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Differential geometric invariants for time-reversal symmetric Bloch bundles II: The low-dimensional "quaternionic" case

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This paper is devoted to the construction of differential geometric invariants for the classification of "quaternionic" vector bundles. Provided that the base space is a smooth manifold of dimension two or three endowed with an involution that leaves fixed only a finite number of points, it is possible to prove that the *Wess–Zumino term* and the *Chern–Simons invariant* yield topological invariants able to distinguish between inequivalent realizations of "quaternionic" structures. This is a nontrivial generalization of results previously known only in the case of tori with time-reversal involution.

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1 Introduction

The present paper continues the study of the classification of *"quaternionic" vector bundles* started in [8; 10; 11]. The main novelty with respect to the previous papers consists of the use of differential geometric invariants to classify inequivalent isomorphism classes of "quaternionic" structures. In this sense, as expressed by the title, this paper represents a continuation of [9] where differential geometric techniques have been used to classify "real" vector bundles.

At a topological level, "quaternionic" vector bundles, or *Q–bundles* for short, are complex vector bundles defined over spaces with involution and endowed with a further structure at the level of the total space. An involution τ on a topological space X is a homeomorphism of period 2, ie $\tau^2 = \mathrm{Id}_X$. The pair (X,τ) will be called an involutive space. The *fixed point* set of the involutive space (X,τ) is by definition

$$X^\tau := \{x \in X \mid \tau(x) = x\}.$$

A Q–bundle over (X, τ) is a pair $(\mathscr{E}, \Theta)$, where $\mathscr{E} \to X$ denotes the underlying complex vector bundle and $\Theta\colon \mathscr{E} \to \mathscr{E}$ is an *antilinear* map which covers the action of τ on the base space and such that Θ^2 acts fiberwise as multiplication by -1. A more precise description is given in Definition 2.2. Q–bundles were introduced for the first time by J L Dupont in [12] (under the name of symplectic vector bundles). They form a category of topological objects which is significantly different from the category of complex vector bundles. For this reason the problem of the classification of Q–bundles over a given involutive space requires the use of tools which are structurally different from those typically used in the classification of complex vector bundles. The aim of the present work is to define differential geometric invariants able to distinguish the elements of $\mathrm{Vec}^m_Q(X, \tau)$, where the latter symbol denotes the set of isomorphism classes of rank m Q–bundles over (X, τ).

The interest for the classification of Q–bundles has increased in the last years because of the connection with the study of *topological insulators*. Although this work does not focus on the theory of topological insulators — the interested reader is referred to the recent reviews by Ando and Fu [2] and Hasan and Kane [25] — it is worth mentioning that the first example of topological effects in condensed matter related to a "quaternionic" structure dates back to the seminal works by L Fu, C L Kane and E J Mele [18; 31]. The existence of distinguished topological phases for the so-called Kane–Mele model is the result of the simultaneous presence of two symmetries. The first symmetry is given by the invariance of the system under spatial translations. This fact allows the use of the *Bloch–Floquet theory* — see Kuchment [36] — for the analysis of the spectral properties of the system. As a result, a well-established procedure provides the construction of a vector bundle, usually known as *Bloch bundle*, from each gapped energy band of the system. Even though the details of the construction of the Bloch bundle will be omitted in this work — the interested reader is referred to Panati [42] or the authors [7, Section 2] — it is important to remark that the Bloch bundle is a complex vector bundle over the torus $\mathbb{T}^d \simeq \mathbb{R}^d/(2\pi\mathbb{Z})^d$. The integer d represents the dimensionality of the system and the physically relevant dimensions are $d = 2, 3$. The second crucial ingredient for the topology of the Kane–Mele model is the fermionic (or odd) *time-reversal symmetry* (TRS). In terms of the Bloch bundle the TRS translates into the involution $\tau_{\mathrm{TR}}\colon \mathbb{T}^d \to \mathbb{T}^d$ of the base space given by

$$\tau_{\mathrm{TR}}(k_1, \dots, k_d) := (-k_1, \dots, -k_d)$$

and into an antilinear map Θ of the total space such that $\Theta^2 = -1$ fiberwise. Therefore, one concludes that the different topological phases of the Kane–Mele model are labeled by the inequivalent realization of Q–bundles over the torus $\mathbb{T}^d$ with involution τ_{TR}, namely by the distinct elements of $\mathrm{Vec}^m_Q(\mathbb{T}^d, \tau_{\mathrm{TR}})$.

The classification of the topological phases of the Kane–Mele model given in [18; 31] is summarized by

$$\mathrm{Vec}^m_Q(\mathbb{T}^d, \tau_{\mathrm{TR}}) = \begin{cases} \mathbb{Z}_2 & \text{if } d = 2, \\ \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^3 & \text{if } d = 3, \end{cases} \tag{1-1}$$

where $\mathbb{Z}_2 := \{\pm 1\}$ is the cyclic group of order 2 presented in multiplicative notation. The topological classification (1-1) has been rigorously derived with the use of different techniques in various papers — see eg [8], Fiorenza, Monaco and Panati [14], and Graf and Porta [24] — and generalized to any (low-dimensional) involutive space (X, τ) by Lawson, Lima-Filho, Michelsohn and dos Santos [37; 45] and in [10; 11], independently. However, the topological classification based on the construction of homotopy invariants (such as characteristic classes) has the disadvantage of being difficult to compute. For this reason one is naturally inclined to look for different types of invariants.

A special role in the classification of complex vector bundles is played by the *Chern classes*. The latter, in view of the Chern–Weil homomorphism, can be represented via differential forms and integrated over suitable cocycles. The resulting *Chern numbers* are enough to provide a complete classification of complex vector bundles in several situations of interest in condensed matter. This is, for instance, the case of the *quantum Hall effect* and the related *TKNN numbers*; see Thouless, Kohmoto, Nightingale and den Nijs [46]. Using this observation as Ariadne's thread, one expects to find differential and integral invariants able to classify Q–bundles at least under some reasonable hypotheses. Indeed, "gauge-theoretic invariants" have already been used to reproduce the classification (1-1). The first pioneering works in this direction are Essin, Moore and Vanderbilt [13], Fu and Kane [17], and Qi, Hughes, Wang and Zhang [44; 47], where the Chern–Simons field theory has been used to relate the topological phases of the Kane–Mele model in $2 + 1$ and $3 + 1$ space-time dimensions with integral quantities like the (time-reversal) *polarization*. Afterwards, these results have been revisited and put in a solid mathematical background in various works like Carpentier, Delplace, Fruchart, Gawędzki, Monaco and Tauber [6; 5; 21; 22; 41], Freed and Moore [16], and Kaufmann, Li and Wehefritz-Kaufmann [32], just to mention some of them. If one ignores the differences due to the use of distinct mathematical techniques, it is possible to recognize a common outcome from all the papers listed above: the topological phases of the two-dimensional Kane–Mele model are governed by the *Wess–Zumino term* [15; 20; 21] while in the three-dimensional case the relevant object is the *Chern–Simons invariant* [15; 21; 28]. The present work is inspired by the latter consideration and it aims to provide a general and rigorous description of the relation between the classification of Q–bundle and the Wess–Zumino term, or the Chern–Simons invariant. The main achievements are presented below.

The two-dimensional case will be described first. In this case the relevant family of base spaces is restricted by the following:

Definition 1.1 (oriented two-dimensional FKMM–manifold) An *oriented two-dimensional FKMM–manifold* is an involutive space (Σ, τ) subject to the following conditions:

(a′) Σ is an oriented two-dimensional compact Hausdorff manifold without boundary.

(b′) The involution τ preserves the manifold structure and the orientation.

(c′) The fixed point set $\Sigma^\tau \neq \varnothing$ consists of a finite collection of points.

Let us point out that *manifold structure* in (b′) shall be eventually assumed to be a smooth one as is stated at the beginning of Section 3. An example of oriented two-dimensional FKMM–manifold is provided by the torus $\mathbb{T}^2$ with the involution τ_{TR}. The set of oriented two-dimensional FKMM–manifolds forms a subclass of the *FKMM–spaces* defined in Definition 2.8 below. Q–bundles over these spaces are completely classified by a characteristic class called *FKMM–invariant*; see Theorem 2.9.

The crucial result for the classification of Q–bundles over two-dimensional FKMM–manifolds is expressed by the chain of isomorphisms

$$\mathrm{Vec}_Q^{2m}(\Sigma, \tau) \overset{\imath_1}{\simeq} [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} / [\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \overset{\imath_2}{\simeq} \mathbb{Z}_2. \tag{1-2}$$

The first isomorphism $\imath_1$ is essentially proved in Theorem 2.13 for $m = 1$ and justified in Remark 2.16 for every $m \in \mathbb{N}$. Elements of $[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ are $\mathbb{Z}_2$–homotopy classes of $\mathbb{Z}_2$–equivariant maps $\xi\colon \Sigma \to \mathbb{SU}(2)$ constrained by the *equivariance* condition $\xi(\tau(x)) = \xi(x)^{-1}$ for all $x \in \Sigma$. The set $[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$ consists of $\mathbb{Z}_2$–homotopy classes of $\mathbb{Z}_2$–equivariant maps $\phi\colon X \to \mathbb{U}(1)$ such that $\phi(\tau(x)) = \overline{\phi(x)} = \phi(x)^{-1}$. The action of $[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$ over $[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ is specified in the statement of Theorem 2.13. The second isomorphism $\imath_2$ is described in Section 2.7 and is given by the composition of two identifications: The first isomorphism,

$$[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} / [\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \overset{\Phi_\kappa}{\simeq} \mathrm{Map}(\Sigma^\tau, \{\pm 1\}) / [\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2},$$

proved in Proposition 2.18, shows that the "new" description of Q–bundles in terms of maps $\xi\colon\Sigma\to\mathbb{SU}(2)$ agrees with the "old" description in terms of the FKMM–invariant given in Proposition 2.10. The second identification,

$$\mathrm{Map}(\Sigma^\tau,\{\pm1\})/[\Sigma,\mathbb{U}(1)]_{\mathbb{Z}_2}\overset{\Pi}{\simeq}\mathbb{Z}_2,$$

is described in Theorem 2.11 and it is induced by the *product sign* map (also known as the *Fu–Kane–Mele index*).

The isomorphism $\imath_1$ in (1-2) expresses the fact that an element of $\mathrm{Vec}^{2m}_Q(\Sigma,\tau)$ can be completely identified with an equivariant map $\xi\colon\Sigma\to\mathbb{SU}(2)$ that, in many situations, can be built explicitly; see Remark 2.19. Therefore, the relevant question is whether there is a way to access directly the isomorphism $\imath_2$ from the knowledge of the classifying map ξ without passing through the FKMM–invariant and the product sign map. The answer is positive. First of all it is important to point out that, without loss of generality, the map ξ can be chosen smooth. This allows us to define the *Wess–Zumino term*

$$\mathcal{WZ}_\Sigma(\xi):=-\frac{1}{24\pi^2}\int_{X_\Sigma}\mathrm{Tr}(\tilde{\xi}^{-1}\cdot\mathrm{d}\tilde{\xi})^3\mod\mathbb{Z},\tag{1-3}$$

where X_Σ is any compact three-dimensional oriented manifold whose boundary coincides with Σ and $\tilde{\xi}\colon X_\Sigma\to\mathbb{SU}(2)$ is any smooth extension of ξ; see Definition 3.16 for more details. The first main result of this paper is:

Theorem 1.2 *Let (Σ,τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Let $(\mathscr{E},\Theta)$ be a Q–bundle of rank $2m$ over (Σ,τ) and $\xi\in\mathrm{Map}(\Sigma,\mathbb{SU}(2))_{\mathbb{Z}_2}$ any map which represents $(\mathscr{E},\Theta)$ in the sense of the isomorphism $\imath_1$ in (1-2). Then the map*

$$\mathrm{Vec}^{2m}_Q(\Sigma,\tau)\ni[(\mathscr{E},\Theta)]\to\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_\Sigma(\xi)}\in\mathbb{Z}_2$$

provides a realization of the isomorphism $\mathrm{Vec}^{2m}_Q(\Sigma,\tau)\simeq\mathbb{Z}_2$ in (1-2).

The proof of Theorem 1.2 is postponed to Section 3.6. Theorem 1.2 clearly applies to the classification of Q–bundles over the involutive torus $(\mathbb{T}^2,\tau_{\mathrm{TR}})$, reproducing in this way results already existing in the literature. In this regard the result [21, (2.9)], previously announced in [22, II.25, page 19], deserves a special mention. The latter is in agreement with Theorem 1.2 above in view of the equality $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_\Sigma(w)}=\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_\Sigma(\xi)}$ (justified by the Polyakov–Wiegmann formula, see Lemma 3.17) where the map w employed in [22] is related to the map ξ of Theorem 1.2 by the relation $w=\xi Q$, with Q the constant matrix in (2-2). However, it is worth pointing out that the validity of

Theorem 1.2 goes far beyond the standard case $(\mathbb{T}^2, \tau_{\mathrm{TR}})$. For instance, Theorem 1.2 extends the classification of Q–bundles over Riemann surfaces of genus g endowed with an orientation-preserving involution with a finite set of fixed points [8, Appendix A] and this application seems to be new in the literature.

In order to describe the three-dimensional case it is worth mentioning that any Q–bundle $(\mathscr{E}, \Theta)$ over the involutive space (X, τ) can be equivalently described by a *principal Q–bundle* $(\mathscr{P}, \widehat{\Theta})$ over the same base space (see Section 3.1) and that for principal Q–bundles there exists a notion of *equivariant Q–connection* (see Section 3.2). Given a Q–connection $\omega \in \Omega^1_Q(\mathscr{P}, \mathfrak{u}(2m))$ one can define the associated *Chern–Simons 3–form*

$$\mathcal{CS}(\omega) := \frac{1}{8\pi^2}\operatorname{Tr}\Big(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\Big)$$

and the *intrinsic Chern–Simons invariant*

$$\mathfrak{cs}(\mathscr{P}, \widehat{\Theta}) := \int_X s^*\mathcal{CS}(\omega) \mod \mathbb{Z} \tag{1-4}$$

as specified in Definitions 3.9 and 3.14. Remarkably, under the hypotheses stipulated in Proposition 3.12, the quantity in the right-hand side of (1-4) turns out to be independent of the choice of the invariant connection ω or of the global section $s\colon X \to \mathscr{P}$, and therefore defines an invariant for the underlying principal Q–bundle $(\mathscr{P}, \widehat{\Theta})$, or equivalently for the associated Q–bundle $(\mathscr{E}, \Theta)$.

Let us recall that when (X, τ) is a three-dimensional FKMM–manifold in the sense of Definition 2.8, Proposition 2.10 applies and we have an isomorphism

$$\mathrm{Vec}^{2m}_Q(X, \tau) \overset{\kappa}{\simeq} \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2} \quad \text{for all } m \in \mathbb{N}.$$

In the formula above, $\mathrm{Map}(X^\tau, \{\pm 1\}) \simeq \mathbb{Z}_2^{|X^\tau|}$ denotes the set of maps from X^τ to $\{\pm 1\}$ (recall that X^τ is a set of finitely many points). The group action of $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ on $\mathrm{Map}(X^\tau, \{\pm 1\})$ is given by multiplication and restriction. The map κ which implements the isomorphism is the FKMM–invariant; see Section 2.3. Given a Q–bundle $(\mathscr{E}, \Theta)$ over (X, τ), its FKMM–invariant $\kappa(\mathscr{E}, \Theta)$ can be represented in terms of a map $\phi \in \mathrm{Map}(X^\tau, \{\pm 1\})$ and one can use the product sign map to define the so-called *strong* Fu–Kane–Mele index

$$\kappa_{\mathrm{s}}(\mathscr{E}, \Theta) := \Pi[\phi] = \prod_{x_j \in X^\tau} \phi(x_j) \in \mathbb{Z}_2. \tag{1-5}$$

It turns out that the definition above is well-posed in the sense that $\kappa_{\mathrm{s}}(\mathscr{E}, \Theta)$ only depends on the equivalence class of ϕ in $\mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$; hence it defines a topological invariant for $(\mathscr{E}, \Theta)$. This fact is a consequence of the second main result of this paper:

Theorem 1.3 *Let (X, τ) be a three-dimensional FKMM–manifold in the sense of Definition 2.8 such that $X^\tau \neq \varnothing$. Assume in addition that:*

(e) *X is oriented and τ* **reverses** *the orientation.*

Let $(\mathscr{E}, \Theta)$ be a Q–bundle over (X, τ) with FKMM–invariant

$$\kappa(\mathscr{E}, \Theta) \in \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$$

according to Proposition 2.10. For a given representative $\phi \in \mathrm{Map}(X^\tau, \{\pm 1\})$ of $\kappa(\mathscr{E}, \Theta)$, let $\Pi[\phi]$ be as in (1-5). Then, independent of the choice of ϕ,

$$\mathrm{e}^{\mathrm{i}2\pi\, \mathfrak{cs}(\mathscr{P}, \widehat{\Theta})} = \Pi[\phi], \tag{1-6}$$

where $(\mathscr{P}, \widehat{\Theta})$ is the principal Q–bundle associated to $(\mathscr{E}, \Theta)$ and $\mathfrak{cs}(\mathscr{P}, \widehat{\Theta})$ is the intrinsic Chern–Simons invariant of Definition 3.14.

The proof of Theorem 1.3 is postponed to Section 3.7. Along with Corollary 3.32, it expresses the fact that the strong index

$$\kappa_{\mathrm{s}}(\mathscr{E}, \Theta) = \mathrm{e}^{\mathrm{i}2\pi\, \mathfrak{cs}(\mathscr{P}, \widehat{\Theta})} \tag{1-7}$$

is a topological invariant which allows us to, at least partially, classify Q–bundles. In the case of the involutive torus $(\mathbb{T}^3, \tau_{\mathrm{TR}})$ described by (1-1) the invariant $\kappa_{\mathrm{s}}(\mathscr{E}, \Theta)$ takes values in the first (strong) summand of $\mathbb{Z}_2 \oplus (\mathbb{Z}_2)^3$. For a more recent review of the topological interpretation of the (strong) Fu–Kane–Mele index we refer to [4].

Theorems 1.2 and 1.3 show that the differential geometric gauge invariants (1-3) and (1-4) can be used as tools for the classification of Q–bundles in dimension two and three, provided that the base space meets some restrictive conditions. The results contained in Theorems 1.2 and 1.3 are valid for base spaces which are much more general than the involutive tori $(\mathbb{T}^d, \tau_{\mathrm{TR}})$ usually considered in literature. However, these results are still not completely satisfactory in view of the restrictions on the nature of the base space that we need to assume. There are two questions which are still open, and that it would be interesting to answer: *Is it possible to extend Theorems 1.2 and 1.3 to involutive base spaces (X, τ) such that X^τ is a submanifold of dimension bigger than zero*? In the case of Theorem 1.2, *is it possible to construct the classifying map ξ directly from the projection which represents the Q–bundle in K–theory without relying on the use of a predetermined global frame*?

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2 "Quaternionic" vector bundles from a topological perspective

In this section base spaces will be considered only from a topological point of view. Henceforth, we will assume that:

Assumption 2.1 ($\mathbb{Z}_2$–CW–complex) X is a topological space which admits the structure of a $\mathbb{Z}_2$–CW–complex. The *dimension d* of X is, by definition, the maximal dimension of its cells, and X is called *low-dimensional* if $0 \leqslant d \leqslant 3$.

For the sake of completeness, let us recall that an involutive space (X, τ) has the structure of a $\mathbb{Z}_2$–CW–complex if it admits a skeleton decomposition given by gluing cells of different dimension in ascending order, and the involution permutes the cells. For a precise definition of the notion of $\mathbb{Z}_2$–CW–complex the reader can refer to [7, Section 4.5] or [1; 38]. Assumption 2.1 allows the space X to have several disconnected components. However, in the case of multiple components, we will tacitly assume that vector bundles built over X possess fibers of constant rank on the whole base space. Let us recall that a space with a CW–complex structure is automatically Hausdorff and paracompact, and it is compact exactly when it is constructed out of a finite number of cells [26]. Almost all the examples considered in this paper will concern spaces with a finite CW–complex structure.

2.1 Basic facts about "quaternionic" vector bundles

In this section we recall some basic facts about the topological category of "quaternionic" vector bundles. Furthermore, the necessary notation for the description of the various results will be fixed. We refer to [8; 10; 11; 12] for a more systematic presentation of the subject.

Definition 2.2 (“quaternionic” vector bundles) A “quaternionic” vector bundle, or Q–bundle, over (X,τ) is a complex vector bundle $\pi:\mathscr{E}\to X$ endowed with a homeomorphism $\Theta:\mathscr{E}\to\mathscr{E}$ such that

(Q_1) the projection π is *equivariant* in the sense that $\pi\circ\Theta=\tau\circ\pi$;

(Q_2) Θ is *antilinear* on each fiber, ie $\Theta(\lambda p)=\bar{\lambda}\Theta(p)$ for all $\lambda\in\mathbb{C}$ and $p\in\mathscr{E}$, where $\bar{\lambda}$ is the complex conjugate of λ;

(Q_3) Θ^2 acts fiberwise as multiplication by -1, namely $\Theta^2|_{\mathscr{E}_x}=-\mathbb{1}_{\mathscr{E}_x}$.

Let us recall that it is always possible to endow $\mathscr{E}$ with an (essentially unique) *equivariant* Hermitian metric $\mathfrak{m}$ with respect to which Θ is an *antiunitary* map between conjugate fibers [8, Proposition 2.5]. The equivariance is expressed by

$$\mathfrak{m}(\Theta(p_1),\Theta(p_2))=\mathfrak{m}(p_2,p_1)\quad\text{for all }(p_1,p_2)\in\mathscr{E}\times_\pi\mathscr{E},$$

where $\mathscr{E}\times_\pi\mathscr{E}:=\{(p_1,p_2)\in\mathscr{E}\times\mathscr{E}\mid\pi(p_1)=\pi(p_2)\}$.

A vector bundle *morphism* between two vector bundles $\pi:\mathscr{E}\to X$ and $\pi':\mathscr{E}'\to X$ over the same base space is a continuous map $f:\mathscr{E}\to\mathscr{E}'$ which is fiber preserving in the sense that $\pi=\pi'\circ f$ and that restricts to a *linear* map on each fiber $f|_x:\mathscr{E}_x\to\mathscr{E}'_x$. Complex vector bundles over X together with vector bundle morphisms define a category. The symbol $\mathrm{Vec}^m_{\mathbb{C}}(X)$ is used to denote the set of equivalence classes of isomorphic vector bundles of rank m. From these data, it is possible to define a category of Q–bundles and Q–morphisms. A Q–morphism between two Q–bundles $(\mathscr{E},\Theta)$ and $(\mathscr{E}',\Theta')$ over the same involutive space (X,τ) is a vector bundle morphism f commuting with the involutions, ie $f\circ\Theta=\Theta'\circ f$. The set of equivalence classes of isomorphic Q–bundles of rank m over (X,τ) will be denoted by $\mathrm{Vec}^m_Q(X,\tau)$.

Remark 2.3 (“real” vector bundles) By changing condition (Q_3) in Definition 2.2 to

(R) Θ^2 acts fiberwise as the multiplication by 1, namely $\Theta^2|_{\mathscr{E}_x}=\mathbb{1}_{\mathscr{E}_x}$,

one ends in the category of *“real” vector bundles*, or *R–bundles*. The set of isomorphism classes of rank m R–bundles over the involutive space (X,τ) is denoted by $\mathrm{Vec}^m_R(X,\tau)$. For more details we refer to [3; 7].

In the case of a trivial involutive space (X,Id_X), one has bijections

$$\mathrm{Vec}^{2m}_Q(X,\mathrm{Id}_X)\simeq\mathrm{Vec}^m_{\mathbb{H}}(X),\quad \mathrm{Vec}^m_R(X,\mathrm{Id}_X)\simeq\mathrm{Vec}^m_{\mathbb{R}}(X),\quad m\in\mathbb{N},\tag{2-1}$$

where $\mathrm{Vec}^m_{\mathbb{F}}(X)$ is the set of equivalence classes of vector bundles over X with typical fiber $\mathbb{F}^m$ and $\mathbb{H}$ denotes the skew field of quaternions. The first isomorphism in (2-1) is proved in [12] — see also [8, Proposition 2.2] — while the proof of the second is provided in [3] — see also [7, Proposition 4.5]. These two results justify the names “quaternionic” and “real” for the related categories.

Let $x \in X^\tau$ and $\mathscr{E}_x \simeq \mathbb{C}^m$ be the related fiber. In this case the restriction $\Theta|_{\mathscr{E}_x} \equiv J$ defines an *antilinear* map $J\colon \mathscr{E}_x \to \mathscr{E}_x$ such that $J^2 = -\mathbb{1}_{\mathscr{E}_x}$. Said differently, the fibers $\mathscr{E}_x$ over fixed points $x \in X^\tau$ are endowed with a *quaternionic* structure; see [8, Remark 2.1]. This fact has an important consequence [8, Proposition 2.1]:

Proposition 2.4 *If $X^\tau \neq \varnothing$, then every Q–bundle over (X,τ) has even rank.*

The set $\mathrm{Vec}_Q^{2m}(X,\tau)$ is nonempty since it contains at least the *trivial* element in the “quaternionic” category. The rank $2m$ *product Q–bundle* over the involutive space (X,τ) is the complex vector bundle

$$X \times \mathbb{C}^{2m} \to X$$

endowed with the *product* Q–structure

$$\Theta_0(x,\mathrm{v}) = (\tau(x), Q\bar{\mathrm{v}}), \quad (x,\mathrm{v}) \in X \times \mathbb{C}^{2m},$$

where the matrix Q is given by

$$Q := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1}_m = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}. \tag{2-2}$$

A “quaternionic” vector bundle is called *Q–trivial* if it is isomorphic to the product Q–bundle.

A *section* of a complex vector bundle $\pi\colon \mathscr{E} \to X$ is a continuous map $s\colon X \to \mathscr{E}$ such that $\pi \circ s = \mathrm{Id}_X$. The set $\Gamma(\mathscr{E})$ of sections of $\mathscr{E}$ has the structure of a left $C(X)$–module with multiplication given by the pointwise product $(fs)(x) := f(x)s(x)$ for any $f \in C(X)$ and $s \in \Gamma(\mathscr{E})$ and for all $x \in X$. If $(\mathscr{E},\Theta)$ is a Q–bundle over (X,τ) then $\Gamma(\mathscr{E})$ is endowed with a natural antilinear antiinvolution $\tau_\Theta\colon \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ given by

$$\tau_\Theta(s) := \Theta \circ s \circ \tau.$$

The compatibility with the $C(X)$–module structure is given by

$$\tau_\Theta(fs) = \tau_*(f)\tau_\Theta(s), \quad f \in C(X),\ s \in \Gamma(\mathscr{E}),$$

where the antilinear involution $\tau_*\colon C(X) \to C(X)$ is defined by $\tau_*(f)(x) := \overline{f(\tau(x))}$. The triviality of a "quaternionic" vector bundle can be characterized in terms of global Q–frames of sections [8, Definition 2.1 and Theorem 2.1].

2.2 Stable range in low dimension

The stable rank condition for vector bundles expresses the pretty general fact that the nontrivial topology can be concentrated in a subvector bundle of *minimal* rank. This minimal value depends on the dimensionality of the base space and on the category of vector bundles under consideration. For complex (as well as real or quaternionic) vector bundles the stable rank condition is a well-known result; see eg [29, Chapter 9, Theorem 1.2]. The proof of the latter is based on an "obstruction-type argument" which provides the construction of a certain *maximal* number of global sections [29, Chapter 2, Theorem 7.1].

The latter argument can be generalized to vector bundles over spaces with involution by means of the notion of $\mathbb{Z}_2$–CW–complex [1; 38] — see also [7, Section 4.5]. A $\mathbb{Z}_2$–CW–complex is a CW–complex with a $\mathbb{Z}_2$–action that permutes the cells. The action of $\mathbb{Z}_2$ on each cell is either trivial or free. Since this construction is modeled after the usual definition of CW–complex, just by replacing "points" with "$\mathbb{Z}_2$–points", (almost) all topological and homological properties valid for CW–complexes have their natural counterpart in the equivariant setting. The use of this technique is essential for the determination of the stable rank condition in the case of R–bundles [7, Theorem 4.25] and Q–bundles [10, Theorems 4.2 and 4.5].

In this section we recall the results about the stable range for R–bundles and (even rank) Q–bundles over low-dimensional base spaces. Indeed, these are the only cases of interest in the present work.

Theorem 2.5 (stable condition in low dimension) *Let (X,τ) be an involutive space such that X has a finite $\mathbb{Z}_2$–CW–complex decomposition of dimension d. Assume that X^τ is discrete. Then*:

- ***Stable condition for R–bundles*** *For all $m \in \mathbb{N}$,*

$$\begin{aligned} &\mathrm{Vec}^m_R(X,\tau) = 0 && \text{if } d = 0, 1, \\ &\mathrm{Vec}^m_R(X,\tau) \simeq \mathrm{Vec}^1_R(X,\tau) && \text{if } 2 \leqslant d \leqslant 3. \end{aligned}$$

- ***Stable condition for Q–bundles*** *For all* $m \in \mathbb{N}$,

$$\begin{aligned} \mathrm{Vec}^{2m}_{Q}(X,\tau) &= 0 && \textit{if } d = 0,1, \\ \mathrm{Vec}^{2m}_{Q}(X,\tau) &\simeq \mathrm{Vec}^{2}_{Q}(X,\tau) && \textit{if } 2 \leqslant d \leqslant 5. \end{aligned}$$

In particular, under the hypotheses of validity of Theorem 2.5, the dimensions $d = 0,1$ are trivial since in these cases only the trivial R– and Q–bundles (up to isomorphism) exist. In the cases $d = 2,3$, which are the really interesting cases for this work, it is enough to study the sets $\mathrm{Vec}^{1}_{R}(X,\tau)$ and $\mathrm{Vec}^{2}_{Q}(X,\tau)$.

2.3 The FKMM–invariant

Q–bundles can be classified, at least partially, by means of a characteristic class called *FKMM–invariant*. This topological object was first introduced in [19] and then studied and generalized in [8; 10; 11]. In this section we review the main properties of the FKMM–invariant.

Let (X,τ) be an involutive space and $X^{\tau} \subseteq X$ its fixed point subset. In order to introduce the FKMM–invariant one needs the *equivariant Borel cohomology* group of (X,τ) with coefficients in the local system $\mathbb{Z}(1)$; ie

$$H^{\bullet}_{\mathbb{Z}_2}(X,\mathbb{Z}(1)) := H^{\bullet}(X_{\sim\tau},\mathbb{Z}(1)). \tag{2-3}$$

More precisely, each equivariant cohomology group $H^{j}_{\mathbb{Z}_2}(X,\mathbb{Z}(1))$ is given by the singular cohomology group $H^{j}(X_{\sim\tau},\mathbb{Z}(1))$ of the homotopy quotient

$$X_{\sim\tau} := X \times \mathbb{S}^{\infty}/(\tau \times \theta_{\infty}),$$

where θ_{∞} is the antipodal map on the infinite sphere $\mathbb{S}^{\infty}$. The local system $\mathbb{Z}(1)$ over (X,τ) can be identified with the product space $\mathbb{Z}(1) \simeq X \times \mathbb{Z}$ made equivariant by the $\mathbb{Z}_2$–action $(x,l) \mapsto (\tau(x),-l)$. The fixed point subset X^{τ} is closed in X and τ–invariant. The inclusion $\imath\colon X^{\tau} \hookrightarrow X$ extends to an inclusion $\imath\colon X^{\tau}_{\sim\tau} \hookrightarrow X_{\sim\tau}$ of the respective homotopy quotients. The *relative* equivariant cohomology can be defined as usual by the identification

$$H^{\bullet}_{\mathbb{Z}_2}(X|X^{\tau},\mathbb{Z}(1)) := H^{\bullet}(X_{\sim\tau}|X^{\tau}_{\sim\tau},\mathbb{Z}(1)).$$

For a more detailed description of equivariant Borel cohomology we refer to Section 3.1 of [8].

The FKMM–invariant is a map

$$\kappa\colon \mathrm{Vec}^{2m}_{Q}(X,\tau) \to H^{2}_{\mathbb{Z}_2}(X|X^{\tau},\mathbb{Z}(1)) \tag{2-4}$$

which associates the isomorphism class $[(\mathscr{E},\Theta)]$ of the Q–bundle $(\mathscr{E},\Theta)$ to a cohomology class $\kappa(\mathscr{E},\Theta)$ in the relative equivariant cohomology group $H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))$. The construction of the map κ was first described in [8, Section 3.3] and then generalized in [10, Section 2.5]. In this section we will skip the details of the construction of the FKMM–invariant and we will focus only on the relevant properties of the map (2-4):

(a) Isomorphic Q–bundles define the same FKMM–invariant.

(b) The FKMM–invariant is *natural* with respect to equivariant maps.

(c) If $(\mathscr{E},\Theta)$ is Q–trivial, then $\kappa(\mathscr{E},\Theta)=0$.

(d) The FKMM–invariant is additive with respect to the Whitney sum and the abelian structure of $H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))$. More precisely,
$$\kappa(\mathscr{E}_1\oplus\mathscr{E}_2,\Theta_1\oplus\Theta_2)=\kappa(\mathscr{E}_1,\Theta_1)\cdot\kappa(\mathscr{E}_2,\Theta_2)$$
for each pair of Q–bundles $(\mathscr{E}_1,\Theta_1)$ and $(\mathscr{E}_2,\Theta_2)$ over the same involutive space (X,τ).

For the justification of these properties we refer to [10, Section 2.6].

2.4 Topological classification over low-dimensional FKMM–spaces

The FKMM–invariant is an extremely efficient tool for the classification of Q–bundles in low dimensions. The first observation is that, in great generality, the FKMM–invariant is injective in low dimensions, ie when the base space has dimension $0\leqslant d\leqslant 3$. More precisely, as a consequence of [10, Theorems 4.7 and 4.9] one has that:

Theorem 2.6 (injectivity in low dimensions) *Let (X,τ) be an involutive space of dimension $d=0,1,2,3$ which satisfies Assumption 2.1. Then the map (2-4) is injective.*

This result suggests that in low dimensions the invariant κ can be used to label inequivalent classes of Q–bundles by means of elements of the cohomology group $H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))$. The next natural question is about the surjectivity of the map κ. In this case it is possible to provide a general positive answer only if $0\leqslant d\leqslant 2$. As proved in [11, Corollary 4.2 and Proposition 4.9] one has that:

Theorem 2.7 (surjectivity in dimension two) *Let (X,τ) be an involutive space of dimension $d=2$ which satisfies Assumption 2.1. Then*
$$\mathrm{Vec}^{2m}_Q(X,\tau)\simeq H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1))\quad \text{for all } m\in\mathbb{N},$$
namely the map (2-4) is bijective.

Theorem 2.7 can be juxtaposed with the stable condition described in Theorem 2.5,

$$\mathrm{Vec}_Q^{2m}(X,\tau)=0 \quad \text{for all } m\in\mathbb{N} \text{ if } d=0,1,$$

to obtain a complete classification of Q–bundles in dimension $d=0,1,2$.

In the case $d=3$, the surjectivity of the FKMM–invariant can be recovered by requiring some extra properties for the base space (X,τ). In the next part of this work we will mainly focus on spaces of the following type:

Definition 2.8 (FKMM–manifold) An involutive space (X,τ) is called an FKMM–manifold if

(a) X is a compact Hausdorff manifold without boundary;

(b) the involution τ preserves the manifold structure;

(c) the fixed point set X^τ consists at most of a finite collection of points;

(d) $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1))=0$.

Let us observe that an involutive space (X,τ) which fulfills conditions (a) and (b) in Definition 2.8 is a *closed manifold* which automatically admits the structure of a $\mathbb{Z}_2$–CW–complex; see eg [39, Theorem 3.6]. Then an FKMM–manifold meets all the requirements stated in Assumption 2.1. The conditions (c) and (d) are the crucial ingredients for the definition of a *topological* FKMM–space according to the original definition [8, Definition 1.1]. The requirement of a manifold structure has a twofold justification: first of all it allows the use of a technical tool (the slice theorem) in the proof of the crucial result [11, Proposition 4.13]; second, the main aim of this work is the study of the classification of Q–bundles over involutive manifolds (see Section 3). The manifold structure and the map τ are tacitly assumed to be of some given regularity (eg C^r or smooth). The next result provides the topological classification of Q–bundles over low-dimensional FKMM–manifolds.

Theorem 2.9 (classification of FKMM–manifolds) *Let (X,τ) be an FKMM–manifold of dimension $0\leqslant d\leqslant 3$. Then, for all $m\in\mathbb{N}$,*

$$\begin{aligned}&\mathrm{Vec}_Q^{2m}(X,\tau)=0 && \text{if } d=0,1,\\ &\mathrm{Vec}_Q^{2m}(X,\tau)\simeq H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1)) && \text{if } d=2,3,\end{aligned}$$

and the isomorphism (in the nontrivial cases) is given by the FKMM–invariant κ.

The cases $d = 0, 1$ are a consequence of the stable condition described in Theorem 2.5. The case $d = 2$ follows from Theorem 2.7. Finally the new case $d = 3$ is proved in [11, Proposition 4.13].

Let us observe that Theorem 2.9 also holds trivially in the case of a free involution, that is, when $X^\tau = \varnothing$. In this case, as a consequence of condition (d) in Definition 2.8 one has that $H^2_{\mathbb{Z}_2}(X|\varnothing, \mathbb{Z}(1)) \simeq H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$. Therefore, as a consequence of Theorem 2.9, one concludes that an FKMM–manifold with free involution only supports the trivial Q–bundle. In order to focus on the nontrivial situations we will assume henceforth that $d = 2, 3$ and $X^\tau \neq \varnothing$.

When (X, τ) is an FKMM–manifold, the cohomology group $H^2_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1))$ has an explicit representation in terms of equivalence classes of maps. As proved in [8, Lemma 3.1] one has the isomorphism

$$H^2_{\mathbb{Z}_2}(X|X^\tau, \mathbb{Z}(1)) \simeq \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}, \tag{2-5}$$

where $\mathrm{Map}(X^\tau, \{\pm 1\}) \simeq \mathbb{Z}_2{}^{|X^\tau|}$ is the set of maps from X^τ to $\{\pm 1\}$ (recall that X^τ is a set of finitely many points) and $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ denotes the set of classes of $\mathbb{Z}_2$–homotopy equivalent equivariant maps between the involutive space (X, τ) and the group $\mathbb{U}(1)$ endowed with the involution given by complex conjugation. The group action of $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ on $\mathrm{Map}(X^\tau, \{\pm 1\})$ is given by restriction and multiplication. More precisely, let $[u] \in [X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ and $s \in \mathrm{Map}(X^\tau, \{\pm 1\})$. Then the action of $[u]$ on s is given by $[u](s) := u|_{X^\tau} \cdot s$. By combining Theorem 2.9 with the isomorphism (2-5) one gets the following result:

Proposition 2.10 *Let (X, τ) be an FKMM–manifold of dimension $d = 2, 3$ and assume that $X^\tau \neq \varnothing$. Then, the FKMM–invariant κ induces the isomorphism*

$$\mathrm{Vec}^{2m}_Q(X, \tau) \simeq \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2} \quad \textit{for all } m \in \mathbb{N}.$$

In summary, the content of Theorem 2.9 and Proposition 2.10 is the following: Every Q–bundle $(\mathscr{E}, \Theta)$ over an FKMM–space (X, τ) of dimension $d = 2, 3$ such that $X^\tau \neq \varnothing$ is classified by its FKMM–invariant $\kappa(\mathscr{E}, \Theta)$. The latter can be represented as a map

$$s_{(\mathscr{E},\Theta)} \colon X^\tau \to \{\pm 1\}$$

modulo the (right) multiplication by the restriction over X^τ of an equivariant function $u \colon X \to \mathbb{U}(1)$. The map $s_{(\mathscr{E},\Theta)}$ is called the *canonical section* associated to $(\mathscr{E}, \Theta)$ and its construction is described in [8, Section 3.2] or [10, Section 2.2].

2.5 The Fu–Kane–Mele index

Let us focus on the nontrivial case of an FKMM–manifold (X,τ) of dimension $d=2,3$ such that $X^\tau \neq \varnothing$. At the end of last section we observed that every Q–bundle $(\mathscr{E},\Theta)$ over (X,τ) is classified by the canonical section $s_{(\mathscr{E},\Theta)} \in \mathrm{Map}(X^\tau,\{\pm 1\})$ modulo the action (multiplication and restriction) of an equivariant map $u\colon X \to \mathbb{U}(1)$. Clearly $(\mathscr{E},\Theta)$ is equivalently classified by any other map $\phi \in \mathrm{Map}(X^\tau,\{\pm 1\})$ in the same equivalence class of $s_{(\mathscr{E},\Theta)}$, namely by any representative of

$$[s_{(\mathscr{E},\Theta)}] \in \mathrm{Map}(X^\tau,\{\pm 1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}.$$

Consider now the *product sign* map

$$\Pi\colon \mathrm{Map}(X^\tau,\{\pm 1\}) \to \{\pm 1\} \tag{2-6}$$

defined by

$$\Pi(\phi) := \prod_{x_j \in X^\tau} \phi(x_j), \qquad \phi \in \mathrm{Map}(X^\tau,\{\pm 1\}). \tag{2-7}$$

The value $\Pi(\phi)$ is called the *Fu–Kane–Mele index* of ϕ. There is no reason to suspect a priori that the Fu–Kane–Mele index is well defined on the equivalence classes in $\mathrm{Map}(X^\tau,\{\pm 1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$. In fact, if ϕ_1 and ϕ_2 were two representatives of the same class $[\phi] \in \mathrm{Map}(X^\tau,\{\pm 1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$ related by an equivariant function $u\colon X \to \mathbb{U}(1)$ which takes an odd number of times the value -1 on X^τ, then one would have that $\Pi[\phi_1] = -\Pi[\phi_2]$. For this reason the following result, proved in [8, Proposition 4.5 and Theorem 4.2] is quite surprising, at first glance, from a topological point of view.

Theorem 2.11 (Fu–Kane–Mele formula, $d=2$) *Let (X,τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Then (X,τ) is an FKMM–manifold according to Definition 2.8. Moreover,*

$$H^2_{\mathbb{Z}_2}(X|X^\tau,\mathbb{Z}(1)) \simeq \mathbb{Z}_2, \tag{2-8}$$

where $\mathbb{Z}_2$ is identified with the multiplicative group $\{\pm 1\}$. Moreover, any Q–bundle $(\mathscr{E},\Theta)$ over (X,τ) is classified by the FKMM–invariant $\kappa(\mathscr{E},\Theta) \in \{\pm 1\}$ which can be computed by $\kappa(\mathscr{E},\Theta) = \Pi(\phi)$, where Π is the product sign map (2-6) and ϕ is any representative of the class $[s_{(\mathscr{E},\Theta)}] \in \mathrm{Map}(X^\tau,\{\pm 1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$ of the canonical section.

Proof (sketch) Clearly conditions (a′), (b′) and (c′) of Definition 1.1 imply conditions (a), (b) and (c) of Definition 2.8. Moreover, Proposition 4.4 of [8] assures that (a′), (b′)

and (c′) imply condition (d) of Definition 2.8, ie $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1))=0$ along with isomorphism (2-8). The rest of the claim is proved in [8, Proposition 4.5 and Theorem 4.2]. □

As a byproduct of Theorem 2.11 one has that the Fu–Kane–Mele index is unambiguously defined on the whole equivalence class $[s_{(\mathscr{E},\Theta)}]$, and the Q–bundle $(\mathscr{E},\Theta)$ is classified, up to isomorphism, by the sign $\Pi(\phi)\in\{\pm 1\}$ where $\phi\in\mathrm{Map}(X^\tau,\{\pm 1\})$ is any map which differs from $s_{(\mathscr{E},\Theta)}$ by the multiplication with the restriction of an equivariant map $u\colon X\to\mathbb{U}(1)$.

Although with some differences, the next result pairs Theorem 2.11 in dimension $d=3$. It can be considered one of the main achievements of this work.

Theorem 2.12 (Fu–Kane–Mele formula, $d=3$) *Let (X,τ) be an FKMM–manifold of dimension $d=3$ with $X^\tau\neq\varnothing$. Assume in addition that:*

(e) *X is oriented and τ **reverses** the orientation.*

Let $(\mathscr{E},\Theta)$ be a Q–bundle over (X,τ) with FKMM–invariant $\kappa(\mathscr{E},\Theta)$ represented by the class $[s_{(\mathscr{E},\Theta)}]\in\mathrm{Map}(X^\tau,\{\pm 1\})/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$ according to Proposition 2.10. Then the sign

$$\kappa_{\mathrm{s}}(\mathscr{E},\Theta):=\Pi[\phi] \tag{2-9}$$

is independent of the choice of the representative $\phi\in[s_{(\mathscr{E},\Theta)}]$ and provides a topological invariant for $(\mathscr{E},\Theta)$.

Theorem 2.12 is a direct consequence of Theorem 1.3, which will be proved in Section 3.7. It is worth noting that even though Theorems 2.11 and 2.12 seem to be of topological nature, they need the manifold structure of X. In particular, Theorem 1.3 relies on differential geometric techniques.

In general the quantity $\kappa_{\mathrm{s}}(\mathscr{E},\Theta)$ in Theorem 2.12 does not completely specify the FKMM–invariant of $(\mathscr{E},\Theta)$, but only a part of it. We refer to $\kappa_{\mathrm{s}}(\mathscr{E},\Theta)$ as the *strong component* of the FKMM–invariant.

2.6 Alternative presentation of "quaternionic" vector bundles in low dimensions

This section is focused on an alternative description of rank 2 Q–bundles over low-dimensional involutive spaces (X,τ) such that $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1))=0$. It is worth mentioning that under these conditions the complex vector bundle underlying each Q–bundle is necessarily trivial [8, Proposition 4.1].

Let $\mathrm{Map}(X, \mathbb{SU}(2))$ be the space of (smooth) maps from X into $\mathbb{SU}(2)$. Given $\xi \in \mathrm{Map}(X, \mathbb{SU}(2))$, let $\tau^*\xi$ be the map defined by $\tau^*\xi(x) := \xi(\tau(x))$ for all $x \in X$. The space of equivariant maps from X into $\mathbb{SU}(2)$ is defined by

$$\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2} := \{\xi \in \mathrm{Map}(X, \mathbb{SU}(2)) \mid \tau^*\xi = \xi^{-1}\}. \tag{2-10}$$

The set of $\mathbb{Z}_2$–homotopy classes of $\mathbb{Z}_2$–equivariant maps will be denoted by

$$[X, \mathbb{SU}(2)]_{\mathbb{Z}_2}.$$

Let us consider also the groups

$$\begin{aligned} \mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2} &:= \{\psi \in \mathrm{Map}(X, \mathbb{U}(2)) \mid \det(\tau^*\psi) = \det(\bar{\psi})\}, \\ \mathrm{Map}(X, \mathbb{U}(1))_{\mathbb{Z}_2} &:= \{\phi \in \mathrm{Map}(X, \mathbb{U}(1)) \mid \tau^*\phi = \bar{\phi}\}, \end{aligned} \tag{2-11}$$

where $\bar{\psi}$ and $\bar{\phi}$ are the complex conjugates of ψ and ϕ, respectively, and the group structures are given by pointwise multiplication. The related sets of equivalence classes under $\mathbb{Z}_2$–homotopy are denoted by $[X, \mathbb{U}(2)]'_{\mathbb{Z}_2}$ and $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$, respectively.

By construction one has an inclusion $\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2} \subset \mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2}$. Moreover, the group $\mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2}$ acts on $\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2}$ as follows: given $\psi \in \mathrm{Map}(X, \mathbb{U}(2))'_{\mathbb{Z}_2}$ let G_ψ be the automorphism of $\mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2}$ given by

$$G_\psi(\xi) := -(\tau^*\psi^{-1})\xi Q \bar{\psi} Q, \quad \xi \in \mathrm{Map}(X, \mathbb{SU}(2))_{\mathbb{Z}_2} \tag{2-12}$$

where Q is the (size 2×2) matrix (2-2). In fact, given that

$$\det(\tau^*\psi^{-1}) = \det(\tau^*\psi)^{-1} = \det(\bar{\psi})^{-1},$$

it follows that $\det(G_\psi(\xi)) = \det(\xi) = 1$. Moreover, the equality $\tau^* G_\psi(\xi) = G_\psi(\xi)^{-1}$ follows from a direct calculation along with the equality $Q\xi = \bar{\xi} Q$ valid for maps with values in $\mathbb{SU}(2)$.

The main aim of this section is to prove the following result:

Theorem 2.13 *Let (X, τ) be an involutive space of dimension $0 \leqslant d \leqslant 2$ satisfying Assumption 2.1. Assume in addition that $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$ in the case $d = 2$. Then there is a natural bijection*

$$\mathrm{Vec}^2_Q(X, \tau) \simeq [X, \mathbb{SU}(2)]_{\mathbb{Z}_2} / [X, \mathbb{U}(1)]_{\mathbb{Z}_2},$$

where the action of $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ on $[X, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ is defined as follows: given $[\phi]$ in $[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$, let $L_{[\phi]}$ be the automorphism of $[X, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ defined by

$$L_{[\phi]}([\xi]) := \left[\begin{pmatrix} \tau^*\bar{\phi} & 0 \\ 0 & 1 \end{pmatrix} \xi \begin{pmatrix} 1 & 0 \\ 0 & \bar{\phi} \end{pmatrix} \right].$$

We start with a couple of preliminary results which are valid in dimension $0 \leqslant d \leqslant 3$.

Lemma 2.14 *Let (X,τ) be a low-dimensional involutive space satisfying Assumption 2.1. Assume in addition that $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1)) = 0$ in the case $d = 2,3$. Then there is a natural bijection*

$$\mathrm{Vec}^2_Q(X,\tau) \simeq \mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}/\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2} \tag{2-13}$$

where the action of $\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}$ on $\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}$ is given by the automorphisms (2-12).

Proof Let $\pi\colon \mathscr{E} \to X$ be a rank 2 Q–bundle. The low-dimensionality of the base space implies that the underlying complex vector bundle $\mathscr{E}$ is isomorphic to the product bundle $X \times \mathbb{C}^2$ [8, Proposition 4.1]. The induced Q–structure Θ on $X \times \mathbb{C}^2$ is then expressed through a function $\xi\colon X \to \mathbb{U}(2)$ of the form $\Theta\colon (x,v) \mapsto (\tau(x), \xi(x)Q\bar{v})$ and the "quaternionic" condition is guaranteed by the constraint $\tau^*\xi = -Q\bar{\xi}^{-1}Q$. Let us introduce the subset

$$\mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2} := \{\xi \in \mathrm{Map}(X,\mathbb{U}(2)) \mid \tau^*\xi = -Q\bar{\xi}^{-1}Q\} \subset \mathrm{Map}(X,\mathbb{U}(2)).$$

Two Q–structures Θ and Θ' on $X \times \mathbb{C}^2$, induced respectively by the maps ξ and ξ' in $\mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2}$, are isomorphic if there exists a map $\psi \in \mathrm{Map}(X,\mathbb{U}(2))$ such that $\tau^*\psi\xi' Q = \xi Q\bar{\psi}$. Consider the action of $\mathrm{Map}(X,\mathbb{U}(2))$ on $\mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2}$ defined as follows: for any $\psi \in \mathrm{Map}(X,\mathbb{U}(2))$ let G_ψ be the automorphism of $\mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2}$ given by the formula (2-12). From the argument above it follows that

$$\mathrm{Vec}^2_Q(X,\tau) \simeq \mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2}/\mathrm{Map}(X,\mathbb{U}(2))$$

where the equivalence relation is induced by the action of the automorphisms G_ψ. Since $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1)) = 0$ by hypothesis, any "real" line bundle over X is automatically trivial [30]. This applies in particular to the determinant line bundle of the Q–bundle $(\mathscr{E},\Theta)$. The triviality of the "real" structure $(x,u) \mapsto (\tau(x), \det(\xi)(x)\bar{u})$ on $X \times \mathbb{C}$ implies the existence of a map $\phi\colon X \to \mathbb{U}(1)$ such that $\det(\xi) = \tau^*\phi\phi$. Consider the map $\psi_0 \in \mathrm{Map}(X,\mathbb{U}(2))$ given by

$$\psi_0(x) := \begin{pmatrix} \phi(x) & 0 \\ 0 & 1 \end{pmatrix}.$$

A direct computation shows that

$$\det(G_{\psi_0}(\xi)) = \det(\tau^*\psi_0)^{-1}\det(\xi)\det(\psi_0)^{-1} = 1. \tag{2-14}$$

As a result, it is possible to choose $\xi \in \mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2} \cap \mathrm{Map}(X,\mathbb{SU}(2))$ as the representative for the element of $\mathrm{Vec}^2_Q(X,\tau)$. Since it holds that $-Q\bar{\xi}Q = \xi$ for maps with values in $\mathbb{SU}(2)$, one has that the intersection $\mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2}\cap\mathrm{Map}(X,\mathbb{SU}(2))$ coincides with the set $\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}$ as described by (2-10). Finally, it is straightforward to see that $\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}$ described by (2-11) is the maximal subgroup of $\mathrm{Map}(X,\mathbb{U}(2))_{\mathbb{Z}_2}$ preserving such representatives. □

Lemma 2.15 *Under the hypotheses of Lemma 2.14 there is a natural bijection*

$$\mathrm{Vec}^2_Q(X,\tau)\simeq[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}. \tag{2-15}$$

Proof Consider the natural surjection onto the equivalence classes

$$\varpi\colon \mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}\hookrightarrow[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}.$$

The action of $\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}$ on $\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}$ given by (2-12) induces an action of the group $[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}$ on $[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}$. Under these actions, ϖ is equivariant, and one gets

$$\mathrm{Vec}^2_Q(X,\tau)\simeq\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}/\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}\xrightarrow{\varpi}[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}.$$

The latter is a bijection. Indeed, given $\xi\in\mathrm{Map}(X,\mathbb{SU}(2))_{\mathbb{Z}_2}$, let $\mathscr{E}_\xi=X\times\mathbb{C}^2$ be the Q–bundle of rank 2 with Q–structure given by $(x,v)\mapsto(\tau(x),\xi(x)Q\bar{v})$. In view of the homotopy property of Q–bundles [8, Theorem 2.3], if ξ and ξ' are $\mathbb{Z}_2$–homotopy equivalent, then $\mathscr{E}_\xi$ and $\mathscr{E}_{\xi'}$ are isomorphic. □

We are now in position to complete the proof of Theorem 2.13. For this purpose the restriction to dimensions $d\leqslant 2$ will be crucial.

Proof of Theorem 2.13 We will begin with the case $m=1$. Consider the exact sequence

$$1\to[X,\mathbb{SU}(2)]\xrightarrow{\imath}[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}\xrightarrow{\det}[X,\mathbb{U}(1)]_{\mathbb{Z}_2}\to 1$$

where $\imath$ is induced by the natural inclusion $\mathrm{Map}(X,\mathbb{SU}(2))\hookrightarrow\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}$ and det stands for the determinant. The latter sequence is right-split in view of the map $s\colon[X,\mathbb{U}(1)]_{\mathbb{Z}_2}\to[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}$ induced (with a slight abuse of notation) by

$$\mathrm{Map}(X,\mathbb{U}(1))_{\mathbb{Z}_2}\ni\phi\overset{s}{\longmapsto}\begin{pmatrix}\phi&0\\0&1\end{pmatrix}\in\mathrm{Map}(X,\mathbb{U}(2))'_{\mathbb{Z}_2}. \tag{2-16}$$

Indeed, it is straightforward to check $\det\circ s=\mathrm{Id}$. Consequently, one has a group isomorphism

$$[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}\simeq[X,\mathbb{SU}(2)]\rtimes[X,\mathbb{U}(1)]_{\mathbb{Z}_2},$$

where $\rtimes$ denotes the semidirect product. Since $\pi_k(\mathbb{SU}(2))=0$ if $k=0,1,2$, it follows that $[X,\mathbb{SU}(2)]=0$ whenever X has dimension $0\leqslant d\leqslant 2$. In these three cases, the isomorphism above reduces to $[X,\mathbb{U}(2)]'_{\mathbb{Z}_2}\simeq[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$ and the combination of the action G described by (2-12) with the homomorphism s in (2-16) produces the action L of $[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$ on $[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}$ as described in the claim. In view of the stable rank condition described in Theorem 2.5, the bijection generalizes to

$$\mathrm{Vec}^{2m}_{Q}(X,\tau)\simeq[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[X,\mathbb{U}(1)]_{\mathbb{Z}_2},\quad m\in\mathbb{N}$$

and this concludes the proof for the general case. □

Remark 2.16 (higher rank case) A representative map $\xi\colon X\to\mathbb{SU}(2)$ for a given Q–bundle $(\mathscr{E},\Theta)$ of rank $2m$ can be constructed in this way: The Q–structure of $(\mathscr{E},\Theta)$ is coded in an equivariant map $\xi'\colon X\to\mathbb{SU}(2m)$ which, for instance, can be constructed from a global frame according to the prescription described in Remark 2.19. The stable rank condition implies that ξ' can be always reduced to the form

$$\xi'\simeq\begin{pmatrix}\xi & 0\\ 0 & \mathbb{1}_{\mathbb{C}^{2(m-1)}}\end{pmatrix}$$

up to conjugation with an equivariant map with values in $\mathbb{U}(2m)$. The reduced map $\xi\colon X\to\mathbb{SU}(2)$ obtained in this way provides a representative of the Q–bundle $(\mathscr{E},\Theta)$ as an element of the group $[X,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[X,\mathbb{U}(1)]_{\mathbb{Z}_2}$.

2.7 The FKMM–invariant for oriented two-dimensional FKMM–manifolds

Throughout this section we will assume that the pair (Σ,τ) is an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. The use of the letter Σ instead of X is motivated to easier connect the results discussed here with the theory developed in Section 3.4 and 3.6

When (Σ,τ) is an oriented two-dimensional FKMM–manifold, two presentations for $\mathrm{Vec}^2_Q(\Sigma,\tau)$ are available. The first description,

$$\mathrm{Vec}^2_Q(\Sigma,\tau)\simeq\mathrm{Map}(\Sigma^\tau,\{\pm1\})/[\Sigma,\mathbb{U}(1)]_{\mathbb{Z}_2},$$

was proved in Proposition 2.10 and uses the FKMM–invariant. The second presentation,

$$\mathrm{Vec}^2_Q(\Sigma,\tau)\simeq[\Sigma,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma,\mathbb{U}(1)]_{\mathbb{Z}_2},$$

comes from Theorem 2.13. Therefore, there must exist an isomorphism of groups

$$[\Sigma,\mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma,\mathbb{U}(1)]_{\mathbb{Z}_2}\simeq\mathrm{Map}(\Sigma^\tau,\{\pm1\})/[\Sigma,\mathbb{U}(1)]_{\mathbb{Z}_2}$$

which associates the map $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ with the FKMM–invariant of the Q–bundle $\mathscr{E}_\xi$ classified by ξ. Such a map can be constructed by means of the *Pfaffian* Pf; see Proposition 2.18.

The evaluation of a map $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ on a fixed point $x \in \Sigma^\tau$ is an element of $\mathbb{SU}(2)$ which satisfies $\xi(x) = \xi(x)^{-1}$. This implies that $\xi(x) = \pm\mathbb{1}_{\mathbb{C}^2}$ if $x \in \Sigma^\tau$. Moreover, every matrix $\xi(x) \in \mathbb{SU}(2)$ satisfies the identity $Q\overline{\xi(x)} = \xi(x)Q$. Then, on a fixed point $x \in \Sigma^\tau$, the matrix $\xi(x)Q = \pm Q$ turns out to be skew-symmetric and the Pfaffian $\mathrm{Pf}(\xi(x)Q)$ is well defined. In particular one has that

$$-\mathrm{Pf}(\xi(x)\cdot Q) = \begin{cases} +1 & \text{if } \xi(x) = +\mathbb{1}_{\mathbb{C}^2}, \\ -1 & \text{if } \xi(x) = -\mathbb{1}_{\mathbb{C}^2}. \end{cases}$$

This suggests studying the mapping

$$\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2} \ni \xi \xrightarrow{\Phi_\kappa} -\mathrm{Pf}(\xi Q|_{\Sigma^\tau}) \in \mathrm{Map}(\Sigma^\tau, \{\pm 1\}). \tag{2-17}$$

Lemma 2.17 *Let (Σ, τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Then there is a bijection*

$$\Phi_\kappa \colon [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} \to \mathrm{Map}(\Sigma^\tau, \{\pm 1\})$$

defined by $[\xi] \mapsto -\mathrm{Pf}(\xi Q)|_{\Sigma^\tau}$.

Proof We will start by proving the injectivity of Φ_κ. Suppose $\xi, \xi' \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ are such that $\Phi_\kappa(\xi) = \Phi_\kappa(\xi')$. We want to show the existence of a $\mathbb{Z}_2$–equivariant homotopy $\tilde{\xi} \colon \Sigma \times [0,1] \to \mathbb{SU}(2)$ such that $\tilde{\xi}|_{\Sigma\times\{0\}} = \xi$ and $\tilde{\xi}|_{\Sigma\times\{1\}} = \xi'$. This can be done by a standard argument in homotopy theory. Let Σ_j be the j–skeleton of Σ with respect to a $\mathbb{Z}_2$–CW decomposition. The 0–skeleton Σ_0 consists of the 0–cells of the form e^0 (a fixed cell) or $\mathbb{Z}_2 \times e^0$ (a free cell), where $e^0 = *$ is a standard 0–cell. Accordingly, we can express Σ_0 as the disjoint union $\Sigma_0 = \Sigma_0^{\mathrm{fix}} \sqcup \Sigma_0^{\mathrm{free}}$. By assumption, we have $\Sigma_0^{\mathrm{fix}} = X^\tau$. Notice that the map Φ_κ factors through

$$[\Sigma, \mathbb{SU}(2))]_{\mathbb{Z}_2} \to \mathrm{Map}(\Sigma^\tau, \mathbb{SU}(2)^\vartheta) \to \mathrm{Map}(\Sigma^\tau, \{\pm 1\}),$$

where the involution ϑ on $\mathbb{SU}(2)$ is $\vartheta(\xi) = \xi^{-1}$, so that the fixed point set $\mathbb{SU}(2)^\vartheta = \{\pm\mathbb{1}_{\mathbb{C}^2}\}$ consists of two points. The first map is induced from the restriction $\xi \mapsto \xi|_{\Sigma^\tau}$. The second map is the bijection induced from the obvious identification $\mathbb{SU}(2)^\vartheta \simeq \{\pm 1\}$. It follows that $\xi|_{\Sigma_0^{\mathrm{fix}}} = \xi'|_{\Sigma_0^{\mathrm{fix}}}$. On the other hand, for each free 0–cell $\mathbb{Z}_2 \times e^0$ we can find a homotopy connecting $\xi|_{\{1\}\times e^0}$ and $\xi'|_{\{1\}\times e^0}$ because $\mathbb{SU}(2)$ is path connected. This homotopy extends to a $\mathbb{Z}_2$–equivariant homotopy connecting $\xi|_{\mathbb{Z}_2\times e^0}$ and $\xi'|_{\mathbb{Z}_2\times e^0}$ since the action of $\mathbb{Z}_2$ on $\mathbb{Z}_2 \times e^0$ is free. In this way, we get a $\mathbb{Z}_2$–equivariant homotopy $\tilde{\xi}_0 \colon \Sigma_0 \times [0,1] \to \mathbb{SU}(2)$ such that $\tilde{\xi}_0|_{\Sigma_0\times\{0\}} = \xi|_{\Sigma_0}$ and $\tilde{\xi}_0|_{\Sigma_0\times\{1\}} = \xi'|_{\Sigma_0}$.

By assumption again, the 1–skeleton Σ_1 is given by attaching only free 1–cells of the form $\mathbb{Z}_2 \times e^1$ to Σ_0. We already have a homotopy $\tilde{\xi}_0|_{\{1\}\times\partial e^1\times[0,1]}$. This homotopy, together with $\xi|_{\{1\}\times e^1}$ and $\xi'|_{\{1\}\times e^1}$, gives a map from

$$\partial(\{1\}\times e^1 \times [0,1]) = (\{1\}\times \partial e^1 \times [0,1]) \cup (\{1\}\times e^1 \times \partial[0,1])$$

which can be extended to a map from $\{1\}\times e^1 \times [0,1]$ in view of $\pi_1(\mathbb{SU}(2)) = 0$. Extending this map equivariantly, and gathering together the maps constructed in this way for each free 1–cell, one gets a $\mathbb{Z}_2$–equivariant homotopy $\tilde{\xi}_1 \colon \Sigma_1 \times [0,1] \to \mathbb{SU}(2)$ which extends $\tilde{\xi}_0$ and connects $\xi|_{\Sigma_1}$ with $\xi'|_{\Sigma_1}$. Finally, the 2–skeleton $\Sigma_2 = \Sigma$ is given by attaching only free 2–cells of the form $\mathbb{Z}_2 \times e^2$ to Σ_1. We already have a homotopy $\tilde{\xi}_1|_{\{1\}\times\partial e^2\times[0,1]}$. This homotopy, together with $\xi|_{\{1\}\times e^2}$ and $\xi'|_{\{1\}\times e^2}$, provides a map from $\partial(\{1\}\times e^2\times[0,1])$. This extends to a map from $\{1\}\times e^2 \times [0,1]$, since $\pi_2(\mathbb{SU}(2)) = 0$. Extending this equivariantly and gathering together the resulting maps for each free 2–cell, one gets a $\mathbb{Z}_2$–equivariant homotopy $\tilde{\xi} \colon \Sigma \times [0,1] \to \mathbb{SU}(2)$ connecting ξ with ξ'.

Now the surjectivity. The idea is to construct an element $\xi_\epsilon \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ for each $\epsilon \in \mathrm{Map}(\Sigma^\tau, \mathbb{Z}_2)$ such that $\Phi_\kappa(\xi_\epsilon) = \epsilon$. A preliminary fact is necessary. Let $D \subset \mathbb{C}$ be the closed unit disk endowed with the involution $z \mapsto -z$. Then, the map $\xi_D \in \mathrm{Map}(D, \mathbb{SU}(2))_{\mathbb{Z}_2}$ given by

$$\xi_D(z) := \frac{1}{2(|z|^2 - |z|) + 1} \begin{pmatrix} 2|z|-1 & -2\bar{z}(|z|-1) \\ 2z(|z|-1) & 2|z|-1 \end{pmatrix}$$

satisfies $\xi_D(0) = -\mathbb{1}_{\mathbb{C}^2}$ and $\xi_D(z) = +\mathbb{1}_{\mathbb{C}^2}$ if $z \in \partial D$. Let $\Sigma^\tau = \{x_1, \dots, x_n\}$ be a given labeling for the fixed points. The *slice theorem* [27, Chapter I, Section 3] assures that for each x_i there exists a closed disk $D_i \subset \Sigma$ such that $\tau(D_i) = D_i$, $x_i \in D_i$ and $D_i \cap D_j = \varnothing$ when $i \neq j$. Let $x_{i_1}, \dots, x_{i_k} \in \Sigma^\tau$ be the set of points such that $\epsilon(x_{i_j}) = -1$. Using an equivariant diffeomorphism $D \cong D_{i_j}$ one can induce the equivariant map $\xi_{D_{i_j}}$ on D_{i_j} from ξ_D. Extending these maps by $\mathbb{1}_{\mathbb{C}^2}$ outside of $D_{i_1} \cup \dots \cup D_{i_k}$ one gets an equivariant map $\xi_\epsilon \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ such that $\xi_\epsilon(x) = \epsilon(x)\mathbb{1}_{\mathbb{C}^2}$ for every $x \in \Sigma^\tau$. This ensures that $\Phi_\kappa(\xi_\epsilon) = \epsilon$. □

Proposition 2.18 *Let (Σ, τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Then the bijection of Lemma 2.17 induces the bijection*

$$\Phi_\kappa \colon [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} / [\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \to \mathrm{Map}(\Sigma^\tau, \{\pm 1\}) / [\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}.$$

Proof Lemma 2.17 asserts the bijectivity of the homomorphism

$$\Phi_\kappa \colon [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} \to \mathrm{Map}(\Sigma^\tau, \{\pm 1\}).$$

The same group $[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$ acts on both sides and Φ_κ is equivariant. An inspection of the group actions shows that Φ_κ descends to a bijective homomorphism between the quotients. □

In view of Theorem 2.13, one can think of a map $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ as a rank 2 Q–bundle on Σ. Then it makes sense to talk about the "FKMM–invariant of the map ξ". Proposition 2.18 shows that such an invariant is indeed built through the isomorphism Φ_κ. More precisely, by combining Proposition 2.18 with Theorem 2.11 one obtains

$$\kappa(\xi) := \Pi \circ \Phi_\kappa(\xi) \in \mathbb{Z}_2, \tag{2-18}$$

where $\kappa(\xi)$ represents the FKMM–invariant of the Q–bundle defined by the map ξ.

Remark 2.19 (construction of the classifying map from a frame) Let $(\mathscr{E}, \Theta)$ be a Q–bundle of rank 2 over an oriented two-dimensional FKMM–manifold. If the map $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ classifies $(\mathscr{E}, \Theta)$ according to Theorem 2.11 then formula (2-18) provides the computation of the FKMM–invariant of $(\mathscr{E}, \Theta)$. Therefore, the relevant problem is how to extract ξ from the knowledge of $(\mathscr{E}, \Theta)$. This problem has a simple solution when a global trivializing frame of sections $t_1, t_2 \colon \Sigma \to \mathscr{E}$ of the underlying (trivial) complex vector bundle is known. This situation has been described in detail [8, Section 4.2]. By a Gram–Schmidt orthonormalization if necessary, one can assume without loss of generality that the frame t_1, t_2 is orthonormal, ie $\mathfrak{m}(t_i, t_j) = \delta_{i,j}$ where $\mathfrak{m}$ is the (unique) Θ–equivariant Hermitian metric on $\mathscr{E}$. Then the classifying map $\xi = \{\xi_{ij}\}$ is given by the formula

$$\xi_{ij}(x) := \mathfrak{m}(\tau^* t_i(x), \Theta t_j(x)),$$

where $\tau^* t_i(x) := t_i(\tau(x))$ and $\Theta t_j(x) := \Theta(t_j(x))$ are short notations.

3 Differential geometric classification of "quaternionic" vector bundles

In this section we provide differential geometric realizations of the FKMM–invariant. However, this require more structure on the involutive space (X, τ). More properly, we need to pass from the *topological category* to the *smooth category*. In this section the quite general Assumption 2.1 will be replaced by the more restrictive:

Assumption 3.1 (smooth category) X is a compact, path-connected, Hausdorff *smooth* d–dimensional manifold without boundary and with a *smooth* involution τ.

In particular, a space X which fulfills Assumption 3.1 is a *closed* manifold and the pair (X,τ) automatically admits the structure of a $\mathbb{Z}_2$–CW–complex; see eg [39, Theorem 3.6]. Observe that the notion of FKMM–manifold given in Definition 2.8 is compatible with Assumption 3.1. It is worth pointing out that the *smooth* condition can be relaxed to a less demanding regularity condition; for instance it is sufficient to assume that the manifold structure is C^r–regular for some $r\in\mathbb{N}$. Anyway, this is only a technical detail and for a simpler presentation it is enough to focus only on the smooth case.

Let us point out that in Section 2.1 we introduced the notion of Q–bundle in the topological category meaning that all the maps involved in the various definitions are continuous functions between topological spaces. However, when the involutive space (X,τ) has an additional smooth manifold structure one can equivalently define Q–bundles in the smooth category by requiring that all spaces involved in the definitions carry a smooth manifold structure and maps are smooth functions. However, for what concerns the problem of the classification, the two categories are equivalent [9, Theorem 2.1], namely

$$^{\mathrm{top}}\,\mathrm{Vec}^m_Q(X,\tau)\simeq{}^{\mathrm{smooth}}\mathrm{Vec}^m_Q(X,\tau).$$

Clearly, the same holds true also in the "real" category. For more details on this point we refer to [9, Section 2].

3.1 Principal "quaternionic" bundles and related FKMM–invariant

The next definition was introduced in [9, Section 2.1].

Definition 3.2 (principal R– and Q–bundle) Let (X,τ) be an involutive space which satisfies Assumption 3.1 and $\pi:\mathscr{P}\to X$ a (smooth) principal $\mathbb{U}(m)$–bundle. We say that $\mathscr{P}$ has a *"real" structure* if there is a homeomorphism $\widehat{\Theta}:\mathscr{P}\to\mathscr{P}$ such that:

(Eq.) The bundle projection π is *equivariant* in the sense that $\pi\circ\widehat{\Theta}=\tau\circ\pi$.

(Inv.) $\widehat{\Theta}$ is an *involution*, ie $\widehat{\Theta}^2(p)=p$ for all $p\in\mathscr{P}$.

($\widehat{R}$) The right $\mathbb{U}(m)$–action on the fibers and the homeomorphism $\widehat{\Theta}$ fulfill the condition

$$\widehat{\Theta}(R_u(p))=R_{\bar{u}}(\widehat{\Theta}(p)),\quad \text{for all } p\in\mathscr{P}\text{ and } u\in\mathbb{U}(m),$$

where $R_u(p)=p\cdot u$ denotes the right $\mathbb{U}(m)$–action and $\bar{u}$ is the complex conjugate of u.

We say that $\mathscr{P}$ has a *"quaternionic" structure* if the structure group $\mathbb{U}(2m)$ has even rank and condition $(\hat{R})$ is replaced by:

($\hat{Q}$) The right $\mathbb{U}(2m)$–action on the fibers and the homeomorphism $\hat{\Theta}$ fulfill the condition
$$\hat{\Theta}(R_u(p)) = R_{\sigma(u)}(\hat{\Theta}(p)), \quad \text{for all } \ p \in \mathscr{P} \text{ and } u \in \mathbb{U}(2m),$$
where $\sigma\colon \mathbb{U}(2m) \to \mathbb{U}(2m)$ is the involution given by
$$\sigma(u) := Q \cdot \bar{u} \cdot Q^{-1} = -Q \cdot \bar{u} \cdot Q$$
and Q is the matrix (2-2).

We will often refer to principal "real" and "quaternionic" bundles with the abbreviations principal R–bundles and principal Q–bundles, respectively.

Remark 3.3 Let us notice that both the "real" and the "quaternionic" case require that $\hat{\Theta}$ has to be an involution. This means that both principal R– and Q–bundles are examples of $\mathbb{Z}_2$*–equivariant* principal bundles (indeed, properties (Eq.) and (Inv.) define these objects). This is indeed a difference with respect to the vector bundle case; cf Definition 2.2.

Morphisms (and isomorphisms) between principal R– and Q–bundles are defined in a natural way: if $(\mathscr{P}, \hat{\Theta})$ and $(\mathscr{P}', \hat{\Theta}')$ are two such principal bundles over the same involutive space (X, τ) then an R– or Q–morphism is a principal bundle morphism $f : \mathscr{P} \to \mathscr{P}'$ such that $f \circ \hat{\Theta}' = \hat{\Theta} \circ f$. We will use the symbols $\mathrm{Prin}_R^{\mathbb{U}(m)}(X, \tau)$ and $\mathrm{Prin}_Q^{\mathbb{U}(2m)}(X, \tau)$ for the sets of equivalence classes of principal "real" and "quaternionic" bundles over (X, τ), respectively. A principal R–bundle over (X, τ) is called *trivial* if it is isomorphic to the product bundle $X \times \mathbb{U}(m)$ with *trivial* R–structure $\hat{\Theta}_0\colon (x, u) \mapsto (\tau(x), \bar{u})$. In much the same way, a *trivial* principal Q–bundle is isomorphic to the product bundle $X \times \mathbb{U}(2m)$ endowed with the *trivial* Q–structure $\hat{\Theta}_0\colon (x, u) \mapsto (\tau(x), \sigma(u))$.

A standard result says that there is an equivalence of categories between principal $\mathbb{U}(m)$–bundles and complex vector bundles. This equivalence is realized by the *associated bundle* construction along its inverse, called *orthonormal frame bundle* construction; see [9, Appendix B] for more details. A similar result extends to the "real" and the "quaternionic" categories [9, Proposition 2.4] leading to

$$\mathrm{Prin}_R^{\mathbb{U}(m)}(X, \tau) \simeq \mathrm{Vec}_R^m(X, \tau), \quad \mathrm{Prin}_Q^{\mathbb{U}(2m)}(X, \tau) \simeq \mathrm{Vec}_Q^{2m}(X, \tau). \tag{3-1}$$

We can take advantage of the above isomorphisms to carry the notion of FKMM–invariant from vector bundles to principal bundles.

Definition 3.4 (FKMM–invariant: principal bundle version) Let $(\mathscr{P}, \widehat{\Theta})$ be a rank $2m$ principal Q–bundle over the involutive space (X, τ). Let $[(\mathscr{E}, \Theta)] \in \mathrm{Vec}_Q^{2m}(X, \tau)$ be the unique class associated with $[(\mathscr{P}, \widehat{\Theta})] \in \mathrm{Prin}_Q^{\mathbb{U}(2m)}(X, \tau)$ by the isomorphism (3-1). One defines the *FKMM–invariant* of $(\mathscr{P}, \widehat{\Theta})$ as the FKMM–invariant of the associated Q–bundle $(\mathscr{E}, \Theta)$, namely

$$\kappa(\mathscr{P}, \widehat{\Theta}) := \kappa(\mathscr{E}, \Theta).$$

Remark 3.5 Let us briefly discuss the consistency of Definition 3.4 with the construction of the FKMM–invariant presented in [10]. In view of the isomorphisms (3-1) to each $\mathbb{U}(2m)$ principal Q–bundle $(\mathscr{P}, \widehat{\Theta})$, one can associate a unique (up to isomorphism) $\mathbb{U}(1)$ principal R–bundle $(\det(\mathscr{P}), \det(\widehat{\Theta}))$ which is defined as the unique (up to isomorphism) $\mathbb{U}(1)$ principal R–bundle associated with the rank one R–bundle $(\det(\mathscr{E}), \det(\Theta))$. Moreover, there is a one-to-one correspondence between sections of a $\mathbb{U}(1)$ principal R–bundle and sections of a rank one R–bundle of unit norm. Then, the quantity $\kappa(\mathscr{P}, \widehat{\Theta})$ turns out to be determined by the equivalence class of the pair $(\det(\mathscr{P}), s_{\mathscr{P}})$ where $s_{(\mathscr{P},\widehat{\Theta})} \equiv s_{(\mathscr{E},\Theta)}$ is the canonical section associated to $(\mathscr{E}, \Theta)$. For more details about the relation between the FKMM–invariant and the canonical section we refer to [8, Section 3.2] or [10, Section 2.2].

3.2 "Quaternionic" connections and curvatures

Connections with "quaternionic" and "real" structures have been studied in Section 2.2 of [9]. We review here the basic definitions and the main properties of these objects. For a reminder about the theory of connections we refer to the classic monographs [33; 34]; see also [9, Appendix B].

We consider principal bundles in the smooth category $\pi\colon \mathscr{P} \to X$ endowed with a "real" or "quaternionic" structure $\widehat{\Theta}\colon \mathscr{P} \to \mathscr{P}$ over the involutive space (X, τ). The structure group is $\mathbb{U}(m)$ (m even in the "quaternionic" case) and $\mathfrak{u}(m)$ is the related Lie algebra. The symbol $\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(m))$ will be used for the *connection* 1–forms associated to given horizontal distributions $p \mapsto H_p$ of $\mathscr{P}$. We observe that the Lie algebra $\mathfrak{u}(m)$ has two natural involutions: a *real* involution $\mathfrak{u}(m) \ni \xi \mapsto \bar{\xi} \in \mathfrak{u}(m)$ and a *quaternionic* involution $\mathfrak{u}(2m) \ni \xi \mapsto \sigma(\xi) := -Q \cdot \bar{\xi} \cdot Q \in \mathfrak{u}(2m)$. Here $\xi \in \mathfrak{u}(m)$ is any anti-Hermitian matrix of size m and the matrix Q was defined in (2-2). Finally, given a k–form $\phi \in \Omega^k(\mathscr{P}, \mathscr{A})$ with values in some structure $\mathscr{A}$ (module, ring, algebra, group, etc) and a smooth map $f\colon \mathscr{P} \to \mathscr{P}$, we denote by $f^*\phi := \phi \circ f_*$ the *pullback* of ϕ with respect to the map f (and $f_*\colon T\mathscr{P} \to T\mathscr{P}$ is the differential, or *pushforward*, of vector fields). Given a $\mathfrak{u}(m)$–valued k–form $\phi \in \Omega^k(\mathscr{P}, \mathfrak{u}(m))$, we define the *complex conjugate* form $\bar{\phi}$ pointwise, ie $\bar{\phi}_p(\mathrm{w}^1_p, \dots, \mathrm{w}^k_p) := \overline{\phi_p(\mathrm{w}^1_p, \dots, \mathrm{w}^k_p)}$ for every k–tuple $\{\mathrm{w}^1_p, \dots, \mathrm{w}^k_p\}$ of tangent vectors at $p \in \mathscr{P}$. It follows that $f^*\bar{\phi} = \overline{f^*\phi}$ for every smooth map $f : \mathscr{P} \to \mathscr{P}$. Similarly, if $\phi \in \Omega^k(\mathscr{P}, \mathfrak{u}(2m))$, we define $\sigma(\phi)$ pointwise by $\sigma(\phi)_p(\mathrm{w}^1_p, \dots, \mathrm{w}^k_p) := -Q \cdot \overline{\phi_p(\mathrm{w}^1_p, \dots, \mathrm{w}^k_p)} \cdot Q$. Hence, one has that $\sigma(f^*\phi) = f^*\sigma(\phi)$.

Definition 3.6 (“real” and “quaternionic” equivariant connections) Let (X, τ) be an involutive space that satisfies Assumption 3.1 and $\pi : \mathscr{P} \to X$ a smooth principal $\mathbb{U}(m)$–bundle over X endowed with a “real” or a “quaternionic” structure $\widehat{\Theta} : \mathscr{P} \to \mathscr{P}$ as in Definition 3.2. A connection 1–form $\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(m))$ is said to be *equivariant* if $\bar{\omega} = \widehat{\Theta}^*\omega$ in the “real” case or $\sigma(\omega) = \widehat{\Theta}^*\omega$ in the “quaternionic” case. Equivariant connections in the “real” case are called “real” connections (or R–connections). Similarly, the “quaternionic” connections (or Q–connections) are the equivariant connections in the “quaternionic” category.

Let $\mathfrak{A}_R(\mathscr{P}) \subset \Omega^1(\mathscr{P}, \mathfrak{u}(m))$ be the space of R–connections on the principal R–bundle $(\mathscr{P}, \widehat{\Theta})$. Similarly, $\mathfrak{A}_Q(\mathscr{P}) \subset \Omega^1(\mathscr{P}, \mathfrak{u}(2m))$ will denote the space of Q–connections on the principal Q–bundle $(\mathscr{P}, \widehat{\Theta})$. Let us introduce the sets of equivariant 1–forms

$$\begin{aligned} \Omega^1_R(\mathscr{P}, \mathfrak{u}(m)) &:= \{\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(m)) \mid \bar{\omega} = \widehat{\Theta}^*\omega\}, \\ \Omega^1_Q(\mathscr{P}, \mathfrak{u}(2m)) &:= \{\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(2m)) \mid \sigma(\omega) = \widehat{\Theta}^*\omega\}. \end{aligned} \tag{3-2}$$

A 1–form is called *horizontal* if it vanishes on vertical vectors. The set of $\mathfrak{u}(m)$–valued 1–forms on $\mathscr{P}$ which are horizontal and which transform according to the adjoint representation of the structure group is denoted by $\Omega^1_{\mathrm{hor}}(\mathscr{P}, \mathfrak{u}(m), \mathrm{Ad})$. Let us introduce the sets

$$\begin{aligned} \mathcal{V}^1_R(\mathscr{P}) &:= \Omega^1_{\mathrm{hor}}(\mathscr{P}, \mathfrak{u}(m), \mathrm{Ad}) \cap \Omega^1_R(\mathscr{P}, \mathfrak{u}(m)), \\ \mathcal{V}^1_Q(\mathscr{P}) &:= \Omega^1_{\mathrm{hor}}(\mathscr{P}, \mathfrak{u}(2m), \mathrm{Ad}) \cap \Omega^1_Q(\mathscr{P}, \mathfrak{u}(2m)). \end{aligned}$$

Proposition 3.7 [9, Propositions 2.11 and 2.12] *The sets $\mathfrak{A}_R(\mathscr{P})$ and $\mathfrak{A}_Q(\mathscr{P})$ are nonempty and are closed under convex combinations with real coefficients. Moreover, they are affine spaces modeled on the vector spaces $\mathcal{V}^1_R(\mathscr{P})$ and $\mathcal{V}^1_Q(\mathscr{P})$, respectively.*

Let F_ω be the curvature associated to the equivariant connection ω by the *structural equation*

$$F_\omega := \mathrm{d}\omega + \tfrac{1}{2}[\omega \wedge \omega].$$

According to [9, Proposition 2.22] one has that F_ω obeys the equivariant constraints

$$\begin{aligned} \overline{F}_\omega &= \widehat{\Theta}^* F_\omega \quad (\text{“real” case}), \\ \sigma(F_\omega) &= \widehat{\Theta}^* F_\omega \quad (\text{“quaternionic” case}). \end{aligned} \tag{3-3}$$

Let $\{\mathcal{F}_\alpha \in \Omega^2(\mathcal{U}_\alpha, \mathfrak{g})\}$ be the collection of local 2–forms which provides the local description of the curvature F_ω — in the sense of [9, Theorem C.2]. When ω is equivariant, it holds true that

$$\begin{aligned} \overline{\mathcal{F}}_\alpha &= \tau^* \mathcal{F}_\alpha \quad (\text{“real” case}), \\ \sigma(\mathcal{F}_\alpha) &= \tau^* \mathcal{F}_\alpha \quad (\text{“quaternionic” case}). \end{aligned} \tag{3-4}$$

3.3 Chern–Simons form and “quaternionic” structure

In this section we discuss some aspects of Chern–Simons theory defined over (compact) manifolds without boundary in the presence of a Q–structure. For a comprehensive introduction to Chern–Simons theory we refer to [15; 28].

Let $\pi : \mathscr{P} \to X$ be a (smooth) principal $\mathbb{U}(m)$–bundle and $\omega \in \Omega^1(\mathscr{P}, \mathfrak{u}(m))$ a connection 1–form. The *Chern–Simons* 3*–form* $\mathcal{CS}(\omega) \in \Omega^3(\mathscr{P})$ associated to ω is defined by

$$\mathcal{CS}(\omega) := \frac{1}{8\pi^2} \operatorname{Tr}\Big(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\Big), \tag{3-5}$$

where Tr is the usual trace on $m \times m$ matrices. The 3–form $\mathcal{CS}(\omega)$ is sometimes called *Chern–Simons Lagrangian*. A direct computation shows that the result of applying the exterior differential to $\mathcal{CS}(\omega)$ can be expressed in terms of the curvature $F_\omega \in \Omega^2(\mathscr{P}, \mathfrak{u}(m))$ according to

$$\mathrm{d}\mathcal{CS}(\omega) := \frac{1}{4\pi^2} \operatorname{Tr}(F_\omega \wedge F_\omega) \in \Omega^4(\mathscr{P}). \tag{3-6}$$

The following result will be used several times in the continuation of this work.

Lemma 3.8 *Assume that $\pi : \mathscr{P} \to X$ admits a (smooth) section $s : X \to \mathscr{P}$ and let $g : X \to \mathbb{U}(m)$ be a (smooth) map. Define a new section $s_g : X \to \mathscr{P}$ using the right action of $\mathbb{U}(m)$, that is, $s_g(x) := s(x) \cdot g(x)$. Then the two pullbacks $s_g^*\mathcal{CS}(\omega), s^*\mathcal{CS}(\omega) \in \Omega^3(X)$ are related by the equation*

$$s_g^*\mathcal{CS}(\omega) = s^*\mathcal{CS}(\omega) + \frac{1}{8\pi^2} \mathrm{d}\operatorname{Tr}(s^*\omega \wedge \mathrm{d}g^{-1} g) + \Lambda(g), \tag{3-7}$$

where $\Lambda(g) \in \Omega^3(X)$ is given by

$$\Lambda(g) := -\frac{1}{24\pi^2} \operatorname{Tr}((g^{-1}\mathrm{d}g)^{\wedge 3}). \tag{3-8}$$

Proof The proof is essentially a computation which is based on the two relations $s_g^*\mathcal{CS}(\omega) = \mathcal{CS}(s_g^*\omega)$ and $s_g^*\omega = g^{-1}(s^*\omega)g + g^{-1}\mathrm{d}g$. Therefore, by exploiting the cyclicity of the trace, one can check that

$$\mathcal{CS}(g^{-1}(s^*\omega)g + g^{-1}\mathrm{d}g) \\ = \mathcal{CS}(s^*\omega) - \frac{1}{8\pi^2}\,\mathrm{dTr}(s^*\omega \wedge g^{-1}\mathrm{d}g) - \frac{1}{24\pi^2}\,\mathrm{Tr}((g^{-1}\mathrm{d}g)^{\wedge 3}).$$

The identity $0 = \mathrm{d}(g^{-1}g) = \mathrm{d}g^{-1}g + g^{-1}\mathrm{d}g$ concludes the computation. □

Definition 3.9 (Chern–Simons invariant) Let X be a compact oriented 3–dimensional manifold without boundary and $\pi\colon \mathscr{P} \to X$ a principal $\mathbb{U}(m)$–bundle equipped with a connection ω. Assume that there is a global section $s\colon X \to \mathscr{P}$. Then the quantity

$$\mathfrak{cs}(\omega) := \int_X s^*\mathcal{CS}(\omega) \mod \mathbb{Z}$$

is called the *Chern–Simons invariant* $\mathfrak{cs}(\omega) \in \mathbb{R}/\mathbb{Z}$ associated to ω.

The following result shows that the Chern–Simons invariant is well defined.

Proposition 3.10 *The Chern–Simons invariant does not dependent on the choice of a particular global section $s\colon X \to \mathscr{P}$, and depends only on the equivalence class of ω up to gauge transformations.*

Proof Two global sections of s_1 and s_2 of $\mathscr{P}$ are related by a unique map $g\colon X \to \mathbb{U}(m)$ such that $s_2(x) = s_1(x)\cdot g(x)$. Lemma 3.8, Stokes' theorem and the fact that X has no boundary imply

$$\int_X (s_1^*\mathcal{CS}(\omega) - s_2^*\mathcal{CS}(\omega)) = \int_X \Lambda(g) =: N_g \in \mathbb{Z},$$

where the integer N_g defines the "degree" of the map g. With a similar argument one can show that $\mathfrak{cs}(\omega) = \mathfrak{cs}(\omega')$ if ω and ω' are related by the transformation induced by an element of the gauge group. □

When a principal $\mathbb{U}(2m)$–bundle $\pi\colon \mathscr{P} \to X$ is endowed with a Q–structure $\widehat{\Theta}$, it is natural to use an equivariant Q–connection $\omega \in \mathfrak{A}_Q(\mathscr{P})$ to define the Chern–Simons 3–form $\mathcal{CS}(\omega)$. The Q–structure $\widehat{\Theta}$ induces a symmetry of $\mathcal{CS}(\omega)$.

Lemma 3.11 *Let $(\mathscr{P}, \widehat{\Theta})$ be a $\mathbb{U}(2m)$ Q–bundle over the involutive manifold (X, τ) which satisfies Assumption 3.1. Let $\omega \in \mathfrak{A}_Q(\mathscr{P})$ be an equivariant connection and $\mathcal{CS}(\omega) \in \Omega^3(\mathscr{P})$ the associated Chern–Simons 3–form. Then*

$$\widehat{\Theta}^*\mathcal{CS}(\omega) = \mathcal{CS}(\omega).$$

Proof The equivariance of ω means that $\widehat{\Theta}^*\omega = Q\bar{\omega}Q^{-1} = -Q({}^t\omega)Q^{-1}$, where we used $\bar{\omega} = -{}^t\omega$ since the form ω takes value in the Lie algebra $\mathfrak{u}(2m)$. The cyclicity of the trace provides

$$\widehat{\Theta}^*\mathcal{CS}(\omega) = \mathcal{CS}(\widehat{\Theta}^*\omega) = \frac{1}{8\pi^2}\operatorname{Tr}\Big({}^t\omega\wedge \mathrm{d}{}^t\omega + \frac{2}{3}\,{}^t\omega\wedge{}^t\omega\wedge{}^t\omega\Big).$$

The identity ${}^t\omega_1\wedge{}^t\omega_2 = (-1)^{q_1q_2}\,{}^t(\omega_2\wedge\omega_1)$ is valid for each pair $\omega_1\in\Omega^{q_1}(\mathscr{P},\mathfrak{u}(2m))$ and $\omega_2\in\Omega^{q_2}(\mathscr{P},\mathfrak{u}(2m))$ and the invariance of the trace under the operation of taking the *transpose* imply

$$\begin{aligned}\widehat{\Theta}^*\mathcal{CS}(\omega) &= \frac{1}{8\pi^2}\operatorname{Tr}\Big(\mathrm{d}\omega\wedge\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega\Big)\\ &= \mathcal{CS}(\omega) + \frac{1}{8\pi^2}\operatorname{Tr}(\mathrm{d}\omega\wedge\omega - \omega\wedge\mathrm{d}\omega)\\ &= \mathcal{CS}(\omega) + \frac{1}{8\pi^2}\,\mathrm{d}\operatorname{Tr}(\omega\wedge\omega).\end{aligned}$$

To conclude the proof it is enough to observe that $\operatorname{Tr}(\omega\wedge\omega)=0$ due to the anticommutation relation of 1–forms. □

The invariance of $\mathcal{CS}(\omega)$ expressed in Lemma 3.11 has an important implication on the Chern–Simons invariant in low dimensions, provided that certain conditions are met.

Proposition 3.12 *Let $(\mathscr{P},\widehat{\Theta})$ be a $\mathbb{U}(2m)$ Q–bundle over the involutive manifold (X,τ) which satisfies Assumption 3.1. Assume in addition that*

(a) *X has dimension $d=3$ and τ reverses the orientation of X;*

(b) *there is a global section $s\colon X\to\mathscr{P}$ (not necessarily equivariant).*

Then

(i) *if $\omega\in\mathfrak{A}_Q(\mathscr{P})$ is an equivariant connection then the associated Chern–Simons invariant $\mathfrak{cs}(\omega)$ takes values in the set $\{0,\frac{1}{2}\}$;*

(ii) *$\mathfrak{cs}(\omega)=\mathfrak{cs}(\omega')$ for each pair of equivariant connections $\omega,\omega'\in\mathfrak{A}_Q(\mathscr{P})$;*

(iii) *if $(\mathscr{P},\widehat{\Theta})$ admits a global equivariant section then $\mathfrak{cs}(\omega)=0$ for any $\omega\in\mathfrak{A}_Q(\mathscr{P})$.*

Proof (i) Let $s\colon X\to\mathscr{P}$ be a global section. Since $\tau_\Theta(s) := \widehat{\Theta}\circ s\circ\tau$ generally differs from s, there is a (unique) map $g\colon X\to\mathbb{U}(2m)$ such that $\tau_\Theta(s)=s\cdot g$. Then

$$\tau^*(s^*\mathcal{CS}(\omega)) = (s\circ\tau)^*\mathcal{CS}(\omega) = (s\cdot g)^*(\widehat{\Theta}^*\mathcal{CS}(\omega)) = (s\cdot g)^*\mathcal{CS}(\omega),$$

where in the last equality we used the result of Lemma 3.11. By exploiting the fact that τ reverses the orientation of X, one has

$$\int_X s^*\mathcal{CS}(\omega) = -\int_X \tau^*(s^*\mathcal{CS}(\omega)) = -\int_X (s\cdot g)^*\mathcal{CS}(\omega) = -\int_X s^*\mathcal{CS}(\omega) + N_g,$$

where $N_g := \int_X \Lambda(g) \in \mathbb{Z}$. This implies that $2\mathfrak{cs}(\omega) = 0$, ie $\mathfrak{cs}(\omega) \in \{0, \frac{1}{2}\}$.

(ii) Let ω' be a second equivariant connection and consider the map

$$[0,1] \ni t \mapsto \omega_t := (1-t)\omega + t\omega' \in \mathfrak{A}_Q(\mathscr{P}).$$

Clearly $\mathfrak{cs}(\omega_t)$ is a polynomial (hence continuous) function in t. On the other hand $\mathfrak{cs}(\omega_t) \in \{0, \frac{1}{2}\}$ since ω_t is equivariant. This implies that $\mathfrak{cs}(\omega_{t_1}) = \mathfrak{cs}(\omega_{t_2})$ for all $t_1, t_2 \in [0,1]$ and in particular $\mathfrak{cs}(\omega) = \mathfrak{cs}(\omega')$.

(iii) If s is a global equivariant section, one has

$$\tau^*(s^*\mathcal{CS}(\omega)) = \tau^*(s^*\mathcal{CS}(\omega)) = \tau^*(s^*(\widehat{\Theta}^*\mathcal{CS}(\omega))) = \tau_\Theta(s)^*\mathcal{CS}(\omega) = s^*\mathcal{CS}(\omega).$$

Hence,

$$\int_X s^*\mathcal{CS}(\omega) = \int_X \tau^*(s^*\mathcal{CS}(\omega)) = -\int_X s^*\mathcal{CS}(\omega),$$

which implies $\int_X s^*\mathcal{CS}(\omega) = 0$. □

Remark 3.13 Due to the low-dimensional assumption (a) in Proposition 3.12, the assumption (b) about the existence of a global section is completely equivalent to the condition of vanishing of the first Chern class of the principal bundle. This condition is guaranteed by the stronger requirements: (1) $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$, or (2) $H^2(X, \mathbb{Z}) = 0$.

The following definition is justified by item (ii) of Proposition 3.12.

Definition 3.14 (intrinsic Chern–Simons invariant) Let $(\mathscr{P}, \widehat{\Theta})$ be a $\mathbb{U}(2m)$ Q–bundle over the involutive manifold (X, τ) such that X has dimension $d = 3$, τ reverses the orientation of X and $\mathscr{P}$ admits a global section. Then the quantity

$$\mathfrak{cs}(\mathscr{P}, \widehat{\Theta}) := \mathfrak{cs}(\omega) \quad \text{for some } \omega \in \mathfrak{A}_Q(\mathscr{P})$$

does not depend on the choice of $\omega \in \mathfrak{A}_Q(\mathscr{P})$ and defines an *intrinsic* (Chern–Simons) invariant for $(\mathscr{P}, \widehat{\Theta})$.

Remark 3.15 (a formula for the Chern–Simons invariant) Let (X,τ) be a three-dimensional involutive manifold satisfying the assumption $H^2_{\mathbb{Z}_2}(X,\mathbb{Z}(1))=0$. As a consequence of Lemma 2.14 and the isomorphism (3-1), any $\mathbb{U}(2m)$ Q–bundle $(\mathscr{P},\widehat{\Theta})$ over (X,τ) can be represented by a smooth map $\xi\colon X\to\mathbb{U}(2m)$ such that $\tau^*\xi=-Q\bar{\xi}^{-1}Q$. The average construction applied to the trivial connection on the product bundle [9, Example 2.15] gives an equivariant connection ω, whose pullback under the trivial section s is $s^*\omega=\frac{1}{2}\sigma(\xi^{-1}d\xi)$. We then have $s^*\mathcal{CS}(\omega)=\frac{1}{2}\Lambda(\xi)$, and hence the formula

$$\mathfrak{cs}(\mathscr{P},\widehat{\Theta})=\frac{1}{2}\int_X\Lambda(\xi)\mod\mathbb{Z}.$$

This formula can be compared with [16, Proposition 11.21].

3.4 Wess–Zumino term in absence of boundaries

In the last section we described the Chern–Simons invariant in the case of three-dimensional base manifolds without boundary. In the case of manifolds with boundary the Chern–Simons invariant itself depends on the choice of a section while the difference of the values of the Chern–Simons invariants depends only on the topological information on the boundary. This information is detected by the so-called *Wess–Zumino term*. The latter is a topological quantity initially defined in the context of certain two-dimensional conformal field theories known as Wess–Zumino–Witten models. An excellent introduction to the theory of Wess–Zumino–Witten models is provided by the lecture notes [20]. The presentation of the properties of the Wess–Zumino term given here follows mainly [15].

Definition 3.16 (Wess–Zumino term) Let Σ be a compact oriented manifold of dimension $d=2$ without boundary. For any map $\xi\colon\Sigma\to\mathbb{SU}(2)$, the *Wess–Zumino term* $\mathcal{WZ}_\Sigma(\xi)\in\mathbb{R}/\mathbb{Z}$ is defined by

$$\mathcal{WZ}_\Sigma(\xi):=\int_{X_\Sigma}\Lambda(\tilde{\xi})\mod\mathbb{Z},$$

where

$$\Lambda(\tilde{\xi}):=-\frac{1}{24\pi^2}\operatorname{Tr}(\tilde{\xi}^{-1}\mathrm{d}\tilde{\xi})^3$$

according to the notation (3-8), X_Σ is any compact three-dimensional oriented manifold whose boundary coincides with Σ, ie $\partial X_\Sigma=\Sigma$, and $\tilde{\xi}\colon X_\Sigma\to\mathbb{SU}(2)$ is any extension of ξ.

Notice that the extended manifold X_Σ and the extended section $\tilde{\xi}$ in Definition 3.16 always exist. The existence of X_Σ follows from the vanishing of the second *bordism group*,[1] $\Omega_2 = 0$ [40, Section 7]. The existence of $\tilde{\xi}$ is due to $\pi_k(\mathbb{SU}(2)) = 0$ for $k = 0, 1, 2$ plus a standard application of the Oka's (type) principle to pass from continuous sections to smooth sections. Finally, the condition $\xi\colon \Sigma \to \mathbb{SU}(2)$ can be relaxed by asking that the section $\xi\colon \Sigma \to \mathbb{U}(2)$ possesses a determinant section $\det(\xi)\colon \Sigma \to \mathbb{U}(1)$ which is nullhomotopic.

The well-posedness of Definition 3.16 is justified in the following result.

Lemma 3.17 (Polyakov–Wiegmann formula) *The Wess–Zumino term is independent of the choice of the extensions X_Σ and $\tilde{\xi}$. Moreover, for every pair of sections $\xi_j : \Sigma \to \mathbb{SU}(2)$, $j = 1, 2$, the* ***Polyakov–Wiegmann formula***

$$\mathcal{WZ}_\Sigma(\xi_1\xi_2) = \mathcal{WZ}_\Sigma(\xi_1) + \mathcal{WZ}_\Sigma(\xi_2) + \frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(\xi_1^{-1}\mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2\xi_2^{-1})$$

holds in $\mathbb{R}/\mathbb{Z}$.

Proof Given Σ and $\xi\colon \Sigma \to \mathbb{SU}(2)$ as in Definition 3.16, consider two extended manifolds X_Σ and X'_Σ such that $\partial X_\Sigma = \Sigma = \partial X'_\Sigma$, and two extended sections $\tilde{\xi}$ and $\tilde{\xi}'$ such that $\tilde{\xi}|_\Sigma = \xi = \tilde{\xi}'|_\Sigma$. By reversing the orientation of X'_Σ and then gluing it with X_Σ along Σ one obtains a compact oriented three-dimensional manifold $X := (-X'_\Sigma) \sqcup X_\Sigma$, where the minus sign indicates the reversal of the orientation. Similarly, $\tilde{\xi}$ and $\tilde{\xi}'$ can be glued together to define a section $\xi_X := (\tilde{\xi} \sqcup \tilde{\xi}')\colon X \to \mathbb{SU}(2)$. It is well known that

$$\int_X \Lambda\,(\xi_M) = -\frac{1}{24\pi^2}\int_X \mathrm{Tr}\left(\xi_X^{-1}\mathrm{d}\xi_X\right)^{\wedge 3} \in \mathbb{Z}.$$

On the other hand, one has that

$$\int_X \Lambda(\xi_X) = \int_{X_\Sigma} \Lambda(\tilde{\xi}) - \int_{X'_\Sigma} \Lambda(\tilde{\xi}') \in \mathbb{Z},$$

where the minus sign is justified by the inversion of the orientation. Thus, since the Wess–Zumino term $\mathcal{WZ}_\Sigma(\xi)$ is defined modulo an integer, it can be computed equivalently through the pair $X_\Sigma, \tilde{\xi}$ or the pair $X'_\Sigma, \tilde{\xi}'$. The Polyakov–Wiegmann formula for $\mathcal{WZ}_\Sigma(\xi_1\xi_2)$ follows from an explicit computation. By taking extensions of ξ_1 and ξ_2 one computes $\Lambda(\xi_1\xi_2) - \Lambda(\xi_1) - \Lambda(\xi_2)$ directly. Then, integration over

[1]The existence of X_Σ can be also justified by observing that closed oriented two-dimensional manifolds are classified by the genus, and a genus g surface is always the boundary of a three-dimensional manifold. For instance the sphere $\mathbb{S}^2$ is the boundary of the three-dimensional disk $\mathbb{D}^3$. Similarly the torus $\mathbb{T}^2$ is the boundary of the manifold $\mathbb{S}^1 \times \mathbb{D}^2$. The same occurs for higher genus surfaces.

X_Σ and an application of Stokes' theorem to obtain the integral on the boundary Σ provide the final result. □

From formula (3-7) and Stokes' theorem one immediately deduces the following result:

Lemma 3.18 *Let X be a compact oriented manifold of dimension $d = 3$ with nonempty boundary $\Sigma := \partial X$. Let $\pi : \mathscr{P} \to X$ be a principal $\mathbb{U}(2)$–bundle equipped with a connection ω and a global (smooth) section $s\colon X \to \mathscr{P}$. Let $g\colon X \to \mathbb{U}(2)$ be any (smooth) map such that $\det(g)\colon X \to \mathbb{U}(1)$ is nullhomotopic. Then*

$$\int_X s_g^* \mathcal{CS}(\omega) - \int_X s^* \mathcal{CS}(\omega) = -\frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(s^*\omega \wedge \mathrm{d}g^{-1} g) + \mathcal{WZ}_\Sigma(g|_\Sigma) \mod \mathbb{Z}.$$

3.5 Wess–Zumino term in presence of boundaries

In the rest of this work we will be interested in calculating the Wess–Zumino term through "cutting and pasting". To set up the machinery, we need to extend the definition of the Wess–Zumino term for two-dimensional manifolds with boundary. To do that, let us observe that associated to a compact oriented one-dimensional manifold S without boundary (a union of circles), there exists a Hermitian line bundle $\mu\colon \mathscr{L}_S \to \mathrm{Map}(S, \mathbb{SU}(2))$. The specific structure of this line bundle will be not used in this work and for this reason the details of the construction of $\mathscr{L}_S$ will be only sketched. The interested reader can refer to [15, Appendix A] or to [35, Section 1.3] for a more rigorous presentation.

