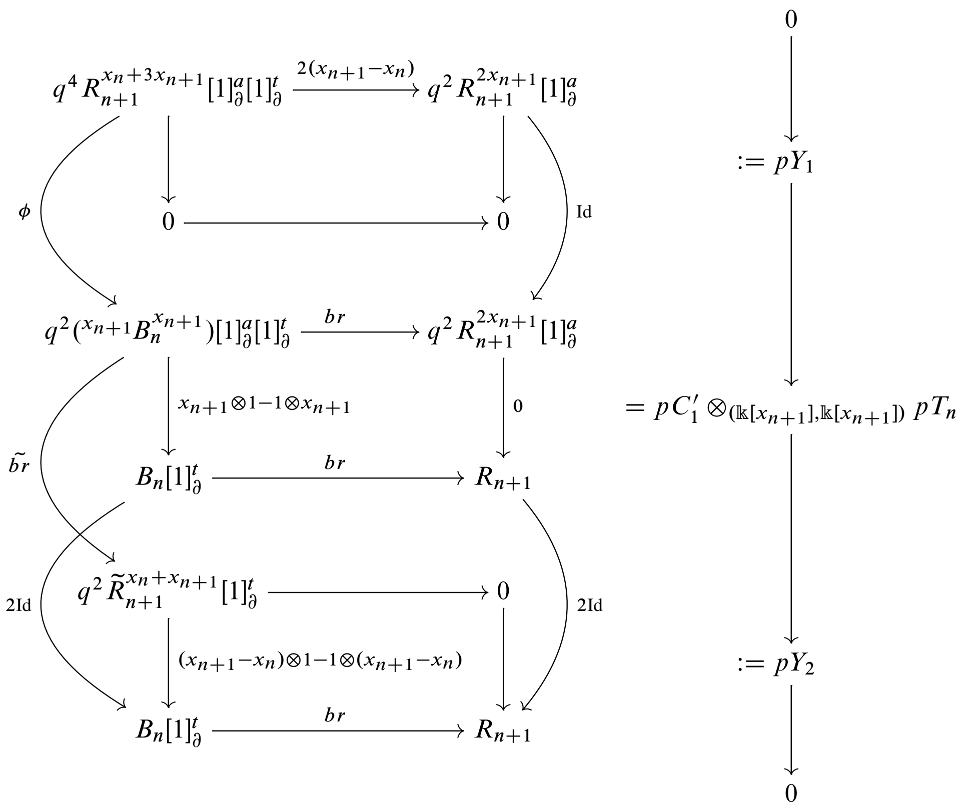


Figure 1

and $pC_1' \otimes_{\Bbbk[x_{n+1}],\Bbbk[x_{n+1}]} pT_n'$:

$$\begin{array}{ccc} a^{-u}t^{-v}q^{2-w}R^{2x_{n+1}}[1-k]^a_\partial[-m]^t_\partial & \longrightarrow & a^{-u}t^{-v}q^{2-w}({}^{x_{n+1}}B_n^{-x_n})[1-k]^a_\partial[-1-m]^t_\partial \\ \big\downarrow & & \big\downarrow \\ a^{-u}t^{-v}q^{-w}[-k]^a_\partial[-m]^t_\partial & \longrightarrow & a^{-u}t^{-v}q^{-w-2}B_n^{-x_i-x_{i-1}}[-k]^a_\partial[-1-m]^t_\partial \end{array} \tag{3-9b}$$

Let us begin by studying the first p–complex square (3-9a). We will exhibit a sub–p–complex pY_1 of (3-9a), whose quotient will be denoted by pY_2. Ignoring for the moment the overall grading shift $a^u t^v q^w [k]^a_\partial [m]^t_\partial$ for simplicity, we have a filtration of the square given by a short exact sequence of (p–complexes) of bimodules as in Figure 1, where by definition the first square is pY_1 and the third square is pY_2. Here ϕ is the map that sends 1 to $(x_{n+1}-x_n)\otimes 1 + 1\otimes(x_{n+1}-x_n)$.

It is not hard to show that $pC_n \otimes_{(R_n,R_n)} (M \otimes_{R_n} pY_2)$ is annihilated by taking first the vertical p–Hochschild homology and then the horizontal topological homology.

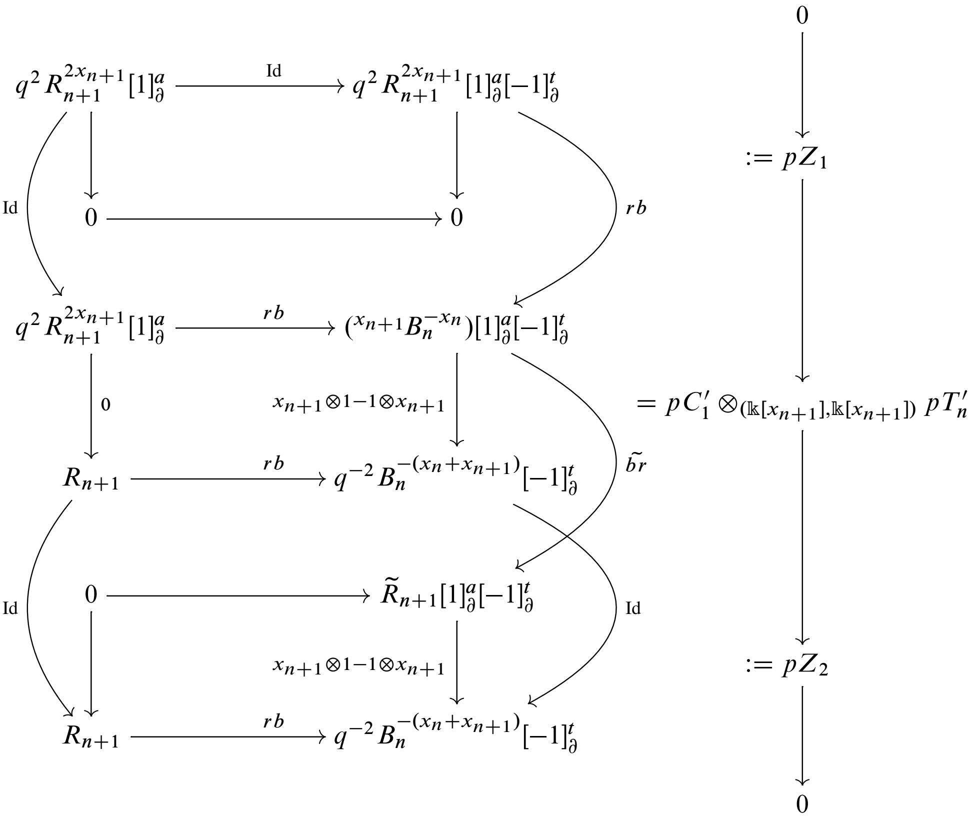
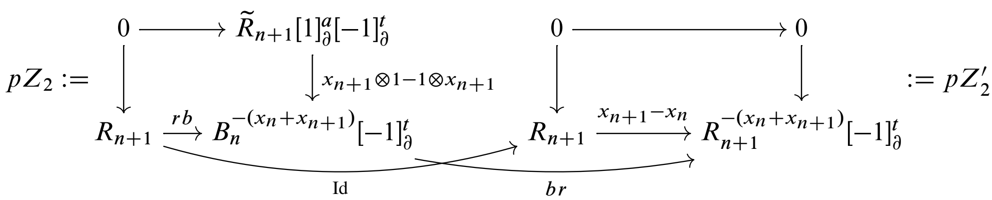


Figure 2

Further, the p–complex pY_1 is quasi-isomorphic to $q^2 R_n^{2x_n}[1]^a_\partial$. Putting back the grading shifts ignored earlier, we obtain the isomorphism

$$
\begin{aligned}
\text{(3-10)}\quad p\widehat{\mathrm{HHH}}^{\partial_q}((M\otimes\Bbbk[x_{n+1}])\otimes_{R_{n+1}} pT_n) &\cong \mathrm{H}^{/}_{\bullet}(p\mathrm{HH}^{\partial_q}((M\otimes_{R_n} pY_1)) \\
&\cong p\widehat{\mathrm{HHH}}^{\partial_q}(a^u t^v q^{w+2} M[k+1]^a_\partial[m]^t_\partial)^{2x_n}.
\end{aligned}
$$

For the second square (3-9b), again there is a short exact sequence of bicomplexes of (R_{n+1}, R_{n+1})–bimodules. Ignoring the overall grading shift $a^{-u}t^{-v}q^{-w}[-k]^a_\partial[-m]^t_\partial$, it is as in Figure 2. Next, consider the morphism of bicomplexes



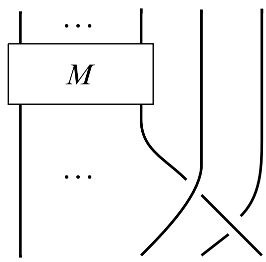
whose kernel is isomorphic to the contractible p–complex

$$pZ_2'' := \quad \begin{array}{c} \widetilde{R}_{n+1}[1]^a_\partial[-1]^t_\partial \\ \downarrow \scriptstyle\mathrm{Id} \\ \widetilde{R}_{n+1}[-1]^t_\partial \end{array}$$

Upon taking pHH, the contribution from pZ_2'' vanishes. Taking back into account the overall grading shift, it follows that we have

$$\begin{aligned} \text{(3-11)}\quad p\widehat{\mathrm{HHH}}^{\partial_q}((M\otimes \Bbbk[x_{n+1}])\otimes_{R_{n+1}} pT_n') &\cong \mathrm{H}^/_\bullet(p\mathrm{HH}^{\partial_q}((M\otimes_{R_n} pZ_2')) \\ &\cong p\widehat{\mathrm{HHH}}^{\partial_q}(a^{-u}t^{-v}q^{-w-2}M[-k]^a_\partial[-m-1]^t_\partial)^{-2x_n}. \end{aligned}$$

Now, let us observe that taking closure of the following diagram of p–DG bimodules

(3-12) 

introduces a canceling pair of Markov II moves. By (3-10) and (3-11), we obtain that

$$\begin{aligned} \text{(3-13)}\quad p\widehat{\mathrm{HHH}}^{\partial_q}\big((M\otimes \Bbbk[x_{n+1},x_{n+2}])\otimes_{R_{n+2}} pT_{n+1}\otimes_{R_{n+2}} pT_{n+2}'\big) \\ \cong q^{f(n+2)}\mathrm{H}^/_\bullet(p\mathrm{HH}_\bullet(M[1]^a_\partial[-1]^t_\partial)). \end{aligned}$$

For the last term to be isomorphic to

$$\text{(3-14)}\qquad p\widehat{\mathrm{HHH}}^{\partial_q}(M) \cong q^{f(n)}\mathrm{H}^/_\bullet(p\mathrm{HH}_\bullet(M)),$$

we need to require the functor isomorphism

$$\text{(3-15)}\qquad [1]^a_\partial[-1]^t_\partial = q^{f(n)-f(n+2)}.$$

We are therefore forced to collapse the a grading onto the t grading such that $a=q^r t$, where $r=f(n)-f(n+2)\in\mathbb{Z}$. For simplicity, let us assume that

$$\text{(3-16)}\qquad K=f(n)-f(n+1)$$

is a constant independent of n. Then $r=2K\in 2\mathbb{Z}$, and we have $a=q^{2K}t$ such that $[1]^a_\partial = q^{2K}[1]^t_\partial$.

Revisiting (3-4), we now set

$$pT_i := q^{-K-2}[-1]^t_\partial\big(B_i[1]^t_\partial \xrightarrow{br_i} R\big), \tag{3-17a}$$

$$pT_i' := q^{K+2}[1]^t_\partial\big(R \xrightarrow{rb_i} q^{-2}B_i^{-x_i-x_{i+1}}[-1]^t_\partial\big), \tag{3-17b}$$

and

$$p\widehat{\mathrm{HHH}}^{\partial_q}(\beta, K+1) := q^{-Kn}\mathrm{H}^{/}_{\bullet}\big(\cdots \to p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i+1}_\beta) \xrightarrow{\partial_t} p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i}_\beta) \xrightarrow{\partial_t} p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i-1}_\beta) \to \cdots\big).$$

Recall from Section 2.2 — see the discussion around (2-5a) and (2-5b) — that, for a given linear polynomial $f = \sum_i a_i x_i$, $a_i \in \mathbb{F}_p$, and a p–DG R_n–module M, we can twist the H_q–module structure on M by f. The resulting p–DG module is denoted by M^f.

Theorem 3.6 *Let β_1 and β_2 be two braids whose closures represent the same link L of r components up to framing. Suppose the framing numbers of the closures $\hat\beta_1$ of β_1 and $\hat\beta_2$ of β_2 differ by $\mathtt{f}_i(\hat\beta_1) - \mathtt{f}_i(\hat\beta_2) = a_i$, $i = 1, \dots, r$. Then*

$$\widehat{\mathrm{HHH}}^{\partial_q}(\beta_1) \cong \widehat{\mathrm{HHH}}^{\partial_q}(\beta_2)^{2\sum_{i=1}^r a_i x_i}$$

and

$$p\widehat{\mathrm{HHH}}^{\partial_q}(\beta_1, K+1) \cong p\widehat{\mathrm{HHH}}^{\partial_q}(\beta_2, K+1)^{2\sum_{i=1}^r a_i x_i}$$

where the generator of the polynomial action for the i^{th} component is denoted by x_i and $\widehat{\mathrm{HHH}}^{\partial_q}(\beta_2)^{2\sum_i a_i x_i}$ means that we twist the H_q–module structure on the i^{th} component by $2a_i x_i$.

Proof The topological invariance follows from Theorem 2.10 and the proof above of invariance under the Markov moves. □

3.3 Unlinks and twistings

In this section, we compute $\widehat{\mathrm{HHH}}^{\partial_q}$ and $p\widehat{\mathrm{HHH}}^{\partial_q}$ for the identity element of the braid group Br_n, and define an unframed link invariant in $\mathbb{R}^3$ by correcting the framing factors appearing in Theorem 3.6.

For the unknot, the Koszul resolution C_1 of $\Bbbk[x]$ as bimodules is given by

$$q^2 a\Bbbk[x]^x \otimes \Bbbk[x]^x \xrightarrow{x\otimes 1 - 1\otimes x} \Bbbk[x] \otimes \Bbbk[x].$$

Tensoring this complex with $\Bbbk[x]$ as a bimodule yields

$$q^2 a\Bbbk[x]^{2x} \xrightarrow{0} \Bbbk[x].$$

Thus the homology of the unknot (up to shift) is identified with the bigraded H_q–module

$$\Bbbk[x] \oplus q^2 a \Bbbk[x]^{2x}.$$

More generally, via the Koszul complex $C_n = C_1^{\otimes n}$, we have that the homology of the n–component unlink L_0 is equal to

$$\widehat{\mathrm{HHH}}^{\partial_q}(L_0) \cong a^{-\frac{n}{2}} t^{\frac{n}{2}} \mathrm{HH}_\bullet(R_n) \cong a^{-\frac{n}{2}} t^{\frac{n}{2}} \bigotimes_{i=1}^{n} (\Bbbk[x_i] \oplus q^2 a \Bbbk[x_i]^{2x_i}). \tag{3-18}$$

Alternatively, up to the grading shift $a^{-\frac{n}{2}} t^{\frac{n}{2}}$, we may identify $\widehat{\mathrm{HHH}}^{\partial_q}(L_0)$ with the exterior algebra over R_n generated by the differential forms dx_i of bidegree aq^2 for $i = 1, \dots, n$, subject to the condition that each dx_i accounts for a twisting of the H_q–module structure by $2x_i$.

It follows that, as for the ordinary HOMFLYPT homology, given a framed link L of ℓ components arising as a braid closure $\hat{\beta}$, its untwisted HOMFLYPT H_q–homology $\widehat{\mathrm{HHH}}^{\partial_q}(\beta)$ is a module over

$$\widehat{\mathrm{HHH}}^{\partial_q}_{0,0,\bullet}(L_0) \cong R_\ell,$$

and thus one may consider a twisting of the H_q–module structure on $\widehat{\mathrm{HHH}}^{\partial_q}(\beta)$ by the functor $R_\ell^f \otimes_{R_\ell} (\cdot)$, where f is a linear polynomial in $x_1, \dots, x_\ell$; see Section 2.2.

Definition 3.7 Let L be a framed link arising from the closure of an n–strand braid β. Label the components of L by 1 through ℓ, and set the (*linear*) *framing factor* of β to be the linear polynomial

$$\mathtt{f}_\beta = -\sum_{i=1}^{\ell} 2\mathtt{f}_i x_i.$$

(1) The *H_q–HOMFLYPT homology* of β is the triply graded H_q–module

$$\mathrm{HHH}^{\partial_q}(\beta) := \widehat{\mathrm{HHH}}^{\partial_q}(\beta)^{\mathtt{f}_\beta} \cong R_\ell^{\mathtt{f}_\beta} \otimes_{R_\ell} \widehat{\mathrm{HHH}}^{\partial_q}(\beta).$$

(2) Likewise, the *H_q–HOMFLYPT p–homology* is the doubly graded H_q–module

$$p\mathrm{HHH}^{\partial_q}(\beta, K+1) := p\mathrm{HHH}^{\partial_q}(\beta, K+1)^{\mathtt{f}_\beta} \cong R_\ell^{\mathtt{f}_\beta} \otimes_{R_\ell} p\widehat{\mathrm{HHH}}^{\partial_q}(\beta, K+1).$$

Corollary 3.8 *Given a braid β, both $\mathrm{HHH}^{\partial_q}(\beta)$ and $p\mathrm{HHH}^{\partial_q}(\beta)$ are link invariants that only depend on the closure of β as a link in $\mathbb{R}^3$. Moreover, these invariants satisfy the following properties*:

(i) *The slash homologies of* $\mathrm{HHH}^{\partial_q}(\beta)$ *and* $p\mathrm{HHH}^{\partial_q}(\beta, K+1)$ *are finite-dimensional.*

(ii) *Furthermore, the Euler characteristic of* $\mathrm{HHH}^{\partial_q}(\beta)$ *is equal to the HOMFLYPT polynomial of* $\hat{\beta}$ *in the formal variables* q *and* a*, while the Euler characteristic of* $p\mathrm{HHH}^{\partial_q}(\beta, K+1)$ *is equal to the* $\mathfrak{sl}_{K+1}$*–polynomial of* $\hat{\beta}$ *in a formal* q*–variable.*

(iii) *The Euler characteristic of the slash homology of* $\mathrm{HHH}^{\partial_q}(\beta)$ *is equal to the specialization of the HOMFLYPT polynomial of* $\hat{\beta}$ *at a root of unity* q*, while the Euler characteristic of the slash homology of* $p\mathrm{HHH}^{\partial_q}(\beta, K+1)$ *is the equal to the specialization of the* $\mathfrak{sl}_{K+1}$*–polynomial of* $\hat{\beta}$ *at a root of unity* q*.*

Proof For the first statement, we note that the twisting of the p–DG structure by the framing factor takes care of the Markov II move.

Next, the finite-dimensionality of the homology theories follows, by construction, from the fact that ${}^{f_{i_1}}B_{i_1}^{g_{i_1}} \otimes_R \cdots \otimes_R {}^{f_{i_m}}B_{i_m}^{g_{i_m}}$ is an H_q–module with 2^m–step filtration whose subquotients are isomorphic to R^f as left $R\#H_q$–modules; thus Corollary 2.4 applies. □

Remark 3.9 The previous discussion in Section 3.2 forces us to make a specialization $a = q^r t$ in the homotopy category of t and q–bigraded p–complexes to obtain a framed Markov II invariance. In particular, when $r = K = 0$, this forces the relation, on the Grothendieck group level, that $a = t = -1$. This specialization leads to a categorification of the Alexander skein relation.