Given S, consider a two-dimensional manifold D_S (a disjoint union of disks) with boundary $\partial D_S = S$ along with the space $\mathrm{Map}(D_S, \mathbb{SU}(2))$. Given an element $\tilde{\gamma}$ in $\mathrm{Map}(D_S, \mathbb{SU}(2))$, its restriction $\gamma := \tilde{\gamma}|_S$ defines an element in $\mathrm{Map}(S, \mathbb{SU}(2))$. Let $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathrm{Map}(D_S, \mathbb{SU}(2))$ be two maps which agree on the boundary S, namely such that $\gamma_1 = \gamma_2$. Such two maps can be glued together to produce a map $\xi_{(1,2)} := \tilde{\gamma}_1 \sqcup \tilde{\gamma}_2$ on the two-dimensional manifold without boundary $\Sigma_S := (-D_S) \sqcup D_S$ obtained by gluing two copies of D_S (with opposite orientation) along the common boundary. As a consequence the quantity $\mathcal{WZ}_{\Sigma_S}(\xi_{(1,2)})$ turns out to be well defined according to Definition 3.16. Consider now the space

$$\mathscr{L}_S := (\mathrm{Map}(D_S, \mathbb{SU}(2)) \times \mathbb{C})/\sim,$$

where the equivalence relation $\sim$ is defined as follows: let $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathrm{Map}(D_S, \mathbb{SU}(2))$ and $z_1, z_2 \in \mathbb{C}$; then

$$(\tilde{\gamma}_1, z_1) \sim (\tilde{\gamma}_2, z_2) \iff \gamma_1 = \gamma_2,\ z_1 = z_2 \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_S}(\xi_{(1,2)})}.$$

The space $\mathscr{L}_S$ defined in this way turns out to be the total space of a complex line bundle over $\mathrm{Map}(S, \mathbb{SU}(2))$ with projection $\mu\colon \mathscr{L}_S \to \mathrm{Map}(S, \mathbb{SU}(2))$ given by

$$\mu\colon [\tilde{\gamma}, z] \mapsto \gamma := \tilde{\gamma}|_S,$$

where $\gamma := \tilde{\gamma}|_S$ is independent of the choice of the representative by construction.

Henceforth, only the following properties of the line bundle $\mu\colon \mathscr{L}_S \to \mathrm{Map}(S, \mathbb{SU}(2))$ will be relevant [15, Proposition A.1]:

(i) For $\gamma_1, \gamma_2 \in \mathrm{Map}(S, \mathbb{SU}(2))$ let $\gamma_1\gamma_2 \in \mathrm{Map}(S, \mathbb{SU}(2))$ defined by pointwise multiplication. Then, there is an isometry

$$\mu^{-1}(\gamma_1) \otimes \mu^{-1}(\gamma_2) \to \mu^{-1}(\gamma_1\gamma_2). \tag{3-9}$$

(ii) The *product of fibers* (3-9) defined by the isometry is associative.

(iii) If $\gamma_0 \in \mathrm{Map}(S, \mathbb{SU}(2))$ is the constant map, there is a trivialization $\mu^{-1}(\gamma_0) \simeq \mathbb{C}$ which respects (3-9).

All the ingredients are now available for extending Definition 3.16 to manifolds with boundary.

Definition 3.19 (Wess–Zumino term with boundary) Let Σ be a compact oriented manifold of dimension $d = 2$ with one-dimensional (compact and oriented) boundary $S := \partial\Sigma$. Let $\mu\colon \mathscr{L}_S \to \mathrm{Map}(S, \mathbb{SU}(2))$ be the associated line bundle. Every $\xi\colon \Sigma \to \mathbb{SU}(2)$ gives rise to a point $\xi|_S \in \mathrm{Map}(S, \mathbb{SU}(2))$ and an associated fiber $\mu^{-1}(\xi|_S) \subset \mathscr{L}_S$. Let D_S be a disjoint union of disks (contractible two-dimensional manifolds) with boundary $\partial D_S = S = \partial\Sigma$. Given any $\zeta_{D_S}\colon D_S \to \mathbb{SU}(2)$ such that $\zeta_{D_S}|_S = \xi|_S$, let $\xi \sqcup \zeta_{D_S}$ be the map defined on the closed manifold $\Sigma_D := \Sigma \sqcup (-D_S)$ by the gluing of the functions ζ_{D_S} and ξ along the common boundary S. The Wess–Zumino term $\mathcal{WZ}_\Sigma(\xi)$ is then defined by the equation

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi)} := [\zeta_{D_S}, \mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_{\Sigma_D}(\xi \sqcup \zeta_{D_S})}] \in \mu^{-1}(\xi|_S).$$

To introduce the next result it is worth mentioning that given a complex vector bundle $\mathscr{E} \to X$, its *conjugate* $\overline{\mathscr{E}} \to X$ is the complex vector bundle whose underlying total space agrees with $\mathscr{E}$ as a set, but with inverted complex structure with respect to the multiplication by scalars $z \in \mathbb{C}$. If $\mathscr{E}$ is endowed with a Hermitian metric, then so is $\overline{\mathscr{E}}$. This allows the identification of $\overline{\mathscr{E}}$ with the *dual* vector bundle $\mathscr{E}^*$.

Proposition 3.20 (orientation) (i) *Let S be a compact oriented one-dimensional manifold without boundary, and $-S$ the same manifold with reversed orientation. Then there exists a natural isometric isomorphism*

$$\mathscr{L}_{-S} \simeq \overline{\mathscr{L}}_S.$$

(ii) *Let Σ be a compact oriented two-dimensional manifold with boundary, and $-\Sigma$ the same manifold with reversed orientation. Then for any $\xi\colon \Sigma \to \mathbb{SU}(2)$,*

$$\mathcal{WZ}_{-\Sigma}(\xi) = -\mathcal{WZ}_{\Sigma}(\xi).$$

Property (i) of Proposition 3.20 is a direct consequence of the construction of the space $\mathscr{L}_S$. Property (ii) follows from Definition 3.19 under the isometry described in (i).

Remark 3.21 (central extension of the loop group) Definition 3.19 will be mainly applied to two-dimensional manifolds Σ such that $\partial\Sigma \simeq \mathbb{S}^1$. In this case we will write $\mathscr{L}_{\mathbb{S}^1}$ instead of $\mathscr{L}_{\partial\Sigma}$. The set $\mathrm{Map}(\mathbb{S}^1, \mathbb{SU}(2))$ endowed with the pointwise multiplication is known as the *loop group* of $\mathbb{SU}(2)$ [43], and will be denoted here by $\mathrm{Loop}_{\mathbb{SU}(2)}$. The total space $S(\mathscr{L}_{\mathbb{S}^1})$ of the principal $\mathbb{U}(1)$–bundle (also known as *circle bundle*) associated to $\mathscr{L}_{\mathbb{S}^1}$ inherits a group structure from the product of fibers (3-9). This gives rise to a central extension of $\mathrm{Loop}_{\mathbb{SU}(2)}$,

$$1 \to \mathbb{U}(1) \to S(\mathscr{L}_{\mathbb{S}^1}) \to \mathrm{Loop}_{\mathbb{SU}(2)} \to 1.$$

Let $\xi_0\colon \Sigma \to \mathbb{SU}(2)$ be the constant map with value the identity matrix $\mathbb{1}_{\mathbb{C}^2} \in \mathbb{SU}(2)$. By definition of the product of fibers (3-9) one has that $[\xi_0, \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_D}(\xi_0 \sqcup \xi_0)}]$ acts as the unit of the group $S(\mathscr{L}_{\mathbb{S}^1})$. Therefore, by invoking Definition 3.19 one obtains that $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma}(\xi_0)} \in \mathscr{L}_{\mathbb{S}^1}$ provides the unit of the central extension $S(\mathscr{L}_{\mathbb{S}^1})$. For a more complete description of this central extension the reader is referred to [15; 35; 43].

The link between Definitions 3.16 and 3.19 is provided by the following result.

Proposition 3.22 (gluing property) *Let Σ be a compact oriented two-dimensional manifold without boundary. Assume that Σ can be cut along an embedded circle $\mathbb{S}^1$ to get two compact oriented two-dimensional manifolds Σ_1 and Σ_2 such that $\partial\Sigma_1 \simeq -\mathbb{S}^1$ and $\partial\Sigma_2 \simeq \mathbb{S}^1$ in such a way that $\Sigma = \Sigma_1 \sqcup \Sigma_2$. Then for any $\xi\colon \Sigma \to \mathbb{SU}(2)$,*

$$\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma}(\xi)} = \langle \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_1}(\xi|_{\Sigma_1})}; \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_2}(\xi|_{\Sigma_1})} \rangle, \tag{3-10}$$

where $\langle \cdot\,;\cdot \rangle$ denotes the contraction between

$$\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_1}(\xi|_{\Sigma_1})} \in \mathscr{L}_{\mathbb{S}^1} \quad \textit{and} \quad \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_2}(\xi|_{\Sigma_2})} \in \mathscr{L}^*_{\mathbb{S}^1}.$$

Equation (3-10) can be reformulated in the suggestive formula

$$\mathcal{WZ}_{\Sigma}(\xi) = \mathcal{WZ}_{\Sigma_1}(\xi|_{\Sigma_1}) - \mathcal{WZ}_{\Sigma_2}(\xi|_{\Sigma_2}) \mod \mathbb{Z}.$$

A proof of a generalized version of Proposition 3.22 can be found in [35, Section 1.3].

Although simplified, the version of the gluing property described in Proposition 3.22 is sufficient for the purposes of this work. Indeed, the gluing property will be mainly applied to the situation described below.

Remark 3.23 Let Σ_1 and Σ_2 be compact oriented two-dimensional manifolds without boundary. Assume that an embedded disk D can be cut out from both the manifolds in such a way that $\Sigma_1 = \Sigma_1' \sqcup D$ and $\Sigma_2 = \Sigma_2' \sqcup D$ where Σ_1' and Σ_2' are two-dimensional manifolds with boundaries $\partial\Sigma_1 \simeq \partial\Sigma_2 \simeq -\partial D \simeq -\mathbb{S}^1$. Let $\xi_1\colon \Sigma_1 \to \mathbb{SU}(2)$ and $\xi_2\colon \Sigma_2 \to \mathbb{SU}(2)$ be two maps such that $\xi_1|_D = \xi_2|_D$ and both ξ_1 and ξ_2 have constant value $\mathbb{1}_{\mathbb{C}^2}$ on a neighborhood of $\Sigma_1' \subset \Sigma_1$ and $\Sigma_2' \subset \Sigma_2$, respectively. Under this setting it holds that

$$\mathcal{WZ}_{\Sigma_1}(\xi_1) = \mathcal{WZ}_{\Sigma_2}(\xi_2) \mod \mathbb{Z}. \tag{3-11}$$

In fact both $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_1'}(\xi_1|_{\Sigma_1'})} \in \mathscr{L}^*_{\mathbb{S}^1}$ and $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_2'}(\xi_2|_{\Sigma_2'})} \in \mathscr{L}^*_{\mathbb{S}^1}$ describe the unit of the central extension $S(\mathscr{L}_{\mathbb{S}^1})$ as discussed in Remark 3.21. Therefore,

$$\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_1'}(\xi_1|_{\Sigma_1'})} = \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_2'}(\xi_2|_{\Sigma_2'})}, \quad \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{D}(\xi_1|_{D})} = \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{D}(\xi_2|_{D})},$$

where the second equality follows from the assumption $\xi_1|_D = \xi_2|_D$. By applying the gluing property (3-10) one gets $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_1}(\xi_1)} = \mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_{\Sigma_2}(\xi_2)}$ which justifies (3-11).

3.6 Classification via Wess–Zumino term in dimension two

In this section the description of rank 2 Q–bundles over an oriented two-dimensional FKMM–manifold (Σ, τ) obtained in Sections 2.6 and 2.7 will be combined with the theory of the Wess–Zumino term described in Sections 3.4 and 3.5 in order to prove that the Wess–Zumino term completely classifies $\mathrm{Vec}^2_Q(\Sigma, \tau)$.

The following three preliminary results are needed.

Lemma 3.24 *Let (Σ, τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. Let $\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ be the set of equivariant maps described by (2-10) and $[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}$ be the set of equivalence classes under the $\mathbb{Z}_2$–homotopy equivalence. Then:*

(i) *The exponentiated Wess–Zumino term of $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ takes values in $\mathbb{Z}_2$, so one gets a map*

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma} \colon \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2} \to \mathbb{Z}_2.$$

(ii) *The map above is invariant under $\mathbb{Z}_2$–homotopy, and hence induces a map*

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma} \colon [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2} \to \mathbb{Z}_2.$$

Proof (i) For every $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ the quantity $\mathcal{WZ}_\Sigma(\xi) \in \mathbb{R}/\mathbb{Z}$ is defined according to Definition 3.16. Since ξ satisfies $\tau^*\xi = \xi^{-1}$, the diffeo-invariance (functoriality) of the Wess–Zumino term [15] implies

$$\mathcal{WZ}_\Sigma(\xi) = \mathcal{WZ}_\Sigma(\tau^*\xi) = \mathcal{WZ}_\Sigma(\xi^{-1}).$$

From the relation $\zeta^{-1}\mathrm{d}\zeta = -\zeta\mathrm{d}\zeta^{-1}$, valid for generic maps with values in $\mathbb{SU}(2)$, it follows that $\mathrm{Tr}(\zeta^{-1}\mathrm{d}\zeta)^n = (-1)^n\,\mathrm{Tr}(\zeta\mathrm{d}\zeta^{-1})^n$. The application of this identity to the Wess–Zumino term implies $\mathcal{WZ}_\Sigma(\xi^{-1}) = -\mathcal{WZ}_\Sigma(\xi)$. In conclusion, one obtains that $\mathcal{WZ}_\Sigma(\xi) = -\mathcal{WZ}_\Sigma(\xi)$ modulo $\mathbb{Z}$, ie $2\mathcal{WZ}_\Sigma(\xi) \in \{0, 1\}$. This proves that the exponential map in (i) takes values in $\mathbb{Z}_2$.

(ii) If $\hat{\xi} \colon \Sigma \times [0,1] \to \mathbb{SU}(2)$ is a $\mathbb{Z}_2$–homotopy, then the map

$$[0,1] \ni t \mapsto \mathcal{WZ}_\Sigma(\hat{\xi}|_{\Sigma\times\{t\}}) \in \mathbb{R}/\mathbb{Z}$$

is continuous. Hence, the value of the exponential $\mathrm{e}^{\mathrm{i}2\pi\mathcal{WZ}_\Sigma}(\hat{\xi}|_{\Sigma\times\{t\}})$ must be constant for all t in view of the discreteness of the target space. This concludes the proof. □

Lemma 3.25 *Let (Σ, τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. For each $\epsilon \in \mathrm{Map}(\Sigma^\tau, \{\pm 1\})$ there exists $\xi_\epsilon \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ such that $\Phi_\kappa(\xi_\epsilon) = \epsilon$ and*

$$\mathrm{e}^{\mathrm{i}2\pi \mathcal{WZ}_\Sigma(\xi_\epsilon)} = \Pi(\epsilon),$$

where Π is the product sign map defined by (2-7).

Proof The proof of Lemma 2.17 contains the recipe to construct a map

$$\xi_\epsilon \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$$

for each $\epsilon \in \mathrm{Map}(\Sigma^\tau, \{\pm 1\})$ such that $\Phi_\kappa(\xi_\epsilon) = \epsilon$. Let $\Sigma^\tau = \{x_1, \dots, x_n\}$ be a labeling for the fixed point set. Let $\epsilon_i \in \mathrm{Map}(\Sigma^\tau, \{\pm 1\})$ be defined by $\epsilon_i(x_j) = 1 - 2\delta_{ij}$. Let $\xi_i := \xi_{\epsilon_i}$ be the element in $\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ such that $\Phi_\kappa(\xi_i) = \epsilon_i$. Note that ξ_i takes the value $\mathbb{1}_{\mathbb{C}^2}$ outside the disk D_i. It follows that $\xi_1, \dots, \xi_n$ commute pointwise, and the pointwise product of the ξ_i is in $\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$. Then, by construction, each ξ_ϵ can be expressed as the pointwise product of a certain number of ξ_i. Let us assume that $\xi_\epsilon = \xi_{i_1} \cdots \xi_{i_k}$. Since the supports of the differential forms $\xi_i^{-1}\mathrm{d}\xi_i$ are pairwise disjoint, the Polyakov–Wiegmann formula (see Lemma 3.17) provides

$$\mathcal{WZ}_\Sigma(\xi_\epsilon) = \mathcal{WZ}_\Sigma(\xi_{i_1}) + \cdots + \mathcal{WZ}_\Sigma(\xi_{i_k}) \mod \mathbb{Z}.$$

The next task is to evaluate the generic term $\mathcal{WZ}_\Sigma(\xi_i)$. For that, the construction in Remark 3.23 will be applied. Given $x_i \in \Sigma^\tau$, consider a small disk $D_i \subset \Sigma$ such that $\tau(D_i) = D_i$ and $x_i \in D_i$ is the only fixed point. The restriction $\xi_i|_{D_i}$ has by construction the following property: $\xi_i|_{D_i}(x_i) = -\mathbb{1}_{\mathbb{C}^2}$ and $\xi_i|_{D_i}(x) = +\mathbb{1}_{\mathbb{C}^2}$ if $x \in \partial D_i$. By an equivariant diffeomorphism, D_i can be identified with the closed unit disk $D \subset \mathbb{C}$ endowed with the involution $z \mapsto -z$ and the map $\xi_i|_{D_i}$ can be identified with the map ξ_D described in the proof of Lemma 2.17. By gluing two copies D and D' of the same disk along the common boundary $\mathbb{S}^1$ one obtains that $D \sqcup D'$ is identifiable with the equivariant sphere $\mathbb{S}^2$ with involution $(k_0, k_1, k_2) \mapsto (k_0, -k_1, -k_2)$ which fixes only the two poles $(\pm 1, 0, 0)$. Consequently, given the constant map $\xi_0 \colon D' \to \mathbb{1}_{\mathbb{C}^2}$, one has that the map $\xi_D \sqcup \xi_0 \colon D \sqcup D' \to \mathbb{SU}(2)$ can be identified with the equivariant map $\chi \colon \mathbb{S}^2 \to \mathbb{SU}(2)$ such that $\chi(\pm 1, 0, 0) = \pm \mathbb{1}_{\mathbb{C}^2}$. Since the conditions described in Remark 3.23 are met, one has that

$$\mathcal{WZ}_\Sigma(\xi_i) = \mathcal{WZ}_{\mathbb{S}^2}(\chi) \mod \mathbb{Z}.$$

A possible realization for χ is

$$\chi(k_0, k_1, k_2) = \begin{pmatrix} k_0 & -k_1 + \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & k_0 \end{pmatrix}. \tag{3-12}$$

Recall that $[\mathbb{S}^2, \mathbb{U}(1)]_{\mathbb{Z}_2} \simeq H^1_{\mathbb{Z}^2}(\mathbb{S}^2, \mathbb{Z}(1))$ [23, Proposition A.2] and

$$H^1_{\mathbb{Z}_2}(\mathbb{S}^2, \mathbb{Z}(1)) \simeq H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \oplus H^{-1}_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \simeq H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \simeq \mathbb{Z}_2$$

by [7, Lemma 5.6]. Since $H^1_{\mathbb{Z}_2}(*, \mathbb{Z}(1)) \simeq [*, \mathbb{U}(1)]_{\mathbb{Z}_2}$, it follows that $[\mathbb{S}^2, \mathbb{U}(1)]_{\mathbb{Z}_2}$ is represented by constant maps. Then the bijection $[\mathbb{S}^2, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[\mathbb{S}^2, \mathbb{U}(1)]_{\mathbb{Z}_2} \simeq \mathbb{Z}_2$ obtained from Proposition 2.18 assures that, up to a $\mathbb{Z}_2$–homotopy if necessary, one can always choose the equivariant map χ as given in (3-12). The computation of $\mathcal{WZ}_{\mathbb{S}^2}(\chi)$ with χ given by (3-12) is as follows: Consider the map $\tilde{\chi} \colon \mathbb{S}^3 \to \mathbb{SU}(2)$ defined by

$$\tilde{\chi}(k_0, k_1, k_2, k_3) = \begin{pmatrix} k_0 + \mathrm{i}k_3 & -k_1 + \mathrm{i}k_2 \\ k_1 + \mathrm{i}k_2 & k_0 - \mathrm{i}k_3 \end{pmatrix}. \tag{3-13}$$

Let $\mathbb{S}^3_+ := \{k \in \mathbb{S}^3 \mid k_3 \geqslant 0\}$ be the upper hemisphere. Then $\partial \mathbb{S}^3_+ \simeq \mathbb{S}^2$ and $\tilde{\chi}|_{\partial \mathbb{S}^3_+} = \chi$. Since $\mathbb{S}^3_+$ is just a half-sphere, one gets by a direct computation that

$$\mathcal{WZ}_{\mathbb{S}^2}(\chi) = \frac{-1}{48\pi^2} \int_{\mathbb{S}^3_+} \mathrm{Tr}(\tilde{\chi}^{-1}\mathrm{d}\tilde{\chi})^3 = \frac{1}{2}.$$

As a consequence, $e^{i2\pi\mathcal{WZ}_\Sigma(\xi_i)} = e^{i2\pi\mathcal{WZ}_{\mathbb{S}^2}(\chi)} = -1$ and

$$e^{i2\pi\mathcal{WZ}_\Sigma(\xi_\epsilon)} = \prod_{x_{i_1},\dots,x_{i_k}} (-1) = \Pi(\epsilon). \qquad \square$$

Lemma 3.26 *Let (Σ,τ) be an oriented two-dimensional FKMM–manifold in the sense of Definition 1.1. The Wess–Zumino term induces a well-defined map*

$$e^{i2\pi\mathcal{WZ}_\Sigma} \colon [\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2} \to \mathbb{Z}_2.$$

Proof The claim is proved if one can show that for any $\xi \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ and $\phi \in \mathrm{Map}(\Sigma, \mathbb{U}(1))_{\mathbb{Z}_2}$, it holds that $e^{i2\pi\mathcal{WZ}_\Sigma(\xi)} = e^{i2\pi\mathcal{WZ}_\Sigma(\xi')}$ where

$$\xi' = \begin{pmatrix} \tau^*\phi & 0 \\ 0 & 1 \end{pmatrix} \cdot \xi \cdot \begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix}.$$

Let $\epsilon := \Phi_\kappa(\xi)$ and $\epsilon' := \Phi_\kappa(\xi')$. Associated with the maps $\epsilon, \epsilon' \in \mathrm{Map}(\Sigma^\tau, \mathbb{Z}_2)$ one can construct the associated maps $\xi_\epsilon, \xi_{\epsilon'} \in \mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$ according to Lemma 3.25. Lemma 2.17 assures that ξ and ξ' are $\mathbb{Z}_2$–homotopy equivalent to ξ_ϵ and $\xi_{\epsilon'}$, respectively. Thus,

$$e^{i2\pi\mathcal{WZ}_\Sigma(\xi)} = e^{i2\pi\mathcal{WZ}_\Sigma(\xi_\epsilon)} = \Pi(\epsilon) = \Pi(\Phi_\kappa(\xi))$$

and similarly $e^{i2\pi\mathcal{WZ}_\Sigma(\xi')} = \Pi(\Phi_\kappa(\xi'))$. Since Proposition 2.18 assures that $\Phi_\kappa(\xi) = \Phi_\kappa(\xi')$, it follows that $e^{i2\pi\mathcal{WZ}_\Sigma(\xi)} = e^{i2\pi\mathcal{WZ}_\Sigma(\xi')}$. $\square$

We are now in position to prove the first main result of this work.

Proof of Theorem 1.2 The case $m = 1$ will be treated first. In view of the bijection proved in Theorem 2.13 and the resulting equality (2-18), it is enough to show that $e^{i2\pi\mathcal{WZ}_\Sigma} = \Pi \circ \Phi_\kappa$ maps from $[\Sigma, \mathbb{SU}(2)]_{\mathbb{Z}_2}/[\Sigma, \mathbb{U}(1)]_{\mathbb{Z}_2}$ into $\mathbb{Z}_2$. By Proposition 2.18 and Theorem 2.11, $\Pi \circ \Phi_\kappa$ is a bijection. Thus, it is enough to prove the equality $e^{i2\pi\mathcal{WZ}_\Sigma} = \Pi \circ \Phi_\kappa$ on $\mathrm{Map}(\Sigma, \mathbb{SU}(2))_{\mathbb{Z}_2}$. However, this is clear from Lemma 3.25. The generalization to the case of Q–bundles of rank $2m$ follows by using the arguments in Remark 2.16. $\square$

3.7 Classification via Chern–Simons invariant in dimension three

The main aim of this section is to provide the proof of Theorem 1.3. This proof is facilitated by a particular presentation of principal Q–bundles over (X,τ). Suppose that $X^\tau = \{x_1, \dots, x_n\}$ consists of n points. Thanks to the *slice theorem* [27, Chapter I, Section 3], for each $i = 1, \dots, n$ one can find a closed τ–invariant disk D_i centered at x_i such that $D_i \cap D_j = \varnothing$ for $i \neq j$ and each D_i is equivariantly homotopic to the standard unit disk in $\mathbb{R}^3$ with antipodal involution $\tau(x) = -x$. Define

$$X_D := \bigsqcup_{i=1,\dots,n} D_i, \quad X' := X \setminus \mathrm{Int}(X_D),$$

so that $X = X' \cup X_D$. Given any map $\varphi : X' \cap X_D \to \mathbb{U}(2)$ one can glue together the product bundles over X' and X_D to form a principal $\mathbb{U}(2)$–bundle over X,

$$\mathscr{P}_\varphi := (X' \times \mathbb{U}(2)) \sqcup_\varphi (X_D \times \mathbb{U}(2)). \tag{3-14}$$

Assume that $\varphi \in \mathrm{Map}(X' \cap X_D, \mathbb{U}(2))_{\mathbb{Z}_2}$; namely, φ is equivariant with respect to the involution $\tau^*\varphi = -Q\bar{\varphi}Q$. Then the principal $\mathbb{U}(2)$–bundle $\mathscr{P}_\varphi$ gives rise to a principal Q–bundle.

Lemma 3.27 *Assume that the hypotheses of Theorem 1.3 are met. Any principal $\mathbb{U}(2)$ Q–bundle $(\mathscr{P}, \widehat{\Theta})$ over (X, τ) is isomorphic to a principal $\mathbb{U}(2)$ Q–bundle $\mathscr{P}_\varphi$ of the type (3-14) for a map $\varphi \in \mathrm{Map}(X' \cap X_D, \mathbb{U}(2))_{\mathbb{Z}_2}$ which satisfies the following property: Let $\varphi_i := \varphi|_{\partial D_i}$ be the restriction of φ on the boundary $\partial D_i \simeq \mathbb{S}^2$ of the disk D_i for every $i = 1, \dots, n$. Then, either φ_i is equivariantly homotopic to the equivariant map $\varphi_* : \mathbb{S}^2 \to \mathbb{U}(2)$ defined by*

$$\varphi_*(x_1, x_2, x_3) := \mathrm{i} \begin{pmatrix} x_1 & -x_2 + \mathrm{i}x_3 \\ x_2 + \mathrm{i}x_3 & x_1 \end{pmatrix},$$

where $\mathbb{S}^2$ is a $\mathbb{Z}_2$–space with the antipodal involution, or φ_i is the constant map at $\mathbb{1}_{\mathbb{C}^2} \in \mathbb{U}(2)$.

Proof Since each connected component D_i of X_D is equivariantly contractible, the principal Q–bundle $\mathscr{P}|_{X_D}$ is trivial. By construction, the involution on X' is free; thus $\mathscr{P}|_{X'}$ is trivial as well. This fact follows from [10, Theorem 4.7(2)] along with the assumption $H^2_{\mathbb{Z}_2}(X, \mathbb{Z}(1)) = 0$ which implies the triviality of even rank Q–bundles over spaces with free involutions. The passage from vector bundles to principal bundles is then justified by the isomorphism (3-1). Let s_{X_D} and $s_{X'}$ be global sections (ie trivializations) of $\mathscr{P}|_{X_D}$ and $\mathscr{P}|_{X'}$, respectively. From these sections one gets the map $\varphi : X' \cap X_D \to \mathbb{U}(2)$ defined by the restriction on $X' \cap X_D$ of the (pointwise) product $s_{X_D}^{-1} s_{X'}$. The map φ is equivariant by construction and defines the principal Q–bundle $\mathscr{P}_\varphi$ as given in (3-14). The isomorphism $\mathscr{P} \simeq \mathscr{P}_\varphi$ is a manifestation of the fact that $\mathscr{P}$ and $\mathscr{P}_\varphi$ have the same system of transition functions. By the homotopy property of Q–bundles, the Q–isomorphism class of $\mathscr{P}_\varphi$ only depends on the $\mathbb{Z}_2$–homotopy class of φ. By [8, Corollary 4.1] one has $[\mathbb{S}^2, \mathbb{U}(2)]_{\mathbb{Z}_2} \simeq \mathbb{Z}_2$, meaning that every equivariant map from the sphere $\mathbb{S}^2$ with the antipodal involution into the space $\mathbb{U}(2)$ with involution $g \mapsto -Q\bar{g}Q$ is $\mathbb{Z}_2$–homotopy equivalent to the constant map at $\mathbb{1}_{\mathbb{C}^2}$ or to the map φ_*. Since $X' \cap X_D$ is a disjoint union of antipodal spheres, the map φ restricted to each disconnected component can be equivariantly deformed to one of these two maps. □

Remark 3.28 Lemma 3.27 deserves two comments. First of all it is worth noticing that the map φ constructed in the proof of the lemma can be always deformed to a smooth map providing in this way a smooth principal Q–bundle $\mathscr{P}_\varphi$ which represents $\mathscr{P}$ in the smooth category. This is a manifestation of the equivalence between continuous and smooth category discussed in [9, Theorem 2.1]. The second observation refers to the content of Remark 2.16. In fact in view of the stable rank condition described in Theorem 2.5 one has that the representation (3-14) must be valid also for principal $\mathbb{U}(2m)$ Q–bundle. In the higher rank case the isomorphism reads

$$\mathscr{P} \simeq \mathscr{P}_\varphi := (X' \times \mathbb{U}(2m)) \sqcup_{\varphi'} (X_D \times \mathbb{U}(2m)), \tag{3-15}$$

where the equivariant map $\varphi' \colon X' \cap X_D \to \mathbb{U}(2m)$ factors as

$$\varphi' \simeq \begin{pmatrix} \varphi & 0 \\ 0 & \mathbb{1}_{\mathbb{C}^{2(m-1)}} \end{pmatrix}$$

and the map $\varphi \colon X' \cap X_D \to \mathbb{U}(2)$ in the upper-left corner satisfies the properties of Lemma 3.27.

In view of the Lemma 3.27 one can assume that $\mathscr{P}$ has been of the form (3-14) since the beginning. With this presentation in hand, the next task is to compute the FKMM–invariant of $\mathscr{P}$. As a preliminary fact, let us recall that the FKMM–invariant of a principal Q–bundle $(\mathscr{P}, \widehat{\Theta})$ is defined as the FKMM–invariant of the associated Q–bundle $(\mathscr{E}, \Theta)$; see Definition 3.4. The FKMM–invariant measures the difference of two trivializations of the sphere bundle of $\det(\mathscr{E})|_{X^\tau}$. This is the same as measuring the difference of two trivializations of $\det(\mathscr{E})|_{X^\tau}$.

Lemma 3.29 *Assume that the hypotheses of Theorem 1.3 are met. Let $(\mathscr{P}, \widehat{\Theta})$ be a principal $\mathbb{U}(2)$ Q–bundle and $\varphi \in \mathrm{Map}(X' \cap X_D, \mathbb{U}(2))_{\mathbb{Z}_2}$ the equivariant map which represents the principal Q–bundle according to Lemma 3.27. Then the FKMM–invariant of $(\mathscr{P}, \widehat{\Theta})$ is represented by the function $\phi := \det(\varphi)|_{X^\tau}$. More precisely, one has that*

$$\kappa(\mathscr{P}, \widehat{\Theta}) = [\phi] \in \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}.$$

Proof Starting from the representation (3-14), one has that

$$\det(\mathscr{P}) = (X' \times \mathbb{U}(1)) \sqcup_{\det(\varphi)} (X_D \times \mathbb{U}(1)).$$

From this expression one infers that the canonical invariant section $s_{(\mathscr{P},\widehat{\Theta})}$ of $\det(\mathscr{P})|_{X^\tau}$ is given by

$$s_{(\mathscr{P},\widehat{\Theta})} = (x,1) \in X^\tau \times \mathbb{U}(1) \subset \det(\mathscr{P}),$$

while a global invariant section s of $\det(\mathscr{P})$ is given by

$$s(x) = \begin{cases} (x, u_{X'}(x)) & \text{if } x \in X', \\ (x, u_D(x)) & \text{if } x \in X_D, \end{cases}$$

where $u_{X'}\colon X' \to \mathbb{U}(1)$ and $u_D\colon X_D \to \mathbb{U}(1)$ are two equivariant maps satisfying $u_{X'} = u_D \cdot \det(\varphi)$ on $X' \cap X_D$. Accordingly, it follows from Lemma 3.27 that $\det(\varphi_i)\colon X' \cap D_i \to \mathbb{U}(1)$ is a constant map at 1 or -1, where $\varphi_i := \varphi|_{D_i}$. Therefore, one can choose $u_{X'}$ to be the constant map at 1 and u_D to be the locally constant map such that $u_D|_{D_i} = \pm 1$ if $\det(\varphi_i) = \pm 1$. Then, it follows that the FKMM–invariant is represented by $u_D|_{X^\tau} = \det(\varphi)|_{X^\tau}$. □

The next goal is to compute the Chern–Simons invariant of $(\mathscr{P}, \widehat{\Theta})$. Let $s_{X'}$ and s_{X_D} be the invariant sections of $\mathscr{P}|_{X'} = X' \times \mathbb{U}(2)$ and $\mathscr{P}|_{X_D} = X_D \times \mathbb{U}(2)$ defined by

$$\begin{aligned} s_{X'}(x) &= (x, \mathbb{1}_{\mathbb{C}^2}) \quad \text{if } x \in X', \\ s_{X_D}(x) &= (x, \mathbb{1}_{\mathbb{C}^2}) \quad \text{if } x \in X_D, \end{aligned} \tag{3-16}$$

respectively. Then, any section s of $\mathscr{P}$ is described as

$$s(x) = \begin{cases} s_{X'}(x)\psi_{X'}(x)^{-1} = (x, \psi_{X'}(x)^{-1}) & \text{if } x \in X', \\ s_{X_D}(x)\psi_D(x)^{-1} = (x, \psi_D(x)^{-1}) & \text{if } x \in X_D, \end{cases} \tag{3-17}$$

for a pair of maps $\psi_{X'}\colon X' \to \mathbb{U}(2)$ and $\psi_D\colon X_D \to \mathbb{U}(2)$ such that $\psi_{X'} = \psi_D\varphi$ on $X' \cap X_D$. The maps $\psi_{X'}$ and ψ_D can be chosen smooth in such a way that the section s is smooth as well. Moreover, the choice of $\psi_{X'}$ and ψ_D can be further specified in view of the following result:

Lemma 3.30 *The smooth maps $\psi_{X'}$ and ψ_D in (3-17) can be chosen such that $\psi_D = \mathbb{1}_{\mathbb{C}^2}$ is the constant map.*

Proof By construction, $\psi_{X'} = \psi_D\varphi$ on $X' \cap X_D$. Thus, the proof of the claim reduces to the problem of extending $\varphi\colon \partial X' \to \mathbb{U}(2)$ to a smooth map $\tilde{\varphi}\colon X' \to \mathbb{U}(2)$ such that $\tilde{\varphi}|_{\partial X'} = \varphi$. Indeed, given such a $\tilde{\varphi}$, the proof can be completed by setting

$\psi_D = \mathbb{1}_{\mathbb{C}^2}$ and $\psi_{X'} = \tilde{\varphi}$. To prove the existence of $\tilde{\varphi}$, notice that the three-manifold X' admits a CW decomposition in which the dimension of each cell is at most 3. The homotopy groups $\pi_i(\mathbb{U}(2))$ are trivial for $i = 0, 2$. The map $\det\colon \mathbb{U}(2) \to \mathbb{U}(1)$ induces an isomorphism $\pi_1(\mathbb{U}(2)) \simeq \pi_1(\mathbb{U}(1)) \simeq \mathbb{Z}$. Since $\det(\varphi)$ is nullhomotopic by construction, one concludes that φ extends to a continuous map $\tilde{\varphi}'\colon X' \to \mathbb{U}(2)$. However, the isomorphism between continuous category and smooth category ensures the existence of a smooth map $\tilde{\varphi}\colon X' \to \mathbb{U}(2)$, approximating the continuous map $\tilde{\varphi}'$, that satisfies $\tilde{\varphi}|_{\partial X'} = \varphi$. □

Given an invariant connection ω on $(\mathscr{P}, \widehat{\Theta})$, one sets

$$\omega_{X'} := s_{X'}^{*}\omega, \qquad \omega_{X_D} := s_{X_D}^{*}\omega.$$

The two local expressions are related by

$$\omega_{X'} = \varphi^{-1}\omega_{X_D}\varphi + \varphi^{-1}d\varphi. \tag{3-18}$$

The following result contains the key computation for the proof of Theorem 1.3.

Lemma 3.31 *Assume that the hypotheses of Theorem 1.3 are met. Let $(\mathscr{P}, \widehat{\Theta})$ be a principal $\mathbb{U}(2)$ Q–bundle and $\varphi \in \mathrm{Map}(X' \cap X_D, \mathbb{U}(2))_{\mathbb{Z}_2}$ the equivariant map which represents the principal Q–bundle according to Lemma 3.27. Then the Chern–Simons invariant of $(\mathscr{P}, \widehat{\Theta})$ is given by*

$$\mathfrak{cs}(\mathscr{P}, \widehat{\Theta}) = \mathcal{WZ}_{\partial X_D}(\varphi) + \frac{1}{8\pi^2}\int_{\partial X_D} \mathrm{Tr}(\omega_{X_D} \wedge d\varphi\varphi^{-1}) \mod \mathbb{Z},$$

where ω_{X_D} is defined by (3-18) from any invariant connection ω.

Proof Let us start with an observation. By construction, $\varphi = \bigsqcup_{i=1,\dots,n}\varphi_i$ and each $\det(\varphi_i)\colon \partial D_i \to \mathbb{U}(1)$ is constant at ± 1. Hence, $\det(\varphi)$ is nullhomotopic and $\mathcal{WZ}_{\partial D}(\varphi)$ makes sense. Now, the computation. Given the section described in (3-17), one has that

$$\begin{aligned}\int_X s^*\mathcal{CS}(\omega) &= \int_{X'} s^*\mathcal{CS}(\omega) + \int_{X_D} s^*\mathcal{CS}(\omega)\\ &= \int_{X'} (s_{X'}\psi_{X'}^{-1})^*\mathcal{CS}(\omega) + \int_{X_D} (s_{X_D}\psi_D^{-1})^*\mathcal{CS}(\omega).\end{aligned}$$

With the help of formula (3-7) one has that

$$\begin{aligned}&(s_{X'}\psi_{X'}^{-1})^*\mathcal{CS}(\omega)\\ &\quad= s_{X'}^*\mathcal{CS}(\omega) + \frac{1}{8\pi^2}\,\mathrm{dTr}(s_{X'}^*\omega \wedge \psi_{X'}\mathrm{d}\psi_{X'}^{-1}) - \frac{1}{24\pi^2}\,\mathrm{Tr}((\psi_{X'}\mathrm{d}\psi_{X'}^{-1})^{\wedge 3}).\end{aligned}$$

Since

$$\int_{X'} s_{X'}^* \mathcal{CS}(\omega) = \int_{X'} \mathcal{CS}(\omega_{X'}) = 0,$$

in view of Proposition 3.12(iii) one gets

$$\int_{X'} (s_{X'}\psi_{X'}^{-1})^* \mathcal{CS}(\omega) = \frac{1}{8\pi^2}\int_{X'} \mathrm{dTr}(\omega_{X'} \wedge \mathrm{d}\psi_{X'}\psi_{X'}^{-1}) + \mathcal{WZ}_{\partial X'}(\psi_{X'}^{-1}|_{\partial X'}) \mod \mathbb{Z},$$

where Definition 3.16 has been used. With a similar computation one gets also

$$\int_{X_D} (s_{X_D}\psi_D^{-1})^* \mathcal{CS}(\omega) \\ = \frac{1}{8\pi^2}\int_{X_D} \mathrm{dTr}(\omega_{X_D} \wedge \mathrm{d}\psi_D\psi_D^{-1}) + \mathcal{WZ}_{\partial X_D}(\psi_D^{-1}|_{\partial X_D}) \mod \mathbb{Z}$$

and, after putting all the pieces together, one obtains

$$\int_X s^* \mathcal{CS}(\omega) = \frac{1}{8\pi^2}\int_{X'} \mathrm{dTr}(\omega_{X'} \wedge \mathrm{d}\psi_{X'}\psi_{X'}^{-1}) + \frac{1}{8\pi^2}\int_{X_D} \mathrm{dTr}(\omega_{X_D} \wedge \mathrm{d}\psi_D\psi_D^{-1}) \\ + \mathcal{WZ}_{\partial X'}(\psi_{X'}^{-1}|_{\partial X'}) + \mathcal{WZ}_{\partial X_D}(\psi_D^{-1}|_{\partial X_D}) \mod \mathbb{Z}.$$

Notice that the orientation on $\partial X' = X' \cap X_D$ induced from X is opposite to that on ∂X_D. Therefore, modulo $\mathbb{Z}$, one gets the equality

$$\mathcal{WZ}_{\partial X'}(\psi_{X'}^{-1}|_{\partial X'}) \\ = -\mathcal{WZ}_{\partial X_D}((\psi_D|_{\partial X_D}\varphi)^{-1}) \\ = -\mathcal{WZ}_{\partial X_D}(\varphi^{-1}) - \mathcal{WZ}_{\partial X_D}(\psi_D|_{\partial X_D}^{-1}) - \frac{1}{8\pi^2}\int_{\partial X_D} \mathrm{Tr}(\varphi \mathrm{d}\varphi^{-1} \wedge \mathrm{d}\psi_D^{-1}\psi_D),$$

which is justified by the relation $\psi_{X'} = \psi_D\varphi$ on $\partial X' = \partial X_D$ and by the use of the Polyakov–Wiegmann formula proved in Lemma 3.17. The local relation between $\psi_{X'}$ and ψ_D also implies

$$\mathrm{Tr}(\omega_{X'} \wedge \mathrm{d}\psi_{X'}\psi_{X'}^{-1}) = \mathrm{Tr}(\omega_{X_D} \wedge \psi_D^{-1}\mathrm{d}\psi_D + \omega_{X_D} \wedge \mathrm{d}\varphi\varphi^{-1} + \mathrm{d}\varphi\varphi^{-1} \wedge \psi_D^{-1}\mathrm{d}\psi_D).$$

Summarizing, one finally gets

$$\int_X s^* \mathcal{CS}(\omega) = -\mathcal{WZ}_{\partial X_D}(\varphi^{-1}) + \frac{1}{8\pi^2}\int_{X_D} \mathrm{dTr}(\omega_{X_D} \wedge \mathrm{d}\varphi\varphi^{-1}) \mod \mathbb{Z}.$$

The proof is completed by the general equality $\mathcal{WZ}_{\partial X_D}(\varphi^{-1}) = -\mathcal{WZ}_{\partial X_D}(\varphi)$ and the use of Definition 3.14. □

We are now in position to provide the proof of the second main result of this work.

Proof of Theorem 1.3 Let us choose the maps $\psi_{X'}$ and ψ_D as in Lemma 3.30. Then $\omega_{X_D} := s_{X_D}^* \omega = (s\psi_D)^*\omega = 0$, since ψ_D is constant. Thus, from the formula in Lemma 3.31 and the definition of the map φ, one gets

$$\mathfrak{cs}(\mathscr{P}, \widehat{\Theta}) = \mathcal{WZ}_{\partial X_D}(\varphi) = \sum_{i=1}^{n} \mathcal{WZ}_{\partial D_i}(\varphi_i) \mod \mathbb{Z}.$$

It holds that $\mathcal{WZ}_{\partial D_i}(\varphi_i) = 1$ when $\varphi_i = \mathbb{1}_{\mathbb{C}^2}$ (obvious!) and $\mathcal{WZ}_{\partial D_i}(\varphi_i) = \frac{1}{2}$ when φ_i is homotopic to the map φ_* in Lemma 3.27. The proof of the latter equality is contained in the proof of Lemma 3.25. In fact the map φ_* coincides with the map (3-12) and a possible extension $\tilde{\varphi}_*$ on the upper hemisphere of $\mathbb{S}^3$ can be realized by the prescription (3-13). In conclusion, one obtains that

$$\mathrm{e}^{\mathrm{i}2\pi\,\mathfrak{cs}(\mathscr{P},\widehat{\Theta})} = \Pi(\det(\varphi)|_{X^\tau}).$$

The proof is finally completed by the result in Lemma 3.29. □

Theorem 1.3 has a surprising consequence.

Corollary 3.32 *Under the assumptions in Theorem 1.3, the homomorphism*

$$\Pi\colon \mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2} \to \mathbb{Z}_2$$

induced by the product sign map (2-7) is well defined.

Proof One needs to shows that the homomorphism $\Pi\colon \mathrm{Map}(X^\tau, \{\pm 1\}) \to \mathbb{Z}_2$ given by the product sign map satisfies $\Pi(\phi\psi|_{X^\tau}) = \Pi(\phi)$ for any map $\phi\colon X^\tau \to \mathbb{Z}_2$ and any equivariant map $\psi\colon X \to \mathbb{U}(1)$. Consider the principal $\mathbb{U}(2)$ Q–bundle $\mathscr{P}_\varphi$ generated according to (3-14) where the map φ is related to ϕ as follows: φ is the constant map at $\mathbb{1}_{\mathbb{C}^2}$ on the disk D_i if $\phi(x_i) = 1$ or φ agrees with φ_* on the boundary of D_i if $\phi(x_i) = -1$. By construction the map ϕ provides a representative of the FKMM–invariant of $\mathscr{P}_\varphi$; see Lemma 3.27. In a similar way the map $\phi' := \phi\psi$ represents the FKMM–invariant of an associated principal $\mathbb{U}(2)$ Q–bundle $\mathscr{P}_{\varphi'}$. Since φ and φ' belong to the same class in $\mathrm{Map}(X^\tau, \{\pm 1\})/[X, \mathbb{U}(1)]_{\mathbb{Z}_2}$ it follows that $\mathscr{P}_\varphi$ and $\mathscr{P}_{\varphi'}$ have the same FKMM–invariant. However, under the hypotheses of Theorem 1.3 the FKMM–invariant is an isomorphism (Proposition 2.10); hence $\mathscr{P}_\varphi$ and $\mathscr{P}_{\varphi'}$ are isomorphic. By the naturality of the Chern–Simons invariant, $\mathfrak{cs}(\mathscr{P}_\varphi, \widehat{\Theta}_\varphi) = \mathfrak{cs}(\mathscr{P}_{\varphi'}, \widehat{\Theta}_{\varphi'})$. The proof of the claim then follows in view of formula (1-6). □

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Detecting isomorphisms in the homotopy category

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We show that no generalization of Whitehead's theorem holds for unpointed spaces. More precisely, we show that the homotopy category of unpointed spaces admits no set of objects jointly reflecting isomorphisms. We give an explicit counterexample involving infinite symmetric groups. In contrast, we prove that the spheres do jointly reflect equivalences in the homotopy 2–category of spaces. We also show that homotopy colimits of transfinite sequential diagrams of spaces are not generally weak colimits in the homotopy category, and furthermore exhibit such a diagram with the property that none of its weak colimits is *privileged*, which means, roughly, that it sees the spheres as compact objects. The nonexistence of a set jointly reflecting isomorphisms in the homotopy category was originally claimed by Heller, but our results on weak colimits show that his argument had an inescapable gap, leading to the need for the new proof given here.

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1 Introduction

Let **Hot** denote the homotopy category of spaces, and let $\mathbf{Hot}_{*,c}$ denote the homotopy category of pointed, connected spaces. Whitehead's theorem says that, in $\mathbf{Hot}_{*,c}$, the set of spheres jointly reflects isomorphisms. One is naturally led to wonder whether there is a set of spaces in **Hot** which jointly reflects isomorphisms.

Brown [1] proved that a functor $\mathbf{Hot}_{*,c}^{\mathrm{op}} \to \mathbf{Set}$ is representable if and only if it is half-exact, in the sense that it sends coproducts and weak pushouts in $\mathbf{Hot}_{*,c}$ to products and weak pullbacks in **Set**. Heller [4] proved an abstract representability theorem: if $\boldsymbol{C}$ is a category with coproducts and weak pushouts and $\boldsymbol{C}$ contains a "bounded" set $\mathcal{G}$

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of objects that jointly reflects isomorphisms (see Definition 1.1 below), then a functor $\boldsymbol{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable if and only if it is half-exact. Heller also gave an example of a half-exact functor $\mathbf{Hot}^{\mathrm{op}} \to \mathbf{Set}$ which is not representable. He then claimed without proof [4, Proposition 1.2] that every set of spaces in **Hot** is bounded, and concluded [4, Corollary 2.3] that no set of spaces jointly reflects isomorphisms in **Hot**.

We show that it is not true that every set of spaces is bounded, reopening the question of whether there is a set of spaces that jointly reflects isomorphisms in **Hot**. We thus also give an independent proof that no set of spaces jointly reflects isomorphisms.

We now give the definitions needed in order to precisely state our results.

Definition 1.1 Let $\boldsymbol{C}$ be any category and let $\mathcal{G} \subseteq \boldsymbol{C}$ be a set of objects.

(1) We say that $\mathcal{G}$ *jointly reflects isomorphisms* if a morphism $f: X \to Y$ in $\boldsymbol{C}$ is an isomorphism whenever $\boldsymbol{C}(S, f): \boldsymbol{C}(S, X) \to \boldsymbol{C}(S, Y)$ is a bijection for every $S \in \mathcal{G}$.[1]

(2) A *weak colimit* of a diagram $D: \boldsymbol{I} \to \boldsymbol{C}$ is a cocone through which every cocone factors, not necessarily uniquely.

(3) A cocone W of $D: \boldsymbol{I} \to \boldsymbol{C}$ is *$\mathcal{G}$–privileged* if the canonical map
$$\operatorname*{colim}_{\alpha \in \boldsymbol{I}} \boldsymbol{C}(S, D(\alpha)) \to \boldsymbol{C}(S, W)$$
is a bijection for every $S \in \mathcal{G}$.

(4) For an ordinal β, we say that $\mathcal{G}$ is *β–bounded* if every diagram $D: \beta \to \boldsymbol{C}$ has a $\mathcal{G}$–privileged weak colimit.

(5) We say that $\mathcal{G}$ is *left cardinally bounded*, or just *bounded*, if it is β–bounded for each sufficiently large regular cardinal β.

We use the word "set" to mean what is sometimes called a "small set", i.e. an object of the category **Set**. All of our ordinals and cardinals are "small". We regard a cardinal as an ordinal which is least in its cardinality class. The *cofinality* of an ordinal α is the smallest ordinal that is the order type of a cofinal subset of α. A cardinal is *regular* if it is equal to its cofinality.

As mentioned above, **Hot** denotes the homotopy category of spaces, by which we mean the localization of the category of spaces at the weak homotopy equivalences, or, equivalently, the category whose objects are CW–complexes and whose morphisms are

[1] Other terminology is in use, such as "$\mathcal{G}$ is a set of (weak) generators" or "the functors $\boldsymbol{C}(S, -)$ are jointly conservative". Heller says that "$\mathcal{G}$ is left adequate".

homotopy classes of continuous maps. It is well known that every small diagram in **Hot** has a weak colimit, and that weak colimits are not unique.

We can now state our main results more precisely. First we give the result that shows that [4, Proposition 1.2] is false:

Theorem 3.1 *The set $\mathcal{G} = \{S^n \mid n \geq 0\}$ of spheres in* **Hot** *is not κ–bounded for any ordinal κ of uncountable cofinality. That is, for each such κ, there exists a diagram $D\colon \kappa \to$* **Hot** *that admits no $\mathcal{G}$–privileged weak colimit.*

Note that Theorem 3.1 applies to all uncountable regular cardinals, showing that the set of spheres is not left cardinally bounded. By adding one more space to the set, we can remove the uncountability assumption:

Corollary 3.2 *Let T denote a countably infinite, discrete space. Then the set $\{S^n \mid n \geq 0\} \cup \{T\}$ is not κ–bounded in* **Hot** *for any limit ordinal κ.*

The proof of Theorem 3.1 is somewhat involved and forms the bulk of the paper. We first show that it is sufficient to find a counterexample in the homotopy category **HoGpd** of groupoids. Then, given κ as in the statement, we consider the diagram $D\colon \kappa \to$ **HoGpd** sending α to the free group on $2+\alpha$ generators. We make use of the theory of graphs of groups (see Serre [7]) and the associated fundamental groupoid (see Higgins [5]) in order to construct a sufficiently pathological cocone $D \to Z$, which we use to show that D admits no $\mathcal{G}'$–privileged weak colimit, where $\mathcal{G}' = \{\boldsymbol{B}\mathbb{Z}\}$. This involves a detailed understanding of the morphisms in Z and how they are expressed as words in the given generators. It follows that the diagram $\kappa \to$ **Hot** sending α to the wedge of α circles has no $\mathcal{G}$–privileged weak colimit, where $\mathcal{G}$ is as in the statement of Theorem 3.1.

Heller's argument for his claim [4, Proposition 1.2] that any set $\mathcal{G}$ of objects in **Hot** is bounded was to take the cocone W to be the homotopy colimit, i.e. a generalized telescope. Since such homotopy colimits are $\mathcal{G}$–privileged, our result above implies that they are not, in general, even weak colimits in **Hot**. This is in contrast to the situation for telescopes of sequences indexed by ω, and for other homotopy colimits of diagrams indexed by freely generated categories. In the introduction to [2], Franke suggests using a Bousfield–Kan spectral sequence to show that Heller's claim is false, by comparing weak colimits to homotopy colimits, but we were unable to find an example in which we could prove that a certain differential was nonzero.

In the homotopy category of pointed, connected spaces, the set of spheres jointly reflects isomorphisms — this is the classical form of Whitehead's theorem. However, we conjecture that the set of spheres is not bounded in $\mathbf{Hot}_{*,c}$. If this is true, it means that Heller's abstract representability theorem, as stated, does not imply Brown's representability theorem. That said, Heller's argument only requires a set of objects that jointly reflects isomorphisms and is β–bounded for *some* regular cardinal β. Thus, since the set of spheres is $\aleph_0$–bounded, the proof of Heller's theorem goes through in $\mathbf{Hot}_{*,c}$.

Next we state the result that shows that the statement of [4, Corollary 2.3] is nevertheless correct:

Theorem 2.1 *The category* **Hot** *contains no set $\mathcal{G}$ of spaces that jointly reflects isomorphisms. That is, there exists no set $\mathcal{G}$ of spaces such that, if $f : X \to Y$ is a map of spaces and $f_* : \mathbf{Hot}(S, X) \to \mathbf{Hot}(S, Y)$ is a bijection for every $S \in \mathcal{G}$, then f is an isomorphism in* **Hot**.

This second result is easier to prove, and so we prove it first, in Section 2. Our method is a generalization of Proposition 4.1 of Matumoto, Minami and Sugawara [6], which gives a "phantom homotopy equivalence", that is, a map in **Hot** which, while not an isomorphism, is seen as one by all finite complexes. Our proof also shows that there is no set of connected spaces that jointly reflects isomorphisms in the homotopy category of connected spaces. Moreover, Theorem 2.1 implies similar results in other settings. For example, since **Hot** is a reflective subcategory of the homotopy category of $(\infty, 1)$–categories, it follows that there is no set of $(\infty, 1)$–categories that jointly reflects isomorphisms in that category.

Since the $(\infty, 1)$–category $\mathcal{S}$ of spaces certainly contains a set of objects jointly reflecting equivalences — namely the set whose only element is the one-point space — while its 1–categorical truncation **Hot** does not, one might ask which behaviours the n–categorical truncations of $\mathcal{S}$ exhibit for larger values of n. In fact, we show in Theorem 4.3 that, in the 2–category $\underline{\mathbf{Hot}}$ of spaces, morphisms and homotopy classes of homotopies between them, the set of spheres *does* jointly reflect equivalences, which is the natural generalization of joint reflection of isomorphisms to 2–category theory. Intuitively, the reason for the divergent behaviour of **Hot** and $\underline{\mathbf{Hot}}$ is that the 2–morphisms of $\underline{\mathbf{Hot}}$ retain the information about based homotopies that is lost in **Hot**.

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2 Hot admits no set that jointly reflects isomorphisms

We make the following definitions. For an ordinal α, write Σ_α for the group of all bijections of the set α, ignoring order. When $\beta < \alpha$, there is a natural inclusion $\Sigma_\beta \hookrightarrow \Sigma_\alpha$, and we define $\Sigma_\alpha^{\mathrm{c}}$ to be the union of the images of Σ_β for all $\beta < \alpha$. We typically consider $\Sigma_\alpha^{\mathrm{c}}$ when α is a cardinal, considered as the smallest ordinal with that cardinality, and we call the elements of $\Sigma_\alpha^{\mathrm{c}}$ *essentially constant* permutations.

Theorem 2.1 *The category* **Hot** *contains no set $\mathcal{G}$ of spaces that jointly reflects isomorphisms.* (See Definition 1.1.)

Proof Let $\mathcal{G}$ be a set of spaces and let α be a regular cardinal larger than the cardinality of $\pi_1(S, s_0)$ for each $S \in \mathcal{G}$ and each $s_0 \in S$. We must construct a map $f : X \to Y$ which is not a homotopy equivalence but which induces bijections on homotopy classes of maps from spaces in $\mathcal{G}$.

Our example will be $Bs : B\Sigma_\alpha^{\mathrm{c}} \to B\Sigma_\alpha^{\mathrm{c}}$, where $s : \Sigma_\alpha^{\mathrm{c}} \to \Sigma_\alpha^{\mathrm{c}}$ is the shift homomorphism given by

$$(s\sigma)(\gamma) = \begin{cases} \sigma(\gamma') + 1 & \text{if } \gamma = \gamma' + 1, \\ \gamma & \text{if } \gamma \text{ is a limit ordinal,} \end{cases}$$

for $\sigma \in \Sigma_\alpha^{\mathrm{c}}$. (Here and in what follows, if γ is a successor ordinal, we write γ' for its predecessor.) We must check that $s\sigma \in \Sigma_\alpha^{\mathrm{c}}$. First, it is essentially constant: if $\beta < \alpha$ and σ fixes each $\gamma \geq \beta$, then, for $\gamma > \beta$, we have $(s\sigma)(\gamma) = \gamma$ if γ is a limit ordinal, and $(s\sigma)(\gamma) = \sigma(\gamma') + 1 = \gamma' + 1 = \gamma$ if γ is a successor. Next, we see that s is a homomorphism: $s(\sigma\tau)$ and $(s\sigma)(s\tau)$ both fix all limit ordinals, while for successors we have

$$(s\sigma)((s\tau)(\gamma)) = \sigma\big([\tau(\gamma') + 1]'\big) + 1 = \sigma\tau(\gamma') + 1 = s(\sigma\tau)(\gamma),$$

as desired. It follows that $s\sigma$ is a bijection, with inverse $s(\sigma^{-1})$.

Let H be a group with classifying space BH and let X be a connected space. If **Gp** denotes the category of groups, recall that $\mathbf{Hot}(X, BH)$ is isomorphic to $\mathbf{Gp}(\pi_1(X), H)$ modulo conjugation by elements of H. (See for example [8, Corollary V.4.4].) In particular, we have a natural isomorphism $\mathbf{Hot}(X, BH) \cong \mathbf{Hot}(B\pi_1(X), BH)$. It also follows that, for groups G and H, $\mathbf{Hot}(BG, BH)$ is isomorphic to $\mathbf{Gp}(G, H)$ modulo conjugation by elements of H, and that an element of $\mathbf{Hot}(BG, BH)$ is a homotopy equivalence if and only if it is represented by an isomorphism.

Note that s is not surjective, since $s\sigma$ always preserves limit ordinals. Therefore, $Bs : B\Sigma_\alpha^{\mathrm{c}} \to B\Sigma_\alpha^{\mathrm{c}}$ is not a homotopy equivalence. However, we will show that it induces an isomorphism on $\mathcal{G}$. First observe that it suffices to prove this for connected components of spaces in $\mathcal{G}$. It follows that it is enough to prove this for spaces of the form BG, where G is a group of cardinality less than α.

Any map $BG \to B\Sigma_\alpha^{\mathrm{c}}$ arises from a homomorphism $\varphi\colon G \to \Sigma_\alpha^{\mathrm{c}}$, well defined up to conjugation. Since α is regular, there is a limit ordinal $\beta < \alpha$ such that $\varphi(g) \in \Sigma_\beta$ for every $g \in G$. We claim that $s \circ \varphi$ is conjugate to φ by an element $\tau \in \Sigma_\alpha^{\mathrm{c}}$ defined by

$$\tau(\gamma) = \begin{cases} \gamma' & \text{if } \gamma < \beta \text{ is a successor ordinal,} \\ \beta + \gamma & \text{if } \gamma < \beta \text{ is a limit ordinal,} \\ \gamma + 1 & \text{if } \beta \le \gamma < \beta + \beta, \\ \gamma & \text{otherwise.} \end{cases}$$

It is straightforward to check that τ is a permutation, and it clearly fixes ordinals greater than or equal to $\beta + \beta$, which is less than α. For $g \in G$, let $\sigma = \varphi(g)$. Then, noting that $\tau^{-1}(\gamma) = \gamma + 1$ for any $\gamma < \beta$, we have

$$\begin{aligned} (\tau^{-1}\sigma\tau)(\gamma) &= \begin{cases} \tau^{-1}(\sigma(\gamma')) & \text{if } \gamma < \beta \text{ is a successor ordinal,} \\ \tau^{-1}(\sigma(\beta + \gamma)) & \text{if } \gamma < \beta \text{ is a limit ordinal,} \\ \tau^{-1}(\sigma(\gamma + 1)) & \text{if } \beta \le \gamma < \beta + \beta, \\ \tau^{-1}(\sigma(\gamma)) & \text{otherwise,} \end{cases} \\ &= \begin{cases} \tau^{-1}(\sigma(\gamma')) & \text{if } \gamma < \beta \text{ is a successor ordinal,} \\ \tau^{-1}(\beta + \gamma) & \text{if } \gamma < \beta \text{ is a limit ordinal,} \\ \tau^{-1}(\gamma + 1) & \text{if } \beta \le \gamma < \beta + \beta, \\ \tau^{-1}(\gamma) & \text{otherwise,} \end{cases} \\ &= \begin{cases} \sigma(\gamma') + 1 & \text{if } \gamma < \beta \text{ is a successor ordinal,} \\ \gamma & \text{if } \gamma < \beta \text{ is a limit ordinal,} \\ \gamma & \text{if } \beta \le \gamma < \beta + \beta, \\ \gamma & \text{otherwise,} \end{cases} \\ &= s(\sigma)(\gamma). \end{aligned}$$

We have used that, if $\gamma \ge \beta$, then $\sigma(\gamma) = \gamma$, and the consequence that, if $\gamma < \beta$, then $\sigma(\gamma) < \beta$.

In summary, we have shown that Bs induces the identity on $\mathbf{Hot}(S, B\Sigma_\alpha^{\mathrm{c}})$ for every $S \in \mathcal{G}$, proving the claim. □

Remark 2.2 Since the map $Bs\colon B\Sigma_\alpha^{\mathrm{c}} \to B\Sigma_\alpha^{\mathrm{c}}$ used in the proof has connected domain and codomain, it follows that there is no set of connected spaces that jointly reflects isomorphisms in the homotopy category of connected spaces.

We explain the origin of the maps s and τ. Morally, s is conjugation by the successor operation on ordinals, with limit ordinals handled specially. The map τ implements this by "making room" for the relevant limit ordinals in a range outside of the support of a particular permutation σ. In fact, if we denote the map τ above by τ_β, then s itself is conjugation by τ_α in $\Sigma_\gamma^{\mathrm{c}}$ for a regular cardinal $\gamma > \alpha$.

Remark 2.3 The referee pointed out an alternative proof of Theorem 2.1, which makes use of the techniques employed in [4, Lemma 2.2], namely the use of HNN extensions. It also involves a map between classifying spaces, but is less explicit. In addition, the referee and N Kuhn pointed out that the case when $\alpha = \omega$ was proved in [6, Proposition 4.1], using an approach very similar to ours.

3 The lack of privileged weak colimits

In this section, we give an example showing that Heller's privileged weak colimits do not generally exist.

Theorem 3.1 *The set $\mathcal{G} = \{S^n \mid n \geq 0\}$ of spheres in* **Hot** *is not κ–bounded for any ordinal κ of uncountable cofinality*, *e.g. for any uncountable regular cardinal. That is*, *for each such κ*, *there exists a diagram $D\colon \kappa \to$* **Hot** *that admits no $\mathcal{G}$–privileged weak colimit.*

In particular, D admits no $\mathcal{G}$–privileged weak colimit for any set $\mathcal{G}$ containing the spheres. Note that the set of spheres is $\aleph_0$–bounded, so we learn that boundedness for one ordinal does not imply it for ordinals with larger cofinality.

Corollary 3.2 *Let T denote a countably infinite*, *discrete space. Then the set $\{S^n \mid n \geq 0\} \cup \{T\}$ is not κ–bounded in* **Hot** *for any limit ordinal κ.*

Proof Since κ is a limit ordinal, it has infinite cofinality. If κ has uncountable cofinality, then Theorem 3.1 applies. If κ has countable cofinality, then $\{T\}$ is not κ–bounded. □

In Section 3.1, we reduce the problem to finding a counterexample in the homotopy category of groupoids. In Section 3.2, we recall the theory of graphs of groups, and prove some general results about the word problem in the fundamental groupoid of a graph of groups. Finally, in Section 3.3, we give a counterexample in the homotopy category of groupoids and complete the proof of Theorem 3.1.

3.1 Reducing from spaces to groupoids

To prove Theorem 3.1 we will work primarily in the homotopy category **HoGpd** of groupoids, that is, the category of groupoids and isomorphism classes of functors. It is well known that the geometric realization of groupoids induces a reflective embedding $B\colon \mathbf{HoGpd} \to \mathbf{Hot}$ whose left adjoint is the fundamental groupoid functor Π_1 and whose image consists of the 1–types, i.e. those spaces X such that $\pi_n(X,x) = 0$ for every $n > 1$ and every $x \in X$. This follows from the adjunction between π_1 and the classifying space functor B that was used in the proof of Theorem 2.1.

Lemma 3.3 *Suppose given a diagram $D\colon J \to \mathbf{HoGpd}$, a set $\mathcal{G}'$ of groupoids and a set $\mathcal{G}$ of spaces containing $B\mathcal{G}'$ as well as S^n for all n. If D admits no $\mathcal{G}'$–privileged weak colimit in* **HoGpd**, *then $B \circ D\colon J \to \mathbf{Hot}$ admits no $\mathcal{G}$–privileged weak colimit in* **Hot**.

Proof We prove the contrapositive. Let $\lambda\colon B \circ D \to X$ be a $\mathcal{G}$–privileged weak colimit, with $X \in \mathbf{Hot}$. Then, since left adjoints preserve weak colimits, $\Pi_1(\lambda)\colon D \to \Pi_1 X$ is a weak colimit. We will show that it is $\mathcal{G}'$–privileged.

First, since λ is $\mathcal{G}$–privileged, every map $a\colon S^n \to X$ factors through a 1–type $BD(j)$ for some j. Thus, when $n > 1$, a is freely homotopic to a constant, which implies that $\pi_n(X,x)$ is trivial for all $x \in X$. We conclude that X is a 1–type itself, so that $X \simeq B(\Pi_1 X)$.

Since B is fully faithful, we see that $\Pi_1(\lambda)\colon D \to \Pi_1 X$ is $\mathcal{G}'$–privileged. Indeed, if $G \in \mathcal{G}'$, then

$$\begin{aligned}\mathbf{HoGpd}(G, \Pi_1 X) &\cong \mathbf{Hot}(BG, B(\Pi_1 X)) \cong \mathbf{Hot}(BG, X)\\ &\cong \operatorname*{colim}_j \mathbf{Hot}(BG, BD(j)) \cong \operatorname*{colim}_j \mathbf{HoGpd}(G, D(j)).\end{aligned}$$

One can show that the composite isomorphism is induced by $\Pi_1(\lambda)$. □

Thus, it suffices to exhibit appropriately pathological diagrams in **HoGpd**, and then to upgrade them to **Hot**. We aim to give a diagram in **HoGpd** admitting no weak colimit privileged with respect to the set $\mathcal{G}' = \{\boldsymbol{B}\mathbb{Z}\}$. Here $\boldsymbol{B}\mathbb{Z}$ denotes the groupoid freely generated by an automorphism, i.e. the groupoid with one object $*$ whose endomorphism group is the integers. Of course, $B(\boldsymbol{B}\mathbb{Z})$ is homotopy equivalent to S^1, so $\mathcal{G}$ in Lemma 3.3 can be taken to be the set of spheres.

Remark 3.4 For any groupoid G, a functor $f\colon \boldsymbol{B}\mathbb{Z} \to G$ corresponds to an object $f(*)$ of G and an automorphism $f_*\colon f(*) \to f(*)$. Furthermore, two such functors $f, g\colon \boldsymbol{B}\mathbb{Z} \to G$ are naturally isomorphic if and only if the automorphisms f_* and g_* are conjugate in G. In particular, a functor $f\colon \boldsymbol{B}\mathbb{Z} \to G$ factors through $h\colon H \to G$ in **HoGpd** if and only if f_* is conjugate to an automorphism in the image of h.

3.2 Graphs of groups

To construct our example, we recall the notion of a graph of groups and prove Corollaries 3.7 and 3.8, and Lemma 3.9, which will be used in the next section.

Definition 3.5 A *graph of groups* Γ is given by

- a graph, i.e. a set X of vertices, a set Y of oriented edges, functions $s, t\colon Y \rightrightarrows X$, and an involution $\overline{(-)}\colon Y \to Y$ interchanging s and t;
- groups G_x and G_y for $x \in X$ and $y \in Y$ equipped with monomorphisms $\mu_y\colon G_y \to G_{s(y)}$ such that $G_y = G_{\bar{y}}$.

For simplicity, we assume that the groups G_x are disjoint. For more on graphs of groups, see [7, Section I.5; 3, Section 1.B].

Higgins [5] defined the fundamental groupoid $\Pi_1\Gamma$ of a graph of groups. The groupoid $\Pi_1\Gamma$ is the groupoid on objects X with generating morphisms the elements of the groups G_x, endowed with x as domain and codomain, together with the elements of Y viewed as morphisms $y\colon s(y) \to t(y)$. These generators are subject to the relations holding in the groups G_x, as well as new relations

$$\mu_{\bar{y}}(a) = y\mu_y(a)\bar{y}$$

for every y and every $a \in G_y$. Note in particular that $\bar{y} = y^{-1}$, and we shall use both notations. It may aid the intuition to consider $\Pi_1\Gamma$ as the fundamental groupoid of the space built from $\coprod_X BG_x$ with cylinders $BG_y \times I$ glued in for each set $\{y, \bar{y}\}$ of elements of Y related by the involution.

By definition, the groupoid $\Pi_1\Gamma$ is a quotient of the groupoid $\boldsymbol{K}$ with object set X and with morphisms freely generated by $\left(\coprod G_x\right) \amalg Y$, subject to the relations holding in the groups G_x. A morphism $x_0 \to x_n$ in $\boldsymbol{K}$ is given by a word $(a_n, y_n, \dots, y_1, a_0)$, with $y_i \in Y$, $s(y_1) = x_0$, $t(y_n) = x_n$, and $s(y_{i+1}) = t(y_i) =: x_i$ for $1 \le i < n$, while $a_i \in G_{x_i}$ for $0 \le i \le n$.

The natural realization functor $\boldsymbol{K} \to \Pi_1\Gamma$ will be denoted by $|(a_n, y_n, \dots, y_1, a_0)| = a_n \circ y_n \circ \dots \circ y_1 \circ a_0$. Higgins proved that every morphism of $\Pi_1\Gamma$ is *uniquely* the image under $|\cdot|$ of a so-called "normal" word. We will not recall this concept, as we need only Higgins' corollary regarding the less rigid *irreducible* words.

A morphism $(a_n, y_n, \dots, y_1, a_0)$ in $\boldsymbol{K}$ is called *reducible* if $n > 1$ and, for some i, $y_{i-1} = \bar{y}_i$ and $a_{i-1} \in \mu_{y_i}(G_{y_i})$. Otherwise, the morphism is said to be *irreducible*. Note that a reducible word can be shortened by the move

$$(\dots, a_i, y_i, \mu_{y_i}(\hat{a}_{i-1}), \bar{y}_i, a_{i-2}, \dots) \mapsto (\dots, a_i \mu_{\bar{y}_i}(\hat{a}_{i-1}) a_{i-2}, \dots)$$

to a word with the same realization. Therefore, every element of $\Pi_1\Gamma$ is the realization of an irreducible word. We will use a key result of [5]:

Proposition 3.6 [5, Corollary 5] *Let w be an irreducible word in $\boldsymbol{K}$. If $|w|$ is an identity morphism in $\Pi_1\Gamma$, then $w = (e)$, where e is an identity element of some G_x.*

Define the *length* $\ell(w)$ of the word $w = (a_n, y_n, \dots, y_1, a_0)$ to be n. We deduce the following:

Corollary 3.7 *Let Γ be a graph of groups and consider a word w in the groupoid $\boldsymbol{K}$. If $\ell(w) > 0$ and $|w|$ is equal to the realization of a zero-length word, then w is reducible.*

Proof Suppose that $w = (a_n, y_n, \dots, y_1, a_0)$ for $n > 0$ and that $|w| = |(a)|$ for some a in some G_x. Let $w' = (a_n, y_n, \dots, y_1, a_0 a^{-1})$. Then $|w'|$ is an identity morphism in $\Pi_1\Gamma$, so, by Proposition 3.6, w' is reducible. Since reduction occurs at interior points, w must be reducible as well. □

Corollary 3.8 *Given a graph of groups Γ and a vertex x, the vertex group G_x embeds in the automorphism group of x in the fundamental groupoid $\Pi_1\Gamma$.*

Because of this, we regard elements of the vertex groups as elements of the fundamental groupoid without explicitly naming the inclusion map.

Proof The map sends $a \in G_x$ to the realization of the word (a). Since the word (a) is irreducible, if the realization is an identity in $\Pi_1\Gamma$, Proposition 3.6 tells us that a is the identity element of G_x. Therefore, this map is injective. □

We next record some facts about free groups:

Lemma 3.9 *Let $A \subseteq B$ be nonabelian free groups, with A free on generators $\{a_i\}$ and B free on $\{a_i\} \cup \{b_j\}$.*

(1) *If $b \in B$ and, for all $a \in A$, we have $bab^{-1} = a$, then b is the identity.*

(2) *If $b \in B$ satisfies $bab^{-1} \in A$ for some $a \in A$, then either a is the identity or $b \in A$.*

Proof Fix $b \in B$. For part (1), if we take $a = a_i$ then the assumption that $ba_i b^{-1} = a_i$ shows that an irreducible word for b must have last letter a_i or a_i^{-1} for every i, which is absurd since there are at least two i's.

For part (2), we assume a is nontrivial and $b \notin A$. Factor b as $b'b''$, where $b'' \in A$ while b' is represented by an irreducible word with rightmost letter some b_j. Then $bab^{-1} = b'a'b'^{-1}$, where $a' := b''ab''^{-1}$ is a nontrivial element of A. The conclusion now follows from the observation that no reductions are possible in the concatenation of the irreducible words for b', a' and b'^{-1}, since concatenating those words gives no letter adjacent to its inverse. □

3.3 A counterexample in the homotopy category of groupoids

We now apply the generalities above to the problem of weak colimits in **HoGpd**.

We fix for the rest of the paper an ordinal κ of uncountable cofinality, and introduce the main characters in our counterexample. Note that Theorem 3.1 will follow if we replace $\kappa = [0, \kappa)$ by the interval $[2, \kappa)$, since the two categories are isomorphic. We use the latter because it allows us to use simple indexing while ensuring that all of the vertex groups below are nonabelian.

Definition 3.10 Define a graph of groups Γ with object set $[2, \kappa)$, vertex group G_α free on α generators, edge set $\{y_\alpha^\beta : \beta \to \alpha \mid \alpha \neq \beta \in [2, \kappa)\}$ and involution $y_\alpha^\beta \mapsto y_\beta^\alpha$. The edge group $G_{y_\alpha^\beta}$ is just $G_{\min(\beta,\alpha)}$. The edge morphism $\mu_{y_\alpha^\beta} : G_{\min(\beta,\alpha)} \to G_\beta$ is the natural inclusion. Let $Z = \Pi_1 \Gamma$.

Next, define a diagram $D : [2, \kappa) \to \mathbf{HoGpd}$ by letting $D(\alpha) = G_\alpha$, with action on morphisms the natural inclusions, denoted by $D_\alpha^\beta : D(\beta) \to D(\alpha)$. We have a cocone $A : D \to Z$ with $A_\alpha : D(\alpha) \to Z$ the natural inclusion of the vertex group. To see that these maps do constitute a cocone, we note that y_α^β is the unique component of a natural isomorphism $A_\beta \cong A_\alpha \circ D_\alpha^\beta$.

We do not need this fact, but it may provide motivation to the reader to know that Z is the "standard" weak colimit of the diagram D, defined as the homotopy coequalizer of the natural diagram

$$\coprod_{\beta<\alpha} D(\beta) \rightrightarrows \coprod_{\beta} D(\beta).$$

Critically, we do *not* have the relations $y_\alpha^\beta y_\beta^\gamma = y_\alpha^\gamma$ in Z which would allow us to lift A into a cocone in the 2–category of groupoids. We now intend to show that D admits no privileged weak colimit by, roughly, showing that this failure is unavoidable: no choice of isomorphisms $A_\beta \cong A_\alpha \circ D_\alpha^\beta$ can give A such a lift.

Write Z_Y for the subgroupoid of Z generated by the edges of the graph. Any morphism of Z_Y can be uniquely written as a reduced word in the generators y_α^β. We say that such a morphism *passes through* a vertex α if this unique word involves a generator with source or target α. The identity id_α is said to *pass through* α and no other vertex.

Lemma 3.11 *Let $u\colon \beta \to \alpha$ in Z and let $2 \le \gamma \le \min(\alpha, \beta)$. Then u is in Z_Y and does not pass through any vertex less than γ if and only if u is the unique component of a natural isomorphism $A_\beta \circ D_\beta^\gamma \cong A_\alpha \circ D_\alpha^\gamma$ between functors $D(\gamma) \to Z$. Explicitly, for all $a \in D(\gamma)$, we must have $D_\alpha^\gamma(a) = uD_\beta^\gamma(a)u^{-1}$ in Z.*

Proof Suppose that u is in Z_Y and does not pass through any vertex less than γ. It suffices to show that y_α^β conjugates D_β^γ into D_α^γ when $\gamma \le \beta \le \alpha$. In this case, $\mu_{y_\alpha^\beta}$ is an identity map, and so the claim follows from the defining relations of Z:

$$y_\alpha^\beta D_\beta^\gamma(a)\bar{y}_\alpha^\beta = y_\alpha^\beta \mu_{y_\alpha^\beta}(D_\beta^\gamma(a))\bar{y}_\alpha^\beta = \mu_{\bar{y}_\alpha^\beta}(D_\beta^\gamma(a)) = D_\alpha^\beta(D_\beta^\gamma(a)) = D_\alpha^\gamma(a).$$

For the converse, let u be the realization of an irreducible word $w = (a_n, y_n, \dots, y_1, a_0)$. We proceed by induction on n. If $n = 0$, then $\alpha = \beta$ and $u = |(a_0)| \in G_\beta$. The assumption that $D_\beta^\gamma(a) = uD_\beta^\gamma(a)u^{-1}$ shows that u centralizes a nonabelian subgroup of a free group. By Lemma 3.9(1), we see that u is trivial, as desired. And clearly u does not pass through a vertex less than γ; indeed, it passes through only β, and $\beta \ge \gamma$.

For the inductive step, assume $n > 0$. Then $s(y_1) = \beta$ and $t(y_n) = \alpha$. Let $t(y_1) = \delta$, and note that $\delta \ne \beta$. In terms of w, the assumption on u is that the word

$$w' = (a_n, y_n, \dots, y_1, a_0 D_\beta^\gamma(a) a_0^{-1}, y_1^{-1}, a_1^{-1}, \dots, y_n^{-1}, a_n^{-1})$$

has realization $D_\alpha^\gamma(a)$ for every $a \in G_\gamma$. Thus, by Corollary 3.7, w' is reducible. Since, by assumption, w is irreducible, any reduction must occur at the central entry.

So, letting $\varepsilon := \min(\beta,\delta)$, we must have $a_0 D^\gamma_\beta(a) a_0^{-1} \in \mu_{y_1}(G_\varepsilon) = D^\varepsilon_\beta(G_\varepsilon)$. In particular, $a_0 D^\varepsilon_\beta(\hat{a}) a_0^{-1} \in D^\varepsilon_\beta(G_\varepsilon)$ for some nonidentity element $\hat{a}$ in $G_{\min(\gamma,\varepsilon)}$. So, by Lemma 3.9(2), we see that $a_0 \in D^\varepsilon_\beta(G_\varepsilon) \subseteq G_\beta$; that is, $a_0 = D^\varepsilon_\beta(\hat{a}_0)$ for some $\hat{a}_0 \in G_\varepsilon$. It then follows that $D^\gamma_\beta(a)$ is in the image of D^ε_β for every $a \in G_\gamma$, which means that $\gamma \le \varepsilon$, since the inclusions of vertex groups are strict.

The reduction of w at its central entry is

$$(a_n, y_n, \dots, y_2, a_1 D^\varepsilon_\delta(\hat{a}_0) D^\gamma_\delta(a) D^\varepsilon_\delta(a_0)^{-1} a_1^{-1}, y_2^{-1}, a_2^{-1}, \dots, a_n^{-1}).$$

Thus, if we define $u' \colon \delta \to \alpha$ to be $|w''|$, where $w'' = (a_n, y_n, \dots, y_2, a_1 D^\varepsilon_\delta(\hat{a}_0))$, then $\ell(w'') < n$ and u' conjugates D^γ_δ to D^γ_α. By induction, $u' \in Z_Y$. Since

$$u' y_1 = a_n y_n \cdots y_2 a_1 D^\varepsilon_\delta(\hat{a}_0) y_1 = a_n y_n \cdots y_2 a_1 y_1 D^\varepsilon_\beta(\hat{a}_0) = u,$$

u is in Z_Y as well. Finally, recall that we observed that $\gamma \le \varepsilon = \min(\beta,\delta)$. By induction, u' does not pass through any vertex less than γ. So the same is true of $u = u' y_1$. $\square$

Let Z_X denote the subgroupoid of Z containing those morphisms in the image of G_x for some x. By Corollary 3.8, Z_X is isomorphic to the disjoint union of the groups G_x.

Lemma 3.12 *Consider a morphism $z \colon \alpha \to \alpha$ in Z. If there are morphisms $u \colon \alpha \to \beta$ and $v \colon \alpha \to \gamma$ in Z such that uzu^{-1} is in Z_X and vzv^{-1} is in Z_Y, then $z = \mathrm{id}_\alpha$.*

Proof Let $y = vzv^{-1}$. Note that the inclusion $Z_Y \to Z$ has a retraction $r \colon Z \to Z_Y$ defined by sending the generators of each vertex group to identity elements. Since $uv^{-1} y v u^{-1}$ is in Z_X, we have that $r(uv^{-1} y v u^{-1}) = r(uv^{-1}) y r(uv^{-1})^{-1}$ is an identity, and so y is an identity. Since $y = vzv^{-1}$ is an identity, we have that z is an identity as well. $\square$

The following is the key technical result:

Lemma 3.13 *Suppose given a family $u^\beta_\alpha \colon \beta \to \alpha$ of morphisms of Z_Y for all $\beta < \alpha \in [2,\kappa)$ such that $u^\gamma_\alpha = u^\beta_\alpha u^\gamma_\beta$ for all triples $\gamma < \beta < \alpha$. Then there exists a pair $\beta < \alpha$ such that u^β_α passes through some γ with $\gamma < \beta$.*

Proof Assume that this is not the case. Let $\delta_0 = 2$ and $\delta_1 = 3$. Inductively, for each $n \in \omega$, let δ_n be an ordinal exceeding every vertex that $u^{\delta_{n-2}}_{\delta_{n-1}}$ passes through. This is possible because κ is a limit ordinal.

For each n, $u^{\delta_{n-1}}_{\delta_n}$ can be written uniquely as a reduced word in the free groupoid Z_Y. Let y_n be a letter in this word which is of the form y^β_α with $\beta < \delta_n \le \alpha$. Such a letter must exist since $u^{\delta_{n-1}}_{\delta_n}$ starts at a vertex less than δ_n and ends at δ_n. Note that y_n cannot occur in the reduced form of any $u^{\delta_{k-1}}_{\delta_k}$ with $k \ne n$. For $k < n$, this holds by definition of δ_n, and, for $k > n$, this holds by our assumption that each u^β_α only passes through γ with $\gamma \ge \beta$. In particular, the y_n are distinct.

Using that κ has uncountable cofinality, choose $\delta_\omega < \kappa$ to be an ordinal exceeding every δ_n. Consider the decompositions

$$u^{\delta_0}_{\delta_\omega} = u^{\delta_1}_{\delta_\omega} u^{\delta_0}_{\delta_1} = u^{\delta_2}_{\delta_\omega} u^{\delta_1}_{\delta_2} u^{\delta_0}_{\delta_1} = u^{\delta_3}_{\delta_\omega} u^{\delta_2}_{\delta_3} u^{\delta_1}_{\delta_2} u^{\delta_0}_{\delta_1} = \cdots .$$

In the expression $u^{\delta_1}_{\delta_\omega} u^{\delta_0}_{\delta_1}$, a y_1 occurs in the reduced form of the right-hand factor, and does not occur in the left-hand factor, so the reduced form of $u^{\delta_0}_{\delta_\omega}$ must contain a y_1. Similarly, the second decomposition involves a y_2, which can't be cancelled from either side, so the reduced form of $u^{\delta_0}_{\delta_\omega}$ must contain a y_2. Continuing, we see that the reduced form of $u^{\delta_0}_{\delta_\omega}$ must contain countably many distinct letters, a contradiction. □

Recall that κ is an arbitrary ordinal of uncountable cofinality.

Proposition 3.14 *There exists a diagram $C : [2,\kappa) \to$ **HoGpd** valued in the homotopy category of groupoids such that, for any weak colimit with cocone $F : C \to W$, there exists an automorphism in W which is not conjugate to any morphism in the image of any leg $F_\alpha : C(\alpha) \to W$ of F.*

Proof We claim that the diagram D (see Definition 3.10) is an example of such a C.

Towards a contradiction, suppose $F : D \to W$ is a weakly colimiting cocone such that every automorphism in W is conjugate to one in the image of some component of F. Write F_α for functors representing the maps $D(\alpha) \to W$. Since F is a cocone in **HoGpd**, for each $\beta < \alpha \in [2,\kappa)$ we may choose a natural isomorphism

$$h^\beta_\alpha : F_\beta \cong F_\alpha \circ D^\beta_\alpha$$

between functors $D(\beta) \to W$ in **Gpd**. Denote by $\hat{h}^\beta_\alpha$ the unique component of h^β_α. As usual we shall denote $(h^\beta_\alpha)^{-1}$ by h^α_β, and similarly for $\hat{h}$, as well as u below.

Recall the natural cocone $A : D \to Z$ from Definition 3.10 and suppose given a representative $f : W \to Z$ of a factorization of the cocone A through F. For each α, pick a natural isomorphism $k_\alpha : A_\alpha \cong f \circ F_\alpha$ with unique component $\hat{k}_\alpha$. For $\beta < \alpha$, let $u^\beta_\alpha = \hat{k}^{-1}_\alpha f(\hat{h}^\beta_\alpha)\hat{k}_\beta$, the unique component of the natural transformation $A_\beta \to A_\alpha \circ D^\beta_\alpha$

defined by $(k_\alpha^{-1} * D_\alpha^\beta) \circ (f * h_\alpha^\beta) \circ k_\beta$, where $*$ denotes whiskering.[2] By Lemma 3.11, we see that each u_α^β is in Z_Y, so the same holds for the morphism $u_{\alpha\beta\gamma}\colon \gamma \to \gamma$ defined as $u_\gamma^\alpha u_\alpha^\beta u_\beta^\gamma$ for $\gamma < \beta < \alpha$. Furthermore, the same lemma guarantees that no u_α^β passes through a vertex less than $\min(\beta, \alpha)$.

For each $\gamma < \beta < \alpha$, denote by $w_{\alpha\beta\gamma} \in W$ the unique component of the composite natural transformation

$$h_\gamma^\alpha \circ (h_\alpha^\beta * D_\beta^\gamma) \circ h_\beta^\gamma \colon F_\gamma \to F_\gamma.$$

We have $w_{\alpha\beta\gamma} = \hat{h}_\gamma^\alpha \hat{h}_\alpha^\beta \hat{h}_\beta^\gamma$, so

$$\hat{k}_\gamma^{-1} f(w_{\alpha\beta\gamma}) \hat{k}_\gamma = \hat{k}_\gamma^{-1} f(\hat{h}_\gamma^\alpha) \hat{k}_\alpha \hat{k}_\alpha^{-1} f(\hat{h}_\alpha^\beta) \hat{k}_\beta \hat{k}_\beta^{-1} f(\hat{h}_\beta^\gamma) \hat{k}_\gamma = u_{\alpha\beta\gamma}.$$

In particular, $u_{\alpha\beta\gamma}$ is conjugate to $f(w_{\alpha\beta\gamma})$.

On the other hand, by assumption on F, $w_{\alpha\beta\gamma}$ is conjugate to a morphism in the image of some $F_\theta\colon D(\theta) \to W$, say to $F_\theta(w'_{\alpha\beta\gamma})$. Composing with f, we see that $u_{\alpha\beta\gamma}$ is conjugate to $f(F_\theta(w'_{\alpha\beta\gamma}))$. Finally, using $\hat{k}_\gamma$, we see $u_{\alpha\beta\gamma}$ is conjugate to $A_\theta(w'_{\alpha\beta\gamma})$, so, in particular, to an element of Z_X. Since we saw above that $u_{\alpha\beta\gamma}$ is in Z_Y, Lemma 3.12 shows that $u_{\alpha\beta\gamma} = \mathrm{id}_\gamma$.

Finally, Lemma 3.13 implies that at least one u_α^β passes through a vertex less than β, contradicting what we saw above. □

Proof of Theorem 3.1 By Proposition 3.14 and Remark 3.4, the diagram D admits no weak colimit privileged with respect to the set $\mathcal{G}' = \{\boldsymbol{B}\mathbb{Z}\}$. Thus, by Lemma 3.3, $B \circ D$ admits no weak colimit in **Hot** which is privileged with respect to the set of spheres. □

4 The spheres reflect equivalences in the 2–category of spaces

We saw in Theorem 2.1 that in the homotopy category of spaces there is no set of objects that jointly reflects isomorphisms. In this section, we show that in the homotopy 2–*category* of spaces, the spheres do jointly reflect equivalences. We first define the terms we are using.

Definition 4.1 By $\underline{\textbf{Hot}}$, we mean the 2–category whose objects are spaces of the homotopy type of a CW–complex and whose hom categories are the fundamental groupoids of mapping spaces; that is, $\underline{\textbf{Hot}}(X, Y) = \Pi_1(Y^X)$.

[2] For instance, $f * h_\alpha^\beta \colon f \circ F_\beta \cong f \circ F_\alpha \circ D_\alpha^\beta$ has unique component $f(\hat{h}_\alpha^\beta)$.

Definition 4.2 A set $\mathcal{G}$ of objects in a 2–category $\mathcal{K}$ *jointly reflects equivalences* if, whenever $f : X \to Y$ is a morphism in $\mathcal{K}$ such that, for every $S \in \mathcal{G}$, the induced functor $\mathcal{K}(S, f) \colon \mathcal{K}(S, X) \to \mathcal{K}(S, Y)$ is an equivalence of categories, f itself must be an equivalence in $\mathcal{K}$.

We shall show in Theorem 4.3 that the 2–category $\underline{\mathbf{Hot}}$ admits a set $\mathcal{G}$ of objects that jointly reflects equivalences, namely $\mathcal{G} = \{S^n \mid n \geq 0\}$. Note that a map f in $\underline{\mathbf{Hot}}$ is an equivalence if and only if it is a homotopy equivalence. This theorem is a corollary of [6, Theorem 1], which shows that, for a map $f : X \to Y$ of (arcwise connected) spaces which is surjective on all fundamental groups, bijectivity of f on higher homotopy groups is equivalent to that on free homotopy classes of maps from spheres.

With this, we are prepared to show that the spheres satisfy the analogue of Whitehead's theorem for $\underline{\mathbf{Hot}}$:

Theorem 4.3 *The set $\mathcal{G} = \{S^n\}$ of spheres jointly reflects equivalences in the* 2–*category* $\underline{\mathbf{Hot}}$ *of spaces.*

Proof Let $f : X \to Y$ be such that $\underline{\mathbf{Hot}}(S^n, f) \colon \underline{\mathbf{Hot}}(S^n, X) \to \underline{\mathbf{Hot}}(S^n, Y)$ is an equivalence of groupoids for every n. Consider an inclusion of $*$ into $S^0 \cong * \sqcup *$. Since this has a retraction, the functor $\underline{\mathbf{Hot}}(*, X) \to \underline{\mathbf{Hot}}(*, Y)$ is a retract of the equivalence $\underline{\mathbf{Hot}}(S^0, X) \to \underline{\mathbf{Hot}}(S^0, Y)$ and is therefore also an equivalence. That is, f induces an equivalence $\Pi_1(X) \to \Pi_1(Y)$ of fundamental groupoids. Thus, f induces an isomorphism on π_0 and on every π_1.

Therefore, we can apply [6, Theorem 1], so that f will be a homotopy equivalence as soon as it induces a bijection on free homotopy classes of maps from S^n. Now, the set of free homotopy classes of maps $S^n \to X$ is simply the set of connected components in the groupoid $\underline{\mathbf{Hot}}(S^n, X)$. Since f induces an equivalence $\underline{\mathbf{Hot}}(S^n, X) \to \underline{\mathbf{Hot}}(S^n, Y)$, a fortiori it induces an isomorphism on connected components, and the theorem is proven. □

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Mod 2 power operations revisited

DYLAN WILSON

In this mostly expository note we take advantage of homotopical and algebraic advances to give a modern account of power operations on the mod 2 homology of $\mathbb{E}_\infty$–ring spectra. The main advance is a quick proof of the Adem relations utilizing the Tate-valued Frobenius as a homotopical incarnation of the total power operation. We also give a streamlined derivation of the action of power operations on the dual Steenrod algebra.

55P43, 55S12

Introduction

As someone who entered college at about the time that Netflix started automatically playing the next episode of a series, I cannot imagine discovering or verifying the Adem relations using the tools available to Adem [1].[1] I even find it hard to *remember* the Adem and Nishida relations.

[1] It was precisely while trying and failing multiple times to prove the Adem relations in equivariant homotopy theory that, in an act of true laziness, I stumbled upon the technique explained in this note.

Luckily, there is a useful mnemonic device which utilizes the *total power operation*

$$Q(t) := \sum_{i\in\mathbb{Z}} Q^i t^i.$$

Here t is an indeterminate, and the operation $Q^i \colon A_* \to A_{*+i}$ acts on the homotopy of any $\mathbb{E}_\infty$–$\mathbb{F}_2$–algebra A. The total power operation then produces a map

$$Q(t)\colon A_* \to A_*((t)).$$

We extend $Q(t)$ to a ring map

$$Q(t)\colon A_*((s)) \to A_*[\![s,t]\!][s^{-1},t^{-1}]$$

by requiring that

$$Q(t)(s) = s + s^2 t^{-1}.$$

With this convention, it is possible to restate the Adem relations, following Bullett and Macdonald [5], Steiner [16], and Bisson and Joyal [3], as:

- **Adem relations** For any $x \in A_*$, $Q(t)Q(s)x$ is symmetric in s and t.

The usual Adem relations are recovered using a trick with residues which we will review in Section 4.3. Steiner's proof that the above identity holds is to reduce it to one of the expressions met in the proof of the Adem relations as in Steenrod [15, page 119] and May [12, 4.7(e,g,i)].

In the case of Steenrod operations acting on the cohomology of a space X, there is a more conceptual argument due to Segal [5, Section 4]. One can use the diagonal map to produce a version of the total power operation taking values in $H^*(X \times \mathrm{B}\Sigma_2)$. Indeed, this is one of the earlier constructions of Steenrod operations [15, Chapter VII]. The iterated total square then takes values in $H^*(X \times \mathrm{B}\Sigma_2 \times \mathrm{B}\Sigma_2) = H^*(X)[s,t]$ but factors through the total fourth power which takes values in $H^*(X \times \mathrm{B}\Sigma_4)$. The automorphism swapping s and t arises as an inner automorphism of Σ_4 so the formula for the iterated square must be symmetric in s and t.

Our primary goal is to explain how the Tate diagonal (Section 2.3) on spectra allows for a similar argument for general power operations. The reader could probably reconstruct the argument themselves just from the observation that the total power operation is the effect on homotopy of the (non–$\mathbb{F}_2$–linear) map of spectra

$$A \xrightarrow{\Delta} (A \otimes_{\mathbb{F}_2} A)^{t\Sigma_2} \to A^{t\Sigma_2}.$$

In fact, we take this as a definition and develop all the basic properties of power operations efficiently from there. We hope that this note will give a mnemonic for the *proofs* of the standard identities for power operations in much the same way that the work of Steiner [16], Bisson and Joyal [3], and Baker [2] has provided mnemonics for their *statements*.

Outline

In Sections 1 and 2 we review the facts we need about the Tate construction and the Tate diagonal, following Nikolaus and Scholze [14]. In Section 3 we give three definitions of the operations Q^i: the classical one, one due to Lurie [10, Section 2.2], and one in terms of the Tate-valued Frobenius. We then explain how to recover the first properties of power operations.

In Section 4 we turn to the Adem relations. The key thing to prove is that having a Σ_4–equivariant map $A^{\otimes 4} \to A$ produces a lift of the iterated total power operation through the Frobenius $A \to A^{t\Sigma_4}$. This takes a little bit of work but the reader could come up with the argument themselves if they remember to use the universal property of the Tate diagonal amongst natural transformations of exact, lax symmetric monoidal functors over and over again. Indeed, this proof is an excellent illustration of the computational utility of establishing such universal properties in the first place.

Finally, in Section 5, we show how the Bisson–Joyal and Baker formulations of the Nishida relations arise naturally from the perspective of the Tate-valued Frobenius. We end by explaining how to recover Steinberger's formulas [4, Section III.2] for the action of power operations on the dual Steenrod algebra. This last step is mostly algebraic, and essentially due to Bisson and Joyal, but we have included it for completeness.

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1 The Tate construction

We review the Tate construction (Section 1.1) and its universal property (Section 1.3) as well as the important Warwick duality (Section 1.2) of Greenlees [8] which allows an alternative computation of the Tate construction. We end (Section 1.4) by spelling out what happens in the case $G = \Sigma_2$.

1.1 Definitions

Let G be a finite group and k an $\mathbb{E}_\infty$–ring, and denote by

$$\mathsf{Mod}_k^{hG} := \mathsf{Psh}(\mathrm{B}G; \mathsf{Mod}_k)$$

the ∞–category of *Borel G–modules*. There is a fully faithful embedding

$$\mathsf{Mod}_k^{hG} \to \mathsf{Mod}_k^{G}$$

from Borel G–modules to modules over k in *genuine* G–spectra whose essential image consists of the *Borel complete G–modules*, ie those X such that $X \to F(\mathrm{E}G_+, X)$ is an equivalence. Let $\mathcal{F}$ be a collection of subgroups closed under subconjugacy, and $\mathrm{E}\mathcal{F}$ the G–space characterized up to homotopy by the requirement

$$\mathrm{E}\mathcal{F}^H = \begin{cases} * & \text{if } H \in \mathcal{F}, \\ \varnothing & \text{if } H \notin \mathcal{F}, \end{cases}$$

and define $\widetilde{\mathrm{E}\mathcal{F}}$ as the cofiber of $\mathrm{E}\mathcal{F}_+ \to S^0$. Then the *$\mathcal{F}$–Tate spectrum* of a Borel G–spectrum can be computed as [7, page 443]

$$X^{t\mathcal{F}} = (\widetilde{\mathrm{E}\mathcal{F}} \wedge F(\mathrm{E}G_+, X))^G,$$

where the right-hand side is computed in genuine G–spectra.

It will be more convenient for us to think of the above as a computation and not a definition. Instead, we opt to define the Tate construction by a universal property, following [14].

To that end, let

$$(\mathsf{Mod}_k^{hG})_{\mathcal{F}-\mathrm{ind}} \subseteq \mathsf{Mod}_k^{hG}$$

be the smallest full, stable subcategory containing all objects which are left Kan extended from diagrams $\mathrm{B}H \to \mathsf{Sp}$ for some $H \in \mathcal{F}$.

Recall [14, Section I.3] that, associated to any exact functor $F\colon \mathsf{Mod}_k^{hG} \to \mathcal{E}$ to a presentable stable ∞–category $\mathcal{E}$, there is a natural transformation

$$F \to L_{\mathcal{F}} F$$

which is initial amongst natural transformations to exact functors which annihilate the subcategory $(\mathsf{Mod}_k^{hG})_{\mathcal{F}-\mathrm{ind}}$. Concretely, $L_{\mathcal{F}}F$ is specified by the formula [14, I.3.3]

$$L_{\mathcal{F}}F(X) = (\mathsf{Mod}_k^{hG})_{\mathcal{F}-\mathrm{ind}/X} Y \operatorname{colim} F(\mathrm{cofib}(Y \to X)).$$

Definition 1.1.1 With notation as above, we define

$$(-)^{t\mathcal{F}} = L_{\mathcal{F}}((-)^{hG}) \colon \mathsf{Mod}_k^{hG} \to \mathsf{Mod}_k.$$

More generally, if $G \subseteq G'$, we define

$$(-)^{t\mathcal{F}} = L_{\mathcal{F}}((-)^{hG}) \colon \mathsf{Mod}_k^{hG'} \to \mathsf{Mod}_k^{hW_{G'}G}$$

where $W_{G'}G = N_{G'}G/G$ is the Weyl group of G in G'.

Example 1.1.2 When $\mathcal{F}$ consists only of the trivial subgroup, we denote $X^{t\mathcal{F}}$ by X^{tG}. This can be computed as the cofiber of the trace map $X_{hG} \to X^{hG}$.

Example 1.1.3 Suppose $G \subseteq \Sigma_n$ is a subgroup and let $\mathcal{F} = \mathcal{T}$ be the family of subgroups of G which do *not* act transitively on $\{1, \dots, n\}$. When $G = C_n$ this coincides with the more commonly seen family of proper subgroups, and when $G = C_p$ this coincides with the family consisting of only the trivial subgroup.

1.2 Warwick duality

We can dualize the construction in the previous section and define the *opposite $\mathcal{F}$–Tate spectrum*[2] as

$$X^{t^{\mathrm{op}}\mathcal{F}} := \operatorname*{holim}_{((\mathsf{Mod}_k^{hG})_{\mathcal{F}-\mathrm{ind}})_{X/} \ni Y} \mathrm{fib}(X \to Y)_{hG}.$$

Greenlees proved [8, Section B] that this construction is not really new:

Theorem 1.2.1 (Warwick duality) *There is a canonical equivalence*

$$X^{t^{\mathrm{op}}\mathcal{F}} \simeq \Sigma^{-1} X^{t\mathcal{F}}.$$

In particular, we obtain extra functoriality: if $\mathcal{F} \subseteq \mathcal{F}'$, then the original construction produces a canonical map $(-)^{t\mathcal{F}'} \to (-)^{t\mathcal{F}}$ while the opposite construction, composed with suspension, produces a map $(-)^{t\mathcal{F}} \to (-)^{t\mathcal{F}'}$.

1.3 Monoidal structure

We will make much use of the following excellent description of the lax symmetric monoidal structure on the Tate construction.

[2]We stole this name from [6].

Proposition 1.3.1 *There is a natural transformation of lax symmetric monoidal functors*

$$(-)^{hG} \to (-)^{t\mathcal{F}}$$

which is initial amongst natural transformations of lax symmetric monoidal functors with target an exact functor that annihilates $(\mathsf{Mod}_k^{hG})_{\mathcal{F}-\mathrm{ind}}$.

This follows from the more general result [14, I.3.6] about the relationship between Verdier quotients and lax symmetric monoidal structures.

1.4 An example

Let k be a field of characteristic 2. Then

$$\pi_* k^{h\Sigma_2} \simeq \mathrm{H}^{-*}(\mathrm{B}\Sigma_2, k) = k[\![t]\!],$$

where $t \in \pi_{-1} k^{h\Sigma_2}$ is the Stiefel–Whitney class of the canonical line bundle. The Tate construction has the effect of inverting t and we can compute

$$\pi_* k^{t\Sigma_2} = k(\!(t)\!),$$

the algebra of Laurent series over k.

On the other side, the homotopy orbits $k_{h\Sigma_2}$ have a dual basis on homotopy

$$\pi_* k_{h\Sigma_2} = k\{e_0, e_1, \dots\},$$

where e_i is the linear dual of t^i. The trace map

$$k_{h\Sigma_2} \to k^{h\Sigma_2}$$

is zero on homotopy groups and so we have a short exact sequence

$$0 \to k[\![t]\!] \to k(\!(t)\!) \to \pi_* \Sigma k_{h\Sigma_2} \to 0$$

which identifies the last term as the quotient $k(\!(t)\!)/k[\![t]\!]$. This provides another basis for the homotopy of $k_{h\Sigma_2}$, and the two are related by the correspondence

$$e_i \leftrightarrow t^{-i-1}.$$

Under this interpretation, the composite map

$$k^{t\Sigma_2} \to \Sigma k_{h\Sigma_2} \to \Sigma k$$

is given by sending a Laurent series $g(t) = \sum a_i t^i$ to the *residue* a_{-1}.

Finally, Warwick duality in this context translates to the computation [9, 16.1]

$$\Sigma^{-1}k^{t\Sigma_2} = \operatorname*{holim}_n (\Sigma^{-n\tau}k)_{h\Sigma_2} = \operatorname*{holim}_n k \wedge (\mathbb{R}P^\infty)^{-n\tau} = \operatorname*{holim}_n k \wedge \mathbb{R}P^\infty_{-n},$$

where τ is the sign representation.