4 Specialized homology theories

4.1 A singly graded homology

Fix $k \in \mathbb{N}$. Consider the H_q–Koszul complex in one-variable,

$$C_1 : 0 \to aq^2 \Bbbk[x]^x \otimes \Bbbk[x]^x \xrightarrow{d_C} \Bbbk[x] \otimes \Bbbk[x] \to 0, \tag{4-1}$$

where d_C is the map $d_C(f) = (x^{kp+2} \otimes 1 + 1 \otimes x^{kp+2}) f$ and $k \in \mathbb{N}$. We regard the differential on the arrow as an endomorphsim of the Koszul complex, of (a, q)–bidegree $(-1, 2kp+2)$.

Lemma 4.1 *The commutator of the endomorphisms* d_C *and* $\partial_q \in H_q$ *is nullhomotopic on the Koszul complex* C_1*.*

Proof The commutator map $\phi := [d_C, \partial_q]$ is given by

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \Bbbk[x]^x \otimes \Bbbk[x]^x & \xrightarrow{x\otimes 1 - 1\otimes x} & \Bbbk[x]\otimes\Bbbk[x] & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\phi} & & & & \\ 0 & \longrightarrow & \Bbbk[x]^x \otimes \Bbbk[x]^x & \xrightarrow{x\otimes 1 - 1\otimes x} & \Bbbk[x]\otimes\Bbbk[x] & \longrightarrow & 0 & & \end{array}$$

where ϕ maps the bimodule generator $1 \otimes 1 \in \Bbbk[x]^x \otimes \Bbbk[x]^x$ as follows:

$$\begin{aligned} \phi(1\otimes 1) &= d_C(\partial_q(1\otimes 1)) - \partial_q d_C(1\otimes 1) \\ &= d_C(x\otimes 1 + 1\otimes x) - \partial_q(x^{kp+2}\otimes 1 + 1\otimes x^{kp+2}) \\ &= (x\otimes 1 + 1\otimes x)(x^{kp+2}\otimes 1 + 1\otimes x^{kp+2}) - 2(x^{kp+3}\otimes 1 + 1\otimes x^{kp+3}) \\ &= -x^{kp+3}\otimes 1 + x^{kp+2}\otimes x + x\otimes x^{kp+2} - 1\otimes x^{kp+3} \\ &= x^{kp+2}\otimes 1(1\otimes x - x\otimes 1) + (x\otimes 1 - 1\otimes x)1\otimes x^{kp+2} \\ &= (x\otimes 1 - 1\otimes x)(1\otimes x^{kp+2} - x^{kp+2}\otimes 1). \end{aligned}$$

We may thus choose a nullhomotopy to be

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \Bbbk[x]^x \otimes \Bbbk[x]^x & \xrightarrow{x\otimes 1 - 1\otimes x} & \Bbbk[x]\otimes\Bbbk[x] & \longrightarrow & 0 \\ & & & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle\phi} & & & & \\ 0 & \longrightarrow & \Bbbk[x]^x \otimes \Bbbk[x]^x & \xrightarrow{x\otimes 1 - 1\otimes x} & \Bbbk[x]\otimes\Bbbk[x] & \longrightarrow & 0 & & \end{array}$$

where h is given by multiplication by the element $1 \otimes x^{kp+2} - x^{kp+2} \otimes 1$, and acts on the rest of the complex by zero. □

The Koszul complex C_n inherits the endomorphism d_C by forming the n–fold tensor product from the one-variable case. It follows, that for a given p–DG bimodule M over R_n, there is an induced differential, still denoted by d_C, given via the identification

$$\mathrm{HH}_\bullet^{\partial_q}(M) \cong \mathrm{H}_\bullet(M \otimes_{(R_n,R_n)} C_n), \tag{4-2}$$

where the induced differential acts on the right hand side by $\mathrm{Id}_M \otimes d_C$. By construction, d_C has Hochschild degree -1 and q–degree $2kp+2$.

Lemma 4.1 immediately implies the following.

Corollary 4.2 *The induced differential d_C on $\mathrm{HH}_\bullet^{\partial_q}(M)$ commutes with the H_q–action.* □

Remark 4.3 The differential d_C, first observed by Cautis [1], has the following more algebrogeometric meaning. Identifying $\mathrm{HH}^1(R_n)$ as vector fields on $\mathrm{Spec}(R_n) = \mathbb{A}^n$, $\mathrm{HH}^1(R_n)$ acts as differential operators on $\mathrm{HH}_\bullet(M)$ for any (R_n, R_n)–bimodule M, regarded as a coherent sheaf on $\mathbb{A}^n \times \mathbb{A}^n \cong T^*(\mathbb{A}^n)$. Under this identification d_C is given by, up to scaling by a nonzero number, contraction with the vector field

$$\zeta_C := \sum_{i=1}^n x_i^{kp+2} \frac{\partial}{\partial x_i}.$$

On the other hand, ∂_q is given by the polynomial derivation by the vector field

$$\zeta_q := \sum_{i=1}^n x_i^2 \frac{\partial}{\partial x_i}.$$

Since these two vector fields satisfy

$$[\zeta_C, \zeta_q] = \sum_{i,j} \Big[x_i^{kp+2} \frac{\partial}{\partial x_i}, x_j^2 \frac{\partial}{\partial x_j} \Big] = \sum_i \Big(2x_i^{kp+3} \frac{\partial}{\partial x_i} - (kp+2) x_i^{kp+3} \frac{\partial}{\partial x_i} \Big) = 0,$$

the two actions naturally commute with each other on $\mathrm{HH}_\bullet(M)$ via the Gerstenhaber module structure on $\mathrm{HH}_\bullet(M)$.

In a more general context, Hochschild homology is a Gerstenhaber module over Hochschild cohomology viewed as a Gerstenhaber algebra. We may view d_C and ∂_q as commuting elements in Hochschild cohomology ring but the element d_C acts on homology via cap product $\zeta_C \cap (\cdot)$ and the element ∂_q acts via a Lie algebra action $\mathcal{L}_{\zeta_q}(\cdot)$. The compatibility of these actions is given by the equation

$$\zeta_C \cap \mathcal{L}_{\zeta_q}(x) = [\zeta_C, \zeta_q] \cap x + \mathcal{L}_{\zeta_q}(\zeta_C \cap x).$$

Since $[\zeta_C, \zeta_q] = 0$, these actions commute.

Now we are ready to introduce a further collapsed p–homology theory of a braid closure. Let $\beta \in \mathrm{Br}_n$ be an n–stranded braid. We have associated to β a usual chain complex of H_q–equivariant Soergel bimodules T_β as in (3-2), of which we take $p\mathrm{HH}_\bullet^{\partial_q}$ for each term:

(4-3)
$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \partial_C} & & \uparrow{\scriptstyle \partial_C} & & \uparrow{\scriptstyle \partial_C} & & \\
\cdots & \xrightarrow{\partial_t} & p\mathrm{HH}_i^{\partial_q}(pT_\beta^{m+1}) & \xrightarrow{\partial_t} & p\mathrm{HH}_i^{\partial_q}(pT_\beta^{m}) & \xrightarrow{\partial_t} & p\mathrm{HH}_i^{\partial_q}(pT_\beta^{m-1}) & \xrightarrow{\partial_t} & \cdots \\
 & & \uparrow{\scriptstyle \partial_C} & & \uparrow{\scriptstyle \partial_C} & & \uparrow{\scriptstyle \partial_C} & & \\
\cdots & \xrightarrow{\partial_t} & p\mathrm{HH}_{i+1}^{\partial_q}(pT_\beta^{m+1}) & \xrightarrow{\partial_t} & p\mathrm{HH}_{i+1}^{\partial_q}(pT_\beta^{m}) & \xrightarrow{\partial_t} & p\mathrm{HH}_{i+1}^{\partial_q}(pT_\beta^{m-1}) & \xrightarrow{\partial_t} & \cdots \\
 & & \uparrow{\scriptstyle \partial_C} & & \uparrow{\scriptstyle \partial_C} & & \uparrow{\scriptstyle \partial_C} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Here, ∂_C is a p–differential arising from d_C as follows. By [11, Proposition 4.8], the p–Hochschild homology groups in a column above are identified with the terms in

$$\cdots \xrightarrow{d_C} \mathrm{HH}^{\partial_q}_{2i+1}(pT^m_\beta) = \cdots \\ \cdots = \mathrm{HH}^{\partial_q}_{2i+1}(pT^m_\beta) \xrightarrow{d_C} \mathrm{HH}^{\partial_q}_{2i}(pT^m_\beta) \xrightarrow{d_C} \mathrm{HH}^{\partial_q}_{2i-1}(pT^m_\beta) = \cdots, \tag{4-4}$$

where each term in odd Hochschild degree is repeated $p-1$ times. Here the horizontal differential is the p–Hochschild induced map of the topological differential, which we have denoted by ∂_t to indicate its origin. On the arrows connecting even and odd Hochschild degree terms, we put the map d_C while keeping the repeated terms connected by identity maps. This defines a p–complex structure, denoted by ∂_C, in each column in diagram (4-3). The p–differential ∂_C commutes with the H_q–action on each term by Corollary 4.2. Denote the total p–differential by $\partial_T := \partial_t + \partial_C + \partial_q$, which collapses the double grading into a single q–grading.

Remark 4.4 We would like to emphasize an important point about the vertical grading collapse. In order to p–extend the Koszul complex (4-1) into a p–Koszul complex with ∂_C of degree two, we are forced to make the functor specialization from $[1]^a_d = a$ into $q^{2kp+2}[1]^q_\partial$, so that the p–extended complex looks like

$$pC_1 : 0 \to q^{2kp+4}\Bbbk[x]^x \otimes \Bbbk[x]^x[1]^q_\partial \xrightarrow{d_C} \Bbbk[x] \otimes \Bbbk[x] \to 0. \tag{4-5}$$

Taking tensor products of pC_1, this determines the correct vertical q–degree shifts in each column of diagram (4-3) of the p–Hochschild homology groups.

Notice that, on the level of Grothendieck groups, this has the effect of specializing the formal variable a into $-q^{2kp+2}$.