2 Tate powers

The source of power operations is the symmetry present on $X^{\otimes n}$. In Section 2.1 we review several constructions based on this symmetry. In Section 2.2 we explain how the construction $X \mapsto (X^{\otimes n})^{t\mathcal{T}}$ arises as a Goodwillie derivative; in particular this construction is exact. In Section 2.3, following [14], we describe the spectral analog of the diagonal map we will use when defining power operations.

2.1 Variants of extended powers

Let $\mathcal{C}$ be a symmetric monoidal ∞–category. Then there is a natural functor

$$\mathcal{C} \to \mathcal{C}^{h\Sigma_n} = \mathsf{Fun}(\mathrm{B}\Sigma_n, \mathcal{C})$$

given as the composite

$$\mathcal{C} \xrightarrow{\delta} (\mathcal{C}^{\times n})^{h\Sigma_n} \to \mathcal{C}^{h\Sigma_n}$$

where the latter map is a choice of tensor product. In other words, for every $X \in \mathcal{C}$, the object $X^{\otimes n}$ has a Σ_n–action.

If $\mathcal{C}$ admits homotopy limits and colimits, we can form both a "symmetric" power of an object and a "divided" power of an object. We do this more generally for a fixed subgroup $G \subseteq \Sigma_n$.

Definition 2.1.1 We define symmetric and divided power functors as

$$\mathrm{Sym}^G(X) := (X^{\otimes n})_{hG}, \quad \Gamma^G(X) := (X^{\otimes n})^{hG}.$$

Finally, if $\mathcal{C} = \mathsf{Mod}_k$ is the ∞–category of k–modules over an $\mathbb{E}_\infty$–ring k, then:

Definition 2.1.2 Let $G \subseteq \Sigma_n$ be a subgroup. We define the *Tate power* of X as

$$\mathrm{T}_G(X) := (X^{\otimes n})^{t\mathcal{T}}$$

where $\mathcal{T}$ is the family of nontransitive subgroups of G.

In each case we abbreviate G as n if $G = \Sigma_n$.

2.2 Tate powers as a Goodwillie derivative

Let $\mathcal{C}$ and $\mathcal{D}$ be stable, presentable ∞–categories. Then the full subcategory

$$\mathsf{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{D}) \subseteq \mathsf{Fun}(\mathcal{C},\mathcal{D})$$

admits a left adjoint [11, 6.1.1.10], the 1–excisive approximation

$$P_1 \colon \mathsf{Fun}(\mathcal{C},\mathcal{D}) \to \mathsf{Fun}^{\mathrm{ex}}(\mathcal{C},\mathcal{D}).$$

In the case where $F(0)=0$, we may compute P_1F as [11, 6.1.1.23 and 6.1.1.27]

$$P_1F(X) = \operatorname*{hocolim}_n \Omega^n_{\mathcal{D}} F(\Sigma^n_{\mathcal{C}} X).$$

I believe the following is well known but do not know a reference.

Proposition 2.2.1 *With notation as in Section 2.1, there is an equivalence*

$$P_1\Gamma^G \simeq T_G.$$

Proof Let V denote the standard representation of Σ_n on $\mathbb{R}^n$ and $\overline{V}$ the reduced standard representation. By the formula above,

$$\begin{aligned}
P_1\Gamma^G(X) &= \operatorname*{hocolim}_j \Omega^j \Gamma^G(\Sigma^j X) \\
&\simeq \operatorname{hocolim} \Omega^j (\Sigma^{jV} X^{\otimes n})^{h\Sigma_n} \\
&\simeq \operatorname{hocolim} (\Sigma^{j\overline{V}} X^{\otimes n})^{h\Sigma_n} \\
&\simeq \operatorname{hocolim} (S^{j\overline{V}} \wedge F(\mathrm{E}G_+, X^{\otimes n}))^G \\
&\simeq (S^{\infty\overline{V}} \wedge F(\mathrm{E}G_+, X^{\otimes n}))^G.
\end{aligned}$$

The last identification used that genuine fixed points commute with all homotopy limits and colimits. Finally, observe that $S^{\infty\overline{V}}$ is a model for $\widetilde{\mathrm{E}\mathcal{T}}$. □

The same argument computes the Goodwillie *co*derivative of Sym^G:

Proposition 2.2.2 *The Goodwillie coderivative of* Sym^G *is* $((-)^{\otimes n})^{t^{\mathrm{op}}\mathcal{T}} = \Sigma^{-1} T_G$.

This last observation motivates the excellent account of stable power operations given by Glasman and Lawson [6].

2.3 The Tate diagonal

Recall the following result of Nikolaus [13, Corollary 6.9]:

Proposition 2.3.1 *The forgetful functor* $U \colon \mathsf{Mod}_k \to \mathsf{Sp}$ *is initial amongst exact, lax symmetric monoidal functors to spectra.*

In the previous section we identified T_G as a Goodwillie derivative. In particular, T_G is exact. It also has a lax symmetric monoidal structure, being a composite of lax symmetric monoidal functors. So we get the following:

Corollary 2.3.2 *There is an essentially unique natural transformation of lax symmetric monoidal functors $U \to UT_G$.*

We refer to this map $\Delta_G \colon M \to T_G(M)$ as the *Tate diagonal.*

Remark 2.3.3 This is not the same as the Tate diagonal in [14] unless $k = S^0$, since we use the tensor product in Mod_k. Of course there is an evident relationship between the two: the Tate diagonal above is just the composite

$$M \to (M^{\wedge n})^{t\mathcal{T}} \to (M^{\otimes n})^{t\mathcal{T}}.$$

Warning 2.3.4 The Tate diagonal is *not* k–linear.

3 Power operations

We now fix a field k of characteristic 2 and let Mod_k be the ∞–category of k–module (spectra). In Section 3.1 we serve up power operations three ways, and then verify they agree in Section 3.5. In between we verify the first properties of power operations up to the Cartan formula. We emphasize that this section does not show off the utility of the approach via the Tate-valued Frobenius, but we have included the proofs since they are still pleasant.

3.1 Three definitions of operations

First we specify the objects on which power operations will act.

Definition 3.1.1 We say that $A \in \mathsf{Mod}_k$ is *equipped with a symmetric multiplication* if we have specified a map $\mathrm{Sym}^2(A) \to A$ of k–modules. Equivalently, if we have specified a map $A^{\otimes 2} \to A$ in $\mathsf{Mod}_k^{h\Sigma_2}$.

Remark 3.1.2 A k–module with a symmetric multiplication is the same as an object of $\mathscr{C}(2,\infty)$ in the notation of [12].

To give the classical construction of power operations we'll need a computation.

Lemma 3.1.3 *For any integer n there is a canonical equivalence*

$$\mathrm{Sym}^2(\Sigma^n k) \simeq \Sigma^{2n} k_{h\Sigma_2}.$$

Proof The object $(\Sigma^n k)^{\otimes 2} = \Sigma^{nV} k$ in $\mathsf{Mod}_k^{h\Sigma_2}$ corresponds to a map $\mathrm{B}\Sigma_2 \to \mathsf{Mod}_k^{h\Sigma_2}$ which is determined by a map

$$\Sigma_2 \to \mathsf{End}_k(\Sigma^{2n} k) \simeq \mathsf{End}_k(k,k) \simeq k$$

of $\mathbb{E}_1$–monoids. The map factors through the units $k^\times$, but k has characteristic 2 and hence no nontrivial square roots of unity. So the action is trivial and the result follows. □

The following construction is the current standard definition of power operations.

Construction 3.1.4 (hands-on power operations) Let A be a k–module equipped with a symmetric multiplication. Given $x \in \pi_n A$ and $i \geq n$, define $Q^i(x) \in \pi_{n+i} A$ as the composite

$$S^{n+i} \xrightarrow{\Sigma^{2n} e_{i-n}} \Sigma^{2n} k_{h\Sigma_2} \simeq \mathrm{Sym}^2(\Sigma^n k) \xrightarrow{\mathrm{Sym}^2(x)} \mathrm{Sym}^2(A) \to A.$$

This has the benefit of generalizing well to power operations for other cohomology theories, but in the case of mod 2 cohomology there is a more uniform option. The author learned this next approach from [10, Section 2.2] and has not found an earlier reference, but a more recent and detailed account can be found in [6].

First we need a preliminary observation. Let $T_2' \colon \mathsf{Mod}_k \to \mathsf{Mod}_k$ denote the left Kan extension of the restriction of T_2 to the full subcategory of compact objects. This endomorphism commutes with all colimits and so — see [11, 7.1.2.4] — there is a bimodule B and an equivalence $T_2'(M) \simeq B \otimes M$. By evaluating on $M = k$ we deduce that $B = k^{t\Sigma_2}$ as a left k–module. Notice, by construction, we have a natural map $B \otimes M \to T_2(M)$.

Construction 3.1.5 (stable power operations) Let A be a k–module equipped with a symmetric multiplication. The element $t^{-i-1} \in \pi_{i+1} k^{t\Sigma_2}$ extends to a *right* module map $\Sigma^i k \to \Sigma^{-1} B$. We now define $Q^i \colon \Sigma^i A \to A$ as the (non–k–linear!) composite

$$\Sigma^i A = \Sigma^i k \otimes A \to \Sigma^{-1} B \otimes A \to \Sigma^{-1} T_2(A) \to \mathrm{Sym}^2(A) \to A.$$

This construction emphasizes the role of $\Sigma^{-1} k^{t\Sigma_2}$ as acting on A, but we can also record this information in a kind of coaction. For that we first need a computation.

Lemma 3.1.6 *For any k–module M equipped with the trivial Σ_2–action, there is a canonical equivalence of $\pi_* k^{t\Sigma_2}$–modules*

$$\pi_* M^{t\Sigma_2} \simeq M_*((t)).$$

Proof It suffices to prove $\pi_* M^{h\Sigma_2} \simeq M_*[\![t]\!]$. From the skeletal filtration on $\mathrm{B}\Sigma_2$,

$$M^{h\Sigma_2} \simeq \operatorname{holim} F(\mathrm{sk}_j \mathrm{B}\Sigma_2, k) \otimes M$$

and $\pi_* F(\mathrm{sk}_j \mathrm{B}\Sigma_2, k) \otimes M = M_*[t]/t^{j+1}$. The transition maps are surjective so there is no $\lim^1$ term in the Milnor exact sequence and the result follows. □

Construction 3.1.7 (Tate-valued Frobenius) Let A be a k–module equipped with a symmetric multiplication. Define the *total power operation* as the composite

$$Q(t)\colon A \xrightarrow{\Delta_2} T_2(A) = (A^{\otimes 2})^{t\Sigma_2} \to A^{t\Sigma_2}.$$

We then define $Q^i : A \to \Sigma^{-i} A$ as the composite

$$A \to A^{t\Sigma_2} \xrightarrow{t^{-i-1}} \Sigma^{-i-1} A^{t\Sigma_2} \to \Sigma^{-i} A_{h\Sigma_2} \to \Sigma^{-i} A.$$

In Section 3.5 we will verify that the two definitions of the endomorphism $Q^i : \Sigma^i A \to A$ coincide and that each induce the operation $Q^i : \pi_n A \to \pi_{n+i} A$ on homotopy. For now we will assume this compatibility.

Remark 3.1.8 (naturality of Frobenius) The Tate-valued Frobenius can be defined for any *spectrum* equipped with a symmetric multiplication, as the composite

$$A \to (A \wedge A)^{t\Sigma_2} \to A^{t\Sigma_2}.$$

Since the k–module Tate diagonal factors through the spectrum Tate diagonal, we learn that the Tate-valued Frobenius only depends on the underlying $\mathbb{E}_\infty$–ring. In particular, the Tate-valued Frobenius is natural for maps $A \to B$ of $\mathbb{E}_\infty$–rings, independent of any compatibility with k–module structures.

3.2 First properties

The first properties follow easily from the Tate-valued Frobenius description, with the exception of the squaring property, which is most readily seen through the classical definition.

Proposition 3.2.1 *The operations Q^i satisfy the following properties*:

(i) ***Additivity*** $Q^i(x+y) = Q^i(x) + Q^i(y)$.

(ii) ***Suspension*** *$\Omega Q^i(x) = Q^i(\Omega x)$.*

(iii) ***Squaring*** *$Q^{|x|}(x) = x^2$.*

(iv) ***Instability*** *$Q^i(x) = 0$ if $i < |x|$.*

(v) ***Action on cohomology*** *If $A = F(X,k)$, where X is a pointed space, then $Q^i(x) = 0$ for $i > 0$ and $Q^0(x) = x$.*

Proof (i) **Additivity** Since Q^i is induced by a map of spectra, it is automatically additive.

(ii) **Suspension** The Tate diagonal is a natural transformation of exact functors, so $\Delta_{2,\Omega A} \simeq \Omega\Delta_2$. Exactness of T_2 then ensures that $\Omega T_2(A) \to T_2(\Omega A)$ is an equivalence, and composing with the multiplication on ΩA identifies $\Omega Q(t)$ with the total power operation for ΩA, which was to be shown.

(iii) **Squaring** Using Construction 3.1.4, observe that $Q^{|x|}(x)$ is the image of the bottom class in $\mathrm{Sym}^2(\Sigma^n k)$, which is the left vertical arrow in the diagram

$$\begin{tikzcd}
\Sigma^n k \otimes \Sigma^n k \arrow[r, "x\otimes x"] \arrow[d] & M \otimes M \arrow[d] \\
\mathrm{Sym}^2(\Sigma^n k) \arrow[r] & \mathrm{Sym}^2(M)
\end{tikzcd}$$

The result follows by chasing the diagram clockwise.

(iv) **Instability** By (ii) we may replace A by $\Omega^{|x|-i} A$ and thereby reduce to the case that $A = \Omega B$ and $i = |x|$. By (iii), $Q^i x = x^2$, but the multiplication on ΩB is always trivial, since $S^1 \to S^1 \wedge S^1$ is null.

(v) **Action on cohomology** By naturality we may replace X with $K(k,n)$ and x with the fundamental class. The vanishing now follows for degree reasons. To check that $Q^0(x) = x$ we may reduce, by naturality, to the case $X = S^n$ and then, by stability, to $X = S^0$. The result now follows from the equivalence $F(S^0, k) = k$. □

3.3 Cartan formula

If A and A' are equipped with symmetric multiplications then $A \otimes A'$ inherits a canonical symmetric multiplication as well. In this case we have an external Cartan formula:

Proposition 3.3.1 (Cartan formula)

$$Q(t)(x \otimes y) = Q(t)(x) \otimes Q(t)(y) \in (A \otimes A')((t)).$$

Proof The formula is equivalent to commutativity of the square

$$\begin{tikzcd} A\otimes A' \arrow[r] \arrow[d, equal] & T_2(A)\otimes T_2(A') \arrow[r] \arrow[d] & A^{t\Sigma_2}\otimes (A')^{t\Sigma_2} \arrow[d] \\ A\otimes A' \arrow[r] & T_2(A\otimes A') \arrow[r] & (A\otimes A')^{t\Sigma_2} \end{tikzcd}$$

The left square commutes because the Tate diagonal is a transformation of lax symmetric monoidal functors. The right-hand square commutes by naturality of the lax structure map

$$(-)^{t\Sigma_2}\otimes(-)^{t\Sigma_2}\to(-\otimes-)^{t\Sigma_2}$$

applied to $(A\otimes A')^{\otimes 2}\simeq A^{\otimes 2}\otimes A'^{\otimes 2}\to A\otimes A'$. □

Corollary 3.3.2 $$Q^n(x\otimes y)=\sum_{i+j=n} Q^i(x)\otimes Q^j(y).$$

As a corollary of the proof, we see:

Corollary 3.3.3 *If $A\otimes A\to A$ is a map of objects equipped with symmetric multiplications, then $Q(t)\colon A\to A^{t\Sigma_2}$ is also a map of objects equipped with symmetric multiplications.*

3.4 An example

We revisit our example $k^{t\Sigma_2}$, but to avoid confusion we change the name of the generator: $k_*^{t\Sigma_2}=k((s))$. From the equivalence $k^{h\Sigma_2}=F(\mathrm{B}\Sigma_{2+},k)$ together with properties (iii), (iv), and (v), we see that

$$Q(t)(s)=s+s^2t^{-1}.$$

The Cartan formula now determines the behavior of $Q(t)$ in general:

$$Q(t)\sum_i a_is^i=\sum_i a_i(s+s^2t^{-1})^i.$$

3.5 Comparing the definitions

Let B denote the bimodule from Construction 3.1.5, which is equivalent to $k^{t\Sigma_2}$ as a left k–module. Let $k\to B$ extend $1\in\pi_0k^{t\Sigma_2}$ as a right module map.

Lemma 3.5.1 *The composite*

$$A\to B\otimes A\to T_2(A)$$

above is equivalent to the Tate diagonal Δ_2.

Proof Indeed, first observe that by the universal property of spectra [11, 1.4.2.23],

$$\Omega^\infty\colon \mathsf{Fun}^{\mathrm{ex}}(\mathsf{Mod}_k, \mathsf{Sp}) \xrightarrow{\simeq} \mathsf{Fun}^{\mathrm{lex}}(\mathsf{Mod}_k, \mathsf{Spaces}).$$

Now let $U\colon \mathsf{Mod}_k \to \mathsf{Sp}$ be the forgetful functor. Then $\Omega^\infty U$ is corepresented by k, so the Yoneda lemma applied to the previous observation implies that

$$\mathrm{Map}_{\mathsf{Fun}^{\mathrm{ex}}(\mathsf{Mod}_k,\mathsf{Sp})}(U, UT_2) \simeq \Omega^\infty k^{t\Sigma_2}.$$

Since the Tate diagonal is a transformation of lax symmetric monoidal functors, the transformation $U \to UT_2$ evaluates on k to the unit $k \to k^{t\Sigma_2}$. Combining this with the previous observation we learn that the Tate diagonal is the unique transformation $U \to UT$ which corresponds to the element $1 \in \pi_0 k^{t\Sigma_2}$. $\square$

Thus the map

$$A \to B \otimes A \to T_2(A) \to A^{t\Sigma_2}$$

coincides with the Tate-valued Frobenius. Now observe that the last three terms are left modules over $k^{t\Sigma_2}$, so multiplication by t^{-i-1} and naturality of $(-)^{t\Sigma_2} \to \Sigma(-)_{h\Sigma_2}$ gives a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & B \otimes A \arrow[r] \arrow[d] & T_2(A) \arrow[r] \arrow[d] & A^{t\Sigma_2} \arrow[d] \\
& \Sigma^{-i-1} B \otimes A \arrow[r] & \Sigma^{-i-1} T_2(A) \arrow[r] \arrow[d] & \Sigma^{-i-1} A^{t\Sigma_2} \arrow[d] \\
& & \Sigma^{-i}\,\mathrm{Sym}^2(A) \arrow[r] & \Sigma^{-i} A
\end{tikzcd}$$

Chasing the diagram around clockwise gives the definition of Q^i in terms of the total power operation. Chasing the diagram around counterclockwise gives the definition of Q^i in terms of Construction 3.1.5. So these two constructions agree.

Now we compare with the classical construction. The equivalence $(\Sigma^n k)^{\otimes 2} \simeq \Sigma^{2n} k$ in $\mathsf{Mod}_k^{h\Sigma_2}$ gives a commutative diagram

$$\begin{tikzcd}
\Sigma^{-1} T_2(\Sigma^n k) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Sym}^2(\Sigma^n k) \arrow[d, "\simeq"] \\
\Sigma^{2n-1} k^{t\Sigma_2} \arrow[r] & \Sigma^{2n} k_{h\Sigma_2}
\end{tikzcd}$$

Since the bottom horizontal map is surjective on homotopy, so is the top, and we see that $\Sigma^{2n} e_{i-n}$ on the lower right corresponds to $t^{-i-1} y$ on the top left, where $y \in \pi_n \Sigma^n k$ is the generator. Now let $x\colon S^n \to A$ be a class and form the diagram

$$\begin{tikzcd}
S^{i+n} \arrow[r, "t^{-i-1}y"] \arrow[dr, "t^{-i-1}x"'] & \Sigma^{-1} T_2(\Sigma^n k) \arrow[r] & \mathrm{Sym}^2(\Sigma^n k) \arrow[d] \\
 & \Sigma^{-1} T_2(A) \arrow[r] & \mathrm{Sym}^2(A)
\end{tikzcd}$$

Traversing clockwise gives $Q^i(x)$ as in Construction 3.1.4 and traversing counterclockwise gives the image of x under Q^i as in Construction 3.1.5, and this completes the argument.

4 Adem relations

The Adem relations arise from relating the iterated total power operation to a total fourth power operation. In Section 4.1 we first explain how to lift the iterated total power operation to an intermediate Tate spectrum. In Section 4.2 we show that the existence of extra symmetry on iterated multiplication allows us to factor further through a total fourth power operation. This implies a version of the Adem relations as an identity between formal Laurent series in two variables, and in Section 4.3 we essentially perform the maneuver from [5] to recover the usual Adem relations.

For notational ease we adopt the following convention in this section:

Convention 4.0.1 If $G \subseteq \Sigma_n$ is a subgroup, and $\mathcal{T}$ denotes the family of nontransitive subgroups of G, then we denote $(-)^{t\mathcal{T}}$ by $(-)^{\tau G}$.

4.1 Iterated power operations

Suppose A is a k–module equipped with a symmetric multiplication. Iterating the multiplication gives a map

$$A^{\otimes 4} \to A$$

which need not admit an Σ_4–equivariant structure. However, it can be made $\Sigma_2 \wr \Sigma_2$–equivariant, so we may define a map

$$A \to T_{\Sigma_2 \wr \Sigma_2}(A) \to A^{\tau \Sigma_2 \wr \Sigma_2}.$$

Our first goal is to show that this lifts the iterated total power operation.

Proposition 4.1.1 *Let A be a k–module equipped with a symmetric multiplication. Then there is a canonical commutative diagram*

$$\begin{tikzcd} & A^{\tau\Sigma_2\wr\Sigma_2} \arrow[d] \\ A \arrow[ur] \arrow[r, "Q(t)\circ Q(s)"'] & (A^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

Proof First consider the diagram

$$\begin{tikzcd} T_2(A) \arrow[r] \arrow[d] & T_2(T_2(A)) \arrow[r] \arrow[d] & ((A^{\otimes 4})^{t\Sigma_2})^{t\Sigma_2} \arrow[d] \\ A^{t\Sigma_2} \arrow[r] & T_2(A^{t\Sigma_2}) \arrow[r] & (A^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

The first square commutes by naturality of the Tate diagonal applied to the map $T_2(A)\to A$. The second square commutes by naturality of the lax structure map for $(-)^{t\Sigma_2}$.

It follows that $Q(t)\circ Q(s)$ can be written as the composite

$$A\to T_2(T_2(A))\to((A^{\otimes 4})^{t\Sigma_2})^{t\Sigma_2}\to(A^{t\Sigma_2})^{t\Sigma_2}.$$

Now consider both $(-)^{\tau\Sigma_2\wr\Sigma_2}$ and $((-)^{t\Sigma_2})^{t\Sigma_2}$ as exact functors $\mathsf{Mod}_k^{h\Sigma_4}\to\mathsf{Mod}_k$. We have a natural transformation

$$(-)^{h\Sigma_2\wr\Sigma_2}\to(-)^{h\Sigma_2\times\Sigma_2}=((-)^{h\Sigma_2})^{h\Sigma_2}\to((-)^{t\Sigma_2})^{t\Sigma_2},$$

where the first map is induced by the inclusion

$$\Sigma_2\times\Sigma_2\to(\Sigma_2\times\Sigma_2)\rtimes\Sigma_2=\Sigma_2\wr\Sigma_2$$

given by the diagonal on the first factor. By the universal property of the Tate construction (Section 1.1), we get a natural transformation $(-)^{\tau\Sigma_2\wr\Sigma_2}\to((-)^{t\Sigma_2})^{t\Sigma_2}$. In particular, applied to the multiplication map $A^{\otimes 4}\to A$, we get a commutative diagram

$$\begin{tikzcd} T_{\Sigma_2\wr\Sigma_2}(A) \arrow[r] \arrow[d] & A^{\tau\Sigma_2\wr\Sigma_2} \arrow[d] \\ ((A^{\otimes 4})^{t\Sigma_2})^{t\Sigma_2} \arrow[r] & (A^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

Finally, the composite

$$\Gamma^{\Sigma_2\wr\Sigma_2}\to\Gamma^{\Sigma_2\times\Sigma_2}\simeq\Gamma^2\circ\Gamma^2\to T_2\circ T_2$$

yields a natural transformation $T_{\Sigma_2 \wr \Sigma_2} \to T_2 \circ T_2$ from the universal property of $T_{\Sigma_2 \wr \Sigma_2}$ as the Goodwillie derivative of $\Gamma^{\Sigma_2 \wr \Sigma_2}$. The diagram

$$\begin{tikzcd} T_{\Sigma_2 \wr \Sigma_2} \arrow[d] \arrow[dr] & \\ T_2 \circ T_2 \arrow[r] & (((-)^{\otimes 4})^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

commutes by the same universal property, and the result follows. □

4.2 Adem objects

For the Adem relations to hold we need the symmetric multiplication to satisfy an extra condition.

Definition 4.2.1 We say that a k–module A equipped with a symmetric multiplication is an *Adem object* if there exists a map $\mathrm{Sym}^4(A) \to A$ such that the diagram

$$\begin{tikzcd} \mathrm{Sym}^2(\mathrm{Sym}^2(A)) \arrow[r] \arrow[d] & \mathrm{Sym}^2(A) \arrow[d] \\ \mathrm{Sym}^4(A) \arrow[r] & A \end{tikzcd}$$

commutes up to homotopy.

Proposition 4.2.2 *If A is an Adem object, then we have a commutative diagram*

$$\begin{tikzcd} & A^{\tau\Sigma_4} \arrow[d] \\ & A^{\tau\Sigma_2 \wr \Sigma_2} \arrow[d] \\ A \arrow[uur] \arrow[ur] \arrow[r, "Q(t)\circ Q(s)"'] & (A^{t\Sigma_2})^{t\Sigma_2} \end{tikzcd}$$

Proof By Proposition 4.1.1, the bottom triangle commutes. Factor the top triangle as

$$\begin{tikzcd} & T_4(A) \arrow[r] \arrow[d] & A^{\tau\Sigma_4} \arrow[d] \\ A \arrow[ur] \arrow[r] & T_{\Sigma_2 \wr \Sigma_2}(A) \arrow[r] & A^{\tau\Sigma_2 \wr \Sigma_2} \end{tikzcd}$$

The triangle commutes because each arrow is a transformation of exact, lax symmetric monoidal functors, and $U:\mathsf{Mod}_k \to \mathsf{Sp}$ is initial amongst such functors (Proposition 2.3.1). The square commutes by the definition of an Adem object, ie the structure of a Σ_4–equivariant map $A^{\otimes 4} \to A$ refining the given $(\Sigma_2 \wr \Sigma_2)$–equivariant structure. □

Theorem 4.2.3 (Adem relations) *If A is an Adem object and $x \in \pi_* A$ is an element, then $Q(t)(Q(s)x)$ is symmetric in the variables s and t. Explicitly,*

$$\sum_{i,j} (Q^i Q^j x)(s+s^2t^{-1})^j t^i = \sum_{i,j} (Q^i Q^j x)(t+t^2s^{-1})^j s^i.$$

Proof By Proposition 4.2.2, the iterated total power operation factors through $A^{\tau\Sigma_4}$ and the operation which swaps s and t arises from an inner automorphism of Σ_4 which thus acts trivially on the Tate construction, whence the claim. The explicit formula follows from the basic properties of power operations, the Cartan formula, and the computation in Section 3.4. □

4.3 Residues and relations

Now we recall how to recover the individual Adem relations using the power series identity above.

Proposition 4.3.1 *Let A be an Adem object and $x \in A_*$ a homotopy class. Then*

$$Q^i Q^j(x) = \sum_{\ell} \binom{\ell-j-1}{2\ell-i} Q^{i+j-\ell} Q^{\ell}(x).$$

Proof In the previous section we showed

$$\sum_j Q(t)(Q^j x)(s+s^2t^{-1})^j = \sum_{k,j} (Q^k Q^j x)(t+t^2s^{-1})^j s^k.$$

Let $u = s+s^2t^{-1}$ and observe that this is composition invertible as a power series in s with coefficients in $k((t))$. Now,

$$Q(t)(Q^j x) = \sum_i (Q^i Q^j x)t^i$$

is the coefficient of u^j on the left-hand side, so we would like to compute the coefficient of u^j on the right-hand side. It will be convenient to reindex the right-hand side, for fixed j, as

$$\sum_{i,\ell} (Q^{i+j-\ell} Q^{\ell} x)(t+t^2s^{-1})^{\ell} s^{i+j-\ell}.$$

Observe that $du = ds$ since $2 = 0$ in k, and hence

$$\operatorname{res}(u^{-j-1}(Q^{i+j-\ell}Q^{\ell}x)(t+t^2s^{-1})^{\ell}s^{i+j-\ell}du) \\ = \operatorname{res}(u^{-j-1}(Q^{i+j-\ell}Q^{\ell}x)(t+t^2s^{-1})^{\ell}s^{i+j-\ell}ds).$$

Fixing i and ℓ and writing $u = st^{-1}(t+s)$ and $(t+t^2s^{-1}) = s^{-1}t(t+s)$, we have

$$u^{-j-1}(t+t^2s^{-1})^{\ell}s^{i+j-\ell} = t^{\ell+j+1}s^{i-2\ell-1}(t+s)^{\ell-j-1}.$$

The coefficient of s^{-1} in the previous expression is then

$$\binom{\ell-j-1}{2\ell-i}t^i$$

and the result follows. □

5 Relationship to the Steenrod algebra

In this section we restrict to the case $k = \mathbb{F}_2$ for ease of exposition. In Section 5.1 we recall the Steenrod coaction on the Tate spectrum, then in Section 5.2 we use this to give a succinct proof of the Nishida relations. Finally, in Section 5.3 we show how this determines the action of $Q(t)$ on the dual Steenrod algebra, following an idea of Bisson and Joyal.

5.1 Coaction on the Tate spectrum

The map $k = S^0 \wedge k \to k \wedge k$ gives rise to a map $k^{t\Sigma_2} \to (k \wedge k)^{t\Sigma_2}$ if we equip the source and target with trivial Σ_2–action.

This induces a completed coaction

$$\psi_R \colon k((t)) \to \mathcal{A}_*((t)).$$

More generally, for any spectrum X, the composite

$$k \wedge X \simeq X \wedge k = S^0 \wedge X \wedge k \to k \wedge X \wedge k$$

gives a completed coaction $\psi_R \colon H_*(X)((t)) \to (H_*(X) \otimes \mathcal{A}_*)((t))$. Now recall that Milnor defined generators[3] of the dual Steenrod algebra by the identity

$$\psi_R(t) = \sum \zeta_i t^{2^i}.$$

[3]We are following Milnor's convention and not the more recent trend of using ζ_i to denote the *conjugates* of Milnor's generators.

5.2 Nishida relations

The easier version of the Nishida relations in this context is in terms of the coaction.

Theorem 5.2.1 (Bisson–Joyal, Baker) *Let X be a spectrum equipped with an equivariant symmetric multiplication $X^{\wedge 2}_{h\Sigma_2} \to X$. Then*

$$\sum_i \psi_R(Q^i x)t^i = Q(\bar{\zeta}(t))\psi_R(x) \in (\mathrm{H}_* X \otimes \mathcal{A}_*)((t)).$$

Proof The right coaction $k \wedge X \to (k \wedge X) \otimes_k (k \wedge k)$ is a map of *spectra* equipped with symmetric multiplications (though it is not a map of k–modules equipped with symmetric multiplications). By Remark 3.1.8 this yields a commutative diagram

$$\begin{tikzcd} k \wedge X \arrow[r, "\psi_R"] \arrow[d] & (k \wedge X) \otimes_k (k \wedge k) \arrow[d] \\ (k \wedge X)^{t\Sigma_2} \arrow[r, "(\psi_R)^{t\Sigma_2}"'] & ((k \wedge X) \otimes_k (k \wedge k))^{t\Sigma_2} \end{tikzcd}$$

The bottom map is the completed coaction defined in the previous subsection. Thus,

$$\psi_R(Q(t)x) = Q(t)(\psi_R(x)).$$

Since ψ_R is a ring map, and $\psi_R(t) = \zeta(t)$, this becomes

$$\sum \psi_R(Q^i x)\zeta(t)^i = Q(t)(\psi_R(x)).$$

Now substitute the conjugate series $\bar{\zeta}(t)$ for t and use the relation $\zeta(\bar{\zeta}(t)) = t$. □

5.3 Action on the dual Steenrod algebra

The following description of the action of the Q^i on $\mathcal{A}_*$ is essentially that of Bisson and Joyal [3, Section 1, Proposition 6].

Theorem 5.3.1 (Bisson–Joyal) *The total power operation on the Milnor generators ζ_i is determined implicitly by the identity*

(1) $$\zeta(s) + \zeta(s)^2\zeta(t)^{-1} = \sum_i (Q(t)\zeta_i)(s^{2^i} + s^{2^{i+1}}t^{-2^i}),$$

(2) $$t^{2^n} Q(t)\zeta_n = \Big(\sum_{i \geq n+1} \zeta_i t^{2^i}\Big) + \zeta(t)^{-1}\Big(\sum_{i \geq n} \zeta_i^2 t^{2^{i+1}}\Big).$$

Proof Write $\pi_* k^{h\Sigma_2} = k[\![s]\!]$. Then

$$\psi_R(Q(t)s) = Q(t)\psi_R(s).$$

Now use the identities $Q(t)s = s + s^2 t^{-1}$ and $\psi_R(s) = \zeta(s)$. Comparing coefficients for s^{2^n} gives a recursion for $Q(t)\zeta_n$ starting with $Q(t)\zeta_0 = Q(t)1 = 1$ and (2) solves the recursion. $\square$

It is not difficult to extract the earlier results of Steinberger [4, Section III.2].

Corollary 5.3.2 (Steinberger) *For $i \geq 2$, $Q^{2^i-2}\zeta_1 = \bar{\zeta}_i$.*

Proof From Theorem 5.3.1(2) above in the case $n = 1$,

$$Q(t)\zeta_1 = t^{-1} + \zeta_1 + \zeta(t)^{-1}.$$

So, for $i \geq 2$, change of variables and a quick computation gives

$$Q^{2^i-2}\zeta_1 = \operatorname{res}(t^{-2^i+1}\zeta(t)^{-1}dt) = \operatorname{res}(\bar{\zeta}(u)^{-2^i+1}u^{-1}du) = \bar{\zeta}_i. \qquad \square$$

Corollary 5.3.3 (Steinberger) *We have $Q^{2^i}\zeta_i = \zeta_{i+1} + \zeta_i^2\zeta_1$ and $Q^{2^i}\bar{\zeta}_i = \bar{\zeta}_{i+1}$.*

Proof The case $i = 0$ is evident, so assume $i \geq 1$. The coefficient of $t^0 s^{2^{i+1}}$ on the right-hand side of Theorem 5.3.1(1) is visibly $Q^{2^i}\zeta_i + Q^0(\zeta_i) = Q^{2^i}\zeta_i$. The constant term of $\zeta(t)^{-1}$ is ζ_1, so the coefficient of $t^0 s^{2^{i+1}}$ on the left-hand side is $\zeta_{i+1} + \zeta_i^2\zeta_1$. The other identity follows from this one by induction and the defining relation for conjugation. $\square$

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The Devinatz–Hopkins theorem via algebraic geometry

ROK GREGORIC

We show how a continuous action of the Morava stabilizer group $\mathbb{G}_n$ on the Lubin–Tate spectrum E_n, satisfying the conclusion $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S$ of the Devinatz–Hopkins theorem, may be obtained by monodromy on the stack of oriented deformations of formal groups in the context of formal spectral algebraic geometry.

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A classical and computationally invaluable result in chromatic homotopy theory, the Morava change-of-rings theorem — see for instance Devinatz [5] — identifies the second page of the $K(n)$–local Adams spectral sequence for the Lubin–Tate spectrum E_n as continuous group cohomology,

$$E_2^{s,t} \simeq \mathrm{H}^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_t(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}S).$$

A conceptual spectrum-level explanation for this isomorphism is given by the Devinatz–Hopkins theorem [6]. It asserts the existence of a (suitably interpreted) continuous action of the Morava stabilizer group $\mathbb{G}_n$ on the Lubin–Tate spectrum E_n, such that its continuous homotopy fixed points are

$$E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S. \tag{1}$$

The proof of the equivalence (1) has by now become largely standard, using nilpotence technology applied to the $K(n)$–local Amitsur complex of E_n, and ultimately stemming from the key observation of Hopkins and others that the Adams spectral sequence of E_n possesses a horizontal vanishing line. The somewhat less straightforward part is instead identifying said Amitsur complex with the simplicial bar resolution of a suitably interpreted continuous action of $\mathbb{G}_n$ on E_n. That was accomplished in a somewhat ad hoc manner in [6], and in various contexts of continuous group actions of spectra such as Behrens and Davis [2] and Quick [21]; though these approaches ostensibly amount to enriching the construction from [6]. A formalization using the condensed set technology of [23] to tackle continuity has also been announced by Clausen and Scholze.

In contrast, we propose to side-step the issue of continuous actions altogether. Instead, we exhibit the action in an appropriate context of formal spectral algebraic geometry. Our results may be summarized as follows.

Theorem *The Morava stabilizer group $\mathbb{G}_n$ admits a canonical action on the formal spectral stack* $\mathrm{Spf}(E_n)$. *For continuous homotopy fixed points of this action defined as* $E_n^{h\mathbb{G}_n} := \mathcal{O}(\mathrm{Spf}(E_n)/\mathbb{G}_n)$, *there is a canonical equivalence* $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S$. *Furthermore, the three resulting spectral sequences coincide*:

(1) *The descent spectral sequence for the structure sheaf on* $\mathrm{Spf}(E_n)/\mathbb{G}_n$,

$$E_2^{s,t} = \mathrm{H}^s(\mathrm{Spf}(E_n)/\mathbb{G}_n; \pi_t(\mathcal{O})) \Rightarrow \pi_{t-s}(L_{K(n)}S).$$

(2) *The homotopy fixed point spectral sequence for the* $\mathbb{G}_n$*–action on* E_n,

$$E_2^{s,t} = \mathrm{H}^s(\mathrm{B}\mathbb{G}_n; \pi_t(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}S).$$

(3) *The* $K(n)$*–local Adams spectral sequence for* E_n,

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*(L_{K(n)}(E_n\otimes E_n))}(\pi_*(E_n), \pi_*(E_n)) \Rightarrow \pi_{t-s}(L_{K(n)}S).$$

Our approach is based on a theorem of Lurie [17, Theorem 5.1.5], identifying $\mathrm{Spf}(E_n)$ with the moduli stack of oriented deformation of a height n formal group. We show that the Morava stabilizer group action arises as an instance of monodromy actions on de Rham spaces. To establish the above version of the Devinatz–Hopkins theorem in our setting, we employ similar arguments to the analogous considerations in classical formal algebraic geometry from Goerss [8, Chapter 7].

The computational underpinning of the proof (somewhat obscured in our account) is the fundamental observation from [6] that the $K(n)$–local Adams spectral sequence for E_n possesses a horizontal vanishing line. Ours is in that sense analogous to all of the currently known approaches to the Devinatz–Hopkins theorem, including, to the best of the author's understanding, the forthcoming work of Clausen and Scholze. The latter construct the continuous (or in their setting, more precisely, condensed) Morava stabilizer group action similarly to us, in that they employ results[1] from [17].

[1]Though unlike our account, where the algebrogeometric aspect of the results in [17] are center-stage, the approach of Clausen and Scholze only relies on the more flexible functoriality of Lubin–Tate theory (in particular, that its base can be taken to be an arbitrary perfect $\boldsymbol{F}_p$–algebra as base, as opposed to only a perfect field) afforded by Lurie's construction, as compared to the traditional one by Goerss, Hopkins and Miller.

In particular, we wish to make it clear that the majority of our proof of the Devinatz–Hopkins theorem follows the same reasoning and insights as the original account in [6].

In general, many of the results in this paper follow without much difficulty from the existing literature. We nonetheless believe that a streamlined conceptual proof of the Devinatz–Hopkins theorem, which this paper provides, is worthwhile. Other than in the presentation, our primary contribution is a novel way to obtain the Morava stabilizer group action by way of formal spectral algebraic geometry, building on Lurie's work in [15; 17]. Related applications of those results to topics in chromatic homotopy theory, primarily concerning Gross–Hopkins duality, are considered by Devalapurkar [4].

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1 Background on formal spectral algebraic geometry

We begin by summarizing some notions and results from [15; 17] which are key for the purpose of this note.

1.1 Adic $\mathbb{E}_\infty$–rings and formal SAG

From the functor of points perspective, formal spectral algebraic geometry, in the form relevant to us and in [17] (but slightly differently from [15, Definition 8.1.1.5], where a connectivity assumption is imposed throughout), concerns functors $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$.

Here $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ denotes the ∞–category of *complete adic* $\mathbb{E}_\infty$*–rings* in the sense of [17, Definition 0.0.11]. That is, an object of $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ consists of an $\mathbb{E}_\infty$–ring A, together with a topology on $\pi_0(A)$ which admits a finitely generated ideal of definition $I \subseteq \pi_0(A)$, such that the topology on $\pi_0(A)$ is equivalent to the I–adic topology, and finally such that the $\mathbb{E}_\infty$–ring A is I–complete, in the sense of [14, Definition 7.2.3.22].

Since the notion of completeness for $\mathbb{E}_\infty$–rings and modules over them features prominently in this note, let us recall (an equivalent rephrasing of) the definition:

Definition 1.1 [15, Proposition 7.3.2.1, Corollary 7.3.3.3] Let A be an adic $\mathbb{E}_\infty$–ring with an ideal of definition $I \subseteq \pi_0(A)$. Then an A–module M is *I–complete* if for every element $a \in I$, the canonical map

$$M \to \varprojlim_n M/a^n$$

is an equivalence of A–modules, where $M/a^n = \mathrm{cofib}(M \xrightarrow{a^n} M)$. Let $\mathrm{Mod}_A^{\mathrm{cplt}} \subseteq \mathrm{Mod}_A$ denote the full subcategory spanned by I–complete A–modules. The adic $\mathbb{E}_\infty$–ring A is *complete* if it is I–complete as a module over itself.

Given a complete adic $\mathbb{E}_\infty$–ring A in the sense discussed above, we define its *formal spectrum* to be the corepresentable functor $\mathrm{Spf}(A)\colon \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$ given by

$$B \mapsto \mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(A,B) := \mathrm{Map}_{\mathrm{CAlg}}(A,B) \times_{\mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(\pi_0(A),\pi_0(B))} \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{CAlg}^\heartsuit}(A,B).$$

Of course, this (Yoneda) embedding $(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}, \mathcal{S})$ is fully faithful, and its codomain is a convenient place to do formal spectral algebraic geometry.

1.2 Formal groups over $\mathbb{E}_\infty$–rings

As an instance of that motto, the theory of formal groups over $\mathbb{E}_\infty$–rings is developed in [17, Chapter 1]. We give a slightly informal account, and refer to [loc. cit.] for a precise and detailed account.

Definition 1.2 A *formal group over an $\mathbb{E}_\infty$–ring A* is an abelian group object in the ∞–category of 1–dimensional fiber-smooth formal spectral A–schemes.

Remark 1.3 There are a number of caveats concerning the above definition:

(1) The notion of an *abelian group object* must be understood in the sense of Section 1.2 of [16]. That is to say, we must equip its Yoneda presheaf with a factorization through the functor $\Omega^\infty\colon \mathrm{Mod}^{\mathrm{cn}}_{\mathbf{Z}} \to \mathcal{S}$, or equivalently, the forgetful functor $\mathcal{T}\mathrm{op}\mathcal{A}\mathrm{b} \to \mathcal{S}$. This is a *strictified* version of the more familiar notion of a grouplike $\mathbb{E}_\infty$–algebra objects, since the Yoneda presheaf is in the latter case asked to factor through $\Omega^\infty\colon \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{S}$, or equivalently, the forgetful functor $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S}) \to \mathcal{S}$.

(2) The requirement of *fiber-smoothness* on a formal A–scheme X is taken in the sense of [15, Definition 11.2.3.1], and roughly amounts to asking for X to be étale-

locally isomorphic to the formal affine space $\widehat{A}^n_A = \mathrm{Spf}(A[\![t_1, \dots, t_n]\!])$ (since we are working in the 1–dimensional case in Definition 1.2, it suffices to take $n = 1$). In particular, this implies that X is a flat over A. This differs from the notion of differential smoothness in the sense of [15, Definition 11.2.2.2], which imposes conditions on the cotangent complex $L_{X/A}$, but is incompatible with flatness unless A is a $\boldsymbol{Q}$–algebra. Since we want ordinary formal groups over commutative rings to be special cases of Definition 1.2, and they are indeed flat, we therefore have no choice but to use fiber-smoothness instead of differential smoothness.

(3) Definition 1.2 is really correct *as stated* when the $\mathbb{E}_\infty$–ring A is connective. For a nonconnective $\mathbb{E}_\infty$–ring A, we should instead define formal groups over A to be formal groups in the above sense over the connective cover $\tau_{\geq 0}(A)$ — see [17, Variant 1.6.2]. However, certain constructions associated to a formal group $\widehat{\boldsymbol{G}}$, for instance the $\mathbb{E}_\infty$–algebra of functions $\mathcal{O}_{\widehat{\boldsymbol{G}}}$ of [17, Notation 1.5.12] and Remark 1.6, depend on whether we are considering it as existing over A or over $\tau_{\geq 0}(A)$. See the thorough treatment in [17, Section 1.2] for precise details.

Example 1.4 The following are the only classes of formal groups that we will be concerned with in this note:

- Over a commutative ring A, viewed as a discrete $\mathbb{E}_\infty$–ring, Definition 1.2 reproduces the usual meaning of (as always, 1–dimensional smooth) formal groups over A.
- Let A be a *complex periodic* $\mathbb{E}_\infty$–ring, ie complex orientable and $\pi_2(A)$ is a locally free $\pi_0(A)$ module of rank 1. Then the *Quillen formal group of A* is
$$\widehat{\boldsymbol{G}}{}^{\mathcal{Q}}_A := \mathrm{Spf}(C^*(\boldsymbol{CP}^\infty; A)),$$
which indeed gives rise to a formal group over A by [17, Section 4.1.3].

Formal groups over A form an ∞–category $\mathcal{M}_{\mathrm{FG}}(A)$, and this construction is functorial in A by base change:

Definition 1.5 Let $f \colon A \to B$ be a map of $\mathbb{E}_\infty$–rings, and $\widehat{\boldsymbol{G}}$ a formal group over A. The pullback of formal spectral schemes along $\mathrm{Spec}(f) \colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives rise to a formal group over B, which we denote by $f^*\widehat{\boldsymbol{G}}$.

There is also another slightly different form of functoriality afforded to formal groups. Sending
$$\widehat{\boldsymbol{G}} \mapsto \widehat{\boldsymbol{G}}{}^0 := \mathrm{Spf}(\pi_0(\mathcal{O}_{\widehat{\boldsymbol{G}}}))$$

gives rise to a functor $\mathcal{M}_{\mathrm{FG}}(A) \to \mathcal{M}_{\mathrm{FG}}(\pi_0(A))$. Informally, this sends a spectral formal group to its underlying ordinary formal group.

Remark 1.6 When the $\mathbb{E}_\infty$–ring A is connective, the preceding construction is a special case of Definition 1.5. Indeed, in that case there exists a map of $\mathbb{E}_\infty$–rings $t\colon A \to \pi_0(A)$, and $\widehat{\boldsymbol{G}}^0 \simeq t^*\widehat{\boldsymbol{G}}$. For a nonconnective $\mathbb{E}_\infty$–ring A on the other hand, the connection between A and $\pi_0(A)$ is only through the span $A \leftarrow \tau_{\geq 0}(A) \to \pi_0(A)$, and so $\widehat{\boldsymbol{G}} \mapsto \widehat{\boldsymbol{G}}^0$ is not merely an instance of base change. This is closely related to the subtleties alluded to in item (3) of Remark 1.3.

1.3 Orientations and deformations of formal groups

The class of formal groups singled out by the following definition is of special importance in relation to chromatic homotopy theory. Here an $\mathbb{E}_\infty$–ring A is called *complex periodic* [15, Definition 4.1.8] if it is both complex orientable and weakly 2–periodic.

Definition 1.7 [17, Proposition 4.3.23] A formal group $\widehat{\boldsymbol{G}}$ over an $\mathbb{E}_\infty$–ring A is *oriented* if and only if A is complex periodic and $\widehat{\boldsymbol{G}} \simeq \widehat{\boldsymbol{G}}{}^{\mathcal{Q}}_A$ is its Quillen formal group. We denote by $\mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}(A) \subseteq \mathcal{M}_{\mathrm{FG}}(A)$ the subspace of oriented formal groups over A.

Remark 1.8 Though the above form is the most practical for our purposes, we would be remiss not to summarize an equivalent but better-motivated approach to defining oriented formal groups [17, Definition 4.3.9]. To any formal group $\widehat{\boldsymbol{G}}$ over an $\mathbb{E}_\infty$–ring A we may by [17, Sections 5.2.1–5.2.3] associate an A–module $\omega_{\widehat{\boldsymbol{G}}}$, its *dualizing line*, and the analogue of the module of invariant differentials on a classical formal group. An *orientation of* $\widehat{\boldsymbol{G}}$ then amounts to an A–linear equivalence $\omega_{\widehat{\boldsymbol{G}}} \simeq \Sigma^{-2}(A)$. This is in spirit a 2–shifted analogue of the various notions of orientation in classical geometric contexts, where it usually means some kind of trivialization of a bundle of volume forms.

The space of *deformations of* $\widehat{\boldsymbol{G}}_0$ *over* A is defined as

$$\mathrm{Def}_{\widehat{\boldsymbol{G}}_0}(A) := \varinjlim_I \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(\kappa, \pi_0(A)/I) \times_{\mathcal{M}_{\mathrm{FG}}(\pi_0(A)/I)} \mathcal{M}_{\mathrm{FG}}(A),$$

with the colimit ranging over all the ideals of definition $I \subseteq \pi_0(A)$. Informally, this consists of a ring homomorphism $f\colon \kappa \to \pi_0(A)/I$, a formal group $\widehat{\boldsymbol{G}}$ over the $\mathbb{E}_\infty$–ring A, and an isomorphism $f^*\widehat{\boldsymbol{G}}_0 \simeq q^*\widehat{\boldsymbol{G}}^0$ of formal groups over $\pi_0(A)/I$, where

$q\colon \pi_0(A) \to \pi_0(A)/I$ is the quotient projection. *Oriented deformations* are defined analogously as

$$\mathrm{Def}_{\widehat{\boldsymbol{G}}_0}(A) := \varinjlim_{I} \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(\kappa, \pi_0(A)/I) \times_{\mathcal{M}_{\mathrm{FG}}(\pi_0(A)/I)} \mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}(A).$$

Both of these construction respect pullback along maps of adic $\mathbb{E}_\infty$–rings, and as such give rise to functors $\mathrm{Def}_{\widehat{\boldsymbol{G}}_0}, \mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} : \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}} \to \mathcal{S}$.

The following theorem of Lurie, a cousin of the Goerss–Hopkins–Miller theorem, may be taken as the definition of Lubin–Tate spectra, and is the bedrock of this note.

Theorem 1.9 [17, Theorem 5.1.5, Remark 6.0.7] *Let $\widehat{\boldsymbol{G}}_0$ be a formal group of finite height over a perfect field κ of characteristic $p > 0$. Let $E(\kappa, \widehat{\boldsymbol{G}})$ be the Lubin–Tate spectrum of $\widehat{\boldsymbol{G}}_0$, viewed as an adic $\mathbb{E}_\infty$–ring with respect to the n^{th} Landweber ideal $\mathfrak{I}_n \subseteq \pi_0(E(\kappa, \widehat{\boldsymbol{G}}_0))$. There is a natural equivalence*

$$\mathrm{Spf}(E(\kappa, \widehat{\boldsymbol{G}}_0)) \simeq \mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$$

in the ∞–category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}, \mathcal{S})$.

Remark 1.10 Lurie formulates his result (which also works over more general perfect base rings than a field) in terms of deformations of p–divisible groups instead of formal groups. This has the advantage of being more general, applying for instance also to étale p–divisible groups, and is crucial in the follow-up paper [18] on Hopkins–Kuhn–Ravenel character theory and transchromatic ambidexterity. Alas, for our purposes, since all the p–divisible groups in sight would be connected, the analogue of Tate's theorem in [17, Section 2.3], allows us to restrict to formal groups instead. Ultimately however, this is nothing more than an aesthetic preference, and this note could well have been written with the functor $\mathcal{M}_{\mathrm{BT}}$ everywhere in place of $\mathcal{M}_{\mathrm{FG}}$.

2 Morava stabilizer group action and fixed points

2.1 Complete Noetherian local $\mathbb{E}_\infty$–rings

For the remainder of this note, κ will be a perfect field of characteristic $p > 0$. We find it convenient to restrict to a smaller subcategory of $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$, consisting roughly of complete Noetherian local $\mathbb{E}_\infty$–rings with residue field κ.

Definition 2.1 Let $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ denote the subcategory spanned by complete adic $\mathbb{E}_\infty$–rings A for which the commutative ring $\pi_0(A)$ is a local Noetherian ring with maximal ideal $\mathfrak{m}$, topologized with respect to the $\mathfrak{m}$–adic topology, and such that there exists an abstract (ie nonspecified) isomorphism $\pi_0(A)/\mathfrak{m} \simeq \kappa$.

Remark 2.2 The notation $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ is potentially misleading. Indeed, unlike what it may seem to indicate, said ∞–category *is not* equivalent to a subcategory of the overcategory $\mathrm{CAlg}_{/\kappa}$. That would hold if we restricted to connective object, but we cannot do so, since our primary interest rests with the nonconnective complex periodic $\mathbb{E}_\infty$–rings.

Remark 2.3 Similarly to the preceding remark, the connective objects in $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ are not the Noetherian $\mathbb{E}_\infty$–rings in the sense of [14, Definition 7.2.4.30]. We could not have used that notion of Noetherianness in the above definition, since it again only applies to connective $\mathbb{E}_\infty$–rings. It would be possible to imitate such a definition, by also imposing finiteness assumptions on the homotopy groups $\pi_i(A)$ in Definition 2.1 for $i \neq 0$. But since we can make do without, we choose to only impose the (unavoidable) π_0–level assumption. That is to say, the notion of a complete Noetherian local $\mathbb{E}_\infty$–ring from Definition 2.1 is only guaranteed to be adequate for the purposes of this paper. For most other purposes in spectral algebraic geometry where a Noetherian assumption might be desirable, stronger finiteness assumptions would probably need to be imposed.

From here on, we will consider the ∞–category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$ as the setting for formal spectral algebraic geometry. In particular, we will implicitly restrict the domain of the functor $\mathrm{Spf}(A)$ to the subcategory $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ for any adic $\mathbb{E}_\infty$–ring A.

Remark 2.4 The restriction functor $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$, induced from the subcategory inclusion $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$, preserves both limits and colimits. The Yoneda embedding $(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}, \mathcal{S})$ also preserves limits, and the coproduct in the ∞–category $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ is given by the completed smash product of [14, Corollary 7.3.5.2]. It follows that we have for any pair of complete adic $\mathbb{E}_\infty$–rings A and B a canonical equivalence

$$\mathrm{Spf}(A) \times \mathrm{Spf}(B) \simeq \mathrm{Spf}(A \mathbin{\widehat{\otimes}} B)$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$. That is to say, restriction to complete Noetherian local $\mathbb{E}_\infty$–rings does not change the products of affine formal spectral schemes.

The functors of the *ring of functions* $\mathcal{O}\colon \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})^{\mathrm{op}} \to \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ and the ∞–category of *quasicoherent sheaves* $\mathrm{QCoh}\colon \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ are defined by right Kan extension from the subcategory of affines $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})^{\mathrm{op}}$ (ie representable functors) on which they are defined as

$$\mathcal{O}(\mathrm{Spf}(A)) := A, \quad \mathrm{QCoh}(\mathrm{Spf}(A)) := \mathrm{Mod}^{\mathrm{cpl}}_A.$$

For a detailed treatment of such an approach to quasicoherent sheaves (in a slightly different but closely related setting), see [15, Section 6.2.2].

Remark 2.5 Because we are not equipping $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ with a Grothendieck topology, questions of descent are beyond our reach. Fortunately, as explained for QCoh in [15, Proposition 6.2.3.1] (in only a slightly different setting), both $\mathcal{O}$ and QCoh are agnostic regarding sheafification, making their definition unambiguous.

Remark 2.6 In defining the functors $\mathcal{O}$ and QCoh by Kan extension, we are being slightly imprecise regarding set-theoretical considerations. The issue is that the category $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ is not small. This may be circumvented by the usual trick of universe enlargement, at the cost of eg the ∞–category $\mathrm{QCoh}(X)$ being not necessarily small. For a precise treatment along those lines in a closely related setting, see [15, Section 6.2]. On the other hand, all functors which we will ultimately be interested in will all be given in explicit ways as small colimits of representables. In principle, we could in each individual such case redefine the functors $\mathcal{O}$ and QCoh by indexing them on appropriate small indexing categories, and verify *post factum* that the choice didn't matter. With this understanding, we will ignore questions of smallness, and set-theoretical technicalities alike, from now on.

Noting that we may have equivalently replaced the ∞–category $\mathcal{C}\mathrm{at}_\infty$ with $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ in the definition of quasicoherent sheaves (with the caveat of Remark 2.6 in mind), we see that any map of functors $f\colon X \to Y$ induces adjoint functors

$$f^*\colon \mathrm{QCoh}(Y) \rightleftarrows \mathrm{QCoh}(X) : f_*,$$

the familiar pullback and pushforward functoriality. In particular, we call pushforward along the terminal map $p\colon X \to *$ *global sections* and denote $\Gamma(X;\mathscr{F}) := p_*(\mathscr{F})$ for any $\mathscr{F} \in \mathrm{QCoh}(X)$. For the structure sheaf $\mathscr{F} = \mathcal{O}_X$, global sections $\Gamma(X;\mathcal{O}_X) \simeq \mathcal{O}(X)$ recover the ring of functions.

Remark 2.7 The pushforward functor $f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is not necessarily very well behaved without some additional assumptions on the morphism $f\colon X \to Y$ (such as being quasicompact and separated); eg the Beck–Chevalley push–pull formula for base change, and the projection formula may both fail in general. In particular, this "functor-of-points" pushforward it might in that case not coincide with a "ringed space" pushforward, if such exists. See also [15, Warning 6.3.4.2].

Remark 2.8 It follows from the definition of global sections that there is a chain of homotopy equivalences

$$\Gamma(X;\mathscr{F}) \simeq \mathrm{Map}_{\mathrm{Sp}}(S, p_*(\mathscr{F})) \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(p^*(S),\mathscr{F}) \simeq \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X,\mathscr{F}).$$

As consequence, we cannot expect the global sections functor $\mathscr{F} \mapsto \Gamma(X;\mathscr{F})$ to preserve filtered colimits unless the structure sheaf $\mathcal{O}_X$ is a compact object of the ∞–category $\mathrm{QCoh}(X)$. Since $\Gamma(X;-) = p_*$ is a pushforward along the terminal map, this ties into the more general ill-behavedness of the pushforward discussed in Remark 2.7. Fortunately, such issues will not arise in the (rather simplistic) applications we discuss in this paper.

2.2 A short digression on monodromy

In the proof of Proposition 2.12 in the next subsection, we will need a certain result, which becomes particularly simple and natural when viewed in a slightly more general context than strictly necessary for our purposes.

Recall that monodromy is classically understood to be the action of the fundamental group $\pi_1(X,x)$ of a base space X on the fiber $\mathcal{L}_x$ of a local systems $\mathcal{L}$ on X, acting through parallel transport around loops. The following is a simple incarnation of that idea in the setting of an ∞–topos, but with the notion of a "point" being understood in the generalized sense of algebraic geometry.

Lemma 2.9 *Let $x\colon P \to X$ be a morphism in an ∞–topos $\mathcal{X}$.*

(i) *The "based loop space" $\Omega_x(X) := P \times_X P$ admits a canonical group structure in the overtopos $\mathcal{X}_{/P}$, exhibiting it as an object $\Omega_x(X) \in \mathrm{Grp}(\mathcal{X}_{/P})$. There is a canonical equivalence of simplicial objects*

$$\mathrm{B}^{\bullet}_{\Omega_x(X)}(P,P) \simeq \check{\mathrm{C}}^{\bullet}(P \xrightarrow{x} X)$$

between its bar construction in $\mathcal{X}_{/P}$ and the Čech nerve of x in $\mathcal{X}$.

(ii) *For any object $Y \in \mathcal{X}_{/X}$, we define its "fiber over x" through the pullback square*

$$\begin{tikzcd} x^*(Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ P \arrow[r, "x"] & X \end{tikzcd} \tag{2}$$

in $\mathcal{X}$. This "fiber" $x^*(Y) \in \mathcal{X}_{/P}$ admits a canonical $\Omega_x(X)$–action, whose bar constriction in $\mathcal{X}_{/P}$ is equivalent to the Čech nerve in $\mathcal{X}$,

$$\mathrm{B}^\bullet_{\Omega_x(X)}(P, x^*(Y)) \simeq \check{\mathrm{C}}^\bullet(x^*(Y) \to Y).$$

Proof Recall from [12, Section 6.1.2; 14, Proposition 2.4.2.5] that group objects and group actions (or their common generalization, groupoid objects) in an ∞–topos are completely and equivalently encoded by their bar constructions. Thus it is necessary and sufficient to verify that the Čech complexes in question are of the appropriate forms for a group object and group action respectively.

For (i), we rewrite the Čech complex of the morphism x as

$$\check{\mathrm{C}}^\bullet(x) \simeq \underbrace{P \times_X \cdots \times_X P}_{\bullet+1} \simeq \underbrace{(P \times_X P) \times_P \cdots \times_P (P \times_X P)}_{\bullet} \simeq \underbrace{\Omega_x(X) \times_P \cdots \times_P \Omega_x(X)}_{\bullet}.$$

It follows clearly that it satisfies the Segal condition and exhibits $\Omega_x(X) \in \mathrm{Grp}(\mathcal{X}_{/P})$.

For (ii), observe that we may compare the two Čech nerves in sight via (degreewise) pullback of simplicial objects. Combining that with point (i), we get equivalences of simplicial objects

$$\begin{aligned}\check{\mathrm{C}}^\bullet(x^*(Y) \to Y) &\simeq \check{\mathrm{C}}^\bullet(P \to X) \times_X Y \\ &\simeq \mathrm{B}^\bullet_{\Omega_x(X)}(P, P) \times_X Y \\ &\simeq \mathrm{B}^\bullet_{\Omega_x(X)}(P, P \times_X Y) \\ &\simeq \mathrm{B}^\bullet_{\Omega_x(X)}(P, x^*(Y)),\end{aligned}$$

exhibiting the desired $\Omega_x(X)$–action on the fiber $x^*(Y)$. □

Remark 2.10 In the setting of Lemma 2.9, passage to geometric realizations from (i) gives an equivalence $\mathrm{B}\Omega_x(X) \simeq X^\wedge_x$ between the classifying space for $\Omega_x(X)$ (in the overtopos $\mathcal{X}_{/P}$) and the so-called *nilpotent completion* of X at x, defined as $X^\wedge_x = |\check{\mathrm{C}}^\bullet(x)|$. This is a not necessarily affine variant of the notion of nilpotent completion of ring spectra, first introduced by Bousfield [3, Theorem 6.5]. When $x\colon P \to X$ is an effective epimorphism, we have $X^\wedge_x \simeq X$. Then Lemma 2.9(ii) shows that $Y \simeq x^*(Y)/\Omega_x(X)$, generalizing the classical fact that a local system on a connected base space is completely determined by its monodromy representation.

Remark 2.11 Let us take for $\mathcal{X}$ the presheaf ∞–topos $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$, the usual setting for "functor of points" nonconnective spectral algebraic geometry (once again ignoring questions of descent). An $\mathbb{E}_\infty$–ring A gives rise to the terminal map of nonconnective affines $x_A\colon \mathrm{Spec}(A)\to\mathrm{Spec}(S)$, which we may view as an A–point of $\mathrm{Spec}(S)$. It follows from Lemma 2.9(i) that the loop space $\Omega_{x_A}(\mathrm{Spec}(S))$ admits a group structure over $\mathrm{Spec}(A)$. That amounts to an appropriately interpreted (see [24] for a thorough discussion of appropriate coalgebras in this setting) Hopf algebroid structure on $\mathcal{O}(\Omega_{x_A}(\mathrm{Spec}(S))\simeq A\otimes A$ over A. Upon passage to homotopy groups, this recovers the usual "generalized dual Steenrod algebra" Hopf algebroid

$$(\pi_*(A),\pi_*(A\otimes A))=(A_*,A_*A).$$

Similarly, given any $\mathbb{E}_\infty$–ring A, the $\Omega_{x_A}(\mathrm{Spec}(S))$–action on the fiber $x_A^*(\mathrm{Spec}(X))$ described in Lemma 2.9(ii), gives rise on homotopy groups to the usual "generalized Steenrod comodule" structure on $\pi_*(A\otimes X)=A_*(X)$. This hints at the relationship between the monodromy construction of Lemma 2.9 and generalized Adams spectral sequences, which we partly elucidate in Section 3.2, and in Remark 3.4 in a bit more detail in the case of the Adams–Novikov spectral sequence.

2.3 Morava stabilizer group action on oriented deformations

Fix a formal group $\widehat{\boldsymbol{G}}_0$ of finite height over $\kappa=\overline{\boldsymbol{F}}_p$ and let $\mathbb{G}(\kappa,\widehat{\boldsymbol{G}}_0)$ be its (big, ie extended) Morava stabilizer group, viewed as an algebraic group, and hence a functor $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}\to\mathcal{S}$, as explained in [8, Remark 5.29] and reviewed in Remark 2.17.

Proposition 2.12 *There exists a canonical action of the Morava stabilizer group* $\mathbb{G}(\kappa,\widehat{\boldsymbol{G}}_0)$ *on the oriented deformations* $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$ *in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa},\mathcal{S})$, *whose two-sided bar construction is equivalent as a simplicial object in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa},\mathcal{S})$,

$$\check{\mathrm{C}}^\bullet(\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}\to *)\simeq \mathrm{B}^\bullet_{\mathbb{G}(\kappa,\widehat{\boldsymbol{G}}_0)}(*,\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}),$$

to the Čech nerve of (*the terminal map of*) $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$.

Before embarking on the proof, let us outline its logical structure. We successively reduce the statement to simpler ones, until we end up with an explicit verification. The first reduction, from the oriented statement of Proposition 2.12 to a nonoriented version, Lemma 2.13, is completely formal. The proof of Lemma 2.13 is where we use the monodromy ideas from the previous subsection. Using them, or more precisely Lemma 2.9, we are reduced to identifying the naturally occurring automorphism group with the Morava stabilizer group. That is something of a classical observation, eg [13, Lectrure 19] or [8, Theorem 7.18], and is the content of Lemma 2.16. Its proof, after reducing from the ∞–categorical to a classical 1–categorical setting, is an explicit point-set-level comparison.

Proof of Proposition 2.12 By the definition of oriented deformations,

$$\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \simeq \mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \times_{\mathcal{M}_{\mathrm{FG}}} \mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}. \tag{3}$$

The factor $\mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}$ in this fibered product may be replaced with $\{\widehat{\boldsymbol{G}}{}^{\mathcal{Q}}_{A}\}$ when A is complex oriented, and with $\varnothing$ when A is not. It follows from this observation that

$$\check{\mathrm{C}}^\bullet(\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \to *) \simeq \check{\mathrm{C}}^\bullet(\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \to \mathcal{M}_{\mathrm{FG}}) \times_{\mathcal{M}_{\mathrm{FG}}} \mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}$$

as the base change of simplicial objects. Consequently, pulling back the equivalence of simplicial objects from the next Lemma 2.13 along the inclusion $\mathcal{M}^{\mathrm{or}}_{\mathrm{FG}} \to \mathcal{M}_{\mathrm{FG}}$ gives rise to a $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$–action on $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$ with the desired bar construction. □

Lemma 2.13 *There exists a canonical action of the Morava stabilizer group* $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$ *on the unoriented deformations* $\mathrm{Def}_{\widehat{\boldsymbol{G}}_0}$ *in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$, *whose two-sided bar construction is equivalent as a simplicial object in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$,

$$\check{\mathrm{C}}^\bullet(\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \to \mathcal{M}_{\mathrm{FG}}) \simeq \mathrm{B}^\bullet_{\mathbb{G}(\kappa,\widehat{\boldsymbol{G}}_0)}(*, \mathrm{Def}_{\widehat{\boldsymbol{G}}_0}),$$

to the Čech nerve of the map $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \to \mathcal{M}_{\mathrm{FG}}$.

Proof Unlike oriented deformations, unoriented deformations of formal groups are as a functor determined (as Kan extension) by its restriction to connective $\mathbb{E}_\infty$–rings by [17, Proof of Theorem 3.4.1]. Therefore, let us implicitly restrict all functors to the full subcategory $(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}} \subseteq \mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ spanned by connective $\mathbb{E}_\infty$–rings for the rest of this proof.

There, we have by [17, Proof of Proposition 3.4.3] a natural identification

$$\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \simeq (\mathrm{Spec}(\kappa)/\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}$$

with the relative de Rham space of the morphism $\mathrm{Spec}(\kappa) \to \mathcal{M}_{\mathrm{FG}}$ classifying $\widehat{\boldsymbol{G}}_0$. Recall from [15, Definition 18.2.1.1] that the *relative de Rham space* of a map of functors $X \to Y$ is defined as the pullback

$$(X/Y)_{\mathrm{dR}} \simeq X_{\mathrm{dR}} \times_{Y_{\mathrm{dR}}} Y, \tag{4}$$

where the *absolute de Rham space* of a functor X is given by[2] $X_{\mathrm{dR}}(A) = X(\pi_0(A)/\mathfrak{m})$.

[2]Restricting to the subcategory $\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa} \subseteq \mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ helps substantially here, as no colimiting over nilpotent ideals of definition is necessary.

Observe that we have at this point found ourselves in the setting of Lemma 2.9, with the pullback square

$$\begin{tikzcd}
\mathrm{Def}_{\widehat{G}_0} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{FG}} \arrow[d] \\
\mathrm{Spec}(\kappa)_{\mathrm{dR}} \arrow[r] & (\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}
\end{tikzcd}$$

playing the role of (2). More precisely, we have

- an ambient ∞–topos $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})$,
- a "point" $\mathrm{Spec}(\kappa)_{\mathrm{dR}} \to (\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}$,
- an object $\mathcal{M}_{\mathrm{FG}}$ over the "base space" $(\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}$,
- and its "fiber" $\mathrm{Spec}(\kappa)_{\mathrm{dR}} \times_{(\mathcal{M}_{\mathrm{FG}})_{\mathrm{dR}}} \mathcal{M}_{\mathrm{FG}} \simeq \mathrm{Def}_{\widehat{G}_0}$.

Lemma 2.9(i) thus exhibits the "based loop space", which is the de Rham space $\underline{\mathrm{Aut}}(\widehat{G}_0)_{\mathrm{dR}}$ of

$$\mathrm{Spec}(\kappa) \times_{\mathcal{M}_{\mathrm{FG}}} \mathrm{Spec}(\kappa) \simeq \Omega_{\widehat{G}_0}(\mathcal{M}_{\mathrm{FG}}) \simeq \underline{\mathrm{Aut}}(\widehat{G}_0),$$

the automorphism group of the formal group $\widehat{G}_0$, as a group object in the overcategory $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})_{/\mathrm{Spec}(\kappa)_{\mathrm{dR}}}$. Thus Lemma 2.9(ii) equips the "fiber" $\mathrm{Def}_{\widehat{G}_0}$ with the "monodromy" $\underline{\mathrm{Aut}}(\widehat{G}_0)$–action over $\mathrm{Spec}(\kappa)_{\mathrm{dR}}$, whose bar construction is

$$\mathrm{B}^{\bullet}_{\underline{\mathrm{Aut}}(\widehat{G}_0)_{\mathrm{dR}}}(\mathrm{Spec}(\kappa)_{\mathrm{dR}}, \mathrm{Def}_{\widehat{G}_0}) \simeq \check{\mathrm{C}}^{\bullet}(\mathrm{Def}_{\widehat{G}_0} \to \mathcal{M}_{\mathrm{FG}}).$$

In light of Lemma 2.16, this $\underline{\mathrm{Aut}}(\widehat{G}_0)_{\mathrm{dR}}$–action on the deformation (pre)stack $\mathrm{Def}_{\widehat{G}_0}$ in the overcategory $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})_{/\mathrm{Spec}(\kappa)_{\mathrm{dR}}}$ is equivalent to a $\mathbb{G}(\kappa, \widehat{G}_0)$–action on it in $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})$, exhibited on the level of bar constructions (see Remark 2.14) by the equivalence

$$\mathrm{B}^{\bullet}_{\underline{\mathrm{Aut}}(\widehat{G}_0)_{\mathrm{dR}}}(\mathrm{Spec}(\kappa)_{\mathrm{dR}}, \mathrm{Def}_{\widehat{G}_0}) \simeq \mathrm{B}^{\bullet}_{\mathbb{G}(\kappa,\widehat{G}_0)}(*, \mathrm{Def}_{\widehat{G}_0}). \tag{5}$$

□

Remark 2.14 We must clarify that the two bar constructions appearing on each side of the equivalence (5) are formed in different ∞–categories. That is to say, the products comprising the simplices on the left-hand side are all taken over $\mathrm{Spec}(\kappa)_{\mathrm{dR}}$, while on the right-hand side, the products are absolute, ie taken over the terminal object $*$.

Remark 2.15 The de Rham space $\mathrm{Spec}(\kappa)_{\mathrm{dR}}$ that we encountered above in the proof of Lemma 2.13 is equivalent to the affine formal scheme $\mathrm{Spf}(W^+(\kappa))$, where $W^+(\kappa)$ the $\mathbb{E}_\infty$–ring of *spherical Witt vectors over* κ, as defined in [17, Example 5.2.7]. Indeed, in [17, Proof of Theorem 5.2.5] the spherical Witt vectors are defined to corepresent as an affine formal scheme the relative de Rham space $(\mathrm{Spec}(\kappa)/\mathrm{Spec}(S))_{\mathrm{dR}}$. But since clearly $\mathrm{Spec}(S) \simeq \mathrm{Spec}(S)_{\mathrm{dR}}$, it follows that $(\mathrm{Spec}(\kappa)/\mathrm{Spec}(S))_{\mathrm{dR}} \simeq \mathrm{Spec}(\kappa)_{\mathrm{dR}}$, as claimed. More concretely, the universal property of the spherical Witt vectors may be written as

$$\mathrm{Map}^{\mathrm{cont}}_{\mathrm{CAlg}}(W^+(\kappa), A) \simeq \varinjlim_{I} \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit}(\kappa, \pi_0(A)/I)$$

for any adic $\mathbb{E}_\infty$–ring A, and with the colimit ranging over all of the finitely generated ideals of definition in $\pi_0(A)$. Another characterization of it is that $W^+(\kappa)$ is a flat p–complete $\mathbb{E}_\infty$–ring and $\pi_0(W^+(\kappa)) = W(\kappa)$ recovers the usual ring of (p–typical) Witt vectors.

Lemma 2.16 *There is a canonical equivalence* $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}} \simeq \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spf}(W^+(\kappa))$ *of group objects in* $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}}, \mathcal{S})_{/\,\mathrm{Spf}(W^+(\kappa))}$.

Proof By unwinding the definitions, we find for any connective $A \in \mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ that

$$\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}(A) \simeq \mathrm{Spec}(\kappa)(\pi_0(A))/\mathfrak{m}) \times_{\mathcal{M}_{\mathrm{FG}}(\pi_0(A)/\mathfrak{m})} \mathrm{Spec}(\kappa)(\pi_0(A)/\mathfrak{m})$$

consists of a pair of maps $f_1, f_2 \colon \kappa \to \pi_0(A)/\mathfrak{m}$ and an isomorphism $\varphi\colon f_1^*\widehat{\boldsymbol{G}}_0 \to f_2^*\widehat{\boldsymbol{G}}$ of formal groups over $\pi_0(A)/\mathfrak{m}$. In particular, it is a discrete space — indeed, this follows from the fact that the ordinary moduli stack of formal groups,

$$\mathcal{M}_{\mathrm{FG}}|_{\mathrm{CAlg}^\heartsuit} \colon \mathrm{CAlg}^\heartsuit \to \mathcal{S},$$

is actually a 1–stack, ie a groupoid-valued functor $\mathrm{CAlg}^\heartsuit \to \tau_{\le 1}(\mathcal{S}) \hookrightarrow \mathcal{S}$. Since the functor $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}$ amounts, as observed above, to passing to internal automorphisms of this stack, and the essential image of the based loops functor $\Omega\colon \tau_{\le 1}(\mathcal{S}_*) \to \mathcal{S}$ belongs to the full subcategory of discrete spaces $\mathcal{S}\mathrm{et} \simeq \tau_{\le 0}(\mathcal{S}) \hookrightarrow \mathcal{S}$, it is a set-valued functor itself. In conclusion, the functor $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}} \colon (\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}} \to \mathcal{S}$ factors through $\mathcal{S}\mathrm{et} \hookrightarrow \mathcal{S}$ in the target, and through $\pi_0\colon (\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}} \to (\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^\heartsuit$ in the source. The same holds for $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spec}(\kappa)_{\mathrm{dR}}$ by definition of the de Rham space. Hence to prove the lemma, it suffices to exhibit an isomorphism between the two functors $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}$ and $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spec}(\kappa)_{\mathrm{dR}}$ as group objects in the ordinary category $\mathrm{Fun}((\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^\heartsuit, \mathcal{S}\mathrm{et})_{/\mathrm{Spec}(\kappa)_{\mathrm{dR}}}$.

Let us therefore construct a natural transformation $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}} \to \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spec}(\kappa)_{\mathrm{dR}}$ as functors $(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^\heartsuit \to \mathcal{S}\mathrm{et}$ over $\mathrm{Spf}(W^+(\kappa)) \simeq \mathrm{Spec}(\kappa)_{\mathrm{dR}}$. Fix a complete Noetherian local commutative ring A with residue field κ. Recall from the above discussion that elements of $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}(A)$ consist of triples (f_1, f_2, φ) as above, and the base map $\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}(A) \to \mathrm{Spec}(\kappa)_{\mathrm{dR}}(A)$ is given by $(f_1, f_2, \varphi) \mapsto f_1$. Thus fixing the element $f_1 \in \mathrm{Spec}(\kappa)_{\mathrm{dR}}(A)$ (since we wish to work over $\mathrm{Spf}(W^+(\kappa)) \simeq \mathrm{Spec}(\kappa)_{\mathrm{dR}}$), we obtain an element

$$(g, \psi) \in \mathrm{Gal}(\kappa/\boldsymbol{F}_p) \ltimes \mathrm{Aut}_{\mathrm{FGrp}(\kappa)}(\widehat{\boldsymbol{G}}_0) = \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$$

as follows. Thanks to the hypothesis that $A/\mathfrak{m} \simeq \kappa$, the field map f_1 may be abstractly identified with a field endomorphism of $\kappa = \overline{\boldsymbol{F}}_p$. Any such endomorphism must fix the prime subfield $\boldsymbol{F}_p$, and since the inclusion $\boldsymbol{F}_p \subseteq \overline{\boldsymbol{F}}_p$ is algebraic, this implies that it is actually an automorphism. It follows that $f_1 : \kappa \to A/\mathfrak{m}$ is a field isomorphism, so we can set $g := f_1^{-1} \circ f_2$. We obtain the formal group isomorphism over κ as

$$\psi : \widehat{\boldsymbol{G}}_0 \xrightarrow{\varphi} (f_1^{-1})^* f_2^* \widehat{\boldsymbol{G}}_0 \simeq g^* \widehat{\boldsymbol{G}}_0.$$

Sending $(f_1, f_2, \varphi) \mapsto ((g, \varphi), f_1)$ gives the desired map of sets

$$\underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}(A) \to \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Spec}(\kappa)_{\mathrm{dR}}(A).$$

It is clear from the description that this procedure is bijective, compatible with the group structure, functorial in A, and compatible with the maps to $\mathrm{Spec}(\kappa)_{\mathrm{dR}}$; hence it gives rise to an equivalence of functors as claimed. □

Remark 2.17 The matter of viewing $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$ as a profinite group scheme here comes from the classical observation that topology coincides with the usual Zariski topology on automorphisms. Indeed, as we noted in the proof, all the functors involved in Lemma 2.16 factor through the functor $\pi_0 : \mathrm{CAlg} \to \mathrm{CAlg}^\heartsuit$, and are as such a matter of classical algebraic geometry. In that context, see [8, Theorem 7.18], or [13, Lecture 19].

On the other hand, let us explain where the profinite structure on $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$ *is coming from* from the algebrogeometric perspective. Let us view the fixed formal group as a functor $\widehat{\boldsymbol{G}}_0 : (\mathrm{CAlg}^{\mathrm{Art}}_{/\kappa})^\heartsuit \to \mathcal{S}\mathrm{et}$ from Artinian local rings with residue field κ; ie infinitesimal extensions of the point $\mathrm{Spec}(\kappa)$. Consider the subcategory $\mathrm{Nil}^{\le n}_{/\kappa} \subseteq (\mathrm{CAlg}^{\mathrm{Art}}_{/\kappa})^\heartsuit$ of local Artinian rings with $\mathfrak{m}^{n+1} = 0$. Restriction and Kan extension back along this inclusion produces a functor $\widehat{\boldsymbol{G}}_0^{\le n} : (\mathrm{CAlg}^{\mathrm{Art}}_{/\kappa})^\heartsuit \to \mathcal{S}\mathrm{et}$, which we may view as the n^{th} infinitesimal neighborhood; Goerss calls this the *n–bud* of the formal group $\widehat{\boldsymbol{G}}_0$, see in particular [8, Remark 3.24]. Since every ideal in an Artinian local ring is nilpotent, the tower

$$\mathrm{Nil}^{\le 0}_{/\kappa} \subseteq \mathrm{Nil}^{\le 1}_{/\kappa} \subseteq \mathrm{Nil}^{\le 2}_{/\kappa} \subseteq \mathrm{Nil}^{\le 3}_{/\kappa} \subseteq \cdots \subseteq (\mathrm{CAlg}^{\mathrm{Art}}_{/\kappa})^\heartsuit$$

is exhaustive and the canonical map $\varinjlim \widehat{\boldsymbol{G}}_0^{\le n} \to \widehat{\boldsymbol{G}}_0$ is an equivalence. Furthermore, any morphism of formal groups $\widehat{\boldsymbol{G}} \to \widehat{\boldsymbol{G}}'$ induces a family of maps $\widehat{\boldsymbol{G}}^{\le n} \to \widehat{\boldsymbol{G}}'^{\le n}$ for all $n \ge 0$, which induces an isomorphism

$$\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) = \mathrm{Aut}(\widehat{\boldsymbol{G}}_0) \simeq \varprojlim \mathrm{Aut}(\widehat{\boldsymbol{G}}_0^{\le n}).$$

Each factor in this filtered limit is finite, recovering the usual profinite structure on the Morava stabilizer group. In particular, this implies that the product

$$\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \simeq \mathrm{Spf}\big(C^*_{\mathrm{cont}}(\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0); E(\kappa, \widehat{\boldsymbol{G}}_0))\big)$$

is the formal spectrum of an incarnation of continuous $E(\kappa, \widehat{\boldsymbol{G}}_0)$–valued cochains on the profinite group $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$.

Remark 2.18 Let us indicate an alternative approach to proving Proposition 2.12. Instead of using the identification (3), we can rather observe that we have for any $A \in \mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ a natural equivalence

$$\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}(A) \simeq \mathrm{Def}_{\widehat{\boldsymbol{G}}_0}(\pi_0(A)) \times_{\mathcal{M}_{\mathrm{FG}}(\pi_0(A))} \mathcal{M}^{\mathrm{or}}_{\mathrm{FG}}(A).$$

In light of that, it suffices to establish an appropriate $\mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0)$–action on unoriented deformations, when all functors in sight are postcomposed with the functor $A \mapsto \pi_0(A)$. That involves only classical (ie nonspectral) algebraic geometry, and as such avoids coherence issues. Therefore the desired bar construction claim follows inductively from finding an appropriately equivariant equivalence

$$\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \times_{\mathcal{M}_{\mathrm{FG}}} \mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \simeq \mathbb{G}(\kappa, \widehat{\boldsymbol{G}}_0) \times \mathrm{Def}_{\widehat{\boldsymbol{G}}_0}, \tag{6}$$

with both sides restricted to the subcategory $(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\heartsuit} \subseteq \mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}$ of discrete objects. Since any complete Noetherian local ring may be written as a filtered limit of Artinian ones, and we are working in the "continuous" category, it further suffices to prove the result upon the further restriction to local Artinian rings; see [8, Remark 7.3]. For that, we can reference [8, Theorem 7.18].

There is one final small hitch: Goerss's analogue of (6) takes the fiber product over a moduli functor $\widehat{\mathcal{H}}(n) = (\mathcal{M}^{=n}_{\mathrm{FG}}/\mathcal{M}^{\le n}_{\mathrm{FG}})_{\mathrm{dR}}$ instead of over $\mathcal{M}_{\mathrm{FG}}$. But since the forgetful functor $\mathrm{Def}_{\widehat{\boldsymbol{G}}_0} \to \mathcal{M}_{\mathrm{FG}}$ naturally factors through the substack inclusion $\widehat{\mathcal{H}}(n) \hookrightarrow \mathcal{M}_{\mathrm{FG}}$, this turns out not to effect the result. See [10, Section 3.5] for further discussion.

2.4 The Devinatz–Hopkins theorem

As before, fix $\kappa = \overline{\boldsymbol{F}}_p$ and let $\widehat{\boldsymbol{G}}_0$ be of height n, (which specifies it up to isomorphism). We denote by E_n and $\mathbb{G}_n$ the associated Lubin–Tate spectrum and Morava stabilizer group, respectively. Proposition 2.12 equips $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \simeq \mathrm{Spf}(E_n)$ with an action[3] of $\mathbb{G}_n$ on $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \simeq \mathrm{Spf}(E_n)$ in the ∞–topos $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})$. Let $\mathrm{Spf}(E_n)/\mathbb{G}_n$ denote the quotient of this action in this ∞–topos. We view its ring of functions

$$E_n^{h\mathbb{G}_n} := \mathcal{O}(\mathrm{Spf}(E_n)/\mathbb{G}_n)$$

as the *continuous homotopy fixed points* of the corresponding action of $\mathbb{G}_n$ on the Lubin–Tate spectrum E_n. See Remark 3.5 for some further justification of this terminology.

Theorem 2.19 (Devinatz–Hopkins) *With continuous homotopy fixed points defined as above, the initial map $L_{K(n)}S \to E_n$ in $L_{K(n)}\,\mathrm{Sp}$ induces an equivalence $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S$.*

Proof By definition of the ring of functions, we have $\mathcal{O}(\mathrm{Spf}(E_n)) \simeq E_n$. Similarly, for products we have $\mathcal{O}(\mathrm{Spf}(E_n)^{\times\bullet}) \simeq E_n^{\widehat{\otimes}\bullet}$, where $\widehat{\otimes}$ denotes the completed smash product of [14, Corollary 7.3.5.2], ie the coproduct in the ∞–category $\mathrm{CAlg}^{\mathrm{ad}}_{\mathrm{cpl}}$ of complete adic $\mathbb{E}_\infty$–rings — see Remark 2.4. Therefore Proposition 2.12 implies that

$$E_n^{h\mathbb{G}_n} \simeq \mathcal{O}(\mathrm{Spf}(E_n)^{\times(\bullet+1)}) \simeq \mathrm{Tot}(E_n^{\widehat{\otimes}(\bullet+1)}).$$

It follows from [17, Corollary 4.5.4] that completion in the ∞–category of E_n–modules coincides with $K(n)$–localization, and so $E_n^{\widehat{\otimes}\bullet} \simeq L_{K(n)}(E_n^{\otimes\bullet})$. Thus it suffices to show that $L_{K(n)}S \to E_n$ induces an equivalence

$$\mathrm{Tot}(L_{K(n)}(E_n^{\otimes(\bullet+1)})) \simeq L_{K(n)}S. \tag{7}$$

That is a standard result, stemming from the nilpotence of $L_{K(n)}S$ in the ∞–category $L_{K(n)}\,\mathrm{Mod}_{E_n}$, and ultimately, the horizontal vanishing line in the $K(n)$–local Adams spectral sequence for E_n; see for instance [6, Proposition AI.3]. But for completeness, we sketch an argument anyway, following the account in [19].

The smashing product theorem of Hopkins and Ravenel [22, Theorem 7.5.6] asserts that the Bousfield localization functor $L_n := L_{E_n}$ is smashing, which, by Proposition 8.2.4

[3] Of course this is just the action of $\mathbb{G}_n$ on $\mathrm{Spf}(E_n)$ induced by the identification of the Morava stabilizer group as $\mathbb{G}_n \simeq \mathrm{Aut}(E_n) \simeq \mathrm{Aut}(\mathrm{Spf}(E_n))$, as observed in [17, Remark 5.0.8]. But from our way of obtaining it, its bar construction is more transparent.

of [22], is equivalent to $L_n S$ being E_n–nilpotent. That is further equivalent, by standard nilpotence technology, eg [13, Lectures 30 and 31], to the cosimplicial object $(E_n^{\otimes(\bullet+1)})$, whose totalization is $L_n S$, being pro-constant. Applying the functor $L_{K(n)}$ to this cosimplicial object then gives the desired equivalence. $\square$

Remark 2.20 An explicit analysis of how the horizontal vanishing line in the $K(n)$–local Adams spectral sequence for E_n gives rise to the equivalence (7) is given in [6, Section 4 and Appendix I]. The argument that we gave, following [19], while phrased slightly differently, is merely a repackaging of the same fundamental idea — indeed, the proof of the Hopkins–Ravenel smashing product theorem is based on the existence of a uniform vanishing line; see [20, Section 3.4] for a sketch and relationship to the "standard nilpotence technology" referred to in the proof above.

Remark 2.21 The equivalence of Theorem 2.19 is a purely function-level statement. Indeed, the quotient $\mathrm{Spf}(E_n)/\mathbb{G}_n$ is not equivalent to the affine formal scheme $\mathrm{Spf}(L_{K(n)} S)$. The value of $\mathrm{Spf}(E_n)/\mathbb{G}_n$ on any noncomplex-periodic $K(n)$–local $\mathbb{E}_\infty$–ring is the empty set, while the value of $\mathrm{Spf}(L_{K(n)} S)$ is contractible for all $K(n)$–local $\mathbb{E}_\infty$–rings.

Despite the preceding remark, we may view quasicoherent sheaves on the quotient $\mathrm{Spf}(E_n)/\mathbb{G}_n$, which are by definition a derived version of Morava modules, as a natural incarnation in spectral formal algebraic geometry of the $K(n)$–local stable category.

Corollary 2.22 *There is a canonical equivalence of symmetric monoidal ∞–categories*

$$\mathrm{QCoh}(\mathrm{Spf}(E_n)/\mathbb{G}_n) \simeq L_{K(n)}\,\mathrm{Sp}.$$

Proof It follows from the proof of Theorem 2.19 that

$$\mathrm{QCoh}(\mathrm{Spf}(E_n)/\mathbb{G}_n) \simeq \mathrm{Tot}(\mathrm{Mod}^{\mathrm{cpl}}_{E_n^{\hat{\otimes}(\bullet+1)}}) \simeq \mathrm{Tot}(L_{K(n)}\,\mathrm{Mod}_{L_{K(n)}(E_n^{\otimes(\bullet+1)})}),$$

which is equivalent to the $K(n)$–local stable ∞–category in [19, Proposition 10.10]. $\square$

2.5 Analogue over a general base

At the cost of replacing the Morava stabilizer group with the more involved algebrogeometric group $\mathscr{G} := \underline{\mathrm{Aut}}(\widehat{\boldsymbol{G}}_0)_{\mathrm{dR}}$, the contents of this section still hold after dropping the assumption that $\kappa = \overline{\boldsymbol{F}}_p$.

Proposition 2.23 *Let $\widehat{\boldsymbol{G}}_0$ be a formal group of finite height over a perfect field κ of positive characteristic. Then there exists a canonical $\mathscr{G}$–action on $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})_{/\mathrm{Spec}(W^+(\kappa))}$, whose two-sided bar construction in said ∞–category is equivalent as a simplicial object in the ∞–category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$,*

$$\check{\mathrm{C}}^\bullet(\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0} \to *) \simeq \mathrm{B}^\bullet_{\mathscr{G}}\big(\mathrm{Spf}(W^+(\kappa)), \mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}\big),$$

to the Čech nerve of (the terminal map of) $\mathrm{Def}^{\mathrm{or}}_{\widehat{\boldsymbol{G}}_0}$.

Proof The only step of the proof of Proposition 2.12 that employs the assumption $\kappa = \overline{\boldsymbol{F}}_p$ is in the proof of Lemma 2.16. The rest of the argument, including the proof of Lemma 2.13, goes through for any perfect field κ of positive characteristic, giving the stated result. □

Proposition 2.23 equips $\mathrm{Spf}(E(\kappa, \widehat{\boldsymbol{G}}))$ with a $\mathscr{G}$–action, though this time we need to be working in the relative setting over $\mathrm{Spf}(W^+(\kappa))$. This may be viewed as an incarnation of a $\mathscr{G}$–action on the Lubin–Tate spectrum $E(\kappa, \widehat{\boldsymbol{G}}_0)$ in the ∞–category $\mathrm{CAlg}^{\mathrm{ad}}_{/W^+(\kappa)}$. Just as before, we obtain a workable definition of continuous homotopy fixed points by setting

$$E(\kappa, \widehat{\boldsymbol{G}}_0)^{h\mathscr{G}} := \mathcal{O}\big(\mathrm{Spf}(E(\kappa, \widehat{\boldsymbol{G}}_0))/\mathscr{G}\big),$$

and the analogue of the Devinatz–Hopkins theorem holds as follows.

Proposition 2.24 *Let $\widehat{\boldsymbol{G}}_0$ be a formal group of height $n < \infty$ over a perfect field κ of positive characteristic. With notation as above, the initial map $L_{K(n)}S \to E(\kappa, \widehat{\boldsymbol{G}}_0)$ in $L_{K(n)}\,\mathrm{Sp}$ induces an equivalence of spectra $E(\kappa, \widehat{\boldsymbol{G}}_0)^{h\mathscr{G}} \simeq L_{K(n)}S$.*

Proof The proof of Theorem 2.19 works just as well in this setting, provided we use [11, Proposition 5.2.6] for the nilpotence claim. □

Remark 2.25 As explained in [11, Notation 2.1.10], every Lubin–Tate spectrum $E(\kappa, \widehat{\boldsymbol{G}}_0)$ gives rise to a Morava K–theory $K(\kappa, \widehat{\boldsymbol{G}}_0)$. It might seem like we should have used the localization functor $L_{K(\kappa, \widehat{\boldsymbol{G}}_0)}$ in Proposition 2.24, but alas this does not matter, since even though the spectra $K(\kappa, \widehat{\boldsymbol{G}}_0)$ do depend on the base field and formal group used to define them, the induced localization functor does not. By [11, Remark 2.1.14], the Bousfield localization functor $L_{K(\kappa, \widehat{\boldsymbol{G}}_0)} \simeq L_{K(n)}$ only depends on the characteristic of the field κ and the height n of the formal group $\widehat{\boldsymbol{G}}_0$.

In particular, we obtain by the same proof as Corollary 2.22 a "derived Morava module" presentation of the $K(n)$–local stable category for every height n formal group $\widehat{\boldsymbol{G}}_0$ over a perfect field κ of positive characteristic.

Corollary 2.26 *Keeping all the notation from Proposition 2.24, there is a canonical equivalence of symmetric monoidal ∞–categories,*

$$\mathrm{QCoh}\big(\mathrm{Spf}(E(\kappa, \widehat{G}_0)/\mathcal{G})\big) \simeq L_{K(n)}\,\mathrm{Sp}.$$

3 Spectral sequences

The goal of this section is to prove a version of the Morava change-of-rings theorem, identifying the $K(n)$–local Adams spectral sequence for E_n with the continuous fixed-point spectral sequence for the $\mathbb{G}_n$–action on E_n. The Devinatz–Hopkins Theorem 2.19 already guarantees that they converge to (filtrations on) the homotopy groups on the same spectrum, but the actual comparison of the spectral sequences (and interpretation of the second one) will take a little more work.

3.1 The descent spectral sequence

Unlike the fundamentally nonconnective $\mathrm{Spf}(E_n)$ and its intimidating-looking quotient $\mathrm{Spf}(E_n)/\mathbb{G}_n$, the classifying (pre)stack $\mathrm{B}\mathbb{G}_n = */\mathbb{G}_n$ is quite well behaved. In particular, it (or better, its sheafification; but since the difference between them does not matter for quasicoherent sheaves or functions, we will freely switch between them) is representable by a formal spectral stack which, while not quite Deligne–Mumford, is nonetheless quite manageable.

For instance, $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_n)$ admits an accessible t–structure by the (formal geometry analogue of) [15, Proposition 6.2.5.8]. Similarly, the descent spectral sequence, a piece of technology familiar from the theory of topological modular forms, applies to $\mathrm{B}\mathbb{G}_n$. The following proof is essentially a repetition of the one in [7, Chapter 5, Section 3], but since the setting is slightly different, we have opted to spell it out.

Lemma 3.1 *Let $X\colon (\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa})^{\mathrm{cn}} \to \mathcal{S}$ be a formal spectral fpqc stack*[4] *that admits a flat cover $U \to X$, such that all the (nontrivial) fiber products $U \times_X \cdots \times_X U$ are affine formal spectral schemes. For any quasicoherent sheaf $\mathcal{F}$ on X, there exists a canonical Adams-graded spectral sequence*

$$E_2^{s,t} = \mathrm{H}^s(X; \pi_t(\mathcal{F})) \Rightarrow \pi_{t-s}(\Gamma(X; \mathcal{F})),$$

called the ***descent spectral sequence****.*

[4] Here we are following [15], in that the absence of the adjective "nonconnective" automatically implies connectivity.

Proof Let $U \to X$ be a flat cover by an affine formal spectral scheme as postulated in the statement of the lemma. It gives rise to a Čech nerve $\check{\mathrm{C}}^\bullet(U \to X)$ and hence a cosimplicial spectrum $\Gamma(\check{\mathrm{C}}^\bullet(U \to X); \mathscr{F}|_{\check{C}^\bullet(U\to X)})$ with totalization $\Gamma(X; \mathscr{F})$. We claim that the Bousfield–Kan spectral sequence of this cosimplicial spectrum, see for instance [14, Remark 1.2.4.4], is the desired spectral sequence.

It converges (albeit only conditionally) to the homotopy groups of the totalization of the cosimplicial spectrum by (an opposite variant of) [14, Proposition 1.2.2.14]. Hence it remains to show that the E_2 page is of the desired form. If $C^*\colon \mathrm{Fun}(\Delta, \mathcal{A}\mathrm{b}) \to \mathrm{Ch}(\mathcal{A}\mathrm{b})^{\geq 0}$ denotes the cochain complex associated to a cosimplicial abelian group[5], then the second page of the Bousfield–Kan spectral sequence of a cosimplicial spectrum $M^\bullet$ may be expressed as cochain complex cohomology,

$$E_2^{s,t} = \mathrm{H}^s(C^*(\pi_t(M^\bullet))).$$

To apply this to the cosimplicial object in question, we must therefore determine the homotopy groups

$$\pi_t\Gamma(\check{\mathrm{C}}^\bullet(U \to X); \mathscr{F}|_{\check{C}^\bullet(U\to X)}) \simeq \pi_t\Gamma(U \times_X \cdots \times_X U; f^*(\mathscr{F})),$$

where $f\colon U \times_X \cdots \times_X U \to X$ is the canonical map. Since $U \to X$ is flat by assumption, the same holds for f, and so $f^* \circ \pi_t \simeq \pi_t \circ f^*$ — see [14, Proposition 7.2.2.13] for the affine case, from which it follows for an arbitrary flat morphism by the yoga of [15, Section 6.2.5]. Secondly, the fiber product $U \times_X \cdots \times_X U$ is affine by hypothesis, from which it follows that its global sections functor is t–exact. Putting all that together, we find that

$$\pi_t\Gamma(U \times_X \cdots \times_X U; f^*(\mathscr{F})) \simeq \Gamma\big(U \times_X \cdots \times_X U; f^*(\pi_t(\mathscr{F}))\big),$$

and so the E_2 page of the spectral sequence in question is just the standard Čech cohomology procedure for computing the s^{th} sheaf cohomology group of the quasicoherent sheaf $\pi_t(\mathscr{F})$ on X. □

Remark 3.2 Though the approach using a cover that we sketched above will be the most convenient for us in what follows, the descent spectral sequence does not depend on that choice from the second page onwards. It may alternatively even be obtained in an invariant way: the assumptions on the stack X ensure that $\mathrm{QCoh}(X)$ admits a well-behaved t–structure. Then the spectral sequence associated to the filtered object

[5]This is the functor that participates in one direction of the Dold–Kan correspondence; see [14, Definition 1.2.3.8] for the opposite version.

$\mathrm{N}(\boldsymbol{Z}) \ni n \mapsto \Gamma(\tau_{\geq -n}(X); \mathscr{F}) \in \mathrm{Sp}$ by [14, Definition 1.2.2.9] again gives rise to the descent spectral sequence after an appropriate reindexing; see [9, Construction 1.5.7] for details.

3.2 The Adams spectral sequence

We wish to apply the descent spectral sequence on the quotient stack $\mathrm{B}\mathbb{G}_n$, for which we need a quasicoherent sheaf on it. Consider the map $q\colon \mathrm{Spf}(E_n)/\mathbb{G}_n \to \mathrm{B}\mathbb{G}_n$, induced on quotients by the terminal projection $p\colon \mathrm{Spf}(E_n) \to *$. Using the pushforward functionality of quasicoherent sheaves, we define the desired sheaf as

$$\mathscr{E}_n := q_*(\mathcal{O}_{\mathrm{Spf}(E_n)/\mathbb{G}_n}) \in \mathrm{QCoh}(\mathrm{B}\mathbb{G}_n).$$

As we will need it in the subsequent proposition, let us identify the fiber of this quasicoherent sheaf at the point $i : * \to */\mathbb{G}_n \simeq \mathrm{B}\mathbb{G}_n$. By invoking base change along the pullback square

$$\begin{tikzcd} \mathrm{Spf}(E_n) \arrow[r, "p"] \arrow[d] & * \arrow[d, "i"] \\ \mathrm{Spf}(E_n)/\mathbb{G}_n \arrow[r, "q"] & \mathrm{B}\mathbb{G}_n \end{tikzcd}$$

we find this fiber to be

$$i^*(\mathscr{E}_n) \simeq i^* q_*(\mathcal{O}_{\mathrm{Spf}(E_n)/\mathbb{G}_n}) \simeq p_*(\mathcal{O}_{\mathrm{Spf}(E_n)}) \simeq E_n.$$

Proposition 3.3 *The descent spectral sequence for the quasicoherent sheaf $\mathscr{E}_n$ on $\mathrm{B}\mathbb{G}_n$ is isomorphic to*

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\pi_*(L_{K(n)}(E_n \otimes E_n))}(\pi_*(E_n), \pi_*(E_n)) \Rightarrow \pi_*(L_{K(n)}S),$$

the $K(n)$–local Adams spectral sequence for E_n.

Proof Observe that both spectral sequences in question may be obtained as Bousfield–Kan spectral sequences of certain cosimplicial spectra. Thus it suffices to exhibit an equivalence between those.

For the descent spectral sequence, we choose the flat cover $i : * \to \mathrm{B}\mathbb{G}_n$; indeed, this is a cover by the usual yoga of classifying stacks, and it is flat thanks to the Morava stabilizer group $\mathbb{G}_n$ being pro-étale and as such flat. Then the Čech nerve of i is given by $\check{\mathrm{C}}^\bullet(* \to \mathrm{B}\mathbb{G}_n) \simeq \mathbb{G}_n^{\times\bullet}$, and coincides with the bar construction of $\mathbb{G}_n$. Let $p_\bullet\colon \mathbb{G}_n^{\times\bullet} \to *$ denote the terminal map. Then it follows from the computation preceding the statement of the proposition that $\mathscr{F}|_{\check{\mathrm{C}}^\bullet(*\to \mathrm{B}\mathbb{G}_n)} \simeq p_\bullet^*(E_n)$, and so the cosimplicial spectrum that gives rise to the relevant descent spectral sequence is $\Gamma(\mathbb{G}^{\times\bullet}; p_\bullet^*(E_n))$, with the cosimplicial structure inherited from the bar construction of $\mathbb{G}_n$.

For the Adams spectral sequence, let us apply the functor $\mathcal{O}$ to the equivalence of simplicial objects of Proposition 2.12. We obtain an equivalence of cosimplicial spectra

$$L_{K(n)}(E_n^{\otimes(\bullet+1)}) \simeq \mathcal{O}(\mathbb{G}_n^\bullet \times \mathrm{Spf}(E_n)).$$

The left-hand side (for recognizing which, we have made use of a calculation from the proof of Theorem 2.19), gives rise to the $K(n)$–local Adams spectral sequence for E_n. To tackle the left-hand side, consider the Cartesian diagram

$$\begin{tikzcd} \mathbb{G}_n^{\times\bullet} \times \mathrm{Spf}(E_n) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{Spf}(E_n) \arrow[d, "p"] \\ \mathbb{G}_n^{\times\bullet} \arrow[r, "p_\bullet"] & * \end{tikzcd}$$

Using base change along it, we have a series of equivalences

$$\begin{aligned} \mathcal{O}(\mathbb{G}_n^{\times\bullet} \times \mathrm{Spf}(E_n)) &\simeq \Gamma(\mathbb{G}_n^{\times\bullet} \times \mathrm{Spf}(E_n); \mathcal{O}_{\mathbb{G}_n^{\times\bullet}\times\mathrm{Spf}(E_n)}) \\ &\simeq \Gamma(\mathbb{G}_n^{\times\bullet}; (\mathrm{pr}_1)_*\mathrm{pr}_2^*(\mathcal{O}_{\mathrm{Spf}(E_n)})) \\ &\simeq \Gamma(\mathbb{G}_n^{\times\bullet}; p_\bullet^* p_*(\mathcal{O}_{\mathrm{Spf}(E_n)})) \\ &\simeq \Gamma(\mathbb{G}_n^{\times\bullet}; p_\bullet^*(E_n)), \end{aligned}$$

and because the cosimplicial structure comes at each step from the bar construction on $\mathbb{G}_n$, this is an equivalence of cosimplicial spectra. Since we already saw that the thus-obtained cosimplicial spectrum gives rise to the descent spectral sequence for $\mathrm{B}\mathbb{G}_n$, we are done. □

Remark 3.4 By working in a nonformal setting, we may argue similarly to the above in order to obtain the Adams–Novikov spectral sequence as a special case of a descent spectral sequence — this is also explained in [15, Remark 9.3.1.9]. Indeed, consider the $\mathbb{E}_\infty$–ring MP, the periodic complex bordism spectrum. As we saw in Remark 2.11, it gives rise to a "based loop space" $\Omega_{\mathcal{X}_{\mathrm{MP}}}(\mathrm{Spec}(S))$ in nonconnective spectral stacks over Spec(MP). Let $\mathcal{X}$ denotes the classifying (pre)stack of this nonconnective spectral group scheme. Its underlying ordinary stack is given by

$$\mathcal{X}^\heartsuit \simeq \mathrm{Spec}(\pi_0(\mathrm{MP}))/\mathrm{Spec}(\pi_0(\mathrm{MP}\otimes\mathrm{MP})) \simeq \mathcal{M}_{\mathrm{FG}}^\heartsuit,$$

which is identified with the ordinary stack of formal groups by a celebrated theorem of Quillen. On the other hand, its (derived) $\mathbb{E}_\infty$–ring of functions is given by

$$\mathcal{O}(\mathcal{X}) \simeq \mathrm{Tot}(\mathrm{MP}^{\otimes(\bullet+1)}) = S^{\wedge}_{\mathrm{MP}} \simeq S,$$

which is by definition the MP–*nilpotent completion* of the sphere spectrum of [3], as already discussed in Remark 2.10. This nilpotent completion is well-known to agree with the sphere spectrum itself. Since the smash product $\mathrm{MP} \otimes \mathrm{MP}$ is a flat MP–module (see [17, Theorem 5.3.13]), a variant of Lemma 3.1 applies to the cover $\mathrm{Spec}(\mathrm{MP}) \to \mathcal{X}$. The resulting descent spectral sequence converges to $\pi_*(S)$, while by an argument analogous to our proof of Proposition 3.3, its second page is

$$E_2^{s,t} = \mathrm{H}^s(\mathcal{M}^{\heartsuit}_{\mathrm{FG}}; \pi_t(\mathcal{O}_{\mathcal{X}})) \simeq \mathrm{Ext}^{s,t}_{\pi_*(\mathrm{MP}\otimes\mathrm{MP})}(\pi_*(\mathrm{MP}), \pi_*(\mathrm{MP})),$$

viewable either as sheaf cohomology on the underlying ordinary stack, or as the Adams–Novikov spectral sequence. See [9] (where the content of this remark is expanded upon in [9, Section 2.6]) for a further development of these ideas.

3.3 Homotopy fixed-point spectral sequence

Let us say a few words about the interpretation of Proposition 3.3. We may view $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_n)$ as a version of continuous discrete representations of the Morava stabilizer group over the sphere spectrum. From that perspective, the underlying spectrum of a quasicoherent sheaf $\mathscr{F}$ on $\mathrm{B}\mathbb{G}_n$ is given by the fiber $i^*(\mathscr{F}) = M$ (keeping the notation $i : * \to \mathrm{B}\mathbb{G}_n$ from the previous subsection), and the sheaf structure on $\mathscr{F}$ witnesses the $\mathbb{G}_n$–action on M. The (continuous) fixed points of this action are incarnated as global sections $M^{h\mathbb{G}_n} := \Gamma(\mathrm{B}\mathbb{G}_n; \mathscr{F})$, and continuous group cohomology is given in terms of sheaf cohomology as

$$\mathrm{H}^i_{\mathrm{cont}}(\mathbb{G}_n; M) := \mathrm{H}^i(\mathrm{B}\mathbb{G}_n; \mathscr{F}) \simeq \pi_{-i}(M^{h\mathbb{G}_n}).$$

Under these identifications, the descent spectral sequence for $\mathrm{B}\mathbb{G}_n$ corresponds to the fixed-point spectral sequence for $\mathbb{G}_n$,

$$E_2^{s,t} = \mathrm{H}^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_t(E_n)) \Rightarrow \pi_{t-s}(E_n^{h\mathbb{G}_n}).$$

Remark 3.5 In line with the preceding discussion, the sheaf $\mathscr{E}_n$ on $\mathrm{B}\mathbb{G}_n$ encodes a continuous $\mathbb{G}_n$–action on the Lubin–Tate spectrum E_n. Its continuous homotopy fixed-points, in the above sense, are given by

$$E_n^{h\mathbb{G}_n} \simeq \Gamma(\mathrm{B}\mathbb{G}_n; \mathscr{E}_n) \simeq p_*q_*(\mathcal{O}_{\mathrm{Spf}(E_n)/\mathbb{G}_n}) \simeq \Gamma(\mathrm{Spf}(E_n)/\mathbb{G}_n; \mathcal{O}_{\mathrm{Spf}(E_n)/\mathbb{G}_n}).$$

That agrees with (and perhaps justifies) our definition in Section 2.4, and its use in the Devinatz–Hopkins Theorem 2.19 in particular.

Remark 3.6 There exist a number of precise incarnations of the ∞–category of continuous $\mathbb{G}_n$–spectra in the literature, eg of [2] or [21]. In each, the construction from [6] is enhanced (relying heavily on Devinatz and Hopkins's detailed study of finite subgroup actions) to produce a version of E_n in the respective category. Instead, we claim that $\mathrm{QCoh}(\mathrm{B}\mathbb{G}_n)$ should be viewed as an incarnation of continuous $\mathbb{G}_n$–spectra, sufficient for our purposes, but not intended to supplant the more sophisticated theories mentioned above (a careful comparison with which we decline to carry out).

Remark 3.7 In spite of the preceding remark, let us observe that our model at least gives rise to spectra with a $\mathbb{G}_n$–action in the sense of [1, Definition 2.2], referred to there as "a simple sense of continuity". Indeed, in light of Remark 2.17, a $\mathbb{G}_n$–action on M in our sense gives rise to an augmented cosimplicial diagram $M \to C^*_{\mathrm{cont}}(\mathbb{G}_n^{\times(\bullet+1)}; M)$. In fact, our approach to continuous $\mathbb{G}_n$–actions is, via the bar resolution $\mathrm{B}\mathbb{G}_n \simeq |\mathbb{G}_n^{\times\bullet}|$, essentially equivalent to the one of [1]. Their restriction to the $K(n)$–local setting is mirrored in our setup by working in the setting of formal algebraic geometry, ie inside the ∞–category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cN}}_{/\kappa}, \mathcal{S})$ instead of say $\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})$.

Remark 3.8 With M as in the previous remark, we find by unwinding the proof of Proposition 3.3 that the descent spectral sequence for the corresponding sheaf on $\mathrm{B}\mathbb{G}_n$ is obtained as the Bousfield–Kan spectral sequence of the cosimplicial object $C^*_{\mathrm{cont}}(\mathbb{G}_n^{\times\bullet}; M)$. That is also one traditional approach to defining the homotopy fixed-point spectral sequence (for a compact Lie group, say), somewhat justifying our identification of the two.

With all the notation in place, the following is a formal consequence of Proposition 3.3.

Corollary 3.9 (Morava's change-of-rings isomorphism) *The second page of the $K(n)$–local Adams spectral sequence of the Lubin–Tate spectrum E_n may be expressed as continuous group cohomology $E_2^{s,t} = \mathrm{H}^s_{\mathrm{cont}}(\mathbb{G}_n; \pi_t(E_n))$.*

Remark 3.10 One difference between our approach and [6] is that they make use of a form of Morava's change-of-rings isomorphism from [5] to set up their theory. For us, on the other hand, that result did not feed into the construction of $E_n^{h\mathbb{G}_n}$ nor its identification with $L_{K(n)}S$, and we could instead derive it from our considerations.

Of course, that is largely a cosmetic difference; Morava's theorem, even if classically phrased differently, ultimately boils down to algebrogeometric considerations regarding the moduli of formal groups of the sort that we based our approach on.

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Neighboring mapping points theorem

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We introduce and study a new family of theorems extending the class of Borsuk–Ulam and topological Radon type theorems. The defining idea for this new family is to replace requirements of the form "the image of a subset that is large in some sense is a singleton" with requirements of the milder form "the image of a subset that is large in some sense is a subset that is small in some sense". This approach covers the case of mappings $\mathbb{S}^m \to \mathbb{R}^n$ with $m < n$ and extends to wider classes of spaces.

An example of a statement from this new family is the following theorem. Let f be a continuous map of the boundary $\partial\Delta^n$ of the n–dimensional simplex Δ^n to a contractible metric space M. Then $\partial\Delta^n$ contains a subset E such that E (is "large" in the sense that it) intersects all facets of Δ^n and the image $f(E)$ (is "small" in the sense that it) is either a singleton or a subset of the boundary ∂B of a metric ball $B \subset M$ whose interior does not meet $f(\partial\Delta^n)$.

We generalize this theorem to noncontractible normal spaces via covers and deduce a series of its corollaries. Several of these corollaries are similar to the topological Radon theorem.

55M20, 55M25, 55P05

1 Introduction

We introduce and study a new family of theorems extending the class of Borsuk–Ulam- and topological Radon-type theorems (though none of our theorems is a generalization of the Borsuk–Ulam or topological Radon theorem itself). By the Borsuk–Ulam- and topological Radon-type theorems we mean those stating that a continuous map takes a "wide" set of some specific kind to a point. Let us list several of the most influential examples:

- *The Borsuk–Ulam theorem* itself says that every continuous map of a Euclidean n–sphere $\mathbb{S}^n$ into Euclidean n–space $\mathbb{R}^n$ identifies two antipodes.

- *The Hopf theorem* states that, if X is a compact Riemannian n–manifold and $f : X \to \mathbb{R}^n$ is a continuous map, then, for any $\delta > 0$, there exists a geodesic $\gamma : [0,\delta] \to X$ of length δ such that $f(\gamma(0)) = f(\gamma(\delta))$.
- *The topological Radon theorem* says that if P is a convex n–polytope, then any continuous map $\partial P \to \mathbb{R}^{n-1}$ identifies two points from disjoint faces.
- *The topological Tverberg theorem* says that, if $d \geq 1$ is an integer, r is a prime power and P is a convex $(r-1)(d+1)$–polytope, then any continuous map $\partial P \to \mathbb{R}^d$ identifies r points from r pairwise disjoint faces.

See eg Steinlein [35; 36], Matoušek [22], Karasev [17], Akopyan, Karasev and Volovikov [3], Frick [11], Bárány, Blagojević and Ziegler [5], Skopenkov [33] and Bárány and Soberón [6] for more examples, including various extensions and generalizations for $\mathbb{Z}_p$–spaces, maps between manifolds, matroids and colored versions. Another related family is Knaster's conjecture-type theorems (see Matschke [23]).

All of these examples involve rigid dimensional restrictions. It is a natural question whether the maps not satisfying these restrictions have any properties of the Borsuk–Ulam kind. In particular, we are interested in whether the Borsuk–Ulam theorem has any reasonable extensions to the case of mappings $\mathbb{S}^m \to \mathbb{R}^n$ with $m < n$ (a related idea appears in Adams, Bush and Frick [2]).

Extensions of this kind are found in a new class we study. This class emerges by replacing conditions of the form "*the image of a subset that is large in some sense is a singleton*" with conditions of the milder form "*the image of a subset that is wide in some sense is a subset that is restricted in some sense*". This approach covers the case of mappings $\mathbb{S}^m \to \mathbb{R}^n$ with $m < n$ and extends to wider classes of spaces.

Here is an example for the simplest nondegenerate case $\mathbb{S}^1 \to \mathbb{R}^2$:

Proposition 1 (Malyutin [20]) *Let a, b and c be three closed arcs covering the circle $\mathbb{S}^1$ such that no two of them cover $\mathbb{S}^1$, and let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a continuous map. Then either $f(a) \cap f(b) \cap f(c) \neq \varnothing$ or each of $f(a)$, $f(b)$ and $f(c)$ touches a closed Euclidean disk $D^2 \subset \mathbb{R}^2$ whose interior does not meet $f(\mathbb{S}^1)$.*

Proposition 1 works for plane curves and knot diagrams and has a corollary with applications in knot theory (see [20]). We formulate this corollary here. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ be a regular smooth plane curve in general position (that is, its only singularities are transversal double points). By an *edge* of γ we mean the closure of a component of the set $\gamma(\mathbb{S}^1) \setminus V$, where V is the set of double points of γ. We say that two edges I and J

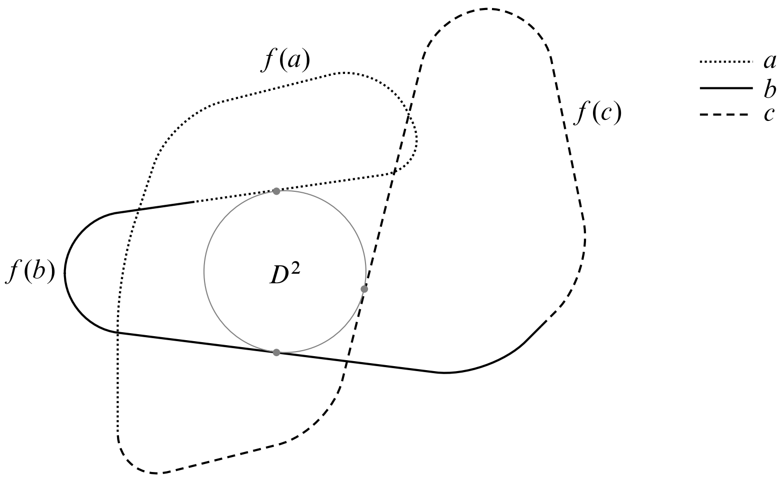

Figure 1: For Proposition 1. The circle ∂D^2 touches $f(a)$, $f(b)$ and $f(c)$

of γ are *neighboring edges* or *neighbors* if there exists a component Q of $\mathbb{R}^2 \setminus \gamma(\mathbb{S}^1)$ such that the boundary ∂Q contains both I and J. We say that two edges I and J of γ are *consecutive* if the union $I \cup J$ coincides with the image $\gamma(\alpha)$ of a (connected) arc α in $\mathbb{S}^1$. We denote by ρ the maximal metric on the set $E(\gamma)$ of edges of γ in the class of metrics satisfying the condition "$\rho(I, J) = 1$ whenever I and J are consecutive edges of γ".

Proposition 2 [20] *If the curve γ has k double points, then γ has a pair of neighboring edges I and J with $\rho(I, J) \geq \frac{2}{3}k$.*

Proposition 1 readily implies Proposition 2 if we choose the arcs a, b and c appropriately. Proposition 2 appears in [20] as an auxiliary lemma (Lemma 5.1) needed to obtain a series of statements related to knot theory. In [20], this lemma is deduced from the topological Helly theorem (see Bogatyĭ [7] and Montejano [24]). The statement of Proposition 2 was one of the starting points for our study.

Here, we generalize Proposition 1 to noncontractible normal spaces via covers. The generalizations and their corollaries will be formulated in the next sections, after definitions. Our method is based on obstruction theory and uses a variation of the concept of non-nullhomotopic covers introduced by Musin [27; 28].

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2 Definitions and results

Throughout this paper we mainly consider normal topological spaces,[1] all simplicial complexes and covers will be finite, all manifolds will be both compact and PL, $\mathbb{S}^n$ will denote the n–dimensional sphere, Δ^n will denote the n–dimensional simplex and $\mathrm{sk}_k(\Delta^n)$ will denote the k–skeleton of Δ^n. We shall denote the set of homotopy classes of continuous maps $V \to W$ by $[V, W]$. The nerve of a (finite) collection $\mathcal{S}$ of sets will be denoted by $\mathcal{N}(\mathcal{S})$. When this does not cause confusion we use the same notation for an abstract simplicial complex and its underlying space (carrier).

The further exposition in this section is structured as follows: first we give a chain of successively stronger generalizations of Proposition 1 (Theorem 4 is the weakest, Theorem 36 is the strongest); then we present a family of corollaries (all but one of which follow from Theorem 4).

2.1 Spherical f–neighbors

All of the following generalizations and corollaries replace the condition “the image is a singleton” appearing in the Borsuk–Ulam-type theorems with the following milder condition of “spherical neighboring”:

Definition 3 (spherical f–neighbors) Let X be a set, let Y be a metric space and let $f\colon X \to Y$ be a map. We say that a subset $N \subset X$ is a *set of spherical f–neighbors* if N contains at least two points and the image $f(N)$ is either a point or a subset of the boundary ∂B of a metric ball[2] $B \subset Y$ whose interior does not meet $f(X)$. If a two-point set $\{p, q\}$ is a set of spherical f–neighbors, we say that p and q are *spherical f–neighbors*. (See Figure 2.)

The first extension generalizes Proposition 1 to the case of spheres of arbitrary dimension and replaces Euclidean spaces with arbitrary contractible metric spaces. (Recall that *facets* of a polytope of dimension n are its faces of dimension $n-1$.)

[1] A topological space X is *normal* if any two disjoint closed sets of X are contained in disjoint open sets of X; see [32, page 446] for equivalent definitions via the Urysohn and shrinking lemmas.

[2] By a *metric ball* in a metric space (Y, d) with metric d we mean a subset of the form $\{y \in Y \mid d(y, x) \le R\}$ with $x \in Y$ and $R \ge 0$.

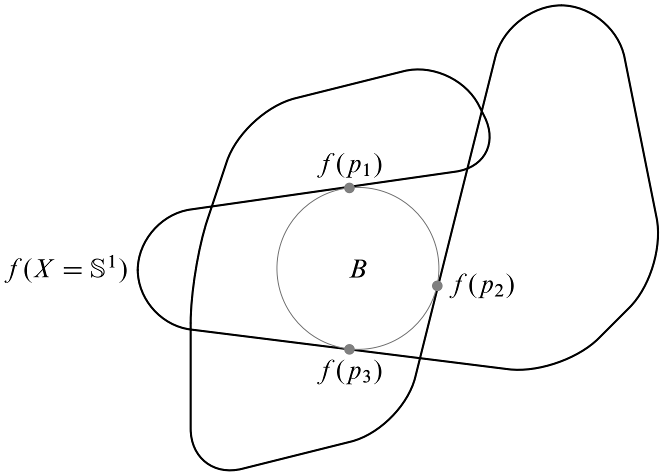

Figure 2: The image of a set $\{p_1, p_2, p_3\}$ of spherical f–neighbors

Theorem 4 *Let f be a continuous map of the boundary $\partial\Delta^n$ of the n–dimensional simplex Δ^n to a contractible metric space M. Then a set of spherical f–neighbors intersects all facets of Δ^n.*

Proof Let d denote the metric on M. If z is a point and N is a subset in M, we write

$$d(z, N) := \inf_{p \in N} d(z, p).$$

Let $\Delta_1, \dots, \Delta_{n+1}$ be the facets of Δ^n. For each $i \in \{1, \dots, n+1\}$, we set

$$E_i := \{z \in M \mid d(z, f(\Delta_i)) = d(z, f(\partial\Delta^n))\}.$$

Observe that E_i contains $f(\Delta_i)$ and is closed, so $\{E_1, \dots, E_{n+1}\}$ is a closed cover of M. Since M is contractible, f extends to a continuous map $F\colon \Delta^n \to M$. Then $\{F^{-1}(E_1), \dots, F^{-1}(E_{n+1})\}$ is a closed cover of Δ^n extending the closed cover $\{\Delta_1, \dots, \Delta_{n+1}\}$ of $\partial\Delta^n$. By the Knaster–Kuratowski–Mazurkiewicz (KKM) lemma, the elements of $\{F^{-1}(E_1), \dots, F^{-1}(E_{n+1})\}$ have a common point p. Then $x = F(p)$ lies in $E_1 \cap \dots \cap E_{n+1}$. Then either $x \in f(\partial\Delta^n)$, so that x belongs to all of $f(\Delta_i)$ by the definition of E_i; or $r = d(x, f(\partial\Delta^n)) > 0$ and the ball $B_r(x)$ of radius r centered at x touches all of $f(\Delta_i)$ while its interior does not meet $f(\partial\Delta^n)$. □

We generalize Theorem 4 by replacing the set of facets with a more general class of covers as in the KKM lemma.

2.2 KKM covers and spherical f–neighbors

Definition 5 (KKM covers) Let Δ^{n+1} be an $(n+1)$–dimensional simplex with vertices labeled $v_1, \dots, v_{n+2}$. A closed cover $\{C_1, \dots, C_{n+2}\}$ of the n–sphere $\mathbb{S}^n$ is called a *KKM cover* if there exists a homeomorphism $h\colon \mathbb{S}^n \to \partial\Delta^{n+1}$ such that, for each $J \subset \{1, \dots, n+2\}$, the convex hull of the vertices v_j with $j \in J$ is covered by the union $\bigcup_{j\in J} h(C_j)$.

The argument in the proof of Theorem 4 also proves the following theorem:

Theorem 6 *Let $\mathcal{C}$ be a KKM cover of the n–sphere $\mathbb{S}^n$, and let $f\colon \mathbb{S}^n \to M$ be a continuous map to a contractible metric space M. Then a set of spherical f–neighbors intersects all elements of $\mathcal{C}$.*

The key role in Theorem 6 is played by the properties of the cover, and not by the fact that the underlying space is a sphere. To move on to the next generalization, we define non-nullhomotopic covers (we generalize the concept of non-nullhomotopic covers given in [27; 28]).

2.3 Non-nullhomotopic covers and spherical f–neighbors

Let X be a normal topological space and let $\mathcal{U} = \{U_1, \dots, U_n\}$ be an open cover of X. Let $\mathcal{N}(\mathcal{U})$ be the nerve of $\mathcal{U}$. Let $\Phi = \{\varphi_1, \dots, \varphi_n\}$ be a partition of unity subordinate to $\mathcal{U}$. Let $v_1, \dots, v_n$ be the vertices of the $(n-1)$–dimensional unit simplex Δ^{n-1}, where

$$\Delta^{n-1} := \{x \in \mathbb{R}^n \mid x_i \geq 0,\ x_1 + \dots + x_n = 1\}.$$

For each i, we identify the vertex of $\mathcal{N}(\mathcal{U})$ corresponding to U_i with v_i, so that $\mathcal{N}(\mathcal{U})$ becomes a subcomplex of Δ^{n-1}. We set

$$h_{\mathcal{U},\Phi}(x) := \sum_{i=1}^{n} \varphi_i(x) v_i.$$

Then $h_{\mathcal{U},\Phi}$ is a continuous map from X to $\mathcal{N}(\mathcal{U}) \subset \Delta^{n-1}$. Since the linear homotopy $\Theta(t) = (1-t)\Phi + t\Psi$ of two partitions of unity Φ and Ψ subordinate to $\mathcal{U}$ induces a homotopy between the corresponding maps, it follows that the homotopy class $[h_{\mathcal{U},\Phi}]$ in $[X, \mathcal{N}(\mathcal{U})]$, where by $[V, W]$ we denote the set of homotopy classes of continuous maps $V \to W$, does not depend on Φ (see [27, Lemma 1.6]). We denote this class in $[X, \mathcal{N}(\mathcal{U})]$ by $[\mathcal{U}]$.

The homotopy classes of covers are also well defined for closed sets. Indeed, in a normal space, any finite closed cover has an open extension with the same nerve (see eg [25, Theorem 1.3; 16, pages 31–33]). Furthermore, if $\mathcal{C} = \{C_1, \dots, C_n\}$ is a closed cover of a normal space X and $\mathcal{S} = \{S_1, \dots, S_n\}$ and $\mathcal{U} = \{U_1, \dots, U_n\}$ are two open covers such that $S_i \cap U_i$ contains C_i for all i that have the same nerve $\mathcal{N}(\mathcal{S}) = \mathcal{N}(\mathcal{U}) = \mathcal{N}(\mathcal{C})$, then each partition of unity subordinate to the open cover

$$\mathcal{T} := \{S_1 \cap U_1, \dots, S_n \cap U_n\}$$

is also subordinate to both $\mathcal{S}$ and $\mathcal{U}$. This implies that $[\mathcal{S}] = [\mathcal{T}] = [\mathcal{U}]$ in $[X, \mathcal{N}(\mathcal{C})]$ due to the independence of the choice of partition of unity mentioned above. Then we set

$$[\mathcal{C}] := [\mathcal{S}] = [\mathcal{T}] = [\mathcal{U}].$$

Definition 7 (non-nullhomotopic covers) We say that an open or closed cover $\mathcal{C}$ of a normal topological space X is *non-nullhomotopic* if the corresponding homotopy class $[\mathcal{C}]$ in $[X, \mathcal{N}(\mathcal{C})]$ contains no constant map.

Remark 8 Any non-nullhomotopic map $X \to K$ to a finite simplicial complex yields non-nullhomotopic covers on X; to obtain an example, take the inverse images of all elements in one of the collections

- open stars of vertices of K,
- stars of vertices of K in its first barycentric subdivision,
- maximal simplexes of K.

Definition 9 (homotopy ranks of maps) Let X be a topological space, let K be a finite simplicial complex and let $h\colon X \to K$ be a continuous map. Let Δ_K be the simplex spanned by the vertices of K, so K is a subcomplex of Δ_K. We define the *homotopy rank* $\operatorname{rk}(h)$ of h to be the least nonnegative integer k such that h is nullhomotopic in $K \cup \operatorname{sk}_k(\Delta_K)$, where sk_k stands for the k–skeleton.[3] (Since Δ_K is contractible, the homotopy rank is well defined and does not exceed the dimension of Δ_K.)

Remark 10 In terms of Definition 9, h is nullhomotopic if and only if $\operatorname{rk}(h) = 0$.

Definition 11 (ranks of covers) We define the (*homotopy*) *rank* $\operatorname{rk}(\mathcal{C})$ of a (closed or open) finite cover $\mathcal{C}$ of a normal space X to be the homotopy rank of maps $X \to \mathcal{N}(\mathcal{C})$ in the class $[\mathcal{C}]$ determined by $\mathcal{C}$.

Remark 12 A cover is non-nullhomotopic if and only if it is of nonzero rank.

[3] We say that $K \cup \operatorname{sk}_k(\Delta_K)$ is the *k–exoskeleton* of K.

Remark 13 Since $\mathrm{sk}_m(\Delta^m) = \Delta^m$ is contractible, it follows that the rank of an n–element cover does not exceed $n-1$.

Definition 14 (principal covers) An n–element cover ($n \geq 2$) of rank $n-1$ is said to be *principal*.

Remark 15 Since any proper nonempty subcomplex of $\partial\Delta^m$ is contractible in $\partial\Delta^m$, it follows that a cover is principal if and only if it is non-nullhomotopic and its nerve is the boundary of a simplex.

Remark 16 Remark 15 implies that no principal cover has a proper subcollection of elements with empty intersection; in particular, no principal cover has disjoint elements.

Remark 17 Any non-nullhomotopic map $X \to \mathbb{S}^k$ to the k–sphere yields a principal cover of X of rank $k+1$ (see Remark 8 and [27, Theorem 1.5]). Thus, a space can have principal covers of distinct ranks.

Remark 18 (conditions for cover non-nullhomotopicity) If the composition of continuous maps is non-nullhomotopic, then each of them is non-nullhomotopic.

- On the one hand, this implies that any refinement of a cover of rank k has rank at least k. In particular, any refinement of a non-nullhomotopic cover is non-nullhomotopic.
- On the other hand, this implies that, if $f\colon X \to Y$ is a continuous map of normal spaces and $\mathcal{C} = \{C_1, \dots, C_n\}$ is a closed cover of Y such that the dimension of the nerve $\mathcal{N}(\mathcal{C})$ is less than the rank $\mathrm{rk}(\widetilde{\mathcal{C}})$ of the induced cover $\widetilde{\mathcal{C}} = \{f^{-1}(C_1), \dots, f^{-1}(C_n)\}$, then $\mathrm{rk}(\mathcal{C}) \geq \mathrm{rk}(\widetilde{\mathcal{C}})$. In particular, if the induced cover is principal and $\bigcap_{i=1}^n C_i = \varnothing$, then $\mathcal{C}$ is principal. (See Lemma 52.)

We now have all the definitions needed to replace spheres in Theorem 6 with general "noncontractible" spaces.

Theorem 19 *Let X be a compact normal space, let M be a contractible metric space and let $f\colon X \to M$ be a continuous map. Then, for any non-nullhomotopic cover $\mathcal{C}$ of X, a set of spherical f–neighbors intersects at least $\mathrm{rk}(\mathcal{C})+1$ elements of $\mathcal{C}$. In particular, for any principal cover, a set of spherical f–neighbors intersects all elements of the cover.*

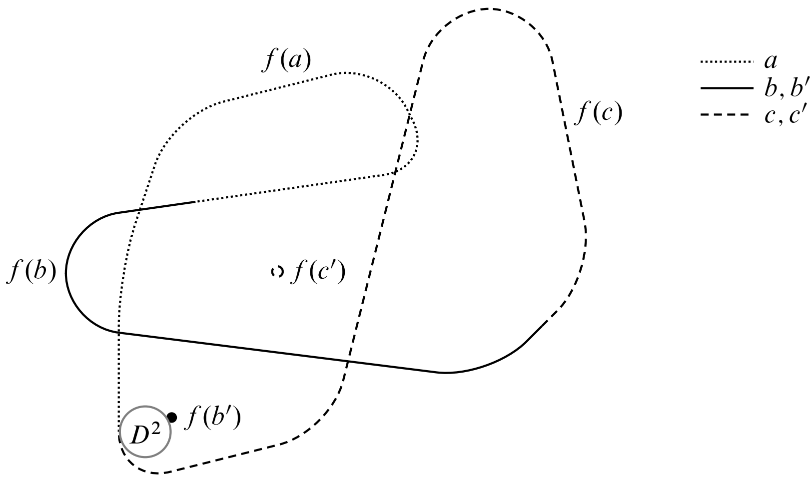

Figure 3: An example with disconnected $X = \mathbb{S}^1 \cup \{b', c'\}$.

Theorem 19 implies Theorem 6 because each KKM cover either is principal or all of its elements have a common point; furthermore, the maps in the homotopy class $[\mathcal{C}]$ corresponding to each principal KKM cover $\mathcal{C}$ are of degree one, so $[\mathcal{C}]$ contains a homeomorphism (see [28, Corollaries 2.1–2.3]).

Remark 20 X in Theorem 19 is not assumed to be connected. Figure 3 shows an example with $X = \mathbb{S}^1 \cup \{b', c'\}$ (compare Figure 1).

Remark 21 Combining the idea that X in Theorem 19 is not necessarily connected with switching attention to the image of the cover leads to generalizations of Helly's theorem and the KKM lemma. See also Lemma 52 below. We do not develop this line here.

2.4 EP triples and ranks, and spherical f–neighbors

We are going to upgrade Theorem 19 to the more general Theorem 36, which covers the case of maps to not necessarily contractible spaces. In order to state and prove Theorem 36, we introduce concepts of Eilenberg–Pontryagin and Knaster–Kuratowski–Mazurkiewicz ranks.

Definition 22 (Eilenberg–Pontryagin triples and ranks) Let Z be a topological space with a subspace A, let K be a finite simplicial complex and let $[h]$ be a homotopy class in $[A, K]$. We say that $(Z, A, [h])$ is an *Eilenberg–Pontryagin triple* if no map in $[h]$ extends to a continuous map $Z \to K$.

We define the *Eilenberg–Pontryagin rank* (*EP rank*) $\mathrm{rk}(Z, A, [h])$ of the triple $(Z, A, [h])$ to be the least nonnegative integer k such that there exists a continuous map $H\colon Z \to K \cup \mathrm{sk}_k(\Delta_K)$ whose restriction $H|_A$ is homotopic in $K \cup \mathrm{sk}_k(\Delta_K)$ to the maps of $[h]$, where Δ_K is the simplex spanned by the vertices of K and containing K as a subcomplex. (Since Δ_K is contractible, the EP rank is well defined and does not exceed the dimension of Δ_K.)

Remark 23 In terms of Definition 22, a triple $(Z, A, [h])$ is Eilenberg–Pontryagin if and only if it is of nonzero EP rank (because $K \cup \mathrm{sk}_0(\Delta_K) = K$).

Remark 24 Since any constant map extends to any ambient space, it follows that in terms of Definitions 9 and 22, for any Z, A, K and h,

$$\mathrm{rk}(Z, A, [h]) \le \mathrm{rk}(h).$$

Furthermore, if A is contractible in Z, then

$$\mathrm{rk}(Z, A, [h]) = \mathrm{rk}(h).$$

In particular, if $\mathcal{C}$ is a finite closed cover of A and A is contractible in Z, then

$$\mathrm{rk}(Z, A, [\mathcal{C}]) = \mathrm{rk}(\mathcal{C}).$$

Example 25

- If $Z = \Delta^n$, $A = K = \partial\Delta^n$ and $h = \mathrm{id}$, then $\mathrm{rk}(Z, A, [h]) = \mathrm{rk}(h) = n$.
- If $Z = K = \Delta^n$, $A = \partial\Delta^n$, and $h = \mathrm{id}$, then $\mathrm{rk}(Z, A, [h]) = \mathrm{rk}(h) = 0$.

Example 26 We have $\mathrm{rk}(Z, A, [h]) = 0$ whenever A is a retract of Z.

Example 27 Let W be an orientable, compact PL m–manifold with connected nonempty boundary ∂W, and let $h\colon \partial W \to \partial\Delta^n$ be a continuous map. Then $\mathrm{rk}(h) \in \{0, n\}$, $\mathrm{rk}(W, \partial W, [h]) \in \{0, n\}$ and $\mathrm{rk}(W, \partial W, [h]) \le \mathrm{rk}(h)$.

- If $m = n$, then $\mathrm{rk}(W, \partial W, [h]) = \mathrm{rk}(h)$ (this follows from the Hopf degree theorem; see the proof of Corollary 39 below).
- If $W = \Delta^m$, then $\mathrm{rk}(W, \partial W, [h]) = \mathrm{rk}(h)$ (because Δ^m is contractible; see Remark 24).
- Results of [29] imply however that, for any m and n with nontrivial $\pi_{m-1}(\mathbb{S}^{n-1})$ and for any non-nullhomotopic $h\colon \mathbb{S}^{m-1} \to \partial\Delta^n$, there exists an m–manifold W with $\partial W = \mathbb{S}^{m-1}$ such that h extends to a continuous map $W \to \partial\Delta^n$, so $\mathrm{rk}(W, \partial W, [h]) = 0$ and $\mathrm{rk}(h) = n$.

Definition 28 (Knaster–Kuratowski–Mazurkiewicz rank) Let Z be a topological space and let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a collection of subsets in Z. We say that the pair $(Z, \mathcal{S})$ is a *Knaster–Kuratowski–Mazurkiewicz* (*KKM*) *system* if no closed cover $\{E_1, \dots, E_n\}$ of Z with $S_i \subset E_i$ for all i has the same nerve as $\mathcal{S}$.

We define the *KKM rank* $\operatorname{rk}(Z, \mathcal{S})$ of the pair $(Z, \mathcal{S})$ to be the least integer k such that there exists a closed cover $\mathcal{E} = \{E_1, \dots, E_n\}$ of Z with $S_i \subset E_i$ for all i such that the dimension of $\mathcal{N}(\mathcal{E}) \setminus \mathcal{N}(\mathcal{S})$ is k.

Remark 29 In terms of Definition 28, a pair $(Z, \mathcal{S})$ is a KKM system if and only if it is of nonzero KKM rank.

Example 30 If $Z = \Delta^m$ and $\mathcal{S} = \{S_1, \dots, S_{m+1}\}$ is a KKM cover of $\partial\Delta^m$, then $\operatorname{rk}(Z, \mathcal{S}) = m$ by the KKM lemma.

Example 31 We have $\operatorname{rk}(Z, \mathcal{S}) = 0$ whenever $\mathcal{S}$ is a closed cover of a retract of Z.

Example 32 We have $\operatorname{rk}(Z, \{S_1, \dots, S_n\}) = 0$ whenever $\bigcap_{i=1}^n S_i \neq \varnothing$.

Lemma 33 *Let a normal space Z contain a normal space A as a subspace, let $\mathcal{C}$ be a closed cover of A, and let $[\mathcal{C}]$ be the corresponding homotopy class in $[A, \mathcal{N}(\mathcal{C})]$, where $\mathcal{N}(\mathcal{C})$ is the nerve. Then the EP rank of the triple $(Z, A, [\mathcal{C}])$ does not exceed the KKM rank of the system $(Z, \mathcal{C})$:*

$$\operatorname{rk}(Z, A, [\mathcal{C}]) \le \operatorname{rk}(Z, \mathcal{C}).$$

Furthermore, if A is closed in Z, then

$$\operatorname{rk}(Z, A, [\mathcal{C}]) = \operatorname{rk}(Z, \mathcal{C}).$$

Lemma 33 is proved in the next section.

Example 34 (showing that the closedness requirement of A in the second part of Lemma 33 is essential) If X is a compact normal space, $\mathcal{C}' = \{C'_1, \dots, C'_n\}$ is a closed cover of X with $\bigcap_{i=1}^n C_i = \varnothing$ and each C'_i nonempty, Z is the cone over X, z_0 is the top of Z, $A = Z \setminus \{z_0\}$, C''_i is the subcone in Z over C'_i, $C_i = C''_i \setminus \{z_0\}$, and $\mathcal{C} = \{C_1, \dots, C_n\}$, then the KKM rank $\operatorname{rk}(Z, \mathcal{C})$ is $n-1$ and the EP rank $\operatorname{rk}(Z, A, [\mathcal{C}])$ is one more than the dimension of the nerve $\mathcal{N}(\mathcal{C}')$. (See Figure 4 with $X = \mathbb{S}^1$.) For example, if $n > 2$ and the elements of $\mathcal{C}'$ are pairwise disjoint, then

$$\operatorname{rk}(Z, A, [\mathcal{C}]) = 1 < n - 1 = \operatorname{rk}(Z, \mathcal{C}).$$

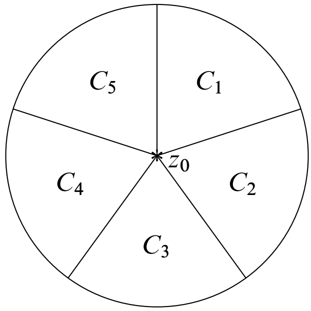

Figure 4: A disk with $\mathrm{rk}(Z, A, [\mathcal{C}]) = 2 < 4 = \mathrm{rk}(Z, \mathcal{C})$.

Remark 35 Lemma 33 implies (see Remark 24) that, given a compact normal space A with a finite closed cover $\mathcal{C}$, for any ambient normal space $Z \supset A$, $\mathrm{rk}(Z, A, [\mathcal{C}]) = \mathrm{rk}(Z, \mathcal{C}) \le \mathrm{rk}(\mathcal{C})$, while $\mathrm{rk}(Z, A, [\mathcal{C}]) = \mathrm{rk}(Z, \mathcal{C}) = \mathrm{rk}(\mathcal{C})$ if A is contractible in Z. (This generalizes [27, Theorem 2.2].)

Theorem 36 *Let A be a compact normal space, let $\mathcal{C}$ be a closed cover of A, and let $[\mathcal{C}]$ be the corresponding homotopy class in $[A, \mathcal{N}(\mathcal{C})]$, where $\mathcal{N}(\mathcal{C})$ is the nerve. Let Z be a normal space containing A as a subspace. If the triple $(Z, A, [\mathcal{C}])$ is Eilenberg–Pontryagin, with EP rank $\mathrm{rk}(Z, A, [\mathcal{C}]) > 0$, then, for any metric space M and any continuous map $f : A \to M$ that extends to a continuous map $Z \to M$, a set of spherical f–neighbors intersects at least $\mathrm{rk}(Z, A, [\mathcal{C}]) + 1$ elements of $\mathcal{C}$.*

Theorem 36 is proved in the next section.

Proof of Theorem 19 We deduce Theorem 19 from Theorem 36. Let X, M, f and $\mathcal{C}$ be as in Theorem 19. Set $\mathrm{Cone}(X) := (X \times [0, 1])/(X \times \{0\})$ and identify X with $X \times \{1\} \subset \mathrm{Cone}(X)$. (The cone is normal because X is compact and normal; see eg [30].) Definitions of ranks imply (see Remark 24) that

$$\mathrm{rk}(\mathrm{Cone}(X), X, [\mathcal{C}]) = \mathrm{rk}(\mathcal{C}). \tag{$***$}$$

In particular, the triple $(\mathrm{Cone}(X), X, [\mathcal{C}])$ is Eilenberg–Pontryagin since $\mathcal{C}$ is non-nullhomotopic. Since M is contractible, it follows that f extends to a continuous map $F : \mathrm{Cone}(X) \to M$. We apply Theorem 36 to the Eilenberg–Pontryagin triple $(\mathrm{Cone}(X), X, [\mathcal{C}])$, with $Z = \mathrm{Cone}(X)$ and $A = X$ in the notation of Theorem 36, and see that a set of spherical f–neighbors intersects at least $\mathrm{rk}(\mathrm{Cone}(X), X, [\mathcal{C}]) + 1$ elements of $\mathcal{C}$. Then Theorem 19 follows by $(***)$. $\square$

Remark 37 Theorem 36 has further refinements regarding the number of distinct sets of spherical f–neighbors intersecting the prescribed number of cover elements, but we do not develop this line here.

2.5 Corollaries

Next, we list several corollaries of Theorems 4, 6, 19 and 36. In fact, all of the following corollaries, except for Corollary 39, follow from Theorem 4.

Definition 38 A continuous map $f\colon A \to Y$ of an orientable, connected, closed PL manifold A to a space Y is said to be *null-cobordant* if there exists an orientable, compact PL manifold W with $\partial W = A$ and a continuous map $F\colon W \to Y$ such that $F|_A = f$.

Corollary 39 (see [27, Theorem 2.6]) *Let A be an orientable, connected, closed PL n–manifold and let $\mathcal{C}$ be a non-nullhomotopic cover of A such that the nerve of $\mathcal{C}$ is homeomorphic to the n–sphere. Then, for any metric space M and any null-cobordant map $f\colon A \to M$, a set of spherical f–neighbors intersects at least $n+2$ elements of $\mathcal{C}$. In particular, if $\mathcal{C}$ is principal and contains precisely $n+2$ elements, then a set of spherical f–neighbors intersects all elements of $\mathcal{C}$.*

Proof If $f\colon A \to M$ is null-cobordant, then there is an orientable, compact PL $(n+1)$–manifold Z with $\partial Z = A$ and a continuous map $F\colon Z \to M$ with $F|_A = f$. A homological argument shows that, for each continuous map $H\colon Z \to \mathcal{N}(\mathcal{C}) \cong \mathbb{S}^n$, the restriction $H|_A$ is of zero degree. Then the Hopf degree theorem implies that $H|_A$ is nullhomotopic. This means that the triple $(Z, A, [\mathcal{C}])$ is Eilenberg–Pontryagin and the statement follows by Theorem 36. □

Remark 40 (the dimensional restriction in Corollary 39 is essential) It is shown in [29] that any continuous map $\mathbb{S}^m \to \mathbb{S}^n$ is null-cobordant if $m > n$. Let m and n be such that $m > n$ and $\pi_m(\mathbb{S}^n)$ is nontrivial, and let $h\colon \mathbb{S}^m \to \partial\Delta^{n+1}$ be a non-nullhomotopic continuous map. Then there exists an orientable, compact PL $(m+1)$–manifold W with $\partial W = \mathbb{S}^m$ and a continuous map $H\colon W \to \partial\Delta^{n+1}$ such that $H|_{\partial W} = h$. Let $\mathcal{C}$ be the closed cover of ∂W composed of the inverse images of the facets of Δ^{n+1}. Then $[\mathcal{C}] = [h]$ and $\mathcal{C}$ is principal. We embed W into a Euclidean ball B^N of large dimension and "tiny" diameter, then embed W into the product $\partial\Delta^{n+1} \times B^N$ such that the projection of this embedding to $\partial\Delta^{n+1}$ yields H, and take the induced metric on W.

Now, let $f\colon \partial W \to W$ be the identity map. Then f is null-cobordant but no set of spherical f–neighbors intersects all elements of $\mathcal{C}$ if the diameter of B^N is sufficiently small.

Corollary 41 *Let M be a contractible metric space, let $\mathbb{S}^n$ be the Euclidean unit n–sphere in Euclidean $(n+1)$–space $\mathbb{R}^{n+1}$ and let $f\colon \mathbb{S}^n \to M$ be a continuous map. Then there exists a pair $\{p,q\}$ of spherical f–neighbors such that the Euclidean distance $\|p-q\|$ is at least $\sqrt{(n+2)/n}$.*

Corollary 41 is proved in the next section.

Remark 42 In [21] we show that, if $M=\mathbb{R}^m$ with $m>n$, then the constant $\sqrt{(n+2)/n}$ in Corollary 41 (the Euclidean distance between the centers of adjacent $(n-1)$–simplices of the regular triangulation of $\mathbb{S}^n$) can be replaced with $\sqrt{2(n+2)/(n+1)}$ (the Euclidean distance between vertices of the regular triangulation of $\mathbb{S}^n$), which is the best possible. Our proof for the Euclidean case $M=\mathbb{R}^m$ is based on the Delaunay triangulations and we do not know whether it extends to all contractible M.

Corollary 43 *Let M be a contractible metric space, let P be a convex n–polytope and let $f\colon \partial P \to M$ be a continuous map. Then a set of spherical f–neighbors intersects at least $n+1$ facets of P.*

Proof via Theorem 4 Corollary 43 is an "equivalent generalization" of Theorem 4 because the $(n-2)$–skeleton of any convex n–polytope contains the $(n-2)$–skeleton of the n–simplex as a topological subspace (see [14]). □

Proof via Theorem 19 Clearly, the cover $\mathcal{C}$ of ∂P composed of the facets of P is non-nullhomotopic of rank n because $\mathcal{C}$ is a *good* cover (that is, any intersection of elements in $\mathcal{C}$ is contractible), so the nerve of $\mathcal{C}$ has homotopy type of $\partial P \cong \mathbb{S}^{n-1}$ by the nerve theorem, while the maps in the class $[\mathcal{C}]$ are homotopy equivalences. Then a set of spherical f–neighbors intersects at least $n+1$ facets of P by Theorem 19. □

Since any collection of $n+1$ facets of the n–cube contains a pair of antipodal facets, Corollary 43 implies the following:

Corollary 44 *Let M be a contractible metric space, let $\partial[0,1]^m$ be the boundary of the m–dimensional cube $[0,1]^m$ and let $f\colon \partial[0,1]^m \to M$ be a continuous map. Then there is a pair of spherical f–neighbors intersecting antipodal facets of $[0,1]^m$.*

There exists an example of continuous map $\mathbb{S}^2 \to \mathbb{R}^3$ showing that the statement of Corollary 44 about spherical f–neighbors lying on antipodal facets holds for neither regular octahedra nor regular dodecahedra nor regular icosahedra. A weaker version of Corollary 44 where "antipodal" is replaced with "disjoint" holds for many polytopes.

2.6 Radon-type theorems

Definition 45 (weak Radon polytopes) We say that an n–polytope P is *weakly Radon* if, for any continuous map $f : \partial P \to M$ into any contractible metric space M, there is a pair of spherical f–neighbors intersecting two disjoint faces of P.

We recall some standard definitions. A *flag* polytope is a convex polytope such that every collection of pairwise intersecting facets has a nonempty intersection. A (*combinatorial*) *fullerene* is a simple 3–polytope with all facets pentagons and hexagons.

A "visual" simply checked sufficient condition for weakly Radon polytopes is provided by the so-called *belts*. A *k–belt* (or a *prismatic k–circuit*) in a 3–polytope is a cyclic sequence $(F_1, \dots, F_k)$ of $k \geq 3$ facets in which pairs of consecutive facets (including $\{F_k, F_1\}$) are adjacent, other pairs of facets do not intersect, and no three facets have a common vertex.

Corollary 46 (1) *If the* $(n-2)$*–skeleton of a convex* n*–polytope* P *contains the* $(n-2)$*–skeleton of the* n*–cube as a topological subspace, then* P *is weakly Radon.*

(2) *Each convex* 3*–polytope having a* k*–belt with* $k \geq 4$ *is weakly Radon.*

(3) *Each flag* 3*–polytope is weakly Radon.*

(4) *Each fullerene is weakly Radon.*

(5) *The regular dodecahedron and the regular icosahedron are weakly Radon.*

Proof Assertion (1) follows from Corollary 44 in an obvious way. Assertions (2) and (5) directly follow from assertion (1). Assertion (3) follows from Corollary 51(2) below. Assertion (4) is a particular case of assertion (3). □

Definition 47 (weak Radon rank) If P is an n–polytope, Y is a metric space and $f : \partial P \to M$ is a map, we say that two facets F_1 and F_2 of P are *spherical f–neighbors* (or that the pair $\{F_1, F_2\}$ is a *pair of spherical f–neighbors*) if there is a pair $\{p, q\}$ of spherical f–neighbors with $p \in F_1$ and $q \in F_2$. We say that $f : \partial P \to M$ *has weak Radon rank m* if there are exactly m distinct pairs of facets of P such that each of these pairs is a pair of disjoint spherical f–neighbors. By the *weak Radon rank* of a polytope P we mean the least of the weak Radon ranks of continuous maps $f\colon \partial P \to M$ into contractible metric spaces.

Corollary 44 allows us to obtain rough lower bounds on the weak Radon rank.

Definition 48 (cubic hemisphere) Let H be a subset of the boundary ∂P of a convex n–polytope P. We say that H is a *cubic hemisphere* if there exists a homeomorphism $h\colon [0,1]^n \to P$ such that the restriction of h to the $(n-2)$–skeleton of $[0,1]^n$ is a topological embedding to the $(n-2)$–skeleton of P and H is the image of the union of n facets of $[0,1]^n$ that have a common vertex.

Definition 49 (lighthouse independence number) We say that a set Z of vertices of an n–polytope is *lighthouse independent* if no two vertices in Z share a facet (equivalently, the corresponding facets of the dual polytope are pairwise disjoint). The *lighthouse independence* number $\operatorname{lin}(P)$ of an n–polytope P is the cardinality of a largest lighthouse independent set of P.

Remark 50 The lighthouse independence number of an n–polytope equals the cardinality of a largest set of pairwise disjoint facets of the dual polytope.

Corollary 51 (1) *Let P be a convex n–polytope. If ∂P contains k cubic hemispheres with pairwise disjoint interiors, then the weak Radon rank of P is at least $\frac{1}{2}k$.*

(2) *Let P be a flag 3–polytope (eg a fullerene). Then the weak Radon rank of P is at least half the lighthouse independence number of P.*

(3) *If P is a flag simple 3–polytope with ψ facets and g is the largest number of edges in a facet of P, then the weak Radon rank of P is at least*

$$\frac{1}{2}\left\lfloor \frac{2\psi-7}{3g-8} \right\rfloor,$$

where $\lfloor\cdot\rfloor$ stands for the floor function.

(4) *If P is a fullerene with ψ facets, then the weak Radon rank of P is at least*

$$\tfrac{1}{2}\left\lfloor \tfrac{1}{5}(\psi-3) \right\rfloor.$$

(5) *The weak Radon rank of the regular dodecahedron is at least 2.*

(6) *The weak Radon rank of the regular icosahedron is at least 2.*

(7) *The weak Radon rank of the cube is 1.*

Corollary 51 is proved in the next section.

3 Proofs

Proof of Lemma 33 (1) We show that $\operatorname{rk}(Z, A, [\mathcal{C}]) \le \operatorname{rk}(Z, \mathcal{C})$.

Let $\mathcal{C} = \{C_1, \dots, C_n\}$, let Δ^{n-1} denote the simplex spanned by the vertices of the nerve $\mathcal{N}(\mathcal{C})$ of $\mathcal{C}$, so $\mathcal{N}(\mathcal{C})$ is a subcomplex of Δ^{n-1}, and let $r := \operatorname{rk}(Z, \mathcal{C})$. By the definition of the KKM rank, there exists a closed cover $\mathcal{E} = \{E_1, \dots, E_n\}$ of Z with $C_i \subset E_i$ for all i such that the dimension of $\mathcal{N}(\mathcal{E}) \setminus \mathcal{N}(\mathcal{C})$ is r. Therefore, the union $\mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_r(\Delta^{n-1})$ contains $\mathcal{N}(\mathcal{E})$. Set

$$\mathcal{E}_A := \{E_1 \cap A, \dots, E_n \cap A\}.$$

Since $C_i \subset E_i$ for all i, it follows that the nerve $\mathcal{N}(\mathcal{E}_A)$ contains $\mathcal{N}(\mathcal{C})$. We have

$$\mathcal{N}(\mathcal{C}) \subset \mathcal{N}(\mathcal{E}_A) \subset \mathcal{N}(\mathcal{E}) \subset \mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_r(\Delta^{n-1}).$$

Let $[\mathcal{E}]$ be the homotopy class in $[Z, \mathcal{N}(\mathcal{E})]$ determined by $\mathcal{E}$ and let $F \colon Z \to \mathcal{N}(\mathcal{E})$ be a map in $[\mathcal{E}]$. Let $[\mathcal{E}_A]$ be the homotopy class in $[A, \mathcal{N}(\mathcal{E}_A)]$ determined by $\mathcal{E}_A$ and let $f' \colon A \to \mathcal{N}(\mathcal{E}_A)$ be a map in $[\mathcal{E}_A]$. Let $f \colon A \to \mathcal{N}(\mathcal{C})$ be a map in $[\mathcal{C}] \in [A, \mathcal{N}(\mathcal{C})]$.

Since $E_i \cap A$ contains C_i for each i, the argument preceding Definition 7 shows that f and f' are homotopic in $\mathcal{N}(\mathcal{E}_A)$. By construction, $F|_A$ and f' are homotopic in $\mathcal{N}(\mathcal{E})$. Thus, $F|_A$ and f are homotopic in $\mathcal{N}(\mathcal{E})$ and hence in $\mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_r(\Delta^{n-1})$ as well. By the definition of the EP rank this means that $\operatorname{rk}(Z, A, [\mathcal{C}]) \le r = \operatorname{rk}(Z, \mathcal{C})$.

(2) We show that $\operatorname{rk}(Z, \mathcal{C}) \le \operatorname{rk}(Z, A, [\mathcal{C}])$ whenever A is closed in Z.

We start by constructing a specific map $A \to \mathcal{N}(\mathcal{C})$ from the class $[\mathcal{C}]$. Let $\mathcal{C} = \{C_1, \dots, C_n\}$ and let $\mathcal{N}(\mathcal{C})$ be a subcomplex in Δ^{n-1} (as in the first part of the proof). Since A is normal, there exists an open cover $\mathcal{U} = \{U_1, \dots, U_n\}$ of A such that U_i contains C_i for each i and the nerve of $\mathcal{U}$ coincides with that of $\mathcal{C}$ (see eg [25, Theorem 1.3; 16, pages 31–33]). The Urysohn lemma for normal spaces implies that, for each i, there exists a continuous function $f_i \colon A \to [0, 1]$ with $f_i(C_i) = 1$ and $f_i(A \setminus U_i) = 0$. Then $\Phi = \{\varphi_1, \dots, \varphi_n\}$, where $\varphi_i := f_i / \sum_j f_j$, is a partition of unity subordinate to $\mathcal{U}$ such that $\varphi_i^{-1}[1/n, 1]$ contains C_i for all i. Let

$$h_{\mathcal{U},\Phi}(x) := \sum_{i=1}^{n} \varphi_i(x) v_i$$

be the corresponding map $A \to \mathcal{N}(\mathcal{C})$ representing the class $[\mathcal{C}] = [\mathcal{U}]$ (see the construction preceding Definition 7).

Now, let $p := \operatorname{rk}(Z, A, [\mathcal{C}])$. Then, by the definition of the EP rank, there exists a continuous map $F\colon Z \to \mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_p(\Delta^{n-1})$ such that the restriction $F|_A$ is homotopic to $h_{\mathcal{U},\Phi}$ in $\mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_p(\Delta^{n-1})$. The generalizations of Borsuk's homotopy extension theorem obtained in [26; 34] imply that, since A is closed in Z, there exists a continuous map $G\colon Z \to \mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_p(\Delta^{n-1})$ with $G|_A = h_{\mathcal{U},\Phi}$. Then the collection of subsets

$$\mathcal{G} := \{G_1^{-1}[1/n, 1], \dots, G_n^{-1}[1/n, 1]\},$$

where $G_1, \dots, G_n$ are the coordinate functions of G, is a closed cover of Z such that $G_i^{-1}[1/n, 1]$ contains C_i for all i and the nerve $\mathcal{N}(\mathcal{G})$ is contained in $\mathcal{N}(\mathcal{C}) \cup \operatorname{sk}_p(\Delta^{n-1})$, so the dimension of $\mathcal{N}(\mathcal{G}) \setminus \mathcal{N}(\mathcal{C})$ is at most p. By the definition of the KKM rank, this means that $\operatorname{rk}(Z, [\mathcal{C}]) \le p = \operatorname{rk}(Z, A, [\mathcal{C}])$. $\square$

Now we state and prove Lemmas 52 and 54, and then deduce Theorem 36 from Lemmas 33, 52 and 54.

Lemma 52 *Let $(Z, \mathcal{C} = \{C_1, \dots, C_n\})$ be a KKM system of rank $r > 0$, let $f : Z \to Z'$ be a continuous map to a topological space Z', and let $\mathcal{C}' = \{C_1', \dots, C_n'\}$ be a family of subsets in Z' such that $f(C_i) \subset C_i'$ for all i. Then either $\{1, \dots, n\}$ contains a subset J of cardinality $r+1$ such that $\bigcap_{j\in J} C_j = \varnothing$ and $\bigcap_{j\in J} C_j' \ne \varnothing$ or $(Z', \mathcal{C}')$ is a KKM system of rank at least r.*

Remark 53 In Lemma 52, two key special cases are $C_i' = f(C_i)$ and $f = \mathrm{id}$.

Proof If neither $\operatorname{rk}(Z', \mathcal{C}') \ge r$ nor $\{1, \dots, n\}$ contains J with $|J| = r + 1$ such that $\bigcap_{j\in J} C_j = \varnothing$ and $\bigcap_{j\in J} C_j' \ne \varnothing$, then

- (i) there exists a closed cover $\mathcal{E}' = \{E_1', \dots, E_n'\}$ of Z' with $C_i' \subset E_i'$ for all i such that the dimension of $\mathcal{N}(\mathcal{E}') \setminus \mathcal{N}(\mathcal{C}')$ is less than r (by definition), and
- (ii) the dimension of $\mathcal{N}(\mathcal{C}') \setminus \mathcal{N}(\mathcal{C})$ is less than r.

Consequently, the dimension of $\mathcal{N}(\mathcal{E}') \setminus \mathcal{N}(\mathcal{C})$ is less than r. The collection $\mathcal{E} = \{E_1, \dots, E_n\}$ with $E_i := f^{-1}(E_i')$ is a closed cover of Z such that $C_i \subset E_i$ for all i. The nerve $\mathcal{N}(\mathcal{E}')$ contains $\mathcal{N}(\mathcal{E})$. Therefore, the dimension of $\mathcal{N}(\mathcal{E}) \setminus \mathcal{N}(\mathcal{C})$ is less than r. This contradicts the assumption that $r = \operatorname{rk}(Z, \mathcal{C})$. $\square$

Lemma 54 *Let $(Z, \mathcal{C} = \{C_1, \dots, C_n\})$ be a KKM system of rank $r > 0$ with metrizable Z and all C_i compact, and let d be a metric on Z. Then there exists a closed metric ball*

$$B_R(x) := \{z \in Z \mid d(z, x) \le R\}, \quad x \in Z,\ R > 0,$$

whose interior intersects no element of $\mathcal{C}$ and whose boundary sphere touches at least $r+1$ elements of $\mathcal{C}$.

Proof If z is a point and N is a subset in Z, we write

$$d(z,N) := \inf_{p \in N} d(z,p).$$

Let A denote the union $\bigcup_{i=1}^{n} C_i$. For each $i \in \{1,\dots,n\}$, we set

$$E_i := \{z \in Z \mid d(z,C_i) = d(z,A)\}.$$

Observe that E_i contains C_i and is closed, so $\{E_1,\dots,E_n\}$ is a closed cover of Z. Since $(Z,\mathcal{C})$ is a KKM system of rank $r>0$, it follows by the definition of KKM rank that the set $\{1,\dots,n\}$ contains a subset J of cardinality $r+1$ such that $\bigcap_{j\in J} C_j = \varnothing$ and $\bigcap_{j\in J} E_j \neq \varnothing$. Let x be a point in $\bigcap_{j\in J} E_j \neq \varnothing$. Then the ball $B_{d(x,A)}(x)$ of radius $d(x,A)$ centered at x meets the requirements of the lemma (since each of C_i is compact). □

Proof of Theorem 36 Since $(Z,A,[\mathcal{C}])$ is an Eilenberg–Pontryagin triple, it follows by Lemma 33 that $(Z,\mathcal{C})$ is a KKM system of rank $\operatorname{rk}(Z,\mathcal{C}) = \operatorname{rk}(Z,A,[\mathcal{C}])$.

Let $\mathcal{C} = \{C_1,\dots,C_n\}$. Set $\mathcal{F} := \{F(C_1),\dots,F(C_n)\}$. Then Lemma 52 implies that we have two possibilities:

(1) The dimension of $\mathcal{N}(\mathcal{F})\setminus\mathcal{N}(\mathcal{C})$ is at least $\operatorname{rk}(Z,\mathcal{C})$, so $\{1,\dots,n\}$ contains a subset J of cardinality $\operatorname{rk}(Z,\mathcal{C})+1$ such that $\bigcap_{j\in J} C_j = \varnothing$ and $\bigcap_{j\in J} F(C_j) \neq \varnothing$.

(2) The pair $(M,\mathcal{F})$ is a KKM system of rank at least $\operatorname{rk}(Z,\mathcal{C})$.

In case (1), for any point $x \in \bigcap_{j\in J} F(C_j)$, the set $F^{-1}(x)$ is a set of spherical $F|_A$–neighbors that intersects all elements of $\{C_j\}_{j\in J}$, which proves the theorem.

In case (2), the required statement follows by Lemma 54 applied to $(M,\mathcal{F})$. □

Proof of Corollary 41 We use a spherical version of Theorem 4. Let T be a regular triangulation of the unit sphere $\mathbb{S}^n$ and let $\widetilde{\Delta}_1, \dots, \widetilde{\Delta}_{n+2}$, be the n–simplices of T; all of $\widetilde{\Delta}_i$ are regular spherical simplices with Euclidean distances between vertices

$$d_{n,\mathrm{Eu}} = \sqrt{\frac{2(n+2)}{n+1}} \tag{1}$$

and angular edge length

$$d_{n,\mathrm{A}} = 2\arcsin\bigl(\tfrac{1}{2}d_{n,\mathrm{Eu}}\bigr) = 2\arcsin\sqrt{\frac{n+2}{2(n+1)}} = \arccos\Bigl(\frac{-1}{n+1}\Bigr). \tag{2}$$

We recall that the *circumradius* of a compact set Q in a metric space is defined as the radius of a least metric ball containing Q. If Q is a compact subset of $\mathbb{S}^n$ we denote by $\operatorname{circ}_{\mathrm{A}} Q$ and $\operatorname{diam}_{\mathrm{A}} Q$, respectively, the circumradius and diameter of Q with respect to the angular metric, and $\operatorname{diam}_{\mathrm{Eu}} Q$ will stand for the Euclidean diameter of Q in $\mathbb{R}^{n+1} \supset \mathbb{S}^n$. Under this notation, Dekster's extension [9] of the Jung theorem says that, for any compact subset Q of $\mathbb{S}^n$,

$$2\arcsin\left(\sqrt{\frac{n+1}{2n}}\sin(\operatorname{circ}_{\mathrm{A}} Q)\right) \le \operatorname{diam}_{\mathrm{A}} Q.$$

This immediately implies that, in the case where $\operatorname{circ}_{\mathrm{A}} Q \le \frac{\pi}{2}$,

$$\sqrt{\frac{2(n+1)}{n}}\sin(\operatorname{circ}_{\mathrm{A}} Q) \le \operatorname{diam}_{\mathrm{Eu}} Q. \tag{3}$$

Another auxiliary fact we need is that

$$\operatorname{diam}_{\mathrm{A}} \widetilde{\Delta}_i = \tfrac{1}{2}(\pi - d_{n,\mathrm{A}}). \tag{4}$$

Indeed, observe that $\widetilde{\Delta}_i$ is the intersection of a finite number of closed hemispheres and hence its boundary is composed of fragments of great hyperspheres, which are geodesic in $\mathbb{S}^n$. Therefore, if a and b are two points in $\widetilde{\Delta}_i$ such that neither a nor b is a vertex of $\widetilde{\Delta}_i$, then $\operatorname{diam}_{\mathrm{A}}\{a,b\} < \operatorname{diam}_{\mathrm{A}} \widetilde{\Delta}_i$ because $\widetilde{\Delta}_i$ contains two geodesic arcs[4] α and β such that α contains a in its relative interior and β contains b in its relative interior. Since $\widetilde{\Delta}_i$ is contained in the interior of a hemisphere, so a and b are not antipodal, it follows that there exist $a' \in \alpha \subset \widetilde{\Delta}_i$ and $b' \in \beta \subset \widetilde{\Delta}_i$ with $\operatorname{diam}_{\mathrm{A}}\{a,b\} < \operatorname{diam}_{\mathrm{A}}\{a',b'\}$ (imagine the interposition of α, β and the metric ball $D \subset \mathbb{S}^n$ of diameter $\operatorname{diam}_{\mathrm{A}}\{a,b\}$ containing a and b). Thus, if a and b are two points in $\widetilde{\Delta}_i$ such that $\operatorname{diam}_{\mathrm{A}}\{a,b\} = \operatorname{diam}_{\mathrm{A}} \widetilde{\Delta}_i$, then one of a and b is a vertex of $\widetilde{\Delta}_i$ and we easily obtain (4) by considering the regular triangulation of $\mathbb{S}^n$ dual (antipodal) to T.

Now, we pass to the proof of Corollary 41. If we have a continuous map $f\colon \mathbb{S}^n \to M$, then Theorem 4 implies that a finite set $\mathscr{P}$ of spherical f–neighbors intersects all of $\widetilde{\Delta}_i$. We need to prove that

$$\operatorname{diam}_{\mathrm{Eu}} \mathscr{P} \ge \sqrt{\frac{n+2}{n}}. \tag{5}$$

Let $B \subset \mathbb{S}^n$ be a metric ball with angular radius $\operatorname{circ}_{\mathrm{A}} \mathscr{P}$ containing $\mathscr{P}$, let $C \in \mathbb{S}^n$ be the center of B, let $A \in \mathbb{S}^n$ be the antipode of C, let $\widetilde{\Delta}_k$ be a simplex of T containing A and

[4] By geodesic arcs in $\mathbb{S}^n$ we mean arcs of great circles.

let $B_2 \subset \mathbb{S}^n$ be the metric ball centered at A of angular radius $\operatorname{diam}_{\mathrm{A}} \widetilde{\Delta}_k = \frac{1}{2}(\pi - d_{n,\mathrm{A}})$ (see (4)). Then B_2 contains $\widetilde{\Delta}_k$. Since $\mathscr{P}$ intersects $\widetilde{\Delta}_k$ while $\mathscr{P} \subset B$ and $\widetilde{\Delta}_k \subset B_2$, it follows that B intersects B_2. Therefore,

$$\operatorname{circ}_{\mathrm{A}} \mathscr{P} = \operatorname{circ}_{\mathrm{A}} B \geq \pi - \operatorname{circ}_{\mathrm{A}} B_2 = \tfrac{1}{2} d_{n,\mathrm{A}}. \tag{6}$$

The situation splits into two cases:

(i) $\operatorname{circ}_{\mathrm{A}} \mathscr{P} > \frac{\pi}{2}$ (ie no hemisphere contains $\mathscr{P}$).

(ii) $\operatorname{circ}_{\mathrm{A}} \mathscr{P} \leq \frac{\pi}{2}$.

In case (i), we observe that, since no hemisphere contains $\mathscr{P}$, it follows that no Euclidean ball in $\mathbb{R}^n$ of radius less than 1 contains $\mathscr{P}$. Then the Jung theorem[5] says that $\operatorname{diam}_{\mathrm{Eu}} \mathscr{P} \geq d_{n,\mathrm{Eu}}$, which implies the required (5).

In case (ii), (3) is applicable and yields

$$\begin{aligned}
\operatorname{diam}_{\mathrm{Eu}} \mathscr{P} &\geq \sqrt{\frac{2(n+1)}{n}} \cdot \sin(\operatorname{circ}_{\mathrm{A}} \mathscr{P}) && \text{(by (3))} \\
&\geq \sqrt{\frac{2(n+1)}{n}} \cdot \sin\big(\tfrac{1}{2} d_{n,\mathrm{A}}\big) && \text{(by (6) and (ii))} \\
&= \sqrt{\frac{2(n+1)}{n}} \cdot \tfrac{1}{2} d_{n,\mathrm{Eu}} && \text{(by (2))} \\
&= \sqrt{\frac{2(n+1)}{n}} \cdot \sqrt{\frac{n+2}{2(n+1)}} = \sqrt{\frac{n+2}{n}} && \text{(by (1)).}
\end{aligned}$$

□

Remark 55 It would be interesting to find a way to upgrade the above proof of Corollary 41 by considering the family of all regular triangulations of the unit sphere $\mathbb{S}^n$.

Proof of Corollary 51 (1) Corollary 44 implies that, if M is a contractible metric space and $f : \partial P \to M$ is a continuous map, then each cubic hemisphere in ∂P contains a facet that is a member of a pair of disjoint facets that are spherical f–neighbors. The statement follows.

(2) Proposition 56 below implies that, if the lighthouse independence number of a flag 3–polytope P is k, then ∂P contains k cubic hemispheres with pairwise disjoint interiors. This implies the required assertion by assertion (1) of the corollary.

(3)–(4) These follow from assertion (2) and Proposition 57 below.

[5] For a discussion and materials concerning the Jung theorem and containment in hemispheres, see [39; 15; 8, pages 112, 113, 132–136, 38; 19; 4; 1, Proposition 2.4].

(5)–(6) These follow from (2) because a direct check shows that the lighthouse independence number of the regular dodecahedron is 4 and the lighthouse independence number of the regular icosahedron is 3.

(7) Corollary 44 shows that the weak Radon rank of the cube is at least 1 and an example where $\partial[0,1]^3$ is mapped to an oblate spheroid in $\mathbb{R}^3$ shows that it is at most 1. □

We say that a vertex v of a polytope is *cubical* if the union of the facets containing v is a cubic hemisphere.

Proposition 56 *All vertices of a flag 3–polytope are cubical.*

Proof Let v be a vertex of a flag 3–polytope P. Observe that no facet of P is a triangle (because any triangular facet together with the three adjacent ones form a collection of four pairwise intersecting facets with no common point). Therefore, each facet of P containing v has a vertex that is not adjacent to v. Let v_1, v_2 and v_3 be three such vertices lying on three distinct facets containing v. Let D denote the union of the facets of P that do not contain v. Then D is a topological disk with the points v_1, v_2 and v_3 on its boundary. Since P is flag, we see that

- no facet contained in D intersects three of the facets not contained in D,
- no facet of P splits D (in the sense that $D \setminus F$ is connected for each facet F).

This implies that

- each of the vertices v_1, v_2 and v_3 is incident to an edge of P whose second endpoint is contained in the interior of D (in particular, the interior of D contains at least one vertex of P); and
- the subgraph G_D in the 1–skeleton P_1 of P induced by the vertices of P contained in the interior of D is connected.

Thus, each of v_1, v_2 and v_3 is adjacent to a vertex of the connected subgraph G_D in P_1. This easily implies that P_1 contains a Y–homeomorphic subgraph Y' that is contained in D and intersects the boundary ∂D exactly in the set $\{v_1, v_2, v_3\}$.

Furthermore, since v_1, v_2 and v_3 belong to three distinct facets containing v, it follows that there exists a triple of edges in P_1 incident to v whose endpoints split ∂D into three arcs each of which contains exactly one of v_1, v_2 and v_3. Clearly, the union of these edges with ∂D and Y' is a graph homeomorphic to the cube 1–skeleton. This shows that v is cubical. □

Proposition 57 (1) *Let P be a flag simple 3–polytope with ψ facets and let g be the largest number of edges in a facet of P. Then the lighthouse independence number of P is at least*
$$\left\lfloor \frac{2\psi-7}{3g-8} \right\rfloor.$$

(2) *If P is a fullerene with ψ facets, then the lighthouse independence number of P is at least*
$$\left\lfloor \tfrac{1}{5}(\psi-3) \right\rfloor.$$

Proof In the proof, if v is a vertex of P, we denote by $L(v)$ the union of facets of P that contain v.

We construct a lighthouse independent set by the following algorithm. First we choose a vertex v_1 of P such that the number of vertices in $L(v_1)$ is the least possible and set $W_1 = L(v_1)$. The number of vertices in $L(v_1)$ is at most $3g-5$.

At each next step, being given $W_i \subset P$ such that a vertex of P is not in W_i, we take a vertex v_{i+1} of P in $P \setminus W_i$ such that the number of vertices in $L(v_{i+1}) \setminus W_i$ is the least possible and set $W_{i+1} = W_i \cup L(v_{i+1})$. Observe that, if a vertex v of P is not in W_i and adjacent to a vertex in W_i, then $L(v)$ shares at least three vertices with W_i. This implies that the number of vertices in $L(v_{i+1}) \setminus W_i$ is at most $3g-8$.

Therefore, if P has N vertices, this algorithm produces a lighthouse independent set $v_1, v_2, \dots$ with at least
$$1 + \left\lfloor \frac{N-(3g-5)}{3g-8} \right\rfloor = \left\lfloor \frac{N-3}{3g-8} \right\rfloor$$
elements. Since P is simple, Euler's formula yields $N = 2\psi-4$. This proves the required estimate.

The case of fullerenes follows if we observe that, when v_1 is a vertex of a pentagon, the number of vertices in $L(v_1)$ is at most 12. □

4 Concluding remarks

Now we discuss several concepts and open questions.

(1) **The Hopf theorem** The trefoil curve in Figure 5 shows that there exists a continuous map $f : \mathbb{S}^1 \to \mathbb{R}^2$ with no pair of spherical f–neighbors having distance less than $\sqrt{3}$ between them. This means that the direct analog of the aforementioned Hopf theorem for spherical f–neighbors does not hold for small distances. It would be

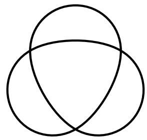

Figure 5

interesting to find more properties of the set of distances between spherical f–neighbors for a continuous map f of given metric spaces. For example:

Question *Is it true that, for any continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^{n+k}$, the set*

$$\Gamma_f := \{\delta \in \mathbb{R} \mid \delta = d(p,q) \text{ for a pair } \{p,q\} \text{ of spherical } f\text{–neighbors}\}$$

contains a nondegenerate interval? Is there a nonzero lower bound for the diameter of Γ_f?

(2) **Topological Tverberg theorems** Projecting a Euclidean n–sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ into a hyperplane in $\mathbb{R}^{n+1}$ shows that there exists a continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$ with no set of spherical f–neighbors of cardinality exceeding 2. Consequently, each convex n–polytope P has a map $f\colon \partial P \to \mathbb{R}^{n-1}$ with no set of spherical f–neighbors intersecting three disjoint faces of P. This means that no direct analog of the topological Tverberg theorems with three or more disjoint faces holds for spherical f–neighbors. This correlates with the property (see Remark 16) that no principal cover has disjoint elements. Nevertheless, we have some analogs of the topological Radon theorem, which is the topological Tverberg theorem for two disjoint faces; see Corollaries 44, 46 and 51.

Problem *Find extensions of topological Tverberg theorems for spherical f–neighbors with additional restrictions.*

(See also Van Kampen–Flores- and Conway–Gordon–Sachs-type results [33].)

(3) **Weak Radon rank** It would be interesting to:

Problem *Describe the set of polyhedra that are not weakly Radon. Find the weak Radon rank for fullerenes.*

(4) **Minimaxes, I** Let (X,ρ) and (M,d) be metric spaces and let $f\colon X \to M$ be a continuous map. Let P_f be the set of all pairs of spherical f–neighbors in X. We set

$$D_f := \sup_{\{x,y\}\in P_f} \rho(x,y), \qquad \mu(X,M) := \inf_{f\in C(X,M)} D_f,$$

where $C(X, M)$ stands for continuous maps. Suppose $X = \mathbb{S}^n$ and $M = \mathbb{R}^m$. If $m \le n$, then $\mu(\mathbb{S}^n, \mathbb{R}^m) = 2$ by the Borsuk–Ulam theorem. For $n < m$, it is shown in [21] that

$$\mu(\mathbb{S}^n, \mathbb{R}^m) = \sqrt{\frac{2(n+2)}{n+1}}.$$

Problem *Find $\mu(X, M)$ and its lower bounds in general and some special cases. In particular, find D_f and μ for the case where $M = \mathbb{R}^n$ and X is an n–dimensional Riemannian manifold.*

(5) **Minimaxes, II** Let us fix $[\mathcal{C}]$ in $[X, \mathbb{S}^{n-2}]$ (see Definition 7), for instance $[\mathcal{C}] \neq 0$ in $\pi_3(\mathbb{S}^2)$.

Question *What is the min–max distance between the points of a set intersecting each element of a cover of this class*?

(6) **Widths, distortion, filling radius, etc** Similarly to $\mu(X, M)$, we consider infima of D_f over families of homotopic maps, over all continuous maps of a given space to certain classes of spaces (eg contractible spaces), etc. This generates a series of new metric "μ–invariants" of maps and metric spaces. These μ–invariants are similar to such invariants as distortion, filling radius and various widths (see [37; 12; 13; 10; 31; 18; 3]).

Problem *Find and describe relations between μ–invariants and classical ones.*

(7) **Topological and visual f–neighbors** Let $f: X \to Y$ be a map of topological spaces. We say that two points a and b in X are *topological f–neighbors* if $f(a)$ and $f(b)$ belong to the boundary of the same connected component of the complement $Y \setminus f(X)$. If Y is a geodesic metric space, we say that a and b in X are *visual f–neighbors* if $f(a)$ and $f(b)$ are connected by a geodesic, in Y, whose interior does not meet $f(X)$.

Problem *Translate the above constructions and questions to these new types of f–neighbors.*

(8) **Helly-type sufficient conditions for principal covers** Remark 18 implies some Helly-type sufficient conditions for principal covers. For example, if $\mathcal{C} = \{C_1, \dots, C_n\}$ is a closed cover of a normal space Y such that $\mathcal{N}(\mathcal{C}) = \partial\Delta^{n-1}$ and, for each $J \subset \{1, \dots, n\}$ with $|J| \le n-2$, any continuous map $\mathbb{S}^{n-2-|J|} \to \bigcap_{j \in J} C_j$ is null-homotopic, then there exists a map $f: \partial\Delta^{n-1} \to Y$ such that the image of each

facet is contained in an element of $\mathcal{C}$, so $\mathcal{C}$ is principal. (See the proofs of Theorems 5 and 6 in [7].)

Question *Which of the other versions of topological Helly theorem* (see eg [7; 24]) *give sufficient conditions for principal and non-nullhomotopic covers*?

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Stable cohomology of the universal degree d hypersurface in $\mathbb{P}^n$

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We consider the universal hypersurface of degree d in $\mathbb{CP}^n$ and compute its stable cohomology (with respect to d). We describe the stable classes geometrically.

14F25, 14J70, 55R80

1 Introduction

Let $U_{d,n}$ be the *parameter space* of smooth degree d hypersurfaces in $\mathbb{P}^n$. There is a natural inclusion $U_{d,n} \subseteq \mathbb{P}^{\binom{n+d}{d}} = \mathbb{P}(V_{d,n})$, where $V_{d,n}$ is the vector space of homogenous degree d complex polynomials in $n+1$ variables. Let

$$U^*_{d,n} := \{(f, p) \in U_{d,n} \times \mathbb{P}^n \mid f(p) = 0\}.$$

Let $\phi\colon U^*_{d,n} \to U_{d,n}$ be defined by $\phi(f, p) = f$. The map $\phi\colon U^*_{d,n} \to U_{d,n}$ is the *universal family* of smooth degree d hypersurfaces in $\mathbb{P}^n$; it satisfies the following property: given a family $\pi: E \to B$ of smooth degree d hypersurfaces in $\mathbb{P}^n$, there is a unique diagram

$$\begin{tikzcd} E \arrow[r, "\exists!"] \arrow[d] & U^*_{d,n} \arrow[d] \\ B \arrow[r, "\exists!"] & U_{d,n} \end{tikzcd}$$

In other words, any family of smooth degree d hypersurfaces is pulled back from this one. Our main result is as follows:

Theorem 1.1 *Let $d, n \geq 1$. Then there is an embedding of graded algebras,*

$$\phi\colon \mathrm{H}^*(\mathrm{PGL}_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n) \hookrightarrow \mathrm{H}^*(U^*_{d,n}; \mathbb{Q}),$$

where $|x| = 2$. Here $|\cdot|$ denotes the cohomological degree. Let $c_1(E)$ denote the first Chern class of the line bundle E.

(1) *The element* $\phi(x) = c_1(\mathcal{L})$, *where* $\mathcal{L}$ *is the fiberwise canonical bundle* (defined in Section 2).

(2) *Suppose* $d \geq 4n+1$. *Then* ϕ *is surjective in degree less than* $(d-1)/2$.

Let $X_{d,n} \subseteq V_{d,n}$ be the open subspace of polynomials defining a nonsingular hypersurface. The complement of $X_{d,n}$ in $V_{d,n}$ is known as the *discriminant hypersurface*; it is the zero locus of the classical discriminant polynomial. It is known to be highly singular.

A point of $X_{d,n}$ determines a projective hypersurface up to a scalar. There is a natural action of $\mathbb{C}^*$ on $X_{d,n}$ such that the quotient $X_{d,n}/\mathbb{C}^*$ is $U_{d,n}$. Let

$$X^*_{d,n} := \{(f, p) \mid f \in X_{d,n}, p \in \mathbb{P}^n, f(p) = 0\}.$$

There is a forgetful map $\pi\colon X^*_{d,n} \to X_{d,n}$ defined by $\pi(f, p) = f$. The fibres of π are

$$Z(f) := \pi^{-1}(f) = \{p \in \mathbb{P}^n \mid f(p) = 0\} \subseteq \mathbb{P}^n.$$

It is well known that the map π is a fibre bundle.

$X^*_{d,n}$ also has several interesting quotients. The action of GL_{n+1} on $X_{d,n}$ lifts to one on $X^*_{d,n}$. We obtain $U^*_{d,n} = X^*_{d,n}/\mathbb{C}^*$. The map $\pi\colon X^*_{d,n} \to X_{d,n}$ is $\mathbb{C}^*$–equivariant and descends to the map $\phi\colon U^*_{d,n} \to U_{d,n}$.

We define $M_{d,n} := U_{d,n}/\mathrm{PGL}_{n+1}(\mathbb{C})$, the *moduli space* of degree d smooth hypersurfaces in $\mathbb{P}^n$. We also define $M^*_{d,n} = X^*_{d,n}/\mathrm{GL}_{n+1}(\mathbb{C})$.

We can rewrite our result in terms of $X^*_{d,n}$ and $M^*_{d,n}$ as well. This is important to us as our proof will mostly involve understanding the space $X^*_{d,n}$. The space $M^*_{d,n}$ is important conceptually.

Theorem 1.2 *Let* $d, n \geq 1$.

(1) *There is an embedding of graded algebras*,

$$\psi\colon (\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}[x]/(x^n)) \hookrightarrow \mathrm{H}^*(X^*_{d,n}; \mathbb{Q}),$$

where $|x| = 2$.

(2) *There is an embedding of graded algebras*,

$$\varphi\colon \mathbb{Q}[x]/(x^n) \hookrightarrow \mathrm{H}^*(M^*_{d,n}; \mathbb{Q}),$$

where $|x| = 2$.

Suppose that $d \geq 4n+1$. *Then the maps* ψ *and* φ *are surjective in degree* $\leq (d-1)/2$.

Theorem 1.2 is equivalent to Theorem 1.1 after applying Theorem 2 of Peters and Steenbrink [6].

Nature of stable cohomology Throughout the course of the proof of Theorem 1.2 we also obtain the following description of the stable cohomology classes of $X^*_{d,n}$. The stable classes are tautological in the following sense: There is a line bundle $\mathscr{L}$ on $M^*_{d,n}$ defined by taking the canonical bundle fibrewise (we rigorously define $\mathscr{L}$ in Section 2). We will show that $c_1(\mathscr{L}), \dots, c_1(\mathscr{L})^{n-1}$ are nonzero in $H^*(M^*_{d,n};\mathbb{Q})$ and that stably the entire cohomology ring of $M^*_{d,n}$ is just the algebra generated by $c_1(\mathscr{L})$. By [6],

$$\mathrm{H}^*(X^*_{d,n};\mathbb{Q}) \cong \mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n};\mathbb{Q}).$$

In this way we have some qualitative understanding of the stable cohomology of $X^*_{d,n}$.

Both the statement of Theorem 1.2 and our proof of it are heavily influenced by [8], in which Tommasi proves analogous theorems for $X_{d,n}$. Our techniques and approach are also similar to that of Das in [3], where he proves

$$\mathrm{H}^*(X^*_{3,3};\mathbb{Q}) \cong \mathrm{H}^*(\mathrm{GL}_3(\mathbb{C});\mathbb{Q}) \otimes \mathbb{Q}[x]/x^3$$

with $|x| = 2$. We would also like to mention the paper by Tommasi [7] where $\mathrm{H}^*(X_{2,4};\mathbb{Q})$ is computed.

In some sense, this paper shows that in a stable range, something similar to Das's theorem is true for marked hypersurfaces in general.

Some motivation and historical comments At this point we'd like to make some remarks on historical motivations for computing and understanding stable cohomology of moduli spaces of algebraic varieties.

The cohomology of moduli spaces are often interesting because they provide us with invariants for families of varieties. However, in many interesting cases the entire cohomology ring of the moduli space may be difficult to understand and compute. An example of such a phenomenon is the moduli space of curves of genus g, $\mathcal{M}_g$. In this setting, $H^*(\mathcal{M}_g;\mathbb{Q})$ is a huge ring which is not fully understood. However, the spaces $\mathcal{M}_g$ are known to satisfy homological stability and the stable cohomology ring can be explicitly described. For a survey, see Cohen [2].

Another motivation for computing the stable cohomology of moduli spaces has to do with arithmetic statistics. Let X be an algebraic variety over $\mathbb{Z}$. Often one would like to compute $\#X(\mathbb{F}_p)$ by studying the eigenvalues of Frob_p on $H^*_{\mathrm{et}}(X;\mathbb{Q}_l)$. There are often comparison theorems which relate the étale cohomology with the singular cohomology of $X(\mathbb{C})$ and computations of $H^*(X(\mathbb{C});\mathbb{Q})$ can often imply bounds on $\#(X(\mathbb{F}_p))$. For an introduction to this topic, see for instance Sections 1 and 2 of Church, Ellenberg and Farb [1].

Method of proof One could attempt to prove Theorem 1.2 by applying the Serre spectral sequence to the fibration $\pi\colon X^*_{d,n}\to X_{d,n}$. To successfully do this however, one would need to understand the groups $\mathrm{H}^p(X_{d,n};\mathrm{H}^q(Z(f);\mathbb{Q}))$. While we do a priori understand what the groups $\mathrm{H}^p(X_{d,n};\mathbb{Q})$ are (this is the main theorem of [8]), this is not sufficient for us to understand what the groups $\mathrm{H}^p(X_{d,n};\mathrm{H}^q(Z(f);\mathbb{Q}))$ are, since $\mathrm{H}^q(Z(f);\mathbb{Q})$ is a *nontrivial* local coefficient system. Instead we use an idea of Das and compute $\mathrm{H}^*(X^p_{d,n};\mathbb{Q})$, where $X^p_{d,n}:=\{f\in X_{d,n}\mid f(p)=0\}$ to avoid any computations with nontrivial coefficient systems. After we have proved Theorem 1.2 we can use it to deduce what these twisted cohomology groups are.

Corollary 1.3 *Let $d,n>0$. Suppose $d\geq 4n+1$ and $k<(d-1)/2$. Then*

$$\mathrm{H}^k(X_{d,n};\mathrm{H}^{n-1}(Z(f);\mathbb{Q}))=\begin{cases}\mathrm{H}^k(X_{d,n};\mathbb{Q}) & \text{if } n \text{ is odd},\\ 0 & \text{if } n \text{ is even}.\end{cases}$$

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2 A lower bound on $\mathrm{H}^k(X^*_{d,n})$

We begin by noting that there is an embedding of algebras

$$\mathrm{H}^k(\mathrm{GL}_{n+1}(\mathbb{C}))\otimes\mathbb{Q}[x]/(x^n)\hookrightarrow \mathrm{H}^k(X^*_d)$$

in the stable range. More precisely, we have the following:

Proposition 2.1 *Let $n\geq 0$ and $d>n+1$. There is a natural embedding of algebras,*

$$i\colon H^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q})\otimes\mathbb{Q}[x]/(x^n)\hookrightarrow H^*(X^*_{d,n};\mathbb{Q}),$$

where $|x|=2$.

Proof We first define the *fiberwise canonical bundle* $\mathscr{L}$ over $M^*_{d,n}$ as

$$\mathscr{L}=\{(f,p,v)\mid (f,p)\in M^*_d,\ v\in\wedge^{n-1}T^*_p(Z(f))\}.$$

We can pull back $\mathscr{L}$ to a bundle on $X^*_{d,n}$, which we will also denote by $\mathscr{L}$. By the same argument as in Theorem 1 of [6],

$$\mathrm{H}^*(X^*_{d,n};\mathbb{Q}) \cong \mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n}(\mathbb{C});\mathbb{Q}).$$

Let $f \in X_{d,n}$. Let $i\colon \mathrm{GL}_{n+1}(\mathbb{C}) \to X_{d,n}$ be the orbit map defined by $i(g) = g \cdot f$. More precisely, Theorem 1 of [6] states that the natural map

$$\pi^*\colon \mathrm{H}^*(M^*_{d,n};\mathbb{Q}) \to \mathrm{H}^*(X^*_{d,n};\mathbb{Q})$$

makes $\mathrm{H}^*(X^*_{d,n};\mathbb{Q})$ a free $\mathrm{H}^*(M^*_{d,n};\mathbb{Q})$–module with a basis given by some set $\{\alpha_i\}$ such that the pullbacks $\{i^*(\alpha_i)\}$ give a basis of $\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q})$. But since $\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q})$ is a free graded commutative algebra, this forces $H^*(X^*_{d,n};\mathbb{Q})$ to be isomorphic to $\mathrm{H}^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathrm{H}^*(M^*_{d,n}(\mathbb{C});\mathbb{Q})$ as an algebra.

If we restrict $\mathscr{L}$ to a particular hypersurface Z, the bundle $\mathscr{L}|_Z = \mathscr{O}_Z(d-n-1)$. The Chern class of $\mathscr{L}|_Z$ satisfies the equality

$$c_1(\mathscr{O}_Z(d-n-1)) = (d-n-1)c_1(\mathscr{O}_Z(1)) = d(d-n-1)\omega_Z,$$

where ω_Z is the Kähler class of the variety Z. This implies that for $d > n+1$, the classes $c_1(\mathscr{L})|_Z, \dots, c_1^{n-1}(\mathscr{L})|_Z$ are nonzero since $\omega_Z, \dots, \omega_Z^{n-1}$ are nonzero. Now taking $x = c_1(\mathscr{L})$, this implies that $\mathrm{H}^*(M;\mathbb{Q})$ contains a subalgebra isomorphic to $\mathbb{Q}[x]/x^n$. □

3 The space X^p_d and the Vassiliev method

Given a space X, the n^{th} *ordered configuration space* of X, denoted by $\mathrm{PConf}_n X$, is

$$\mathrm{PConf}_n X := \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for all } i \neq j\}.$$

There is a natural action of the symmetric group on n letters S_n on X by permuting the coordinates. The quotient $\mathrm{PConf}_n X/S_n$ is called the n^{th} *unordered configuration space* and denoted by $\mathrm{UConf}_n X$. In order to understand $X_{d,n}$ we will first look at the cohomology of a related space. For a fixed point $p \in \mathbb{P}^n$, we set

$$X^p_d = \{f \in X_{d,n} \mid f(p) = 0\}.$$

Then

$$X^p_d \subseteq V^p_d = \{f \in V_d \mid f(p) = 0\}.$$

The space V_d^p is a vector space. The complement of X_d^p in V_d^p will be called $\Sigma_{d,p}$. We will compute its Borel–Moore homology and use Alexander duality to compute $H^*(X_d^p)$.

Let $p \in \mathbb{P}^n$. By definition, p is a one-dimensional subspace $p \subseteq \mathbb{C}^{n+1}$. Choose a complementary subspace $W \subseteq \mathbb{C}^{n+1}$ (it is not unique, but we will fix a particular one). We define $G_p := \mathrm{GL}(W)$.

Let $x_1, \dots, x_n$ be local coordinates in a neighbourhood U containing p. Pick a local trivialization s of the line bundle $\mathbb{O}(d)$ in U. There is an induced map

$$f^* \colon T_0^*(\mathbb{O}(d)_p) \to T_p^*(\mathbb{P}^n).$$

Let us use our local coordinates to identify $T_0^*(\mathbb{O}(d)_p)$ with $\mathbb{C}$ and $T_p^*(\mathbb{P}^n)$ with $\mathbb{C}^n$.

Suppose $f \in X_d^p$. Then the map f^* is nonzero because f has a regular zero locus. This defines a map

$$\pi \colon X_d^p \to T_p^*(\mathbb{P}^n) - \{0\} \cong \mathbb{C}^n - \{0\}$$

given by $\pi(f) = f^*(1)$.

Proposition 3.1 *The map $\pi \colon X_d^p \to \mathbb{C}^n - \{0\}$ is a fibration.*

Proof The group G_p acts on $\mathbb{P}^n$ fixing p. Therefore it acts on both X_d^p and $\mathbb{C}^n - \{0\}$. The map π is equivariant with respect to these actions. The map π is therefore the pullback of a map π' from X_d^p/G_p to $\mathbb{C}^n - \{0\}/G_p$. But $\mathbb{C}^n - \{0\}/G_p$ is a point, and since π' is surjective it is a fibration. Since pullbacks of fibrations are fibrations, π is a fibration. □

Let $X_v := \pi^{-1}(v)$ and let

$$V_v := \{f \in V_d \mid f^*(1) = v\}.$$

Clearly, $X_v \subseteq V_v$. Let $\Sigma_v := V_v - X_v$. We will try to understand the Borel–Moore homology of Σ_v.

To accomplish this, the Vassiliev method [10] will be applied. The Vassiliev method to compute Borel–Moore homology involves stratifying a space and using the associated spectral sequence to compute its Borel–Moore homology. The space Σ_v will be stratified based on the points at which a section f is singular. The techniques used are very similar to that in [8] which contains many of the technical details.

We denote the k–simplex with vertex set $\{a_0, \ldots, a_k\}$ by $\Delta_{\{a_0,\ldots,a_k\}}$. We denote a k–simplex by Δ_k and an open k–simplex by Δ_k°

We will now construct a cubical space $\mathcal{C}$ which will be involved in understanding Σ_v. Let $N = (d-1)/2$. Let I be a subset of $\{1, \ldots, N-1\}$. For $k < N$, let

$$\mathcal{C}_I := \{(f, p) \mid f \in \Sigma_v, p \colon I \to \mathbb{P}^n, p(I) \subseteq \text{singular zeroes of } f\}.$$

We define $\Sigma_v^{\geq N} = \{f \in \Sigma_v \mid f \text{ has at least } N \text{ singular zeroes}\}$. We define

$$\mathcal{C}_{I \cup \{N\}} := \{(f, p) \mid f \in \Sigma_v, p \colon I \to \mathbb{P}^n, p(I) \subseteq \text{singular zeroes of } f, f \in \overline{\Sigma}^{\geq N}\}.$$

If $I \subseteq J$ then we have a natural map from $\mathcal{C}_J \to \mathcal{C}_I$ defined by restricting p. This gives $\mathcal{C}$ the structure of a cubical space over the set $\{1, \ldots, N\}$. We can take the geometric realization of $\mathcal{C}$, denoted by $|\mathcal{C}|$. Then there is a map $\rho \colon |\mathcal{C}| \to \Sigma_v$, induced by the forgetful maps $\mathcal{C}_I \to \Sigma_v$.

$|\mathcal{C}|$ is topologized in a nonstandard way so as to make ρ proper. We topologize it as follows: in [8], a space $|\mathscr{X}|$ is constructed with a map $\rho \colon |\mathscr{X}| \to \Sigma$. Here, $\Sigma = V_d - X_d$. The topology on $|\mathscr{X}|$ is chosen carefully so as to make ρ proper. The construction of $|\mathscr{X}|$ as a set identical to that of $|\mathcal{C}|$ except we replace Σ_v with Σ. There is a natural inclusion $|\mathcal{C}| \to |\mathscr{X}|$. We give $|\mathcal{C}|$ the subspace topology along this map.

Proposition 3.2 *The map $\rho \colon |\mathcal{C}| \to \Sigma_v$ is a proper homotopy equivalence.*

Proof This proof is nearly identical to that of Lemma 15 in [8]. The properness of $\rho \colon |\mathcal{C}| \to \Sigma_v$ follows from the properness of $\rho \colon |\mathscr{X}| \to \Sigma$ and the properties of the subspace topology. In our setting, having contractible fibres implies that the map ρ is a homotopy equivalence; this follows by combining Theorems 1.1 and 1.2 of [5]. We will now prove that the fibres are contractible. If $f \notin \overline{\Sigma}_v^{\geq N}$, let $\{p_1, \ldots, p_k\}$ be the singular zeroes of f. In this case the fibre $\rho^{-1}(f)$ is a simplex with vertices given by the images of the points $(f, x_i) \in \mathcal{C}_{\{1\}} \times \Delta_{\{1\}}$. Now suppose $f \in \overline{\Sigma}_v^{\geq N}$. In this case the fibre $\rho^{-1}(f)$ is a cone over the point $f \in \mathcal{C}_N \times \Delta_{\{N\}}$. $\square$

Now as in any geometric realization, $|\mathcal{C}|$ is filtered by

$$F_n = \operatorname{im}\biggl(\coprod_{|I| \leq n} \mathcal{C}_I \times \Delta_k\biggr).$$

The F_n form an increasing filtration of $|\mathcal{C}|$, ie $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq F_{n+1} \subseteq \cdots$ and $\bigcup_{n=1}^\infty F_n = |\mathcal{C}|$.

Proposition 3.3 *Let $d, n \geq 1$ and $N = (d-1)/2$. For $k < N$, the space $F_k - F_{k-1}$ is a Δ_k°–bundle, over a vector bundle B_k over $\mathrm{UConf}_k(\mathbb{P}^n - p)$.*

Proof The space $F_k - F_{k-1}$ consists of the interiors of k simplices, labelled by $\{f, p_0, \dots, p_k\}$. Let

$$B_k = \{(f, \{p_0, \dots, p_k\}) \in \Sigma_v \times \mathrm{UConf}_k(\mathbb{P}^n - p) \mid p_i \text{ are singular zeroes of } f\}.$$

We have a map $\phi \colon F_k - F_{k-1} \to B_k$, defined by

$$\phi((f, \{p_0, \dots, p_k\}), s_0, \dots, s_k) = (f, \{p_0, \dots, p_k\}).$$

The map ϕ expresses $F_k - F_{k-1}$ as a fibre bundle over B_k with Δ_k° fibres, ie we have a diagram

$$\begin{tikzcd} \Delta_k^\circ \arrow[r] & F_k - F_{k-1} \arrow[d] \\ & B_k \end{tikzcd}$$

We have a map $B_k \to \mathrm{UConf}_k(\mathbb{P}^n - p)$ defined by $\{f, p_0, \dots, p_k\} \mapsto \{p_0, \dots, p_k\}$. This is a vector bundle by Lemma 3.2 in [9]. $\square$

We have a one-dimensional local coefficient system denoted by $\pm\mathbb{Q}$ on $\mathrm{UConf}_k(\mathbb{P}^n - p)$ defined in the following way: Let S_k be the symmetric group on k letters. We have a homomorphism $\pi_1 \mathrm{UConf}_k(\mathbb{P}^n - p) \to S_k$ associated to the covering space $\mathrm{PConf}_k(\mathbb{P}^n - p) \to \mathrm{UConf}_k(\mathbb{P}^n - p)$. Compose this homomorphism with the sign representation $S_k \to \pm 1 = \mathrm{GL}_1(\mathbb{Q})$ to obtain our local system.

Proposition 3.4 *Let $d, n \geq 1$ and $e_d = \dim_{\mathbb{C}}(V_v)$. For $k < (d-1)/2$,*

$$\overline{H}_*(F_k - F_{k-1}) \cong H_{*-(k+2e_d-2(n+1)(k+1))}(\mathrm{UConf}_k(\mathbb{P}^n - p), \pm\mathbb{Q}).$$

Proof By Proposition 3.3 the space $F_k - F_{k-1}$ is a bundle over $\mathrm{UConf}_k(\mathbb{P}^n - p)$. This fact implies that

$$\overline{H}_*(F_k - F_{k-1}) \cong H_{*-(k+2e_d-2(n+1)(k+1))}(\mathrm{UConf}_k(\mathbb{P}^n - p), \mathbb{Q}(\sigma)).$$

Here $\mathbb{Q}(\sigma)$ is the local system obtained by the action of $\pi_1(\mathrm{UConf}_k(\mathbb{P}^n - p))$ on the fibres $\overline{H}_k(\Delta_k^\circ)$, where in this case Δ_k° is the open k–simplex corresponding to the fibres of the map $F_k - F_{k-1} \to B_k$. But one observes that the action of $\pi_1(\mathrm{UConf}_k(\mathbb{P}^n - p))$ on this open simplex is by permutation of the vertices, which implies $\mathbb{Q}(\sigma) = \pm\mathbb{Q}$. $\square$

As with any filtered space, we have a spectral sequence with

$$E_1^{p,q} = \overline{H}_{p+q}(F_p - F_{p-1}; \mathbb{Q})$$

converging to $\overline{H}_*(Y; \mathbb{Q})$. Now for $p < N$, by Proposition 3.4,

$$E_1^{p,q} = \overline{H}_{q-(2e_d-2(n+1)(p+1))}(\mathrm{UConf}_p(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

We would like to claim that $E_1^{N,q}$ doesn't matter in the stable range. To be more precise, we have the following:

Lemma 3.5 *Let $d, n \geq 1$, let $N = (d-1)/2$, and let $k > 2e_d - N$. Then*

$$\overline{H}_k(|\mathcal{C}| - F_N; \mathbb{Q}) \cong \overline{H}_k(|\mathcal{C}|; \mathbb{Q}).$$

Proof We first will try to bound the $\overline{H}_*(F_N; \mathbb{Q})$ and then use the long exact sequence of the pair. F_N is the union of locally closed subspaces

$$\phi_k = \{(f, x_1, \dots, x_k), p \mid f \in \Sigma^{\geq N}, x_i \text{ are singular zeroes of } f, p \in \Delta_k\}.$$

We have a surjection $\pi: \phi_k \to \mathrm{UConf}_k(\mathbb{P}^n - p)$. This map π is in fact a fibre bundle with fibres $\Delta^k \times \mathbb{C}^{e_d - N(n+1)}$. The space $\mathrm{UConf}_k(\mathbb{P}^n - p)$ is kn–dimensional. Therefore,

$$\overline{H}_*(\phi_k; \mathbb{Q}) = 0 \quad \text{if } * > 2(e_d - (n+1)N) + kn < 2e_d - N.$$

This implies that for all k, $\overline{H}_*(\phi_k; \mathbb{Q}) = 0$ if $* > 2e_d - N$. This implies $\overline{H}_*(F_N; \mathbb{Q}) = 0$ if $* > 2e_d - N$. By the long exact sequence in Borel–Moore homology associated to the pair $F_N \hookrightarrow Y$, $\overline{H}_k(Y - F_N; \mathbb{Q}) \cong \overline{H}_k(Y; \mathbb{Q})$ for $k > 2e_d - N$. □

4 Interlude

In [8], Tommasi proves the following result:

Theorem 4.1 [8] *Let $d, n \geq 1$, let $f \in X_{d,n}$, and let $\psi: \mathrm{GL}_{n+1}(\mathbb{C}) \to X_{d,n}$ be the orbit map defined by $\psi(g) = g \cdot f$. Then $\psi^*: H^k(X_{d,n}, \mathbb{Q}) \to H^k(\mathrm{GL}_{n+1}(\mathbb{C}), \mathbb{Q})$ is an isomorphism for $k < (d+1)/2$.*

In this section we shall look at the proof of Theorem 4.1 in [8] and use it to prove an identity used later on in this paper. One of the ingredients in the proof of Theorem 4.1 is a Vassiliev spectral sequence. We introduce a new convention, by letting h denote the dimension of H. We also define $\mathrm{Gr}(p, n)$ to be the Grassmannian of p–planes in $\mathbb{C}^n$. In what follows we shall need a few basic facts about $H_*(\mathrm{Gr}(p,n);\mathbb{Q})$ and Schubert symbols. Let

$$0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = \mathbb{C}^n$$

be a complete flag. Given $U \in \mathrm{Gr}(p,n)$, we can associate to it a sequence of numbers, $a_i = \dim U \cap E_i$. These a_i satisfy the conditions

$$0 \le a_{i+1} - a_i \le 1, a_0 = 0 \quad \text{and} \quad a_n = p.$$

Such sequences are called *Schubert symbols*. Let $\boldsymbol{a} = (a_0, \ldots, a_n)$. We call $\boldsymbol{a}$ a Schubert symbol if $0 \le a_{i+1} - a_i \le 1$, $a_0 = 0$ and $a_n = p$. Associated to each Schubert symbol $\boldsymbol{a}$ we have a subvariety $W_{\boldsymbol{a}} \subseteq \mathrm{Gr}(p,\mathbb{C}^n)$ defined as

$$W_{\boldsymbol{a}} := \overline{\{U \subseteq \mathbb{C}^n \mid \dim(U \cap \mathbb{C}^i) = a_i\}}.$$

The main result we will be using is the following.

Theorem 4.2 *Let $\boldsymbol{a}$ be a Schubert symbol. The classes $[W_{\boldsymbol{a}}] \in H_*(\mathrm{Gr}(p,n);\mathbb{Q})$ form a basis.*

For a proof of Theorem 4.2 see page 1071 of [4].