When $[1]^t_\partial = [1]^q_\partial$ and $a = q^{2kp+2}[1]^q_\partial$, the braiding complexes (3-17) specialize to

$$\begin{aligned} pT_i &:= q^{-kp-3}\big(B_i \xrightarrow{br_i} R[-1]^q_\partial\big), \\ pT'_i &:= q^{kp+3}\big(R[1]^q_\partial \xrightarrow{rb_i} q^{-2} B_i^{-(x_i+x_{i+1})}\big). \end{aligned} \tag{4-6}$$

Comparing (3-17) with (4-6), this forces

$$K = kp + 1. \tag{4-7}$$

This also explains the necessity of p–extension in the collapsed t and a direction in pHHH in the previous section: the homological shift in that direction needs to be p–extended to agree with the homological shift in the q–direction.

Furthermore, the bigrading in diagram (4-3) is now interpreted as a single grading, with both ∂_C and ∂_t raising q–degree by two.

Definition 4.5 Let β be an n stranded braid. The *untwisted* $\mathfrak{sl}_{kp+2}$ *p–homology* of β is the slash homology group

$$p\widehat{\mathrm{H}}(\beta, kp+2) := q^{-n(kp+1)}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}^{\partial_q}_{\bullet}(pT_\beta), \partial_T),$$

viewed as an object in $\mathcal{C}(\Bbbk, \partial_q)$. We will drop the $kp+2$ decoration whenever k is fixed and clear from context.

The homology group $p\widehat{\mathrm{H}}(\beta)$ is only singly graded as an object in $\mathcal{C}(\Bbbk, \partial_q)$. By construction, $p\widehat{\mathrm{H}}(\beta)$ is the slash homology with respect to the ∂_T action on $\bigoplus_{i,j} p\mathrm{HH}^{\partial_q}_i(pT^j_\beta)$; see diagram (4-4). The latter space is doubly graded by the topological degree and q–degree with values in $\mathbb{Z}\times\mathbb{Z}$ (the Hochschild a degree is already forced to be collapsed with the q degree to make the Cautis differential ∂_C homogeneous). However, as in Section 3.2, the Markov II invariance for the homology theory already requires one to collapse the t–grading onto the a–grading, thus also onto the q–grading. We will use $p\widehat{\mathrm{H}}_i(\beta)$ to stand for the homogeneous subspace sitting in some q–degree i.

Remark 4.6 This approach to a categorification of the Jones polynomial, at generic values of q, was first developed by Cautis [1]. We follow the exposition of Robert and Wagner from [13] and the closely related approach of Queffelec, Rose and Sartori [12].

4.2 Topological invariance

In this subsection, we establish the topological invariance of the untwisted homology theory.

Theorem 4.7 *The homology $p\widehat{\mathrm{H}}(\beta, kp+2)$ is a finite-dimensional framed link invariant depending only on the braid closure of β.*

Proof The proof of the theorem is similar to [11, Theorem 5.6]. It amounts to showing that taking slash homology of $p\mathrm{HH}^{\partial}_{\bullet}(\beta)$ with respect to ∂_T satisfies the Markov II move.

We start by discussing the normal H_q–equivariant Hochschild homology version. Let L be a link in $\mathbb{R}^3$ obtained as a braid closure $\hat{\beta}$, where $\beta \in \mathrm{Br}_n$ is an n–stranded braid. Recall that the homology groups $\mathrm{HH}^{\partial}_{\bullet}(L)$ are defined by tensoring a complex of Soergel bimodules M determined by β with the Koszul complex C_n and computing its termwise vertical (Hochschild) homology. The differential d_C is defined on the Koszul complex C_n. To emphasize its dependence on n, we will write d_C on C_n as d_n in this proof, and likewise write ∂_n for the p–extended differential on pC_n.

Since

$$C_{n+1} = C_n \otimes C_1' = C_n \otimes \Bbbk[x_{n+1}] \otimes \Lambda\langle dx_{n+1}\rangle \otimes \Bbbk[x_{n+1}],$$

the vertical differential may be inductively defined as

$$d_{n+1} = d_n \otimes \mathrm{Id} + \mathrm{Id} \otimes d_1'. \tag{4-8}$$

Here we have set $C_1' = \Bbbk[x_{n+1}] \otimes \Lambda\langle dx_{n+1}\rangle \otimes \Bbbk[x_{n+1}]$ equipped with part of the Cautis differential

$$d_1' := x_{n+1}^{kp+2} \otimes \iota_{\frac{\partial}{\partial x_{n+1}}} \otimes 1 + 1 \otimes \iota_{\frac{\partial}{\partial x_{n+1}}} \otimes x_{n+1}^{kp+2}.$$

The notation ι denotes the contraction of dx_{n+1} with $\frac{\partial}{\partial x_{n+1}}$. Under p–extension, write ∂_C for the p–extended Cautis differential and ∂_1' as the p–extended differential of d_1'.

We start by reexamining the diagram in Figure 1 with the shifts in (4-6). It will be helpful to keep the a and t gradings separate for the proof, with it understood that $[1]_\partial^a = q^{2kp+2}[1]_\partial^q$ and $[1]_\partial^t = [1]_\partial^q$. Thus we have a short exact sequence as in Figure 3. Further, the sequence splits as bimodules over (R_n, R_n) (see the proof of [11, Proposition 4.12] for an explicit splitting).

We claim that, as modules over $\Bbbk[\partial_T]/(\partial_T^p)$, the p–homology groups

$$p\mathrm{HH}_\bullet^{\partial_q}((M \otimes \Bbbk[x_{n+1}]) \otimes_{R_{n+1}} T_n)$$

fit into a distinguished triangle

$$\begin{aligned}\mathrm{H}_\bullet^{/}(pC_n \otimes_{(R_n,R_n)} (M \otimes_{R_n} pY_2)) &\to \mathrm{HH}_\bullet^{\partial_q}((M \otimes \Bbbk[x_{n+1}]) \otimes_{R_{n+1}} pT_n) \\ &\to \mathrm{H}_\bullet^{/}(pC_n \otimes_{(R_n,R_n)} (M \otimes_{R_n} pY_1)) \xrightarrow{[1]}\end{aligned} \tag{4-9}$$

after taking vertical slash (p–Hochschild) homology. Note that this p–complex triangle is in reverse order of the filtration in Figure 3.

Indeed, since ∂_C acts on the pY_1 and pY_2 tensor factors via ∂_1', it suffices to check that ∂_1' preserves the submodule arising from pY_2 and presents the part arising from pY_1 as a quotient. To do this, we reexamine the sequence in Figure 1 under vertical slash (p–Hochschild) homology, with the auxiliary a and t–gradings. The part pY_2, under vertical homotopy equivalence, contributes to the horizontal (topological) complex

$$pY_2' := \left(q^{-kp-3} R_{n+1} \xrightarrow{\partial_t = \mathrm{Id}} q^{-kp-3} R_{n+1}[-1]_\partial^t\right) \tag{4-10a}$$

sitting entirely in p–Hochschild degree 0. Likewise, the part pY_1 contributes to the horizontal

$$pY_1' := \left(q^{-kp+1} R_{n+1}^{x_n+3x_{n+1}}[1]_\partial^a \xrightarrow{\partial_t = 2(x_{n+1}-x_n)} q^{-kp-1} R_{n+1}^{2x_{n+1}}[1]_\partial^a[-1]_\partial^t\right) \tag{4-10b}$$

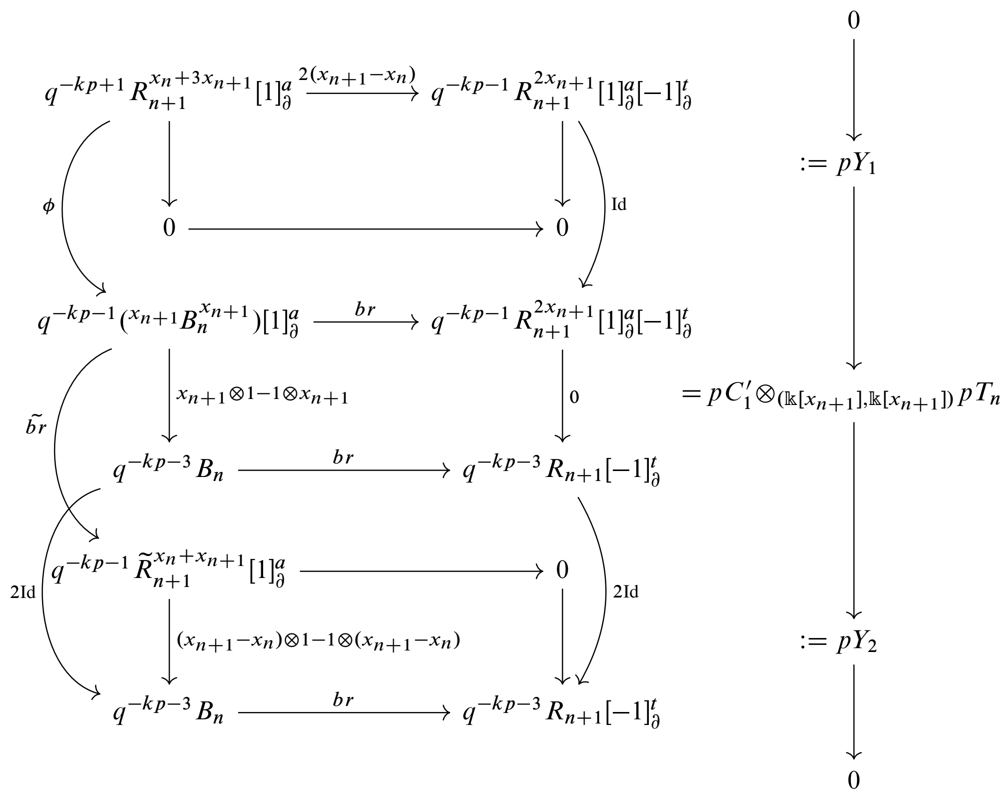


Figure 3

sitting entirely in p–Hochschild degrees $1,\dots,p-1$. Since ∂_1' decreases the a–degree by one (ie acting vertically downwards), pY_2' must be preserved under ∂_1', acting upon it trivially, and pY_1' is equipped with the quotient action of ∂_1'.

By the above discussion, $\partial_T=\partial_t+\partial_C+\partial_q$ acts on the term containing pY_2' only through $\partial_t+\partial_q$. Since this term is the cone of the identity map, it is nullhomotopic and thus

$$\mathrm{H}^{/}_{\bullet}(pC_n\otimes_{(R_n,R_n)}(M\otimes_{R_n}pY_2))\cong 0.$$

Consequently, using that $[1]^a_\partial=q^{2kp+2}[1]^t_\partial$, we have an isomorphism

$$\begin{aligned}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_\bullet((M\otimes\Bbbk[x_{n+1}])\otimes_{R_{n+1}}pT_n),\partial_T)&\cong \mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_\bullet(M\otimes_{R_n}pY_1'),\partial_T)\\&\cong q^{kp+1}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_\bullet(M),\partial_T)^{2x_n}.\end{aligned}$$

The q^{kp+1} factor is canceled out in the overall shift of $p\widehat{\mathrm{H}}$. This finishes the first part of Markov II move.

The other case of the Markov II move is entirely similar, which we omit.

Finally, the finite-dimensionality of $p\widehat{\mathrm{H}}(\beta)$ follows from Corollary 2.4. □

To obtain a categorical link invariant, we need to introduce a p–differential twisting to correct the framing factor occurring in Theorem 4.7, as done in [11, Section 5.3]. For a braid $\beta \in \mathrm{Br}_n$ whose closure is a framed link with ℓ components, choose for each framed component of $\hat{\beta}$ in β a single strand in β that lies in that component after closure, say, the i_r^{th} strand is chosen for the r^{th} component. Then define the polynomial ring $\Bbbk[x_{i_1},\dots,x_{i_\ell}]$ as a subring of $\Bbbk[x_1,\dots,x_n]$ generated by the chosen variables. Set

$$\Bbbk[x_{i_1},\dots,x_{i_\ell}]^{\mathtt{f}\beta} := \Bbbk[x_{i_1},\dots,x_{i_\ell}]\cdot 1_\beta, \quad \partial(1_\beta) := -\sum_{r=1}^{\ell} 2\mathtt{f}_r x_{i_r} 1_\beta. \tag{4-11}$$

Then we make the twisting of H_q–modules on the $p\mathrm{HH}_\bullet$–level termwise on $p\mathrm{HH}_\bullet(pT_\beta^i)$,

$$p\mathrm{HH}_\bullet^{\mathtt{f}\beta}(pT_\beta) := p\mathrm{HH}_\bullet(pT_\beta)\otimes_{\Bbbk[x_{i_1},\dots,x_{i_\ell}]}\Bbbk[x_{i_1},\dots,x_{i_\ell}]^{\mathtt{f}\beta}. \tag{4-12}$$

Definition 4.8 Given $\beta \in \mathrm{Br}_n$ whose closure is a framed link with ℓ components, the $\mathfrak{sl}_{kp+2}$ *p–homology* is the object

$$p\mathrm{H}(\beta, kp+2) := q^{-n(kp+1)}\mathrm{H}_\bullet^{/}(p\mathrm{HH}_\bullet^{\mathtt{f}\beta}(pT_\beta), \partial_T)$$

in the homotopy category $\mathcal{C}(\Bbbk, \partial_q)$.

As done for pHHH, we will often drop $kp+2$ in the notation of the homology.

Theorem 4.9 *The $\mathfrak{sl}_{kp+2}$ p–homology $p\mathrm{H}(\beta, kp+2)$ is a singly graded, finite-dimensional link invariant depending only on the braid closure of β as a link in $\mathbb{R}^3$. Furthermore, when $k \in 2\mathbb{Z}$, its graded Euler characteristic*

$$\chi(p\mathrm{H}(L, kp+2)) := \sum_i q^i \dim_{\Bbbk}(p\mathrm{H}_i(L, kp+2))$$

is equal to the Jones polynomial evaluated at a $2p^{th}$ root of unity.

Proof The above framing twisting compensates for the linear factors appearing in Markov II moves, thus establishing the topological invariance of $p\mathrm{H}(\beta)$.

For the last statement, we will use the fact that the Euler characteristic does not change before or after taking slash homology. This is because, as with the usual chain complexes, taking slash homology only gets rid of acyclic summands whose Euler characteristics are zero.

Let us revisit diagram (4-4). Before collapsing the t and q–gradings, the diagram arises by p–extending $\mathrm{HH}_\bullet(T_\beta)$ in the vertical (t–)direction. Let $P_\beta(v,t)$ be the Poincaré polynomial of the bigraded complex $\mathrm{HH}_\bullet(T_\beta)$ where, for now, v and t are treated as formal variables coming from q and t grading shifts. As shown by Cautis [1], $P_\beta(v,-1)$ is the $\mathfrak{sl}_{kp+2}$ polynomial of the link $\hat{\beta}$ in the variable v.

The p–extension in the topological direction is equivalent to categorically specializing $[1]_d^t$ to $[1]_\partial^q$. It has the effect, on the Euler characteristic level, of specializing $t=-1$. Thus we obtain that the Euler characteristic of $p\mathrm{H}(\beta)$ is equal to $P_\beta(v=q,t=-1)$. This the evaluation of the $\mathfrak{sl}_{kp+2}$ polynomial evaluated at a $2p^{\text{th}}$ root of unity q. When $k\in 2\mathbb{Z}$, we have $q^{kp+2}=q^2$ in

$$\mathbb{O}_p := K_0(\mathcal{C}(\Bbbk,\partial_q)) \cong \frac{\mathbb{Z}[q]}{(1+q^2+\cdots+q^{2(p-1)})},$$

so this evaluation is equal to the value of the Jones polynomial in $\mathbb{O}_p$. □

5 Examples

In this section we compute the various homologies constructed earlier for $(2,n)$ torus links $T_{2,n}$. Note that there are no framing factors to incorporate in this family of examples. The calculations are straightforward modifications of the computations made in [6] and adjusted for p–DG notions in [11, Section 6]. We refer the reader to [11] and just state the modified results here with minimal explanation.

Throughout the remainder of this subsection, let $R=\Bbbk[x_1,x_2]$, $B=B_1$, and $T=T_1$.

5.1 The HOMFLYPT homology of the $(2,n)$ torus link

First note that the homology of the n–component unlink L_0 is

$$p\mathrm{HHH}^{\partial_q}(L_0,K+1)\cong \bigotimes_{i=1}^{n} q^{-K}(\Bbbk[x_i]\oplus q^{2K+2}[1]_\partial^t \Bbbk[x_i]^{2x_i}).$$

The following simplification of $T^{\otimes n}$ is proved in the same way as [11, Lemma 6.1].

Lemma 5.1 *In $\mathcal{C}^{\partial_q}(R,R,\partial_0)$, one has* $T^{\otimes n} \cong (q^{-K-1}[-1]^t_\partial)^n$

$$\big(q^{2(n-1)}B^{(n-1)e_1}[n]^t_\partial \xrightarrow{p_n} q^{2(n-2)}B^{(n-2)e_1}[n-1]^t_\partial \xrightarrow{p_{n-1}} \cdots \cdots \xrightarrow{p_3} q^2B^{e_1}[2]^t_\partial \xrightarrow{p_2} B[1]^t_\partial \xrightarrow{br} R\big),$$

where

$$p_{2i} = 1\otimes(x_2-x_1)-(x_2-x_1)\otimes 1, \quad p_{2i+1} = 1\otimes(x_2-x_1)+(x_2-x_1)\otimes 1.$$

The following result is proved in the same way as [11, Proposition 6.3]

Proposition 5.2 *The bigraded H_q–HOMFLYPT p–homology of a $(2,n)$ torus knot, as an H_q–module depends on the parity of n.*

(i) *If n is odd, it is*

$$q^{-nK-2n-2K}[-n]^t_\partial(q^{2K+2}[1]^t_\partial\Bbbk[x]^{2x}\oplus q^{4K+4}[2]^t_\partial\Bbbk[x]^{4x})$$
$$\oplus \bigoplus_{i\in\{2,4,\dots,n-1\}} q^{-nK-2n-2K}[i-n]^t_\partial\left(q^{2(i-1)}\Bbbk[x]^{2(i-1)x}\oplus q^{2K}[1]^t_\partial\begin{pmatrix} q^{2i}\Bbbk[x]\\ \oplus \\ q^{2i+2}\Bbbk[x]\end{pmatrix} \oplus q^{2i+4+4K}[2]^t_\partial\Bbbk[x]^{2(i+1)x}\right)$$

with the H_q–structure on the middle object

$$\begin{pmatrix}\Bbbk[x]\\ \oplus \\ \Bbbk[x]\end{pmatrix}$$

given by

$$\begin{pmatrix} 2ix & 0\\ -2 & (2i+2)x\end{pmatrix}.$$

(ii) *If n is even, it is*

$$q^{-nK-2n-2K}[-n]^t_\partial(q^{2K+2}[1]^t_\partial\Bbbk[x]^{2x}\oplus q^{4K+4}[2]^t_\partial\Bbbk[x]^{4x})$$
$$\oplus \bigoplus_{i\in\{2,4,\dots,n-2\}} q^{-nK-2n-2K}[i-n]^t_\partial\begin{pmatrix} q^{2(i-1)}\Bbbk[x]^{2(i-1)x}\\ \oplus \\ q^{2K}[1]^t_\partial\begin{pmatrix} q^{2i}\Bbbk[x]\\ \oplus \\ q^{2i+2}\Bbbk[x]\end{pmatrix}\\ \oplus \\ q^{2i+4+4K}[2]^t_\partial\Bbbk[x]^{2(i+1)x}\end{pmatrix}$$

$$\oplus\, q^{-nK-2n-2K}[-n]^t_\partial \begin{pmatrix} q^{2(n-1)}\Bbbk[x_1,x_2]^{(n-1)(x_1+x_2)} \\ \oplus \\ q^{2K}[1]^t_\partial \begin{pmatrix} q^{2n}\Bbbk[x_1,x_2] \\ \oplus \\ q^{2n+2}\Bbbk[x_1,x_2] \end{pmatrix} \\ \oplus \\ q^{2n+4+4K}[2]^t_\partial \Bbbk[x_1,x_2]^{(n+2)(x_1+x_2)} \end{pmatrix} [n]^t_\partial$$

with the H_q–structure on the middle object

$$\begin{pmatrix} q^{2i}\Bbbk[x] \\ \oplus \\ q^{2i+2}\Bbbk[x] \end{pmatrix}$$

given by

$$\begin{pmatrix} 2ix & 0 \\ -2 & (2i+2)x \end{pmatrix}$$

and the H_q–structure on the middle object

$$\begin{pmatrix} q^{2n}\Bbbk[x_1,x_2] \\ \oplus \\ q^{2n+2}\Bbbk[x_1,x_2] \end{pmatrix}$$

given by

$$\begin{pmatrix} (n+1)x_1+(n-1)x_2 & 0 \\ -2 & n(x_1+x_2)+2x_2 \end{pmatrix}.$$

Corollary 5.3 *In the stable category of H_q–modules, the slash homology of the H_q–HOMFLYPT p–homology of a $(2,n)$ torus link $p\mathrm{HHH}^{\partial_q}(T_{2,n}, K+1)$ depends on the parity of n.*