Proposition 4.3 *Let n be a positive integer. Then*

$$\sum_{k,p} h_k(\mathrm{Gr}(p,\mathbb{C}^n);\mathbb{Q}) = 2^n.$$

Proof By Theorem 4.2,

$$\begin{aligned}\sum_{k,p} h_k(\mathrm{Gr}(p,\mathbb{C}^n);\mathbb{Q}) &= \sum_p \#\{(a_0,\ldots,a_n) \mid 0 \le a_{i+1}-a_i \le 1, a_0 = 0, a_n = p\} \\ &= \#\{(a_0,\ldots,a_n) \mid 0 \le a_{i+1}-a_i \le 1, a_0 = 0\} \\ &= \#\{(b_1,\ldots,b_n) \in \{0,1\}\}.\end{aligned}$$

The last equality follows because if we are given a sequence of a_i, we can uniquely obtain a sequence of b_i, by letting $b_i = a_i - a_{i-1}$. □

Our main aim of this section is to prove the following technical result.

Theorem 4.4 *The Vassiliev spectral sequence in [8] degenerates in the stable range: if $p < (d+1)/2$ and $q > 0$, then $E_1^{p,q} \cong E_{p,q}^{\infty}$.*

Equivalently, for $k < (d+1)/2$,

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\mathbb{Q}) = h_k(\mathrm{GL}_{n+1};\mathbb{Q}). \tag{1}$$

Remark 4.5 The statements are equivalent because the group $\mathrm{H}^k(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q})$ is a subquotient of

$$\bigoplus \mathrm{H}_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\mathbb{Q}).$$

Proof We already know that

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) \geq h_k(\mathrm{GL}_{n+1};\mathbb{Q})$$

because the left hand side of (1) are the appropriate terms in a spectral sequence converging to the right hand side of (1).

It suffices to prove that

$$\sum_k \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) = \sum_k h_k(\mathrm{GL}_{n+1};\mathbb{Q}) = 2^{n+1}.$$

Lemma 2 in [10] states that

$$h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n),\pm\mathbb{Q}) = h_{2(p+1)(n+1)-p-k-1-p(p-1)}(\mathrm{Gr}(p,\mathbb{C}^{n+1});\mathbb{Q}).$$

Therefore

$$\sum_k \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n);\pm\mathbb{Q}) = \sum_k \sum_p h_k(\mathrm{Gr}(p,\mathbb{C}^{n+1});\mathbb{Q}).$$

By Proposition 4.3, this is equal to 2^{n+1}. □

5 Computation

We would like to know what the groups $\overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n - p);\pm\mathbb{Q})$ are. First note that in [10] Vassiliev proves that:

Proposition 5.1 [10] *Let $k, n > 0$. Then*

$$H_*(\mathrm{UConf}_k(\mathbb{P}^n);\pm\mathbb{Q}) \cong H_{*-(k)(k-1)}(\mathrm{Gr}_k(\mathbb{C}^{n+1});\mathbb{Q}).$$

Also note that in light of Theorem 4.2 the homology of Grassmannians is well understood in terms of Schubert cells.

Consider the long exact sequence in Borel–Moore homology associated to

$$\mathrm{UConf}_{k+1}(\mathbb{P}^n - p) \subseteq \mathrm{UConf}_{k+1}(\mathbb{P}^n) \hookleftarrow \mathrm{UConf}_k(\mathbb{P}^n - p).$$

The last inclusion is defined by the map $\phi\colon \mathrm{UConf}_k(\mathbb{P}^n - p) \to \mathrm{UConf}_{k+1}(\mathbb{P}^n)$, where $\phi(\{x_1, \dots, x_n\}) = \{x_1, \dots, x_n, p\}$.

We consider the long exact sequence in Borel–Moore homology associated to the pair $(\mathrm{UConf}_{k+1}(\mathbb{P}^n), \mathrm{UConf}_{k+1}(\mathbb{P}^n - p))$. Here $\mathrm{UConf}_{k+1}(\mathbb{P}^n - p)$ is an open subset of $\mathrm{UConf}_{k+1}(\mathbb{P}^n)$ with complement homeomorphic to $\mathrm{UConf}_k(\mathbb{P}^n - p)$. A segment of this exact sequence is

$$\begin{aligned}(2)\quad \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \to \overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \\ \to \overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n - p); \pm\mathbb{Q})\end{aligned}$$

Proposition 5.2 *Let $k, n > 0$. Then there is a canonical decomposition*

$$\begin{aligned}&\overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \\ &\qquad \cong \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \oplus \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}),\end{aligned}$$

due to the fact that (2) splits.

Proof Lemma 2 of [10] implies that (2) decomposes into split short exact sequences,

$$\begin{aligned}&\overline{H}_*(\mathrm{UConf}_{k+1}(\mathbb{P}^n); \pm\mathbb{Q}) \\ &\qquad \cong \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}) \oplus \overline{H}_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q}). \qquad \square\end{aligned}$$

Remark 5.3 In fact $H_*(\mathrm{UConf}_k(\mathbb{P}^n - p); \pm\mathbb{Q})$ has a basis given by Schubert symbols with $a_1 = 0$.

Proposition 5.4 *If the Vassiliev spectral sequence has no nonzero differentials and $k < (d-1)/2$, then $H^k(X_v) \cong H^k(G_p)$ as vector spaces.*

Proof Now in our spectral sequence we had

$$E_1^{p,q} = \overline{H}_{q-(2e_d - 2(p+1)(n+1))}(\mathrm{UConf}_{p+1}(\mathbb{P}^n - p); \pm\mathbb{Q}).$$

First collect all terms in the main diagonal, ie

$$V := \bigoplus_{p+q=l} \overline{H}_{q-(2D_n-2(p+1)(n+1))}(\mathrm{UConf}_{p+1}(\mathbb{P}^n - p); \pm\mathbb{Q})$$

It will suffice to prove that

$$\dim V = \sum_{p \le 2D_n - k} h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) \tag{3}$$
$$= h^k(\mathrm{GL}_n; \mathbb{Q}).$$

Theorem 4.4 implies

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = h_k(\mathrm{GL}_{n+1}; \mathbb{Q}). \tag{4}$$

Proposition 5.1 implies

$$h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = 0 \quad \text{if } p > n.$$

So as long as $n < 2(D_n + n + 1) - k$,

$$\sum_{p \le 2(D_n+n+1)-k} h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}).$$

But the condition $n < 2(D_n + n + 1) - k$ is equivalent to $k < 2(D_n + 1) + n$, which is true if $k < N$. We have another equality from Proposition 5.2,

$$h_k(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) + h_k(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}) = h_k(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}).$$

Plugging this into (4),

$$\begin{aligned} h^k(\mathrm{GL}_{n+1}; \mathbb{Q}) &= \sum h_{2(p+1)(n+1)-p-k}(\mathrm{UConf}_p(\mathbb{P}^n); \pm\mathbb{Q}) \\ &= \sum h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) \\ &\qquad + h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}). \end{aligned}$$

We have the identity

$$h^k(\mathrm{GL}_n; \mathbb{Q}) + h^{k-(2n+1)}(\mathrm{GL}_n; \mathbb{Q}) = h^k(\mathrm{GL}_{n+1}; \mathbb{Q}).$$

This implies

$$\begin{aligned} &h^k(\mathrm{GL}_n; \mathbb{Q}) + h^{k-(2n+1)}(\mathrm{GL}_n; \mathbb{Q}) \\ &\quad = \sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n - pt); \pm\mathbb{Q}) \\ &\qquad + h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n - pt); \pm\mathbb{Q}). \end{aligned} \tag{5}$$

Now we will try to prove (3) by induction on k. For $k=0$, (3) is trivial. By induction,

$$h^{k-(2n+1)}(\mathrm{GL}_n;\mathbb{Q})=\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_{p-1}(\mathbb{P}^n-pt);\pm\mathbb{Q}).$$

Putting this into (5), we obtain

$$\sum_p h_{2(p+1)(n+1)-p-k-1}(\mathrm{UConf}_p(\mathbb{P}^n-pt);\pm\mathbb{Q})=h^k(\mathrm{GL}_n;\mathbb{Q}).$$ □

Now we can look at the Serre spectral sequence associated to the fibration

$$X_v\hookrightarrow X_p\to\mathbb{C}^n-0.$$

We observe that if there are no nonzero differentials, then

$$H^*(X_p;\mathbb{Q})\cong H^*(X_v;\mathbb{Q})\otimes\mathbb{Q}[e_{2n-1}]/e_{2n-1}^2.$$

This is because the Serre spectral sequence degenerates and since $\mathbb{Q}[e_{2n-1}]/e_{2n-1}^2$ is a free graded commutative algebra the ring structure of the total space is forced to be the tensor product.

Proposition 5.5 *Let $d>0$ and $p\in\mathbb{P}^n$. Then*

$$H^*(X_{d,p};\mathbb{Q})\cong H^*(G_p;\mathbb{Q})\otimes A,$$

where A is $H^*(X_d^p/G_p;\mathbb{Q})$.

Proof This follows immediately from Theorem 2 in [6]. □

We will also need the following fact, which is a special case of Lemma 2.6 in [3].

Proposition 5.6 *Let $d>0$, let $k<(d-1)/2$, and let $U_d^*=X_d^*/\mathbb{C}^*$. Then*

$$H^*(X_d^*;\mathbb{Q})\cong H^*(U_d^*;\mathbb{Q})\otimes\mathbb{Q}[e_1]/(e_1^2),$$

where $|e_1|=1$.

Proposition 5.6 implies if there are no nonzero differentials in both our Vassiliev spectral sequence and in the Serre spectral sequence associated to the fibration $X_{d,n}^p\to\mathbb{C}^n-0$ then

$$H^*(U_{d,p};\mathbb{Q})\cong H^*(G_p;\mathbb{Q})\otimes\mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2)$$

for $*<(d-1)/2$. In case there are nonzero differentials in either spectral sequence, then $H^*(U_{d,p};\mathbb{Q})\cong H^*(G_p;\mathbb{Q})$ for $*<(d-1)/2$.

6 Comparing fibre bundles

In this section we finish the proof of Theorem 1.2.

Proof of Theorem 1.2 We compare three related fibre bundles and their associated spectral sequences. This is similar to the proof of Theorem 1.1 in [3].

Let $\mathrm{PG}_p := \mathrm{Stab}_{\mathrm{PGL}(n+1)}(p)$:

$$\begin{tikzcd}
\mathrm{PG}_p \arrow[r] \arrow[dr] & U_{d,p} \arrow[r] \arrow[dr] & U_d \arrow[dr] & \\
& \mathrm{PGL}_{n+1}(\mathbb{C}) \arrow[r] \arrow[d] & U_d^* \arrow[r] \arrow[d] & U_d \times \mathbb{P}^n \arrow[d] \\
& \mathbb{P}^n \arrow[r, equal] & \mathbb{P}^n \arrow[r, equal] & \mathbb{P}^n
\end{tikzcd} \tag{6}$$

We denote the exterior algebra on generators $a_1, \dots, a_n$ by

$$\Lambda\langle a_1, \dots, a_n\rangle.$$

By Proposition 5.4 and [6, Theorem 1], there are two possibilities for $H^*(U_{d,p};\mathbb{Q})$: either

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(\mathrm{PG}_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2) \cong \Lambda\langle u_1, u_3, \dots, u_{2n-1}, e_{2n-1}\rangle$$

or

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(\mathrm{PG}_p;\mathbb{Q}) = \Lambda\langle u_1, u_3, \dots, u_{2n-1}\rangle.$$

Suppose for the sake of contradiction $H^*(U_{d,p}) = \Lambda\langle u_3, \dots, u_{2n-1}\rangle$ for $* < (d-1)/2$. In this case, $H^*(U_{d,p};\mathbb{Q}) \cong H^*(\mathrm{PG}_p;\mathbb{Q})$ for $* < (d-1)/2$. Then since the homology of the base and the fibres are isomorphic, $H^*(U_d^*;\mathbb{Q}) \cong H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q})$ for $* < (d-1)/2$. However, by Proposition 2.1,

$$H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathbb{Q}[x]/x^n) \subseteq H^*(U_d^*;\mathbb{Q}).$$

But $H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q})$ does not contain a subalgebra isomorphic to

$$H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes \mathbb{Q}[x]/x^n).$$

This is a contradiction. So we must be in the case where

$$H^*(U_{d,p};\mathbb{Q}) \cong H^*(\mathrm{PG}_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2).$$

Consider the Serre spectral sequence associated to the fibration $U_d^* \to \mathbb{P}^n$. Its E_2 page has terms

$$E_2^{p,q} = H^p(\mathbb{P}^n, H^q(U_d^p;\mathbb{Q})) \cong H^p(\mathbb{P}^n;\mathbb{Q}) \otimes H^q(U_d^p;\mathbb{Q}).$$

Now

$$H^q(U_d^p;\mathbb{Q}) \cong H^q(\mathrm{PG}_p;\mathbb{Q}) \otimes \mathbb{Q}[e_{2n-1}]/(e_{2n-1}^2).$$

Consider the trivial fibre bundle $U_d \times \mathbb{P}^n \to \mathbb{P}^n$. There is a natural inclusion of fibre bundles as shown in (6). This induces a map of spectral sequences between the associated Serre spectral sequences.

Note that any class $\alpha \in H^q(U_d^p;\mathbb{Q})$ that lies in the image of $H^q(U_d;\mathbb{Q})$ is mapped to zero under any differential thanks to the fact that all differentials are zero in the spectral sequence associated to a trivial fibration. The only possible nonzero differential in the E_2 page of the Serre spectral sequence associated to the fibration $U_d^* \to \mathbb{P}^n$ is $d(e_{2n-1})$.

Suppose for contradiction that $d(e_{2n-1}) = 0$. This implies that

$$H^k(U_d^*;\mathbb{Q}) \cong (H^*(U_{d,p};\mathbb{Q}) \otimes H^*(\mathbb{P}^n;\mathbb{Q}))_k = (H^*(\mathrm{PG}_p;\mathbb{Q}) \otimes H^*(\mathbb{P}^n,\mathbb{Q}))_k$$

for $k < (d-1)/2$.

Let $p(t)$ be the Poincaré polynomial of U_d^*. We already know that

$$H^*(U_d^*;\mathbb{Q}) \cong H^*(\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}) \otimes H^*(U_d^*/\mathrm{PGL}_{n+1}(\mathbb{C});\mathbb{Q}).$$

So $(1+t^3)\cdots(1+t^{2n+1}) \mid p(t)$. On the other hand, if $de_{2n-1} = 0$ then

$$p(t) = (1+t^3)\cdots(1+t^{2n-1})(1+t^2+t^4+\cdots+t^{2n}) \mod t^{(d-1)/2}.$$

If $d \geq 4n+1$, then this implies that $(1+t^{2n+1}) \nmid p(t)$. This is a contradiction.

So we must have a differential killing the class in $H^{2n}(\mathbb{P}^n, H^0(U_{d,p});\mathbb{Q})$. The differential must come from e_{2n-1}; ie $d(e_{2n-1}) = ax^n$ for some $a \in \mathbb{Q}^*$. This (along with multiplicativity of differentials) determines all differentials and implies (1). By Proposition 5.6, (1) implies (2). By Theorem 1 of [6],

$$H^*(X_{d,n}^*;\mathbb{Q}) \cong H^*(M_{d,n}^*;\mathbb{Q}) \otimes (H^*(\mathrm{GL}_{n+1})(\mathbb{C});\mathbb{Q}).$$

In light of this, (2) implies (3). □

Having finished the proof of Theorem 1.2 we can prove Corollary 1.3.

Proof of Corollary 1.3 Consider the fibration

$$\begin{tikzcd} Z(f) \arrow[r] & X_d^* \arrow[d] \\ & X_d \end{tikzcd}$$

and its associated Serre spectral sequence whose E_2 page is of the form

$$H^p(X_d; H^q(Z(f);\mathbb{Q})) \Rightarrow H^*(X_d^*;\mathbb{Q}).$$

By Theorem 4.1 for $* < (d+1)/2$,

$$H^*(X_d;\mathbb{Q}) \cong H^*(\mathrm{GL}_{n+1}(\mathbb{C});\mathbb{Q}).$$

By Theorem 1.2, we know that the classes in the E_2 page corresponding to the group $H^p(\mathrm{GL}_{n+1}(\mathbb{C}); c_1(\mathscr{L})^q)$ survive until the E^∞ page, and in the stable range all other terms are killed by differentials.

Now suppose n is even. Then the only other terms in the spectral sequence are of the form $H^p(X_d; H^{n-1}(Z(f);\mathbb{Q}))$. However it is not possible for any such term to be in the image or in the preimage of a nonzero differential. This is because all other terms survive, so any possible nonzero differential must be from $H^{p_1}(X_d; H^{n-1}(Z(f);\mathbb{Q}))$ to $H^{p_2}(X_d; H^{n-1}(Z(f);\mathbb{Q}))$ for some choice of p_1 and p_2. However no differential is of bidegree $(p_2-p_1, 0)$. This implies that

$$H^p(X_d; H^{n-1}(Z(f);\mathbb{Q})) \cong 0.$$

A similar argument shows that if n is odd, $H^p(X_d; H^{n-1}(Z(f);\mathbb{Q})) \cong H^p(X_d;\mathbb{Q})$. Essentially the only difference between the even case and the odd case is that in the odd case we have a class $c_1(\mathscr{L})^{(n-1)/2} \in H^{n-1}(Z(f);\mathbb{Q})$. Let $A = \mathbb{Q}\text{–span}(c_1(\mathscr{L})^{(n-1)/2})$ By Theorem 1.2, we know that $H^p(X_{d,n}; A)$ survives until the E^∞ page. An argument similar to that in the even case shows that

$$H^p(X_d; H^{n-1}(Z(f);\mathbb{Q})) \cong H^p(X_d; A). \qquad \square$$

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On the wheeled PROP of stable cohomology of Aut(F_n) with bivariant coefficients

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We show that the stable cohomology of automorphism groups of free groups with coefficients obtained by applying Hom$(-,-)$ to tensor powers of the abelianization, is equipped with the structure of a wheeled PROP $\mathcal{H}$. We define another wheeled PROP $\mathcal{E}$ by Ext–groups in the category of functors from the category of finitely generated free groups to $\mathbb{k}$–modules. The main result of this paper is the construction of a morphism of wheeled PROPs $\varphi\colon \mathcal{E} \to \mathcal{H}$ such that $\varphi(\mathcal{E})$ is the wheeled PROP generated by the cohomology class h_1 constructed by the first author.

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1 Introduction

This paper concerns the cohomology of automorphism groups of free groups $\mathrm{Aut}(\mathbb{Z}^{*n})$ for $n \in \mathbb{N}$, with coefficients given by the $\mathbb{k}$–modules

$$B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n}) := \mathrm{Hom}_{\mathcal{V}}((\mathbb{k}^n)^{\otimes l}, (\mathbb{k}^n)^{\otimes q}),$$

where $l, q \in \mathbb{N}$, $\mathbb{k}$ is a commutative ring and $\mathcal{V}$ is the category of $\mathbb{k}$–modules, and where the structure of $\mathrm{Aut}(\mathbb{Z}^{*n})$–module on $B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})$ is given by the diagonal action.

In [8] — see also [9] — for $n \geq 2$, the first author introduced a nonzero cohomology class

$$h_1 \in H^1(\mathrm{Aut}(\mathbb{Z}^{*n}), \mathrm{Hom}_{\mathcal{V}}(\mathbb{k}^n, (\mathbb{k}^n)^{\otimes 2}))$$

and constructed, from h_1, cohomology classes

$$h_p \in H^p\big(\mathrm{Aut}(\mathbb{Z}^{*n}), \mathrm{Hom}_{\mathcal{V}}(\mathbb{k}^n, (\mathbb{k}^n)^{\otimes p+1})\big)$$

for $p > 1$ and $\bar{h}_p \in H^p(\mathrm{Aut}(\mathbb{Z}^{*n}), (\mathbb{k}^n)^{\otimes p})$ for $p \geq 1$, even in the unstable range. The construction of these classes is inspired by previous works of Morita [19] and Morita with the first author [10; 11] concerning cohomology classes of the mapping class group with trivial coefficients $\mathbb{Q}$; see Remark 7.3.

By a classical construction (see Section 4), there are group morphisms

$$H^*(\mathrm{Aut}(\mathbb{Z}^{*n+1}); B_{l,q}(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1})) \xrightarrow{\alpha_n} H^*(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})).$$

The stable cohomology of the automorphism groups of free groups with coefficients given by $B_{l,q}$ is defined by

$$H^*_{\mathrm{st}}(B_{l,q}) := \lim_{n\in\mathbb{N}} H^*(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n}))$$

where the limit is taken over the group morphisms α_n.

By a result of Randal-Williams and Wahl [21], this cohomology stabilizes so that the stable cohomology $H^i_{\mathrm{st}}(B_{l,q})$ is isomorphic to $H^i(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n}))$ for n big enough. It follows from the stability that stable cohomology is equipped with a cup product map

$$\cup\colon H^*_{\mathrm{st}}(B_{l_1,q_1}) \otimes H^*_{\mathrm{st}}(B_{l_2,q_2}) \to H^*_{\mathrm{st}}(B_{l_1,q_1} \otimes B_{l_2,q_2})$$

for $l_1, q_1, l_2, q_2 \in \mathbb{N}$.

In Definition 6.1 we define the PROP $\mathcal{H}$, where the morphisms are the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{H}(q,l) = H^*_{\mathrm{st}}(B_{l,q})$$

where the action of symmetric group $\mathfrak{S}_q$ (resp. $\mathfrak{S}_l$) is given by place permutation on the tensor product $(-)^{\otimes q}$ (resp. $(-)^{\otimes l}$) and where the horizontal composition is given by the cup product map for stable cohomology and the vertical composition is induced by the composition in $\mathcal{V}$.

We show that this PROP is equipped with further structure:

Proposition 1 (Proposition 6.2) *The PROP $\mathcal{H}$ is a wheeled PROP, ie it is equipped with contraction maps*

$$\xi^i_j\colon \mathcal{H}(q,l) \to \mathcal{H}(q-1, l-1)$$

for $1 \le i \le q$ and $1 \le j \le l$ compatible with the structure of PROP.

Wheeled PROPs were introduced by Markl, Merkulov and Shadrin in [15] to treat PROPs equipped with trace maps. The typical example of a wheeled PROP is the PROP of endomorphism of a free finitely generated module where the contractions are given by partial trace maps; see Example 2.2. The wheeled PROP structure on the PROP $\mathcal{H}$ should be viewed as a cohomological version of the wheeled endomorphism PROP.

In the stable range, for $p > 1$, the cohomology classes h_p are obtained from h_1 using the horizontal and vertical composition in the PROP $\mathcal{H}$, and for $p \geq 1$, the classes $\bar{h}_p$ are obtained from h_p using the contraction maps. We deduce that the classes h_p and $\bar{h}_p$ are in the subwheeled PROP $\mathcal{K}$ of $\mathcal{H}$ generated by the class h_1.

Understanding the stable cohomology of $\mathrm{Aut}(\mathbb{Z}^{*n})$ with coefficients given by $B_{l,q}$ is equivalent to giving a description of the wheeled PROP $\mathcal{H}$ in terms of generators and relations. This is open; in particular, it is unknown whether the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{H}$ is strict.

By the results of Djament and the second author [1; 3], we know that the stable cohomology of $\mathrm{Aut}(\mathbb{Z}^{*n})$ with nonconstant coefficients is closely related to Ext–groups in the category $\mathcal{F}(\mathbf{gr}; \Bbbk)$ of functors from the category **gr** of finitely generated free groups to the category $\mathcal{V}$ of $\Bbbk$–modules. More precisely, the main result of [3], obtained using functor homology methods, implies that $\mathcal{H}(0, l) = 0$ for $l > 0$ and [1, Théorème 4] gives, for $\Bbbk = \mathbb{Q}$, a natural isomorphism

$$\mathcal{H}(q, 0) \simeq \bigoplus_{j \in \mathbb{N}} \mathrm{Ext}^{*-j}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}), \tag{1-1}$$

where $\mathfrak{a}^{\otimes q}$ is the q^{th} tensor power of the abelianization functor and $\Lambda^j \mathfrak{a}$ is the j^{th} exterior power of the abelianization functor; see Section 3. The Ext–groups on the right-hand side of the isomorphism (1-1) are computed in [22] (the result is recalled in Proposition 10.1) giving the explicit computation of $\mathcal{H}(q, 0)$.

Note that Randal-Williams obtained in [20] the computation of $\mathcal{H}(0, l)$ and $\mathcal{H}(q, 0)$ using geometric techniques independent of the approach in [1; 3].

Inspired by a conjecture given in [1] we define in Section 10, for $\Bbbk = \mathbb{Q}$, a PROP $\mathcal{E}$ where the morphisms are the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{E}(q, l) = \bigoplus_{j \in \mathbb{N}} \mathrm{Ext}^{*-j}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}).$$

We give an explicit description of the PROP $\mathcal{E}$ (see Theorem 5) and deduce from it that the PROP $\mathcal{E}$ inherits a structure of *wheeled* PROP.

The main result of this paper is the following:

Theorem 2 (Theorem 11.1) *There is a morphism of wheeled PROPs*,

$$\varphi : \mathcal{E} \to \mathcal{H},$$

such that $\varphi(\mathcal{E}) \simeq \mathcal{K}$.

Djament's conjecture can be rephrased in the following way:

Conjecture 3 *The morphism* φ *is an isomorphism of wheeled PROPs.*

This conjecture would imply that the inclusion functor $\mathcal{K} \hookrightarrow \mathcal{H}$ is an equivalence of categories, ie the class h_1 would generate the wheeled PROP $\mathcal{H}$.

The sub-PROP $\mathcal{E}_0$ of $\mathcal{E}$ given by

$$\mathcal{E}_0(q,l) = \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$$

has been studied by the second author; in particular, by [22, Proposition 3.5] the PROP $\mathcal{E}_0$ is generated by its underlying operad $\mathcal{P}_0$. In Proposition 9.13 we give an explicit description of this operad by generators and relations. A more conceptual description of the operad $\mathcal{P}_0$ is the following:

Proposition 4 (Proposition 9.14) *The operad* $\mathcal{P}_0$ *is the operadic suspension of the operad* $\mathcal{C}om$ *of* ***nonunital*** *commutative algebras.*

The previous results on $\mathcal{E}_0$ and $\mathcal{P}_0$ are also true for $\Bbbk = \mathbb{Z}$.

The forgetful functor from wheeled PROP to operads has a left adjoint. We denote by $\mathcal{C}_{\mathcal{P}_0^{\circlearrowleft}}$ the wheeled PROP associated to the operad $\mathcal{P}_0$ by this functor. We obtain the following result.

Theorem 5 (Theorem 10.11) *There is an isomorphism of PROPs*

$$\chi : \mathcal{C}_{\mathcal{P}_0^{\circlearrowleft}} \xrightarrow{\simeq} \mathcal{E}.$$

In particular, $\mathcal{E}$ inherits a structure of *wheeled* PROP via this isomorphism. The existence of a wheeled structure on the PROP $\mathcal{E}$ is quite surprising and is very specific to the situation studied in this paper; see Remark 10.12. We deduce from Theorem 5 a description of the wheeled PROP $\mathcal{E}$ by generators and relations.

Let $\mathcal{E}_w$ (resp. $\mathcal{H}'$) be the sub-PROP of $\mathcal{E}$ (resp. $\mathcal{H}$) keeping only the morphisms to 0 and the endomorphisms in degree 0 in $\mathcal{E}$ (resp. $\mathcal{H}$). Djament's result can be rephrased in the following way:

Proposition 6 [1, Théorème 4] *By restriction*, φ *induces an isomorphism of PROPs*, $\varphi' : \mathcal{E}_w \xrightarrow{\simeq} \mathcal{H}'$.

Notation We denote by $\mathbb{N} = \{0, 1, \dots\}$ and by $\boldsymbol{q}$ the set $\{1, \dots, q\}$.

Throughout the paper, $\Bbbk$ is a commutative ring which will be, most of the time, a field of characteristic zero or $\mathbb{Z}$. We denote by $\mathcal{V}$ the category of $\Bbbk$–modules and $\mathcal{V}^f$ its full subcategory of free finitely generated modules.

A *homological* $\mathbb{Z}$–graded $\Bbbk$–module is denoted by $V_\bullet = \bigoplus_n V_n$, and a morphism *of homological degree* d, $f : V_\bullet \to W_\bullet$, is a family of linear maps $f_n : V_n \to W_{n+d}$ for all $n \in \mathbb{Z}$. To $V_\bullet$ a homological $\mathbb{Z}$–graded $\Bbbk$–module, we associate a *cohomological* $\mathbb{Z}$–graded $\Bbbk$–module $V^\bullet$ by $V^n := V_{-n}$. A morphism of homological degree d corresponds to a morphism of *cohomological degree* $-d$.

Graded $\Bbbk$–modules and morphisms of degree 0 form a category denoted by $\mathrm{gr}\,\mathcal{V}$. For $\otimes$, the tensor product of $\mathbb{Z}$–graded $\Bbbk$–modules, the category $(\mathrm{gr}\,\mathcal{V}, \otimes, \Bbbk)$ is equipped with the symmetry given by the maps $\tau : V \otimes W \to W \otimes V$ defined by $\tau(v \otimes w) := (-1)^{pq} w \otimes v$ where $v \in V_p$ and $w \in W_q$.

For V a $\Bbbk$–module we denote again by V the graded $\Bbbk$–module concentrated in degree 0, where it is equal to V.

Let $\Bbbk s$ be the graded $\Bbbk$–module concentrated in degree one and such that $(\Bbbk s)_1$ is spanned by s; the *suspension* of a graded $\Bbbk$–module V is $sV := \Bbbk s \otimes V$, so that $(sV)_i = V_{i-1}$.

The duality functor, denoted by $-^* : (\mathcal{V})^{\mathrm{op}} \to \mathcal{V}$ is defined by $\mathrm{Hom}_{\mathcal{V}}(-, \Bbbk)$.

Nonspecified tensor products are taken over $\Bbbk$.

For objects C and C' of a category $\mathcal{C}$, the set of morphisms from C to C' is denoted by $\mathrm{Hom}_{\mathcal{C}}(C, C')$ or $\mathcal{C}(C, C')$.

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Part I Recollections

2 Recollections on PROPs and operads

PROPs and operads arose in the work of Mac Lane [13]. Since then, they have turned out to be very important algebraic structures, especially in algebraic topology.

In this section we recall some basic facts that we will use in the paper on PROPs and operads, as well as their wheeled versions introduced more recently by Markl, Merkulov and Shadrin [15].

2.1 Classical PROPs and operads

For PROPs, we refer the reader to [14] for further details. The notion of PROP is closely related to the notion of operad. For operads, we refer the reader to [12].

A PROP is a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ with objects the natural numbers whose symmetric monoidal structure $\otimes$ is given by the sum of integers on objects.

In this paper we will consider PROPs enriched over $\mathrm{gr}\,\mathcal{V}$, called graded linear PROPs. Such a PROP $\mathcal{C}$ is a collection $\{\mathcal{C}(m,n)\}_{m,n\in\mathbb{N}}$ of graded $(\mathfrak{S}_m, \mathfrak{S}_n)$–bimodules (ie graded left $\mathfrak{S}_m \otimes (\mathfrak{S}_n)^{\mathrm{op}}$–modules) together with two types of compositions: the horizontal composition

$$\mathcal{C}(m_1,n_1) \otimes \mathcal{C}(m_2,n_2) \to \mathcal{C}(m_1+m_2, n_1+n_2)$$

induced by the monoidal product, and the vertical composition

$$\mathcal{C}(n,l) \otimes \mathcal{C}(m,n) \to \mathcal{C}(m,l)$$

given by the categorical composition.

An *operation of biarity* (m,n) in a PROP $\mathcal{C}$ is an element in $\mathcal{C}(m,n)$.

In the rest of this paper all the operads and PROPs will be *graded linear*. To simplify the terminology we will call them simply operads and PROPs.

An important example of a PROP is the endomorphism PROP of a graded $\mathbb{k}$–module:

Example 2.1 To an object V of $\mathrm{gr}\,\mathcal{V}$, we associate the PROP, denoted by $\mathcal{E}nd_V$, defined by

$$\mathcal{E}nd_V(m,n) = \mathrm{Hom}_{\mathrm{gr}\,\mathcal{V}}(V^{\otimes m}, V^{\otimes n}),$$

with the action of the symmetric groups given by the action on the tensor product by place permutations. The horizontal composition is given by the tensor product of linear maps and the vertical composition by the composition in $\mathcal{V}$.

Every PROP $\mathcal{C}$ has an underlying operad $\mathcal{P}_{\mathcal{C}}$ given by $\mathcal{P}_{\mathcal{C}}(n) = \mathrm{Hom}_{\mathcal{C}}(n,1)$.

Conversely, every operad $\mathcal{P}_0$ generates a PROP $\mathcal{C}_{\mathcal{P}_0}$ where

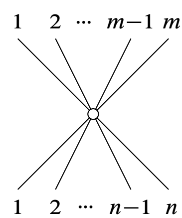
$$\mathcal{C}_{\mathcal{P}_0}(q,l) = \bigoplus_{f:\boldsymbol{q}\to\boldsymbol{l}} \bigotimes_{i=1}^{l} \mathcal{P}_0(f^{-1}(i)).$$

For two PROPs, $\mathcal{C}$ and $\mathcal{C}'$, a morphism of PROPs is a strict monoidal functor $F:\mathcal{C}\to\mathcal{C}'$ which is the identity on the objects and graded linear (ie the maps between the Hom–sets are morphisms of degree 0).

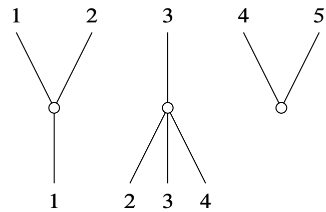
For a PROP $\mathcal{C}$, a morphism in $\mathcal{C}(m,n)$ can be represented by a *directed* (m,n)*–graph*, ie a finite, not necessary connected, graph such that each edge is equipped with a direction, there are no directed cycles and the set of legs is divided into the set of inputs labeled by $\{1,\dots,m\}$ and outputs labeled by $\{1,\dots,n\}$.

In our pictures the graphs are oriented from top to bottom.

Using the horizontal composition in $\mathcal{C}$, each morphism in $\mathcal{C}$ is the disjoint union of (m,n)*–corollas* which are (connected) graphs of the form



For example, the following depicts a morphism in $\mathcal{C}(5,4)$:



2.2 Wheeled PROPs and wheeled operads

In this section we recall the wheeled versions of PROPs and operads, introduced by Markl, Merkulov and Shadrin in [15], in order to encode algebras equipped with traces; see Example 2.2. Note that wheeled PROPs are particular cases of traced monoidal categories introduced by Joyal, Street and Verity in [7].

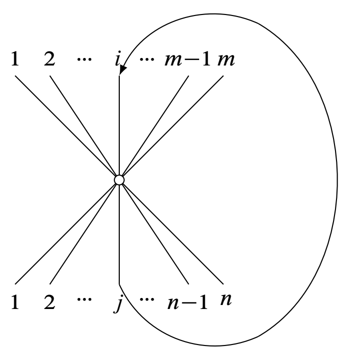
A wheeled PROP is a PROP equipped with *contractions*

$$\xi^i_j : \mathcal{C}(m,n) \to \mathcal{C}(m-1,n-1)$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. These contractions satisfy compatibility axioms.

For a wheeled PROP $\mathcal{C}$, a morphism in $\mathcal{C}(m,n)$ can be represented by a directed (m,n)–graph having possibly wheels and loops.

The contraction ξ^i_j can be viewed as connecting the i^{th} input and the j^{th} output. For example, for an (m,n)–corolla we have the following picture:



In a wheeled PROP, vertical composition is determined by the horizontal composition and the contractions by the formula

$$\mathcal{C}(n,l) \otimes \mathcal{C}(m,n) \to \mathcal{C}(n+m,l+n) \xrightarrow{(\xi^1_{l+1})^n} \mathcal{C}(m,l); \tag{2-1}$$

see [15, (17)].

A fundamental example of a wheeled PROP is the wheeled endomorphism PROP associated with a free finitely generated $\Bbbk$–module where the contractions are given by the trace map:

Example 2.2 By classical linear algebra, for objects E and F in $\mathcal{V}$, we have canonical homomorphisms $E^* \otimes F \to \mathrm{Hom}_{\mathcal{V}}(E, F)$ and $E^* \otimes F^* \to (E \otimes F)^*$ which are isomorphisms if E is a free finitely generated $\Bbbk$–module.

For V an object of $\mathcal{V}^f$, the previous observations give rive to the isomorphisms

$$\theta_{m,n} \colon \mathrm{Hom}_{\mathcal{V}^f}(V^{\otimes m}, V^{\otimes n}) \xrightarrow{\simeq} (V^*)^{\otimes m} \otimes V^{\otimes n}.$$

By Example 2.1, $\mathcal{E}nd_V(m,n) = \mathrm{Hom}_{\mathcal{V}}(V^{\otimes m}, V^{\otimes n})$ defines a PROP. For $1 \le i \le m$ and $1 \le j \le n$, the contractions $\xi^i_j \colon \mathcal{E}nd_V(m,n) \to \mathcal{E}nd_V(m-1,n-1)$ correspond through the previous isomorphisms to the maps

$$\varphi^i_j \colon (V^*)^{\otimes m} \otimes V^{\otimes n} \to (V^*)^{\otimes m-1} \otimes V^{\otimes n-1}$$

given by

$$\begin{aligned}&\varphi^i_j(f_1 \otimes \cdots \otimes f_m \otimes v_1 \otimes \cdots \otimes v_n) \\ &\quad = f_i(v_j)(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_m \otimes v_1 \otimes \cdots \otimes v_{j-1} \otimes v_{j+1} \otimes \cdots \otimes v_n)\end{aligned}$$

where $f_i \in V^*$ and $v_i \in V$.

Note that $\varphi^1_1 \colon V^* \otimes V \to \Bbbk$ is the evaluation and $\varphi^1_1 \circ \theta_{1,1} \colon \mathrm{Hom}_{\mathcal{V}^f}(V, V) \to \Bbbk$ is the trace map Tr which associates to an endomorphism of V its trace.

A morphism of wheeled PROPs is a morphism of PROPs that is compatible with the contractions.

The forgetful functor from the category of wheeled PROPs to the category of PROPs has a left adjoint denoted by $(-)^{\circlearrowleft}$. For $\mathcal{C}$ a PROP, $\mathcal{C}^{\circlearrowleft}$ is called the *wheeled completion* of $\mathcal{C}$; see [15, Definition 2.1.9].

Recall from [15, Definition 5.1.1], that a *wheeled operad* $\mathcal{P} = \{\mathcal{P}(n,m)\}_{m,n}$, where $n \in \mathbb{N}$ and $m \in \{0, 1\}$, consists of

(1) an ordinary operad $\mathcal{P}_0 := \{\mathcal{P}(n,1)\}_{n \ge 0}$;

(2) a right $\mathcal{P}_0$–module $\mathcal{P}_w := \{\mathcal{P}(n,0)\}_{n \ge 0}$;

(3) for $1 \le i \le n$, contractions $\xi^i \colon \mathcal{P}_0(n) \to \mathcal{P}_w(n-1)$, satisfying compatibility conditions with the structures given in (1) and (2).

The operad $\mathcal{P}_0$ is called the *operadic part* of $\mathcal{P}$ and $\mathcal{P}_w$ its *wheeled* part.

Recall that the operad $\mathcal{P}_0$ is itself a right $\mathcal{P}_0$–module.

Every wheeled PROP $\mathcal{C}$ has an underlying wheeled operad $\mathcal{P}^{\mathcal{C}}$ where the operadic part is $\mathcal{P}^{\mathcal{C}}_0 = \{\mathrm{Hom}_{\mathcal{C}}(n,1)\}$, the wheeled part is $\mathcal{P}^{\mathcal{C}}_w = \{\mathrm{Hom}_{\mathcal{C}}(n,0)\}$ and the contractions $\xi^i : \mathcal{P}^{\mathcal{C}}_0(n) \to \mathcal{P}^{\mathcal{C}}_w(n-1)$ are the contractions $\xi^i_1 : \mathrm{Hom}_{\mathcal{C}}(n,1) \to \mathrm{Hom}_{\mathcal{C}}(n,0)$ of the wheeled PROP.

Conversely, every wheeled operad $\mathcal{P}$ generates a wheeled PROP. Let $\mathcal{C}_{\mathcal{P}}$ be the free wheeled PROP generated by the wheeled operad $\mathcal{P}$. The following explicit description of the wheeled PROP $\mathcal{C}_{\mathcal{P}}$ follows from the description of the free wheeled PROP generated by a collection of $(\mathfrak{S}_m, \mathfrak{S}_n)$–bimodules given in [17, Section 2.1.6] — see also [16, Section 2.3].

Proposition 2.3 *The wheeled PROP $\mathcal{C}_{\mathcal{P}}$ associated to a wheeled operad $\mathcal{P}$ is given by the $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules*

$$\mathcal{C}_{\mathcal{P}}(q,l) = \bigoplus_{J \subset \boldsymbol{q}} \Bigg(\bigoplus_{f : J \twoheadrightarrow \boldsymbol{l}} \bigotimes_{i=1}^{l} \mathcal{P}_0(f^{-1}(i)) \Bigg) \otimes \Bigg(\bigoplus_{k \in \mathbb{N}} \Bigg(\bigoplus_{g : \boldsymbol{q} \setminus J \twoheadrightarrow \boldsymbol{k}} \bigotimes_{i=1}^{k} \mathcal{P}_w(g^{-1}(i)) \Bigg)_{\mathfrak{S}_k} \Bigg)$$

where $\mathfrak{S}_k$ acts on $\bigoplus_{g : \boldsymbol{q} \setminus J \twoheadrightarrow \boldsymbol{k}} \bigotimes_{i=1}^{k} \mathcal{P}_w(g^{-1}(i))$ by postcomposition on $g : \boldsymbol{q} \setminus J \twoheadrightarrow \boldsymbol{k}$. The symmetric group $\mathfrak{S}_l$ acts by postcomposition on $f : J \to \boldsymbol{l}$ and $\mathfrak{S}_q$ by precomposition on $f : J \to \boldsymbol{l}$ and on $g : \boldsymbol{q} \setminus J \twoheadrightarrow \boldsymbol{k}$.

Horizontal composition is induced by disjoint union of maps and partitions.

The contractions $\xi^i_j : \mathcal{C}_{\mathcal{P}}(q,l) \to \mathcal{C}_{\mathcal{P}}(q-1,l-1)$ for $1 \le i \le q$ and $1 \le j \le l$ are induced by

(i) *the contractions $\xi^i : \mathcal{P}_0(n) \to \mathcal{P}_w(n-1)$,*

(ii) *the composition in the operad $\mathcal{P}_0$,*

(iii) *the right $\mathcal{P}_0$–module structure on $\mathcal{P}_w$.*

To illustrate the contractions defined in the previous proposition consider in $\mathcal{C}_{\mathcal{P}}(9,2)$ the summand corresponding to $J = \{1,2,3,4,5\}$, $f : J \to \boldsymbol{2}$ given by $f(1) = f(2) = f(3) = 1$ and $f(4) = f(5) = 2$, $g : \{6,7,8,9\} \to \boldsymbol{2}$ given by $g(6) = g(7) = g(8) = 1$ and $g(9) = 2$ and consider the element

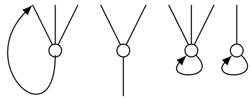
$$X \in \mathcal{P}_0(\{1,2,3\}) \otimes \mathcal{P}_0(\{4,5\}) \otimes \big(\mathcal{P}_w(\{6,7,8\}) \otimes \mathcal{P}_w(\{9\})\big)_{\mathfrak{S}_2}$$

given by the graph

$X =$ 

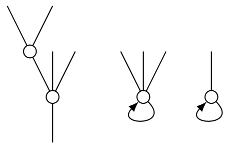
Case (i) is illustrated by

$$\xi_1^1(X) =$$



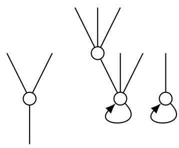
Case (ii) is illustrated by

$$\xi_2^1(X) =$$



Case (iii) is illustrated by

$$\xi_1^6(X) =$$



Remark 2.4 Note that

$$\mathcal{C}_{\mathcal{P}}(n,n) = \bigoplus_{f \in \mathfrak{S}_n} (\mathcal{P}_0(1))^{\otimes n}.$$

Considering the identity operation $\mathrm{Id} \in \mathcal{P}_0(1)$, we obtain a monomorphism of $\mathfrak{S}_n$–bimodules

$$\Bbbk[\mathfrak{S}_n] \to \mathcal{C}_{\mathcal{P}}(n,n).$$

Remark 2.5 Recall that from (2-1) vertical composition in a wheeled PROP is induced by horizontal composition and contractions.

The forgetful functor from the category of wheeled operads to the category of operads has a left adjoint denoted by $(-)^{\circlearrowright}$. For $\mathcal{P}_0$ an operad, $(\mathcal{P}_0)^{\circlearrowright}$ is called the *wheeled completion* of $\mathcal{P}_0$.

Remark 2.6 The wheeled PROP $\mathcal{C}_{\mathcal{P}}$ generated by a wheeled operad $\mathcal{P}$ has two distinguished sub-PROPs:

(1) the sub-PROP $\mathcal{C}_{\mathcal{P}_0}$ generated by the operad $\mathcal{P}_0$, corresponding to forgetting the wheeled part of $\mathcal{P}$;

(2) the sub-PROP denoted by $\mathcal{C}_w$ such that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{C}_w(n,0) &= \mathcal{C}_{\mathcal{P}}(n,0), \\ \mathcal{C}_w(n,n) &= \Bbbk[\mathfrak{S}_n] && \text{for } n \geq 1, \\ \mathcal{C}_w(n,m) &= 0 && \text{for } m \notin \{0,n\}, \end{aligned}$$

corresponding to forgetting the operadic part in the PROP $\mathcal{C}_{\mathcal{P}}$.

3 Recollections on the functor category $\mathcal{F}(\mathbf{gr};\Bbbk)$

The purpose of this section is to review results on covariant and contravariant functors from the category $\mathbf{gr}$ of finitely generated free groups to $\Bbbk$–modules. We refer the reader to [2; 3; 5] for more details.

We denote by $\mathbb{Z}^{*n}$ the free group on n generators. The category $\mathbf{gr}$ has a skeleton with objects $\mathbb{Z}^{*n}$ for $n \in \mathbb{N}$. Consequently, $\mathbf{gr}$ is essentially small and we denote by $\mathcal{F}(\mathbf{gr};\Bbbk)$ (resp. $\mathcal{F}(\mathbf{gr}^{\mathrm{op}};\Bbbk)$) the category of covariant (resp. contravariant) functors from $\mathbf{gr}$ to $\mathcal{V}$.

A fundamental example of functor in $\mathcal{F}(\mathbf{gr};\Bbbk)$ is the abelianization functor $\mathfrak{a}\colon \mathbf{gr} \to \mathcal{V}$ that sends a free group G to $(G/[G,G]) \otimes_{\mathbb{Z}} \Bbbk$.

Composing $\mathfrak{a}$ with the duality functor $-^*\colon \mathcal{V} \to (\mathcal{V})^{\mathrm{op}}$, we obtain a functor from $\mathbf{gr}$ to $\mathcal{V}^{\mathrm{op}}$. The category of functors from $\mathbf{gr}$ to $\mathcal{V}^{\mathrm{op}}$ is equivalent to $(\mathcal{F}(\mathbf{gr}^{\mathrm{op}};\Bbbk))^{\mathrm{op}}$. We will denote by $\mathfrak{a}^\vee\colon \mathbf{gr}^{\mathrm{op}} \to \mathcal{V}$ the functor corresponding to $-^* \circ \mathfrak{a}$ by this equivalence.

By the Yoneda lemma, the category $\mathcal{F}(\mathbf{gr};\Bbbk)$ has enough projective objects and a set of projective generators is given by the functors, for $n \in \mathbb{N}$,

$$P_n := \Bbbk[\mathbf{gr}(\mathbb{Z}^{*n}, -)]$$

where $\Bbbk[-]$ is the linearization functor from the category of sets to $\mathcal{V}$.

Each functor $F\colon \mathbf{gr} \to \mathcal{V}$ can be decomposed naturally as a direct sum $F = F(0) \oplus \bar{F}$ where $F(0)$ is the constant functor equal to $F(0)$ and $\bar{F}$ is a *reduced* functor, ie $\bar{F}(0) = 0$. For simplicity we denote $\bar{P} := \bar{P}_1$.

The notion of cross-effects and polynomial functors introduced by Eilenberg and Mac Lane for categories of modules can be extended to functors on $\mathbf{gr}$ and on $\mathbf{gr}^{\mathrm{op}}$. The d^{th} cross-effect defines an exact functor $\mathrm{cr}_d\colon \mathcal{F}(\mathbf{gr};\Bbbk) \to \mathcal{F}(\mathbf{gr}^{\times n};\Bbbk)$, where $\mathcal{F}(\mathbf{gr}^{\times n};\Bbbk)$ is the category of functors from $\mathbf{gr}^{\times n}$ to $\mathcal{V}$. A functor $F\colon \mathbf{gr} \to \mathcal{V}$ is polynomial of degree d if $\mathrm{cr}_{d+1}(F) = 0$ and $\mathrm{cr}_d(F) \neq 0$. Similarly, we can define polynomial functors on $\mathbf{gr}^{\mathrm{op}}$.

The functors $\mathfrak{a}$ and $\mathfrak{a}^\vee$ are reduced polynomial functors of degree one.

The reduced functor $\bar{P}$ and the cross-effects are related by the following result:

Proposition 3.1 *For $d \in \mathbb{N}$ and $F \in \mathcal{F}(\mathbf{gr};\Bbbk)$, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes d}, F) \simeq \mathrm{cr}_d(F)(\mathbb{Z},\dots,\mathbb{Z}).$$

We deduce the following corollary:

Corollary 3.2 *For* $d \in \mathbb{N}$ *and* $F \in \mathcal{F}(\mathbf{gr}; \Bbbk)$ *a polynomial functor of degree* $< d$,

$$\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes d}, F) = 0.$$

Since the abelianization functor $\mathfrak{a}$ takes its values in $\mathcal{V}^f$, for F a functor from $\mathcal{V}^f$ to $\mathcal{V}$, we can postcompose $\mathfrak{a}$ with F to obtain a functor of $\mathcal{F}(\mathbf{gr}; \Bbbk)$. An important example of functor from $\mathcal{V}^f$ to $\mathcal{V}$ is the d^{th} tensor product functor $T^d : \mathcal{V}^f \to \mathcal{V}$, for $d \in \mathbb{N}$, defined on objects by $V \mapsto V^{\otimes d}$. The symmetric group $\mathfrak{S}_d$ acts by place permutations on T^d. The functor $\mathfrak{a}^{\otimes d} := T^d \circ \mathfrak{a}$ is a polynomial covariant functor of degree d and $(\mathfrak{a}^\vee)^{\otimes d} := T^d \circ \mathfrak{a}^\vee$ is a polynomial contravariant functor of degree d. The notion of exponential functors is a powerful tool for computation; see [4]. A graded exponential functor is a monoidal functor from $(\mathcal{V}^f, \oplus, 0)$ to $(\mathrm{gr}\,\mathcal{V}, \otimes, \Bbbk)$.

If $\Bbbk$ is a field of characteristic 0, the d^{th} exterior power functor is defined, on a vector space V, by $\Lambda^d(V) = (T^d(V) \otimes \mathrm{sgn}_n)_{\mathfrak{S}_d}$ where sgn_n is the signature module and $\mathfrak{S}_d$ acts diagonally. The functor Λ^d is a direct summand of T^d. The functor $\Lambda^d \mathfrak{a} := \Lambda^d \circ \mathfrak{a}$ is a polynomial covariant functor of degree d. The exterior powers define a graded exponential functor $\Lambda^\bullet$. In particular, there are natural commutative products $\Lambda^i \otimes \Lambda^j \to \Lambda^{i+j}$ and cocommutative coproducts $\Lambda^{i+j} \to \Lambda^i \otimes \Lambda^j$, for $i, j \in \mathbb{N}$. Composing with the abelianization functor, we obtain a natural transformation of functors in $\mathcal{F}(\mathbf{gr}, \Bbbk)$,

$$\Lambda^{i+j}\mathfrak{a} \to \Lambda^i \mathfrak{a} \otimes \Lambda^j \mathfrak{a}, \tag{3-1}$$

which will be used later.

For $G, F \in \mathcal{F}(\mathbf{gr}; \Bbbk)$, the *exterior tensor product* of G and F is the functor

$$G \boxtimes F : \mathbf{gr} \times \mathbf{gr} \to \Bbbk\text{–Mod}$$

given on objects by

$$(G \boxtimes F)(\mathbb{Z}^{*n}, \mathbb{Z}^{*m}) = G(\mathbb{Z}^{*n}) \otimes F(\mathbb{Z}^{*m}).$$

Similarly, for $G \in \mathcal{F}(\mathbf{gr}^{\mathrm{op}}; \Bbbk)$ and $F \in \mathcal{F}(\mathbf{gr}; \Bbbk)$ we define $G \boxtimes F : \mathbf{gr}^{\mathrm{op}} \times \mathbf{gr} \to \Bbbk\text{–Mod}$.

We denote by $\amalg_d : \mathbf{gr}^{\times d} \to \mathbf{gr}$ the functor obtained by iteration of the free product (which is the coproduct in $\mathbf{gr}$) and $\delta_d : \mathbf{gr} \to \mathbf{gr}^{\times d}$ the diagonal functor. The functor δ_d is right adjoint to the functor $\amalg_d$. It follows that the functor $\delta_d^* : \mathcal{F}(\mathbf{gr}^{\times d}; \Bbbk) \to \mathcal{F}(\mathbf{gr}; \Bbbk)$ given by precomposition is left adjoint of the functor $\pi_d^* : \mathcal{F}(\mathbf{gr}; \Bbbk) \to \mathcal{F}(\mathbf{gr}^{\times d}; \Bbbk)$ given by precomposition.

Tensor powers $T^\bullet$ do not define an exponential functor but we have a similar property using induction of symmetric groups; see [22, (3)]. In particular,

$$\pi_2^*(\mathfrak{a}^{\otimes q}) \simeq \bigoplus_{J \subset \boldsymbol{q}} \mathfrak{a}^{\otimes |J|} \boxtimes \mathfrak{a}^{\otimes |\boldsymbol{q} \setminus J|}. \tag{3-2}$$

Part II Stable cohomology of $\mathrm{Aut}(\mathbb{Z}^{*n})$ with coefficients given by a bifunctor

4 Definition of stable homology of $\mathrm{Aut}(\mathbb{Z}^{*n})$

Let $I_n\colon \mathrm{Aut}(\mathbb{Z}^{*n}) \to \mathrm{Aut}(\mathbb{Z}^{*n+1})$ be the group monomorphism induced by $- * \mathbb{Z}$. By restriction along I_n we obtain a functor $U^{I_n}\colon \mathrm{Aut}(\mathbb{Z}^{*n+1})$–Mod $\to \mathrm{Aut}(\mathbb{Z}^{*n})$–Mod where $\mathrm{Aut}(\mathbb{Z}^{*n})$–Mod is the category of modules over $\mathrm{Aut}(\mathbb{Z}^{*n})$.

For $B\colon \mathbf{gr}^{\mathrm{op}} \times \mathbf{gr} \to \mathcal{V}$ a functor and $n \in \mathbb{N}$, $B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})$ is an $\mathrm{Aut}(\mathbb{Z}^{*n})^{\mathrm{op}} \times \mathrm{Aut}(\mathbb{Z}^{*n})$–module. Let $p_n\colon \mathbb{Z}^{*n+1} \to \mathbb{Z}^{*n}$ be the group epimorphism given by the projection on the first n variables and $i_n\colon \mathbb{Z}^{*n} \to \mathbb{Z}^{*n+1}$ be the group monomorphisms given by the inclusion of the first n variables.

The previous data give rise to $\mathrm{Aut}(\mathbb{Z}^{*n})$–homomorphisms

$$U^{I_n}(B(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1})) \xrightarrow{B(i_n, p_n)} B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})$$

where the structure of $\mathrm{Aut}(\mathbb{Z}^{*n})$–module on $B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})$ and $U^{I_n}(B(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1}))$ is given by the diagonal action.

This gives group morphisms

$$H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n+1}); B(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1})\big) \xrightarrow{\alpha_n} H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n}); B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})\big).$$

The stable cohomology of the automorphism groups of free groups with coefficients given by B is then defined by

$$H^*_{\mathrm{st}}(B) := \lim_{n \in \mathbb{N}} H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n}); B(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})\big),$$

where the limit is taken over the group morphisms α_n.

In this paper we consider the family of coefficients $B_{l,q} = (\mathfrak{a}^\vee)^{\otimes l} \boxtimes \mathfrak{a}^{\otimes q}$, where $l, q \in \mathbb{N}$ and $\boxtimes$ is the exterior tensor product. By the usual canonical homomorphism

$E^* \otimes F \to \mathrm{Hom}(E, F)$ for two $\Bbbk$–modules E and F, which is an isomorphism if E is free finitely generated, we obtain isomorphisms

$$B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*m}) = ((\Bbbk^n)^*)^{\otimes l} \otimes (\Bbbk^m)^{\otimes q} \simeq \mathrm{Hom}_{\mathcal{V}}((\Bbbk^n)^{\otimes l}, (\Bbbk^m)^{\otimes q}).$$

Based on this isomorphism, $B_{l,q}$ will sometimes be denoted by $\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$.

5 Stability

Let $\mathcal{G}$ be the category having as objects the finitely generated free groups and where a morphism from A to B is a pair (u, H) where $u\colon A \hookrightarrow B$ is an injective homomorphism and H is a subgroup of B such that $B = H * u(A)$. Recall from [3, Définition 4.2] that there is a functor $\iota\colon \mathcal{G} \to \mathbf{gr}^{\mathrm{op}} \times \mathbf{gr}$ sending an object A to (A, A) and a map $(u, H)\colon A \to B$ to $(B = H * u(A) \to u(A) \xrightarrow{u^{-1}} A, u\colon A \to B)$.

The category $\mathcal{G}$ is homogeneous in the sense of [21, Definition 1.3] and the functor $B_{l,q} = (\mathfrak{a}^\vee)^{\otimes l} \boxtimes \mathfrak{a}^{\otimes q}$ is the exterior product between a polynomial contravariant functor of degree l and a polynomial covariant functor of degree q, so the composition $B \circ \iota$ is a coefficient system of degree $l+q$. Hence, by [21, Theorem 5.4], for $i \in \mathbb{N}$ the group morphism

$$H^i(\mathrm{Aut}(\mathbb{Z}^{*n+1}); B_{l,q}(\mathbb{Z}^{*n+1}, \mathbb{Z}^{*n+1})) \xrightarrow{\alpha_n} H^i(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})) \tag{5-1}$$

is an isomorphism for $n \geq N_{l,q,i}$ where $N_{l,q,i} = 2i + l + q + 3$. We deduce that, for n big enough, we have an isomorphism

$$H^i_{\mathrm{st}}((\mathfrak{a}^\vee)^{\otimes l} \otimes \mathfrak{a}^{\otimes q}) \simeq H^i(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})). \tag{5-2}$$

For l_1, q_1, l_2, q_2 in $\mathbb{N}$, the cup product gives morphisms

$$\begin{array}{c} H^i(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l_1,q_1}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})) \otimes H^j(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l_2,q_2}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})) \\ \Big\downarrow \cup \\ H^{i+j}(\mathrm{Aut}(\mathbb{Z}^{*n}); B_{l_1,q_1}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n}) \otimes B_{l_2,q_2}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})) \end{array}$$

For $n > \mathrm{Max}(N_{l_1,q_1,i}, N_{l_2,q_2,j})$, the stability isomorphisms (5-2) give the following cup product map on the stable cohomology:

$$\cup\colon H^i_{\mathrm{st}}(B_{l_1,q_1}) \otimes H^j_{\mathrm{st}}(B_{l_2,q_2}) \to H^{i+j}_{\mathrm{st}}(B_{l_1,q_1} \otimes B_{l_2,q_2}). \tag{5-3}$$

6 The wheeled PROP $\mathcal{H}$ of stable cohomology

The aim of this section is to prove that the stable cohomology of $\mathrm{Aut}(\mathbb{Z}^{*n})$ with coefficients twisted by $B_{l,q} = (\mathfrak{a}^\vee)^{\otimes l} \boxtimes \mathfrak{a}^{\otimes q} = \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$ defines a wheeled PROP. This should be viewed as a cohomological version of the wheeled endomorphism PROP considered in Examples 2.1 and 2.2 using the stability isomorphism (5-2).

Definition 6.1 The PROP $\mathcal{H}$ is defined by the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{H}(q,l) = H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q}))$$

where the action of the symmetric group $\mathfrak{S}_q$ (resp. $\mathfrak{S}_l$) is given by place permutations of the copies of $\mathfrak{a}$ (resp. $(\mathfrak{a}^\vee)$).

The horizontal composition $\otimes\colon \mathcal{H}(q_1,l_1) \otimes \mathcal{H}(q_2,l_2) \to \mathcal{H}(q_1+q_2, l_1+l_2)$ is given by

$$\begin{array}{c} H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_1}, \mathfrak{a}^{\otimes q_1})) \otimes H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_2}, \mathfrak{a}^{\otimes q_2})) \\ \Big\downarrow \cup \\ H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_1}, \mathfrak{a}^{\otimes q_1}) \otimes \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_2}, \mathfrak{a}^{\otimes q_2})) \\ \Big\downarrow \lambda \\ H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l_1+l_2}, \mathfrak{a}^{\otimes q_1+q_2})) \end{array}$$

where $\cup$ is the cup product map given in (5-3) and λ is the map induced by the tensor product of linear maps.

The vertical composition $\circ\colon \mathcal{H}(l,m) \otimes \mathcal{H}(q,l) \to \mathcal{H}(q,m)$ is given by

$$\begin{array}{c} H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes m}, \mathfrak{a}^{\otimes l})) \otimes H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})) \\ \Big\downarrow \cup \\ H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes m}, \mathfrak{a}^{\otimes l}) \otimes \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})) \\ \Big\downarrow \gamma \\ H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes m}, \mathfrak{a}^{\otimes q})) \end{array}$$

where γ is the map induced by the composition in $\mathcal{V}$.

Proposition 6.2 *The PROP $\mathcal{H}$ is a wheeled PROP for the contractions*, *for $1 \le i \le q$ and $1 \le j \le l$*,

$$\xi^i_j : \mathcal{H}(q,l) \to \mathcal{H}(q-1, l-1)$$

induced, for $n \geq N_{l,q,*}$*, by the maps*

$$H^*(\mathrm{Aut}(\mathbb{Z}^{*n}), \varphi_j^i)\colon H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n}), B_{l,q}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})\big) \to H^*\big(\mathrm{Aut}(\mathbb{Z}^{*n}), B_{l-1,q-1}(\mathbb{Z}^{*n}, \mathbb{Z}^{*n})\big),$$

where φ_j^i *are as defined in Example 2.2*

Proof The maps φ_j^i are $\mathrm{Aut}(\mathbb{Z}^{*n})$–equivariant. For $n \geq N_{l,q,*}$, since

$$N_{l,q,*} > N_{l-1,q-1,*},$$

$H^*(\mathrm{Aut}(\mathbb{Z}^{*n}), \varphi_j^i)$ induces a map $\xi_j^i\colon \mathcal{H}(q,l) \to \mathcal{H}(q-1, l-1)$.

We verify that the contraction maps satisfy commutativity conditions and that they are compatible with the horizontal composition.

The biequivariance condition for the contraction maps corresponds to the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{H}(q,l) \arrow[r, "\xi_j^i"] \arrow[d, "(\sigma_1,\sigma_2)"'] & \mathcal{H}(q-1,l-1) \arrow[d, "(\sigma_1^{(\sigma_1^{-1}(i))},\sigma_2^{(j)})"] \\
\mathcal{H}(q,l) \arrow[r, "\xi_{\sigma_2(j)}^{\sigma_1^{-1}(i)}"'] & \mathcal{H}(q-1,l-1)
\end{tikzcd}$$

where $\sigma_1 \in \mathfrak{S}_q$ and $\sigma_2 \in \mathfrak{S}_l$, $\sigma_2^{(j)} \in \mathfrak{S}_{l-1}$ is the permutation induced by σ_2 forgetting j and $\sigma_2(j)$ and reindexing, and $\sigma_1^{(\sigma_1^{-1}(i))} \in \mathfrak{S}_{q-1}$ is the permutation induced by σ_1 forgetting $\sigma_1^{-1}(i)$ and i and reindexing. □

Remark 6.3 For $l > 0$, $(\mathfrak{a}^\vee)^{\otimes l}$ is a reduced contravariant functor which is polynomial of degree l. It follows from the main result of [3] that $H^*_{\mathrm{st}}((\mathfrak{a}^\vee)^{\otimes l}) = 0$, so $\mathcal{H}(0,l) = 0$ for $l > 0$.

In order to relate our results to Djament's result obtained in [1] we introduce the following:

Definition 6.4 Let $\mathcal{H}'$ be the sub-PROP of $\mathcal{H}$ such that, for $n \in \mathbb{N}$

$$\mathcal{H}'(n,0) = \mathcal{H}(n,0), \quad \mathcal{H}'(n,n) = \Bbbk[\mathfrak{S}_n], \quad \mathcal{H}'(n,m) = 0 \text{ if } m \notin \{0,n\}.$$

Remark 6.5 Note that

$$\mathcal{H}(n,n)^0 = H^0_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes n})) \simeq (\mathrm{Hom}_{\mathcal{V}}((\Bbbk^m)^{\otimes n}, (\Bbbk^m)^{\otimes n}))^{\mathrm{Aut}(\mathbb{Z}^{*m})}$$

where the last isomorphism is given by (5-2), for m big enough. Hence

$$\mathcal{H}(n,n)^0 \simeq \Bbbk[\mathfrak{S}_n]$$

and the endomorphisms in the sub-PROP $\mathcal{H}'$ correspond to the endomorphisms in $\mathcal{H}$ in degree 0.

Remark 6.6 The wheeled PROP structure on $\mathcal{H}$ comes from the wheeled endomorphism PROP and does not depend on the family of groups considered. Consequently, there are other families of groups for which we have a wheeled PROP structure on the stable cohomology with coefficients given by $B_{l,q} = \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$. For example, for the braid groups we have a wheeled PROP $\mathcal{H}^{B_\infty}$ and the group morphism $B_n \to \mathrm{Aut}(\mathbb{Z}^{*n})$ induces a morphism of wheeled PROP $\mathcal{H} \to \mathcal{H}^{B_\infty}$. Similarly, for $\Sigma_{g,1}$ a connected and oriented surface of genus g with 1 boundary component and $\mathcal{M}_{g,1}$ its mapping class group, we have a wheeled PROP $\mathcal{H}^{\mathrm{MCG}_{\infty,1}}$ and the group morphism $\mathcal{M}_{g,1} \to \mathrm{Aut}(\mathbb{Z}^{*2g})$ gives a morphism of wheeled PROP $\mathcal{H} \to \mathcal{H}^{\mathrm{MCG}_{\infty,1}}$. The wheeled PROPs $\mathcal{H}^{B_\infty}$ and $\mathcal{H}^{\mathrm{MCG}_{\infty,1}}$ have further structure. This will be developed elsewhere. Similarly, for a symmetric monoidal category $\mathcal{C}$ and a dualizable object in $\mathcal{C}$, the cohomology of the automorphism groups in $\mathcal{C}$ with appropriate coefficients has a wheeled PROP structure.

Remark 6.7 We have also wheeled PROP structures in the unstable ranges. More precisely, for $n \in \mathbb{N}$, we can define a wheeled PROP $\mathcal{H}^n$ given by the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{H}^n(q,l) = H^*(\mathrm{Aut}(\mathbb{Z}^{*n}); \mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})),$$

the wheeled PROP structure being defined in a similar way as in Definition 6.1 and Proposition 6.2. The stabilization morphism (5-1) gives a morphism of wheeled PROPs: $\mathcal{H}^{n+1} \to \mathcal{H}^n$. The PROP $\mathcal{H}$ considered in Definition 6.1 is the limit of these PROP morphisms. The PROPs $\mathcal{H}^n$ are, in general, more complicated than $\mathcal{H}$ since they can contain nonstable cohomological classes.

7 Cohomological classes

In [8] — see also [9] — the first author introduced cohomology classes that give nonzero morphisms in the PROP $\mathcal{H}$. In this section we show that these classes are obtained from the class h_1, recalled below, using the wheeled PROP structure on $\mathcal{H}$.

The q^{th} Johnson map induced by a Magnus expansion θ is a map

$$\tau_q^\theta\colon \mathrm{Aut}(\mathbb{Z}^{*n}) \to \mathrm{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes q+1}).$$

By [8, Lemma 2.1], τ_1^θ is a 1–cocycle and the cohomology class

$$h_1 = [\tau_1^\theta] \in H^1(\mathrm{Aut}(\mathbb{Z}^{*n}), \mathrm{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes 2}))$$

does not depend on the choice of Magnus expansion θ. For n big enough, h_1 gives a nonzero element in $\mathrm{Hom}_{\mathcal{H}}(2,1)$ in cohomological degree 1. Using [8, (4.4)] and the anticommutativity of the cup product we obtain that $\mathfrak{S}_2$ acts on h_1 by the signature.

By [8, (4.11)], we have the relation in $\mathrm{Hom}_{\mathcal{H}}(3,1)$

$$h_1 \circ (h_1 \otimes 1) + h_1 \circ (1 \otimes h_1) = 0 \tag{7-1}$$

where $\otimes$ is the horizontal composition in the PROP $\mathcal{H}$ and $\circ$ is the vertical composition in the PROP $\mathcal{H}$.

Let $\mathcal{K}$ be the subwheeled PROP of $\mathcal{H}$ generated by the class h_1.

Proposition 7.1 *For $p \in \mathbb{N}$, the classes $h_{p+1} \in \mathcal{K}(p+2,1)$, defined inductively by*

$$h_{p+1} = h_1 \circ (h_p \otimes 1),$$

and $\bar{h}_p \in \mathcal{K}(p,0)$, defined by

$$\bar{h}_p = \xi_1^1(h_p),$$

are the cohomological classes introduced in [8], in the stable range.

Proof For $p \geq 2$, using the cup product we obtain classes

$$\begin{aligned}(h_1)^{\cup p} \in H^p\big(&\mathrm{Aut}(\mathbb{Z}^{*n}), \mathrm{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes 2})^{\otimes p}\big)\\ &\simeq H^p\big(\mathrm{Aut}(\mathbb{Z}^{*n}), \mathrm{Hom}_{\mathcal{V}}((\Bbbk^n)^{\otimes p}, (\Bbbk^n)^{\otimes 2p})\big)\end{aligned}$$

where the isomorphism is induced by the canonical homomorphism of $\Bbbk$–modules given by tensor product of linear maps which is an isomorphism for free finitely generated modules.

In the stable range, we obtain $(h_1)^{\cup p} \in \mathcal{H}(2p, p)$ and the previous construction corresponds to the horizontal composition in the PROP $\mathcal{H}$ introduced in Definition 6.1.

Consider the maps

$$\begin{aligned}\varsigma_p\colon ((\Bbbk^n)^* \otimes (\Bbbk^n)^{\otimes 2})^{\otimes p} &\simeq \mathrm{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes 2})^{\otimes p}\\ &\to \mathrm{Hom}_{\mathcal{V}}(\Bbbk^n, (\Bbbk^n)^{\otimes p+1}) \simeq (\Bbbk^n)^* \otimes (\Bbbk^n)^{\otimes p+1}\end{aligned}$$

given by

$$\varsigma_p(u_1\otimes u_2\otimes\cdots\otimes u_p):=(u_1\otimes 1_{(\mathbb{k}^n)^{\otimes p-1}})\circ(u_2\otimes 1_{(\mathbb{k}^n)^{\otimes p-2}})\circ\cdots\circ(u_{p-1}\otimes 1_{\mathbb{k}^n})\circ u_p$$

where $u_i\in\mathrm{Hom}_{\mathcal{V}}(\mathbb{k}^n,(\mathbb{k}^n)^{\otimes 2})$ for $1\le i\le p$; see [8, (4.8)].

The cohomological classes $h_p\in H^p(\mathrm{Aut}(\mathbb{Z}^{*n}),\mathrm{Hom}_{\mathcal{V}}(\mathbb{k}^n,(\mathbb{k}^n)^{\otimes p+1}))$ are defined in [8, Theorem 4.1] by

$$h_p=H^p(\mathrm{Aut}(\mathbb{Z}^{*n}),\varsigma_p)(h_1^{\cup p}).$$

Note that

$$\varsigma_p(u_1\otimes u_2\otimes\cdots\otimes u_p)=(\varsigma_{p-1}(u_1\otimes u_2\otimes\cdots\otimes u_{p-1})\otimes 1_{\mathbb{k}^n})\circ u_p$$

It follows that, for n big enough, the classes h_p can be defined recursively by

$$h_{p+1}=h_1\circ(h_p\otimes 1)\in\mathcal{H}(p+2,1).$$

Consider the map

$$\varphi_1^1\colon(\mathbb{k}^n)^*\otimes(\mathbb{k}^n)^{\otimes p+1}\to(\mathbb{k}^n)^{\otimes p}$$

introduced in Example 2.2. The reduced class $\bar{h}_p\in H^p(\mathrm{Aut}(\mathbb{Z}^{*n}),(\mathbb{k}^n)^{\otimes p})$ is defined, in [8, (4.7)], from the class h_p by

$$\bar{h}_p=H^p(\mathrm{Aut}(\mathbb{Z}^{*n}),\varphi_1^1)(h_p).$$

In the stable range, this corresponds to considering the contraction

$$\xi_1^1\colon\mathcal{H}(p+1,1)\to\mathcal{H}(p,0)$$

introduced in Proposition 6.2, so we have $\bar{h}_p=\xi_1^1(h_p)$. □

Remark 7.2 By the biequivariance condition for the contraction map,

$$\xi_1^2(h_1)=-\bar{h}_1.$$

Remark 7.3 The wheeled PROP $\mathcal{H}^{\mathrm{MCG}_{\infty,1}}$ evoked in Remark 6.6 is related to the graph description of the (twisted) Mumford–Morita–Miller classes by Morita and the first author [11; 19]. Morita [18] extended the first Johnson homomorphism of the Torelli group $\mathcal{I}_{g,1}$ to a twisted cohomology class $\tilde{k}\in H^1\big(\mathcal{M}_{g,1};\frac{1}{2}\Lambda^3\mathfrak{a}(\pi_1(\Sigma_{g,1}))\big)$. The class h_1 restricts to $\tilde{k}$ on the mapping class group $\mathcal{M}_{g,1}$. Morita [19] constructed cohomology classes of the mapping class group with trivial coefficients $\mathbb{Q}$ by contracting a power of the class $\tilde{k}$ in terms of trivalent graphs. More precisely, any trivalent graph Γ with $2n$ vertices defines an $\mathrm{Sp}(\mathfrak{a}(\pi_1(\Sigma_{g,1})))$–invariant linear map $\alpha_\Gamma\colon\Lambda^{2n}(\Lambda^3\mathfrak{a}(\pi_1(\Sigma_{g,1}))\otimes\mathbb{Q})\to\mathbb{Q}$ by using the intersection pairing on $\mathfrak{a}(\pi_1(\Sigma_{g,1}))$. Then we obtain a cohomology class $\alpha_{\Gamma*}(\tilde{k}^{2n})\in H^{2n}(\mathcal{M}_{g,1};\mathbb{Q})$. Morita and the first author proved all these classes are polynomials in the Mumford–Morita–Miller classes [10], and generalized his construction to all finite graphs and twisted Mumford–Morita–Miller classes [11]. Here any graph with n univalent vertices defines a cohomology class of $\mathcal{M}_{g,1}$ with coefficients in $\Lambda^n \mathfrak{a}(\pi_1(\Sigma_{g,1})) \otimes \mathbb{Q}$, which is proved to be a polynomial of twisted Mumford–Morita–Miller classes.

Part III Functor cohomology in $\mathcal{F}(\mathbf{gr}; \Bbbk)$

The aim of this part is to introduce the wheeled PROP $\mathcal{E}$ given by Ext–groups in the functor category $\mathcal{F}(\mathbf{gr})$.

8 Projective resolution of the abelianization functor

The abelianization functor $\mathfrak{a}$ has an explicit projective resolution in $\mathcal{F}(\mathbf{gr}; \Bbbk)$. This resolution occurs in [6, Proposition 5.1] and plays a crucial rôle in [3] and [22]. In this section we recall the construction of this projective resolution.

Recall (see Section 3) that for $n \in \mathbb{N}$, the functors $P_n := \Bbbk[\mathbf{gr}(\mathbb{Z}^{*n}, -)]$ form a set of projective generators of the category $\mathcal{F}(\mathbf{gr}; \Bbbk)$. Consider the simplicial object in $\mathcal{F}(\mathbf{gr}; \Bbbk)$

$$\cdots \overset{\cdots}{\rightrightarrows} P_{n+1} \overset{\cdots}{\rightrightarrows} P_n \overset{\cdots}{\rightrightarrows} \cdots \rightrightarrows P_2 \rightrightarrows P_1 \rightrightarrows P_0 \tag{8-1}$$

where $\delta_i : P_{n+1} \to P_n$ for $0 \leq i \leq n+1$ are defined by

$$\begin{aligned}
\delta_0[g_1, g_2, \dots, g_n, g_{n+1}] &= [g_2, \dots, g_n, g_{n+1}], \\
\delta_i[g_1, g_2, \dots, g_n, g_{n+1}] &= [g_1, \dots, g_i g_{i+1}, \dots, g_n, g_{n+1}] \quad \text{for } 1 \leq i \leq n, \\
\delta_{n+1}[g_1, g_2, \dots, g_n, g_{n+1}] &= [g_1, g_2, \dots, g_n],
\end{aligned}$$

and $\varepsilon_i : P_n \to P_{n+1}$ for $1 \leq i \leq n+1$ are defined by

$$\varepsilon_i[g_1, \dots, g_n] = [g_1, \dots, g_{i-1}, 1, g_i, \dots, g_n].$$

We denote by $C_\bullet$ the unnormalized chain complex associated to this simplicial object and $D_\bullet$ the complex defined by $D_i = C_{i+1}$ for $i \geq 0$ and $D_i = 0$ for $i < 0$.

Since the homology of a free group is naturally isomorphic to its abelianization in degree 1 and is zero in degree > 1, $D_\bullet$ is a resolution of $\mathfrak{a}$ and we obtain that the exact sequence in $\mathcal{F}(\mathbf{gr})$

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_2 \to P_1$$

is a projective resolution of the abelianization functor $\mathfrak{a} : \mathbf{gr} \to \mathrm{Ab}$.

Considering the normalized version we obtain a variant of the previous resolution of the form

$$\cdots \to \overline{P}^{\otimes n+1} \xrightarrow{d_n} \overline{P}^{\otimes n} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \overline{P}^{\otimes 2} \xrightarrow{d_1} \overline{P} \tag{8-2}$$

where $\overline{P}$ is the reduced part of P_1 (see Section 3) and the map $\pi\colon \overline{P} \to \mathfrak{a}$ corresponds to $1 \in \mathbb{Z}$ via the isomorphism $\mathrm{Hom}(\overline{P}, \mathfrak{a}) \simeq \mathbb{Z}$ obtained using Proposition 3.1.

9 The PROP $\mathcal{E}^0$

The aim of this section is to describe the structure of the following graded PROP which can be viewed as an Ext–version of the endomorphism PROP:

Definition 9.1 The graded linear PROP $\mathcal{E}^0$ is defined by the $(\mathfrak{S}_q, \mathfrak{S}_l)$–graded bimodules

$$\mathcal{E}^0(q,l) = \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$$

where the action of the symmetric group $\mathfrak{S}_q$ (resp. $\mathfrak{S}_l$) is given by the permutations of the copies of $\mathfrak{a}$ in the first (resp. second) variable.

The horizontal composition $\mathcal{E}^0(q_1,l_1) \otimes \mathcal{E}^0(q_2,l_2) \to \mathcal{E}^0(q_1+q_2, l_1+l_2)$ is given by the exterior product and the vertical composition $\mathcal{E}^0(q,l) \otimes \mathcal{E}^0(l,m) \to \mathcal{E}^0(q,m)$ is given by the Yoneda product.

Remark 9.2 We warn the reader that $\mathrm{Hom}_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$ should not be confused with $\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})$ introduced at the end of Section 4.

In [22], the second author obtained the following results:

Theorem 9.3 [22, Theorem 2.3] *For $l, q \in \mathbb{N}$, we have an isomorphism*

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q}) \simeq \begin{cases} \Bbbk[\mathrm{Surj}(\boldsymbol{q},\boldsymbol{l})] & \text{if } * = q-l, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{Surj}(\boldsymbol{q},\boldsymbol{l})$ is the set of surjections from $\boldsymbol{q}$ to $\boldsymbol{l}$.

Theorem 9.4 [22, Proposition 2.5] *The symmetric groups $\mathfrak{S}_q$ and $\mathfrak{S}_l$ act on*

$$\mathrm{Ext}^{q-l}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q}) \simeq \Bbbk[\mathrm{Surj}(\boldsymbol{q},\boldsymbol{l})]$$

in the following way: for $\sigma \in \mathfrak{S}_q$, $\tau_{a,b} \in \mathfrak{S}_l$, *the transposition of* a *and* b *where* $a, b \in \{1, \dots, l\}$, *and* $f \in \mathrm{Surj}(\boldsymbol{q}, \boldsymbol{l})$

$$[f].\sigma = \prod_{1 \le i \le l} \varepsilon(\overline{\sigma|_{(f\circ\sigma)^{-1}(i)}})[f \circ \sigma],$$

where $\sigma|_{(f\circ\sigma)^{-1}(i)} \colon (f \circ \sigma)^{-1}(i) \to \sigma((f \circ \sigma)^{-1}(i))$, *and*

$$\tau_{a,b}.[f] = (-1)^{(|f^{-1}(a)|-1)(|f^{-1}(b)|-1)}[\tau_{a,b} \circ f].$$

Proposition 9.5 [22, Proposition 3.1] *The external product*

$$e \colon \mathrm{Ext}^{m-l}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes m}) \otimes \mathrm{Ext}^{n-p}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes p}, \mathfrak{a}^{\otimes n}) \to \mathrm{Ext}^{m+n-l-p}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l+p}, \mathfrak{a}^{\otimes m+n})$$

is induced by the disjoint union of sets via the isomorphism of Theorem 9.3.

For c_{m-l} (resp. c_{n-p}) a cocycle representing a generator of $\mathrm{Ext}^{m-l}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes m})$ (resp. $\mathrm{Ext}^{n-p}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes p}, \mathfrak{a}^{\otimes n})$), we will denote $e([c_{m-l}], [c_{n-p}])$ by $[c_{m-l}] \otimes [c_{n-p}]$.

Note that in the description of the Yoneda product in terms of surjection given in [22, Proposition 3.1] the signs are not correct. One of the aim of Sections 9.1 and 9.2 is to give a corrigendum of this statement.

9.1 Explicit classes in $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$

The aim of this section is to construct explicit cocycles representing the generators in $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$ and to study their behavior via the Yoneda product

$$\mathcal{Y} \colon \mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \otimes \mathrm{Ext}^{1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes n+1}) \to \mathrm{Ext}^{n}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n+1})$$

We begin by introducing explicit classes in $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \simeq \Bbbk$.

Proposition 9.6 *For* $n \in \mathbb{N} \setminus \{0\}$, *the morphism* $\pi^{\otimes n} \colon \bar{P}^{\otimes n} \to \mathfrak{a}^{\otimes n}$ *is a cocycle representing a generator of* $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \simeq \Bbbk$.

Proof Using the normalized bar resolution (8-2), $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$ is the homology of the complex

$$\cdots \to \mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n-1}, \mathfrak{a}^{\otimes n}) \xrightarrow{d} \mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n}, \mathfrak{a}^{\otimes n}) \xrightarrow{d} \mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n+1}, \mathfrak{a}^{\otimes n}) \to \cdots.$$

By Corollary 3.2, $\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n+1}, \mathfrak{a}^{\otimes n}) = 0$ since $\mathfrak{a}^{\otimes n}$ is a polynomial functor of degree n. It follows that $d(\pi^{\otimes n}) = 0$.

Moreover, the morphism $\pi^{\otimes n}$ represents $[\mathrm{Id}_n]$ via the isomorphism

$$\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n}, \mathfrak{a}^{\otimes n}) \simeq \mathrm{cr}_n \mathfrak{a}^{\otimes n}(\mathbb{Z}, \dots, \mathbb{Z}) \simeq \Bbbk[\mathfrak{S}_n]$$

using the external product $\mathrm{Hom}(\bar{P}, \mathfrak{a}) \otimes \cdots \otimes \mathrm{Hom}(\bar{P}, \mathfrak{a}) \to \mathrm{Hom}(\bar{P}^{\otimes n}, \mathfrak{a}^{\otimes n})$. We deduce from the previous complex an exact sequence of $\mathfrak{S}_n$–modules

$$\mathrm{Hom}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\bar{P}^{\otimes n-1}, \mathfrak{a}^{\otimes n}) \to \Bbbk[\mathfrak{S}_n] \to \mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \to 0.$$

It follows that $[\mathrm{Id}_n]$ gives a generator of the $\mathfrak{S}_n$–module $\mathrm{Ext}^{n-1}_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}, \mathfrak{a}^{\otimes n}) \simeq \Bbbk$. □

In the next proposition we introduce particular classes in $\mathrm{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes n+1})$. Let $Q_n^\bullet \to \mathfrak{a}^{\otimes n}$ be a projective resolution of $\mathfrak{a}^{\otimes n}$ and consider the resolution

$$\bar{P}^{\otimes \bullet+1} \otimes \mathfrak{a}^{\otimes n-1} \to \mathfrak{a}^{\otimes n}$$

obtained by tensoring the complex (8-2) with $\mathfrak{a}^{\otimes n-1}$. By standard homological algebra — see [23, Comparison Theorem 2.2.6] — there is a chain map

$$\alpha^\bullet \colon Q_n^\bullet \to \bar{P}^{\otimes \bullet+1} \otimes \mathfrak{a}^{\otimes n-1}$$

lifting $\mathrm{Id}_{\mathfrak{a}^{\otimes n}} : \mathfrak{a}^{\otimes n} \to \mathfrak{a}^{\otimes n}$ which is unique up to chain homotopy equivalence.

Lemma 9.7 *The map*

$$(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-1}}) \circ \alpha^1 \colon Q_n^1 \to \mathfrak{a}^{\otimes n+1}$$

represents the class of $\mathrm{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes n+1})$ *corresponding to the exterior product* $[\pi^{\otimes 2}] \otimes [\mathrm{Id}_{\mathfrak{a}^{\otimes n-1}}]$.

Proof This is a direct consequence of the definition of the exterior product of classes. □

Lemma 9.8 *The functor* $\mathrm{Im}(d_{n-1})$ *has projective resolution*

(9-1) $$\bar{P}^{\otimes \bullet+n} \colon \cdots \to \bar{P}^{\otimes n+1} \xrightarrow{d_n} \bar{P}^{\otimes n}$$

given by truncating (8-2). Moreover, the map $\pi^{\otimes n} \colon \bar{P}^{\otimes n} \to \mathfrak{a}^{\otimes n}$ *factorizes through* $\mathrm{Im}(d_{n-1})$, *giving rise to a morphism* $\bar{\pi}^{\otimes n} \colon \mathrm{Im}(d_{n-1}) \to \mathfrak{a}^{\otimes n}$.

Proof By the projective resolution (9-1) we obtain the complex

$$\cdots \leftarrow \mathrm{Hom}(\bar{P}^{\otimes n+1}, \mathfrak{a}^{\otimes n}) \xleftarrow{d_n^*} \mathrm{Hom}(\bar{P}^{\otimes n}, \mathfrak{a}^{\otimes n}),$$

computing $\mathrm{Ext}^i_{\mathcal{F}(\mathbf{gr})}(\mathrm{Im}(d_{n-1}), \mathfrak{a}^{\otimes n})$. The map $\pi^{\otimes n} \colon \bar{P}^{\otimes n} \to \mathfrak{a}^{\otimes n}$ satisfies $\pi^{\otimes n} \circ d_n = 0$, so it represents a cocycle in $\mathrm{Hom}_{\mathcal{F}(\mathbf{gr})}(\mathrm{Im}(d_{n-1}), \mathfrak{a}^{\otimes n})$. We deduce that $\pi^{\otimes n}$ factorizes through $\mathrm{Im}(d_{n-1})$, giving rise to a morphism $\bar{\pi}^{\otimes n} \colon \mathrm{Im}(d_{n-1}) \to \mathfrak{a}^{\otimes n}$ □

Lemma 9.9 *We have a morphism of exact chain complexes*

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \overline{P}^{\otimes k+1}\otimes\overline{P}^{\otimes n-1} \arrow[r, "d_{n+k-1}"] \arrow[d, "\mathrm{Id}\otimes\pi^{\otimes n-1}"] & \cdots \arrow[r] \arrow[d] & \overline{P}^{\otimes 2}\otimes\overline{P}^{\otimes n-1} \arrow[r, "d_n"] \arrow[d, "\mathrm{Id}\otimes\pi^{\otimes n-1}"] & \overline{P}\otimes\overline{P}^{\otimes n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\mathrm{Id}\otimes\pi^{\otimes n-1}"] & \mathrm{Im}(d_{n-1}) \arrow[r] \arrow[d, "\bar{\pi}^{\otimes n}"] & 0 \\
\cdots \arrow[r] & \overline{P}^{\otimes k+1}\otimes\mathfrak{a}^{\otimes n-1} \arrow[r, "d_k\otimes\mathrm{Id}"] & \cdots \arrow[r] & \overline{P}^{\otimes 2}\otimes\mathfrak{a}^{\otimes n-1} \arrow[r, "d_1\otimes\mathrm{Id}"] & \overline{P}\otimes\mathfrak{a}^{\otimes n-1} \arrow[r, "\pi\otimes\mathrm{Id}"] & \mathfrak{a}^{\otimes n} \arrow[r] & 0
\end{tikzcd}$$

Proof The square on the right commutes since

$$\bar{\pi}^{\otimes n}\circ d_{n-1}=\pi^{\otimes n}=(\pi\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes\pi^{\otimes n-1}).$$

For $k\in\mathbb{N}$, a direct computation using the differential in the reduced bar resolution gives the commutativity of the diagram

$$\begin{tikzcd}
\overline{P}^{\otimes k+2}\otimes\overline{P}^{\otimes n-1} \arrow[r, "d_{n+k}"] \arrow[d, "\mathrm{Id}\otimes\pi^{\otimes n-1}"] & \overline{P}^{\otimes k+1}\otimes\overline{P}^{\otimes n-1} \arrow[d, "\mathrm{Id}\otimes\pi^{\otimes n-1}"] \\
\overline{P}^{\otimes k+2}\otimes\mathfrak{a}^{\otimes n-1} \arrow[r, "d_{k+1}\otimes\mathrm{Id}"'] & \overline{P}^{\otimes k+1}\otimes\mathfrak{a}^{\otimes n-1}
\end{tikzcd}$$

□

Remark 9.10 The reader's attention is drawn to the fact that the following diagram is only commutative *up to a sign*:

$$\begin{tikzcd}
\overline{P}^{\otimes n-1}\otimes\overline{P}^{\otimes k+2} \arrow[r, "d_{n+k}"] \arrow[d, "\pi^{\otimes n-1}\otimes\mathrm{Id}"] & \overline{P}^{\otimes n-1}\otimes\overline{P}^{\otimes k+1} \arrow[d, "\pi^{\otimes n-1}\otimes\mathrm{Id}"] \\
\mathfrak{a}^{\otimes n-1}\otimes\overline{P}^{\otimes k+2} \arrow[r, "\mathrm{Id}\otimes d_k"'] & \mathfrak{a}^{\otimes n-1}\otimes\overline{P}^{\otimes k+1}
\end{tikzcd}$$

Proposition 9.11 *For* $n\geq 2$

$$\mathcal{Y}([\pi^{\otimes n}],[\pi^{\otimes 2}]\otimes[\mathrm{Id}_{\mathfrak{a}^{\otimes n-1}}])=[\pi^{\otimes n+1}].$$

Proof By [23, Comparison Theorem 2.2.6] there is a chain map $\beta^\bullet$ lifting

$$\bar{\pi}^{\otimes n}\colon\mathrm{Im}(d_{n-1})\to\mathfrak{a}^{\otimes n}.$$

We obtain the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \overline{P}^{\otimes n+1} \arrow[r, "d_n"] \arrow[d, "\beta^1"] & \overline{P}^{\otimes n} \arrow[r, "d_{n-1}"] \arrow[d, "\beta^0"] & \mathrm{Im}(d_{n-1}) \arrow[r] \arrow[d, "\bar{\pi}^{\otimes n}"] & 0 \\
\cdots \arrow[r] \arrow[d] & Q_n^1 \arrow[r] \arrow[d, "\alpha^1"] & Q_n^0 \arrow[r] \arrow[d, "\alpha^0"] & \mathfrak{a}^{\otimes n} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
\cdots \arrow[r] & \overline{P}^{\otimes 2}\otimes\mathfrak{a}^{\otimes n-1} \arrow[r] & \overline{P}^{\otimes 1}\otimes\mathfrak{a}^{\otimes n-1} \arrow[r] & \mathfrak{a}^{\otimes n} \arrow[r] & 0
\end{tikzcd}$$

By the construction of the Yoneda product, $(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-i}}) \circ \alpha^1 \circ \beta^1$ is a cocycle representing $\mathcal{Y}([\pi^{\otimes n}], [(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-i}}) \circ \alpha^1])$.

Since the chain map lifting $\bar{\pi}^{\otimes n} \colon \mathrm{Im}(d_{n-1}) \to \mathfrak{a}^{\otimes n}$ is unique up to chain homotopy equivalence, using Lemma 9.9 we have

$$[(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-i}}) \circ \alpha^1 \circ \beta^1] = [(\pi^{\otimes 2} \otimes \mathrm{Id}_{\mathfrak{a}^{\otimes n-i}}) \circ (\mathrm{Id} \otimes \pi^{\otimes n-1})] = [\pi^{\otimes n+1}]. \qquad \square$$

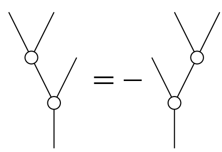
9.2 The operad $\mathcal{P}_0$

In [22, Proposition 3.5] the second author proved that the graded PROP $\mathcal{E}^0$ is freely generated by its underlying operad $\mathcal{P}_0$. Using Theorems 9.3 and 9.4, $\mathcal{P}_0$ is the graded operad such that $\mathcal{P}_0(0) = 0$ and for $k > 0$, $\mathcal{P}_0(k)$ is the sign representation of $\mathfrak{S}_k$ placed in cohomological degree $k-1$ and 0 in other degrees. The aim of this section is to give an explicit description of this operad, in particular to describe the composition which is induced by the Yoneda product.

Definition 9.12 The operad $\mathcal{Q}$ is the quadratic graded operad generated by one antisymmetric binary operation μ in degree 1 subject to the quadratic relation

$$\mu \circ_1 \mu = -\mu \circ_2 \mu.$$

Pictorially, we have



Proposition 9.13 *The operad $\mathcal{P}_0$ is isomorphic to $\mathcal{Q}$.*

Proof We show that there is a morphism of operads $f \colon \mathcal{Q} \to \mathcal{P}_0$ given on the generator μ by $f_2(\mu) = [\pi^{\otimes 2}]$ where $[\pi^{\otimes 2}]$ is a generator, in degree 1, of $\mathcal{E}^0(2,1) = \mathcal{P}_0(2)$ defined in Proposition 9.6 which is antisymmetric by Theorem 9.4.

Before proving that $[\pi^{\otimes 2}]$ satisfies the quadratic relation satisfied by μ, note that the partial composition operations in $\mathcal{P}_0$ are given by the restriction of the categorical composition induced by the Yoneda product

$$\mathcal{Y} \colon \mathcal{P}_0(n) \otimes_{\mathfrak{S}_n} \mathcal{E}^0(n+1, n) \to \mathcal{P}_0(n+1).$$

More explicitly the partial composition $\circ_i$ is obtained by restricting to the inclusion of $\mathbb{k}$–modules

$$\xi_i^n : \mathcal{P}_0(2) \simeq \mathcal{P}_0(1)^{\otimes i-1} \otimes \mathcal{P}_0(2) \otimes \mathcal{P}_0(1)^{\otimes n-i} \hookrightarrow \mathcal{E}^0(n+1,n)$$

given by the external product.

For $c_2 \in \mathfrak{S}_2$ and $c_3 \in \mathfrak{S}_3$ the cyclic permutations given by $i \mapsto i+1$, by Theorem 9.4 we have in $\mathrm{Ext}^1_{\mathcal{F}(\mathbf{gr};\mathbb{k})}(\mathfrak{a}^{\otimes 2}, \mathfrak{a}^{\otimes 3})$,

$$c_2.([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}]).c_3 = [\pi^{\otimes 2}] \otimes [\mathrm{Id}_{\mathfrak{a}}],$$

so

$$\begin{aligned}
\mathcal{Y}([\pi^{\otimes 2}], [\pi^{\otimes 2}] \otimes [\mathrm{Id}_{\mathfrak{a}}]) &= \mathcal{Y}([\pi^{\otimes 2}], c_2.([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}]).c_3) \\
&= \mathcal{Y}([\pi^{\otimes 2}].c_2, ([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}])).c_3 \\
&= \mathcal{Y}([\pi^{\otimes 2}](-1), ([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}])) \\
&= -\mathcal{Y}([\pi^{\otimes 2}], ([\mathrm{Id}_{\mathfrak{a}}] \otimes [\pi^{\otimes 2}]))
\end{aligned}$$

using that $\mathcal{P}_0(k)$ is the sign representation in degree $k-1$. We deduce that $[\pi^{\otimes 2}]$ satisfies the quadratic relation.

The fact that the operad $\mathcal{P}_0$ is binary follows from Proposition 9.11 and the fact that $\mathcal{P}_0(n+1)$ is $\mathbb{k}$ concentrated in degree n.

We deduce that the morphism of operads $f : \mathcal{Q} \to \mathcal{P}_0$ is an isomorphism. □

In Proposition 9.14 we give a more conceptual description of the operad $\mathcal{P}_0$ using the notion of *operadic suspension*, recalled in the next paragraphs following [12, Section 7.2.2].

Let $\mathcal{S}$ be the underlying operad of the endomorphism PROP (see Example 2.1) associated with the graded vector space $s\mathbb{k}$ (ie the graded vector space concentrated in homological degree one and such that $(s\mathbb{k})_1 = \mathbb{k}$). Explicitly, as a representation of $\mathfrak{S}_n$, $\mathcal{S}(n) = \mathrm{Hom}_{\mathrm{gr}\mathcal{V}}((s\mathbb{k})^{\otimes n}, s\mathbb{k})$ is the signature representation concentrated in homological degree $-n+1$.

For $\mathcal{P}$ and $\mathcal{Q}$ two operads, the Hadamard tensor product $\mathcal{P} \otimes_H \mathcal{Q}$ is an operad such that

$$(\mathcal{P} \otimes_H \mathcal{Q})(n) = \mathcal{P}(n) \otimes_H \mathcal{Q}(n),$$

where the action of $\mathfrak{S}_n$ is the diagonal action. The unit of the Hadamard product is the operad $u\mathcal{C}om$ of *unital* commutative algebras. In particular we have $u\mathcal{C}om(0) = \mathbb{k}$.

For $\mathcal{P}$ an operad, the *operadic suspension* of $\mathcal{P}$ is the operad $\mathcal{S} \otimes_H \mathcal{P}$.

The operad $\mathcal{P}_0$ being defined using Ext–groups, it is naturally graded with cohomological degree. So $\mathcal{P}_0(n)$ is a graded $\mathbb{k}$–module concentrated in *cohomological* degree $n-1$ and so in *homological* degree $1-n$.

Let $\mathcal{C}om$ be the operad of *nonunital* commutative algebras (thus $\mathcal{C}om(0)=0$). In the next proposition we consider $\mathcal{P}_0$ with its homological degree.

Proposition 9.14 *The operad $\mathcal{P}_0$ is the operadic suspension of the operad $\mathcal{C}om$. In other words, we have an isomorphism of operads*

$$\mathcal{P}_0 \simeq \mathcal{S} \otimes_H \mathcal{C}om.$$

Proof Recall that $\mathcal{P}_0(0)=0$ and for $n>0$, $\mathcal{P}_0(n)$ is the sign representation of $\mathfrak{S}_n$ placed in homological degree $1-n$ and 0 in other degrees. The underlying $\mathfrak{S}_n$–modules are isomorphic since $\mathcal{C}om(0)=0$ and for $n \geq 1$, $\mathcal{P}_0(n) \simeq \mathcal{S}(n)$.

By Proposition 9.13, $\mathcal{P}_0$ is a quadratic operad generated by μ such that $\mu \circ_1 \mu = -\mu \circ_2 \mu$. Define $\mathcal{P}_0(2) \to \mathcal{S}(2)$ sending μ to the generator ν of $\mathcal{S}(2)$. By a similar argument as in the proof of Proposition 9.13 for the endomorphism PROP associated with the graded vector space $s\mathbb{k}$, one proves that ν satisfies the relation $\nu \circ_1 \nu = -\nu \circ_2 \nu$. This implies the isomorphism in the statement. □

Remark 9.15 We denote by $s_i^n \in \mathrm{Surj}(n, n-1)$ the unique surjection preserving the natural order and such that $s_i^n(i) = s_i^n(i+1) = i$. The Yoneda product gives, via the isomorphism given in Theorem 9.3, a map

$$Y : \mathbb{k}[\mathrm{Surj}(m,l)] \otimes \mathbb{k}[\mathrm{Surj}(n,m)] \to \mathbb{k}[\mathrm{Surj}(n,l)].$$

For $m=2$, $l=1$ and $n=3$, the quadratic relation in the operad $\mathcal{P}_0$ corresponds to $Y([s_1^2] \otimes [s_2^3]) = -Y([s_1^2] \otimes [s_1^3])$ showing that the signs given in [22, Proposition 3.1] are not correct.

The following maps will be used in Proposition 10.7 in order to define the contraction maps in the wheeled operad $\mathcal{P}$.

Definition 9.16 For $1 \leq i \leq n$, let $\xi^i : \mathcal{P}_0(n) \to \mathcal{P}_0(n-1)$ be the morphism of graded vector spaces of degree -1 given by

$$\xi^i([\pi^{\otimes n}]) = (-1)^i [\pi^{\otimes n-1}].$$

Proposition 9.17 *For $1 \leq i \leq n$, the contraction map $\xi^i \colon \mathcal{P}_0(n) \to \mathcal{P}_0(n-1)$ is equivariant; ie for $\sigma \in \mathfrak{S}_n$, the diagram*

$$\begin{tikzcd} \mathcal{P}_0(n) \arrow[r, "\xi^i"] \arrow[d, "\sigma"'] & \mathcal{P}_0(n-1) \arrow[d, "\sigma^{(i)}"] \\ \mathcal{P}_0(n) \arrow[r, "\xi^{\sigma(i)}"'] & \mathcal{P}_0(n-1) \end{tikzcd}$$

is commutative, where $\sigma^{(i)} \in \mathfrak{S}_{n-1}$ is the permutation $\sigma \colon \boldsymbol{n} \backslash \{i\} \to \boldsymbol{n} \backslash \{\sigma(i)\}$ considered as reindexed.

Proof Since $\mathfrak{S}_n$ acts on $\mathcal{P}_0(n)$ by the signature, we need to prove that

$$\varepsilon(\sigma)(-1)^{\sigma(i)} = \varepsilon(\sigma^{(i)})(-1)^i.$$

Let $\alpha \in \mathfrak{S}_n$ be the permutation given by

$$\alpha = (1,2) \circ (2,3) \circ \cdots \circ (\sigma(i)-1, \sigma(i)) \circ \sigma \circ (i-1, i) \circ \cdots \circ (1,2)$$

where $(l, l+1)$ is the permutation exchanging l and $l+1$. We have

$$\varepsilon(\alpha) = (-1)^{\sigma(i)-1}\varepsilon(\sigma)(-1)^{i-1},$$

and for $\alpha^{(1)}$ the permutation $\alpha \colon \boldsymbol{n} \setminus \{1\} \to \boldsymbol{n} \setminus \{\alpha(1) = 1\}$ considered as reindexed. We have $\alpha^{(1)} = \sigma^{(i)}$ and $\alpha = \mathrm{Id}_{\{1\}} \oplus \alpha^{(1)}$. Hence

$$\varepsilon(\sigma^{(i)}) = \varepsilon(\alpha^{(1)}) = \varepsilon(\alpha) = (-1)^{\sigma(i)-1}\varepsilon(\sigma)(-1)^{i-1}. \qquad \square$$

10 The PROP $\mathcal{E}$

In the rest of the paper $\Bbbk$ is a field of characteristic zero. The previous condition on $\Bbbk$ allows us to use the computation of $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q})$ given in [22] (see Proposition 10.1 below) obtained by taking the coinvariants by the action of the symmetric groups, twisted by the signature, in the result of Theorem 9.3.

The aim of this section is to describe the structure of the graded PROP $\mathcal{E}$ introduced in Definition 10.2. We will prove in Theorem 10.11 that the PROP $\mathcal{E}$ is a wheeled PROP. The PROP $\mathcal{E}$ extends the PROP $\mathcal{E}^0$ in the sense that $\mathcal{E}^0$ is a sub-PROP of $\mathcal{E}$ (see Remark 10.3).

Since the exterior powers intervene in the PROP $\mathcal{E}$ we need the following result:

Proposition 10.1 [22, Theorem 4.2] *For $\mathbb{k}$ a field of characteristic 0 and $n, m \in \mathbb{N}$, we have isomorphisms*

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}) \simeq \begin{cases} \mathbb{k}^{S(q,j)} & \text{if } * = q-j, \\ 0 & \text{otherwise,} \end{cases}$$

where $S(q,j)$ denotes the number of ways to partition a set of q elements into j nonempty subsets

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\Lambda^n \mathfrak{a}, \Lambda^m \mathfrak{a}) \simeq \begin{cases} \mathbb{k}^{\rho(m,n)} & \text{if } * = m-n, \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho(m,n)$ denotes the number of partitions of m into n parts.

Since $\mathrm{Hom}_{\mathcal{F}(\mathbf{gr})}(\Lambda^j \mathfrak{a}, \Lambda^j \mathfrak{a}) \simeq \mathbb{k}$, the external product gives a morphism

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m} \otimes \Lambda^i \mathfrak{a}, \mathfrak{a}^{\otimes l}) \xrightarrow{E} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m} \otimes \Lambda^i \mathfrak{a} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}). \tag{10-1}$$

Recall that for $V^\bullet$ a cohomologically graded module, for $i \in \mathbb{N}$ the i^{th} desuspension of $V^\bullet$ is the graded module $s^{-i} V^\bullet$ such that $s^{-i} V^n = V^{n-i}$.