(i) *If n is odd, it is*

$$q^{-nK-2n-2K}[-n]^t_\partial(q^{p+2K}V^q_{p-2}[1]^t_\partial \oplus q^{p+4K}V^q_{p-4}[2]^t_\partial)$$

$$\oplus \bigoplus_{i\in\{2,4,\dots,n-1\}} q^{-nK-2n-2K}[-n]^t_\partial \left(q^p V^q_{p-2(i-1)} \oplus \begin{pmatrix} q^{p+2K}V^q_{p-2i} \\ \oplus \\ q^{p+2K}V^q_{p-2i-2} \end{pmatrix} [1]^t_\partial \oplus q^{p+2+4K}V^q_{p-2(i+1)}[2]^t_\partial \right)[i]^t_\partial.$$

(ii) *If n is even, it is*

$$
\begin{aligned}
&q^{-nK-2n-2K}[-n]^t_\partial(q^{p+2K}V^q_{p-2}[1]^t_\partial \oplus q^{p+4K}V^q_{p-4}[2]^t_\partial)\\
&\quad\oplus \bigoplus_{i\in\{2,4,\dots,n-2\}} q^{-nK-2n-2K}[-n]^t_\partial\left(q^pV^q_{p-2(i-1)}\right.\\
&\qquad\qquad\left.\oplus\begin{pmatrix} q^{p+2K}V^q_{p-2i}\\ \oplus\\ q^{p+2K}V^q_{p-2(i+1)}\end{pmatrix}[1]^t_\partial \oplus q^{p+2+4K}V^q_{p-2(i+1)}[2]^t_\partial\right)[i]^t_\partial\\
&\quad\oplus q^{-nK-2n-2K}[-n]^t_\partial\begin{pmatrix} q^{2p}V^q_{p-(n-1)}\otimes V^q_{p-(n-1)}\\ \oplus\\ (q^{2p+2K}V^q_{p-n-1}\otimes V^q_{p-n+1}\oplus q^{2p+2K}V^q_{p-n}\otimes V^q_{p-n-2})[1]^t_\partial\\ \oplus\\ q^{2p+4K}V^q_{p-(n+2)}\otimes V^q_{p-(n+2)}[2]^t_\partial\end{pmatrix}[n]^t_\partial.
\end{aligned}
$$

5.2 The $\mathfrak{sl}_{kp+2}$–homology of the $(2,n)$ torus link

To compute this homology, we will use the following tool. If

$$M_\bullet = \left(\cdots \xrightarrow{\partial_t} M_{i+1} \xrightarrow{\partial_t} M_i \xrightarrow{\partial_t} M_{i-1} \xrightarrow{\partial_t} \cdots\right),$$

we write $\mathcal{T}(M_\bullet)$ to be the total complex whose p–differential is the sum $\partial_T := \partial_t + \partial_q$.

Proposition 5.4 [11, Proposition 6.6] *Let $M_\bullet$ be a contractible p–complex of $H_q=\Bbbk[\partial_q]/(\partial_q^p)$–modules. Then the complex $(\mathcal{T}(M_\bullet), \partial_T = \partial_t + \partial_q)$ is acyclic.*

We will be applying Proposition 5.4 in the following situation. Suppose $N_\bullet$ is a p–complex of H_q–modules whose boundary maps preserve the H_q–module structure. Further, let $M_\bullet$ be a sub–p–complex that is closed under the H_q–action, and there is a map σ on $M_\bullet$ as in Proposition 5.4 that preserves the H_q–module structure. Then, when totalizing the p–complexes, we have $\mathcal{T}(M_\bullet) \subset \mathcal{T}(N_\bullet)$ and the natural projection map

$$\mathcal{T}(N_\bullet) \to \mathcal{T}(N_\bullet)/\mathcal{T}(M_\bullet)$$

is a quasi-isomorphism. Similarly, if $M_\bullet$ is instead a quotient complex of $N_\bullet$ that satisfies the condition of Proposition 5.4, and $K_\bullet$ is the kernel of the natural projection map

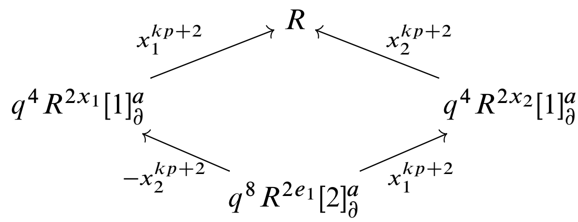
$$0 \to K_\bullet \to N_\bullet \to M_\bullet \to 0,$$

then the inclusion map of totalized complexes $\mathcal{T}(K_\bullet) \to \mathcal{T}(N_\bullet)$ is a quasi-isomorphism.

We modify the calculation in the previous section of the $(2,n)$ torus link to include the Cautis p–differential ∂_C. Recall that in this singly graded theory, $a = q^{2kp+2}[1]^q_\partial$ and $t = [1]^q_\partial$.

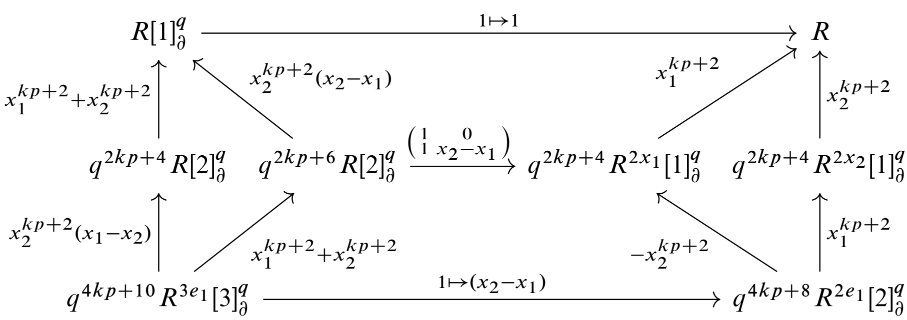
The Hochschild homology $p\mathrm{HH}^{\partial_q}_\bullet(R)$ with the induced Cautis differential ∂_C is given by

(5-1)



First we study $p\mathrm{HH}^{\partial_q}_\bullet(br)\colon p\mathrm{HH}^{\partial_q}_\bullet(B)[1]^q_\partial \to p\mathrm{HH}^{\partial_q}_\bullet(R)$,

(5-2)



where the object $q^{2kp+4}R[2]^q_\partial \oplus q^{2kp+6}R[2]^q_\partial$ in the left square is twisted by the matrix

$$\begin{pmatrix} 2x_1 & 0 \\ 2 & x_1+3x_2 \end{pmatrix}. \tag{5-3}$$

Filtering the total complex (5-2) we obtain that it is quasi-isomorphic to

$$\Bbbk\langle x_1^a x_2^b \mid 0 \le a \le kp+2,\, 0 \le b \le kp+1\rangle[1]^q_\partial \xrightarrow{1} \Bbbk\langle x_1^a x_2^b \mid 0 \le a,b \le kp+1\rangle,$$

which is quasi-isomorphic to

$$\Bbbk\langle x_1^{kp+2}, x_1^{kp+2}x_2, \dots, x_1^{kp+2}x_2^{kp+1}\rangle[1]^q_\partial,$$

where

$$\partial(x_1^{kp+2}x_2^j) = (kp+2+j)x_1^{kp+2}x_2^{j+1}.$$

This is quasi-isomorphic to $q^5 V_1[1]^q_\partial$ if $k=0$. If $k>0$, it's quasi-isomorphic to

$$(q^{3p+2}V_{p-2} \oplus q^{4kp+4}V_2)[1]^q_\partial.$$

Next we analyze

$$p\mathrm{HH}^{\partial_q}_\bullet(p_{2i+1})\colon p\mathrm{HH}^{\partial_q}_\bullet(q^{4i}B^{2ie_1}[2i+1]^q_\partial)\to \mathrm{HH}^{\partial_q}_\bullet(q^{4i-2}B^{(2i-1)e_1}[2i]^q_\partial),$$

$$p\mathrm{HH}^{\partial_q}_\bullet(q^{4i}B^{2ie_1}[2i+1]^q_\partial)=\begin{array}{ccccc} & & q^{4i}R^{2ie_1}[2i+1]^q_\partial & & \\ & \overset{x_1^{kp+2}+x_2^{kp+2}}{\nearrow} & & \overset{x_2^{kp+2}(x_2-x_1)}{\nwarrow} & \\ q^{4i+2kp+4}R[2i+1]^q_\partial[1]^q_\partial & & & & q^{4i+2kp+6}R[2i+1]^q_\partial[1]^q_\partial \\ & \underset{x_2^{kp+2}(x_1-x_2)}{\nwarrow} & & \underset{x_1^{kp+2}+x_2^{kp+2}}{\nearrow} & \\ & & q^{4i+4kp+10}R^{(2i+3)e_1}[2i+1]^q_\partial[2]^q_\partial & & \end{array}$$

$$p\mathrm{HH}^{\partial_q}_\bullet(q^{4i-2}B^{(2i-1)e_1}[2i]^q_\partial)=\begin{array}{ccccc} & & q^{4i-2}R^{(2i-1)e_1}[2i]^q_\partial & & \\ & \overset{x_1^{kp+2}+x_2^{kp+2}}{\nearrow} & & \overset{x_2^{kp+2}(x_2-x_1)}{\nwarrow} & \\ q^{4i+2kp+2}R[2i]^q_\partial[1]^q_\partial & & & & q^{4i+2kp+4}R[2i]^q_\partial[1]^q_\partial \\ & \underset{x_2^{kp+2}(x_2-x_1)}{\nwarrow} & & \underset{x_1^{kp+2}+x_2^{kp+2}}{\nearrow} & \\ & & q^{4i+4kp+8}R^{(2i+2)e_1}[2i]^q_\partial[2]^q_\partial & & \end{array}$$

where the differentials for both objects in the middle horizontal rows of the diagrams above are twisted by (5-3) and $p\mathrm{HH}^{\partial_q}_\bullet(p_{2i+1})=2(x_2-x_1)$ (diagonal multiplication by $2(x_2-x_1)$). Filtering this total complex yields the total complex

$$\begin{array}{c} q^{4i}\Bbbk\langle x_1^a x_2^b \mid 0\le a\le kp+2,\ 0\le b\le kp+1\rangle[2i+1]^q_\partial \\ \big\downarrow{\scriptstyle 2(x_2-x_1)} \\ q^{4i-2}\Bbbk\langle x_1^a x_2^b \mid 0\le a\le kp+2,\ 0\le b\le kp+1\rangle[2i]^q_\partial \end{array} \tag{5-4}$$

This is quasi-isomorphic to

$$q^{4i}\Bbbk\langle x_1^{kp+2}, x_1^{kp+2}x_2,\dots,x_1^{kp+2}x_2^{kp+1}\rangle[2i+1]^q_\partial\oplus q^{4i-2}\Bbbk\langle 1,x_1,\dots,x_1^{kp+1}\rangle[2i]^q_\partial$$

where the differential on the basis elements is given by

$$\begin{array}{ccc} x_1^{kp+2} & & 1 \\ {\scriptstyle kp+4i+2}\big\downarrow & & \big\downarrow{\scriptstyle 4i-2} \\ x_1^{kp+2}x_2 & & x_1 \\ {\scriptstyle kp+4i+3}\big\downarrow & \oplus & \big\downarrow{\scriptstyle 4i-1} \\ \vdots & & \vdots \\ {\scriptstyle kp+4i+kp+2}\big\downarrow & & \big\downarrow{\scriptstyle 4i-2+kp} \\ x_1^{kp+2}x_2^{kp+1} & & x_1^{kp+1} \end{array}$$

Thus the total homology of this complex is isomorphic to X_i, which is defined by

$$
(5\text{-}5)\qquad
\begin{cases}
\begin{pmatrix}
q^{4i}(q^{2(kp+2)+j}V_j \oplus q^{2(kp+2+j+1+(k-1)p)+p-j}V_{p-j})[2i+1]^q_\partial \\
\oplus \\
q^{4i-2}(q^{\bar{j}}V_{\bar{j}} \oplus q^{2((k-1)p+\bar{j}+1)+p-\bar{j}}V_{p-\bar{j}})[2i]^q_\partial
\end{pmatrix} & \text{if } j,\bar{j}\neq 0, \\
\begin{pmatrix}
q^{4i}(q^{2(kp+2)}V_0 \oplus q^{2(kp+2+kp+1)}V_0)[2i+1]^q_\partial \\
\oplus \\
q^{4i-2}(q^{\bar{j}}V_{\bar{j}} \oplus q^{2((k-1)p+\bar{j}+1)+p-\bar{j}}V_{p-\bar{j}})[2i]^q_\partial
\end{pmatrix} & \text{if } j=0,\ \bar{j}\neq 0, \\
\begin{pmatrix}
q^{4i}(q^{2(kp+2)+j}V_j \oplus q^{2(kp+2+j+1+(k-1)p)+p-j}V_{p-j})[2i+1]^q_\partial \\
\oplus \\
q^{4i-2}(V_0 \oplus q^{2(kp+1)}V_0)[2i]^q_\partial
\end{pmatrix} & \text{if } j\neq 0,\ \bar{j}=0,
\end{cases}
$$

where $j \in \{0,\dots,p\}$ such that $4i+2+j$ is divisible by p and $\bar{j} \in \{0,\dots,p\}$ such that $4i-2+\bar{j}$ is divisible by p.

Once again when n is even, the leftmost term in $T^{\otimes n}$ maps by zero into the rest of the complex so we have to understand the total homology of $p\mathrm{HH}_\bullet(q^{2(n-1)}B^{(n-1)e_1}[n]^q_\partial)$. Filtering

$$
\begin{array}{ccccc}
 & & q^{2(n-1)}R^{(n-1)e_1}[n]^q_\partial & & \\
 & \nearrow{\scriptstyle x_1^{kp+2}+x_2^{kp+2}} & & \nwarrow{\scriptstyle x_2^{kp+2}(x_2-x_1)} & \\
q^{2(n+1)+2kp}R^{(n-1)e_1}[n]^q_\partial[1]^q_\partial & & & & q^{2(n+2)+2kp}R^{(n-1)e_1}[n]^q_\partial[1]^q_\partial \\
 & \nwarrow{\scriptstyle x_2^{kp+2}(x_1-x_2)} & & \nearrow{\scriptstyle x_1^{kp+2}+x_2^{kp+2}} & \\
 & & q^{2(n+4)+4kp}R^{(n+2)e_1}[n]^q_\partial[2]^q_\partial & &
\end{array}
$$

where the middle terms $q^{2(n+1)+2kp}R^{(n-1)e_1}[n]^q_\partial[1]^q_\partial \oplus q^{2(n+2)+2kp}R^{(n-1)e_1}[n]^q_\partial[1]^q_\partial$ are further twisted by the matrix (5-3), yields that the diagram above is quasi-isomorphic to

$$
(5\text{-}6)\qquad Y_{\frac{n}{2}} = q^{2(n-1)}\Bbbk\langle x_1^a x_2^b \mid 0 \le a \le kp+2,\ 0 \le b \le kp+1\rangle[n]^q_\partial
$$

with a differential inherited from the polynomial algebra and twisted by $(n-1)e_1$. All of these computations together with an overall shift of $q^{-(n+2)kp-3n-2}[-n]^q_\partial$ yields the slash homology of the $(2,n)$ torus link for $k>0$,

$$
(5\text{-}7)\quad p\mathrm{H}(T_{2,n}, kp+2) \cong
\begin{cases}
q^{-(n+2)kp-3n-2}[-n]^q_\partial\big((q^{3p+2}V_{p-2}\oplus q^{4kp+4}V_2)[1]^q_\partial \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} X_i\big) & \text{if } 2\nmid n, \\
q^{-(n+2)kp-3n-2}[-n]^q_\partial\big((q^{3p+2}V_{p-2}\oplus q^{4kp+4}V_2)[1]^q_\partial \oplus \bigoplus_{i=1}^{\frac{n-2}{2}} X_i \oplus \mathrm{H}^{/}_\bullet(Y_{\frac{n}{2}})\big) & \text{if } 2\mid n,
\end{cases}
$$

where X_i is the p–complex in (5-5) and $Y_{\frac{n}{2}}$ is the p–complex in (5-6).

$$\begin{array}{cccccccccccccc}
1 & \xrightarrow{1} & x_2 & \xrightarrow{2} & x_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_2^{p-1} & \xrightarrow{0} & x_2^p & \xrightarrow{1} & x_2^{p+1} & \xrightarrow{-2} \\
\big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & \\
x_1 & \xrightarrow{1} & x_1x_2 & \xrightarrow{2} & x_1x_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_1x_2^{p-1} & \xrightarrow{0} & x_1x_2^p & \xrightarrow{1} & x_1x_2^{p+1} & \xrightarrow{-2} \\
\big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} & & & & \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\big\downarrow{\scriptstyle p-1} & & \big\downarrow{\scriptstyle p-1} & & \big\downarrow{\scriptstyle p-1} & & & & \big\downarrow{\scriptstyle p-1} & & \big\downarrow{\scriptstyle p-1} & & \big\downarrow{\scriptstyle p-1} & \\
x_1^{p-1} & \xrightarrow{1} & x_1^{p-1}x_2 & \xrightarrow{2} & x_1^{p-1}x_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_1^{p-1}x_2^{p-1} & \xrightarrow{0} & x_1^{p-1}x_2^p & \xrightarrow{1} & x_1^{p-1}x_2^{p+1} & \xrightarrow{-2} \\
\big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & & & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & & \big\downarrow{\scriptstyle 0} & \\
x_1^p & \xrightarrow{1} & x_1^px_2 & \xrightarrow{2} & x_1^px_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_1^px_2^{p-1} & \xrightarrow{0} & x_1^px_2^p & \xrightarrow{1} & x_1^px_2^{p+1} & \xrightarrow{-2} \\
\big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle 1} & \\
x_1^{p+1} & \xrightarrow{1} & x_1^{p+1}x_2 & \xrightarrow{2} & x_1^{p+1}x_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_1^{p+1}x_2^{p-1} & \xrightarrow{0} & x_1^{p+1}x_2^p & \xrightarrow{1} & x_1^{p+1}x_2^{p+1} & \xrightarrow{-2} \\
\big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} & & & & \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle 2} & \\
x_1^{p+2} & \xrightarrow{4} & x_1^{p+2}x_2 & \xrightarrow{5} & x_1^{p+2}x_2^2 & \xrightarrow{6} & \cdots & \xrightarrow{p+2} & x_1^{p+2}x_2^{p-1} & \xrightarrow{p+3} & x_1^{p+2}x_2^p & \xrightarrow{p+4} & x_1^{p+2}x_2^{p+1} &
\end{array}$$

Figure 4

Decategorifying the slash homology, for instance on the Hopf link ($n=2$), we obtain that the Euler characteristic of $p\mathrm{H}(T_{2,2}, kp+2)$ is equal to

$$q^{-8}(q^2+q^4+q^6+q^8).$$

Finding the homology of $Y_{\frac{n}{2}}$ is nontrivial. In the example below we take $n=2$ which means we are computing part of the homology for the Hopf link. We also take $k=1$ just for convenience of notation.