Definition 10.2 The PROP $\mathcal{E}$ is defined by the graded $(\mathfrak{S}_q, \mathfrak{S}_l)$–bimodules

$$\mathcal{E}(q,l) = \bigoplus_{j \in \mathbb{N}} s^{-j}\, \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\mathbb{k})}(\mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}),$$

where the action of the symmetric group $\mathfrak{S}_l$ (resp. $\mathfrak{S}_q$) is given by place permutation of the copies of $\mathfrak{a}$ in the first (resp. second) variable.

The horizontal composition $\otimes : \mathrm{Hom}_{\mathcal{E}}(q_1, l_1) \otimes \mathrm{Hom}_{\mathcal{E}}(q_2, l_2) \to \mathrm{Hom}_{\mathcal{E}}(q_1+q_2, l_1+l_2)$ is given by

$$\begin{array}{c}
\bigoplus_{j \in \mathbb{N}} s^{-j}\, \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_1} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q_1}) \otimes \bigoplus_{i \in \mathbb{N}} s^{-i}\, \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_2} \otimes \Lambda^i \mathfrak{a}, \mathfrak{a}^{\otimes q_2}) \\
\big\downarrow \beta \\
\bigoplus_{i,j \in \mathbb{N}} s^{-j-i}\, \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_1} \otimes \Lambda^j \mathfrak{a} \otimes \mathfrak{a}^{\otimes l_2} \otimes \Lambda^i \mathfrak{a}, \mathfrak{a}^{\otimes q_1+q_2}) \\
\big\downarrow T \\
\bigoplus_{i,j \in \mathbb{N}} s^{-j-i}\, \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_1+l_2} \otimes \Lambda^j \mathfrak{a} \otimes \Lambda^i \mathfrak{a}, \mathfrak{a}^{\otimes q_1+q_2}) \\
\big\downarrow c \\
\bigoplus_{j+i \in \mathbb{N}} s^{-j-i}\, \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l_1+l_2} \otimes \Lambda^{j+i} \mathfrak{a}, \mathfrak{a}^{\otimes q_1+q_2})
\end{array}$$

where β is the exterior product, T is induced by the permutation of $\Lambda^i \mathfrak{a}$ and $\mathfrak{a}^{\otimes l_2}$ and c by the natural transformation $\Lambda^{i+j} \mathfrak{a} \to \Lambda^i \mathfrak{a} \otimes \Lambda^j \mathfrak{a}$ (see (3-1)).

The vertical composition $\circ\colon \mathrm{Hom}_{\mathcal{E}}(q,l)\otimes \mathrm{Hom}_{\mathcal{E}}(l,m)\to \mathrm{Hom}_{\mathcal{E}}(q,m)$ is given by

$$\begin{array}{c}
\bigoplus_{j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l}\otimes \Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\otimes \bigoplus_{i\in\mathbb{N}} s^{-i}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m}\otimes \Lambda^i\mathfrak{a},\mathfrak{a}^{\otimes l})\\
\big\downarrow{\simeq}\\
\bigoplus_{i,j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes l}\otimes \Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\otimes s^{-i}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m}\otimes \Lambda^i\mathfrak{a},\mathfrak{a}^{\otimes l})\\
\big\downarrow\\
\bigoplus_{i,j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*(\mathfrak{a}^{\otimes l}\otimes \Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\otimes s^{-i}\,\mathrm{Ext}^*(\mathfrak{a}^{\otimes m}\otimes \Lambda^i\mathfrak{a}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a})\\
\big\downarrow{\mathcal{Y}}\\
\bigoplus_{i,j\in\mathbb{N}} s^{-i-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m}\otimes \Lambda^i\mathfrak{a}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\\
\big\downarrow{c}\\
\bigoplus_{i+j\in\mathbb{N}} s^{-i-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}^{\otimes m}\otimes \Lambda^{i+j}\mathfrak{a},\mathfrak{a}^{\otimes q})
\end{array}$$

where the second morphism is induced by the map (10-1), the third by the Yoneda product and the last one by the canonical natural transformation $\Lambda^{i+j}\mathfrak{a}\to\Lambda^i\mathfrak{a}\otimes\Lambda^j\mathfrak{a}$ given in (3-1).

Remark 10.3 The PROP $\mathcal{E}^0$ (see Definition 9.1) is the sub-PROP of $\mathcal{E}$ having the same objects and such that $\mathcal{E}^0(q,l)$ is the direct summand of $\mathcal{E}(q,l)$ for $j=0$.

10.1 Calculation of $\mathcal{E}(q,l)$

The aim of this section is to prove the following result:

Proposition 10.4 *For $q,l\in\mathbb{N}$ we have an isomorphism of graded $(\mathfrak{S}_q,\mathfrak{S}_l)$–bimodules*

$$\mathcal{E}(q,l)=\bigoplus_{J\subset \boldsymbol{q}}\Big(\bigoplus_{f\colon J\twoheadrightarrow \boldsymbol{l}}\bigotimes_{i=1}^{l}\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes|f^{-1}(i)|})\Big)\otimes\bigg(\bigoplus_{j\in\mathbb{N}}\Big(\bigoplus_{g\colon \boldsymbol{q}\setminus J\twoheadrightarrow \boldsymbol{j}}\bigotimes_{i=1}^{j}s^{-1}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes|g^{-1}(i)|})\Big)_{\mathfrak{S}_j}\bigg).$$

The proof of this proposition relies on the following lemma:

Lemma 10.5 *For $q,l,j\in\mathbb{N}$, we have the following isomorphisms of graded $(\mathfrak{S}_q,\mathfrak{S}_l)$–bimodules*:

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q})\simeq\bigoplus_{J\subset\boldsymbol{q}}\big(\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l},\mathfrak{a}^{\otimes|J|})\otimes\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes|q\setminus J|})\big),$$

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l},\mathfrak{a}^{\otimes q}) \simeq \bigoplus_{f:\boldsymbol{q}\twoheadrightarrow \boldsymbol{l}} \biggl(\bigotimes_{k=1}^{l} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes |f^{-1}(k)|})\biggr),$$

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q}) \simeq \biggl(\bigoplus_{f:\boldsymbol{q}\twoheadrightarrow \boldsymbol{j}} \biggl(\bigotimes_{k=1}^{j} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes |f^{-1}(k)|})\biggr)\biggr)_{\mathfrak{S}_j},$$

where $\mathfrak{S}_j$ acts on $\bigoplus_{f:\boldsymbol{q}\twoheadrightarrow \boldsymbol{j}} (\bigotimes_{k=1}^{j} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes |f^{-1}(k)|}))$ by postcomposition on $f:\boldsymbol{q}\twoheadrightarrow \boldsymbol{j}$.

Proof For the first isomorphism,

$$\begin{aligned}
\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q}) &\simeq \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\boxtimes\Lambda^j\mathfrak{a},\pi_2^*(\mathfrak{a}^{\otimes q}))\\
&\simeq \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr};\Bbbk)}\Bigl(\mathfrak{a}^{\otimes l}\boxtimes\Lambda^j\mathfrak{a},\bigoplus_{J\subset\boldsymbol{q}}\mathfrak{a}^{\otimes|J|}\boxtimes\mathfrak{a}^{\otimes|\boldsymbol{q}\setminus J|}\Bigr)\\
&\simeq \bigoplus_{J\subset\boldsymbol{q}}\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\boxtimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes|J|}\boxtimes\mathfrak{a}^{\otimes|\boldsymbol{q}\setminus J|}),
\end{aligned}$$

where the first isomorphism is given by the adjunction between δ_2^* and π_2^* (see Section 3) and the second by (3-2).

Using the resolution given in Section III, we obtain that $\mathfrak{a}^{\otimes l}$ and $\Lambda^j\mathfrak{a}$ have resolutions by finitely generated projective functors. Moreover, the values of $\mathfrak{a}^{\otimes n}$ and $\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l},\mathfrak{a}^{\otimes|J|})$ (by Theorem 9.3) are torsion free. It follows, by the Künneth formula, that the graded morphism

$$\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l},\mathfrak{a}^{\otimes|J|})\otimes\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes|\boldsymbol{q}\setminus J|}) \xrightarrow{\simeq} \mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr}\times\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\boxtimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes|J|}\boxtimes\mathfrak{a}^{\otimes|\boldsymbol{q}\setminus J|})$$

is an isomorphism.

For the second and third isomorphisms, we refer the reader to the proof of [22, Theorems 2.3 and 4.2]. □

Remark 10.6 By Proposition 10.4,

$$\mathcal{E}(n,n)\simeq\bigoplus_{\sigma\in\mathfrak{S}_n}\biggl(\bigotimes_{i=1}^{n}\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a})\biggr)\simeq\Bbbk[\mathfrak{S}_n]$$

where the last isomorphism is given by Theorem 9.3. Hence the graded bimodule $\mathcal{E}(n,n)$ is $\Bbbk[\mathfrak{S}_n]$ concentrated in degree 0.

10.2 The wheeled PROP $\mathcal{C}_{\mathcal{P}}$

The aim of this section is to give an isomorphism between the PROP $\mathcal{E}$ and the wheeled PROP associated to the following wheeled operad $\mathcal{P}$.

Proposition 10.7 *The following data define a wheeled operad denoted by* $\mathcal{P}$:

(1) *the operadic part of* $\mathcal{P}$ *is the operad* $\mathcal{P}_0$ *considered in Section 9.2*;

(2) *the wheeled part* $\mathcal{P}_w$ *is given by*

$$\mathcal{P}_w(n) = s^{-1}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes n});$$

(3) *for* $1 \le i \le n$, *the contractions* $\xi^i : \mathcal{P}_0(n) \to \mathcal{P}_w(n-1)$ *are the degree* 0 *maps induced by Definition 9.16.*

Proof We have $\mathcal{P}_w \simeq s^{-1}\mathcal{P}_0$, so $\mathcal{P}_w$ is a right $\mathcal{P}_0$–module. The contraction maps $\xi_i : \mathcal{P}_0(n) \to \mathcal{P}_w(n-1)$ are equivariant by Proposition 9.17. □

Let $\mathcal{Q}^{\circlearrowright}$ be the wheeled completion of the operad $\mathcal{Q}$ given in Definition 9.12. In the following, we give an explicit description of $\mathcal{Q}^{\circlearrowright}$.

Proposition 10.8 *The wheeled operad* $\mathcal{Q}^{\circlearrowright}$ *is given by the following data*:

(1) ***The operadic part*** $(\mathcal{Q}^{\circlearrowright})_0$ $(\mathcal{Q}^{\circlearrowright})_0(n)$ *is the graded vector space concentrated in degree* $n-1$ *and generated by* μ_n *defined inductively by* $\mu_1 = \mathrm{Id}$, $\mu_2 = \mu$ *and for* $n \ge 2$,

$$\mu_{n+1} = \mu \circ_1 \mu_n.$$

(2) ***The wheeled part*** $(\mathcal{Q}^{\circlearrowright})_w$ $(\mathcal{Q}^{\circlearrowright})_w(n)$ *is the graded vector space concentrated in degree* n *generated by*

$$\xi^1(\mu) \circ_1 \mu_n.$$

Proof We have $(\mathcal{Q}^{\circlearrowright})_0 = \mathcal{Q}$, so the description of the operadic part follows from Definition 9.12.

The proof of the description of $(\mathcal{Q}^{\circlearrowright})_w(n)$ is similar to that of $(\mathcal{C}om^{\circlearrowright})_w(n)$ given in [15, Example 5.2.5], replacing the commutativity property by the commutativity up to signs and taking into account the fact that we have graded modules. □

Proposition 10.9 *The wheeled operad* $\mathcal{P}$ *is isomorphic to* $\mathcal{Q}^{\circlearrowright}$. *In particular,* $\mathcal{P}$ *is isomorphic to the wheeled completion of the quadratic operad* $\mathcal{P}_0$, *ie* $\mathcal{P} \simeq \mathcal{P}_0^{\circlearrowright}$.

Proof Recall that the wheelification $(-)^{\circlearrowright}$ is the left adjoint of the forgetful functor F from wheeled operads to operads. Since $\mathcal{P}$ is a wheeled operad whose operadic part is $\mathcal{P}_0$, we have a morphism of operads $\mathcal{P}_0 \to F(\mathcal{P})$. The composition of the isomorphism of operads $f : \mathcal{Q} \to \mathcal{P}_0$ constructed in the proof of Proposition 9.13, with the morphism of operads defined above, gives a morphism of operads $\mathcal{Q} \to F(\mathcal{P}_0)$. By adjunction, this morphism induces a morphism of wheeled operads

$$f^{\circlearrowright} : \mathcal{Q}^{\circlearrowright} \to \mathcal{P}.$$

By Proposition 9.13, the restriction of $f^{\circlearrowright}$ to the operadic parts is an isomorphism of operads given explicitly on the generators of $(\mathcal{Q}^{\circlearrowright})_0(n)$ by

$$f^{\circlearrowright}(\mu_n) = [\pi^{\otimes n}].$$

For the wheeled part, by Theorem 9.3, $\mathcal{P}_w(n) = s^{-1}\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr})}(\mathfrak{a}, \mathfrak{a}^{\otimes n})$ is the graded vector space concentrated in degree n generated by $[\pi^{\otimes n}]$. By Proposition 10.8 it follows that $\mathcal{P}_w(n)$ and $(\mathcal{Q}^{\circlearrowright})_w(n)$ are isomorphic as graded vector spaces.

Since $f^{\circlearrowright}$ is a morphism of wheeled operads, the compatibility with the contractions gives:

$$f^{\circlearrowright}(\xi^1(\mu)) = \xi^1(f^{\circlearrowright}(\mu)) = \xi^1([\pi^{\otimes 2}]) = -[\pi]$$

where the last equality is given by Definition 9.16. We deduce that

$$f^{\circlearrowright} : (\mathcal{Q}^{\circlearrowright})_w(1) \to \mathcal{P}_w(1)$$

is an isomorphism. By Proposition 10.8, the generator of $(\mathcal{Q}^{\circlearrowright})_w(n)$ is obtained by composition of $\xi^1(\mu)$ with μ_n. It follows from the compatibility of $f^{\circlearrowright}$ with the composition that, for all $n \geq 1$,

$$f^{\circlearrowright} : (\mathcal{Q}^{\circlearrowright})_w(n) \to \mathcal{P}_w(n)$$

is an isomorphism. □

Corollary 10.10 *The PROP $\mathcal{C}_{\mathcal{P}}$ is isomorphic to the wheeled PROP generated by one antisymmetric operation μ of biarity $(2,1)$ in degree 1 subject to the quadratic relation*

$$\mu(\mu \otimes 1) = -\mu(1 \otimes \mu).$$

The following theorem relates the PROP $\mathcal{C}_{\mathcal{P}}$ to the PROP $\mathcal{E}$ of extension groups introduced in Section 10.

Theorem 10.11 *There is an isomorphism of PROPs*

$$\chi : \mathcal{C}_{\mathcal{P}} \xrightarrow{\simeq} \mathcal{E}.$$

In particular, $\mathcal{E}$ inherits a structure of wheeled PROP via this isomorphism.

Proof Forgetting the wheeled structure on $\mathcal{C}_{\mathcal{P}}$, the PROP $\mathcal{C}_{\mathcal{P}}$ is generated by one antisymmetric operation μ of biarity $(2,1)$ in degree 1 subject to the quadratic relation

$$\mu(\mu\otimes 1) = -\mu(1\otimes\mu)$$

and one operation $\bar{\mu} = \xi^1(\mu)$ of biarity $(1,0)$ in degree 1.

The functor χ is defined on these generators by

$$\chi(\mu) = [\pi^{\otimes 2}] \quad \text{and} \quad \chi(\bar{\mu}) = [\pi] \in s^{-1}\operatorname{Ext}^*(\Lambda^1\mathfrak{a},\mathfrak{a}),$$

and $[\pi^{\otimes 2}]$ satisfies the quadratic relation by Proposition 9.13. This defines an isomorphism of PROPs since

$$\mathcal{C}_{\mathcal{P}}(n,m) \simeq \mathcal{E}(n,m),$$

comparing the formulas given in Propositions 2.3 and 10.4. □

Remark 10.12 The existence of a wheeled structure on the PROP $\mathcal{E}$ is quite surprising since it is induced by a morphism (of degree 0)

$$\operatorname{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes 2}) \to s^{-1}\operatorname{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}).$$

By Theorem 9.3, $\operatorname{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes 2}) \simeq \Bbbk$, thus the Yoneda product with a generator in $\operatorname{Ext}^1_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes 2})$ gives a morphism

$$\operatorname{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}) \to s^{1}\operatorname{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a},\mathfrak{a}^{\otimes 2}).$$

By Theorem 9.3, this morphism is an isomorphism and the wheeled structure on the PROP $\mathcal{E}$ is given by the inverse on this morphism.

It follows that the existence of a wheeled structure is very specific to the situation studied in this paper (ie Ext–groups in the category $\mathcal{F}(\mathbf{gr};\Bbbk)$ between the tensor powers of the functor $\mathfrak{a}$) and, in general, there is no such natural map.

Remark 10.13 Theorem 10.11 should be viewed as an extension, in the wheeled world, of the isomorphism of PROPs

$$C_{\mathcal{P}_0} \simeq \mathcal{E}^0$$

given in [22, Proposition 3.5]. More precisely there is a commutative diagram

$$\begin{tikzcd} C_{\mathcal{P}} \arrow[r, "\simeq"] & \mathcal{E} \\ C_{\mathcal{P}_0} \arrow[u] \arrow[r, "\simeq"] & \mathcal{E}^0 \arrow[u] \end{tikzcd}$$

where the vertical maps are the inclusion functors.

Part IV Comparaison of the PROPs $\mathcal{H}$ and $\mathcal{E}$

11 The morphism of wheeled PROPs $\varphi\colon \mathcal{E} \to \mathcal{H}$

In this section we define a morphism from the wheeled PROP $\mathcal{E}$ to the wheeled PROP $\mathcal{H}$ of stable cohomology considered in Section 6.

Theorem 11.1 *Let μ be the generator of the wheeled PROP $\mathcal{E}$. There is a morphism of wheeled PROPs $\varphi\colon \mathcal{E} \to \mathcal{H}$ given on generators by*

$$\varphi(\mu) = h_1.$$

Proof By the isomorphism given in Theorem 10.11 the PROP $\mathcal{E}$ is generated by the antisymmetric element μ of biarity $(2,1)$ in degree 1 subject to the quadratic relation

$$\mu(\mu \otimes 1) = -\mu(1 \otimes \mu)$$

and one operation $\bar{\mu} = \xi^1(\mu)$ of biarity $(1,0)$ in degree 1. The functor φ is defined on these generators by

$$\varphi_{2,1}(\mu) = h_1 \quad \text{and} \quad \varphi_{1,0}(\bar{\mu}) = \bar{h}_1.$$

By Section 7, h_1 is antisymmetric and satisfies the quadratic relation.

For $i \in \{1,2\}$, by Proposition 7.1 and Remark 7.2, we have $\xi_1^i(h_1) = (-1)^{i+1}\bar{h}_1$. It follows that the diagram

$$\begin{tikzcd}
\mathcal{E}(2,1) \arrow[r, "\varphi_{2,1}"] \arrow[d, "\xi_1^i"'] & \mathcal{H}(2,1) \arrow[d, "\xi_1^i"] \\
\mathcal{E}(1,0) \arrow[r, "\varphi_{1,0}"'] & \mathcal{H}(1,0)
\end{tikzcd}$$

is commutative, giving the compatibility of the wheeled PROP structures. □

Corollary 11.2 *The subwheeled PROP $\mathcal{K}$ of $\mathcal{H}$ is equivalent to the wheeled PROP associated to the wheeled completion of the operadic suspension of the operad $\mathcal{C}om$.*

Proof By Theorem 10.11 and Proposition 10.9, $\mathcal{E}$ is the wheeled PROP generated by the wheeled completion of the operad $\mathcal{P}_0$, which is the suspension of the operad $\mathcal{C}om$ by Proposition 9.14. By Section 7, $\varphi(\mathcal{E}) \simeq \mathcal{K}$. □

The morphism of wheeled PROPs $\varphi\colon \mathcal{E} \to \mathcal{H}$ induces an explicit graded morphism on Hom–sets,

$$\varphi_{q,l}\colon \bigoplus_{j=0}^{q-l} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l} \otimes \Lambda^j \mathfrak{a}, \mathfrak{a}^{\otimes q}) \to H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l}, \mathfrak{a}^{\otimes q})). \tag{11-1}$$

We denote by $\mathcal{E}_w$ the sub-PROP of $\mathcal{E}$ corresponding to forgetting the operadic part in the wheeled PROP $\mathcal{E}$ (see Remark 2.6). By restriction, the morphism $\varphi\colon \mathcal{E} \to \mathcal{H}$ induces a morphism $\varphi'\colon \mathcal{E}_w \to \mathcal{H}'$, where $\mathcal{H}'$ is defined in Definition 6.4.

One of the main and difficult results of Djament [1, Théorème 4] gives a graded isomorphism

$$\bigoplus_{j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q}) \simeq H^*_{\mathrm{st}}(\mathfrak{a}^{\otimes q}).$$

By [1, Corollaire 3.7] this isomorphism is induced by the morphism

$$\varphi_{q,0}\colon \bigoplus_{j\in\mathbb{N}} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q}) \to H^*_{\mathrm{st}}(\mathfrak{a}^{\otimes q}).$$

We deduce the following result:

Theorem 11.3 [1, Théorème 4] *The functor* $\varphi'\colon \mathcal{E}_w \to \mathcal{H}'$ *is an equivalence of categories.*

Note that [1, Proposition 3.5] corresponds to the compatibility of the isomorphisms $\varphi_{q,0}$ with the horizontal composition in the PROPs $\mathcal{E}_w$ and $\mathcal{H}'$. Theorem 11.3 gives also the compatibility of the isomorphisms $\varphi_{q,0}$ with the action of the symmetric groups $\mathfrak{S}_q$.

For stable cohomology with coefficients given by a bivariant functor, Djament gives a conjecture in [1, Théorème 7.4]. In particular, Djament conjectures that there exist graded isomorphisms

$$\bigoplus_{j=0}^{q-l} s^{-j}\,\mathrm{Ext}^*_{\mathcal{F}(\mathbf{gr};\Bbbk)}(\mathfrak{a}^{\otimes l}\otimes\Lambda^j\mathfrak{a},\mathfrak{a}^{\otimes q}) \simeq H^*_{\mathrm{st}}(\mathrm{Hom}_{\mathcal{V}}(\mathfrak{a}^{\otimes l},\mathfrak{a}^{\otimes q})).$$

Natural candidates for maps giving these isomorphisms are the maps $\varphi_{q,l}$. By functoriality, these maps are compatible with horizontal and vertical compositions in the PROPs and with the contractions.

Djament's conjecture can be rephrased in the following way

Conjecture 11.4 *The morphism* φ *is an isomorphism of wheeled PROPs.*

Remark 11.5 Let $\Gamma_r = \Gamma_r(\mathbb{Z}^{*n})$ for $r \geq 1$ be the lower central series of the free group $\mathbb{Z}^{*n}$; $\Gamma_1 := \mathbb{Z}^{*n}$ and $\Gamma_{r+1} := [\Gamma_r, \Gamma_1]$ for $r \geq 1$. The Andreadakis filtration, $A(r)$ for $r \geq 0$, of the automorphism group $\mathrm{Aut}(\mathbb{Z}^{*n})$ is defined to be the kernel of the natural homomorphism $\mathrm{Aut}(\mathbb{Z}^{*n}) \to \mathrm{Aut}(\Gamma_1/\Gamma_{r+1})$. In particular, $A(0) = \mathrm{Aut}(\mathbb{Z}^{*n})$, and $A(1)$, which is called the IA–automorphism group of the free group $\mathbb{Z}^{*n}$, is the kernel of the abelianization homomorphism $\alpha\colon A(0) \to \mathrm{Aut}(\mathbb{Z}^{\oplus n}) = \mathrm{GL}_n(\mathbb{Z})$ induced by the abelianization $\Gamma_1/\Gamma_2 \cong \mathbb{Z}^{\oplus n}$. For $r \geq 1$, we have a group homomorphism $\tau_r \colon A(r) \to \mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes(r+1)})$ called the r^{th} Johnson homomorphism, which induces a group embedding $A(r)/A(r+1) \hookrightarrow \mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes(r+1)})$. Any linear map $\Bbbk^n \to \bigoplus_{s=2}^{r+1}(\Bbbk^n)^{\otimes s}$ defines an algebra automorphism of the truncated tensor algebra $T_{\leq r+1}(\Bbbk^n) := \bigoplus_{s=0}^{r+1}(\Bbbk^n)^{\otimes s}$ by extending the linear map multiplicatively. This makes the direct sum $\bigoplus_{s=1}^{r+1}\mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes s})$ a subgroup of the algebra automorphism group of the algebra $T_{\leq r+1}(\Bbbk^n)$. The group $\mathrm{GL}_n(\Bbbk)$ acts on the subgroup $\bigoplus_{s=2}^{r+1}\mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes s})$ in an obvious way, so that we can take the semidirect product $\bigl(\bigoplus_{s=2}^{r+1}\mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes s})\bigr) \rtimes \mathrm{GL}_n(\Bbbk)$. As was shown in [8, Theorem 3.1, page 13], there exists a group homomorphism

$$(\tau_1^{\theta}, \dots, \tau_r^{\theta}, \alpha) \colon A(0) \to \left(\bigoplus_{s=2}^{r+1}\mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes s})\right) \rtimes \mathrm{GL}_n(\Bbbk)$$

such that $\tau_s^{\theta}|_{A(s)}$ equals the s^{th} Johnson homomorphism

$$\tau_s \colon A(s) \to \mathrm{Hom}(\Bbbk^n, (\Bbbk^n)^{\otimes(s+1)})$$

for each $1 \leq s \leq r$.

The multiplicativity of the group homomorphism implies that τ_1^{θ} is a cocycle, and τ_2^{θ} defines the same quadratic relation for τ_1^{θ} as in (7-1). The cohomology class h_1 equals that of the cocycle τ_1^{θ}. Hence the class h_1 comes from $A(0)/A(2)$, and the quadratic relation holds on $A(0)/A(3)$. This means the map $\varphi_{q,l}$ lifts to the cohomology group of $A(0)/A(3)$,

$$\begin{tikzcd} & H^*\bigl(A(0)/A(3), ((\Bbbk^n)^*)^{\otimes q} \otimes (\Bbbk^n)^{\otimes l}\bigr) \arrow[d] \\ \mathcal{E}(q,l) \arrow[ur] \arrow[r, "\varphi_{q,l}"] & \mathcal{H}(q,l) \end{tikzcd}$$

where the vertical arrow means the homomorphism induced by the quotient map $\mathrm{Aut}(\mathbb{Z}^{*n}) = A(0) \to A(0)/A(3)$ and n is big enough.

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Anchored foams and annular homology

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We describe equivariant SL(2) and SL(3) homology for links in the thickened annulus via foam evaluation. The thickened annulus is replaced by 3–space with a distinguished line in it. Generators of state spaces for annular webs are represented by foams with boundary that may intersect the distinguished line; intersection points, called anchor points, contribute additional terms, reminiscent of square roots of the Hessian, to the foam evaluation. Both oriented and unoriented SL(3) foams are treated.

57K18; 18N25, 57K16

1 Introduction

Asaeda–Przytycki–Sikora [2] homology of links in the thickened annulus has led to a number of interesting developments — see the first author [1], Baldwin, Beliakova, Grigsby, Licata, Putyra and Wehrli [3; 5; 11; 12; 13] and Roberts [35] — and extensions of their work to SL(N) and GL(N) link homology in the thickened annulus — see Queffelec, Rose, Sartori and Wedrich [30; 31; 32].

GL(N) and SL(N) link homology theories are closely related to foam evaluation. This connection was made the most transparent by the work of Robert and Wagner [34], who wrote down a combinatorial formula for GL(N) closed foam evaluation that allows to build GL(N) link homology from the ground up, bypassing categorical approaches to the latter. A variation of their formula was used by Robert and the second author [18] to evaluate unoriented SL(3) foams, giving a combinatorial approach to some of the structures discovered by Kronheimer and Mrowka [23].

In this paper we extend foam evaluation framework to build equivariant SL(2) and SL(3) state spaces for annular webs and, consequently, equivariant SL(2) and SL(3) homology for links in the thickened annulus. Our construction complements earlier work [30; 32] on the subject. The same approach allows us to define state spaces for unoriented SL(3) annular webs, extending the construction in [18]. As in [18], the unoriented SL(3) theory yields state spaces and skein relations for planar webs but does not extend to a link homology theory.

In the APS (Asaeda–Przytycki–Sikora) annular homology and its equivariant and SL(N) generalizations, one first defines state spaces for annular SL(2) and SL(N) webs, where annular SL(2) webs are just collections of embedded circles in an annulus. See also Boerner [7; 8], where the APS theory is reformulated using embedded surfaces.

Our idea is to think of an open thickened annulus as the complement to a line L in $\mathbb{R}^3$, chosen for convenience to be the z–axis. An annular SL(N) web Γ is then placed into the xy–plane with $(0,0)$ removed. To define its state space $\langle\Gamma\rangle$, we consider SL(N) foams F in the half-space $\mathbb{R}^3_-$ bounded by the xy–plane such that Γ is the boundary of F. These foams may intersect the z–axis, and we refer to the intersection points as *anchor points* and to such foams as *anchored foams*. Anchor points additionally carry a label from 1 to N, and we modify foam evaluation by adding a new type of factors associated to anchor points.

We treat $N=2$ and $N=3$ cases, with modified evaluations given by formulas (2) and (77), respectively; also see (35) for the unoriented SL(3) anchored foam evaluation.

Anchored foam evaluation take values in the ring of polynomials rather than the ring of symmetric polynomials. One starts with an admissible coloring c of facets of a foam F, as usual. An anchor point labeled i lying on a facet of color j contributes $\delta_{i,j}\sqrt{\pm f'(x_i)}$ to the evaluation $\langle F,c\rangle$, where, in the SL(3) case as an example,

$$f(x)=(x-x_1)(x-x_2)(x-x_3)$$

is the polynomial of degree three with roots x_1, x_2 and x_3. The full evaluation $\langle F\rangle$ is given by summing over $\langle F,c\rangle$ for all admissible colorings c. We check integrality of these evaluations, with $\langle F\rangle$ a polynomial in x_1, x_2 and x_3, in the SL(3) case.

Given evaluations of anchored closed foams, one can form state spaces for annular webs. We show that this modified evaluation, with anchor points contributing $\delta_{i,j}\sqrt{\pm f'(x_i)}$, perfectly matches the structure of state spaces of annular homology, in SL(2) and SL(3) cases. The construction also allows us to define unoriented SL(3) homology for annular trivalent graphs, extending [18] to the annular framework.

With state spaces at hand, it is straightforward to define annular SL(2) and SL(3) link homology, by analogy with [1; 2; 4; 14] in the SL(2) setting, with [18] in the unoriented

SL(3) setting, and with [15; 28; 34] in the oriented SL(3) setting. State spaces and link homology carry additional gradings coming from intersection points of foams with the z–axis. We show that the result matches equivariant SL(2) homology [1] of the first author. A simple modification of the construction (truncating the ground ring by sending the x_i to 0 upon evaluation) gives a foam approach to the original APS homology. We expect that the nonequivariant variant of our SL(3) construction recovers the $N = 3$ case of the homology in [30]. It seems that the equivariant annular SL(3) homology, as described in the present paper, is new.

Section 2 describes SL(2) homology via anchored foams. The evaluation is defined in Section 2.1, which also contains the skein relations for anchored SL(2) foams. The state spaces are studied in Section 2.2. The state space of n circles in the annulus is a free module of rank 2^n over the ground ring R_α of polynomials in two variables; see Theorem 2.11. The numbers of contractible and essential circles control the bigraded rank. This section also discusses categories of *anchored* and *annular* cobordisms. Annular cobordisms between annular SL(2) webs are disjoint from the z–axis, while anchored cobordism may intersect it.

Theorem 2.20 identifies the annular cobordism functor with that constructed in [1]. Consequently, equivariant annular SL(2) link homology [1] can be rederived via anchored foams. To obtain the original APS homology, one can use anchored foam evaluation, combined with the homomorphism $R_\alpha \to \mathbb{Z}$ taking α_1 and α_2 to 0 to get state spaces and cobordism maps in the APS theory.

Section 3 constructs the state spaces for the annular unoriented SL(3) foam theory, extending the construction of [18]. We start with the evaluation (Section 3.1), followed by skein relations on annular foams (Section 3.2) and properties of state spaces (Section 3.3). Section 3.4 describes similarities between anchor points contributions and Lee's theory, given by inverting the discriminant in the ground ring. Similar to the planar case [18], we don't know a way to describe the state space of an annular web when regions of valency at most four, allowing an inductive simplification, are absent.

In Section 4 we describe annular equivariant SL(3) link homology, based on anchored (annular) oriented SL(3) foams. This homology extends Mackaay–Vaz [28] equivariant SL(3) homology of links in $\mathbb{R}^3$; also see Clark [10], the second author [15], Morrison and Nieh [29], and Robert [33] for the nonequivariant homology in $\mathbb{R}^3$. We start with a review of oriented SL(3) foams in Section 4.1 and then follow a similar route to that of the earlier sections.

Our constructions of annular equivariant link homology via foam evaluation requires working with $U(1)^{\times N}$–equivariant homology rather than $U(N)$ or $\mathrm{GL}(N)$–equivariant homology. In these G–equivariant theories homology of the empty link is $\mathrm{H}_G(p,\mathbb{Z})$, the G–equivariant cohomology of a point. For $U(1)^{\times N}$ that cohomology consists of polynomials in N variables (denoted here by α_1 and α_2 for $N=2$, and x_1 x_2 and x_3 for $N=3$), which is a larger ring than its subring of symmetric polynomials, which is the corresponding equivariant cohomology of a point for $U(N)$ and $\mathrm{GL}(N)$. Having a larger background ring gives additional freedom and allows a "symmetry breaking" between these polynomial variables, necessary in our case as clear from the evaluation (also see Remark 2.1 below).

Working with that larger ring and $U(1)^{\times N}$–equivariant cohomology is a recent phenomenon. It was used by T Sano [37] in resolving the minus sign ambiguity in the functorial extension of Khovanov homology to link cobordisms, bypassing earlier constructions that required additional decorations of links and cobordisms (see [19] for more references and a short discussion). We expect this symmetry breaking of the ground ring generators to find more applications to link homology in the future.

A recent paper of R Lipshitz and S Sarkar [25] contains an application of annular equivariant link homology. The authors use maps associated to moving a strand across the puncture. These maps come for free from the anchored foam perspective of the present paper; see [25, Remark 3.2].

Unoriented SL(3) homology for planar graphs (webs) is closely related to the 4–color theorem and Kronheimer–Mrowka instanton homology for 3–orbifolds [18; 23]. This homology of webs remains a mysterious structure which has only been computed for reducible webs (see Boozer [9] for a computational approach to homology of the dodecahedron and other nonreducible webs). In the annular case, nonreducible webs have fewer vertices, with the smallest such web shown in Figure 10, and annular homology may shed light on and aid in understanding unoriented SL(3) homology of nonreducible webs and related structures.

We expect that our construction admits a generalization to $\mathrm{SL}(N)$ homology for all N via an extension of the Robert–Wagner formula [34] to the anchored case.

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2 SL(2) anchored homology

2.1 Anchored surfaces and their evaluations

Consider the integral polynomial ring $R_\alpha = \mathbb{Z}[\alpha_1, \alpha_2]$ in two variables α_1, α_2. Define a grading on R_α by setting

$$\deg(\alpha_1) = \deg(\alpha_2) = 2. \tag{1}$$

Denote by τ the nontrivial involution of $\{1, 2\}$. It is given by $\tau(i) = 3-i$ for $i \in \{1, 2\}$. Also denote by τ the induced involution of R_α which permutes α_1 and α_2, so that $\tau(\alpha_i) = \alpha_{\tau(i)} = \alpha_{3-i}$. Let R be the τ–invariant subring of R_α, which consists of symmetric polynomials in α_1 and α_2. The subring R is itself a polynomial ring, $R = \mathbb{Z}[E_1, E_2]$, where E_1 and E_2 are elementary symmetric polynomials in α_1 and α_2,

$$E_1 = \alpha_1 + \alpha_2, \quad E_2 = \alpha_1\alpha_2.$$

Degrees of E_1 and E_2 are 2 and 4, respectively.

Let $L \subset \mathbb{R}^3$ denote the z–axis, $L = (0, 0) \times \mathbb{R}$. Let $S \subset \mathbb{R}^3$ be a closed, smoothly embedded surface which intersects L transversely. The surface S may be decorated by dots, disjoint from L, that can otherwise float freely on components of S. The intersection points $S \cap L$ are called *anchor points*. Fix a labeling ℓ, which is a map from the set of anchor points to $\{1, 2\}$,

$$\ell\colon S \cap L \to \{1, 2\}.$$

Order the anchor points by $1, \dots, 2k$, read from bottom to top, so that the labeling ℓ consists of a choice $\ell(j) \in \{1, 2\}$ for each $1 \le j \le 2k$. We will define an evaluation

$$\langle S \rangle \in R_\alpha$$

for S with the fixed labeling ℓ, which is omitted from the notation.

Let $\mathrm{Comp}(S)$ denote the set of connected components of S. A *coloring* of S is a function $c\colon \mathrm{Comp}(S) \to \{1, 2\}$, and we denote by $\mathrm{adm}(S)$ the set of colorings of S. The surface S has $2^{|\mathrm{Comp}(S)|}$ colorings. Fix a coloring c. For $i = 1, 2$, let $d_i(c)$ denote the number of dots on components colored i. Let S_2 denote the union of the 2–colored components. For $1 \le j \le 2k$, let $c(j)$ denote the color of the j^{th} anchor point, induced by c, which may in general be different from the fixed label $\ell(j)$. Define

$$\langle S, c\rangle = (-1)^{\chi(S_2)/2} \frac{\alpha_1^{d_1(c)}\alpha_2^{d_2(c)}\big(\prod_{j=1}^{2k}(\alpha_{c(j)} - \alpha_{\ell(j)})\big)^{1/2}}{(\alpha_1 - \alpha_2)^{\chi(S)/2}}. \tag{2}$$

Note that $\chi(S_2)$ is even since S_2 is a closed surface in $\mathbb{R}^3$. Let us explain the square root in the above equation.

Each component S' of S can be made disjoint from L via a homotopy. Since the mod 2 intersection number is preserved under homotopy, it follows that S' intersects L at an even number of points $p_1, \dots, p_{2m}$, which can be ordered as encountered along L, from bottom to top. Suppose S' is colored by $c(S') = j$, and moreover S' contains an anchor point labeled j. Then the product $\prod_{j=1}^{2m}(\alpha_{c(j)} - \alpha_{\ell(j)}) = 0$, since it contains a term $\alpha_j - \alpha_j = 0$, and the entire evaluation $\langle S, c\rangle = 0$. Thus, the evaluation (2) is only nonzero when the anchor points on a component S' colored j are all labeled by the complementary color $\tau(j)$. In this case, each component contributes an even number of factors of either $\alpha_1 - \alpha_2$ or $\alpha_2 - \alpha_1$ to the product $\prod_{j=1}^{2m}(\alpha_{c(j)} - \alpha_{\ell(j)})$, and we define the square root to be $(\alpha_1 - \alpha_2)^m$ or $(\alpha_2 - \alpha_1)^m$, respectively. If S' has no anchor points, this term is 1 and can be removed from the product.

Note that the evaluation is the product of evaluations of individual components,

$$\langle S, c\rangle = \prod_{S' \in \mathrm{Comp}(S)} \langle S', c(S')\rangle. \tag{3}$$

Thus, if S' is colored 1 by $c' = c(S')$, has $2k$ anchor points all labeled 2 and carries d dots, then

$$\langle S', c'\rangle = \alpha_1^d (\alpha_1 - \alpha_2)^{k - \chi(S')/2}. \tag{4}$$

If S' is colored 2 by $c' = c(S')$, has $2k$ anchor points all labeled 1 and carries d dots, then

$$\langle S', c'\rangle = (-1)^{\chi(S')/2 + k} \alpha_2^d (\alpha_1 - \alpha_2)^{k - \chi(S')/2} = \alpha_2^d (\alpha_2 - \alpha_1)^{k - \chi(S')/2}. \tag{5}$$

Otherwise, if one of the anchor points has the same label as the color of S', the evaluation $\langle S', c'\rangle = 0$ and $\langle S, c\rangle = 0$.

Define the evaluation of S by

$$\langle S\rangle = \sum_c \langle S, c\rangle, \tag{6}$$

where the sum is over all colorings of S. Note that if $S \cap L = \varnothing$, then $\langle S\rangle$ agrees with the evaluation in [19; 34]. Also note that $\langle S\rangle = 0$ if a component of S has two anchor points with different labels 1, 2.

We have

$$\langle S\rangle = \prod_{S'\in \operatorname{Comp} S} \langle S'\rangle, \tag{7}$$

that is, evaluation of S is the product of evaluations over connected components of S.

We can rewrite $\langle S\rangle$ as follows. First, suppose S is connected, carrying d dots, with $2k \geq 0$ anchor points. For $i = 1, 2$, let c_i denote the coloring of S by i. Define

$$\langle S, c_1\rangle = \frac{\alpha_1^d((\alpha_1-\alpha_{\ell(1)})\cdots(\alpha_1-\alpha_{\ell(2k)}))^{1/2}}{(\alpha_1-\alpha_2)^{\chi(S)/2}}, \tag{8}$$

$$\langle S, c_2\rangle = (-1)^{\chi(S)/2}\frac{\alpha_2^d((\alpha_2-\alpha_{\ell(1)})\cdots(\alpha_2-\alpha_{\ell(2k)}))^{1/2}}{(\alpha_1-\alpha_2)^{\chi(S)/2}}. \tag{9}$$

Again, square roots in the above equations are taken in the natural way. If S has oppositely labeled anchor points then both (8) and (9) are zero. If all anchor points are labeled 1, then (8) is zero, whereas (9) is equal to

$$\langle S, c_2\rangle = (-1)^{\chi(S)/2}\frac{\alpha_2^d(\alpha_2-\alpha_1)^k}{(\alpha_1-\alpha_2)^{\chi(S)/2}}.$$

On the other hand, if all anchor points are labeled by 2 then (9) is zero and (8) equals

$$\frac{\alpha_1^d(\alpha_1-\alpha_2)^k}{(\alpha_1-\alpha_2)^{\chi(S)/2}}.$$

Then for connected S with anchor points,

$$\langle S\rangle = \langle S, c_1\rangle + \langle S, c_2\rangle,$$

where at most one of the summands $\langle S, c_i\rangle$ is nonzero.

Clearly the evaluation is multiplicative under disjoint union. That is, if $S = S_1 \sqcup\cdots\sqcup S_n$, then

$$\langle S\rangle = \langle S_1\rangle\cdots\langle S_n\rangle.$$

Remark 2.1 Unlike closed foam evaluations appearing elsewhere [16; 18; 19; 34; 36], our evaluation does not in general produce a symmetric function. The following examples illustrate this.

Example 2.2 Let S be a sphere intersecting L in two points with labels i and j and carrying d dots. If $i \neq j$, then each coloring c yields $\langle S, c\rangle = 0$. If both anchor points are labeled 1, then only coloring S by 2 contributes to the sum, and we have

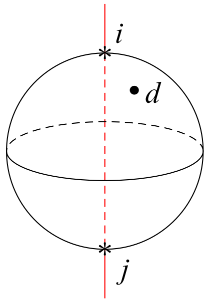
$$\langle S\rangle = \langle S, c_2\rangle = -\frac{\alpha_2^d(\alpha_2-\alpha_1)}{\alpha_1-\alpha_2} = \alpha_2^d.$$

On the other hand, if both anchor points are labeled 2, then

$$\langle S\rangle = \langle S, c_1\rangle = \alpha_1^d.$$

This is summarized pictorially by

(10)



$$= \delta_{ij}\tau(\alpha_i)^d.$$

Both signs are positive since $k + \chi(\mathbb{S}^2)/2 = 1+1 = 2$ is even.

Note that these evaluations are not symmetric in α_1 and α_2.

Example 2.3 More generally, let S be a genus g surface with d dots and $2k$ anchor points. If $k = 0$ (that is, if S is disjoint from L) then the evaluation is

$$\langle S\rangle = \frac{\alpha_1^d + (-1)^{g-1}\alpha_2^d}{(\alpha_1-\alpha_2)^{1-g}}.$$

On the other hand, if $k > 0$, then

$$\langle S\rangle = \begin{cases} \alpha_2^d(\alpha_2-\alpha_1)^{k+g-1} & \text{if } \ell(1) = \cdots = \ell(2k) = 1, \\ \alpha_1^d(\alpha_1-\alpha_2)^{k+g-1} & \text{if } \ell(1) = \cdots = \ell(2k) = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

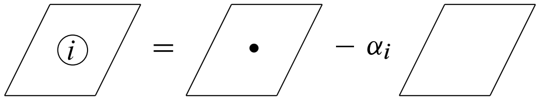
Proposition 2.4 *For any anchored surface $S \subset \mathbb{R}^3$ with d dots and $2k$ anchor points, its evaluation $\langle S\rangle$ is a homogeneous polynomial in α_1 and α_2 of degree*

$$-\chi(S) + 2d + 2k.$$

Proof If S does not intersect L, then this follows from Example 2.3. Suppose that S intersects L. It suffices to verify the statement for connected surfaces. If S intersects L, then the statement follows from (11), since $k > 0$. □

We recall the following notation from [19]. For $i = 1, 2$, we allow surfaces to carry decorations (i) consisting of i inscribed in a small circle. They must be disjoint from

L and are allowed to float freely along the connected component on which they appear. We call these *shifted* dots. Diagrammatically, a shifted dot ⓘ is the difference between a dot and α_i:

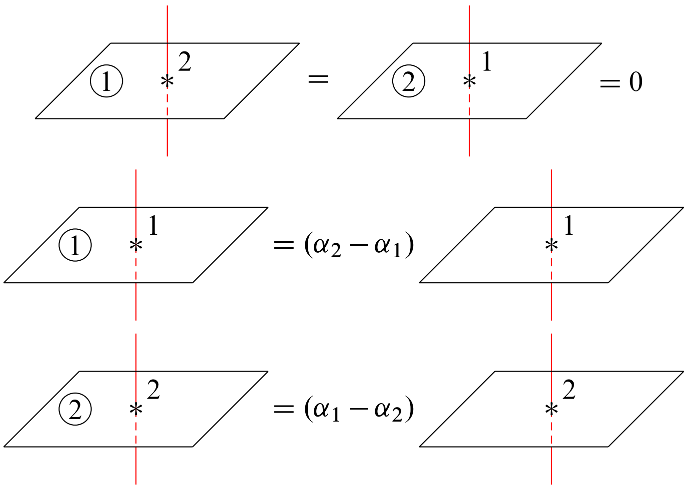
(12) 

Lemma 2.5 *Let S be an anchored foam and let $S \cup ⓘ$ denote the anchored foam obtained by placing a shifted dot ⓘ on some component S' of S. Then*

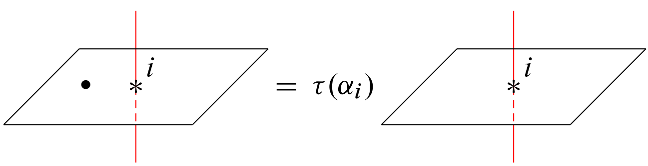
$$\langle S \cup ⓘ \rangle = \begin{cases} 0 & \text{if } S' \text{ has an anchor point labeled } \tau(i), \\ (-1)^i(\alpha_1 - \alpha_2)\langle S \rangle & \text{if all anchor points on } S' \text{ are labeled } i. \end{cases}$$

Proof This is clear from the definitions. □

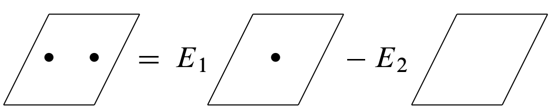
Lemma 2.5 is summarized diagrammatically by

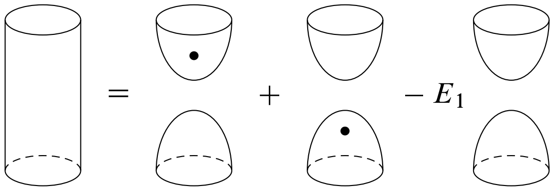
(13) 

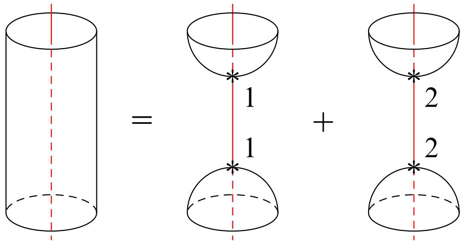
Alternatively, the skein relations (13) may written compactly as

(14) 

Lemma 2.6 *The following local relations hold*:

(15) 

(16) 

(17) 

Proof The relation (15) is straightforward. Let us now verify (16), which is proved in the same way as for nonanchored foams, see [19, Lemma 3.5]. Let S denote the surface on the left, and let F denote the surface obtained by surgering S as shown on the right. Denote by F^t (resp. F^b) the surface obtained from F by placing an additional dot on the top (resp. bottom) depicted disk. Note that anchor points, as well as their labels, are the same for F^t, F^b, and F. Colorings of F, F^t, and F^b are in a canonical bijection. There are four local models for a coloring of F, illustrated in Figure 1.

Let c be a coloring of F of the type shown in Figure 1(c), with the corresponding coloring of F^t and F^b still denoted by c. We have

$$\langle F^t, c\rangle = \alpha_1 \langle F, c\rangle, \quad \langle F^b, c\rangle = \alpha_2 \langle F, c\rangle,$$

hence $\langle F^t, c\rangle + \langle F^b, c\rangle - E_1\langle F, c\rangle = 0$. A similar calculation holds for a coloring c of Figure 1(d) type.

There is a natural bijection between colorings of S and colorings of F of Figures 1(a) and 1(b) types. Let c be a coloring of F of Figure 1(a) type, and continue to denote by

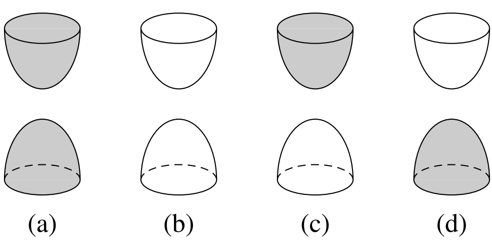


Figure 1: Local models for colorings of F. Shaded indicates color 1 and solid white indicates color 2.

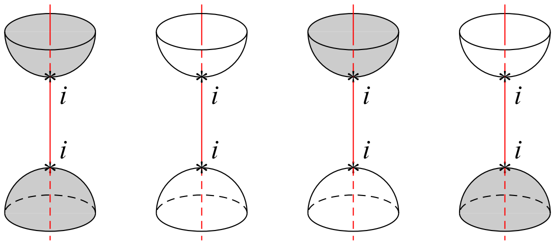


Figure 2: Local models for colorings of F^i. Shaded indicates color 1 and solid white indicates color 2.

c the corresponding coloring of S. Then

$$\chi(F) = \chi(S) + 2, \quad \chi(F_2(c)) = \chi(S_2(c)),$$
$$\langle F^t, c\rangle = \alpha_1 \langle F, c\rangle, \qquad \langle F^b, c\rangle = \alpha_1\langle F, c\rangle,$$

so we have

$$\langle F^t, c\rangle + \langle F^b, c\rangle - E_1\langle F, c\rangle = (\alpha_1 - \alpha_2)\langle F, c\rangle = \langle S, c\rangle.$$

Finally, if c is a coloring of F of the Figure 1(b) type, then

$$\chi(F) = \chi(S) + 2, \qquad \langle F^t, c\rangle = \alpha_2\langle F, c\rangle,$$
$$\chi(F_2(c)) = \chi(S_2(c)) + 2, \quad \langle F^b, c\rangle = \alpha_2\langle F, c\rangle,$$

which yields

$$\langle F^t, c\rangle + \langle F^b, c\rangle - E_1\langle F, c\rangle = (\alpha_2 - \alpha_1)\langle F, c\rangle = -(\alpha_2 - \alpha_1)\frac{\langle S, c\rangle}{\alpha_1 - \alpha_2} = \langle S, c\rangle.$$

We now address (17), where anchor points are present. Let S denote the surface on the left-hand side of the equality. Let F^1 and F^2 denote the two anchored foams obtained by surgery on S in which the new anchor points are both labeled 1 or 2, respectively, so that (17) reads $\langle S\rangle = \langle F^1\rangle + \langle F^2\rangle$. For each $i = 1, 2$ there are four local models for a coloring of F^i, shown in Figure 2. Colorings c in Figures 2(c) and 2(d) evaluate to zero for both $i = 1, 2$,

$$\langle F^1, c\rangle = \langle F^2, c\rangle = 0,$$

and they don't correspond to any colorings of S. There is a natural bijection between colorings of S and colorings of F^i of the types in Figures 2(a) and 2(b).

Let c be a coloring of S in which the depicted region of S in (17) is colored 1, with the corresponding colorings of F^1 and F^2 still denoted by c. We have immediately that $\langle F^1, c\rangle = 0$. On the other hand,

$$\chi(F^2) = \chi(S) + 2, \quad \chi(F_2^2(c)) = \chi(S_2(c)),$$

and F^2 has two additional anchor points compared to S, both labeled 2 and their regions colored 1. Therefore,

$$\langle F^1, c\rangle + \langle F^2, c\rangle = \langle F^2, c\rangle = (\alpha_1 - \alpha_2)\frac{\langle S, c\rangle}{\alpha_1 - \alpha_2} = \langle S, c\rangle.$$

Now let c be a coloring of S in which the depicted region of (17) is colored 2, and continue to denote by c the corresponding colorings of F^1 and F^2. Then $\langle F^2, c\rangle = 0$. Since

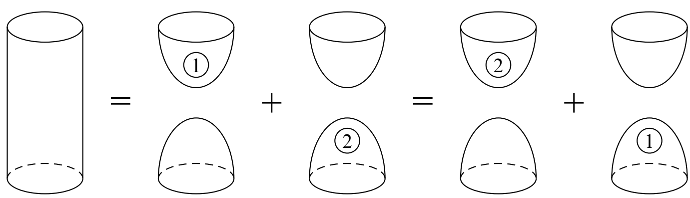
$$\chi(F^1) = \chi(S) + 2, \quad \chi(F_2^1(c)) = \chi(S_2(c)) + 2,$$

and F^1 contains two more anchor points labeled 1 and colored 2 than S does, we obtain

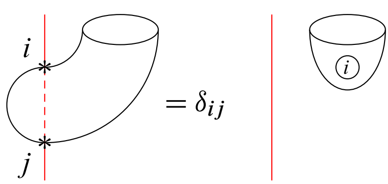
$$\langle F^1, c\rangle + \langle F^2, c\rangle = \langle F^1, c\rangle = -(\alpha_2 - \alpha_1)\frac{\langle S, c\rangle}{\alpha_1 - \alpha_2} = \langle S, c\rangle.$$

Relation $\langle S\rangle = \langle F^1\rangle + \langle F^2\rangle$ in (17) follows. □

Equation (16) can also be written using shifted dots:

(18) 

Corollary 2.7 *The following local relation holds*:

(19) 

Proof This can be seen by applying the neck-cutting relation (16) near the depicted contractible circle and evaluating the resulting anchored sphere according to (10). □

2.2 State spaces

Following [6; 19], we can apply the universal construction to the evaluation described above. Let $\mathcal{P} = \mathbb{R}^2 \setminus (0,0)$ denote the punctured plane. Given a collection C of disjoint simple closed curves in $\mathcal{P}$, let $\mathrm{Fr}(C)$ denote the free R_α–module with a basis consisting of properly embedded compact surfaces $S \subset \mathbb{R}^2 \times (-\infty, 0]$ with $\partial S = C$ and which are transverse to the ray $L_- := (0,0) \times (-\infty, 0]$. The intersection $S \cap L_-$ is a 0–submanifold of L_- and consists of finitely many points. Moreover, each such surface S must carry a labeling, a map

$$\ell = \ell_S \colon S \cap L_- \to \{1,2\}$$

from the set of its intersection points with the ray L_- (its *anchor points*) to $\{1,2\}$. For a basis element $S \in \mathrm{Fr}(C)$, let $\overline{S} \subset \mathbb{R}^2 \times [0,\infty)$ denote its reflection through the plane $\mathbb{R}^2$. Labels of anchor points do not change upon reflection. For two basis elements $S, S' \in \mathrm{Fr}(C)$, denote by $\overline{S}S'$ the closed anchored surface obtained by gluing $\overline{S}$ to S' along their common boundary C.

Define a bilinear form

$$(-,-)\colon \mathrm{Fr}(C) \times \mathrm{Fr}(C) \to R_\alpha \tag{20}$$

by setting $(S, S') = \langle \overline{S}S' \rangle$. A direct computation shows that the form is symmetric, since for a closed surface T the evaluation satisfies $\langle \overline{T} \rangle = \langle T \rangle$.

Define the *state space of* C, denoted by $\langle C \rangle$, to be the quotient of $\mathrm{Fr}(C)$ by the kernel

$$\{x \in \mathrm{Fr}(C) \mid (x,y) = 0 \text{ for all } y \in \mathrm{Fr}(C)\}$$

of this bilinear form. For a basis element $S \in \mathrm{Fr}(C)$, we will write $[S]$ to denote its equivalence class in $\langle C \rangle$.

Equip the ground ring R_α with a bigrading by placing α_1 and α_2 in bidegree $(2,0)$. We extend this bigrading $(\mathrm{qdeg}, \mathrm{adeg})$ to $\mathrm{Fr}(C)$ as follows. For a basis element $S \in \mathrm{Fr}(C)$ with d dots and m anchor points, set the quantum grading $\mathrm{qdeg}(S) \in \mathbb{Z}$ to be

$$\mathrm{qdeg}(S) = -\chi(S) + 2d + m. \tag{21}$$

Note that if S is a closed surface, then $\langle S \rangle \in R_\alpha$ is a homogeneous polynomial of degree $\mathrm{qdeg}(S)$, following the degree convention (1).

	label 1	label 2
i odd	1	-1
i even	-1	1

Table 1: The contribution of the i^{th} anchor point on S to $\operatorname{adeg}(S)$.

Next, let $\ell(1), \dots, \ell(m)$ denote the labels of the anchor points of S, ordered from bottom to top, and define the annular grading $\operatorname{adeg}(S) \in \mathbb{Z}$ by setting

$$\operatorname{adeg}(S) = \sum_{i=1}^{m} (-1)^{i+\ell(i)}. \tag{22}$$

In other words, if the i^{th} anchor point p_i is labeled 1, then it contributes 1 to adeg if i is odd and -1 if i is even. Likewise, if p_i has label 2 then it contributes -1 if i is odd and 1 if i is even; see also Table 1. Multiplication by α_1 or α_2 increases the $(\operatorname{qdeg}, \operatorname{adeg})$–bidegree by $(2, 0)$.

Example 2.8 Let C consist of two noncontractible circles. The bidegree $(\operatorname{qdeg}, \operatorname{adeg})$ of the four anchored surfaces in $\operatorname{Fr}(C)$ whose underlying surface consists of two disks each intersecting L_- once are recorded in Figure 3.

Lemma 2.9 *Let S be an anchored surface. Then $\langle S \rangle = 0$ or $\operatorname{adeg}(S) = 0$.*

Proof If some component of S has anchor points with different labels then $\langle S \rangle = 0$. Assume that all anchor points on any component of S are labeled identically. We also assume that S intersects L, otherwise $\operatorname{adeg}(S) = 0$ is immediate. As usual, order the anchor points $p_1, \dots, p_m$ from bottom to top.

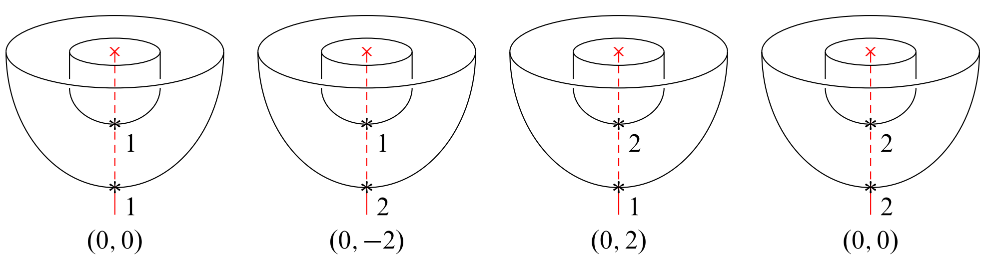


Figure 3: The $(\operatorname{qdeg}, \operatorname{adeg})$–bidegrees of some anchored surfaces whose boundary consists of two noncontractible circles.

Take a generic half-plane P in $\mathbb{R}^3$ containing the anchor line L, so that $P \cap S$ consists of finitely many arcs (with boundary on L) and circles (disjoint from L). For any arc a in $P \cap S$ with boundary $\partial a = \{p_i, p_j\}$, necessarily i and j have opposite parities. To see this, any anchor point between p_i and p_j is one boundary point of an arc in $P \cap S$, and the other boundary point of this arc must also be between p_i and p_j, which shows that the number of anchor points between p_i and p_j is even. Moreover, $\ell(p_i) = \ell(p_j)$ by assumption. Therefore the total contribution of the anchor points p_i and p_j to $\operatorname{adeg}(S)$ is zero. Summing over all arcs in $P \cap S$ yields the statement of the lemma. □

The subspace $\ker((-,-)) \subset \operatorname{Fr}(C)$ respects this bigrading on $\operatorname{Fr}(C)$. Consequently, the bigrading descends to the state space $\langle C \rangle$.

Note that the relations (16) and (17) are bihomogeneous. Let $S \in \operatorname{Fr}(C)$ be a basis element of the form $S = S_1 \sqcup S_2$ where $S_1, S_2 \in \operatorname{Fr}(C)$ are anchored surfaces with S_2 closed. Then in $\langle C \rangle$,

$$[S] = \langle S_2 \rangle [S_1], \quad \langle S_2 \rangle \in R_\alpha. \tag{23}$$

Moreover, the relation (23) is bihomogeneous. That it is homogeneous with respect to qdeg follows from the fact that $\langle S_2 \rangle \in R_\alpha$ is a polynomial of degree $\operatorname{qdeg}(S_2)$. Lemma 2.9 ensures that $\operatorname{adeg}(S_2) = \operatorname{adeg}(\langle S_2 \rangle) = 0$, so $\operatorname{adeg}(S) = \operatorname{adeg}(S_1)$.

Given a bigraded module $M = \bigoplus_{(i,j)\in\mathbb{Z}^2} M_{i,j}$ over a commutative domain such that each $M_{i,j}$ has finite rank, define its *graded rank* to be

$$\operatorname{grank}(M) = \sum_{i,j} \operatorname{rank}(M_{i,j}) q^i a^j.$$

Lemma 2.10 *Let $C \subset \mathcal{P}$ be a single circle. Then the state space $\langle C \rangle$ is a free R_α–module of rank 2. Moreover,*

$$\operatorname{grank}(\langle C \rangle) = \begin{cases} q + q^{-1} & \text{if } C \text{ is contractible,} \\ a + a^{-1} & \text{if } C \text{ is noncontractible.} \end{cases}$$

Proof We consider two cases. If C is contractible, then by applying the neck-cutting relation (16) near C and evaluating closed anchored surfaces as in (23), we see that $\langle C \rangle$ is spanned by the two elements S and $S_\bullet$ shown in Figure 4. Bidegrees of S and $S_\bullet$ are $(-1, 0)$ and $(1, 0)$, respectively. Computing the matrix of the bilinear form (20) for these elements yields

$$\begin{pmatrix} \bar{S}S & \bar{S}S_\bullet \\ \bar{S}_\bullet S & \bar{S}_\bullet S_\bullet \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & E_1 \end{pmatrix},$$

which is invertible; thus S and $S_\bullet$ constitute a basis for $\langle C \rangle$.

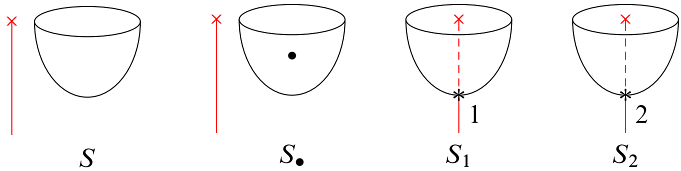


Figure 4: Basis elements for the state space of a single circle C. The first two surfaces form a basis if C is contractible, and the last two form a basis if C is noncontractible.

Now suppose C is noncontractible. Applying the neck-cutting relation (17) near C and evaluating closed anchored surfaces shows that the two elements S_1 and S_2 depicted in Figure 4 span $\langle C\rangle$. Bidegrees of S_1 and S_2 are $(0,1)$ and $(0,-1)$, respectively. The matrix of the bilinear form is

$$\begin{pmatrix} \overline{S}_1 S_1 & \overline{S}_1 S_2 \\ \overline{S}_2 S_1 & \overline{S}_2 S_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence S_1 and S_2 are linearly independent and constitute a basis of $\langle C\rangle$. □

Theorem 2.11 *Let $C \subset \mathcal{P}$ consist of n contractible circles and m noncontractible circles. Then the state space $\langle C\rangle$ is a free R_α–module of rank 2^{n+m}. Moreover,*

$$\operatorname{grank}(\langle C\rangle) = (q+q^{-1})^n (a+a^{-1})^m.$$

Proof Consider a 2^{n+m}–element set $B(C)$ of basis vectors of $\mathrm{Fr}(C)$ consisting of surfaces S satisfying:

- Each component of S is a disk.
- Each disk in S with contractible boundary is disjoint from L_- and carries either zero or one dot.
- Each disk in S with noncontractible boundary intersects L_- exactly once, and its intersection point may be labeled by either 1 or 2.

That $B(C)$ spans $\langle S\rangle$ follows from applying the two neck-cutting relations (16) and (17) near the circles in C and evaluating closed anchored surfaces. Linear independence of $B(C)$ and the statement regarding graded rank follow from the computations in Lemma 2.10. □

Elements of the basis $B(C)$ constructed above are *standard generators*. For such a $\Sigma \in B(C)$ with d dots and anchor points labeled $\ell_1, \dots, \ell_m$, we have

$$\operatorname{qdeg}(\Sigma) = -n + 2d, \quad \operatorname{adeg}(\Sigma) = \sum_{i=1}^{m} (-1)^{i+\ell(i)}. \tag{24}$$

Let $C_0, C_1 \subset \mathcal{P}$ be two collections of disjoint circles in the punctured plane. An *anchored cobordism from C_0 to C_1* is a smoothly and properly embedded compact surface $S \subset \mathbb{R}^2 \times [0,1]$ with boundary $\partial S = C_0 \sqcup C_1$, such that $C_i \subset \mathbb{R}^2 \times \{i\}$ for $i = 0, 1$. Moreover, S is required to intersect the arc $L_{[0,1]} := (0,0) \times [0,1]$ transversely and come equipped with a labeling of these intersection points (called *anchor points*), which is a map

$$\ell = \ell_S \colon S \cap L_{[0,1]} \to \{1, 2\}$$

from the set of its anchor points to $\{1,2\}$. Anchored cobordisms are allowed to carry dots which can float on components but cannot jump to a different component.

We compose anchored cobordisms in the usual manner. For anchored cobordisms $S_1 \colon C_0 \to C_1$ and $S_2 \colon C_1 \to C_2$, let $S_2 S_1 \colon C_0 \to C_2$ denote the anchored cobordism obtained by gluing along the common boundary C_1 and rescaling. Labels of anchor points of $S_2 S_1$ are inherited from labels of S_1 and S_2.

As above, if an anchored cobordism S from C_0 to C_1 has m anchor points and carries d dots, define

$$\operatorname{qdeg}(S) = -\chi(S) + 2d + m.$$

Let $\ell(1), \dots, \ell(m)$ denote the labels of anchor points of S, ordered from bottom to top, and let n be the number of noncontractible circles in C_0. Set

$$\operatorname{adeg}(S) = (-1)^n \sum_{i=1}^{m} (-1)^{i+\ell(i)}.$$

Remark 2.12 If $C_0 = \varnothing$, then S is a basis element of $\operatorname{Fr}(C_1)$, and moreover the two degrees $\operatorname{qdeg}(S)$, $\operatorname{adeg}(S)$ defined above for anchored cobordisms agree with the definitions in (21) and (22) for elements of $\operatorname{Fr}(C_1)$.

An anchored cobordism S from C_0 to C_1 induces an R_α–linear map

$$S \colon \operatorname{Fr}(C_0) \to \operatorname{Fr}(C_1)$$

defined on the basis by gluing along the common boundary C_0. The definition of state spaces via universal construction immediately implies that we have an induced map

$$\langle S \rangle \colon \langle C_0 \rangle \to \langle C_1 \rangle. \tag{25}$$

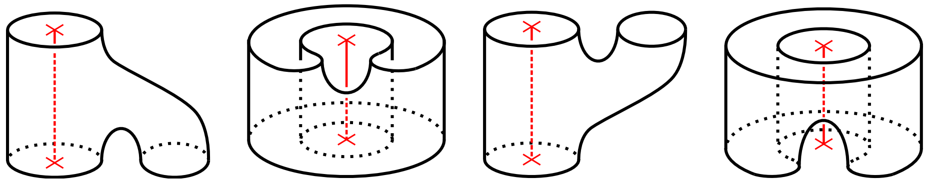


Figure 5: Elementary saddles involving noncontractible circles.

Lemma 2.13 *Let $S_1 : C_0 \to C_1$ and $S_2 : C_1 \to C_2$ be anchored cobordisms. Then*

$$\operatorname{qdeg}(S_2 S_1) = \operatorname{qdeg}(S_2) + \operatorname{qdeg}(S_1), \quad \operatorname{adeg}(S_2 S_1) = \operatorname{adeg}(S_2) + \operatorname{adeg}(S_1).$$

In particular, $\langle S_1 \rangle : \langle C_0 \rangle \to \langle C_1 \rangle$ is a map of bidegree $(\operatorname{qdeg}(S_1), \operatorname{adeg}(S_1))$.

Proof The first equality involving qdeg is straightforward. Let n and m denote the number of noncontractible circles in C_0 and C_1 respectively, and let k denote the number of anchor points of S_1. We have

$$\operatorname{adeg}(S_2 S_1) = \operatorname{adeg}(S_1) + (-1)^{n+m+k} \operatorname{adeg}(S_2).$$

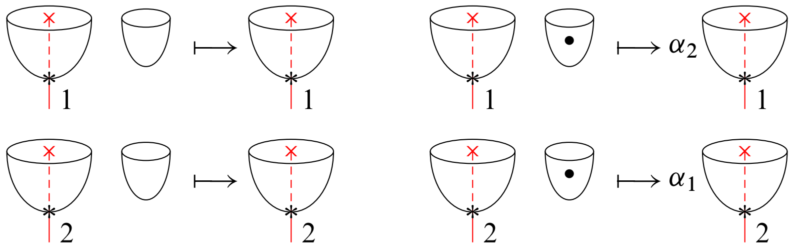
Note $n + m + k$ is even, since it is equal to the number of anchor points of the closed surface obtained by gluing disks to all boundary circles of S_1.

The final statement concerning the bidegree of $\langle S_1 \rangle$ follows from interpreting generators of $\langle C_0 \rangle$ as anchored cobordisms $\varnothing \to C_0$, as in Remark 2.12. □

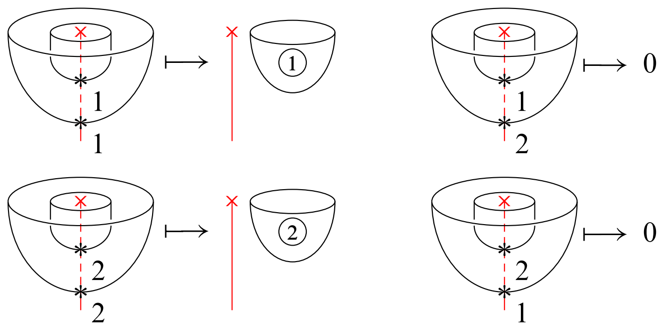
Definition 2.14 An *annular cobordism* is an anchored cobordism $S \subset \mathbb{R}^2 \times [0, 1]$ which is disjoint from the arc $L_{[0,1]} = (0, 0) \times [0, 1]$. An *elementary* annular cobordism is one with a single nondegenerate critical point with respect to the height function $\mathbb{R}^2 \times [0, 1] \to [0, 1]$.

Elementary annular cobordisms consist of a union of a product cobordism with a cup, cap, or saddle. Every annular cobordism may be obtained by composing finitely many elementary ones. Cup and cap annular cobordisms always have contractible boundary. There are four types of elementary annular saddles involving at least one noncontractible circle, illustrated in Figure 5. In the next four examples we write down the maps assigned to these four cobordisms in the standard bases of state spaces, as defined in the proof of Theorem 2.11. We also use the notation of shifted dots from (12).

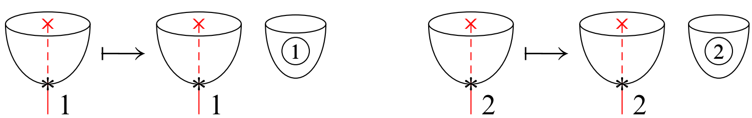
Example 2.15 (Figure 5, type A map) The calculation for this map follows at once from the skein relation (14):



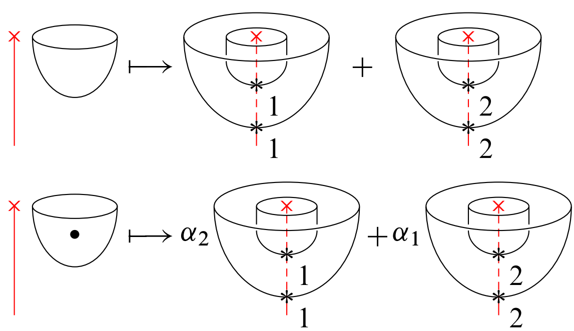
Example 2.16 (Figure 5, type B map) This calculation follows easily from the skein relation (19):



Example 2.17 (Figure 5, type C map) A convenient way to perform this calculation is to use neck-cutting with shifted dots (18) near the contractible circle and then simplify using the relations (13):



Example 2.18 (Figure 5, type D map) The neck-cutting relation (17) is helpful here. For the dotted cup we also use (14) to simplify further:



Recall the involution τ of R_α that transposes α_1 and α_2, and extend it to an antilinear involution, also denoted τ, of the free R_α–module $\mathrm{Fr}(C)$ as follows. Involution τ on $\mathrm{Fr}(C)$ sends a surface S to the same surface with the labeling ℓ of anchor points reversed and acts on linear combinations by

$$\tau\Big(\sum_i \lambda_i S_i\Big) = \sum_i \tau(\lambda_i)\tau(S_i).$$

For a closed surface S we have, by direct computation, $\langle \tau(S)\rangle = \tau(\langle S\rangle)$, showing compatibility of the two involutions. If S, in addition, carries shifted dots, involution τ reverses their labels, so that $\tau(①) = ②$ and $\tau(②) = ①$. Involution τ descends to an involution, also denoted τ, on $\langle C\rangle$. Annular degree is negated under τ: $\mathrm{adeg}(\tau(S)) = -\mathrm{adeg}(S)$ for an anchored cobordism S.

2.3 Annular link homology

Let ACob denote the category whose objects consist of collections of finitely many disjoint simple closed curves in the punctured plane $\mathcal{P}$. A morphism from C_0 to C_1 in ACob is an anchored cobordism from C_0 to C_1, up to ambient isotopy fixing the boundary pointwise and mapping $L_{[0,1]}$ to itself. Let ACob' denote the subcategory of ACob with the same objects as ACob but whose morphisms are isotopy classes of annular cobordisms, disjoint from the anchor line L. The composition of annular cobordisms is again annular.

Let R_α–ggmod denote the category of bigraded R_α–modules and homogeneous maps (of any bidegree) between them. We have a functor

$$\langle - \rangle\colon \mathrm{ACob} \to R_\alpha\text{–ggmod},$$

which sends a collection of circles $C \subset \mathcal{P}$ to the state space $\langle C\rangle$ and sends an anchored cobordism S from C_0 to C_1 to the map $\langle S\rangle\colon \langle C_0\rangle \to \langle C_1\rangle$ as in (25). By Lemma 2.13, $\langle S\rangle$ is a map of bidegree $(\mathrm{qdeg}(S), \mathrm{adeg}(S))$. We can restrict to the category of annular cobordisms to get a functor

$$\langle - \rangle'\colon \mathrm{ACob}' \to R_\alpha\text{–ggmod},$$

which assigns to an annular cobordism S a map $\langle S\rangle' = \langle S\rangle$ of bidegree $(\mathrm{qdeg}(S), 0)$. The restriction $\langle - \rangle'$ does not change the state space assigned to a collection of circles $C \subset \mathcal{P}$.

On the other hand, a functor

$$\mathcal{G}_\alpha \colon \mathrm{ACob}' \to R_\alpha\text{–ggmod}$$

was introduced in [1]. We briefly recall $\mathcal{G}_\alpha$ below.

Consider the algebra

$$A_\alpha = R_\alpha[X]/((X-\alpha_1)(X-\alpha_2)).$$

It is a free R_α–module with basis $\{1, X\}$. The trace $\epsilon_\alpha \colon A_\alpha \to R_\alpha$ given by $1 \mapsto 0$ and $X \mapsto 1$ makes A_α into a Frobenius algebra, which defines a $(1+1)$–dimensional TQFT, a functor $\mathcal{F}_\alpha$ from the category of dotted cobordisms to the category of R_α–modules. A dot on a cobordism is interpreted as multiplication by $X \in A_\alpha$. Define a grading on A_α by setting

$$\mathrm{qdeg}(1) = -1, \quad \mathrm{qdeg}(X) = 1. \tag{26}$$

With this grading, a cobordism S with d dots is assigned by $\mathcal{F}_\alpha$ a map of degree $-\chi(S) + 2d$. Alternatively, the TQFT $\mathcal{F}_\alpha$ is the result of applying the universal construction to the closed surface evaluation (6) when restricted to surfaces disjoint from L and collections of contractible circles in $\mathcal{P}$. See [19] for further details about the Frobenius pair (R_α, A_α).

Let $C \subset \mathcal{P}$ be a collection of n contractible and m noncontractible circles. Define the bigraded R_α–module $\mathcal{G}_\alpha(C)$ as follows. As an R_α–module, we set

$$\mathcal{G}_\alpha(C) = \mathcal{F}_\alpha(C) = A_\alpha^{\otimes(n+m)}.$$

Define the *annular grading*, denoted adeg, on $\mathcal{F}_\alpha(C)$ as follows.

Every tensor factor A_α corresponding to a contractible circle is concentrated in annular degree zero. Order the noncontractible circles in C from outermost (furthest from the puncture) to innermost. Introduce the notation

$$v_0 = 1, \quad v_1 = X - \alpha_1, \quad v_0' = 1, \quad v_1' = X - \alpha_2. \tag{27}$$

Both $\{v_0, v_1\} = \{1, X - \alpha_1\}$ and $\{v_0', v_1'\} = \{1, X - \alpha_2\}$ constitute an R_α–basis for A_α. Set

$$\mathrm{adeg}(v_0) = \mathrm{adeg}(v_0') = -1, \quad \mathrm{adeg}(v_1) = \mathrm{adeg}(v_1') = 1. \tag{28}$$

The annular grading on noncontractible circle is defined by assigning the homogeneous basis $\{v_0, v_1\}$ or $\{v_0', v_1'\}$ to the corresponding tensor factor of A_α in an alternating

	1	X	v_0	v_1	v'_0	v'_1
qdeg′	-1	1	0	0	0	0
adeg	0	0	-1	1	-1	1

Table 2: The $(\mathrm{qdeg}', \mathrm{adeg})$–bidegrees of relevant elements, where $\{1, X\}$ is a basis for a contractible circle and $\{v_0, v_1\}$ and $\{v'_0, v'_1\}$ are bases for noncontractible circles.

manner with respect to nesting in $\mathcal{P}$, with the convention that the outermost circle is assigned $\{v_0, v_1\}$.

It is convenient to distinguish between the modules assigned to different types of circles in $\mathcal{P}$. Let V_α and V'_α denote the R_α–modules A_α with bases $\{v_0, v_1\}$ and $\{v'_0, v'_1\}$, respectively. The notation A_α will be reserved for the module assigned to a contractible circle, with basis $\{1, X\}$.

The R_α–module $\mathcal{G}_\alpha(C)$ also carries a quantum grading qdeg inherited from (26). Define a modified quantum grading qdeg' on $\mathcal{G}_\alpha(C)$ by

$$\mathrm{qdeg}' = \mathrm{qdeg} - \mathrm{adeg}. \tag{29}$$

We will consider $\mathcal{G}_\alpha(C)$ as a bigraded R_α–module with bigrading $(\mathrm{qdeg}', \mathrm{adeg})$. Bidegrees are recorded in Table 2.

Remark 2.19 The modified quantum grading qdeg' appears elsewhere in the literature and is more natural in the context of annular link homology. In [12] this grading was denoted j'. Similarly, the annular link homology defined in [5] carries the modified quantum grading.

We now define $\mathcal{G}_\alpha$ on annular cobordisms. For an annular cobordism $S \subset \mathbb{R}^2 \times [0, 1]$, if the boundary of S is contractible in $\mathcal{P}$ then $\mathcal{G}_\alpha(S) = \mathcal{F}_\alpha(S)$, where $\mathcal{F}_\alpha$ is the TQFT corresponding to the Frobenius algebra A_α as above. Formulas for the maps assigned by $\mathcal{G}_\alpha$ to the four elementary cobordisms in Figure 5 are recorded below. If other essential circles are present, then due to parity the formulas may be slightly different from those below. To obtain the full set of formulas, one interchanges $v_0 \leftrightarrow v'_0$, $v_1 \leftrightarrow v'_1$, and $\alpha_1 \leftrightarrow \alpha_2$:

$$\begin{gathered} V_\alpha \otimes A_\alpha \xrightarrow{\text{(A)}} V_\alpha, \\ v_0 \otimes 1 \mapsto v_0, \quad v_1 \otimes 1 \mapsto v_1, \quad v_0 \otimes X \mapsto \alpha_1 v_0, \quad v_1 \otimes X \mapsto \alpha_2 v_1, \end{gathered} \tag{30}$$

$$(31)\qquad \begin{gathered} V_\alpha \otimes V'_\alpha \xrightarrow{\text{(B)}} A_\alpha, \\ v_0 \otimes v'_0 \mapsto 0, \quad v_1 \otimes v'_0 \mapsto X-\alpha_1, \quad v_0 \otimes v'_1 \mapsto X-\alpha_2, \quad v_1 \otimes v'_1 \mapsto 0, \end{gathered}$$

$$(32)\qquad \begin{gathered} V_\alpha \xrightarrow{\text{(C)}} V_\alpha \otimes A_\alpha, \\ v_0 \mapsto v_0 \otimes (X-\alpha_2), \quad v_1 \mapsto v_1 \otimes (X-\alpha_1), \end{gathered}$$

$$(33)\qquad \begin{gathered} A_\alpha \xrightarrow{\text{(D)}} V_\alpha \otimes V'_\alpha, \\ 1 \mapsto v_0 \otimes v'_1 + v_1 \otimes v'_0, \quad X \mapsto \alpha_1 v_0 \otimes v'_1 + \alpha_2 v_1 \otimes v'_0. \end{gathered}$$

Theorem 2.20 *The functors* $\langle - \rangle' \colon \mathrm{ACob}' \to R_\alpha$–ggmod *and* $\mathcal{G}_\alpha \colon \mathrm{ACob}' \to R_\alpha$–ggmod *are naturally isomorphic via bidegree-preserving maps.*

Proof Let $C \subset \mathcal{P}$ be a collection of circles. We will define an R_α–linear, bidegree preserving isomorphism $\Phi_C \colon \langle C \rangle \to \mathcal{G}_\alpha(C)$ and show that it is natural with respect to annular cobordisms.

Let n and m denote the number of contractible and noncontractible circles in C, respectively. Fix an ordering $1, \dots, n$ of the contractible circles in C. The R_α–module $\mathcal{G}_\alpha(C)$ is free with basis given by elements of the form

$$y_1 \otimes \cdots \otimes y_n \otimes z_1 \otimes \cdots \otimes z_m,$$

where each y_i is in $\{1, X\}$, specifying a basis element of the i^{th} contractible circle, and each z_j is in either $\{v_0, v_1\}$ or $\{v'_0, v'_1\}$, depending on nesting, specifying basis elements of the noncontractible circles. The ordering of factors $z_1 \otimes \cdots \otimes z_m$ corresponding to noncontractible circles is from outermost to innermost as usual, so that the first factor z_1 labels the outermost noncontractible circle.

We now define the isomorphism $\Phi_C \colon \langle C \rangle \to \mathcal{G}_\alpha(C)$. Recall the standard basis $B = B(C)$ for $\langle C \rangle$ defined in the proof of Theorem 2.11. For $\Sigma \in B$ with anchor points labeled $\ell_1, \dots, \ell_m$, read from bottom to top, set

$$\Phi_C(\Sigma) = y_1 \otimes \cdots \otimes y_n \otimes z_1 \otimes \cdots \otimes z_m,$$

where $y_i = 1$ if the corresponding cup in Σ is undotted and $y_i = X$ if the corresponding cup in Σ is dotted. The generators z_j of noncontractible circles are determined using the rule

$$z_j = \begin{cases} v_1 & \text{if } j \text{ is odd and } \ell_j = 1, \\ v_0 & \text{if } j \text{ is odd and } \ell_j = 2, \\ v'_0 & \text{if } j \text{ is even and } \ell_j = 1, \\ v'_1 & \text{if } j \text{ is even and } \ell_j = 2. \end{cases}$$

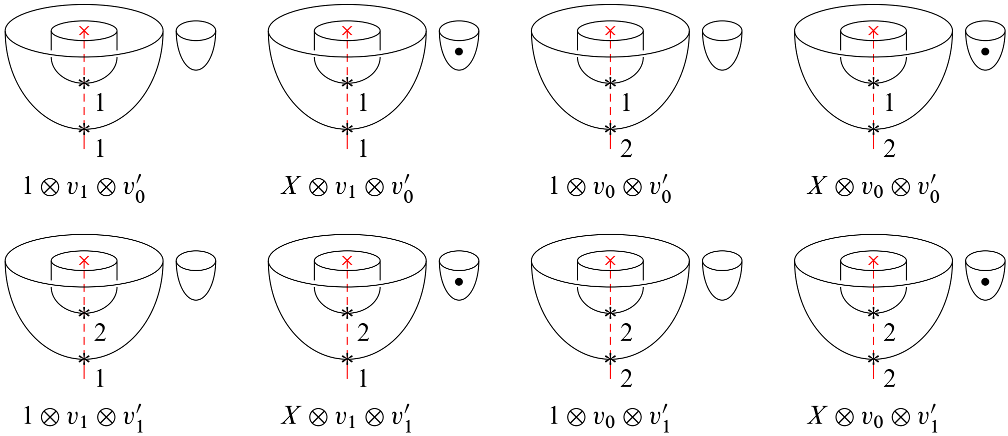


Figure 6: An example of the isomorphism Φ_C when C consists of one contractible circle and two noncontractible circles. Basis elements Σ of $\langle C\rangle$ are drawn with the corresponding basis element $\Phi_C(\Sigma) \in \mathcal{G}_\alpha(C)$ written underneath.

See Figure 6 for an example of the assignment Φ_C when $n = 1$ and $m = 2$. By comparing the bidegree formula (24) for Σ with the bidegree of $\Phi_C(\Sigma)$ (see Table 2), we see that Φ_C is a bidegree-preserving isomorphism. Recall that we use the modified quantum grading (29) for $\mathcal{G}_\alpha(C)$.

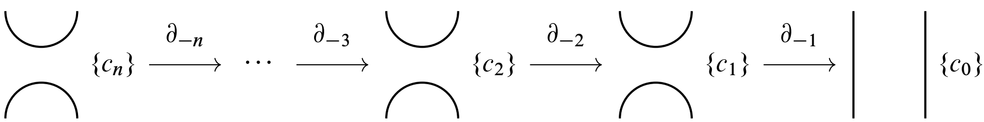
Now let $S\colon C_1 \to C_2$ be an annular cobordism. To complete the proof, we check that the square

$$\begin{tikzcd}
\langle C_1\rangle \arrow[r, "\Phi_{C_1}"] \arrow[d, "\langle S\rangle"'] & \mathcal{G}_\alpha(C_1) \arrow[d, "\mathcal{G}_\alpha(S)"] \\
\langle C_2\rangle \arrow[r, "\Phi_{C_2}"] & \mathcal{G}_\alpha(C_2)
\end{tikzcd}$$

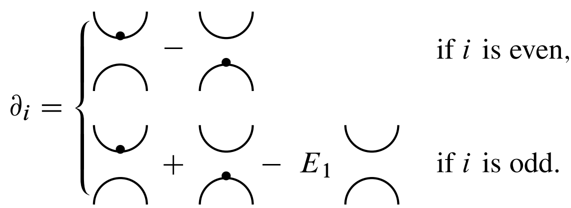
commutes. If all the boundary circles of S are contractible, then commutativity of the square is straightforward. Otherwise, if S has at least one noncontractible boundary circle, it suffices to consider the case where S is one of the elementary annular cobordisms depicted in Figure 5. Formulas for these maps were recorded in Examples 2.15–2.18. Comparing with the formulas (30)–(33) completes the proof. $\square$

Let $\mathbb{A} := S^1 \times [0,1]$ denote the annulus. For an oriented link $L \subset \mathbb{A} \times [0,1]$ in the thickened annulus, a generic projection of L onto $\mathbb{A} \times \{0\}$ yields a link diagram D in the interior of $\mathbb{A}$. Identifying the interior of $\mathbb{A}$ with the punctured plane $\mathcal{P}$, we may form the cube of resolutions of D in the usual way, for instance as described in [4, Section 2], with all smoothings drawn in $\mathcal{P}$. The result is a commutative cube in the category ACob'. Introducing signs to make the cube anticommutative, taking direct sums along diagonals, adding homological and quantum grading shifts, and applying the functor $\langle - \rangle' \colon \mathrm{ACob}' \to R_\alpha$–ggmod, one obtains a chain complex $C(D)$ of bigraded R_α–modules. Diagrams representing isotopic annular links are related by Reidemeister moves away from the puncture. By standard arguments [4; 14], the chain homotopy class of $C(D)$ is an invariant of the annular link L. We write $H(L)$ to denote the homology of $C(D)$, for any diagram D of L. Theorem 2.20 implies that the resulting annular homology is isomorphic to that of [1].

Example 2.21 As an explicit example, let σ denote the positive crossing generator of the 2–strand braid group, and let L_n denote the annular link obtained as the annular closure of σ^{-n}. Consider the complex $C(n)$:



The right-most term is in homological degree zero and the quantum grading shifts c_i are given by $c_0 = n$ and $c_i = n + 2i - 1$ for $1 \le i \le n$. The right-most differential ∂_{-1} is the saddle cobordism, and for $-n \le i \le -2$ the differentials are



The above schematic depiction of ∂_i is interpreted as follows: each ∂_i is an R_α–linear combination of surfaces, each of which is given by the product cobordism on the depicted planar tangle, with a dot on a component of the surface if the corresponding tangle component is dotted. One can show that the chain complex $C(L_n)$ is chain homotopy equivalent to the annular closure of $C(n)$.

Note that the annular closure of chain groups of $C(n)$ in negative homological degree are each a contractible circle, contributing a free module with basis 1 and X (represented by the surfaces S and $S_\bullet$ in Figure 4). In homological degree zero the result is two essential circles. We also see that, upon taking the annular closure, that $\partial_i = 0$ for i even, and that ∂_i for $i \le -3$ odd is given by $\partial_i(1) = 2X - E_1$ and $\partial_i(X) = E_1 X - 2E_2$,

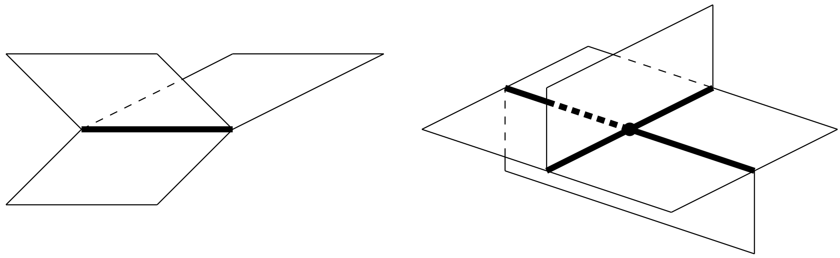


Figure 7: Local model of a prefoam near singular points. Left, seam points where three facets meet. Right, a seam vertex where six facets meet. The singular graph $s(F)$ is drawn bold.

which is injective. The differential ∂_{-1} is the map in Example 2.18, which is also injective. Therefore, in homological degree $i \leq 0$,

$$H^i(L_n) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ \dfrac{R_\alpha\{n-2i-2,0\} \oplus R_\alpha\{n-2i,0\}}{\langle(-E_1,2),(-2E_2,E_1)\rangle} & \text{if } i<0 \text{ and } i \text{ is even,} \\ R_\alpha\{n,-2\} \oplus R_\alpha\{n,2\} \oplus (R_\alpha\{n,0\}/\langle\alpha_2-\alpha_1\rangle) & \text{if } i=0, \end{cases}$$

where the curly brackets $\{j,k\}$ denote an upwards (qdeg, adeg) shift of (j,k), and the angled brackets denote the R_α–submodule generated by the enclosed elements.

3 Unoriented SL(3) anchored homology of planar annular webs

We recall definitions and notations from [18], including that of (unoriented) SL(3) foams and refer the reader to [18, Section 2.1] for more details.

Definition 3.1 A (closed) SL(3) *prefoam* is a compact 2–dimensional CW complex equipped with a PL–structure such that each point has an open neighborhood that is either an open disk, the product of a tripod and an open interval (Figure 7, left), or the cone over the 1–skeleton of a tetrahedron (Figure 7, right). Points of the first type are called *regular*, those of the second are called *seam points*, and those of the third are called *seam vertices*. A (closed) SL(3) *foam* is a closed SL(3) prefoam together with a PL embedding into $\mathbb{R}^3$.

We will simply write *prefoam* and *foam* in place of *closed* SL(3) *(pre)foam*. For a prefoam F, denote by $v(F)$ the set of seam vertices and by $s(F)$ the set of seam points

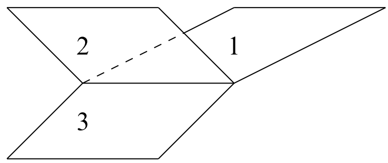


Figure 8: The local model for a preadmissible coloring near a seam point.

and seam vertices. The subspace $s(F)$ is a 4–valent graph which may contain closed loops. Connected components of $s(F) \setminus v(F)$ are called *seams*.

The subspace $F \setminus s(F)$ is a (not necessarily compact) surface, and a connected component of $F \setminus s(F)$ will be called a *facet* of F. The (finite) set of facets of F is denoted by $f(F)$. Facets of prefoams may be decorated by a finite number of dots, which are allowed to float freely on their facets but may not cross seams or enter seam vertices.

A *coloring* of a prefoam F is a map

$$c\colon f(F) \to \{1,2,3\}.$$

That is, a coloring assigns 1, 2 or 3 to each facet of F. A coloring is called *preadmissible* if the three facets meeting at each seam of F have distinct colors; see Figure 8. For a preadmissible coloring c and $1 \le i, j \le 3$ with $i \ne j$, let $F_{ij}(c)$ denote the union of facets colored i or j. The preadmissibility condition guarantees that each $F_{ij}(c)$ is a closed surface; see [18, Proposition 2.2].

A coloring c is called *admissible* if each $F_{ij}(c)$ is orientable. For a foam F (that is, a prefoam embedded in $\mathbb{R}^3$), every preadmissible coloring is admissible, since $F_{ij}(c)$ is a closed surface in $\mathbb{R}^3$.

3.1 Unoriented anchored SL(3) foams and their evaluations

Fix a field $\Bbbk$ of characteristic 2. In this section the following commutative rings will be used:

- $R'_x = \Bbbk[x_1, x_2, x_3]$ is the ring of polynomials in three variables.
- $R_x = \Bbbk[E_1, E_2, E_3]$ the subring of R'_x that consists of symmetric polynomials in x_1, x_2 and x_3, with generators E_i being elementary symmetric polynomials:

$$E_1 = x_1 + x_2 + x_3, \quad E_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad E_3 = x_1x_2x_3.$$

- $R''_x = R'_x[(x_1+x_2)^{-1}, (x_2+x_3)^{-1}, (x_1+x_3)^{-1}]$ is a localization of R'_x given by inverting $x_i + x_j$, for $1 \le i < j \le 3$.

- $\widetilde{R}'_x = \Bbbk[\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}]$ is the extension of R'_x obtained by introducing square roots of x_1, x_2 and x_3.
- $\widetilde{R}''_x = \Bbbk[\sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}, (x_1+x_2)^{-1}, (x_2+x_3)^{-1}, (x_1+x_3)^{-1}]$ is a localization of $\widetilde{R}'_x$ given by inverting $x_i + x_j$, for $1 \le i < j \le 3$.

All five of these rings are graded by setting $\deg(x_i) = 2$ for $i = 1, 2, 3$. Inclusions of the above rings are summarized in the following diagram:

$$\begin{array}{ccccc} & & \widetilde{R}'_x & \subset & \widetilde{R}''_x \\ & & \cup & & \cup \\ R_x & \subset & R'_x & \subset & R''_x \end{array} \tag{34}$$

We follow the notation established in [18] for these rings with the additional subscript x to distinguish from the notation in Section 2.

Definition 3.2 An *anchored* SL(3) *foam* F is an SL(3) foam $F' \subset \mathbb{R}^3$ that may intersect the line L at finitely many points away from the singular graph $s(F')$ of F'. Thus each intersection point belongs to some facet f of F', and intersection of facets with L are required to be transverse. Denote by $p(F) = F \cap L$ the set of intersection points (anchor points) of F. Intersection points carry labels in $\{1, 2, 3\}$; that is, F comes equipped with a fixed map

$$\ell\colon p(F) \to \{1, 2, 3\}.$$

It is convenient to order anchor points $p_1, \dots, p_m$ from bottom to top, with labels $\ell_i = \ell(p_i)$, $i = 1, \dots, m$.

We now refine the notion of admissible coloring of a foam to that of admissible coloring of an anchored foam F. Consider an anchored foam F with the underlying foam F'. A coloring $c \in \mathrm{adm}(F')$ induces a coloring of anchor points in F', by assigning to each point the color of its facet. We say that c is admissible if that's exactly the labeling of anchor points of F, that is, $\ell(p) = c(f)$ for each anchor point p in a facet f, and then set $c(p) = \ell(p)$.

In this way, the set of admissible colorings of F' is in a bijection with the set of admissible colorings of anchored foams F that become F' upon forgetting the labeling of anchor points:

$$\mathrm{adm}(F') \cong \coprod_F \mathrm{adm}(F).$$

Various constructions with SL(3) foams in [18] extend directly to anchored foams. In particular, bicolored surfaces $F_{ij}(c)$ are well defined, associated to an admissible coloring c. We will also call an admissible coloring simply a coloring. We will use i, j and k to denote the three elements of $\{1,2,3\}$, not necessarily in that order.

We refine [18, Definition 2.9] for anchored foams.

Definition 3.3 Let F be an anchored foam, $c \in \mathrm{adm}(F)$ be an admissible coloring, and Σ a connected component of $F_{ij}(c)$ which is disjoint from L. Define a coloring c' of F which swaps the colors i and j on facets of Σ, and leaves all other facets colored according to c. We say that c and c' are related by an *ij–Kempe move along* Σ. Note that since Σ has no anchor points, c' is still an admissible coloring of F.

Kempe moves can be done on components Σ of $F_{ij}(c)$ that intersect L as well, but the resulting anchored foam F_0 is different from F due to carrying different labels on anchor points on Σ.

For $k \in \{1,2,3\}$, denote by k' and k'' its two complementary elements, so that $\{k,k',k''\} = \{1,2,3\}$. Let F be an anchored foam with labeling ℓ. Let $c \in \mathrm{adm}(F)$ be an admissible coloring. For an anchor point $p \in p(F)$ lying on a facet $f \in f(F)$, we set $c(p) = c(f) = \ell(p)$; that is, $c(p)$ is the color of the facet, according to c, on which p lies, which equals $\ell(p)$ since c is admissible. For $1 \le i \le 3$, let $d_i(c)$ denote the number of dots on facets colored i. For $1 \le i \ne j \le 3$, let $F_{ij}(c)$ be the union of facets of F colored i or j. The space $F_{ij}(c)$ is a closed surface in $\mathbb{R}^3$ and hence has even Euler characteristic. Set

$$\langle F, c\rangle = \frac{P(F,c)}{Q(F,c)}, \tag{35}$$

where

$$P(F,c) = \prod_{i=1}^{3} x_i^{d_i(c)} \cdot \Bigl(\prod_{p\in p(F)} (x_{c(p)} + x_{\ell(p)'})(x_{c(p)} + x_{\ell(p)''})\Bigr)^{1/2}, \tag{36}$$

$$Q(F,c) = \prod_{1\le i<j\le 3} (x_i + x_j)^{\chi(F_{ij}(c))/2}. \tag{37}$$

The product of the two terms under the square root, for a given anchor point p, is equal to

$$\begin{aligned} &(x_1+x_2)(x_1+x_3) \quad \text{if } c(p)=1,\\ &(x_2+x_1)(x_2+x_3) \quad \text{if } c(p)=2,\\ &(x_3+x_1)(x_3+x_2) \quad \text{if } c(p)=3. \end{aligned}$$

Remark 3.4 This product is the inverse of the square decoration $\square$ in [18, Section 4.1]. The square decoration was used to study a separable version of the unoriented SL(3) theory, with the discriminant $\mathcal{D} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$ inverted, which is a version of the Lee theory. Here, we use the defect line L rather than freely floating square dots in [18, Section 4.1] in the opposite way, to add factors to the evaluation rather than divide by terms in the discriminant.

Remark 3.5 If c is an admissible coloring of the underlying foam F' of F but not of the anchored foam F, then the evaluation (35) is still defined and equal to zero;

$$\langle F, c\rangle = 0, \quad c \in \operatorname{adm} F' \setminus \operatorname{adm} F. \tag{38}$$

This holds since, for some $p \in p(F)$, its color $c(p)$ differs from its label $\ell(p)$, so that $x_{c(p)} + x_{c(p)} = 0$ appears under the square root in (36) and $P(F, c) = 0$. Thus,

$$(x_{c(p)}+x_{\ell(p)'})(x_{c(p)}+x_{\ell(p)''}) = \begin{cases} (x_{\ell(p)}+x_{\ell(p)'})(x_{\ell(p)}+x_{\ell(p)''}) & \text{if } c(p) = \ell(p), \\ 0 & \text{otherwise.} \end{cases}$$

Define the *evaluation of* F to be

$$\langle F\rangle = \sum_{c\in\operatorname{adm}(F)} \langle F, c\rangle. \tag{39}$$

Alternatively, we can sum over the larger set of $c \in \operatorname{adm}(F')$, due to (38).

Let us explain the square root in (36). The equality $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ holds in a commutative ring of characteristic 2, so $\langle F, c\rangle$ is in the ring $\widetilde{R}''_x$; see (34). We will show in Proposition 3.11 that, in fact, no square roots appear, so that $\langle F, c\rangle \in R''_x$. Likewise, in Proposition 3.12 we show that $\langle F\rangle \in R'_x$.

The evaluation (39) is multiplicative with respect to disjoint union and does not depend on a particular embedding of F into $M = (\mathbb{R}^3, L)$ as long as anchor points on F and their labels are specified.

If an anchored foam F is a disjoint union of anchored foams $F_1 \sqcup \cdots \sqcup F_k$, then

$$\langle F\rangle = \langle F_1\rangle \cdots \langle F_k\rangle.$$

If F is disjoint from L, then $\langle F\rangle$ is equal to the evaluation in [18, Section 2.3].

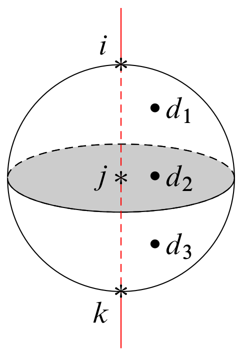
Example 3.6 Let F be a 2–sphere $\mathbb{S}^2$ with two anchor points and d dots. Its evaluation is zero unless both points have the same label $i \in \{1, 2, 3\}$, in which case there is only admissible coloring c which colors F by i. Let $j, k \in \{1, 2, 3\}$ denote the complementary elements to i. The surfaces $F_{ij}(c)$ and $F_{ik}(c)$ are 2–spheres, while $F_{jk}(c)=\varnothing$. Then the evaluation is

$$\langle F\rangle = \frac{x_i^d((x_i+x_j)^2(x_i+x_k)^2)^{1/2}}{(x_i+x_j)(x_i+x_k)} = x_i^d.$$

Example 3.7 More generally, let F be a genus g surface carrying d dots and $2n>0$ anchor points. It evaluates to zero unless all points are labeled by the same $i\in\{1,2,3\}$. In this case, letting $j,k\in\{1,2,3\}$ be the complementary elements to i, the evaluation is

$$\langle F\rangle = \frac{x_i^d((x_i+x_j)(x_i+x_k))^n}{((x_i+x_j)(x_i+x_k))^{1-g}} = x_i^d((x_i+x_j)(x_i+x_k))^{n+g-1}.$$

Example 3.8 Consider the theta foam F whose facets each intersect L once, with anchor points labeled $i,j,k\in\{1,2,3\}$ and facets carrying d_1, d_2 and d_3 dots,



In an admissible coloring of the underlying foam, the three facets must have distinct colors, so $\langle F\rangle=0$ if i, j and k are not distinct. If i, j and k are distinct, then there is one admissible coloring c which colors the top, middle, and bottom facets, respectively, by i, j and k. The surfaces $F_{ij}(c)$, $F_{ik}(c)$, $F_{jk}(c)$ are 2–spheres, and the evaluation is

$$\langle F\rangle = x_i^{d_1}x_j^{d_2}x_k^{d_3}.$$

Remark 3.9 Note that the evaluation of an anchored foam is in general not a symmetric function in x_1, x_2 and x_3, whereas in [18] the evaluation is always an element of R_x.

Let us call a sequence $\ell\in\{1,2,3\}^m$ *preadmissible* if the following holds. Let u_1, u_2 and u_3 be three nonzero elements of the abelian group $\mathbb{Z}/2\times\mathbb{Z}/2$. Sequence ℓ is *preadmissible* if and only if

$$\sum_{i=1}^m u_{\ell(i)} = 0 \in \mathbb{Z}/2\times\mathbb{Z}/2. \tag{40}$$

Proposition 3.10 *If an anchored foam F has an admissible coloring, the sequence ℓ of its anchor points is preadmissible.*

Proof Consider a generic intersection of F with a half-plane in $\mathbb{R}^3$ bounding L. This intersection is a trivalent graph Γ in the half-plane. Coloring c of F induces a coloring c' of edges of Γ such that around each trivalent vertex of Γ the colors of the three edges are distinct (Tait coloring). On the boundary points (one-valent vertices) of Γ the coloring is given by labeling ℓ. The sum on the left hand side of (40) is zero since it can alternatively be written as the sum of triples of vectors $u_1+u_2+u_3=0$ over all trivalent vertices of Γ. Each inner edge of Γ, that bounds two trivalent vertices, contributes $u_i+u_i=0$ to the sum, where i is the color of the edge. An edge with one or both endpoints on the boundary contributes the sum of the u_i over its boundary points. □

For an anchored foam F and $1\le i\le 3$, let $\mathrm{an}(i)$ denote the number of anchor points of F with label i (the dependence on F is omitted).