We thus need to compute the homology of Z_1, given in Figure 4, where the arrows labeled -2 mean that the differential acts by $x_1^j x_2^{p+1} \mapsto -2x_1^{p+2}x_2^j$.

There is a large contractible summand Z_2 in the upper-left corner. Then there is short exact sequence of complexes

$$Z_2 \to Z_1 \to Z_3$$

where Z_3 is as in Figure 5. The second row from the bottom with the rightmost column, along with the third row from the bottom and second column from the right give a contractible summand Z_4 of Z_3:

$$Z_4 = \Bbbk\langle x_1^{p+1}+x_2^{p+1}, \dots, x_1^{p+1}x_2^{p-1}+x_1^{p-1}x_2^{p+1}\rangle \oplus \Bbbk\langle x_1^p+x_2^p, \dots, x_1^p x_2^{p-1}+x_1^{p-1}x_2^p\rangle.$$

$$
\begin{array}{cccccccccccccccc}
 & & & & & & & & & & x_2^p & \xrightarrow{1} & x_2^{p+1} & \xrightarrow{-2} \\
 & & & & & & & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & \\
 & & & & & & & & & & x_1x_2^p & \xrightarrow{1} & x_1x_2^{p+1} & \xrightarrow{-2} \\
 & & & & & & & & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & \\
 & & & & & & & & & & \vdots & & \vdots & \\
 & & & & & & & & & & \downarrow{\scriptstyle p-1} & & \downarrow{\scriptstyle p-1} & \\
 & & & & & & & & & & x_1^{p-1}x_2^p & \xrightarrow{1} & x_1^{p-1}x_2^{p+1} & \xrightarrow{-2} \\
 & & & & & & & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & \\
x_1^p & \xrightarrow{1} & x_1^px_2 & \xrightarrow{2} & x_1^px_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_1^px_2^{p-1} & \xrightarrow{0} & x_1^px_2^p & \xrightarrow{1} & x_1^px_2^{p+1} & \xrightarrow{-2} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & \\
x_1^{p+1} & \xrightarrow{1} & x_1^{p+1}x_2 & \xrightarrow{2} & x_1^{p+1}x_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_1^{p+1}x_2^{p-1} & \xrightarrow{0} & x_1^{p+1}x_2^p & \xrightarrow{1} & x_1^{p+1}x_2^{p+1} & \xrightarrow{-2} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & \\
x_1^{p+2} & \xrightarrow{4} & x_1^{p+2}x_2 & \xrightarrow{5} & x_1^{p+2}x_2^2 & \xrightarrow{6} & \cdots & \xrightarrow{p+2} & x_1^{p+2}x_2^{p-1} & \xrightarrow{p+3} & x_1^{p+2}x_2^p & \xrightarrow{p+4} & x_1^{p+2}x_2^{p+1} &
\end{array}
$$

Figure 5

Then there is a short exact sequence of complexes

$$Z_4 \to Z_3 \to Z_5,$$

where Z_5 is

$$
\begin{array}{cccccccccccccccc}
x_1^p & \xrightarrow{1} & x_1^px_2 & \xrightarrow{2} & x_1^px_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_1^px_2^{p-1} & \xrightarrow{0} & x_1^px_2^p & \xrightarrow{1} & x_1^px_2^{p+1} & \xrightarrow{-2} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & \\
x_1^{p+1} & \xrightarrow{1} & x_1^{p+1}x_2 & \xrightarrow{2} & x_1^{p+1}x_2^2 & \xrightarrow{3} & \cdots & \xrightarrow{p-1} & x_1^{p+1}x_2^{p-1} & \xrightarrow{0} & x_1^{p+1}x_2^p & \xrightarrow{1} & x_1^{p+1}x_2^{p+1} & \xrightarrow{-2} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & \\
x_1^{p+2} & \xrightarrow{4} & x_1^{p+2}x_2 & \xrightarrow{5} & x_1^{p+2}x_2^2 & \xrightarrow{6} & \cdots & \xrightarrow{p+2} & x_1^{p+2}x_2^{p-1} & \xrightarrow{p+3} & x_1^{p+2}x_2^p & \xrightarrow{p+4} & x_1^{p+2}x_2^{p+1} &
\end{array}
$$

Now let Z_6 be the contractible subcomplex of Z_5 generated by x_1^{p+1}. That is

$$Z_6 = \Bbbk\langle x_1^{p+1}, 1!x_1^{p+1}x_2 + a_0x_2^{p+1}, \dots, (p-1)!x_1^{p+1}x_2^{p-1} + a_{p-2}x_1^{p+2}x_2^{p-2}\rangle$$

for some coefficients $a_0, \dots, a_{p-2}$. Then there is a short exact sequence of complexes

$$Z_6 \to Z_5 \to Z_7$$

where Z_7 is

$$\begin{array}{ccccccccccccc}
x_1^p & \xrightarrow{1} & x_1^p x_2 & \xrightarrow{2} & x_1^p x_2^2 & \xrightarrow{3} \cdots \xrightarrow{p-1} & x_1^p x_2^{p-1} & \xrightarrow{0} & x_1^p x_2^p & \xrightarrow{1} & x_1^p x_2^{p+1} & \xrightarrow{-2} \\
 & & & & & & & & \Big\downarrow 1 & & \Big\downarrow 1 & \\
 & & & & & & & & x_1^{p+1} x_2^p & \xrightarrow{1} & x_1^{p+1} x_2^{p+1} & \xrightarrow{-2} \\
 & & & & & & & & \Big\downarrow 2 & & \Big\downarrow 2 & \\
x_1^{p+2} & \xrightarrow{4} & x_1^{p+2} x_2 & \xrightarrow{5} & x_1^{p+2} x_2^2 & \xrightarrow{6} \cdots \xrightarrow{p+2} & x_1^{p+2} x_2^{p-1} & \xrightarrow{p+3} & x_1^{p+2} x_2^p & \xrightarrow{p+4} & x_1^{p+2} x_2^{p+1} &
\end{array}$$

Consider the contractible summand

$$Z_8 = \Bbbk\langle x_1^p, \dots, x_1^p x_2^{p-1}\rangle.$$

Then there is a short exact sequence

$$Z_8 \to Z_7 \to Z_9,$$

where Z_9 is

$$\begin{array}{ccccccccccccc}
 & & & & & & & & x_1^p x_2^p & \xrightarrow{1} & x_1^p x_2^{p+1} & \xrightarrow{-2} \\
 & & & & & & & & \Big\downarrow 1 & & \Big\downarrow 1 & \\
 & & & & & & & & x_1^{p+1} x_2^p & \xrightarrow{1} & x_1^{p+1} x_2^{p+1} & \xrightarrow{-2} \\
 & & & & & & & & \Big\downarrow 2 & & \Big\downarrow 2 & \\
x_1^{p+2} & \xrightarrow{4} & x_1^{p+2} x_2 & \xrightarrow{5} & x_1^{p+2} x_2^2 & \xrightarrow{6} \cdots \xrightarrow{p+2} & x_1^{p+2} x_2^{p-1} & \xrightarrow{p+3} & x_1^{p+2} x_2^p & \xrightarrow{p+4} & x_1^{p+2} x_2^{p+1} &
\end{array}$$

We now easily decompose Z_9 into a sum of complexes

$$Z_9' \oplus Z_9'' \oplus Z_9''' \oplus Z_9'''',$$

where Z_9' comes from the bottom row. More specifically,

$$\begin{aligned}
Z_9' &= \Bbbk\langle x_1^{p+2}, x_1^{p+2}x_2, \dots, x_1^{p+2}x_2^{p-4}\rangle, \\
Z_9'' &= \Bbbk\langle x_1^{p+2}x_2^{p-3}, x_1^{p+2}x_2^{p-2}, x_1^{p+2}x_2^{p-1}, x_1^{p+2}x_2^{p}, x_1^{p+2}x_2^{p+1}\rangle, \\
Z_9''' &= \Bbbk\big\langle x_1^p x_2^p - \tfrac{1}{2}x_1^{p+2}x_2^{p-2}, x_1^{p+1}x_2^p + x_1^p x_2^{p+1} - x_1^{p+2}x_2^{p-1}, -x_1^{p+2}x_2^p + 2x_1^{p+1}x_2^{p+1}\big\rangle, \\
Z_9'''' &= \Bbbk\langle 2x_1^{p+2}x_2^{p-1} - 3x_1^{p+1}x_2^p + 3x_1^p x_2^{p+1}\rangle.
\end{aligned}$$

Thus for $k = 1$ and $n = 2$ we get

$$\mathrm{H}_\bullet^{/}(Y_{\frac{2}{2}}) \cong q^2[2]_\partial^q (q^{3p}V_{p-4} \oplus q^{4p+1}V_3 \oplus q^{4p+2}V_2 \oplus q^{4p+2}V_0).$$

Corollary 5.5 *For distinct $k \in \mathbb{N}$, $p\mathrm{H}(\,\cdot\,, kp+2)$ are distinct as link homology theories.*

Proof If we repeat the above calculation for $k = 2$ and $n = 2$, everything would proceed in the same way. Other than internal q–grading shifts, the homology $\mathrm{H}^{/}_{\bullet}(Y_{\frac{2}{2}})$ would be the same as above and contain objects V_{p-4}, V_3, V_2 and V_0.

The homology of the Hopf link in [11] does not contain objects of the form V_{p-4} or V_3 (see [11, (6.17)], in particular) in this tail part of the calculation. For more general k, these objects appear with different shifts; see (5-7). Thus we obtain here new categorifications of the Jones polynomial at a $2p^{\text{th}}$ root of unity different from the original one constructed in [11]. □

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