Proposition 3.11 *For an anchored foam F and an admissible coloring c, we have $\langle F,c\rangle\in R''_x$.*

Proof Recall the rings R''_x and $\widetilde{R}''_x$ defined in (34). It's clear that $\langle F,c\rangle$ belongs to the larger ring $\widetilde{R}''_x$.

The expression in (35) under the square root is equal to

$$(x_1+x_2)^{\mathrm{an}(1)+\mathrm{an}(2)}(x_2+x_3)^{\mathrm{an}(2)+\mathrm{an}(3)}(x_1+x_3)^{\mathrm{an}(1)+\mathrm{an}(3)}.$$

For $1\le i<j\le 3$, the integer $\mathrm{an}(i)+\mathrm{an}(j)$ is even since it is equal to the number of intersection points of the closed surface $F_{ij}(c)$ with L; see also Proposition 3.10. Consequently, taking the square root produces integral exponent of x_i+x_j, implying that $\langle F,c\rangle$ is in R''_x. □

Using the above notation, the square root term in (36) is equal to

$$\widetilde{Q}(F,c):=\prod_{1\le i<j\le 3}(x_i+x_j)^{(\mathrm{an}(i)+\mathrm{an}(j))/2}, \tag{41}$$

so formula (35) can be rewritten as

$$\langle F,c\rangle=\prod_{i=1}^{3}x_i^{d_i(c)}\prod_{1\le i<j\le 3}(x_i+x_j)^{(\mathrm{an}(i)+\mathrm{an}(j)-\chi(F_{ij}(c)))/2}. \tag{42}$$

Proposition 3.12 *For an anchored foam F, we have $\langle F\rangle \in R'_x = \Bbbk[x_1, x_2, x_3]$.*

Proof The proof of Theorem 2.17 in [18] extends with minor changes to this case. Note that the evaluation is no longer a symmetric function. We must show that positive powers of $x_i + x_j$ for $1 \le i < j \le 3$, do not appear in the denominator of $\langle F\rangle$. Let us specialize to $i = 1$ and $j = 2$. Denominators $x_1 + x_2$ in the evaluations $\langle F, c\rangle$ may appear only from the components of $F_{12}(c)$ that are 2–spheres. If a 2–sphere does not intersect L, the proof in [18] works in this case as well. Suppose a 2–sphere component Σ of $F_{12}(c)$ intersects L in $\mathrm{an}(1)$ points colored 1 and $\mathrm{an}(2)$ points colored 2 (necessarily in the corresponding facets of F carrying those colors under c). These points contribute

$$(x_1+x_2)^{\mathrm{an}(1)+\mathrm{an}(2)}(x_1+x_3)^{\mathrm{an}(1)}(x_2+x_3)^{\mathrm{an}(2)}$$

to the expression under the square root, and $\mathrm{an}(1)+\mathrm{an}(2) \ge 2$, allowing to cancel the denominator term $x_1 + x_2$ that Σ contributes. Summing over all admissible colorings and otherwise following the arguments in [18, Theorem 2.17] implies the result. □

Remark 3.13 Contributions of anchor points to the evaluation $\langle F, c\rangle$ can be interpreted as follows. Consider polynomial $f(x) = (x - x_1)(x - x_2)(x - x_3) \in R'_x[x]$. Then

$$f'(x) = (x-x_2)(x-x_3) + (x-x_1)(x-x_3) + (x-x_1)(x-x_2)$$

and

$$\begin{aligned} f'(x_1) &= (x_1-x_2)(x_1-x_3), \\ f'(x_2) &= (x_2-x_1)(x_2-x_3), \\ f'(x_3) &= (x_3-x_1)(x_3-x_2). \end{aligned}$$

Contribution of an anchor point p with a label $i = \ell(p)$ to the evaluations $\langle F, c\rangle$ and $\langle F\rangle$ is then $\sqrt{f'(x_i)}$, the square root of the derivative of f at the root x_i of the polynomial f. In characteristic two, signs do not matter, but this observation hints how to extend the evaluation to characteristic 0.

Since the labels $i_1, \dots, i_m$ of anchor points are fixed in a given F, these marked points contribute the same term,

$$\sqrt{f'|_{L\cap F}} := \left(\prod_{r=1}^{m} f'(x_{i_r})\right)^{1/2},$$

and we have

$$\langle F, c\rangle = \sqrt{f'|_{L\cap F}}\cdot\langle F', c\rangle, \quad \langle F\rangle = \sqrt{f'|_{L\cap F}}\cdot\langle F'\rangle, \tag{43}$$

where F' is the foam F viewed as a regular foam with anchored points and their labels ignored. When coloring c of F is not compatible with labels of anchor points, though, we should define $\sqrt{f'|_{L\cap F}} = 0$ to match the formula $\langle F, c\rangle = 0$.

Also notice that, switching to characteristic 0 and from the matrix factorization viewpoint [20], $f(x) = w'(x)$ is the derivative of the potential

$$w(x) = \tfrac{1}{4}x^4 - \tfrac{1}{3}E_1x^3 + \tfrac{1}{2}E_2x^2 - E_3x,$$

so the contributions of anchor points are given by square roots of the second derivative $\sqrt{w''(x_i)}$ at critical points of w, analogous to the square root of the Hessian factor that appears, for example, in the steepest descent method formulas.

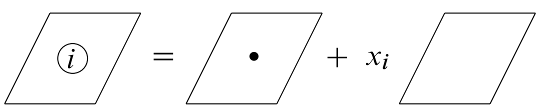
3.2 Skein relations

In this subsection we record several local relations satisfied by the evaluation of anchored SL(3) foams. We start with the following proposition concerning the relations in [18, Section 2.5], which should be understood as occurring away from the anchor line L.

Proposition 3.14 *The twelve local relations in* [18, Propositions 2.22–2.33] *hold.*

Proof The arguments in [18] apply without modification. □

We will use shifted dots in this section, as in (12). For $1 \le i \le 3$, we allow anchored foams to carry decorations of the form $\textcircled{$i$} = \bullet + x_i$ on a facet. They are required to be disjoint from L, float freely on their facets, but cannot move past seams or seam vertices:



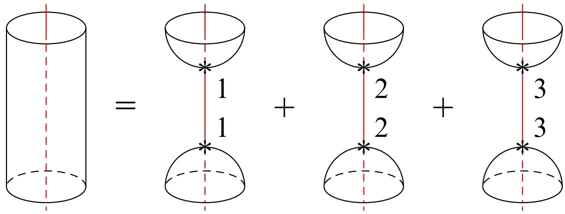
For an anchored foam F carrying $\textcircled{$i$}$ on some facet $f \in f(F)$, any coloring $c \in \mathrm{adm}(F)$ which colors f by i evaluates to zero, $\langle F, c\rangle = 0$. An anchor point labeled i has the same effect as placing

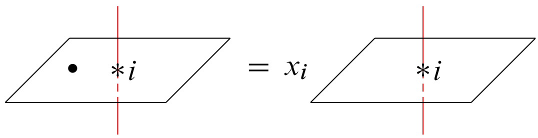
$$\sqrt{\textcircled{j}\,\textcircled{k}} = \sqrt{\textcircled{i'}\,\textcircled{i''}}$$

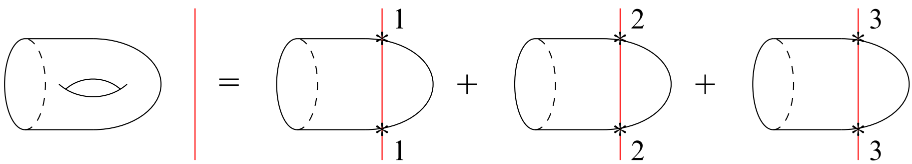
on the facet on which it lies (recall our conventions that $\{1,2,3\} = \{i,j,k\} = \{i,i',i''\}$). See also (47) and the discussion in Section 3.4.

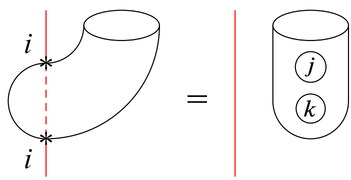
We also have relations involving the anchor line.

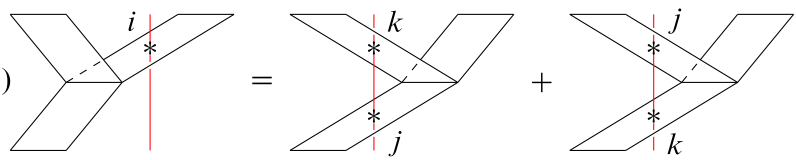
Lemma 3.15 *The following local relations hold*:

(44)


(45)


(46)


(47)


(48) $(x_j + x_k)$


In the last two equations, $\{i, j, k\} = \{1, 2, 3\}$.

Proof Let us verify (44); the other four relations are easier to check and the proof is left to the reader. Denote by F the anchored foam on the left-hand side, and by G^1, G^2 and G^3 the three foams on the right-hand side, with the superscript corresponding to the labels of the depicted anchor points. For $1 \le i \le 3$, let $\mathrm{adm}_i(F)$ be the set of admissible colorings of F in which the depicted tube is colored by i. Admissible colorings of G^i must color the two disks by i, so there is a natural bijection $\mathrm{adm}_i(F) \cong \mathrm{adm}(G^i)$.

For $c \in \mathrm{adm}_i(F)$, let $c' \in \mathrm{adm}(G^i)$ denote the corresponding coloring. We will show that
$$\langle F, c\rangle = \langle G^i, c'\rangle,$$
which completes the proof.

The anchored foam G^i carries two more anchor points, both labeled i, than F does, while the dot placement for G^i and F is the same, so

$$P(G,c') = (x_i + x_j)(x_i + x_k)P(F,c),$$

where $\{i, j, k\} = \{1,2,3\}$. On the other hand,

$$\chi(G^i_{ij}(c')) = \chi(F_{ij}(c)) + 2, \quad \chi(G^i_{ik}(c')) = \chi(F_{ik}(c)) + 2, \quad \chi(G^i_{jk}(c')) = \chi(F_{jk}(c)),$$

which yields

$$Q(G,c') = (x_i + x_j)(x_i + x_k)Q(F,c).$$

Thus $\langle F,c\rangle = \langle Q,c'\rangle$ as desired. Summing over all admissible colorings of F we get

$$\langle F\rangle = \langle G^1\rangle + \langle G^2\rangle + \langle G^3\rangle,$$

completing the proof. □

3.3 State spaces

We generalize the notion of webs and cobordisms between them from [18, Section 3.1] in the presence of the anchor line L.

Definition 3.16 A *web* is a trivalent graph Γ which is PL–embedded into the punctured plane $\mathcal{P} = \mathbb{R}^2 \setminus \{(0,0)\}$. We allow webs to have closed loops with no vertices. A *anchored foam with boundary* V is obtained by intersecting a closed anchored foam $F \subset \mathbb{R}^3$ carrying no dots with a thickened plane $\mathbb{R}^2 \times [0,1]$ such that $F \cap (\mathcal{P} \times \{i\})$ for $i = 0, 1$ is a web (in particular, F is disjoint from the two points $(0,0,0)$ and $(0,0,1)$). A connected component of the complement of singular points in $F \cap (\mathbb{R}^2 \times [0,1])$ is called a *facet*. Each facet may be decorated by finitely many dots which can float freely along the facet but cannot intersect the anchor line or cross singular points.

Foams with boundary are considered equivalent if there is an orientation-preserving homeomorphism of $\mathbb{R}^2 \times [0,1]$ taking one to the other which fixes the boundary of $\mathbb{R}^2 \times [0,1]$ pointwise and maps the line segment $L_{[0,1]} := \{(0,0)\} \times [0,1]$ to itself.

For a foam with boundary V, let

$$p(V) = V \cap L_{[0,1]}$$

denote its intersection points with the anchor line, called *anchor points*. Each anchor point is required to carry a label in $\{1,2,3\}$.

We view V as a cobordism from the web $\partial_0 V := V \cap (\mathbb{R}^2 \times \{0\})$ to the web $\partial_1 V := V \cap (\mathbb{R}^2 \times \{1\})$. A closed foam is then a cobordism from the empty web to itself. We will often refer to foams with boundary simply as *foams* when the meaning is clear from context. Composition WV of foams V and W with $\partial_1 V = \partial_0 W$ is defined in the natural way. We obtain a category AFoam of webs and anchored foams.

The category AFoam has a contravariant involution ω which is the identity on webs and which sends a foam to its reflection about $\mathbb{R}^2 \times \{\frac{1}{2}\}$, preserving the labels of anchor points. As for closed foams, denote by $s(V)$ and $v(V)$ the singular graph and singular vertices, respectively, of a foam with boundary V. Define the *degree* of V to be

$$\deg(V) = 2\big(|d(V)| + |p(V)| - \chi(V)\big) - \chi(s(V)), \tag{49}$$

where $d(V)$ is the set of dots on V.

The definition of admissible colorings extends naturally to anchored foams with boundary. An admissible coloring induces a Tait coloring on the boundary webs. If a foam with boundary V has an admissible coloring c, then by [18, Remark 2.8],

$$\deg(V) = 2|d(V)| + 2|p(V)| - \big(\chi(V_{12}(c)) + \chi(V_{13}(c)) + \chi(V_{23}(c))\big). \tag{50}$$

It follows that for a closed foam F, its evaluation $\langle F \rangle \in R'_x$ is a homogeneous polynomial of degree $\deg(F)$.

Lemma 3.17 *For composable foams V and W,*

$$\deg(WV) = \deg(W) + \deg(V).$$

Proof This follows from [18, Proposition 3.1] and $|p(WV)| = |p(W)| + |p(V)|$. □

We now define state spaces for webs via universal construction and the evaluation formula (39). For a web Γ, let

$$\mathrm{Fr}(\Gamma)$$

denote the free R'_x–module generated by all anchored foams V from the empty web to Γ. Define a bilinear form

$$(-,-)\colon \mathrm{Fr}(\Gamma) \times \mathrm{Fr}(\Gamma) \to R'_x$$

by $(V, W) = \langle \omega(V) W \rangle$. This bilinear form is symmetric since $\langle F \rangle = \langle \omega(F) \rangle$ for any closed anchored foam F. Define the state space $\langle \Gamma \rangle := \mathrm{Fr}(\Gamma)/\ker((-,-))$ to be the

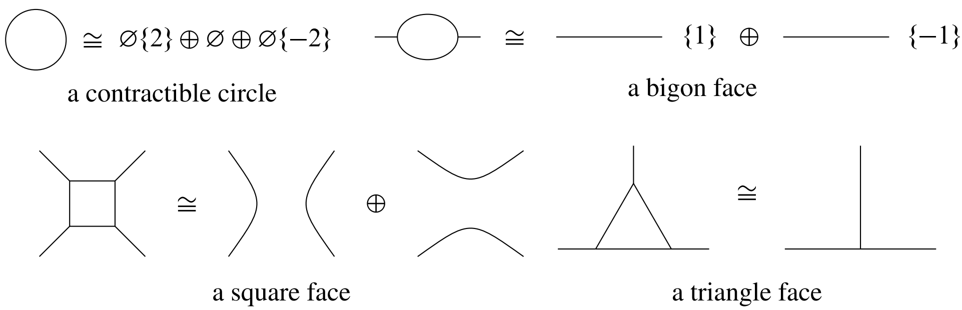


Figure 9: Direct sum decompositions from [18, Section 3.3], where the depicted regions do not contain the puncture.

quotient of $\mathrm{Fr}(\Gamma)$ by the kernel

$$\ker((-,-)) = \{x \in \mathrm{Fr}(\Gamma) \mid (x,y) = 0 \text{ for all } y \in \mathrm{Fr}(\Gamma)\}$$

of the bilinear form. Note that $(-,-)$ is degree-preserving, so its kernel and the state space $\langle \Gamma \rangle$ are graded R'_x–modules.

An anchored foam $V\colon \Gamma_0 \to \Gamma_1$ naturally induces a map

$$\langle V \rangle \colon \langle \Gamma_0 \rangle \to \langle \Gamma_1 \rangle$$

of degree $\deg(V)$, defined by sending the equivalence class of a basis element $U \in \mathrm{Fr}(\Gamma_0)$ to the class of the composition VU. This is functorial with respect to composition of anchored foams, $\langle WV \rangle = \langle W \rangle \langle V \rangle$ for composable anchored foams with boundary V and W.

Remark 3.18 For a web Γ and basis elements $V_1, V_2 \in \mathrm{Fr}(\Gamma)$, an admissible coloring of the closed foam $\omega(V_2)V_1$ induces a Tait coloring of Γ. Thus $\langle \Gamma \rangle = 0$ if Γ has no Tait colorings; see also [18, Proposition 3.16].

Proposition 3.19 *The local*[1] *isomorphisms in* [18, Propositions 3.12–3.15], *also shown in Figure 9, hold.*

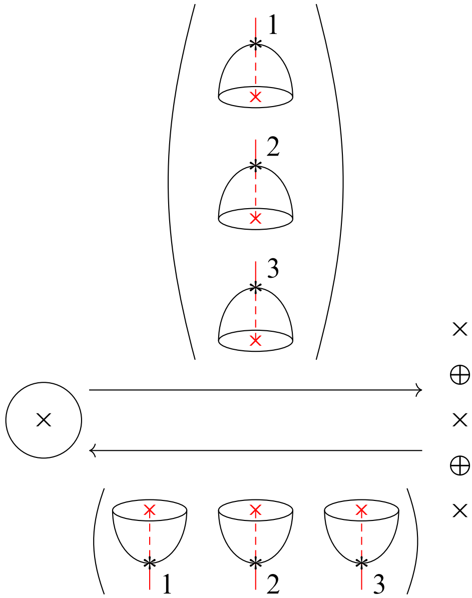
Proof Proposition 3.14 guarantees that the explicit isomorphisms defined in [18] hold in the anchored setting as well. □

[1] Here local means that the webs involved in the isomorphisms are identical outside of a disk which is disjoint from the puncture, and in this disk they are related as in the figures accompanying the statements of the propositions.

Proposition 3.20 *Let $\Gamma \subset \mathcal{P}$ be a web with a noncontractible circle C which bounds a disk in $\mathbb{R}^2 \setminus \Gamma$, and let $\Gamma' = \Gamma \setminus C$ be the web obtained by removing C. Then there is an isomorphism*

$$\langle \Gamma \rangle \cong \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle$$

given by the maps



Proof This follows from Example 3.6 and the relation (44). Note that there are no grading shifts in the three copies of $\langle \Gamma' \rangle$. □

It is an interesting and nontrivial problem to identify the state spaces $\langle \Gamma \rangle$. In the construction in [18] without the anchor line, state spaces can be simplified using the relations in [18, Section 3.3]; see Figure 9. In particular, bipartite webs always contain a contractible circle, bigon, or square, so the state space in the bipartite case is a free module of graded rank equal to the Kuperberg bracket [24], normalized as in [15]; see also [18, Propositions 3.17 and 4.15]. The simplest web which cannot be simplified using the relations in Figure 9 and for which the state space is unknown is the dodecahedral graph, as explored in [9; 17], and, on the gauge theory side, in [21; 22; 23].

One may also ask to identify state spaces in the presence of the anchor line and the modified evaluation considered in this paper. Propositions 3.19 and 3.20 give some ways to simplify state spaces. In general, we are not able to decompose the bigon, square, and triangle regions in Figure 9 if they contain the puncture. An extended evaluation, obtained by introducing additional types of intersection points of L with a foam, is discussed in Section 3.5. The following lemma addresses reducibility of smallest webs.

Lemma 3.21 *Let $\Gamma \subset \mathbb{R}^2$ be a connected, planar, trivalent graph with no edges connecting a vertex to itself.* [2]

(1) *If Γ is bipartite, then Γ has at least two bounded faces with at most four edges each.*

(2) *If at most one of the bounded faces of Γ has fewer than five edges, then Γ has at least eight vertices.*

Proof Let v, e, and f denote the number of vertices, edges, and faces (including the unbounded face) of Γ, respectively. Label the faces $1, \dots, f$, and for $1 \le i \le f$, let r_i denote the number of edges that form the boundary of the i^{th} face. We have

$$\sum_{i=1}^{f} r_i = 2e = 3v, \tag{51}$$

where the second equality holds since Γ is trivalent.

We first prove statement (1). Since Γ is bipartite, each r_i is even. Suppose for the sake of contradiction that at most one bounded face of Γ has four or fewer edges. Then (51) implies

$$\sum_{i=1}^{f} r_i > 6(f-2),$$

so $12 > 6f - 3v$. On the other hand, an Euler characteristic computation gives

$$12 = 6(f - e + v) = 6f - 3v,$$

which is a contradiction.

Let us now address statement (2). From (51) we obtain

$$3v \ge 5(f-2) + 4 = 5f - 6$$

since, by assumption, there are $f-2$ faces with at least five edges each, and the remaining two faces each have at least two edges. This together with an Euler characteristic computation gives $f \ge 6$, and it follows that $v \ge 8$. □

[2] A graph with such an edge has trivial state space; see Remark 3.18.

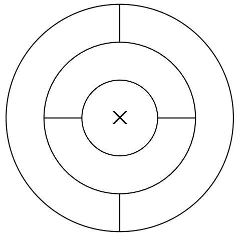


Figure 10: The simplest nonreducible web in the punctured plane.

Corollary 3.22 *Let $\Gamma \subset \mathcal{P}$ be a bipartite web. Then $\langle \Gamma \rangle$ is a free R'_x–module of rank equal to the number of Tait colorings of Γ.*

Proof By statement (1) of Lemma 3.21, any such web has either an innermost noncontractible circle or a region, not containing the puncture, which either bounds a closed loop, or is a bigon or square face. Thus state space can be reduced using Propositions 3.19 and 3.20. Since the resulting web remains bipartite we can continue the procedure until the state space is reduced to a direct sum of empty webs, each of which is free of rank 1. On the other hand, the number of Tait colorings can be computed using the same relations. □

It is natural to ask what is the simplest web for which the state space cannot be reduced using Propositions 3.19 and 3.20. By statement (2) of Lemma 3.21, such a web has at least eight vertices. The web shown in Figure 10 has precisely eight vertices and cannot be simplified using our local relations. We have not identified the state space of this web, but it can be approached via the 4–periodic (and, in general, nonexact) complex described in [18, Section 4.3]. It can be applied along any of the four edges of Figure 10 web near either the marked or the infinite point. One of the other three webs in the complex contains a loop and has trivial homology, but additional computations are needed to identify the state space due to nonexactness of the complex.

An annular graph $\Gamma \subset \mathcal{P}$ is called *reducible* if its state space can be reduced to a sum of those for the empty annular graph by recursively applying the relations in Figure 9 and relation in Proposition 3.20. It may make sense to also allow reductions to annular graphs without Tait colorings (including graphs with loops), since such graphs have trivial state spaces.

A reducible annular graph allows an identification of its state space with a suitable free graded R_x–module by recursively applying the above state sum decompositions. As a special case, we have the following decomposition formula for collections of simple closed curves in an annulus.

Proposition 3.23 *Let $\Gamma \subset \mathcal{P}$ consist of n contractible circles and m noncontractible circles. Then the state space $\langle \Gamma \rangle$ is a free R'_x–module of graded rank $3^m(q^2+1+q^{-2})^n$.*

In particular, for a reducible Γ, the graded rank of the free R'_x–module $\langle \Gamma \rangle$ can be computed recursively.

Anchored foams and state spaces carry an additional $(\mathbb{Z}/2\times\mathbb{Z}/2)$–grading as follows. Recall that u_1, u_2 and u_3 denote the nonzero elements of $\mathbb{Z}/2\times\mathbb{Z}/2$. For a foam V with (possibly empty) boundary, define

$$\operatorname{adeg}(V) = \sum_{p \in p(V)} u_{\ell(p)}.$$

We call adeg the *annular degree*. Clearly adeg is additive under disjoint union and composition.

The annular degree extends to a $(\mathbb{Z}/2\times\mathbb{Z}/2)$–grading on $\mathrm{Fr}(\Gamma)$, for a web $\Gamma \subset \mathcal{P}$, by setting the ground ring R'_x to be concentrated in annular degree zero. Proposition 3.10 implies that $\langle F \rangle = 0$ or $\operatorname{adeg}(\langle F \rangle) = 0$ for any closed foam F. It follows that $(-,-)$ preserves annular degree, so adeg descends to a $(\mathbb{Z}/2\times\mathbb{Z}/2)$–grading on the state space $\langle \Gamma \rangle$. The annular grading is the unoriented version of the grading on state spaces of annular oriented webs by the integral weight lattice of $\mathfrak{sl}_3$ — see Section 4.4 — even though the action of the latter is lacking on the equivariant annular state spaces.

In [18, Section 4] the authors consider localization of the unoriented SL(3) theory given by inverting the discriminant $\mathcal{D} = (x_1+x_2)(x_1+x_3)(x_2+x_3)$. This localization results in a significant simplification of the theory, making it separable, so to speak. In particular, a suitable 4–term sequence of web state spaces in [18, Section 4.3] is exact.

This localization easily extends to the annular case. The corresponding 4–term sequence is exact in the annular case as well. The ground ring for that theory is $R'_{\mathcal{D}} := \Bbbk[x_1, x_2, x_3, \mathcal{D}^{-1}]$, with $\Bbbk$ a characteristic two field. The analogue of [18, Proposition 4.13] holds: the localized state space of an annular web Γ is a projective $R'_{\mathcal{D}}$–module of rank equal to the number of Tait colorings of Γ. The latter is the number of edge colorings of Γ into three colors such that at each vertex the colors are distinct. Proof of this result in [18] easily adapts to the annular case, with the modification that the region around the marked point can be inductively simplified, if necessary, by reducing to the other three terms in the exact sequence, until it has a single edge (a loop around the marked point).

3.4 Remark on Lee's theory

Recall the function

$$f(x) = (x + x_1)(x + x_2)(x + x_3) = x^3 + E_1 x^2 + E_2 x + E_3 \tag{52}$$

(in characteristic 2 signs do not matter) with coefficients in the ring R_x and roots in $R'_x \supset R_x$. One can form the quotient ring $A := R'_x[x]/(f(x))$, naturally isomorphic to the homology of a contractible circle in our theory. Let

$$\mathcal{D} = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = E_1 E_2 + E_3 \tag{53}$$

be the discriminant. Consider the localization

$$R'_{\mathcal{D}} := R'_x[\mathcal{D}^{-1}], \quad A_{\mathcal{D}} := R'_{\mathcal{D}} \otimes_{R'_x} A. \tag{54}$$

Introduce idempotents $e_1, e_2, e_3 \in A_{\mathcal{D}}$,

$$e_i := \frac{(x + x_j)(x + x_k)}{(x_i + x_j)(x_i + x_k)}, \quad \{i, j, k\} = \{1, 2, 3\}. \tag{55}$$

We have

$$1 = e_1 + e_2 + e_3, \quad e_i e_j = \delta_{i,j} e_i. \tag{56}$$

These idempotents decompose the ring $A_{\mathcal{D}}$ into the direct product

$$A_{\mathcal{D}} \cong R'_{\mathcal{D}} e_1 \times R'_{\mathcal{D}} e_2 \times R'_{\mathcal{D}} e_3 \cong R'_{\mathcal{D}} \times R'_{\mathcal{D}} \times R'_{\mathcal{D}}. \tag{57}$$

An idempotent e_i can be visualized as floating on a facet of a foam F, in the localized theory. These idempotents allow us to decompose an evaluation of a foam F with n facets into 3^n terms by summing over all ways to place each of these three idempotents onto facets of F. Each term is straightforward to compute and equals zero unless the idempotents define a Tait coloring (an admissible coloring) of F.

Idempotent e_i bears a close relation to an anchor point labeled i. The anchor point p on a facet f contributes the term $\sqrt{f'(x_{c(f)})} = \sqrt{(x_{c(f)} + x_j)(x_{c(f)} + x_k)}$ to the evaluation $\langle F, c \rangle$. The square of this term is either 0 (if $i \neq c(f)$) or the denominator of e_i, if $i = c(f)$, for any coloring c of F.

Comparing e_i and an anchor point p labeled i, when coloring c associates color $c(f) \neq i$ to the facet f carrying e_i or p, both evaluations are zero. When $c(f) = i$, the idempotented dot e_i contributes 1 to the evaluation, while the anchor point contributes $\sqrt{f'(x_i)}$. The denominator of e_i is $f'(x_i)$.

One can try to unify e_i and anchor points p by considering anchor lines and circles L in $\mathbb{R}^3$ possibly intersecting a foam F. Intersection points (anchor points) carry labels $i \in \{1, 2, 3\}$ and a circle anchor points labeled i is the idempotent e_i. Then a "small" circle intersecting a facet f at two points, both labeled i, can also be converted into e_i. Notice that once e_i are allowed, integrality is lost and an evaluation of such a foam may contain denominators which are products of $x_i + x_j$.

For a different generalization, instead of a single line $L \subset \mathbb{R}^3$ consider a 1–manifold L properly embedded in $\mathbb{R}^3$, say a finite union of lines and circles, possibly knotted. All anchor points (intersection points with L) on a foam F carry labels, with the usual contribution to the evaluation, as in formula (36). The integrality Theorem 4.15 still holds for such generalized evaluation. In particular, given k points on a plane, one can define various state spaces for webs Γ embedded in the plane and disjoint from these marked points. Also note that for $k \geq 2$ punctures, bipartite graphs are in general not reducible, which makes it harder to understand corresponding state spaces in the oriented SL(3) case.

Remark 3.24 A handle next to but disjoint from an anchor line can be written as a sum of three lower genus terms intersecting the line — see (46) — which follows from the formula

$$\begin{aligned} m \circ \Delta(1) &= (x_1 + x_2)(x_1 + x_3) + (x_1 + x_2)(x_2 + x_3) + (x_1 + x_3)(x_2 + x_3) \\ &= f'(x_1) + f'(x_2) + f'(x_3). \end{aligned}$$

3.5 Unlabeled anchor points and bigon decomposition

Direct sum decompositions for webs Γ containing a bigon, triangle, or square face which do not contain the puncture are given in Proposition 3.19. On the other hand, Proposition 3.20 describes how to simplify a web containing an innermost noncontractible circle. In order to have direct sum decompositions for more general regions containing the puncture, we introduce additional types of intersections of the anchor line L with a foam and modify the evaluation $\langle - \rangle$.

In addition to anchor points, which carry labels in $\{1, 2, 3\}$ as in Definition 3.2, we allow finitely many transverse intersections of L with a foam F away from the singular

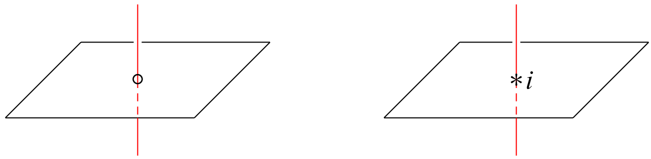


Figure 11: Left, a type 1 anchor point marked $\circ$ and carrying no label. Right, a type 2 anchor point marked $*$ with label $i \in \{1,2,3\}$.

graph $s(F)$, and we do not require labels. We will call the usual (labeled) anchor points *type* 2, and the new (unlabeled) anchor points *type* 1. In the figures, we denote type 2 anchor points by an asterisk $*$ as usual, along with a label in $\{1,2,3\}$, and type 1 anchor points will be indicated by a small unshaded circle $\circ$. Figure 11 illustrates the convention. Let $p_1(F)$ and $p_2(F)$ denote the set of type 1 and type 2 anchor points, respectively (using the notation in Section 3.1, $p(F) = p_2(F)$). The definition of admissible coloring remains the same.

We modify the evaluation in the presence of type 1 points as follows. Let $c \in \operatorname{adm}(F)$. For $p \in p_1(F)$ lying on some facet $f \in f(F)$, let $c(p) := c(f)$ denote the coloring of the facet on which p lies. Also recall that for $i \in \{1,2,3\}$, we write i', i'' and j, k to denote the two complementary elements, so $\{1,2,3\} = \{i,j,k\} = \{i,i',i''\}$.

Define

$$\widetilde{Q}_\circ(F,c) = \prod_{p \in p_1(F)} \sqrt{x_{c(p)'} + x_{c(p)''}}, \tag{58}$$

$$P_\circ(F,c) = P(F,c) \cdot \widetilde{Q}_\circ(F,c), \tag{59}$$

$$\langle F,c \rangle_\circ = \frac{P_\circ(F,c)}{Q(F,c)}, \tag{60}$$

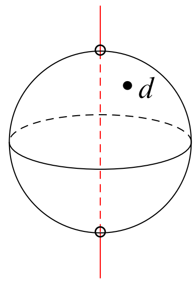
$$\langle F \rangle_\circ = \sum_{c \in \operatorname{adm}(F)} \langle F,c \rangle_\circ, \tag{61}$$

where $P(F,c)$ and $Q(F,c)$ are as defined in (36) and (37). In other words, a type 1 point p on an i–colored facet contributes a factor of $\sqrt{x_j + x_k}$ to the evaluation $\langle F,c \rangle_\circ$.

Remark 3.25 Type 1 intersection points are related to the triangle decoration from [18, Section 4.1]. Precisely, the contribution of a type 1 point p to the square root in (58) equals the inverse of placing a triangle decoration on the facet where p lies. See relation (62), as well as Remark 3.4 for a related discussion.

Note that a type 1 intersection point contributes half the degree of a type 2 point to the degree of the evaluation and, thus, to the degree of a cobordism represented by a foam with boundary.

Example 3.26 Consider a 2–sphere F carrying d dots and intersecting L in two type 1 anchor points,

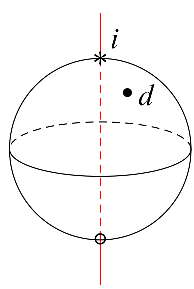


For $1 \le i \le 3$, let $c_i \in \mathrm{adm}(F)$ color F by i. Then

$$\begin{aligned}\langle F, c_i\rangle_\circ &= \frac{x_i^d(x_j+x_k)}{(x_i+x_j)(x_i+x_k)},\\ \langle F\rangle_\circ &= \langle F,c_1\rangle_\circ + \langle F,c_2\rangle_\circ + \langle F,c_3\rangle_\circ\\ &= \frac{x_1^d(x_2+x_3)^2 + x_2^d(x_1+x_3)^2 + x_3^d(x_1+x_2)^2}{(x_1+x_2)(x_1+x_3)(x_2+x_3)}\\ &= \frac{x_1^d(x_2^2+x_3^2) + x_2^d(x_1^2+x_3^2) + x_3^d(x_1^2+x_2^2)}{(x_1+x_2)(x_1+x_3)(x_2+x_3)}.\end{aligned}$$

Thus, $\langle F\rangle_\circ = 0$ if $d = 0, 2$, and $\langle F\rangle_\circ = 1$ if $d = 1$. For $d \ge 3$, the last expression above equals the ratio of the antisymmetrizer with exponent $(d, 2, 0)$ and antisymmetrizer with exponent $(2, 1, 0)$ (up to adding signs, which does not matter in characteristic 2). Thus $\langle F\rangle_\circ$ equals the Schur function $s_\lambda(x_1, x_2, x_3)$ for the partition $\lambda = (d-2, 1, 0)$ when $d \ge 3$.

Example 3.27 Consider a 2–sphere F carrying d dots and intersecting L in one type 1 anchor point and one type 2 anchor point,



Then F has one admissible coloring, and

$$\langle F\rangle_\circ = \frac{x_i^d\sqrt{(x_i+x_j)(x_i+x_k)(x_j+x_k)}}{(x_i+x_j)(x_i+x_k)} = \frac{x_i^d\sqrt{x_j+x_k}}{\sqrt{(x_i+x_j)(x_i+x_k)}}.$$

From Example 3.27 we see that the evaluation $\langle F\rangle_\circ$ in general has denominators and square roots, so we can only conclude that

$$\langle F\rangle_\circ \in \widetilde{R}_\circ := \Bbbk[x_1,x_2,x_3,(x_1+x_2)^{-1/2},(x_2+x_3)^{-1/2},(x_1+x_3)^{-1/2}].$$

Note that $\widetilde{R}_\circ$ is a subring of $\widetilde{R}''_x$; see Section 3.1 and diagram (34).

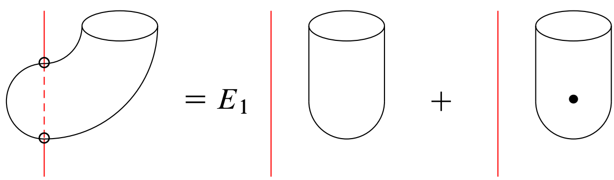
We use $\widetilde{R}_\circ$ as the ground ring of the theory. Evaluations of closed anchored foams F with two types of anchor points belong to this ring. We define the state space $\langle\Gamma\rangle_\circ$ of a trivalent graph $\Gamma\subset\mathcal{P}$ using this evaluation and following the general recipe of Section 3.3. The state space is a graded $\widetilde{R}_\circ$–module, but, due to the presence of invertible elements $(x_i+x_j)^{1/2}$ of degree 1, grading carries little information, and for many purposes one can downsize and consider the degree zero part $\langle\Gamma\rangle_\circ^0$ of the state space, which is a module over the degree 0 subring $\widetilde{R}_\circ^0$ of $\widetilde{R}_\circ$.

This theory is functorial and foams with top and bottom boundary and anchor points of those two different types induce maps between the corresponding state spaces. Various direct sum decompositions that hold for the unoriented SL(3) theory $\langle-\rangle$ hold for this theory as well.

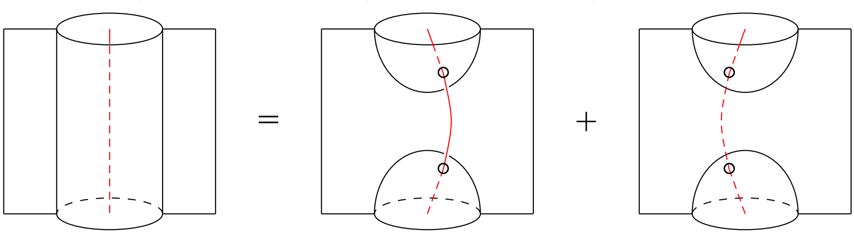
We also have local relations involving type 1 intersection points.

Lemma 3.28 *The following local relations*[3] *hold for the theory* $\langle-\rangle_\circ$:

(62)

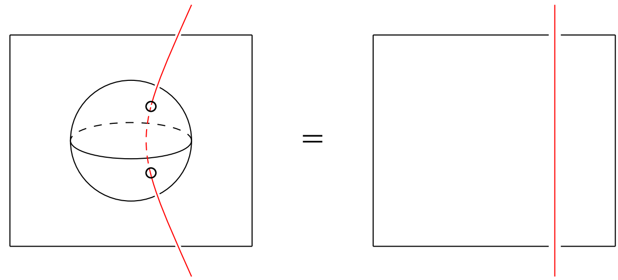


(63)

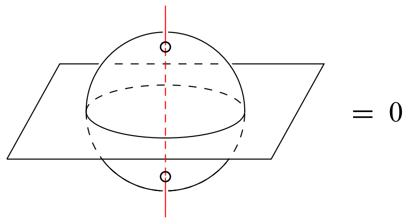


[3]To clarify relation (63): the first term on the right-hand side of the equality has a type 1 anchor point on each of two front-facing half-bubbles, while the second term has a type 1 anchor point on each of the two back-facing half-bubbles.

(64)



(65)



Proof Relation (62) is straightforward and left to the reader. Let us verify relation (63). Denote by F the foam on the left-hand side of the equality, and denote by F^1 and F^2 the two foams on the right-hand side. There is a natural identification $\mathrm{adm}(F^1)=\mathrm{adm}(F^2)$.

Let $c\in\mathrm{adm}(F^1)$ be a coloring in which the front two half-bubble facets are differently colored, say the top front half-bubble is colored j, the bottom front half-bubble is colored k, and the remaining "big" facet is colored i. Continue to denote by $c\in\mathrm{adm}(F^2)$ the corresponding coloring of F^2. The top type 1 intersection point of F^1 contributes $\sqrt{x_i+x_k}$ to $\langle F^1,c\rangle$ and the bottom type 1 intersection point of F^1 contributes $\sqrt{x_i+x_j}$, while the contributions of these points to $\langle F^2,c\rangle$ are reversed. Thus in characteristic two we have

$$\langle F^1,c\rangle+\langle F^2,c\rangle=0.$$

Next, the admissible colorings of F are in natural bijection with the admissible colorings of F^1 (and of F^2) in which the front half-bubbles of F^1 are colored the same. Let $c\in\mathrm{adm}(F)$, and let $c'\in\mathrm{adm}(F^1)\cong\mathrm{adm}(F^2)$ denote the corresponding colorings. Suppose that c' colors the front half-bubbles of F^1 by j, the "big" facet by i, and the back half-bubbles by k. Then

$$\langle F^1,c'\rangle=\frac{x_i+x_k}{x_j+x_k}\langle F,c\rangle \quad \text{and} \quad \langle F^2,c'\rangle=\frac{x_i+x_j}{x_j+x_k}\langle F,c\rangle,$$

from which we obtain

$$\langle F,c\rangle=\langle F^1,c'\rangle+\langle F^2,c'\rangle,$$

which completes the proof of relation (63).

We now address the relation (64). Let G denote the foam on the left-hand side of the equation, and let G' denote the foam on the right-hand side. Let $c \in \mathrm{adm}(G)$, and assume c colors the "big" facet of G by i, the front bubble by j, and the back bubble by k. Let $c' \in \mathrm{adm}(G)$ denote the coloring which is identical to c except the front and back bubbles are colored by k and j, respectively. Let $c'' \in \mathrm{adm}(G')$ denote the coloring of G' in which the depicted facet is colored i, and the remaining facets are colored according to c (equivalently, c'). We claim that

$$\langle G, c\rangle + \langle G, c'\rangle = \langle G', c''\rangle,$$

which completes the proof. To verify the above equality, observe that

$$\langle G, c\rangle = \frac{x_i + x_k}{x_j + x_k}\langle G', c''\rangle \quad \text{and} \quad \langle G, c'\rangle = \frac{x_i + x_j}{x_j + x_k}\langle G', c''\rangle.$$

The proof of relation (65) is similar and left to the reader. □

The previous lemma allows us to simplify the state space $\langle \Gamma\rangle_\circ$ assigned to a web $\Gamma \subset \mathcal{P}$ with a bigon region containing the puncture.

Proposition 3.29 *The two maps shown in Figure 12 are mutually inverse isomorphisms between state spaces of graphs in the theory $\langle - \rangle_\circ$.*

Proof This follows from the relations in Lemma 3.28. □

4 Oriented SL(3) anchored homology

In this section we recall oriented SL(3) foams, which were introduced in [15] in the context of $\mathfrak{sl}(3)$ link homology. An equivariant analogue was defined in [28]; see also [10; 26; 27; 29; 33] for various aspects of SL(3) foams and link homology. In Section 4.1 we define an evaluation of oriented SL(3) foams via colorings in the style of Robert and Wagner [34] and show in Theorem 4.26 that our evaluation agrees with that of [28]. In Section 4.2 we deform the evaluation in the presence of the anchor line L. In Theorem 4.15 we show that our evaluation is always a polynomial.

To avoid introducing new notation, in this section we will reuse the notation for various rings from Section 3:

- $R'_x = \mathbb{Z}[x_1, x_2, x_3]$ is the ring of polynomials in three variables.
- $R_x = \mathbb{Z}[E_1, E_2, E_3]$ is the subring of R'_x that consists of symmetric polynomials in x_1, x_2 and x_3, with generators E_i being the elementary symmetric polynomials

$$E_1 = x_1 + x_2 + x_3, \quad E_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad E_3 = x_1x_2x_3.$$

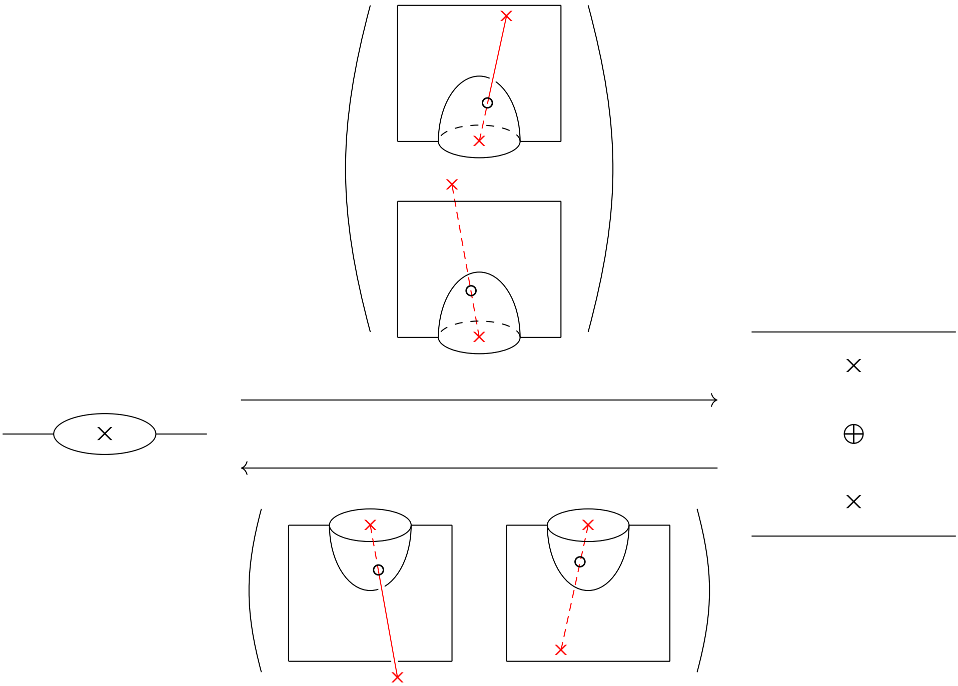


Figure 12: Isomorphisms which simplify a bigon region containing the puncture, for the theory $\langle - \rangle_\circ$. In the top map, the top foam has a type 1 point on the front half-bubble, and the bottom foam has a type 1 point on the back half-bubble. In the bottom map, the first foam has a type 1 point on the front half-bubble, and the second foam has a type 1 point on the back half-bubble.

- $R''_x = R'_x[(x_1-x_2)^{-1}, (x_2-x_3)^{-1}, (x_1-x_3)^{-1}]$ is a localization of R'_x given by inverting $x_i - x_j$, for $1 \le i < j \le 3$.
- $\widetilde{R}'_x = R'_x[\sqrt{x_1-x_3}, \sqrt{x_2-x_3}, \sqrt{x_1-x_3}]$ is the extension of R'_x obtained by introducing square roots of $\sqrt{x_i-x_j}$, for $1 \le i < j \le 3$.
- $\widetilde{R}''_x = \widetilde{R}'_x[(x_1-x_2)^{-1}, (x_2-x_3)^{-1}, (x_1-x_3)^{-1}]$ is a suitable localization of the ring $\widetilde{R}'_x$.

All five of these rings are graded by setting $\deg(x_1) = \deg(x_2) = \deg(x_3) = 2$. Inclusions of the above rings are summarized in the following diagram:

(66)
$$\begin{array}{ccccc} & & \widetilde{R}'_x & \subset & \widetilde{R}''_x \\ & & \cup & & \cup \\ R_x & \subset & R'_x & \subset & R''_x \end{array}$$

4.1 Oriented SL(3) foams and their evaluations

We begin by recalling the definition of oriented SL(3) foams from [15, Section 3.2].

Definition 4.1 A (closed) *oriented* SL(3) *prefoam* F consists of the following data:

- An orientable surface F' with connected components $F_1, \dots, F_k$ and a partition of the boundary components of F' into triples. The underlying CW structure of F is obtained by identifying the three circles in each triple. The image of the three circles in each triple becomes a single circle in F, called a *singular circle*. The image of the surfaces F_i are called *facets*. Three facets meet at each singular circle.
- For each singular circle Z, we fix a cyclic ordering of the three facets meeting at Z. There are two possible choices of cyclic ordering for each Z.
- Each facet may carry some number of dots, which are allowed to float freely along the facet but cannot cross singular circles.

A *oriented* SL(3) *foam* is a prefoam as above equipped with an embedding into $\mathbb{R}^3$, along with an orientation on each facet such that any two of the three facets meeting at each singular circle are incompatibly oriented, as shown in Figure 13, left. Each singular circle Z acquires an induced orientation; see Figure 13, middle. This induced orientation on Z specifies a cyclic ordering of the three facets meeting at Z by following the left-hand rule — Figure 13, right — and we require this to match the cyclic ordering specified by the prefoam F.

Note that unlike unoriented foams considered in Section 3, the oriented SL(3) prefoams in the present section do not contain singular vertices. When there is no risk of confusion

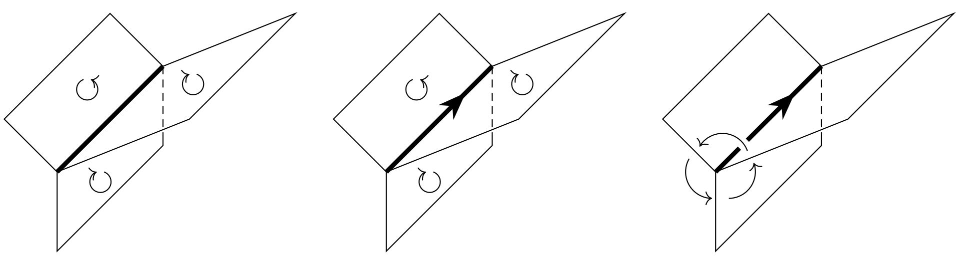


Figure 13: Left: orientations of three facets meeting at a singular circle. Middle: the induced orientation of a singular circle. Right: the induced cyclic ordering.

between the foams introduced in the Definition 4.1 and those of Section 3, in this section we will simply write (pre)foam rather than oriented SL(3) (pre)foam.

For a prefoam F, let $\Theta(F)$ denote the set of its singular circles and $\theta(F) = |\Theta(F)|$ the number of singular circles. Each $Z \in \Theta(F)$ has a neighborhood homeomorphic to the product of a circle $\mathbb{S}^1$ and a tripod. Let $f(F)$ denote the set of facets of F. We use the definitions of preadmissible and admissible colorings of prefoams and foams from Section 3 in the present situation. For a prefoam F, $\mathrm{adm}(F)$ denotes the set of admissible colorings of F. Note that if F is a foam, every preadmissible coloring is also admissible.

Fix a prefoam F and an admissible coloring $c \in \mathrm{adm}(F)$. For $1 \leq i \neq j \leq 3$, bicolored surfaces $F_{ij}(c)$ consist of all facets colored i or j; each $F_{ij}(c)$ is a closed, orientable surface. For $1 \leq i \leq 3$, let $F_i(c)$ be the surface consisting of all facets of F which are colored i by c; the surface $F_i(c)$ is orientable and has $\theta(F)$ boundary components. Denote by $\overline{F}_i(c)$ the closed surface obtained by gluing disks along boundary components of $F_i(c)$. We have

$$\begin{aligned} \chi(\overline{F}_i(c)) &= \chi(F_i(c)) + \theta(F), && 1 \leq i \leq 3, \\ \chi(F_{ij}(c)) &= \chi(F_i(c)) + \chi(F_j(c)), && 1 \leq i < j \leq 3. \end{aligned} \tag{67}$$

The three facets meeting at each singular circle are colored by i, j and k, whereas before we used i, j and k to denote the three elements of $\{1,2,3\}$. We now define quantities $\theta^{\pm}(c)$ and $\theta^{\pm}_{ij}(c)$ associated with the set of singular circles $\Theta(F)$ and the admissible coloring c.

Definition 4.2 Let F be a prefoam with admissible coloring c, and let $1 \leq i < j \leq 3$. A singular circle $Z \in \Theta(F)$ is *positive* with respect to (i, j) if the cyclic ordering of the colors of the three facets meeting at Z is $(i\ k\ j)$. If F is a foam, then an equivalent formulation is as follows: when looking along the orientation of Z with the facet colored k, as in Figure 14, the i–colored facet is to the left of the j–colored facet. Otherwise, we say Z is negative with respect to (i, j). See Figure 14, left, for a pictorial definition. Let $\theta^+_{ij}(c)$ (resp. $\theta^-_{ij}(c)$) denote the number of positive (resp. negative) circles with respect to (i, j). We have

$$\theta^+_{ij}(F,c) + \theta^-_{ij}(F,c) = \theta(F).$$

We say that a singular circle Z is *positive* with respect to c if the colors of the three facets meeting at Z are (1 2 3) in the cyclic ordering, and otherwise Z is negative;

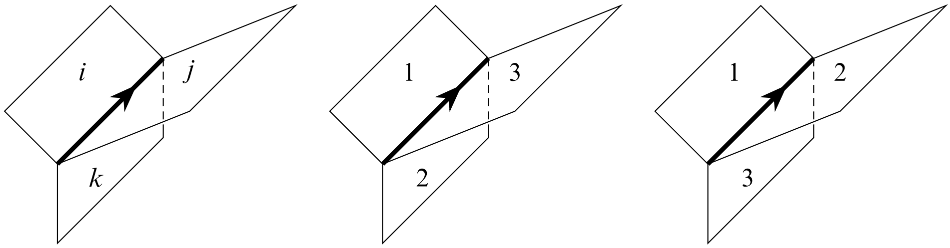


Figure 14: Left: a positive (i,j)–circle, where $i<j$. Middle: a positive singular circle. Right: a negative singular circle.

see Figure 14, middle and right. Let $\theta^+(F,c)$ (resp. $\theta^-(F,c)$) denote the number of positive (resp. negative) circles in F with respect to c. We have

$$\theta^+(F,c)+\theta^-(F,c)=\theta(F). \tag{68}$$

We will often omit F from the notation and simply write θ, $\theta^{\pm}_{ij}(c)$, and $\theta^{\pm}(c)$.

We now define the evaluations $\langle F,c\rangle$ and $\langle F\rangle$. For a prefoam F, $c\in\mathrm{adm}(F)$, and $1\le i\le 3$, let $d_i(c)$ denote the number of dots on facets colored i. Define

$$P(F,c)=\prod_{i=1}^{3} x_i^{d_i(c)}, \tag{69}$$

$$Q(F,c)=\prod_{1\le i<j\le 3}(x_i-x_j)^{\chi(F_{ij}(c))/2}, \tag{70}$$

$$s(F,c)=\sum_{i=1}^{3} i\,\chi(\bar F_i(c))/2+\sum_{1\le i<j\le 3}\theta^+_{ij}(c). \tag{71}$$

Set

$$\langle F,c\rangle=(-1)^{s(F,c)}\frac{P(F,c)}{Q(F,c)}, \tag{72}$$

$$\langle F\rangle=\sum_{c\in\mathrm{adm}(F)}\langle F,c\rangle. \tag{73}$$

A priori, the evaluations $\langle F,c\rangle$ and $\langle F\rangle$ lie in the ring R''_x; see diagram (66).

In what follows, we use the symbol $\equiv$ to mean equality modulo 2. Note that

$$\sum_{i=1}^{3} i\,\chi(\bar F_i(c))/2\equiv\frac{\chi(\bar F_1(c))+\chi(\bar F_3(c))}{2}, \tag{74}$$

since $\chi(\bar{F}_2(c))$ is even. Moreover, from (67) we obtain

$$\sum_{i=1}^{3} i\chi(\bar{F}_i(c))/2 \equiv \theta + \sum_{i=1}^{3} i\chi(F_i(c))/2. \tag{75}$$

Lemma 4.3 *For a prefoam F and $c \in \mathrm{adm}(F)$,*

$$\sum_{1\le i<j\le 3} \theta^+_{ij}(c) \equiv \theta^+(c).$$

It follows that

$$s(F,c) \equiv \sum_{i=1}^{3} i\chi(F_i(c))/2 + \theta^-(c). \tag{76}$$

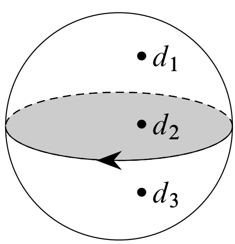
Proof Let $Z \in \Theta(F)$. Observe that if Z is positive with respect to c, then it contributes only to $\theta^+_{13}(c)$. Likewise, if Z is negative then it contributes to both $\theta^+_{12}(c)$ and $\theta^+_{23}(c)$ but not to $\theta^+_{13}(c)$, which verifies the first equality. The second equality follows from (75) and (68). □

Example 4.4 Let F be a 2–sphere $\mathbb{S}^2$ with d dots. For $1 \le i \le 3$, let $c_i \in \mathrm{adm}(F)$ color F by i. We have

$$\begin{aligned}
\langle F\rangle &= \langle F,c_1\rangle + \langle F,c_2\rangle + \langle F,c_3\rangle \\
&= -\frac{x_1^d}{(x_1-x_2)(x_1-x_3)} + \frac{x_2^d}{(x_1-x_2)(x_2-x_3)} - \frac{x_3^d}{(x_1-x_3)(x_2-x_3)} \\
&= \frac{-x_1^d(x_2-x_3) + x_2^d(x_1-x_3) - x_3^d(x_1-x_2)}{(x_1-x_2)(x_2-x_3)(x_1-x_3)} \\
&= -s_{(d-2,0,0)}(x_1,x_2,x_3) = -h_{d-2}(x_1,x_2,x_3) = -\sum_{i+j+k=d-2} x_1^i x_2^j x_3^k,
\end{aligned}$$

where $s_{(d-2,0,0)}(x_1,x_2,x_3)$ is the Schur function of the partition $(d-2,0,0)$, and $h_{d-2}(x_1,x_2,x_3)$ is the complete symmetric function of degree $d-2$. In particular $\langle F\rangle = 0$ if $d=0$ or $d=1$, and $\langle F\rangle = -1$ if $d=2$.

Example 4.5 Let F be the theta foam



Given any $c \in \mathrm{adm}(F)$, each capped-off surface $\bar{F}_i(c)$ and each bicolored surface $F_{ij}(c)$ is a 2–sphere. In particular,

$$s(F,c) \equiv \theta^+(c).$$

For $\sigma \in S_3$, let $c(\sigma) \in \mathrm{adm}(F)$ denote the coloring which colors the top facet by $\sigma(1)$, the middle facet by $\sigma(2)$, and the bottom facet by $\sigma(3)$. We have

$$\langle F\rangle = \sum_{\sigma\in S_3} \langle F, c(\sigma)\rangle = \frac{\sum_{\sigma\in S_3}(-1)^{\theta^+(c(\sigma))} x_{\sigma(1)}^{d_1} x_{\sigma(2)}^{d_2} x_{\sigma(3)}^{d_3}}{(x_1-x_2)(x_1-x_3)(x_2-x_3)},$$

and moreover

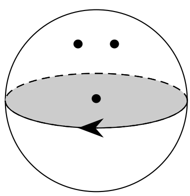
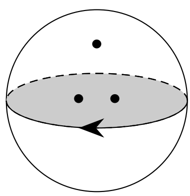
$$\theta^+(c(\sigma)) \equiv |\sigma|,$$

where $|\sigma|$ is the length of σ.

Therefore if $d_1 \geq d_2 \geq d_3$,

$$\langle F\rangle = s_{(d_1-2,d_2-1,d_3)}(x_1,x_2,x_3),$$

the Schur function with partition (d_1-2, d_2-1, d_3). In particular, $\langle F\rangle = 0$ if d_1, d_2 and d_3 are not distinct. If d_1, d_2 and d_3 are distinct and $d_1+d_2+d_3 \leq 3$, then up to cyclic permutation there are two choices:

 $= 1,$  $= -1.$

The symmetric group S_3 naturally acts on $\mathrm{adm}(F)$ and on the five rings in the diagram (66). The following lemma is analogous to [34, Lemma 2.16].

Lemma 4.6 *Let F be a prefoam, $c \in \mathrm{adm}(F)$, and $\sigma \in S_3$. Then*

$$\sigma(\langle F,c\rangle) = \langle F,\sigma(c)\rangle.$$

Proof We may assume that σ is a transposition $(i\ i+1)$ for $i = 1,2$. We have

$$\sigma(P(F,c)) = P(F,\sigma(c)), \quad \sigma(Q(F,c)) = (-1)^{\chi(F_{i(i+1)}(c))/2} Q(F,\sigma(c)).$$

Let $k \in \{1,2,3\}\setminus\{i,i+1\}$. Note that a singular circle Z is positive with respect to c if and only if Z is negative with respect to $\sigma(c)$, so

$$\theta^+(c) + \theta^+(\sigma(c)) = \theta = \theta^-(c) + \theta^-(\sigma(c)).$$

Moreover,

$$F_i(c) = F_{i+1}(\sigma(c)), \quad F_{i+1}(c) = F_i(\sigma(c)), \quad F_k(c) = F_k(\sigma(c)).$$

Therefore

$$\begin{aligned} s(F,c) - s(F,\sigma(c)) &= \frac{\chi(F_{i+1}(c)) - \chi(F_i(c))}{2} + \theta^-(c) - \theta^-(\sigma(c)) \\ &\equiv \frac{\chi(F_{i+1}(c)) - \chi(F_i(c))}{2} + \theta \\ &\equiv \frac{\chi(F_{i+1}(c)) + \chi(F_i(c))}{2} \\ &\equiv \frac{\chi(F_{i(i+1)}(c))}{2}. \end{aligned}$$

□

Corollary 4.7 *The evaluation $\langle F \rangle$ is a symmetric rational function.*

Later we will prove that $\langle F \rangle$ is in fact a polynomial; see Corollary 4.16.

Lemma 4.8 *Let $i \in \{1,2\}$, let F be a prefoam, and let $c \in \mathrm{adm}(F)$ be an admissible coloring. Suppose $c' \in \mathrm{adm}(F)$ is obtained from c by a $(1,2)$–Kempe move along a surface $\Sigma \subset F_{12}(c)$. Then*

$$s(F,c) \equiv s(F,c') + \tfrac{1}{2}\chi(\Sigma).$$

Proof Note that this is analogous to [34, Lemma 2.19]. Letting $\theta(\Sigma)$ denote the number of seam circles on Σ, we have

$$\theta^-(c) + \theta^-(c') \equiv \theta(\Sigma) \equiv \chi(F_1(c) \cap \Sigma).$$

Note also that

$$\begin{aligned} \chi(F_1(c)) - \chi(F_1(c')) &= \chi(F_1(c) \cap \Sigma) - \chi(F_2(c) \cap \Sigma), \\ \chi(F_2(c)) - \chi(F_2(c')) &= \chi(F_2(c) \cap \Sigma) - \chi(F_1(c) \cap \Sigma). \end{aligned}$$

We compute

$$\begin{aligned} s(F,c) - s(F,c') &\equiv \frac{\chi(F_1(c)) - \chi(F_1(c'))}{2} + \frac{2\big(\chi(F_2(c)) - \chi(F_2(c'))\big)}{2} + \theta(\Sigma) \\ &\equiv \frac{\chi(F_2(c) \cap \Sigma) - \chi(F_1(c) \cap \Sigma)}{2} + \chi(F_1(c) \cap \Sigma) \\ &\equiv \tfrac{1}{2}\chi(\Sigma). \end{aligned}$$

□

4.2 Oriented anchored SL(3) foams and their evaluations

Definition 4.9 An *oriented anchored* SL(3) *foam* F is an oriented foam $F' \subset \mathbb{R}^3$ that may intersect the anchor line L at finitely many points away from the singular circles of F', so that each intersection point belongs to some facet of F', and moreover these intersections are required to be transverse. Denote by $p(F) = F' \cap L$ the set of intersection points (anchor points) of F. The anchor points carry labels in $\{1,2,3\}$; that is, F comes equipped with a fixed map

$$\ell\colon p(F) \to \{1,2,3\}.$$

Fix an anchored foam F and an admissible coloring c of the underlying foam F'. Each anchor point $p \in p(F)$ lying on a facet f inherits a color $c(p) := c(f)$. As in Section 3, we say that c is an admissible coloring of the anchored foam F if for each $p \in p(F)$, the color of p equals the label of p, that is, $c(p) = \ell(p)$. Denote by $\mathrm{adm}(F)$ the set of admissible colorings of F.

For $i \in \{1,2,3\}$, let i' and i'' denote the complementary elements, so that $\{i, i', i''\} = \{1,2,3\}$. Define the evaluations

$$\langle F, c\rangle = (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)} \Bigl(\prod_{p \in p(F)} (-1)^{c(p)-1} (x_{c(p)} - x_{\ell(p)'})(x_{c(p)} - x_{\ell(p)''}) \Bigr)^{1/2}, \tag{77}$$

$$\langle F\rangle = \sum_{c \in \mathrm{adm}(F)} \langle F, c\rangle, \tag{78}$$

where $P(F,c)$, $Q(F,c)$ and $s(F,c)$ are as defined in (69), (70) and (71), respectively.

Let us explain the square root in (77). We have $c(p) = \ell(p)$ for every anchor point $p \in p(F)$. If p is labeled i, then it contributes

$$(-1)^{i-1}(x_i - x_j)(x_i - x_k)$$

to the product under the square root. More concretely, the product of the two terms under the square root, for a fixed anchor point p, is equal to

$$\begin{aligned} &(x_1 - x_2)(x_1 - x_3) && \text{if } c(p) = 1,\\ &(x_1 - x_2)(x_2 - x_3) && \text{if } c(p) = 2,\\ &(x_1 - x_3)(x_2 - x_3) && \text{if } c(p) = 3. \end{aligned}$$

Let $\mathrm{an}(i)$ be the number of anchor points p with $c(p) = i$. Then for $1 \le i < j \le 3$ the sum $\mathrm{an}(i) + \mathrm{an}(j)$ is even, which follows from Proposition 3.10.

We define the square root as the product

$$\widetilde{Q}(F,c) := \prod_{1 \le i < j \le 3} (x_i - x_j)^{(\mathrm{an}(i)+\mathrm{an}(j))/2} \tag{79}$$

and rewrite formula (77) as

$$\begin{aligned}\langle F,c\rangle &:= (-1)^{s(F,c)}\frac{P(F,c)\widetilde{Q}(F,c)}{Q(F,c)}\\ &= (-1)^{s(F,c)}P(F,c)\prod_{1\le i<j\le 3}(x_i-x_j)^{(\mathrm{an}(i)+\mathrm{an}(j)-\chi(F_{ij}(c)))/2}.\end{aligned}$$

Note that $\widetilde{Q}(F,c)$ depends only on the labels of anchor points and not on the coloring c, as long as c respects labels of anchor points (otherwise, the evaluation is 0). Consequently, it can also be denoted by $\widetilde{Q}(F)$. Alternatively, it may be useful to allow more general colorings c, with $\widetilde{Q}(F,c)=0$ for c not compatible with the labels of anchor points.

Recall diagram (66) and the surrounding discussion for notations of various rings. The above formula implies the following proposition.

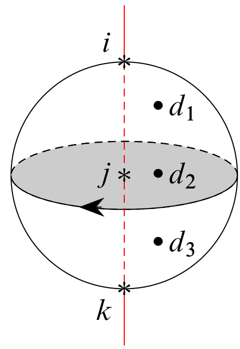
Proposition 4.10 *The evaluation $\langle F,c\rangle$ is an element of R''_x.*

Remark 4.11 As discussed in Remark 3.5, if c is an admissible coloring of the underlying foam F' but not of the anchored foam F, then the evaluation (77) is still well-defined and equal to zero. Even if we don't restrict the notion of admissible colorings of an anchored foam to those which color anchor points according to their labels, additional terms in the evaluation will each be 0, not contributing anything.

Example 4.12 Let F be a 2–sphere $\mathbb{S}^2$ carrying d dots and intersecting L twice. Then $\langle F\rangle = 0$ unless both anchor points are labeled by $i\in\{1,2,3\}$. In this case, there is one admissible coloring c which colors F by i. We see that $s(F,c)\equiv i$, and the evaluation is

$$\langle F\rangle = (-1)^i x_i^d.$$

Example 4.13 Consider the theta foam F whose facets each intersect L exactly once,



There is one admissible coloring c, and we have

$$\langle F\rangle = \langle F, c\rangle = \begin{cases} x_i^{d_1} x_j^{d_2} x_k^{d_3} & \text{if } (i,j,k) = (1,3,2) \text{ or a cyclic permutation,} \\ -x_i^{d_1} x_j^{d_2} x_k^{d_3} & \text{if } (i,j,k) = (1,2,3) \text{ or a cyclic permutation.} \end{cases}$$

The symmetric group S_3 acts on all five of the rings in diagram (66). Recall also that S_3 acts on the set of admissible colorings of an unanchored foam (ie those considered in Section 4.1). However, for an anchored foam F, $c \in \operatorname{adm}(F)$, and $\sigma \in S_3$, the coloring $\sigma(c)$ is in general not admissible for F.

Consider instead the anchored foam $\sigma(F)$ defined as follows. The underlying foam of $\sigma(F)$ agrees with the underlying foam of F. If anchor points of F are labeled by $\ell\colon p(F) \to \{1,2,3\}$, then the anchor points of $\sigma(F)$ are labeled by $\sigma(l)\colon p \mapsto \sigma(\ell(p))$. Note that σ provides a bijection $\operatorname{adm}(F) \cong \operatorname{adm}(\sigma(F))$ via $c \mapsto \sigma(c)$. The following lemma says that the evaluations $\langle F\rangle$ and $\langle\sigma(F)\rangle$ differ by a sign, and moreover the sign depends only on σ and on labels of anchor points of F.

Lemma 4.14 *For an anchored foam F, $c \in \operatorname{adm}(F)$, and $\sigma \in S_3$, we have*

$$\sigma(\langle F, c\rangle) = (-1)^{\varepsilon(F,\sigma)} \langle \sigma(F), \sigma(c)\rangle,$$

where

$$\varepsilon(F,\sigma) = \sum_{\substack{1\le i<j\le 3 \\ \sigma(i)>\sigma(j)}} \frac{\operatorname{an}(i) + \operatorname{an}(j)}{2}. \tag{80}$$

It follows that

$$\sigma(\langle F\rangle) = (-1)^{\varepsilon(F,\sigma)} \langle \sigma(F)\rangle.$$

Proof By Lemma 4.6,

$$\sigma\left((-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)}\right) = (-1)^{s(\sigma(F),\sigma(c))} \frac{P(\sigma(F),\sigma(c))}{Q(\sigma(F),\sigma(c))}.$$

It is clear that

$$\sigma(\widetilde{Q}(F)) = (-1)^{\varepsilon(F,\sigma)} \widetilde{Q}(\sigma(F)),$$

and the first equality follows. For the second equality, we have

$$\begin{aligned} \sigma(\langle F\rangle) &= \sum_{c\in\operatorname{adm}(F)} \sigma(\langle F, c\rangle) \\ &= (-1)^{\varepsilon(F,\sigma)} \sum_{c\in\operatorname{adm}(F)} \langle \sigma(F), \sigma(c)\rangle \\ &= (-1)^{\varepsilon(F,\sigma)} \langle \sigma(F)\rangle. \end{aligned}$$

□

For $1 \le i \neq j \le 3$, consider the ring

$$R''_{ij} := R'_x[(x_i - x_k)^{-1}, (x_j - x_k)^{-1}].$$

Each R''_{ij} is a subring of R''_x. A permutation $\sigma \in S_3$ sends R''_{ij} isomorphically onto $R''_{\sigma(i)\sigma(j)}$.

We are now ready for the main result of this section.

Theorem 4.15 *The evaluation $\langle F \rangle$ of an anchored foam is an element of R'_x, the polynomial ring in variables x_1, x_2 and x_3.*

Proof The proof is similar to that of [18, Theorem 2.17] and [34, Proposition 2.18]. By Lemma 4.14, it suffices to show that $\langle F \rangle \in R''_{12}$ for any anchored foam F. This is because we may take a permutation $\sigma \in S_3$ sending 1 to i and 2 to j, and consider the anchored foam $\sigma^{-1}(F)$. Then $\langle \sigma^{-1}(F) \rangle \in R''_{12}$ implies that

$$\pm\langle F \rangle = \pm\langle \sigma(\sigma^{-1}(F)) \rangle = \pm\sigma(\langle \sigma^{-1}(F) \rangle) \in R''_{ij},$$

where the first equality comes from Lemma 4.14. It follows that

$$\langle F \rangle \in R''_{12} \cap R''_{23} \cap R''_{13} = R'_x.$$

Let us show that $\langle F \rangle \in R''_{12}$. Partition $\mathrm{adm}(F)$ into equivalence classes as follows. For $c \in \mathrm{adm}(F)$, the class C_c containing c consists of colorings obtained from c by performing a sequence of $(1,2)$ Kempe moves along surfaces in $F_{12}(c)$ which are disjoint from L. If $F_{12}(c)$ has n connected components, $k \ge 0$ of which are disjoint from L, then C_c consists of 2^k elements. We will show that

$$\sum_{c' \in C_c} \langle F, c' \rangle \in R''_{12},$$

which will conclude the proof.

Write $\Sigma := F_{12}(c)$ as a disjoint union

$$\Sigma = \Sigma' \cup \Sigma_1 \cup \cdots \cup \Sigma_k,$$

where each Σ_a, for $a = 1, \ldots, k$, is connected and disjoint from L, and where each component of Σ' intersects L. For $i = 1, 2$ and $a = 1, \ldots, k$, let $t_i(a)$ denote the number of dots on i–colored facets (according to c) of Σ_a, and let t_3 denote the number of dots on 3–colored facets (according to c) of F. We claim that

$$\sum_{c'\in C_c} \langle F, c'\rangle = \frac{x_3^{t_3}\cdot\prod_{a=1}^{k}\bigl(x_1^{t_1(a)}x_2^{t_2(a)}+(-1)^{\chi(\Sigma_a)/2}x_2^{t_1(a)}x_1^{t_2(a)}((x_1-x_3)/(x_2-x_3))^{\ell_{\Sigma_a}(c)/2}\bigr)\cdot\widetilde{Q}(F)}{(x_1-x_2)^{\chi(\Sigma)/2}(x_1-x_3)^{\chi(F_{13}(c))/2}(x_2-x_3)^{\chi(F_{23}(c))/2}}, \tag{81}$$

where

- $\ell_{\Sigma_a}(c)\in 2\mathbb{Z}$ is an even integer such that
$$\chi(F_{13}(c')) = \chi(F_{13}(c)) - \ell_{\Sigma_a}(c), \quad \chi(F_{23}(c')) = \chi(F_{23}(c)) + \ell_{\Sigma_a}(c)$$
for the coloring $c'\in C_c$ which is obtained from c by a $(1,2)$ Kempe move along Σ_a. See [18, Lemma 2.12(3)] for details regarding this integer.
- $\widetilde{Q}(F)$ is the contribution from the anchor points of F; see (79).

To verify the claimed equality, expand the product to obtain 2^k terms, each of which corresponds to one of the 2^k colorings in C_c. That the sign is correct follows from Lemma 4.8.

Finally, we argue that $(x_1-x_2)^{\chi(\Sigma)/2}$ divides the numerator of (81). Positive contributions to $\chi(\Sigma)$ come from 2–sphere components of Σ. Each Σ_a which is a 2–sphere contributes one to the exponent $\chi(\Sigma)/2$. On the other hand, the corresponding factor in the product in the numerator of (81) is divisible by x_1-x_2. The remaining positive contributions to $\chi(\Sigma)/2$ come from 2–sphere components of Σ'. Such a component Σ_0 contains at least two anchor points, each labeled 1 or 2, so the contribution from Σ_0 can be canceled with terms in $\widetilde{Q}(F)$. □

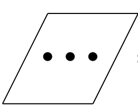
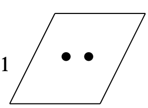
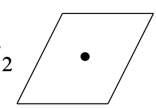
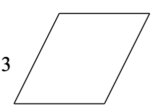
Corollary 4.16 *If F is a prefoam or a foam which is disjoint from L, then $\langle F\rangle\in R_x$, the ring of symmetric polynomials in x_1, x_2 and x_3.*

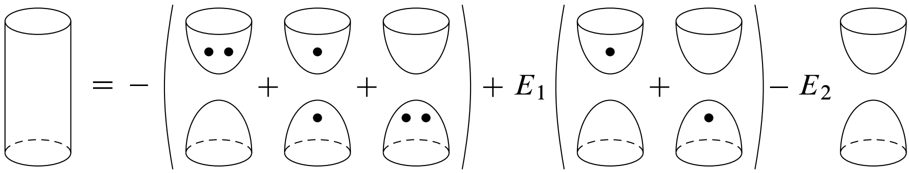
Proof This follows from Lemma 4.14 and Theorem 4.15. □

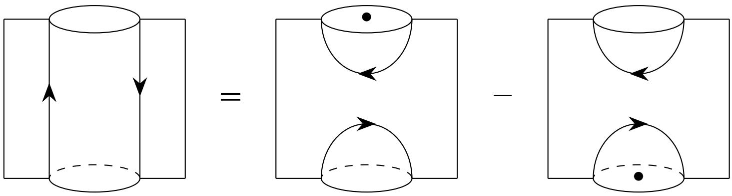
4.3 Skein relations

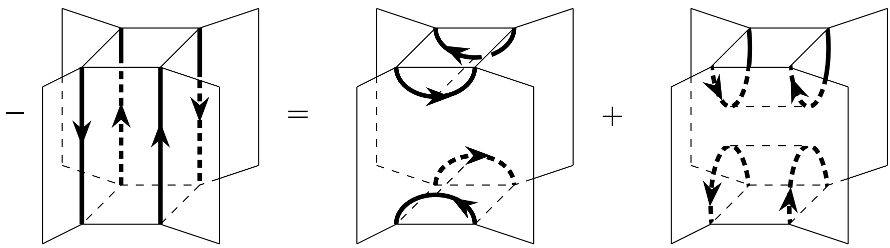
In this section we record several local relations involving oriented anchored SL(3) foams.

Lemma 4.17 *The following local relations hold for anchored foams. Seam lines are drawn in bold in relation (85) to clarify the picture:*

(82)  $= E_1$  $- E_2$  $+ E_3$ 

(83) 

(84) 

(85) 

Proof Proofs of these four relations are similar to Propositions 2.33, 2.22, 2.23 and 2.24 in [18], respectively, with the caveat that we must keep track of the sign (71). Moreover, S_3 symmetry is used in [18] to simplify the calculations. Anchor points and their labels are the same for the foams depicted in each of these four relations, so Lemma 4.14 implies that we may use S_3 symmetry in a similar manner.

We verify relation (83) and leave the remaining three relations to the reader. Let F denote the foam appearing on the left-hand side of the equality. The six foams on the right-hand side are identical except for placement of dots. We denote them by $G^1, \dots, G^6$, so that the relation reads

$$\langle F\rangle = -(\langle G^1\rangle + \langle G^2\rangle + \langle G^3\rangle) + E_1(\langle G^4\rangle + \langle G^5\rangle) - E_2\langle G^6\rangle.$$

Admissible colorings of $G^1, \dots, G^6$ are in canonical bijection. For $c \in \operatorname{adm}(G^1)$, let

$$\langle G, c\rangle := -(\langle G^1, c\rangle + \langle G^2, c\rangle + \langle G^3, c\rangle) + E_1(\langle G^4, c\rangle + \langle G^5, c\rangle) - E_2\langle G^6, c\rangle.$$

There are two types of colorings of G^1: those which color the two depicted disks the same, and those which color them differently. Those of the first type are in canonical bijection with colorings of F.

Suppose $c \in \operatorname{adm}(G^1)$ colors both disks the same color, say i, and denote by $c \in \operatorname{adm}(G^2) \cong \cdots \cong \operatorname{adm}(G^6)$ and $c' \in \operatorname{adm}(F)$ the corresponding colorings. We will show that $\langle F, c'\rangle = \langle G, c\rangle$. We may assume $i = 1$. Then

$$\langle G^1, c\rangle = \langle G^2, c\rangle = \langle G^3, c\rangle = x_1^2\langle G^6, c\rangle, \quad \langle G^4, c\rangle = \langle G^5, c\rangle = x_1\langle G^6, c\rangle,$$

which yields

$$\langle G, c\rangle = -3x_1^2\langle G^6, c\rangle + 2E_1x_1\langle G^6, c\rangle - E_2\langle G^6, c\rangle = -(x_1 - x_2)(x_1 - x_3)\langle G^6, c\rangle.$$

To compare this with $\langle F, c'\rangle$, observe that

$$\chi(F_1(c')) + 2 = \chi(G_1^6(c)), \quad \chi(F_2(c')) = \chi(G_2^6(c)), \quad \chi(F_3(c')) = \chi(G_3^6(c)),$$

which implies $s(F, c') \equiv s(G, c) + 1$. Moreover,

$$\chi(F_{12}(c'))+2 = \chi(G_{12}^6(c)), \quad \chi(F_{13}(c'))+2 = \chi(G_{13}^6(c)), \quad \chi(F_{23}(c')) = \chi(G_{23}^6(c)).$$

Therefore,

$$\langle G^6, c\rangle = -\frac{\langle F, c'\rangle}{(x_1 - x_2)(x_1 - x_3)},$$

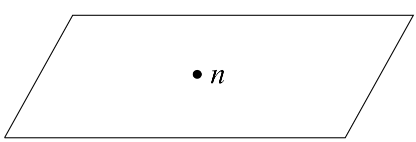
which verifies $\langle F, c'\rangle = \langle G, c\rangle$.

To complete the proof, suppose that c colors the top depicted disk by i and the bottom disk by j, with $i \neq j$. We have

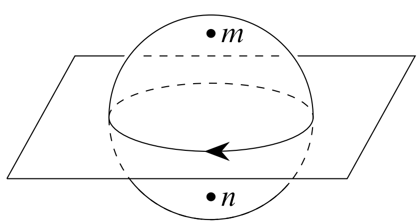
$$\langle G^1, c\rangle = x_i^2\langle G^6, c\rangle, \quad \langle G^2, c\rangle = x_ix_j\langle G^6, c\rangle, \quad \langle G^3, c\rangle = x_j^2\langle G^3, c\rangle,$$
$$\langle G^4, c\rangle = x_i\langle G^6, c\rangle, \quad \langle G^5, c\rangle = x_j\langle G^6, c\rangle.$$

Therefore $\langle G, c\rangle = 0$, which concludes the proof. $\square$

Lemma 4.18 *Let F be an anchored foam. Denote by $F_{n,m}$ the anchored foam obtained from F by adding a bubble (disjoint from L) to some facet in F, with the two new facets carrying n and m dots respectively, such that the facet with n dots directly precedes the facet with m dots in the cyclic ordering. Let F_n denote the foam obtained from F by adding n dots to the same facet,*



F_n



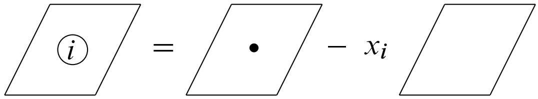
$F_{n,m}$

Then

$$\langle F_{n,n}\rangle = 0, \quad \langle F_{1,0}\rangle = -\langle F_{0,1}\rangle = \langle F\rangle, \quad \langle F_{2,0}\rangle = -\langle F_{0,2}\rangle = E_1\langle F\rangle - \langle F_1\rangle.$$

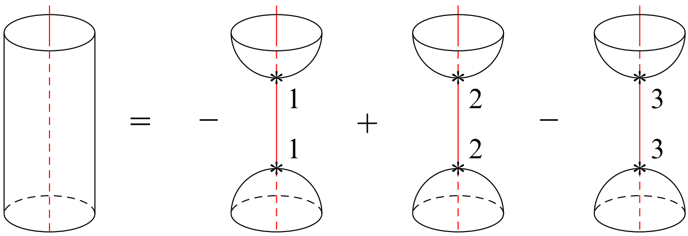
Remark 4.19 The relations in Lemmas 4.17 and 4.18 also hold for prefoams.

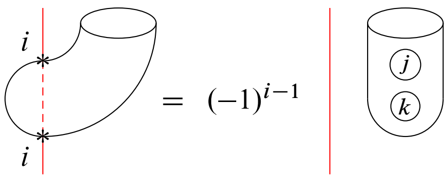
Similar to the SL(2) and unoriented SL(3) setting, for oriented SL(3) foams we allow shifted dots $(i) = \bullet - x_i$ $(1 \le i \le 3)$ on a facet:

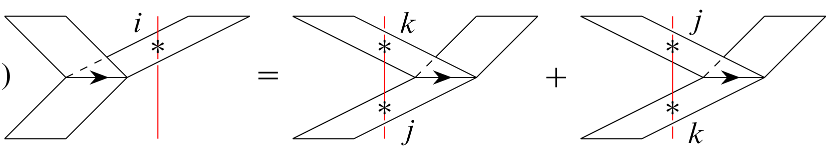


They must be disjoint from L and are allowed to float freely on their facets but cannot cross seam lines.

Lemma 4.20 *The following local relations hold*:

(86)


(87)


(88) $(x_j - x_k)$


In the last equation we assume $j < k$.

Proof We verify (86) and leave the remaining relations to the reader. The argument is similar to that of relation (44) in Lemma 3.15, so we will be brief. Let F denote the foam on the left-hand side, and let G^1, G^2 and G^3 denote the three foams on the right-hand side, with superscript corresponding to labels of the anchor points. For $1 \le i \le 3$, let $\mathrm{adm}_i(F)$ consist of all admissible colorings of F which color the depicted tube by i. There is a natural bijection $\mathrm{adm}_i(F) \cong \mathrm{adm}(G^i)$.

Given $c \in \mathrm{adm}_i(F)$, let $c' \in \mathrm{adm}(G^i)$ denote the corresponding coloring. Arguing as in the proof of Lemma 3.15, we obtain

$$\langle F, c\rangle = \pm\langle G^i, c'\rangle.$$

Figure 15: The orientations at each trivalent vertex of an oriented SL(3) web must be either all outgoing or all incoming.

It remains to show that the above sign is equal to $(-1)^i$. We have

$$\chi(F_j(c)) = \chi(G^i_j(c')), \quad \chi(F_k(c)) = \chi(G^j_k(c')), \quad \chi(F_i(c)) = \chi(G^i_i(c')) - 2,$$
$$\theta^{\pm}(F,c) = \theta^{\pm}(G^i,c'),$$

so $s(F,c) \equiv s(G^i,c') + i$ as needed. □

4.4 State spaces

In this section we define state spaces associated to oriented SL(3) webs. Much of this is analogous to notions in Section 3.3.

Definition 4.21 An *oriented* SL(3) *web* is a planar trivalent graph $\Gamma \subset \mathcal{P}$ in the punctured plane, which may have closed loops with no vertices. Moreover, edges and loops of Γ carry orientations such that each vertex is either a source or a sink, as shown in Figure 15. In this section we will simply write *web* rather than oriented SL(3) web.

The definition of an anchored foam with boundary in the oriented setting is analogous to that of Definition 3.16. The singular graph of a foam with boundary V is a union of finitely many arcs (with boundary in $\mathbb{R}^2 \times \{0,1\}$) and circles (disjoint from $\mathbb{R}^2 \times \{0,1\}$). Intersection points of V with $L_{[0,1]}$ (anchor points) must be disjoint from the singular graph and carry labels in $\{1,2,3\}$. Facets of V are required to carry orientations satisfying the convention in Figure 13, left, near singular points. As usual, we will use the left-hand rule to specify these orientations and cyclic orderings by orienting each singular circle and arc, as shown in Figure 13, middle and right.

As in Section 3.3, let $\partial_i V := V \cap (\mathbb{R}^2 \times \{0\})$ for $i = 0, 1$. The orientation of facets of V induces an orientation on $\partial_0 V$ and $\partial_1 V$ via the convention in Figure 16. We view V as a cobordism from the oriented web $\partial_0 V$ to the oriented web $\partial_1 V$. Composition WV of foams V and W with $\partial_1 V = \partial_0 W$ is defined in the natural way.

Denote by $p(V) = V \cap L_{[0,1]}$ the set of anchor points of V and by $|d(V)|$ the number of dots. The degree of V is defined to be

$$\deg(V) = 2(|d(V)| + |p(V)| - \chi(V)) + \chi(\partial V). \tag{89}$$

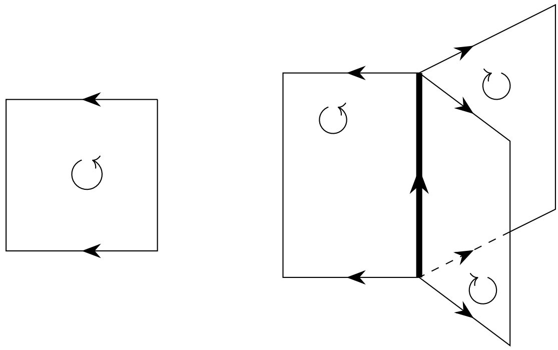


Figure 16: Our convention for the induced orientation on the webs $\partial_0 V$ (bottom) and $\partial_1 V$ (top).

Degree is clearly additive under composition and is compatible with the grading on R'_x, in the sense that if V is a closed foam, then $\deg(V) = \deg(\langle V\rangle)$.

As in Definition 2.14, by an *annular foam* we mean a foam (with boundary) which is disjoint from L. The composition of two annular foams is again annular.

There is an involution ω defined by reflecting a foam with boundary through $\mathbb{R}^2 \times \{1/2\}$. We have $\partial_1 V = \partial_0(\omega(V))$ and $\partial_0 V = \partial_1(\omega(V))$ for any foam with boundary V. Given a web $\Gamma \subset \mathcal{P}$, let $\mathrm{Fr}(\Gamma)$ denote the free R'_x–module generated by foams with boundary V from the empty web to Γ (that is, $\partial_0 V = \varnothing$, $\partial_1 V = \Gamma$). Define a bilinear form

$$(-,-)\colon \mathrm{Fr}(\Gamma) \times \mathrm{Fr}(\Gamma) \to R'_x$$

by $(V, W) = \omega(V)W$. This bilinear form is symmetric since $\langle F\rangle = \langle \omega(F)\rangle$ for any closed foam F. The state space $\langle\Gamma\rangle$ is the quotient of $\mathrm{Fr}(\Gamma)$ by the kernel

$$\ker((-,-)) = \{x \in \mathrm{Fr}(\Gamma) \mid (x, y) = 0 \text{ for all } y \in \mathrm{Fr}(\Gamma)\}$$

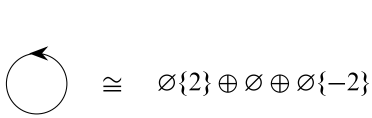
of the bilinear form,

$$\langle\Gamma\rangle := \mathrm{Fr}(\Gamma)/\ker((-,-)).$$

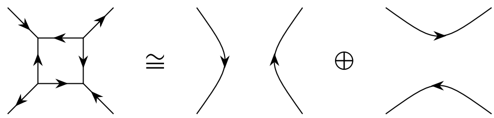
The state space $\langle\Gamma\rangle$ inherits the grading from $\mathrm{Fr}(\Gamma)$ since $(-,-)$ is degree-preserving. A foam with boundary V from Γ_0 to Γ_1 naturally induces a map

$$\langle V\rangle\colon \langle\Gamma_0\rangle \to \langle\Gamma_1\rangle$$

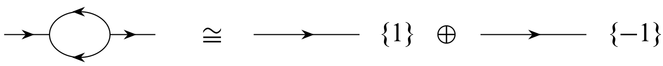
of degree $\deg(V)$, defined by sending the equivalence class of a basis element $U \in \mathrm{Fr}(\Gamma_0)$ to the equivalence class of $VU \in \mathrm{Fr}(\Gamma_1)$. This assignment is functorial with respect to composition of foams, $\langle WV\rangle = \langle W\rangle\langle V\rangle$ for composable V and W.



(a) A contractible circle.



(b) A square face.



(c) A bigon face.

Figure 17: Local relations for state spaces of oriented SL(3) webs, where the depicted regions do not contain the puncture.

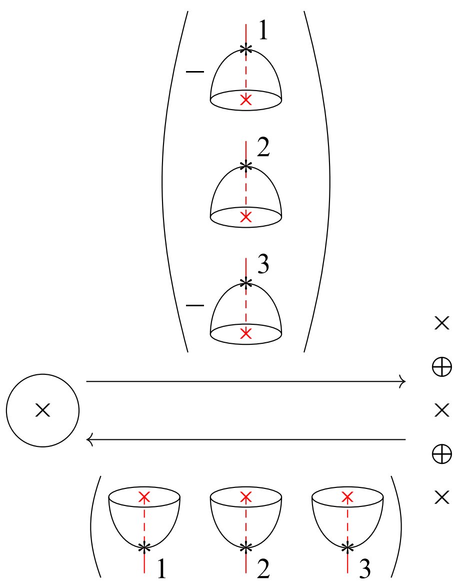
Lemma 4.22 *The three local isomorphisms shown in Figure 17 hold.*

Proof The arguments for relations (a), (b), and (c) of the figure are analogous to Propositions 7, 9, and 8, respectively, of [15]. The relevant relations are Lemmas 4.17 and 4.18. □

Proposition 4.23 *Let $\Gamma \subset \mathcal{P}$ be a web with a noncontractible circle C which bounds a disk in $\mathbb{R}^2 \setminus \Gamma$, and let $\Gamma' = \Gamma \setminus C$ be the web obtained by removing C. Then there is an isomorphism*

$$\langle \Gamma \rangle \cong \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle \oplus \langle \Gamma' \rangle$$

given by the following maps (orientation of the circle is omitted):



Proof This follows from Example 4.12 and the neck-cutting relation (86). □

Theorem 4.24 *For any web $\Gamma \subset \mathcal{P}$, the state space $\langle\Gamma\rangle$ is a free graded R'_x–module of rank equal to the number of Tait colorings of Γ. Moreover, if Γ is contractible, then the graded rank of $\langle\Gamma\rangle$ equals the Kuperberg polynomial [24] of Γ, normalized as in* [15, Section 2].

Proof Lemma 3.21(1) guarantees that we can reduce $\langle\Gamma\rangle$ to a direct sum of empty webs by recursively applying the local isomorphisms in Lemma 4.22 and Proposition 4.23. It is then clear that the rank equals the number of Tait colorings.

If Γ is contractible, $\langle\Gamma\rangle$ can be simplified using only the isomorphisms in Lemma 4.22. Upon taking graded ranks, these isomorphisms recover the recursive relations for computing the Kuperberg polynomial. □

Theorem 4.24 does not address the graded rank of state spaces of noncontractible webs. These may be computed recursively. As a special case, if Γ consists of n contractible and m noncontractible circles, then $\langle\Gamma\rangle$ is free of graded rank $3^m(q^2+1+q^{-2})^n$.

Given a web $\Gamma \subset \mathcal{P}$, we can forget the puncture and the anchor line L and apply the universal construction to the evaluation (73). Precisely, let $\mathrm{Fr}(\Gamma)_{\mathrm{forget}}$ denote the free R_x–module generated by all foams with boundary Γ (forgetting the anchor line). By Corollary 4.16, we can define the bilinear form $(-,-)\colon \mathrm{Fr}(\Gamma)_{\mathrm{forget}} \times \mathrm{Fr}(\Gamma)_{\mathrm{forget}} \to R_x$ and the corresponding state space $\langle\Gamma\rangle_{\mathrm{forget}}$ in the usual way. Thus we obtain state spaces for webs in $\mathbb{R}^2$, functorial with respect to foams in $\mathbb{R}^2\times[0,1]$. These state spaces and maps induced by foams are graded via (89), where $|p(V)|=0$.

Proposition 4.25 *For a contractible web $\Gamma \subset \mathcal{P}$, there is a degree-preserving isomorphism*

$$\langle\Gamma\rangle \cong \langle\Gamma\rangle_{\mathrm{forget}},$$

natural with respect to foams with contractible boundary and which are disjoint from L.

Proof This follows from Theorem 4.24. □

On the other hand, Mackaay and Vaz [28] define an evaluation $\langle-\rangle_{\mathrm{MV}}$ for oriented SL(3) prefoams and use it to define an equivariant (also called *universal*) version of the $\mathfrak{sl}(3)$ link homology introduced in [15]. They work over the ground ring $\mathbb{Z}[a,b,c]$ and associate a state space $\langle\Gamma\rangle_{\mathrm{MV}}$ to each web $\Gamma\subset\mathbb{R}^2$ via the universal construction applied to their prefoam evaluation $\langle-\rangle_{\mathrm{MV}}$. To compare with our situation, identify $\mathbb{Z}[a,b,c]$ with the ring $R_x=\mathbb{Z}[E_1,E_2,E_3]$ of symmetric functions in x_1, x_2 and x_3 via a ring isomorphism φ defined by $\varphi(a)=E_1$, $\varphi(b)=-E_2$ and $\varphi(c)=E_3$.

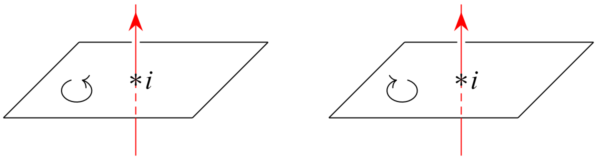


Figure 18: The oriented intersection number between a facet and L.

Theorem 4.26 *For any closed prefoam F,*

$$\langle F\rangle = \varphi(\langle F\rangle_{\mathrm{MV}}).$$

It follows that there are isomorphisms $\langle\Gamma\rangle_{\mathrm{forget}} \cong \langle\Gamma\rangle_{\mathrm{MV}} \otimes_{\mathbb{Z}[a,b,c]} R_x$ for any web $\Gamma \subset \mathbb{R}^2$, natural with respect to maps induced by foams with boundary.

Proof The evaluation $\langle - \rangle_{\mathrm{MV}}$ is defined by applying the local relations (3D), (CN), (S), and (Θ) in [28, Section 2.1] to reduce any foam to an element of $\mathbb{Z}[a,b,c]$. Under the change of variables $a \mapsto E_1$, $b \mapsto -E_2$ and $c \mapsto E_3$, these four relations hold for our evaluation $\langle - \rangle$ by relation (82), relation (83), Example 4.4, and Example 4.5. The statement follows. □

As in the SL(2) and unoriented SL(3) setting considered earlier in the paper, we can define an additional grading on oriented SL(3) foams and state spaces. Define the abelian group

$$\Lambda = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2 \oplus \mathbb{Z}w_3/(w_1+w_2+w_3), \tag{90}$$

on three generators and one relation. Λ is a free abelian group of rank two.

Orient the anchor line L from bottom to top. For an anchored foam V with boundary and $p \in p(V)$ an anchor point lying on some facet f, let $s(p) \in \{\pm 1\}$ denote the oriented intersection number between f and L ($s(p)$ does not depend on the label of p); see Figure 18 for the convention. Define the *annular degree of V* to be

$$\mathrm{adeg}(V) = \sum_{p \in p(V)} s(p) w_{\ell(p)} \in \Lambda. \tag{91}$$

Proposition 4.27 *If F is a closed anchored foam with an admissible coloring c, then* $\mathrm{adeg}(F) = 0$.

Proof The proof is similar to that of Proposition 3.10. The intersection of F with a generic half-plane that bounds L is an oriented web Γ with boundary points on L.

An admissible coloring c of F induces a Tait coloring of Γ. The boundary points (one-valent vertices) of Γ are colored according to their label. The sum in (91) may be rewritten as the sum of terms $\pm(w_1+w_2+w_3)=0$ over all trivalent vertices of Γ, where the sign is $+1$ if all edges are incoming and -1 if all edges are outgoing. Each i–colored inner edge e of Γ bounds two trivalent vertices and contributes $\pm(w_i-w_i)=0$ since e is oriented towards one of its boundary vertices and away from the other. The remaining edges, with one or both endpoints on L, contribute precisely $\operatorname{adeg}(F)$. □

Let $\Gamma\subset\mathcal{P}$ be an (annular oriented) SL(3) web. An anchored foam $F\subset\mathbb{R}^3_-$ with $\partial F=\Gamma$ has a well-defined degree $\operatorname{adeg}(F)\in\Lambda$ via (91). Furthermore, we equip the coefficient ring R'_x with a Λ–grading, with all elements of degree 0. This makes free R'_x–module $\operatorname{Fr}(\Gamma)$ into a Λ–graded module, and Proposition 4.27 implies that the kernel of the bilinear form on $\operatorname{Fr}(\Gamma)$ is Λ–graded as well. Consequently, the grading descends to a Λ–grading on the state space $\langle\Gamma\rangle$. A foam V with boundary induces a map $\langle V\rangle\colon\langle-\partial_0\Gamma\rangle\to\langle\partial_1\Gamma\rangle$ which changes adeg by $\operatorname{adeg}(V)$. If V has no anchor points, it induces an annular degree 0 map between the state spaces of its boundaries. The state space of a contractible web is concentrated in annular degree zero.

Λ–grading on $\langle\Gamma\rangle$ is the analogue of grading on finite-dimensional SL(3) representations by the weight lattice. In fact, in the nonequivariant version of our construction, where all the x_i are set to 0 upon closed foam evaluation (and state spaces are defined accordingly, over a ground field rather than the ring R'_x), the state space $\langle\Gamma\rangle$ is naturally an $\mathfrak{sl}_3$–representation. We also refer the reader to Queffelec and Rose [30] for the construction of sutured annular $\mathfrak{sl}_n$–homology, with state spaces of annular webs carrying an $\mathfrak{sl}_n$–action. In the equivariant case, it is not clear how to define an $\mathfrak{sl}_3$–action or what's the substitute for it.

Denote by $\mathrm{AFoam}_{\mathrm{or}}$ the category whose objects consist of oriented SL(3) webs in $\mathcal{P}$ and whose morphisms are R'_x–linear combinations of anchored cobordisms between webs. Morphism spaces in this category are triply graded via (qdeg, adeg). The state space construction assembles into a functor

$$\langle-\rangle\colon\mathrm{AFoam}_{\mathrm{or}}\to R'_x\text{–}\mathrm{g}_3\mathrm{mod}$$

landing in the category of triply graded R'_x–modules.

This functor respects the trigradings on the hom spaces in the two categories. Restricting to the subcategory of annular cobordisms and their linear combinations, the induced maps have annular degree 0.

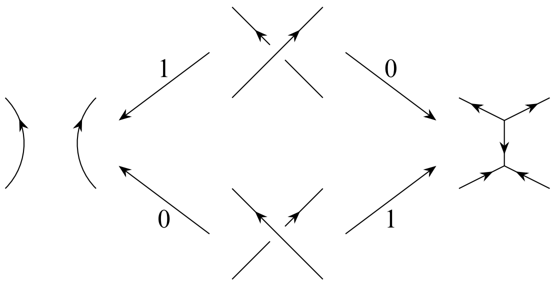


Figure 19: The 0– and 1–smoothings used to define the SL(3) chain complex.

4.5 Annular SL(3)–link homology

Let $L \subset \mathbb{A} \times [0,1]$ be a link in the thickened annulus. Projecting onto $\mathbb{A} \times \{0\} = \mathbb{A}$ and identifying the interior of $\mathbb{A}$ with the punctured plane $\mathcal{P}$, we obtain a link diagram $D \subset \mathcal{P}$. Following [15, Section 4; 28], form the *cube of resolutions* of D. Order the crossings of D by $1, \dots, n$ and use the rule in Figure 19 to decorate each vertex $u \in \{0,1\}^n$ by the corresponding web $D_u \subset \mathcal{P}$.

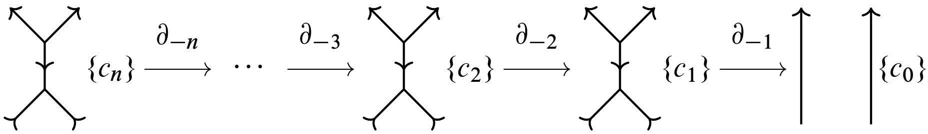
Introducing signs to make the cube anticommute, collapsing the cube to a chain complex, adding internal and homological degree shifts, and applying the functor $\langle - \rangle \colon \mathrm{AFoam}_{\mathrm{or}} \to R'_x\text{–}\mathrm{g}_3\mathrm{mod}$ yields a chain complex $C(D)$ of $\mathbb{Z} \oplus \Lambda$–graded R'_x–modules. In homological degree i, the complex is given by

$$C^i(D) = \bigoplus_{|u|=i+n_+} \langle D_u \rangle \{2(n_+ - n_-) - i\},$$

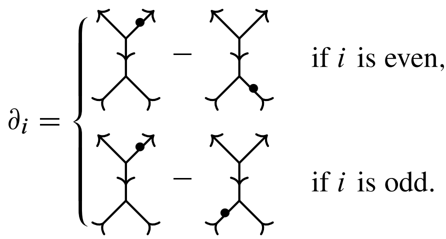
where n_+ and n_- are the number of positive and negative crossings of D. The $\mathbb{Z}$–grading is given by deg — see (89) — and the Λ–grading given by adeg — see (91). Degree shifts in the cube of resolutions are applied only to the $\mathbb{Z}$–degree deg. Diagrams in $\mathcal{P}$ representing isotopic annular links are related by Reidemeister moves away from the puncture. Proofs of Reidemeister invariance in [28] are local, and all local relations (away from the anchor line) on foams in [28] also hold for our evaluation $\langle - \rangle$ by (83), Example 4.4, and Example 4.5. It follows that the chain homotopy class of $C(D)$ is an invariant of the annular link L. We define *equivariant annular* SL(3) *homology* as cohomology groups $H(C(D))$.

Moreover, foams between webs appearing in the cube of resolutions are disjoint from L. Thus the differential preserves annular degree throughout the complex. Consequently, equivariant annular SL(3) link homology carries a homological grading as well as an internal $\mathbb{Z} \oplus \Lambda$–grading $(\deg, \mathrm{adeg})$. Cohomology groups $H(C(D))$ are trigraded R'_x–modules.

Example 4.28 We conclude with an explicit calculation. Let σ denote the positive crossing generator of the 2–strand braid group, let L_n denote the annular link obtained as the annular closure of σ^n, and let $C(L_n)$ denote the corresponding chain complex. Consider the complex $C(n)$,



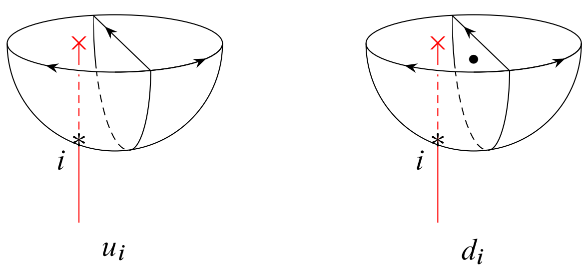
The right-most term is in homological degree zero and the quantum grading shifts c_i are $c_0 = 2n$ and $c_i = 2n + 2i - 1$ for $1 \leq i \leq n$. The right-most differential ∂_{-1} is the unzip cobordism, and for $-n \leq i \leq -2$ the differentials are



One can show that the chain complex $C(L_n)$ is chain homotopy equivalent to the annular closure of $C(n)$.

Upon taking annular closures, the differential ∂_i for even i is zero. Consider the annular closure Γ of the web appearing in negative homological degree. The state space $\langle \Gamma \rangle$ is a free R'_x–module of rank six, and we choose a basis $\{u_1, d_i, u_2, d_2, u_3, d_3\}$ shown in (92). Bidegrees of u_i and d_i are $(-1, -w_i)$ and $(1, -w_i)$, respectively (not accounting for grading shifts):

(92)



After taking the annular closure, the differential ∂_i, for $i \leq -3$ odd, is given as the difference of foams $F - G$, where F puts a dot on the right-most facet and G puts a dot on the middle facet of the depicted generators. We have

$$\begin{aligned} F(u_i) &= (x_j + x_k)u_i - d_i, & F(d_i) &= x_j x_k u_i, \\ G(u_i) &= d_i, & G(d_i) &= (x_j + x_k)d_i - x_j x_k u_i. \end{aligned}$$

In particular, ∂_i for $i \leq -3$ and i odd is injective.

Let us now compute the right-most differential, which is the annular closure of the unzip cobordism. Let Γ_0 denote the web consisting of two essential counterclockwise oriented circles, which is the annular closure of the term in homological degree zero in $C(n)$. For $1 \leq i, j \leq 3$, let $g_{ij} : \varnothing \to \Gamma_0$ be the foam consisting of two cups, each intersecting the anchor line once, with the anchor point of the inner cup labeled i and the anchor point of the outer cup labeled j. By Proposition 4.23, $\{g_{ij}\}_{1 \leq i, j \leq 3}$ is a basis for $\langle \Gamma_0 \rangle$. After introducing the grading shift, the generator g_{ij} is in quantum degree $2n$ and in annular degree $w_i + w_j = -w_k$. Let $Z : \Gamma \to \Gamma_0$ denote the unzip cobordism. By applying the neck-cutting relation (86) near the two circles that constitute Γ_0, we write $\partial_{-1}(u_i)$ as a sum

$$\partial_{-1}(u_i) = \sum_{1 \leq s, t \leq 3} (-1)^{s+t} g_{st} \sqcup \tau_{st},$$

where τ_{st} is a theta foam as in Example 4.13, with no dots, and anchor points labeled i, s and t read from bottom to top. These theta foams evaluate to zero unless $\{i, s, t\} = \{1, 2, 3\}$, and otherwise they evaluate to ± 1. Moreover, $\langle \tau_{st} \rangle = -\langle \tau_{ts} \rangle$. Therefore,

$$\partial_{-1}(u_i) = \pm (g_{jk} - g_{kj}).$$

A similar procedure yields $\partial_{-1}(d_i) = \pm (x_j g_{jk} - x_k g_{kj})$.

Thus, in homological degree $s \leq 0$ and annular degree $-w_i$, the homology of L_n is given by

$$H^{s,-w_i}(L_n) = \begin{cases} 0 & \text{if } s \text{ is odd}, \\ \dfrac{R'_x\{2n-2s-2\} \oplus R'_x\{2n-2s\}}{\langle (x_j + x_k, -2), (2x_j x_k, -(x_j + x_k)) \rangle} & \text{if } s < 0 \text{ and } s \text{ is even}, \\ (R'_x/(x_j - x_k))\{2n\} & \text{if } s = 0. \end{cases}$$

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On a problem of Hopf for circle bundles over aspherical manifolds with hyperbolic fundamental groups

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We prove that a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group admits a self-map of absolute degree greater than one if and only if it is virtually trivial. This generalizes in every dimension the case of circle bundles over hyperbolic surfaces, for which the result was known by the work of Brooks and Goldman on the Seifert volume. As a consequence, we verify the following strong version of a problem of Hopf for the above class of manifolds: every self-map of nonzero degree of a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group is either homotopic to a homeomorphism or homotopic to a nontrivial covering and the bundle is virtually trivial. As another application, we derive the first examples of nonvanishing numerical invariants that are monotone with respect to the mapping degree on nontrivial circle bundles over aspherical manifolds with hyperbolic fundamental groups in any dimension.

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1 Introduction

A long-standing question of Hopf (see Problem 5.26 in Kirby's list [14]) asks:

Problem 1.1 (Hopf) *For a closed oriented manifold M, is every self-map $f: M \to M$ of degree ± 1 a homotopy equivalence?*

A complete solution to Hopf's problem seems to be currently out of reach. Nevertheless, some affirmative answers are known for certain classes of manifolds and dimensions, most notably for simply connected manifolds (by Whitehead's theorem), for manifolds of dimension at most four with Hopfian fundamental groups (see Hausmann [13], and recall that a group is called Hopfian if every surjective endomorphism is an isomorphism), and for aspherical manifolds with hyperbolic fundamental groups (eg negatively curved manifolds). These last groups are Hopfian (see Maltsev [19] and Sela [27]), thus the asphericity assumption, together with the simple fact that any map of degree ± 1 is π_1–surjective, affirmatively answer Problem 1.1 for closed aspherical manifolds with hyperbolic fundamental groups.

In fact, the assumption about degree ± 1 is unnecessary in affirmatively answering Problem 1.1 for aspherical manifolds with nonelementary hyperbolic fundamental groups, because those manifolds cannot admit self-maps of degree other than ± 1 or zero; see Bridson, Hinkkanen and Martin [5], Mineyev [20; 21] and Sela [26; 27], and Section 3.1. Hence, every self-map of nonzero degree of a closed oriented aspherical manifold with nonelementary hyperbolic fundamental group is a homotopy equivalence. Of course, this statement does not hold for all (aspherical) manifolds because, for example, the circle admits self-maps of any degree. Nevertheless, every self-map of the circle of degree greater than one is homotopic to a (nontrivial) covering. The same is true for every self-map of nilpotent manifolds (see Belegradek [3]) and for certain solvable mapping tori of homeomorphisms of the n–dimensional torus; see Neofytidis [23] and Wang [29]. In addition, every nonzero degree self-map of a 3–manifold M is either a homotopy equivalence or homotopic to a covering map, unless the fundamental group of each prime summand of M is finite or cyclic; see Wang [30]. The above results suggest the following question for aspherical manifolds:

Problem 1.2 (strong version of Hopf's problem for aspherical manifolds) *Is every nonzero degree self-map of a closed oriented aspherical manifold either a homotopy equivalence or homotopic to a nontrivial covering*?

In dimension three, hyperbolic manifolds and manifolds containing a hyperbolic piece in their JSJ decomposition do not admit any self-map of degree greater than one (equivalently, of absolute degree greater than one, by taking f^2 whenever $\deg(f) < -1$) due to the positivity of the simplicial volume; see Gromov [11]. (Recall that the simplicial volume $\|\cdot\|$ satisfies $\|M'\| \geq |\deg(f)| \cdot \|M\|$ for every map $f\colon M' \to M$.) The other classes of aspherical 3–manifolds which do not admit self-maps of degree greater than one are $\widetilde{\mathrm{SL}}_2$–manifolds (see Brooks and Goldman [6]) and graph manifolds (see Derbez and Wang [10]), since those manifolds have another (virtually) positive invariant that is monotone with respect to mapping degrees, namely the Seifert volume (introduced in [6] by Brooks and Goldman). In particular, nontrivial circle bundles over closed hyperbolic surfaces (which are modeled on the $\widetilde{\mathrm{SL}}_2$ geometry) do not admit self-maps of degree greater than one. On the other hand, it is clear that trivial circle bundles over (hyperbolic) surfaces, ie products $S^1 \times \Sigma$, admit self-maps of any degree (and those maps are either homotopy equivalences or homotopic to nontrivial coverings [30]).

Recall that a circle bundle $M \xrightarrow{\pi} N$ is classified by its Euler class $e \in H^2(N;\mathbb{Z})$. In particular, M is virtually trivial if and only if e is torsion. For a circle bundle M over a closed oriented surface Σ, its Euler class $e \in H^2(\Sigma) = \mathbb{Z}$ is either zero and the bundle is trivial (ie $M = S^1 \times \Sigma$), or e is not zero and nontorsion and the bundle is not virtually trivial. Our main result is that the nonexistence of self-maps of degree greater than one on nontrivial circle bundles over closed oriented hyperbolic surfaces (ie over closed oriented aspherical 2–manifolds with hyperbolic fundamental groups) can be extended to any dimension. In fact, we prove a stronger statement:

Theorem 1.3 *An oriented circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group admits a self-map of absolute degree greater than one if and only if it is virtually trivial.*

The "if" direction holds more generally without any assumption on the hyperbolicity of the fundamental group of the base:

Example 1.4 Let M be a virtually trivial oriented circle bundle over a closed oriented manifold N. Then its Euler class $e \in H^2(N)$ is k–torsion for some k. Since M is fiberwise oriented, M is a principal $\mathrm{U}(1)$–bundle and thus M can be viewed as the (associated) complex line bundle whose first Chern class is the Euler class e. Consider the tensor product $M \otimes \cdots \otimes M$ of $k+1$ copies of M. Then the first Chern class of $M \otimes \cdots \otimes M$ is

$$c_1(M \otimes \cdots \otimes M) = (k+1)c_1(M) = c_1(M),$$

and so $M \otimes \cdots \otimes M$ is isomorphic to M. Taking the $k+1$ power of a section of M gives us a fiberwise map

$$f \colon M \to M \otimes \cdots \otimes M,$$

which has degree $k+1$ on the fibers and degree one on the base N. Thus $\deg(f) = k+1$.

Outline of the proof of the main theorem

In view of Example 1.4, the proof of Theorem 1.3 amounts to showing that if an oriented circle bundle M over a closed oriented aspherical manifold N with $\pi_1(N)$ hyperbolic admits a self-map f of degree greater than one, then M must be virtually trivial. We will show that such an f is in fact homotopic to a fiberwise nontrivial self-covering of M, and thus the powers of f induce a purely decreasing sequence

$$\pi_1(M) \supsetneq f_*(\pi_1(M)) \supsetneq \cdots \supsetneq f_*^m(\pi_1(M)) \supsetneq f_*^{m+1}(\pi_1(M)) \supsetneq \cdots. \tag{1}$$

Using this sequence, we will obtain an infinite-index subgroup of $\pi_1(M)$ given by

$$G := \bigcap_m f_*^m(\pi_1(M)).$$

The last part of the proof uses the concept of groups infinite-index presentable by products (IIPP) and characterizations of groups fulfilling this condition [22]. More precisely, we will see that the multiplication map

$$\varphi\colon C(\pi_1(M)) \times G \to \pi_1(M)$$

defines a presentation by products for $\pi_1(M)$, where both G and the center $C(\pi_1(M))$ have infinite index in $\pi_1(M)$. This will lead us to the conclusion that $\pi_1(M)$ has a finite-index subgroup isomorphic to a product and M is virtually trivial.

Remark 1.5 In the proof of Theorem 1.3 we will use the fact that the base is an aspherical manifold which does not admit self-maps of degree greater than one, and its fundamental group is Hopfian with trivial center. Thus we can extend Theorem 1.3 (and its consequences; see Section 2) to any circle bundle over a closed oriented manifold N that fulfills the aforementioned properties. For instance, if N is an irreducible locally symmetric space of noncompact type, then it is aspherical, it has positive simplicial volume by Bucher-Karlsson [7] and Lafont and Schmidt [16] (and thus does not admit self-maps of degree greater than one), and $\pi_1(N)$ is Hopfian (see Maltsev [19]) without center; see Raghunathan [25].

Remark 1.6 A decreasing sequence (1) exists whenever an aspherical manifold M admits a self-map f of degree greater than one and $\pi_1(\overline{M})$ is Hopfian for every finite cover $\overline{M}$ of M (which is conjectured to be true; see Section 2). This gives further evidence towards an affirmative answer to Problem 1.2, since the existence of such a sequence is a necessary condition for f to be homotopic to a nontrivial covering. Now, every finite-index subgroup of the fundamental group of a circle bundle over a closed aspherical manifold with hyperbolic fundamental group is indeed Hopfian, and therefore this gives us an alternative way of obtaining sequence (1). We will discuss the Hopf property for those circle bundles and Problem 1.2 more generally in Section 5.

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2 Applications of the main result

Before proceeding to the proof of Theorem 1.3, let us mention a few consequences of Theorem 1.3, or of parts of its proof.

It is a long-standing question (motivated by Problem 1.1) whether the fundamental group of every closed aspherical manifold is Hopfian (see [24] for a discussion). If this is true, then every self-map of an aspherical manifold of degree ± 1 is a homotopy equivalence. In the course of the proof of Theorem 1.3, we will see that every self-map of a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group is homotopic to a fiberwise covering map, and this alone shows that Problem 1.1 and, in most cases, Problem 1.2 indeed have affirmative answers for self-maps of those manifolds. More interestingly, Theorem 1.3 implies the following complete characterization with respect to Problem 1.2:

Corollary 2.1 *A self-map of nonzero degree of an oriented circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group either is a homotopy equivalence or is homotopic to a fiberwise nontrivial covering* (*and to a nontrivial covering in dimensions other than four and five*) *and the bundle is virtually trivial.*

Remark 2.2 (the Borel conjecture: from homotopy equivalences to homeomorphisms) In most cases, an even stronger conclusion holds for the homotopy equivalences of Corollary 2.1. Recall that the Borel conjecture asserts that any homotopy equivalence between two closed aspherical manifolds is homotopic to a homeomorphism. (Note that the Borel conjecture does not hold in the smooth category or for nonaspherical manifolds; see for example the survey paper [18] and the discussion in [28].) A complete affirmative answer to the Borel conjecture is known in dimensions less than four (see again [18] for a survey). Moreover, by [1; 2], the fundamental group of a circle bundle M over a closed aspherical manifold N with $\pi_1(N)$ hyperbolic and $\dim(N) \neq 3, 4$ satisfies the Farrell–Jones conjecture, and therefore the Borel conjecture, and so every homotopy equivalence of such a circle bundle is in fact homotopic to a homeomorphism. (See also [5] for self-maps of the base N.)

Beyond the Seifert volume for nontrivial circle bundles over hyperbolic surfaces [6], no other nonvanishing monotone invariant respecting the degree seems to have been known on higher-dimensional circle bundles over aspherical manifolds with hyperbolic fundamental groups (note that the simplicial volume vanishes as well [11]). A consequence of Theorem 1.3 is that such a monotone invariant exists and is given by the domination seminorm. Recall that the domination seminorm is defined by

$$\nu_M(M') := \sup\{|\deg(f)| \mid f : M' \to M\},$$

and it was introduced in [9]. Theorem 1.3 implies:

Corollary 2.3 *If M is a not virtually trivial circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group*, *then $\nu_M(M) = 1$.*

However, the domination seminorm is not finite in general, because M might admit maps of infinitely many different degrees from another manifold M'. Nevertheless, Theorem 1.3 and the nonvanishing of the Seifert volume for nontrivial circle bundles over hyperbolic surfaces suggest:

Conjecture 2.4 *In every dimension n, there is a homotopy n–manifold numerical invariant I_n satisfying $I_n(M) \geq |\deg(f)| I_n(N)$ for each map $f : M \to N$ which is positive and finite on every not virtually trivial circle bundle over a closed aspherical manifold with hyperbolic fundamental group.*

3 Infinite sequences of coverings

In this section we reduce our discussion to self-coverings of a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group, and thus obtain a purely decreasing sequence of finite-index subgroups of the fundamental group of this bundle.

3.1 Self-maps of aspherical manifolds with hyperbolic fundamental groups

First, we observe that the hyperbolicity of the fundamental group of the base implies strong restrictions on the possible degrees of its self-maps:

Proposition 3.1 [5] *Every self-map of nonzero degree of a closed aspherical manifold with nonelementary hyperbolic fundamental group is a homotopy equivalence.*

Proof There are two ways to see this. The first (given in [5]) is purely algebraic, using the co-Hopf property of torsion-free nonelementary hyperbolic groups [26; 27]. The other way uses bounded cohomology and the simplicial volume; see [11; 20; 21].

Suppose N is a closed oriented aspherical manifold whose fundamental group is nonelementary hyperbolic and $f\colon N \to N$ is a map of nonzero degree. By [26; 27] (see also [5, Lemma 4.2]), $\pi_1(N)$ is co-Hopfian (ie every injective endomorphism is an isomorphism), and so by the asphericity of N it suffices to show that f_* is injective. Suppose the contrary, and let x be a nontrivial element in $\ker(f_*)$. Since $f_*(\pi_1(N))$ has finite index in $\pi_1(N)$, there is some $n \in \mathbb{N}$ such that $x^n \in f_*(\pi_1(N))$, ie there is some $y \in \pi_1(N)$ such that $f_*(y) = x^n$. Clearly $x^n \neq 1$, because $\pi_1(N)$ is torsion-free, and so $y \notin \ker(f_*)$. Now, $f_*^2(y) = f_*(x^n) = 1$, which means that $y \in \ker(f_*^2)$. By iterating this process, we obtain a purely increasing sequence

$$\ker(f_*) \subsetneq \ker(f_*^2) \subsetneq \cdots \subsetneq \ker(f_*^m) \subsetneq \ker(f_*^{m+1}) \subsetneq \cdots.$$

But this sequence contradicts Sela's result [26; 27] that, for every endomorphism ψ of a torsion-free hyperbolic group, there exists $m_0 \in \mathbb{N}$ such that $\ker(\psi^k) = \ker(\psi^{m_0})$ for all $k \geq m_0$. We deduce that f_* is injective, and therefore an isomorphism as required.

Alternatively to the above argument, since $\pi_1(M)$ is nonelementary hyperbolic, the comparison map from bounded cohomology to ordinary cohomology

$$\psi_{\pi_1(M)}\colon H_b^n(\pi_1(M);\mathbb{R}) \to H^n(\pi_1(M);\mathbb{R})$$

is surjective; see [20; 21; 12]. Thus, by the duality of the simplicial ℓ^1–seminorm and the bounded cohomology ℓ^∞–seminorm (see [11]), we deduce that M has positive simplicial volume. This implies that every nonzero degree map $f\colon M \to M$ has degree ± 1. In particular, f is π_1–surjective, and thus an isomorphism, because $\pi_1(M)$ is Hopfian [19; 27]. □

3.2 Fundamental group and finite covers

Let $M \xrightarrow{\pi} N$ be an oriented circle bundle, where N is a closed oriented aspherical manifold with $\pi_1(N)$ hyperbolic. We may assume that $\dim(N) \geq 2$, otherwise we deal with the well-known case of T^2. The fundamental group of M fits into the central extension (see [4; 8])

$$1 \to C(\pi_1(M)) \to \pi_1(M) \xrightarrow{\pi_*} \pi_1(N) \to 1,$$

where $C(\pi_1(M)) = \mathbb{Z}$. (Note that $C(\pi_1(N)) = 1$ because $\pi_1(N)$ is torsion-free nonelementary hyperbolic.)

It is easy to observe that every finite covering of M is of the same type. More precisely:

Lemma 3.2 [22, Lemma 4.6] *Every finite cover $\overline{M} \xrightarrow{p} M$ is a circle bundle $\overline{M} \xrightarrow{\bar{\pi}} \overline{N}$, where $\overline{N} \xrightarrow{\bar{p}} N$ is a finite covering.*

In particular, p is a generalized bundle map covering $\bar{p}$ and the (infinite cyclic) center of $\pi_1(\overline{M})$ is mapped under p_* into the center of $\pi_1(M)$.

3.3 Reduction to fiberwise covering maps

Now, let $f\colon M \to M$ be a map of nonzero degree. We observe that f is homotopic to a fiberwise covering map:

Proposition 3.3 *The map f is homotopic to a fiberwise covering where the induced map $f_{S^1}\colon S^1 \to S^1$ has degree $\pm\deg(f)$.*

Proof Consider the composite map $\pi \circ f\colon M \to N$ and the induced homomorphism

$$(\pi \circ f)_*\colon \pi_1(M) \to \pi_1(N).$$

Since the center of $\pi_1(N)$ is trivial, we derive, after lifting f to a π_1–surjective map $\bar{f}\colon M \to \overline{M}$ (where $\overline{M} \xrightarrow{p} M$ corresponds to $f_*(\pi_1(M))$), that the center of $\pi_1(M)$ is mapped under $(\pi \circ f)_*$ to the trivial element of $\pi_1(N)$; see Lemma 3.2 and the lines above that. Thus f factors, up to homotopy, through a self-map $g\colon N \to N$, ie $\pi \circ f = g \circ \pi$ (up to homotopy).

Clearly $\deg(g) \neq 0$, otherwise f would factor through the degree-zero map from the pullback bundle of g along π to M, which is impossible because $\deg(f) \neq 0$. Now, Proposition 3.1 implies that g is a homotopy equivalence of N (in particular $\deg(g) = \pm 1$). Hence, the induced map f_{S^1} on the S^1 fiber is homotopic to a self-covering of degree

$$\deg(f_{S^1}) = \pm\deg(f). \qquad \square$$

Since every map of degree ± 1 is π_1–surjective, every self-map of M of degree ± 1 is a homotopy equivalence, thus answering Problem 1.1 in the affirmative. More interestingly, the above proposition gives the following strong affirmative answer to Problem 1.2 (see Remark 2.2):

Corollary 3.4 *Let M be an oriented circle bundle over a closed oriented aspherical manifold N with hyperbolic fundamental group and $\dim(N) \neq 3, 4$. Every self-map of M of nonzero degree is either homotopic to a homeomorphism or homotopic to a nontrivial covering.*

Consider now the iterates

$$f^m : M \to M, \quad m \geq 1.$$

By Proposition 3.3, each f^m is homotopic to a fiberwise covering of degree

$$(\deg(f))^m = [\pi_1(M) : f_*^m(\pi_1(M))].$$

That is, for each m, the homomorphism

$$f_*^m : \pi_1(M) \to \pi_1(M)$$

maps every element $x \in C(\pi_1(M)) = \mathbb{Z} = \langle z \rangle$ to $x^{\pm \deg(f^m)} \in C(\pi_1(M))$ and induces an isomorphism on $\pi_1(N) = \pi_*(\pi_1(M))$. In particular, when $\deg(f) > 1$, we obtain:

Corollary 3.5 *If $f : M \to M$ has degree greater than one, then there is a purely decreasing infinite sequence of subgroups of $\pi_1(M)$ given by*

$$\pi_1(M) \supsetneq f_*(\pi_1(M)) \supsetneq \cdots \supsetneq f_*^m(\pi_1(M)) \supsetneq f_*^{m+1}(\pi_1(M)) \supsetneq \cdots. \tag{2}$$

4 Distinguishing between trivial and nontrivial bundles

Now we show that the existence of sequence (2) implies that $\pi_1(M)$ has a finite-index subgroup which is isomorphic to a direct product, and thus M is virtually trivial. To this end, we construct a presentation of $\pi_1(M)$ by a product of two infinite-index subgroups.

4.1 Groups infinite-index presentable by products

An infinite group Γ is said to be *infinite-index presentable by products* or *IIPP* if there exist two infinite subgroups $\Gamma_1, \Gamma_2 \subset \Gamma$ that commute elementwise, such that $[\Gamma : \Gamma_i] = \infty$ for both Γ_i and the multiplication homomorphism

$$\Gamma_1 \times \Gamma_2 \to \Gamma$$

surjects onto a finite-index subgroup of Γ.

The notion of IIPP groups was introduced in [22] in the study of maps of nonzero degree from direct products to aspherical manifolds with nontrivial center. The concept of groups presentable by products (ie without the constraint on the index) was introduced in [15]. It is clear that when Γ is a *reducible* group, that is, virtually a product of two infinite groups, then Γ is IIPP. Thus, a natural problem is to determine when these two properties are equivalent. In general they are not equivalent, as shown in [22, Section 8], however, equivalence is achieved under certain assumptions:

Theorem 4.1 [22, Theorem D] *Suppose Γ fits into a central extension*

$$1 \to C(\Gamma) \to \Gamma \to \Gamma/C(\Gamma) \to 1,$$

where $\Gamma/C(\Gamma)$ is not presentable by products. Then Γ is IIPP if and only if it is reducible.

The following theorem characterizes aspherical circle bundles when the fundamental group of the base is not presentable by products:

Theorem 4.2 [22, Theorem C] *Let $M \xrightarrow{\pi} N$ be a circle bundle over a closed aspherical manifold N whose fundamental group $\pi_1(N)$ is not presentable by products. Then the following are equivalent*:

(i) *M admits a map of nonzero degree from a direct product.*

(ii) *M is finitely covered by a product $S^1 \times \overline{N}$ for some finite cover $\overline{N} \to N$.*

(iii) *$\pi_1(M)$ is reducible.*

(iv) *$\pi_1(M)$ is IIPP.*

Since nonelementary hyperbolic groups are not presentable by products [15], each circle bundle M over a closed aspherical manifold N with $\pi_1(N)$ hyperbolic fulfills the assumptions of Theorems 4.1 and 4.2. Using this, we will be able to deduce that M is virtually trivial. Furthermore, our presentation by products for $\pi_1(M)$ will have trivial kernel; see Remark 4.3.

4.2 An infinite-index presentation by products

Under the assumption of the existence of $f^m : M \to M$ with $\deg(f^m) = (\deg(f))^m > 1$ for all $m \geq 1$, and thus of sequence (2), we consider the subgroup of $\pi_1(M)$ defined by

$$G := \bigcap_m f_*^m(\pi_1(M)).$$

First, we observe the general fact (ie without using the specific forms of the $f_*^m(\pi_1(M))$) that G has infinite index in $\pi_1(M)$. Suppose, to the contrary, that $[\pi_1(M) : G] < \infty$. Then since

$$[\pi_1(M) : f_*^m(\pi_1(M))] \leq [\pi_1(M) : G]$$

for all m, and $\pi_1(M)$ contains only finitely many subgroups of a fixed index, we deduce that there exists n such that $f_*^n(\pi_1(M)) = f_*^k(\pi_1(M))$ for all $k \geq n$. This is, however, impossible by Corollary 3.5. Now we will show that $\pi_1(M)$ admits a presentation by the product $C(\pi_1(M)) \times G$. Let

$$\varphi\colon C(\pi_1(M)) \times G \to \pi_1(M) \tag{3}$$

be the multiplication map. Since each element of $C(\pi_1(M))$ commutes with every element of G, we deduce that φ is in fact a well-defined homomorphism.

We claim that φ surjects onto a finite-index subgroup of $\pi_1(M)$, ie that $C(\pi_1(M))G$ has finite index in $\pi_1(M)$. To this end, we will use the specific description of f^m and $f_*^m(\pi_1(M))$. In Section 3.3 we saw that, for every m, the composite f^m is a fiberwise covering of degree $\deg(f^m)$ on the fibers that induces an isomorphism on $\pi_1(N)$, and even more it induces a homotopy equivalence of N. In particular, for every $m \geq 1$, we obtain a short exact sequence

$$1 \to \langle z^{\deg(f)^m} \rangle \to f_*^m(\pi_1(M)) \to \pi_1(N_m) \to 1,$$

where $\pi_1(N_m) \cong \pi_1(N)$. Hence, $\pi_1(M)/f_*^m(\pi_1(M)) \cong \mathbb{Z}/\deg(f)^m\mathbb{Z}$ for all $m \geq 1$, and so $\pi_1(M)/G \cong \mathbb{Z}$. Thus, we obtain a short exact sequence (induced by π_*),

$$1 \to (C(\pi_1(M))G)/G \to \pi_1(M)/G \xrightarrow{\bar{\pi}_*} \pi_1(N)/\pi_*(G) \to 1.$$

Since $(C(\pi_1(M))G)/G \cong \mathbb{Z}$, we conclude that $\pi_*(G)$ is a finite-index subgroup of $\pi_1(N)$.

Now let $x \in \pi_1(M)$. If $x = z^s \in C(\pi_1(M)) = \langle z \rangle$, then $\varphi(x, 1) = x$. If $x \notin C(\pi_1(M))$, then $\pi_*(x)$ is not trivial in $\pi_1(N)$ and so there exists $t \geq 0$ such that $\pi_*(x^t) \in \pi_*(G)$, that is $\pi_*(x^t) = \pi_*(g)$ for some $g \in G$. Thus $x^t = z^a g$ for some $a \in \mathbb{Z}$, and so $\varphi(z^a, g) = x^t$. We conclude that $\varphi(C(\pi_1(M)) \times G) = C(\pi_1(M))G$ has finite index in $\pi_1(M)$.

Since moreover $C(\pi_1(M))$ and G have infinite index in $\pi_1(M)$, we conclude that the presentation given in (3) is an infinite-index presentation by products. Theorem 4.2 implies that $\pi_1(M)$ is reducible and M is virtually a trivial circle bundle.

This finishes the proof of Theorem 1.3.

Remark 4.3 The kernel of φ must be trivial because it is isomorphic to $C(\pi_1(M)) \cap G$, which is trivial. Thus $C(\pi_1(M))G$ is isomorphic to the fundamental group of a trivial circle bundle that covers M. In particular, the property of $\pi_1(N)$ that is not presentable by products was not necessary for our proof.

An alternative way to see that $C(\pi_1(M)) \cap G$ is trivial is to observe that

$$[C(\pi_1(M)) : C(\pi_1(M)) \cap G] = [\pi_1(M) : G] = \infty.$$

Together with the fact that $C(\pi_1(M)) = \mathbb{Z}$, we conclude that $C(\pi_1(M)) \cap G$ is trivial.

The proof of Corollary 2.1 is now straightforward:

Proof of Corollary 2.1 Let M be a circle bundle over a closed oriented aspherical manifold N with $\pi_1(N)$ hyperbolic and $f : M \to M$ be a map of nonzero degree. As we have seen in Section 3.3, if $\deg(f) = \pm 1$ then f is a homotopy equivalence, and if $\deg(f) \neq \pm 1$ then f is homotopic to a nontrivial fiberwise covering (and to a nontrivial covering when $\dim(N) \neq 3, 4$; see Remark 2.2). In the latter case, Theorem 1.3 implies moreover that M is virtually $S^1 \times \overline{N}$ for some finite covering $\overline{N} \to N$. □

Remark 4.4 When M has torsion Euler class $e \in H^2(N)$, we have seen in Example 1.4 that M admits a self-map f of degree greater than one. Recall that a product finite covering $S^1 \times \overline{N} \to M$ is obtained by pulling back $M \xrightarrow{\pi} N$ along the finite covering $\overline{N} \to N$ that corresponds to the finite-index subgroup

$$H := \ker(\pi_1(N) \xrightarrow{h} H_1(N) \xrightarrow{\pi_T} \operatorname{Tor} H_1(N)) \subseteq \pi_1(N),$$

where h denotes the Hurewicz map and π_T is the projection to the torsion of $H_1(N)$. (Note that e lies in $\operatorname{Tor} H_1(M)$ by the universal coefficient theorem.) The groups H and $\pi_*(G)$ are commensurable in $\pi_1(N)$ because

$$[\pi_*(G) : \pi_*(G) \cap H] \leq [\pi_1(N) : H] < \infty$$

and

$$[H : \pi_*(G) \cap H] \leq [\pi_1(N) : \pi_*(G)] < \infty.$$

5 The Hopf property and strong version of Hopf's problem

In this section we discuss the Hopf property for circle bundles over aspherical manifolds with hyperbolic fundamental groups and Problem 1.2 more generally.

5.1 The Hopf property

First, we show that the fundamental groups of circle bundles over aspherical manifolds with hyperbolic fundamental groups are Hopfian:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C(\pi_1(M)) & \longrightarrow & \pi_1(M) & \xrightarrow{\pi_*} & \pi_1(N) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle \phi|_{C(\pi_1(M))}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \bar{\phi}} & & \\
1 & \longrightarrow & C(\pi_1(M)) & \longrightarrow & \pi_1(M) & \xrightarrow{\pi_*} & \pi_1(N) & \longrightarrow & 1
\end{array}$$

Figure 1: The Hopf property for $\pi_1(M)$.

Proposition 5.1 *If M is a circle bundle over a closed oriented aspherical manifold with hyperbolic fundamental group, then every finite-index subgroup of $\pi_1(M)$ is Hopfian.*

Proof Let $M \xrightarrow{\pi} N$ be a circle bundle where N is a closed oriented aspherical manifold with $\pi_1(N)$ hyperbolic. (As before, we can assume that $\pi_1(N)$ is not cyclic.) Since every finite covering of M is of the same type (see Lemma 3.2), it suffices to show that $\pi_1(M)$ is Hopfian.

Let $\phi\colon \pi_1(M) \to \pi_1(M)$ be a surjective homomorphism. Then $\phi(C(\pi_1(M))) \subseteq C(\pi_1(M))$, and so the composite homomorphism $\pi_* \circ \phi\colon \pi_1(M) \to \pi_1(N)$ maps $C(\pi_1(M))$ to the trivial element of $\pi_1(N)$. In particular, there exists a surjective homomorphism $\bar{\phi}\colon \pi_1(N) \to \pi_1(N)$ such that $\bar{\phi} \circ \pi_* = \pi_* \circ \phi$. Now, $\bar{\phi}$ is injective as well (and so an isomorphism), because $\pi_1(N)$ is Hopfian, being hyperbolic and torsion-free [19; 27]. Then, again using the surjectivity of ϕ, we deduce that

$$\phi|_{C(\pi_1(M))}\colon C(\pi_1(M)) \to C(\pi_1(M))$$

is also surjective. Since $C(\pi_1(M)) = \mathbb{Z}$ is Hopfian, we conclude that $\phi|_{C(\pi_1(M))}$ is in fact an isomorphism. Then the five lemma for the commutative diagram in Figure 1 implies that ϕ is an isomorphism as well. □

In this way, we obtain also an alternative proof of the fact that every self-map of M of degree ± 1 is a homotopy equivalence. Of course, the above group-theoretic argument uses the same line of argument as the proof of Proposition 3.3, with the difference that it starts with a stronger assumption, namely that ϕ is surjective.

5.2 Infinite decreasing sequences and Problem 1.2

The fact that every finite-index subgroup of the fundamental group of a circle bundle over an aspherical manifold N with hyperbolic $\pi_1(N)$ has the Hopf property is actually conjectured to be true for all aspherical manifolds. Not only would this immediately verify Problem 1.1 for every aspherical manifold, it also gives evidence for an affirmative answer to Problem 1.2. Namely, let $f\colon M \to M$ be a map of degree $\deg(f) > 1$ and suppose that every finite-index subgroup of $\pi_1(M)$ is Hopfian. Then, as in the case of nontrivial coverings, there is a purely decreasing infinite sequence

$$\pi_1(M) \supsetneq f_*(\pi_1(M)) \supsetneq \cdots \supsetneq f_*^m(\pi_1(M)) \supsetneq f_*^{m+1}(\pi_1(M)) \supsetneq \cdots.$$

The proof of this claim can be found along the lines of the proof of Theorem 14.40 of [17], but let us give the details for completeness: Suppose to the contrary that there is some n such that $f_*^n(\pi_1(M)) = f_*^k(\pi_1(M))$ for all $k \geq n$. Let $M_n \xrightarrow{p_n} M$ be the finite covering of M corresponding to $f_*^n(\pi_1(M))$ and denote by $\overline{f^n}\colon M \to M_n$ the lift of f^n, which induces a surjection on π_1. Since $f_*^n(\pi_1(M)) = f_*^{2n}(\pi_1(M))$, we deduce that the composite map $\overline{f^n} \circ p_n\colon M_n \to M_n$ induces a surjection

$$(\overline{f^n} \circ p_n)_*\colon \pi_1(M_n) \to \pi_1(M_n).$$

Since $\pi_1(M_n)$ is Hopfian, we deduce that $(\overline{f^n} \circ p_n)_*$ is an isomorphism, and so a homotopy equivalence, because M_n is aspherical. In particular,

$$\deg(\overline{f^n}), \deg(p_n) \in \{\pm 1\},$$

which leads to the absurd conclusion that $\deg(f) = \pm 1$.

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The mod 2 cohomology of the infinite families of Coxeter groups of type B and D as almost-Hopf rings

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We describe a Hopf ring structure on the direct sum of the cohomology groups $\bigoplus_{n\geq 0} H^*(W_{B_n};\mathbb{F}_2)$ of the Coxeter groups of type W_{B_n}, and an almost-Hopf ring structure on the direct sum of the cohomology groups $\bigoplus_{n\geq 0} H^*(W_{D_n};\mathbb{F}_2)$ of the Coxeter groups of type W_{D_n}, with coefficients in the field with two elements $\mathbb{F}_2$. We give presentations with generators and relations, determine additive bases and compute the Steenrod algebra action. The generators are described both in terms of a geometric construction by De Concini and Salvetti and their restriction to elementary abelian 2–subgroups.

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1 Introduction

The Coxeter groups of type W_{B_n} and W_{D_n} are two infinite families of finite reflection groups. Coxeter groups are traditionally described via Coxeter diagrams, ie graphs in which each edge e has a weight $m_e \geq 3$. Given such an object, the associated Coxeter group has a generator s_v for every vertex v, with relations of the form $s_v^2 = 1$, $(s_v s_w)^{m_e} = 1$ for every edge $e = (v, w)$, and $(s_v s_w)^2 = 1$ if v and w are not connected by an edge. For an exhaustive introduction to the geometry and topology of these groups we refer to Davis's book [3]. The reflection groups of type W_{B_n} and W_{D_n} are the finite Coxeter groups associated with the Coxeter diagrams in Figure 1.

The goal of this paper is to provide an effective description of the mod 2 cohomology of these groups. Other authors have previously computed these cohomology groups. Most notably, Swenson, in his thesis [18], adapted techniques used by Hu'ng [12] and Feshbach [5], stemming from the analysis of the restriction maps to elementary abelian 2–subgroups, to compute generators and relations for the mod 2 cohomology algebra of a finite reflection group. However, his presentation is involved and intrinsically recursive. Borrowing ideas from Giusti, Salvatore and Sinha [7; 9], we exploit additional

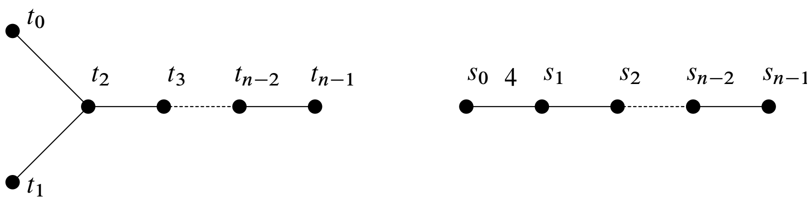

Figure 1: Diagrams of type D_n (left) and B_n (right).

structures to provide a simpler description of the cup product. Our approach also has the advantage of being more easily readable from the well-known chain-level geometric and combinatorial description of a resolution for Coxeter groups by De Concini and Salvetti [4].

The sequences of Coxeter groups of type B and D have standard embeddings

$$W_{B_n} \times W_{B_m} \to W_{B_{n+m}}, \quad W_{D_n} \times W_{D_m} \to W_{D_{n+m}}$$

that are, in a certain sense, compatible. The homomorphisms induced by these maps on mod 2 cohomology define a coproduct Δ. The cohomology transfer maps associated with them determine a product $\odot$. There is also a canonical embedding of W_{D_n} into W_{B_n} as an index-2 subgroup, which induces an involution $\iota\colon H^*(W_{D_n};\mathbb{F}_2) \to H^*(W_{D_n};\mathbb{F}_2)$.

In the B case, the resulting structure is modeled on that of the symmetric groups, the Coxeter groups of type A, as described by Giusti, Salvatore and Sinha [7] (mod 2) and by the author [10] (modulo odd primes). Together with the usual cup product $\cdot$, these maps form a ring in the category of $\mathbb{F}_2$–coalgebras, ie a Hopf ring over $\mathbb{F}_2$. More explicitly, given a ring R, a (graded) Hopf ring over R is a graded R–module with a coproduct Δ and two products, $\odot$ and $\cdot$, such that

- $(A, \Delta, \odot)$ is a Hopf algebra, with an antipode S;
- $(A, \Delta, \cdot)$ is a bialgebras over R;
- if $x, y, z \in A$ and $\Delta(x) = \sum_i x_i' \otimes x_i''$, then the distributivity formula
$$x \cdot (y \odot z) = \sum_i (-1)^{\deg(y)\deg(x_i'')} (x_i' \cdot y) \odot (x_i'' \cdot z)$$
holds.

In the D case, Δ, $\odot$ and $\cdot$ satisfy the last two axioms in the definition of a Hopf ring, and Δ and $\cdot$ form a bialgebra. However, as we will explain later, Δ and $\odot$ do not form a bialgebra. We call this weaker structure an *almost-Hopf ring* over $\mathbb{F}_2$. Due to this fact, the study of the cohomology of W_{D_n}, with the cup product, the transfer product, and the coproduct, is more complicated. The reader will find similarities between the cohomology of W_{D_n} and that of the alternating groups, as described by Giusti and Sinha [9]. Such structures stem from the seminal work of Strickland and Turner [17], in which the authors discovered a Hopf ring structure on the cohomology of symmetric groups, even with generalized cohomology theories.

The main results of this paper are Theorems 5.9 and 5.15, stated in Section 5.2, consisting of a presentation in terms of generators and relations of the mod 2 cohomology of the Coxeter groups of type B_n as a Hopf ring and of type D_n as an almost-Hopf ring respectively. We provide here self-contained statements for clarity and reference.

Theorem 5.9 (main theorem for type B) *The Hopf ring $\bigoplus_{n\geq 0} H^*(W_{B_n};\mathbb{F}_2)$ over $\mathbb{F}_2$ admits a presentation with two families of generators, $\gamma_{k,n} \in H^{n(2^k-1)}(W_{B_{n2^k}};\mathbb{F}_2)$ for $k \geq 0$ and $n > 0$, and $\delta_n \in H^n(W_{B_n};\mathbb{F}_2)$ for $n > 0$, and the following relations:*

- $\Delta(\gamma_{k,m}) = \sum_{i+j=m} \gamma_{k,i} \otimes \gamma_{k,j}$;
- $\gamma_{k,n} \odot \gamma_{k,m} = \binom{n+m}{n}\gamma_{k,n+m}$;
- $\Delta(\delta_n) = \sum_{k+l=n} \delta_k \otimes \delta_l$;
- $\delta_n \odot \delta_m = \binom{n+m}{n}\delta_{n+m}$;
- *the cup product, $\cdot$, of classes in different components is* 0;
- $\gamma_{0,n}$ *is the $\cdot$–unit of $H^*(W_{B_n};\mathbb{F}_2)$.*

The generators are explicitly characterized, both combinatorially at the cochain level (see Definition 5.1) and geometrically, as suitable Thom classes (see Proposition 5.3). The classes $\gamma_{k,n}$ and the relations among them arise from the presentation of the mod 2 cohomology of the symmetric groups as a Hopf ring. The only new generators are δ_n and their behavior is governed by the third and fourth relations above.

The almost-Hopf ring constructed from the cohomology rings of the Coxeter groups of type D is more complicated. The relations are intricate, and the behavior of generators is more easily understood with the aid of a "polarized" basis $\mathcal{B}^+ \sqcup \mathcal{B}^- \sqcup \mathcal{B}^0$ (see Proposition 5.22). For instance, the bialgebra axiom for $\odot$ and Δ is replaced with a different compatibility identity involving the projection p^+ onto the addend $(\mathrm{Span}(\mathcal{B}^+) \otimes A_D^{\otimes 3}) \oplus (\mathrm{Span}(\mathcal{B}^0) \otimes \mathrm{Span}(\mathcal{B}^+) \otimes A_D^{\otimes 2})$:

$$\Delta(x \odot y) = (\odot \otimes \odot)\tau(p^+)(\Delta(x) \otimes \Delta(y)) \quad \text{for all } x, y,$$

where τ is the transposition of the second and third factors. Nevertheless, this surrogate axiom can be expressed directly in terms of the generators, without explicit reference to the additive basis (see Proposition 5.14).

The main presentation theorem in this regard is the following.

Theorem 5.15 (main theorem for type D) *The almost-Hopf ring structure over $\mathbb{F}_2$ of $\bigoplus_{n\geq 0} H^*(W_{D_n};\mathbb{F}_2)$ extends uniquely to a graded almost-Hopf ring structure with components on the $\mathbb{F}_2$–vector space $\mathbb{F}_2 1^+ \oplus \mathbb{F}_2 1^- \oplus \bigoplus_{n\geq 1} H^*(W_{D_n};\mathbb{F}_2)$ such that*

- $1^+ \odot \_ = \mathrm{id}$, $1^- \odot \_ = \iota$, $1^+ \cdot 1^+ = 1^+$, $1^- \cdot 1^- = 1^-$, *and* $1^+ \cdot 1^- = 0$;
- $\Delta(1^\pm) = 1^+ \otimes 1^\pm + 1^- \otimes 1^\mp$;
- $\Delta(x) = 1^+ \otimes x + 1^- \otimes \iota(x) + \bar\Delta(x) + \iota(x) \otimes 1^- + x \otimes 1^+$ *for all x in* $\bigoplus_{n\geq 1} H^*(W_{D_n};\mathbb{F}_2)$, *where $\bar\Delta$ is the reduced coproduct in* $\bigoplus_{n\geq 0} H^*(W_{D_n};\mathbb{F}_2)$.

This almost-Hopf ring admits a presentation with two families of generators,

$$\begin{aligned} \gamma^+_{k,n} &\in H^{n(2^k-1)}(W_{D_{n2^k}};\mathbb{F}_2) && \text{for } k,n>0,\\ \delta^0_{n:m} &\in H^n(W_{D_{n+m}};\mathbb{F}_2) && \text{for } n\neq 1 \text{ and } m\geq 0, \end{aligned}$$

together with 1^-. The compatibility identity above and the following list of equalities provide a complete set of relations, where 1^+ is the $\odot$–unit:

- $1^- \odot 1^- = 1^+$, $1^- \cdot 1^- = 1^-$, $1^+ \cdot 1^- = 0$, and $\Delta(1^-) = 1^+ \otimes 1^- + 1^- \otimes 1^+$;
- $\Delta(\gamma^+_{k,m}) = \sum_{l=0}^m \bigl(\gamma^+_{k,l} \otimes \gamma^+_{k,m-l} + (1^- \odot \gamma^+_{k,l}) \otimes (1^- \odot \gamma^+_{k,m-l})\bigr)$;
- $\Delta(\delta^0_{n:m}) = \sum_{i=0}^n \sum_{j=0}^m \delta^0_{i:j} \otimes \delta^0_{n-i:m-j}$;
- $\gamma^+_{k,a} \odot \gamma^+_{k,b} = \binom{a+b}{a} \gamma^+_{k,a+b}$ *and* $\delta^0_{n:m} \odot 1^- = \delta^0_{n:m}$,
- $b \odot b' = 0$ *if b and b' are cup products of generators of the form* $\delta^0_{n:m}$;
- $\gamma^+_{k,n} \cdot (1^- \odot \gamma^+_{h,m}) = 0$ *for all $n,m,k \geq 1$ and $h \geq 2$*;
- $\gamma^+_{1,m} \cdot (1^- \odot \gamma^+_{1,m}) = (\gamma^+_{1,m-1})^2 \odot \delta^0_{2:0}$ *for all $m \geq 1$*;
- *the cup product, $\cdot$, of generators belonging to different components is 0*;
- $\delta^0_{0:m}$ *is the $\cdot$–product unit of the m^{th} component*;
- $\delta^0_{n:m} \cdot \gamma^+_{k,\frac{n+m}{2^k}} = \delta^0_{n:0} \cdot \gamma^+_{k,\frac{n}{2^k}} \odot \gamma^+_{k,\frac{m}{2^k}}$ *for all $k>0$ and $m,n\geq 0$ with $n \neq 1$.*

In this case, too, the generators are explicitly described (see Definitions 5.4 and 5.5).

The relations are spread out in a few lemmas to prove the identities concerning coproduct, transfer product, and cup product separately. Building on these core theorems, we also describe convenient additive bases for the cohomology of these groups, with a graphical description via skyline diagrams similar to that obtained for the symmetric group in [7], and compute the Steenrod algebra action. Our formulation of the cohomology of W_{B_n} and W_{D_n} yields without additional effort many features of these cohomology algebras. For instance, Hepworth's homological stability results [11] in these particular cases follow directly.

We obtain our presentation via three technical tools. First, we exploit De Concini and Salvetti's geometric combinatorial model to realize such (almost) Hopf rings structures at the cochain level. Specializing their construction to the families of groups of our interest, we observe that a resolution for W_{B_n} is obtained from the symmetrized version of the planar level trees used by Giusti and Sinha [9] for the symmetric groups. The cohomology of W_{D_n} is governed by an oriented version of these objects. We describe cochain representatives of the structural maps in detail. Our treatment follows the paper cited above closely. However, we note that while the transfer product is realized very similarly to the Σ_n case, coproducts are more complicated and require the combinatorial operation of "pruning" symmetric planar level trees. This cochain-level description allows us to quickly retrieve some of our relations and give a more geometric flavor to our generating classes. For instance, they can be interpreted as Thom classes in a suitable sense.

Second, we use the existence of well-behaved maps between W_{B_n}, W_{D_n} and Σ_n. These homomorphisms preserve parts of our structures. Therefore, we exploit them to build our presentations on the known result for the cohomology of the symmetric groups. We provide a cochain-level description of these morphisms, and we determine both their action on generators and their relations to the coproduct and transfer product.

Third, we reconcile with Swenson's approach, and we investigate restrictions to elementary abelian 2–subgroups. The mod 2 cohomology of finite reflection groups is known to be detected by this family of subgroups. We effectively compute the action of these restriction maps on our additive bases. The multiplicative structure on the cohomology of (the invariant subalgebras of) such subgroups is known. Thus, these calculations allow us to deduce cup product relations that would be otherwise difficult to obtain.

We organize the paper as follows. After describing the structures on the cohomology of W_{B_n} and W_{D_n} in Section 2, we devote the following two sections to developing our geometric tools. In Section 3, we review De Concini and Salvetti's construction, and we specialize it to W_{B_n} and W_{D_n}. In Section 4, we investigate the combinatorics of pruning operations, and we retrieve cochain-level representatives of our structural and connecting homomorphisms. Section 5 is devoted to our main theorems. We define generators and we discuss relations between them. In this context, we also deduce from our presentation additive bases, and we discuss the relations between the cohomology of Coxeter groups of type A, B and C. We postpone the proofs of the presentation theorem and some cup product relations. In Section 6, we turn our attention to the restriction to elementary abelian 2–subgroups. We review relevant results from Swenson's thesis, compute restriction maps, and use them to complete the proof of our cup-product relations. Section 7 is devoted to completing the proof of our main theorems. In Section 8, we calculate the Steenrod algebra action.

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2 (Almost) Hopf ring structures for the cohomology of W_{B_n} and W_{D_n}

We begin this paper by describing in detail how the desired algebraic structures on the cohomology of Coxeter groups of type B and D are obtained. Throughout this paper, we use several combinatorial descriptions of the groups W_{B_n} and W_{D_n}. We refer to [2, Chapter 8] for a thorough treatment, and we recall below what we need for our purposes.

With reference to Figure 1, we recall that there is an inclusion $j_n\colon W_{D_n} \hookrightarrow W_{B_n}$ defined by $t_0 \mapsto s_0 s_1 s_0$ and $t_i \mapsto s_i$ if $i > 0$. W_{B_n} can be seen as the group of signed permutation on n numbers, that is, the group of bijective functions f from the set $\{-n, \dots, -1, 1, \dots, n\}$ into itself that satisfy $f(-i) = -f(i)$ for every $1 \le i \le n$. Hence W_{B_n} is naturally a subgroup of Σ_{2n}, the symmetric group on $2n$ objects. The image of j_n is $W_{B_n} \cap \mathrm{Alt}(2n)$, the intersection of W_{B_n} with the alternating group $\mathrm{Alt}(2n)$, the subgroup of even permutations in Σ_{2n}. Note that Σ_n can be identified with the parabolic subgroup of W_{B_n} generated by $s_1, \dots, s_{n-1}$, corresponding to the signed permutations on $\{-n, \dots, n\}$ that preserve signs. There is also a standard projection $W_{B_n} \to \Sigma_n$, of which the previous inclusion is a section, whose kernel is the normal subgroup generated by s_0. We observe that this provides an isomorphism between W_{B_n} and the wreath product $\mathbb{F}_2 \wr \Sigma_n$, a semidirect product of $\mathbb{F}_2^n$ and Σ_n. Therefore, the inclusions $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$ extend naturally to monomorphisms $W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$. These inclusions are associative and commutative up to conjugation.

Let $A_B = \bigoplus_{n \geq 0} H^*(W_{B_n}; \mathbb{F}_2)$. We define a coproduct Δ and two products, $\cdot$ and $\odot$, on A_B in the following way:

- Δ is induced by the obvious monomorphisms $W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$;
- $\odot$ is induced by the cohomology transfer maps associated with these inclusions;
- $\cdot$ is the usual cup product.

Due to the associativity and the commutativity of the natural inclusions, these morphisms define an almost-Hopf ring structure. This is a general fact, as noticed in [9]. In this case, however, A_B is a full Hopf ring.

Proposition 2.1 *A_B, with these structural morphisms, is a Hopf ring.*

Proof The almost-Hopf ring axioms hold by [9, Theorem 2.3]. It remains only to prove that $(A_B, \Delta, \odot)$ forms a bialgebra. This claim follows from the fact — compare with [7, Section 3] — that this diagram is a pullback of finite coverings for all $n, m \in \mathbb{N}$,

$$\begin{tikzcd}
\bigsqcup_{\substack{p+q=n \\ r+s=m}} \dfrac{E(W_{B_{n+m}})}{W_{B_p} \times W_{B_q} \times W_{B_r} \times W_{B_s}} \arrow[r, "\bigsqcup \pi_{p+1,r+s}"] \arrow[d, "\bigsqcup \pi_{p+r,q+s}"'] & \dfrac{E(W_{B_{n+m}})}{W_{B_n} \times W_{B_m}} \arrow[d, "\pi_{n,m}"] \\
\bigsqcup_{k+l=n+m} \dfrac{E(W_{B_{n+m}})}{W_{B_k} \times W_{B_l}} \arrow[r, "\bigsqcup \pi_{k,l}"'] & \dfrac{E(W_{B_{n+m}})}{W_{B_{n+m}}}
\end{tikzcd}$$

where π indicates the projections. □

We remark that, since A_B with Δ and $\odot$ is a conilpotent bialgebra, the existence of the antipode comes for free. This antipodal morphism does not play a role in our treatment; thus, we will not discuss it further.

Similarly, we can construct an additional almost-Hopf ring structure on the cohomology of the Coxeter groups of type D_n. Indeed, on the direct sum $A_D = \bigoplus_{n \geq 0} H^*(W_{D_n}; \mathbb{F}_2)$, we can define a coproduct Δ and two products $\odot$ and $\cdot$ as done for A_B. However, these do not make A_D a full Hopf ring because, as we will see later, $(A_D, \Delta, \odot)$ fails to be a bialgebra.

With essentially the same proof used for A_B, we can prove the following easy proposition, which follows again from [9, Theorem 2.3].

Proposition 2.2 *A_D, with the coproduct and the two products defined before, is an almost-Hopf ring over $\mathbb{F}_2$.*

As we remarked in the introduction, there is a similar result for the mod 2 cohomology of the symmetric groups, obtained by Giusti, Salvatore and Sinha in [7]. We recall their statement here because we will build our computations upon it.

Theorem 2.3 [7, Theorems 1.2 and 3.2] *$A_\Sigma = \bigoplus_{n\geq 0} H^*(\Sigma_n;\mathbb{F}_2)$, together with a coproduct $\Delta\colon A_\Sigma \to A_\Sigma \otimes A_\Sigma$ induced by the obvious inclusions $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, a product $\odot\colon A_\Sigma \otimes A_\Sigma \to A_\Sigma$ given by the transfer maps associated with these inclusions, and a second product $\cdot\colon A_\Sigma \otimes A_\Sigma \to A_\Sigma$ defined as the usual cup product, is a Hopf ring over $\mathbb{F}_2$.*

A_Σ is generated, as a Hopf ring, by classes $\gamma_{k,n} \in H^{n(2^k-1)}(\Sigma_{n2^k};\mathbb{F}_2)$ for $k \geq 0$ and $n \geq 1$. The coproduct of these classes is given by the formula

$$\Delta(\gamma_{k,n}) = \sum_{l=0}^{n} \gamma_{k,l} \otimes \gamma_{k,n-l},$$

the cup product of generators belonging to different components is 0, and

$$\gamma_{k,n} \odot \gamma_{k,m} = \binom{n+m}{n} \gamma_{k,n+m}.$$

There are no more relations between these classes.

The unit of the algebra $H^*(\Sigma_n;\mathbb{F}_2)$ under the cup product is $\gamma_{0,n} \in H^0(\Sigma_n;\mathbb{F}_2)$. For this reason, we will often denote it with the symbol 1_n throughout the paper.

3 Review of a geometric construction of De Concini and Salvetti and Fox–Neuwirth type cell structures

3.1 De Concini and Salvetti resolution

In this section, we recall a geometric construction introduced by De Concini and Salvetti in [4], which we will require to describe the generators of the Hopf ring under consideration.

Given a finite reflection group $G \le \mathrm{Gl}_n(\mathbb{R})$, there is a natural hyperplane arrangement $\mathcal{A}_G$ in $\mathbb{R}^n$ associated with G, whose hyperplanes are the fixed points sets of reflections in G. The choice of a fundamental chamber C_0 of $\mathcal{A}_G$ gives rise to a Coxeter presentation (G, S) for G, whose set of generators S is composed by reflections with respect to hyperplanes that are supports of a face of C_0. Every finite Coxeter group arises this way.

For any $F \subseteq \mathbb{R}^n$, we can define

$$\mathcal{A}_F = \{H \in \mathcal{A}_G : F \subseteq H\}.$$

$\mathcal{A}_F$ gives rise to a stratification $\Phi(\mathcal{A}_F)$ of $\mathbb{R}^n$, in which the strata are the connected components of sets of the form $L \setminus \bigcup_{H \in \mathcal{A}_F, H \not\supseteq L} H$, where L is the intersection of some of the hyperplanes of $\mathcal{A}_F$. Let $\mathbb{R}^\infty$ be the direct limit of $\mathbb{R}^m$ under the inclusions $\mathbb{R}^m \hookrightarrow \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^{m+1}$. For all $m \in \mathbb{N} \cup \{\infty\}$, there is a stratification Φ_m (different from the product stratification) of the topological space,

$$Y_G^{(m)} = \mathbb{R}^n \otimes \mathbb{R}^m \setminus \bigcup_{H \in \mathcal{A}_G} (H \otimes \mathbb{R}^m) = (\mathbb{R}^n)^m \setminus \bigcup_{H \in \mathcal{A}_G} H^m.$$

The strata in Φ_m are defined as sets of the form $F_1 \times \cdots \times F_k \times \cdots$, with $F_k \in \Phi(\mathcal{A}_{F_{k-1}})$ for $k \ge 1$. Here we put, by convention, $F_0 = \{0\}$. In what follows, if there is no ambiguity, we will use the simpler notations $Y^{(m)}$ and Y to indicate $Y_G^{(m)}$ and $Y_G^{(\infty)}$ respectively.

De Concini and Salvetti construct a regular G–equivariant CW–complex $X \subseteq Y$ that is "dual" to the stratification Φ_∞, in the sense that for every stratum $F \in \Phi_\infty$ of codimension d, there exist a unique d–dimensional cell in X that intersects F, and they intersect transversally in a single point. For $m < \infty$, the intersection $X^{(m)}$ of X with $Y^{(m)}$ is a subcomplex of X whose cells are dual to strata in Φ_m. This construction is done equivariantly, in the sense that for every stratum $F \in \Phi_\infty$ and every $g \in G$, if $\varphi\colon D^d \to X$ is the cell dual to F in X, then $(g.\_) \circ \varphi\colon D^d \to X$ is the cell dual to $g.F$. The authors then show that X is a G–equivariant strong deformation retract of Y. Since Y is contractible and G–free, the quotient X/G is a cellular model for the classifying space $B(G)$ and the cellular chain complex $C_*^G = C_*^{\mathrm{CW}}(X)$ is a $\mathbb{Z}[G]$–free resolution of $\mathbb{Z}$.

The strata of Φ_∞ have a more compact combinatorial description in terms of the Coxeter presentation. For every $s \in S$ generating reflection for G, we let H_s be the hyperplane fixed by s. H_s divides the space $\mathbb{R}^n$ into two semispaces, H_s^+ and H_s^-. We let H_s^+ be the semispace that contains the chosen fundamental chamber C_0. To a flag $\underline{\Gamma} = (S \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k = \varnothing)$ of subsets of S we can associate a stratum F of Φ_∞ such that $x = (x_1, \dots, x_n) \in (\mathbb{R}^\infty)^n$ belongs to F if and only if

$$((x_1)_r, \dots, (x_n)_r) \in H_s \quad \text{if } s \in \Gamma_r,$$
$$((x_1)_r, \dots, (x_n)_r) \in H_s^+ \quad \text{if } s \in \Gamma_{r-1} \setminus \Gamma_r$$

is satisfied for every $s \in S$ and every $r \geq 1$. Thus, to a couple $(\underline{\Gamma}, g)$, where $\underline{\Gamma}$ is a flag as before and $g \in G$, we can associate the stratum $g.F$ obtained from the above F by applying g. This construction yields an algebraic-combinatorial description of the cellular chain complex of X. The main theorem of De Concini and Salvetti's paper is the following.

Theorem 3.1 [4, Section 3] *Let (G, S) be a finite Coxeter group, and consider the set*

$$\{(\underline{\Gamma}, \gamma) \mid \gamma \in G, \underline{\Gamma} = (\Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k \supseteq \cdots), \Gamma_1 \subseteq S, \Gamma_k = \varnothing \text{ for some } k\}.$$

The function described above is a bijection between this set and the set of strata in Φ_∞ (and thus, by duality, with the set of cells in X). The codimension of the stratum (and the dimension of the corresponding dual cell) associated with $(\underline{\Gamma}, \gamma)$ is equal to $\sum_{r=1}^\infty |\Gamma_r|$, and the action of an element $g \in G$ on strata and cells corresponds to the function $(\underline{\Gamma}, \gamma) \mapsto (\underline{\Gamma}, g\gamma)$.

Let $c(\underline{\Gamma}, \gamma)$ be the cell dual to the stratum corresponding to $(\underline{\Gamma}, \gamma)$. The boundary homomorphism in $C_*^{\mathrm{CW}}(X)$ is given by the formula

$$\partial c(\underline{\Gamma}, \gamma) = \sum_{i \geq 1} \sum_{\tau \in \Gamma_i} \sum_{\substack{\beta \in W_{\Gamma_i}^{\Gamma_i \setminus \{\tau\}} \\ \beta^{-1}\Gamma_{i+1}\beta \subseteq \Gamma_i \setminus \{\tau\}}} (-1)^{\alpha(\underline{\Gamma}, i, \tau, \beta)} c(\underline{\Gamma}', \gamma\beta),$$

where α is an integer number easily computed in terms of $\underline{\Gamma}$, i, τ, β, $\Gamma_k' = \Gamma_k$ for $k < i$, $\Gamma_i' = \Gamma_i \setminus \{\tau\}$ and $\Gamma_k' = \beta^{-1}\Gamma_k\beta$ if $k > i$, and $W_{T'}^T$, for $T' \subseteq T \subseteq S$ is the set of minimal length coset representatives for the parabolic subgroup $W_{T'}$ in W_T.

We remark that in the case of Coxeter groups of type B or D, minimal coset representatives are explicitly known. For a complete description, we refer, for instance, to [14].

3.2 Alexander duality and Fox–Neuwirth complexes

We recall an alternative description of C_*^G. This description has been exposed in [8], where it is investigated in much detail in the A_n case. As observed in that paper, for every $1 \leq m \leq \infty$, the strata of Φ_m are the interiors of cells in a G–equivariant cell structure on the Alexandroff compactification $(Y^{(m)})^+ = Y^{(m)} \cup \{\infty\}$.

Denote its augmented (G–equivariant) cellular chain complex with the symbol $\widetilde{\mathrm{FN}}_G^m$. Its cells are the closures $e(F)$ of strata $F \in \Phi_m$ (together with the basepoint $\{*\}$) and, from the construction of $X^{(m)}$ as a CW–complex dual to Φ_m — details in [4] — $e(F)$ is contained in the boundary of $e(F')$ if and only if the cell of X dual to F contains the cell dual to F' in its boundary. This fact implies that the complex $\widetilde{\mathrm{FN}}_G^m$ is, up to a shift of degrees, the dual of $C_*^{\mathrm{CW}}(X^{(m)})$, at least modulo 2 (in general, there are differences in some signs due to orientations). Explicitly, the closure in $\widetilde{\mathrm{FN}}_G^m$ of a stratum of dimension d correspond to the dual of a chain in $C_*^{\mathrm{CW}}(X^{(m)})$ of dimension $nm-d$. In the remaining sections of this paper, we will always implicitly assume this shift, and we will grade $\widetilde{\mathrm{FN}}_G^m$ to match the corresponding dimension of the dual cell.

In particular, $\widetilde{\mathrm{FN}}_G^m$ calculates the cohomology of $Y^{(m)}$ and is therefore acyclic up to dimension $nm-2$. Alternatively, we can see this, as explained in [8], by observing that the Atiyah duality theorem implies that the Spanier dual of $Y^{(m)}$ is $(Y^{(m)})^+$.

Passing to the limit for $m \to \infty$, we obtain an acyclic $\mathbb{F}_2$–complex $\widetilde{\mathrm{FN}}_G \otimes \mathbb{F}_2$, dual to $C_*^{\mathrm{CW}}(X) \otimes \mathbb{F}_2$, for which a basis $\{e(S)\}_{S \in \Phi_\infty}$ is parametrized by strata in Φ_∞. The degree of $e(S)$ as a cochain of X is equal to the codimension of F. This is an equivariant cochain model for $E(G)$. In particular, the quotient $\mathrm{FN}_G \otimes \mathbb{F}_2 = \widetilde{\mathrm{FN}}_G / G \otimes \mathbb{F}_2$ calculates $\widetilde{H}^*(G; \mathbb{F}_2)$. In the following, when we need to stress the Coxeter group G involved, we will use the heavier notation $\Phi_{\infty,G}$ instead of Φ_∞.

This description of the cochain complex FN_G calculating the cohomology of G fits particularly well with a chain-level interpretation of duality via intersection theory that we will occasionally use in proofs and that we briefly recall here. Given a manifold X and an immersion $i : W \to X$ of a codimension d manifold in X, we say that a smooth singular chain in X is transverse to i if, for every simplex $\sigma : \Delta^k \to X$ of the chain, σ is transverse on every face of Δ^k and subface, in the sense of manifolds with corners. It can be proved that the subcomplex consisting of chains that are transverse to i is chain equivalent to the full one. To every d–dimensional singular simplex $\sigma : \Delta^d \to X$ transverse to i we can associate the element $\tau_W(\sigma) \in \mathbb{F}_2$ given by the mod 2 cardinality of $\sigma^{-1}(W)$. This procedure defines a cochain dual in the complex dual to the chain complex of singular chains transverse to i. If i is a proper embedding, τ_W is a cocycle and defines a cohomology class. The most important constructions in cohomology can be understood geometrically using this model. In particular, if $f : Y \to X$ is transverse to i, then $f^{\#}(\tau_W) = \tau_{f^{-1}(W)}$. The reader will find a complete reference of this approach to cohomology in [6].

In our particular context, each stratum $S \in \Phi_\infty$ defines such a cochain τ_S. We understand the coboundary of τ_S as $\tau_{\partial(S)}$, so we can identify $\widetilde{\mathrm{FN}}^*_G$, at least modulo 2, with the cochain complex spanned by τ_S for strata $S \in \Phi_\infty$. Suppose $W \subseteq Y_G^{(\infty)}$ is a proper submanifold of codimension d obtained as a union of strata. In that case, its associated cochain τ_W is the sum of τ_S for strata $S \subseteq W$ of minimal codimension, and $\delta(\tau_W) = 0$. If, in addition, the action of G preserves W, then, passing to the quotient, its image $\overline{W} \subseteq Y_G^{(\infty)}/G$ defines a Thom class represented in FN^*_G by the sum of strata contained in $\overline{W}$. This construction is made precise in [7, Definition 4.6].

3.3 The special case of Coxeter groups of type B

We conclude this section by further investigating the cases of our interest $G = W_{B_n}$ and, in the following subsection, $G = W_{D_n}$. The strata of Φ_m for the symmetric group Σ_n can be described in terms of leveled trees, as shown in [8] using ideas dating back to Vassiliev [19]. A straightforward adaptation of these ideas shows that, in the case of the Coxeter groups of type B_n, we can describe them in terms of symmetric leveled trees. This interpretation encodes geometrically and combinatorially the structure of W_{B_n} as a wreath product of Σ_n with a cyclic group of order 2. Below we provide the precise definitions.

First, we observe that, since W_{B_n} is generated by a set $S = \{s_0, \dots, s_{n-1}\}$ of n fundamental reflections as described in Figure 1, the Fox–Neuwirth complex $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ has a $\mathbb{Z}[W_{B_n}]$–basis $\{e(\underline{a})\}$ indexed by n–tuples of nonnegative integer numbers $(a_0, \dots, a_{n-1})$.

The reflection hyperplane arrangement associated with W_{B_n} can be described as $\mathcal{A}_{B_n} = \{H^{\pm}_{i,j}\}_{1 \le i < j \le n} \cup \{H^0_i\}$, where

$$H^{\pm}_{i,j} = \{\underline{x} \in \mathbb{R}^n \mid x_i = \pm x_j\}, \quad H^0_i = \{\underline{x} \in \mathbb{R}^n \mid x_i = 0\}.$$

Moreover, s_0 can be identified with the reflection with respect to H^0_1 and, for every $i > 0$, s_i with the reflection with respect to $H^+_{i,i+1}$. Thus the basis element corresponding to $\underline{a} = (a_0, \dots, a_{n-1})$ is described as the stratum

$$e(\underline{a}) = \big\{(\underline{x}_1, \dots, \underline{x}_n) \in (\mathbb{R}^\infty)^n \mid \forall 1 \le i \le n-1, \forall 1 \le j \le a_i : (x_i)_j = (x_{i+1})_j, \\ (x_i)_{a_i+1} < (x_{i+1})_{a_i+1}, \forall 1 \le k \le a_0 : (x_1)_k = 0, (x_1)_{a_0+1} > 0\big\}.$$

Passing to the quotient by the action of W_{B_n}, we see that $\mathrm{FN}^*_{W_{B_n}}$ has a $\mathbb{Z}$–basis constituted by $[a_0 : \cdots : a_{n-1}] = [e(a_0, \ldots, a_{n-1})]$.

The differential on $\mathrm{FN}^*_{W_{B_n}}$ is complicated, but it is combinatorially accessible via a description of its basis in terms of trees.

Definition 3.2 A *signed depth-ordering* is a sequence of labeled inequalities of the form $\Gamma = (0 <_{a_0} i_1 <_{a_1} \cdots <_{a_{n-1}} i_n)$, where $i_k \in \{-n, \ldots, -1, 1, \ldots, n\}$ for all $1 \leq k \leq n$, and these indices have pairwise different absolute values. By convention, we let $i_0 = 0$.

A *planar level tree* is a planarly embedded tree T satisfying the following conditions:

- it has a root vertex embedded in $(0,0)$ and all the other vertices having their second coordinate (the "height") equal to a positive integer;
- two edges connected by an edge have heights whose difference is 1;
- the height along the unique minimal path from the root to every leaf is always increasing.

A *planar level tree with labels in I* is a couple (T, λ) defined as follows: T is a planar level tree, and λ is a bijective labeling of the leaves of T with elements of I.

A *symmetric planar level tree* is a planar level tree invariant under the reflection r along the $\boldsymbol{y}$–axis and having an odd number of leaves.

An *antisymmetric planar level tree with labels in $\{-n, \ldots, n\}$* is a labeled planar level tree (T, λ) with labels in $\{-n, \ldots, n\}$ such that T is symmetric, and two leaves that correspond to each other under the application of r have labels opposite to each other.

The antisymmetric planar level tree associated with a depth ordering Γ is the antisymmetric planar level tree T_Γ, unique up to isotopy, defined by the following properties:

- the labels of the leaves, from left to right, are $-i_n, \ldots, -i_1, 0, i_1, \ldots, i_n$;
- the leaves labeled i_{k-1}, i_k, for $1 \leq k \leq n$, are separated by a vertex of height a_k but not by vertices of height less than a_k.

Let $k \geq 0$. The *k–symmetrization* $S_k(T)$ (resp. $\widetilde{S}_k(T)$) of a planar level tree T (with labels in $\{1, \ldots, n\}$) is a symmetric planar level tree S (resp. antisymmetric planar level tree with labels in $\{-n, \ldots, n\}$) obtained by the following procedure. Glue T from the right to a vertical linear planar level tree lying into the $\boldsymbol{y}$–axis up to height k.

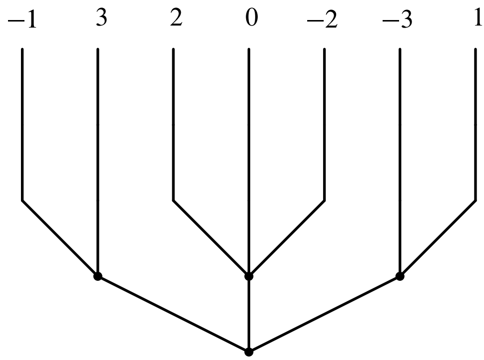

Figure 2: An example of antisymmetric planar level tree with labels in $[-3,3]$.

Then, add the mirror image of such tree under r to obtain a symmetric planar level tree (choosing the unique antisymmetric labeling that extends the labeling of T in the labeled case).

There is a free action of W_{B_n} on the set antisymmetric planar level trees with labels in $\{-n,\dots,n\}$ given by interpreting elements of W_{B_n} as signed permutations and permuting labels accordingly. We always assume that the edges of a level tree are oriented so that there is a unique oriented path from the root vertex to each leaf.

Similarly to the symmetric group case, we have the following immediate proposition.

Proposition 3.3 *The function $\Gamma \mapsto T_\Gamma$ is a bijection between the set of signed depth-orderings with n labels and the set of isotopy classes of antisymmetric planar level trees with labels in $\{-n,\dots,n\}$. Furthermore, to $\Gamma = (0 <_{a_0} i_1 <_{a_1} \cdots <_{a_{n-1}} i_n)$ is associated a stratum $\sigma e(\underline{a}) \in \Phi_{\infty,W_{B_n}}$, where $\sigma(k) = i_k$, $\underline{a} = (a_0,\dots,a_{n-1})$, and this provides a W_{B_n}–equivariant additive basis of $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ labeled by signed depth-orderings or, equivalently, by isotopy classes of antisymmetric planar level trees with labels in $\{-n,\dots,n\}$. W_{B_n} acts on this basis by permuting labels. Consequently, an additive basis for $\mathrm{FN}^*_{W_{B_n}}$ is given by symmetric planar level trees with $2n+1$ leaves.*

An example of an antisymmetric planar level tree (T,λ), with labels in $[-3,3]$, is given in Figure 2. The associated signed depth-ordering is $\Gamma = (0 <_1 -2 <_0 -3 <_1 1)$ and the corresponding stratum is $\sigma e([1,0,1])$, where $\sigma(1) = -2$, $\sigma(2) = -3$ and $\sigma(3) = 1$.

We observe that we can use Proposition 3.3 to reinterpret operations on (symmetric) level trees in terms of n–tuples or (signed) depth-orderings. For instance, the k–

symmetrization of trees provides a linear map $S_k\colon \mathrm{FN}^*_{\Sigma_n} \to \mathrm{FN}^*_{W_{B_n}}$ that we can interpret as $[a_1 : \cdots : a_{n-1}] \mapsto [k : a_1 : \cdots : a_{n-1}]$.

We can now describe the differential in terms of this basis.

Definition 3.4 [8] Let (T, λ) be a planar level tree. Let v be an internal vertex. Let $E(v)$ be the set of edges whose source vertex is v. The planar embedding of T induces an order on $E(v)$, defined from left to right. A vertex permutation of (T, λ) at v is another planar level tree that is isomorphic to (T, λ) as a labeled tree but with a different planar embedding that differs from the original one only by the ordering on $E(v)$.

Given a planar level tree (T, λ) and an internal vertex v, let (e, f), with $e < f$, be a couple of adjacent edges in $E(v)$. Let u_e and u_f be the targets of e and f, respectively. Let σ be a shuffle of the two sets $E(u_e)$ and $E(u_f)$. Let $d_{e,f,\sigma}(T, \lambda)$ be the planar level tree obtained by gluing together e and f, with common target $\bar{u}$, and then applying the vertex permutation that permutes the edges in $E(\bar{u})$ by σ.

Recall that, in the A_n case, the differential in $\widetilde{\mathrm{FN}}^*$ of the basis element corresponding to a planar level tree with labels (T, λ) is given by the sum over (v, σ) as above of $d_{v,\sigma}(T, \lambda)$. Similarly, we have the following proposition, which essentially states that a symmetrization of the previous construction gives the differential in the B_n case.

Proposition 3.5 *With the correspondence provided by Proposition 3.3, the differential of the cochain complex $\widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \mathbb{F}_2$ is given in terms of antisymmetric level trees with labels $d(T, \lambda) = \sum_{(e,f,\sigma)} \sum_{(e',f',\tau)} d_{e,f,\sigma} d_{e',f',\tau}(T, \lambda)$, where the sum is over sextuples $(e, f, \sigma, e', f', \tau)$ such that $d_{e,f,\sigma} d_{e',f',\tau}(T, \lambda)$ is again an antisymmetric planar level tree. Equivalently, $d(T, \lambda)$ is obtained by performing an operation $d_{e,f,\sigma}$ starting from a couple of adjacent vertices (e, f) lying into the positive half-plane $\{(x, y) \mid x \geq 0\}$, and then perform the mirror operation on the mirror pair of adjacent edges (e', f') in the negative half-plane. If we call such symmetric operation $d^S_{e,f,\sigma}$, we have that*

$$d(T, \lambda) = \sum_{(e,f)} \sum_{\sigma} d^S_{e,f,\sigma}(T, \lambda),$$

where the sum is over couples (e, f) of adjacent edges in the positive half-plane and shuffles σ of the two sets of vertices incident to the target of e and f, respectively.

We can equivalently express this construction using planar level trees T with $n+1$ leaves labeled by $(-n, \ldots, -1, 0, 1, \ldots, n)$, with labels having pairwise different absolute values, such that the leftmost leaf has label 0. We recover the corresponding antisymmetric level tree as follows. We choose a representative of the isotopy class of (T,λ) in which the entire oriented path from the root vertex to the label 0 lies on the $\boldsymbol{y}$–axis. Then we merge T with its mirror image along $\boldsymbol{y}$ with opposite labels. In this case, the differential is given by contracting a couple of adjacent edges and shuffling. When the result is a tree whose leftmost leaf is not labeled by 0, we replace the part of the tree belonging to the negative half-plane $\{(x,y) \mid x \leq 0\}$ with its mirror image in the positive half-plane, with opposite labels, and shuffle the corresponding edges in all possible ways.

3.4 The special case of Coxeter groups of type D

We now turn to the description of the complex $\mathrm{FN}^*_{W_{D_n}}$. Once again, since this Coxeter group has n fundamental reflections $t_0,\dots,t_{n-1}$, a $\mathbb{Z}[W_{D_n}]$–basis for $\widetilde{\mathrm{FN}}^*_{W_{D_n}}$ is indexed by n–tuples of nonnegative integers $\underline{a}=(a_0,\dots,a_{n-1})$.

The inclusion $j_n\colon W_{D_n}\to W_{B_n}$ identifies the reflection arrangement associated to W_{D_n} with the subarrangement of $\mathcal{A}_{W_{B_n}}$ composed by the hyperplanes $H^{\pm}_{i,j}$, for $1\leq i<j\leq n$, and $t_i=s_i$ for $1\leq i\leq n$, while t_0 is the reflection along $H^-_{1,2}$. Thus the basis element of $\widetilde{\mathrm{FN}}^*_{W_{D_n}}$ corresponding to $\underline{a}$ is described as the stratum

$$e(\underline{a})=\big\{(\underline{x}_1,\dots,\underline{x}_n)\in(\mathbb{R}^\infty)^n \mid \forall 1\leq i\leq n-1, 1\leq j\leq a_i:(x_i)_j=(x_{i+1})_j,\\ (x_i)_{a_i+1}<(x_{i+1})_{a_i+1}, \forall 1\leq k\leq a_0:(x_2)_k=-(x_1)_k,(x_2)_{a_0+1}>-(x_1)_{a_0+1}\big\}.$$

Passing to the quotient by the action of W_{D_n}, we see that $\mathrm{FN}^*_{W_{D_n}}$ has a $\mathbb{Z}$–basis constituted by $[a_0:\dots:a_{n-1}]=[e(a_0,\dots,a_{n-1})]$.

The complex $\widetilde{\mathrm{FN}}^*_{W_{B_n}}/j_n(W_{D_n})$ also calculates the cohomology of W_{D_n}. Therefore, there is a cochain equivalence $\varphi\colon \mathrm{FN}^*_{W_{D_n}}\to\widetilde{\mathrm{FN}}^*_{W_{B_n}}/j_n(W_{D_n})$ between the two resolutions. In the subsequent section, we compute an explicit formula for φ that we will use to perform cochain-level computation in the following sections. For instance, we will prove the relations for coproduct of transfer products of Hopf ring generators by mapping them to $\widetilde{\mathrm{FN}}^*_{W_{B_n}}/j_n(W_{D_n})$, where their expressions are closer to the B_n case. As a notational convention, we denote this cochain complex by ${\mathrm{FN}'}^*_{W_{D_n}}$.

First, we observe that $[W_{B_n}:j_n(W_{D_n})]=2$; thus $j_n(W_{D_n})$ is a normal subgroup of W_{B_n}. The two cosets of $j_n(W_{D_n})$ in W_{B_n} are represented by the identity and s_0, the only fundamental reflection of W_{B_n} that is not contained in $j_n(W_{D_n})$. Thus, given a $\mathbb{Z}[W_{B_n}]$–basis $\mathcal{B}$ for $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$, the classes of x and $s_0.x$, where $x\in\mathcal{B}$, provide a $\mathbb{Z}$–basis for ${\mathrm{FN}'}^*_{W_{D_n}}$. Let $\mathcal{B}$ be the basis defined above in terms of n–tuples or equivalently of symmetric planar level trees, parametrized by n–tuples of nonnegative integers $\underline{a} = (a_0, \dots, a_{n-1})$. We denote by $[a_0 : \cdots : a_{n-1}]^+$ and $[a_0 : \cdots : a_{n-1}]^-$ the cochains in $\mathrm{FN}'^*_{W_{D_n}}$ arising from the basis element corresponding to $\underline{a}$ and $s_0\underline{a}$.

The complex $\mathrm{FN}'^*_{W_{D_n}}$ also has a description in terms of trees.

Definition 3.6 Let T be a symmetric planar level tree with $2n+1$ leaves. An *orientation* of T is the choice of an element of $L/\sim$, where L is the set of antisymmetric labelings of T with labels in $\{-n, \dots, n\}$, and $\sim$ is the equivalence relation defined by

$$\lambda \sim \lambda' \quad \iff \quad \lambda' = \lambda\sigma \quad \text{for some } \sigma \in \mathrm{Alt}(2n+1).$$

An *oriented symmetric planar level tree* is a couple $(T, \mathcal{O})$, where T is a symmetric planar level tree and $\mathcal{O}$ is an orientation of T.

Note that if two antisymmetric labelings of a symmetric planar level tree T differ by a permutation $\sigma \in \Sigma_{\{-n,\dots,n\}}$, then σ must fix 0 and act as a signed permutation on $\{-n, \dots, -1, 1, \dots, n\}$. Hence, an orientation of T is the choice of an antisymmetric labeling up to the action of $j_n(W_{D_n})$. Since the index $[W_{B_n} : j_n(W_{D_n})]$ is 2, there are two possible orientations for a symmetric planar level tree T, determined by the parity of the number of negative labels of leaves in the positive half-plane. In particular, we can identify an orientation $\mathcal{O}$ with a sign $+$ or $-$, corresponding to labelings with an even or odd number of positively labeled leaves in the positive half-plane, respectively.

Moreover, from the fact that $\mathrm{Alt}(2n+1)$ is normal in Σ_{2n+1}, it follows that if T is a symmetric planar level tree, λ is a labeling of T and $\sigma(T)$ is a vertex permutation of T at a vertex v, then the orientation of the permuted labeled tree $\sigma(T, \lambda)$ only depends on the orientation determined by λ. Therefore, the rule for the differential in $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ induces a formula for the differential in $\mathrm{FN}'^*_{W_{D_n}}$ in terms of trees. Hence, we have the following description.

Proposition 3.7 $\mathrm{FN}'^*_{W_{D_n}}$ *can be described as the cochain complex having additive basis indexed by oriented symmetric planar level trees with $2n+1$ leaves, with differential induced by the symmetric tree differential in $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ by keeping track of orientations.*

The reader is encouraged to compare this description with the notion of "charged" configuration used in [9].

4 Geometry and combinatorics: chain-level formulas

We devote this section to developing some formulas that will allow us to perform calculations at the (co)chain level. These computations will be needed at points, especially when retrieving relations. We first compute some connecting maps between the Fox–Neuwirth complexes of Coxeter groups of type A, B and D. Then, we provide cochain representatives of the structural maps of our almost-Hopf ring structures.

4.1 The connecting homomorphisms

As $\widetilde{\mathrm{FN}}^*_{W_{D_n}}$ and $\widetilde{\mathrm{FN}}'^*_{W_{D_n}}$ are both free resolutions of $\mathbb{Z}$ as a $\mathbb{Z}[W_{D_n}]$–module, they need to be W_{D_n}–equivariantly cochain equivalent. We begin by providing a formula for an explicit equivalence φ relating the two models $\mathrm{FN}^*_{W_{D_n}}$ and ${\mathrm{FN}'}^*_{W_{D_n}}$.

Lemma 4.1 *There is a cochain homotopy equivalence $\varphi^*\colon \mathrm{FN}^*_{W_{D_n}} \to {\mathrm{FN}'}^*_{W_{D_n}}$ defined by the formula*

$$\varphi^*[a_0:\cdots:a_{n-1}] = \begin{cases} [a_0:a_1:a_2:\cdots:a_{n-1}]^+ & \text{if } a_0 < a_1, \\ [a_0:a_1:a_2:\cdots:a_{n-1}]^+ + [a_1:a_0:a_2:\cdots:a_{n-1}]^- & \text{if } a_0 = a_1, \\ [a_1:a_0:a_2:\cdots:a_{n-1}]^- & \text{if } a_0 > a_1, \end{cases}$$

induced by the inclusion $Y^{(\infty)}_{W_{B_n}} \subseteq Y^{(\infty)}_{W_{D_n}}$ and yielding the identity in cohomology.

Proof We observe that the inclusion

$$Y^{(\infty)}_{W_{B_n}} \subseteq Y^{(\infty)}_{W_{D_n}}$$

is a W_{D_n}–equivariant homotopy equivalence. Moreover, the inverse image in $Y^{(\infty)}_{W_{B_n}}$ of each stratum of $\Phi_{\infty,W_{D_n}}$ is a union of strata in $\Phi_{\infty,W_{B_n}}$. Thus, passing to quotients, this yields a map

$$\varphi\colon \frac{Y^{(\infty)}_{W_{B_n}}}{W_{D_n}} \to \frac{Y^{(\infty)}_{W_{D_n}}}{W_{D_n}}$$

that induces a well-defined map between the cochain complexes $\varphi^*\colon \mathrm{FN}^*_{W_{D_n}} \to {\mathrm{FN}'}^*_{W_{D_n}}$.

We now check that φ^* satisfies the given formulas. It is sufficient to consider the finite approximations

$$\varphi^{(d)}\colon \frac{Y^{(d)}_{W_{B_n}}}{W_{D_n}} \to \frac{Y^{(d)}_{W_{D_n}}}{W_{D_n}}.$$

For any given stratum $S = e(a_0, \ldots, a_{n-1})$ for W_{D_n}, since $\varphi^{(d)}$, being a 0–codimensional immersion, is transverse to S, we have that $(\varphi^{(d)})^*(\tau_S) = \tau_{(\varphi^{(d)})^{-1}(S)}$. We now distinguish three cases:

- if $a_0 < a_1$, then $(\varphi^{(d)})^{-1}(S) = e(a_0, a_1, a_2, \ldots, a_{n-1})$;
- if $a_0 > a_1$, then $(\varphi^{(d)})^{-1}(S) = s_0.e(a_1, a_0, a_2, \ldots, a_{n-1})$;
- if $a_0 = a_1$, then $(\varphi^{(d)})^{-1}(S)$ is the union of $e(a_0, \ldots, a_{n-1})$, $s_0.e(a_0, \ldots, a_{n-1})$ and strata of bigger codimension.

This implies that φ^* has the desired form. □

We also consider the following group homomorphisms:

- the standard inclusion $j : \Sigma_n \to W_{B_n}$ already considered in the previous section;
- the involution $c_{s_0} : W_{D_n} \to W_{D_n}$ given by conjugation by s_0, the unique generating reflection of W_{B_n} that does not belong to W_{D_n}, that fixes t_i for $2 \leq i < n$ and switches t_0 and t_1;
- the two inclusions $i_+, i_- : \Sigma_n \to W_{D_n}$ given, in terms of the Coxeter generators $t_0, \ldots, t_n$ of Figure 1, by $i_\pm(i, i+1) = t_i$ if $i \geq 2$, $i_+(1,2) = t_1$ and $i_-(1,2) = t_0$.

We denote by $\iota\colon H^*(W_{D_n}; \mathbb{F}_2) \to H^*(W_{D_n}; \mathbb{F}_2)$ the morphism induced by c_{s_0} on cohomology.

We note that the two following properties hold by construction:

- $\pi j = \mathrm{id}_{\Sigma_n}$;
- $\pi \circ i_+ = \pi \circ i_- = \mathrm{id}_{\Sigma_n}$, where $\pi : W_{D_n} \to \Sigma_n$ is the composition of the inclusion $j : W_{D_n} \to W_{B_n}$ with the projection $W_{B_n} \to \Sigma_n$;
- $c_{s_0} \circ i_+ = i_-$.

We compute cochain representatives of ι in the following lemmas.

Lemma 4.2 *ι is induced by the cochain-level map $\iota^\#\colon \mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}^*_{W_{D_n}}$ defined by*

$$\iota^\#[a_0 : a_1 : a_2 : \cdots : a_{n-1}] = [a_1 : a_0 : a_2 : \cdots : a_{n-1}].$$

Proof Since the image under c_{s_0} of a fundamental reflection for $W = W_{D_n}$ is again a fundamental reflection, for every $\Gamma' \subseteq \Gamma \subseteq \{t_0, \ldots, t_{n-1}\}$, the set of minimal-length coset representatives satisfies $c_{s_0}(W_\Gamma^{\Gamma'}) = W_{c_{s_0}(\Gamma)}^{c_{s_0}(\Gamma')}$. Thus,

$$e(\Gamma_1 \supseteq \cdots \supseteq \Gamma_k \supseteq \cdots) \mapsto e(c_{s_0}(\Gamma_1) \supseteq \cdots \supseteq c_{s_0}(\Gamma_k) \supseteq \cdots)$$

defines a c_{s_0}–equivariant chain map $C_*^{W_{D_n}} \to C_*^{W_{D_n}}$. This yields, dually, the desired cochain map $\mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}^*_{W_{D_n}}$. $\square$

We can also describe ι in terms of $\mathrm{FN}'^*_{W_{D_n}}$. The proof of the following lemma is straightforward.

Lemma 4.3 *ι is induced by the cochain-level map $\iota'^{\#}\colon \mathrm{FN}'^*_{W_{D_n}} \to \mathrm{FN}'^*_{W_{D_n}}$ defined by*

$$\iota'^{\#}[a_0:\cdots:a_{n-1}]^+ = [a_0:\cdots:a_{n-1}]^-, \quad \iota'^{\#}[a_0:\cdots:a_{n-1}]^- = [a_0:\cdots:a_{n-1}]^+.$$

In terms of oriented symmetric planar level trees, the map $\iota'^{\#}$ acts on $(T,\mathcal{O})$ by replacing $\mathcal{O}$ with the opposite orientation.

The following identity is also proved by direct inspection.

Lemma 4.4 *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{FN}^*_{W_{D_n}} \arrow[r, "\varphi^*"] \arrow[d, "\iota^{\#}"'] & \mathrm{FN}'^*_{W_{D_n}} \arrow[d, "\iota'^{\#}"] \\
\mathrm{FN}^*_{W_{D_n}} \arrow[r, "\varphi^*"'] & \mathrm{FN}'^*_{W_{D_n}}
\end{tikzcd}$$

The formulas for the other connecting maps follow from a general remark.

Lemma 4.5 *Let G be a Coxeter group, with Coxeter generators $S = \{s_0,\dots,s_n\}$ and $H \le G$ be a parabolic subgroup, generated by a subset $T = \{s_{i_0},\dots,s_{i_m}\} \subseteq S$. The inclusion $H \hookrightarrow G$ is represented at the chain level by the chain map $C_*^H \to C_*^G$ given by $c(\underline{\Gamma},\gamma) \mapsto c(\underline{\Gamma},\gamma)$, for flags $\underline{\Gamma} = (\Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_k \supseteq \varnothing)$ with $\Gamma_0 \subseteq T \subseteq S$ and elements $\gamma \in H$.*

Dually, it is represented at the cochain level by the cochain map $\mathrm{FN}^*_G \to \mathrm{FN}^*_H$ given by

$$[e(a_0,\dots,a_n)] \in C_*^H \mapsto \begin{cases} [e(a_{i_0},\dots,a_{i_m})] & \textit{if } a_j = 0 \textit{ for all } j \notin \{i_0,\dots,i_m\}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof Since the inclusion of parabolic subgroups preserves minimal coset representatives, the De Concini–Salvetti boundary formula of Theorem 3.1 implies that the given linear morphism $C_*^H \to C_*^G$ is an H–equivariant chain map. Dualizing this yields the cochain formula between Fox–Neuwirth complexes. $\square$

As particular cases of this lemma, we retrieve cochain formulas for our connecting homomorphisms:

Corollary 4.6 *The following statements are true.*

(1) *The linear morphism* $j^{\#}\colon \mathrm{FN}^*_{W_{B_n}} \to \mathrm{FN}^*_{\Sigma_n}$ *given by*

$$[a_0,\dots,a_{n-1}] \mapsto \begin{cases} [a_1,\dots,a_{n-1}] & \text{if } a_0 = 0, \\ 0 & \text{if } a_0 > 0, \end{cases}$$

represents j *at the cochain level.*

(2) *The linear morphism* $i_+^{\#}\colon \mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}^*_{\Sigma_n}$ *given by*

$$[a_0,\dots,a_{n-1}] \mapsto \begin{cases} [a_1,\dots,a_{n-1}] & \text{if } a_0 = 0, \\ 0 & \text{if } a_0 > 0, \end{cases}$$

represents i_+ *at the cochain level.*

(3) *The linear morphism* $i_-^{\#}\colon \mathrm{FN}^*_{W_{D_n}} \to \mathrm{FN}^*_{\Sigma_n}$ *given by*

$$[a_0,\dots,a_{n-1}] \mapsto \begin{cases} [a_0,a_2,\dots,a_{n-1}] & \text{if } a_1 = 0, \\ 0 & \text{if } a_1 > 0, \end{cases}$$

represents i_+ *at the cochain level.*

4.2 Structural morphisms: A_B

We want to describe the almost-Hopf ring structures presented in Section 2 in our geometric context. We begin with the coproduct map in A_B. In contrast with the symmetric group case, the cochain-level map inducing the coproduct is relatively complicated. Its underlying combinatorics is built upon elementary steps that we, mindful of the botanic analogy, suggestively call "prunings".

Definition 4.7 Let T be a planar level tree. An *elementary k–pruning* of T is a planar level tree T' obtained by the following procedure. Choose an internal vertex v of T of height k, and consider on $E(v)$ the order induced by the planar embedding. Let $1 \le l < |E(v)|$, consider the l biggest elements $e_1,\dots,e_l$ of $E(v)$ with respect to this order, and let v_i' be the target of e_i. T'' is the subtree of T spanned by v and all vertices that can be reached from one of the v_i' through an oriented path. T' is the complementary subtree of T'' in T. We call the planarly embedded subtree T'' the *scrap* of the elementary k–pruning. An elementary k–pruning is said to be *minimal* if $l = 1$. A *k–pruning* is a couple (T', T'') constructed as follows:

- T' is obtained from a sequence of elementary k–prunings
$$T \rightsquigarrow T'_1 \rightsquigarrow T'_2 \rightsquigarrow \cdots \rightsquigarrow T'_j = T'$$
performed on pairwise different vertices $v_1, \dots, v_j$ of T, with scraps $T''_1, \dots, T''_j$;
- T'' is a planar level tree obtained by joining these scrap subtrees along a vertex w of height k and performing a vertex permutation at w that shuffles the edges of the scraps.

Let T be a symmetric planar level tree. An *elementary symmetric k–pruning* of T is a tree T' obtained as follows. Apply to T an elementary (nonsymmetric) k–pruning whose scrap T'' does not contain the central leaf belonging to the $\boldsymbol{y}$–axis. Then, remove the image of the subtree of T'' under the reflection r along the vertical axis. T'' is called the *scrap* of the elementary symmetric pruning. An elementary symmetric k–pruning is said to be *minimal* if it is obtained from a minimal elementary k–pruning. A *symmetric k–pruning* is a couple (T', T''), where

- T' is obtained from a sequence of elementary k–prunings
$$T \rightsquigarrow T'_1 \rightsquigarrow T'_2 \rightsquigarrow \cdots \rightsquigarrow T'_j = T'$$
performed on pairwise different vertices of T, with scraps $T''_1, \dots, T''_j$;
- T'' is a nonsymmetric planar level tree obtained by joining the scrap subtrees to a vertex w of height k and performing a vertex permutation at w that shuffles the edges of the scraps.

We note that elementary k–prunings at different vertices commute, both in the symmetric and nonsymmetric cases. Hence, a k–pruning or symmetric k–pruning is uniquely determined by the set of elementary k–prunings that compose it, independently of the order in which they are performed.

There is also an alternative way to define (symmetric) k–prunings in terms of minimal k–prunings instead of elementary ones. A (symmetric) k–pruning is obtained by performing a sequence of minimal elementary (symmetric) k–prunings, not necessarily at pairwise different vertices, and then joining the scraps at a vertex of height k without shuffling the edges.

We now consider three linear morphisms that we will need to define the cochain-level coproduct map:

- the k–pruning map
$$P_k\colon \mathrm{FN}^*_{W_{B_n}} \otimes \mathbb{F}_2 \to \bigoplus_{a+b=n} \mathrm{FN}^*_{W_{B_a}} \otimes \mathrm{FN}^*_{W_{B_b}} \otimes \mathbb{F}_2$$
that maps a symmetric planar level tree T to the sum $\sum T' \otimes S_k(T'')$ over all the possible symmetric k–prunings (T', T'') of T;
- the minimal k–pruning map
$$P_k^{\min}\colon \mathrm{FN}^*_{W_{B_n}} \otimes \mathbb{F}_2 \to \bigoplus_{a+b=n} \mathrm{FN}^*_{W_{B_a}} \otimes \mathrm{FN}^*_{W_{B_b}} \otimes \mathbb{F}_2$$
that maps a symmetric planar level tree T to the sum $\sum T' \otimes S_k(T'')$ over all the possible minimal elementary symmetric k–prunings (T', T'') of T;
- the concatenation map $C\colon \mathrm{FN}^*_{W_{B_n}} \otimes \mathrm{FN}^*_{W_{B_m}} \otimes \mathbb{F}_2 \to \mathrm{FN}^*_{W_{B_{n+m}}}$ such that $C([a_0 : \cdots : a_{n-1}] \otimes [b_0 : \cdots : b_{m-1}]) = [a_0 : \cdots : a_{n-1} : b_0 : \cdots : b_{m-1}]$.

The map P_k is exemplified in Figure 3. We understand C at the level of symmetric planar level trees as the function given by the following procedure. Take a couple of such objects (T, S). Cut S along its central vertical axis. Finally, glue the right piece of S onto the right side of T and the left part onto its left side to obtain a new symmetric planar level tree. We remark that these linear morphisms are degree-preserving, but they are not chain maps.

In the A_n case, we can define a similar k–pruning map P'_k by summing all nonsymmetric k–prunings. For $k = 0$, P'_0 is a chain map, and it is shown in [8] to induce the coproduct in cohomology. This statement is not true in the B_n case because the differential of an antisymmetric planar level tree with labels behaves badly near the central "trunk" labeled 0. Nevertheless, at each level k, away from this central stem, this is essentially true. For this intuitive reason, we must define our cochain-level coproduct map differently: prune a symmetric planar level tree at every level and tensor it with a symmetric planar level tree whose principal k–blocks, as defined below in Definition 4.10, are the scraps of the performed prunings. To prove this statement, we need some preliminary calculations.

Suppose that a symmetric planar level tree T corresponds to $[a_0 : \cdots : a_{n-1}] \in \mathrm{FN}_{W_{B_n}}$. In that case, consider the set of couples of adjacent edges (e, f), with $e < f$, in T having the same source vertex and belonging to the positive half-plane $\{(x, y) \mid x \geq 0\}$. This set in bijective correspondence with $\{0, \ldots, n-1\}$, and the height of the common vertex of the couple (e, f) corresponding to i is a_i. This bijection is explicitly given by counting the leaves in the positive half-plane that lie on the left of e. For $0 \leq i \leq n-1$,

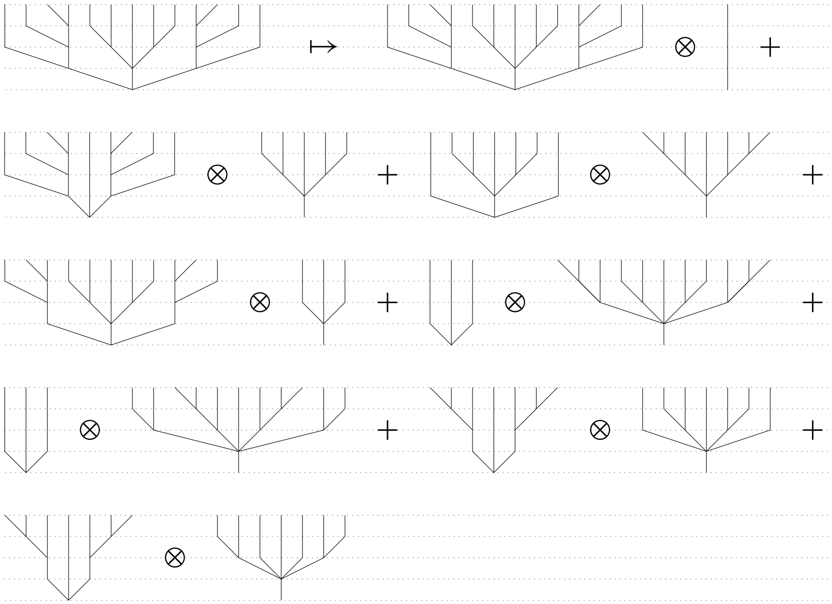

Figure 3: The map P_1, defined as the sum of all possible symmetric 1–pruning, on a given symmetric planar level tree.

we denote by $d_i(T)$ or equivalently by $d_{e,f}$ the sum of the addends $d^S_{e,f,\sigma}$ of the differential d, as expressed in Proposition 3.5, in which a vertex shuffle constructed from the couple (e,f) corresponding to i appear. Thus, $d(T)=\sum_{i=0}^{n-1} d_i(T)$.

Lemma 4.8 *Let T be a symmetric planar level tree corresponding to $[a_0:\cdots:a_{n-1}]$. Let m_k be the smallest index such that $a_{m_k}=k$. Let I be the trivial symmetric planar level tree. Then the following statements are true*:

(1) *the pruning maps and the differential satisfy the equality*

$$P_k d + dP_k + (\mathrm{id}\otimes d_0)(P_k - \mathrm{id}\otimes I) = (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes \mathrm{id})P^{\min}_{k-1};$$

(2) $P_k(T)=T\otimes I$ *if* $a_i<k$ *for all* $0\le i<n$;

(3) *for all* $\boldsymbol{a}=[a_0:\cdots:a_{n-1}]$ *and* $\boldsymbol{b}=[b_0:\cdots:b_{m-1}]$ *with* $b_0<\min\{a_0,\dots,a_{n-1}\}$,

$$d_i C(\boldsymbol{a}\otimes\boldsymbol{b}) = \begin{cases} C(d_i\otimes \mathrm{id})(\boldsymbol{a}\otimes\boldsymbol{b}) & \text{if } 0\le i<n,\\ C(\mathrm{id}\otimes d_{i-n})(\boldsymbol{a}\otimes\boldsymbol{b}) & \text{if } n<i<n+m,\end{cases}$$

and the latter also holds for $i=n$ *if* $b_0<\min\{a_0,\dots,a_{n-1}\}-1$;

(4) $(\mathrm{id}\otimes C)(P^{\min}_k\otimes\mathrm{id})P_k(T)=P_k(T)-T\otimes I$;

(5) $C(C\otimes\mathrm{id})=C(\mathrm{id}\otimes C)$.

Proof The statements from (2) to (5) are easy. Regarding (2), if $a_i < k$ for all i, T has no vertex of height k with more than one outgoing edge. Thus the only possible symmetric k–pruning is the trivial one. Regarding (3), the bijection

$$\varphi\colon \{0,\dots,n-1\} \sqcup \{0,\dots,m-1\} \to \{0,\dots,n+m-1\}$$

that shifts elements of $\{0,\dots,m-1\}$ by n yields a bijection between pairs (e, f) of adjacent edges of the symmetric planar level tree T corresponding to $C(\boldsymbol{a} \otimes \boldsymbol{b})$ and those of the symmetric planar level trees T' and T'' corresponding to $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. If $b_0 < \min\{a_0,\dots,a_{n-1}\}$, then for all $i \in \{0,\dots,n+m-1\}$, with the only possible exception of $i = n$, this bijection preserves $E(v_{e_i})$ and $E(v_{f_i})$, where v_{e_i} and v_{f_i} are the target vertices of the corresponding pair of edges (e_i, f_i). The edges in $E(v_{e_i})$ and $E(v_{f_i})$ of the corresponding pair come either both from T' or both from T''. Hence $d_i C(\boldsymbol{a} \otimes \boldsymbol{b}) = d_{\varphi^{-1}(i)} \boldsymbol{a} \otimes \boldsymbol{b}$. If $b_0 < \min\{a_0,\dots,a_{n-1}\} - 1$ the same is also true for the edges e_n and f_n, so the equality is satisfied also in this case. Statement (4) is immediate from the definition of k–prunings and the combinatorics of shuffles, and (5) is obvious.

On the contrary, (1) is more complicated and requires a more detailed proof. As a notational convention, let $d_l^h = \sum_{i:a_i=l} d_i$, the sum of the contributions to the differential coming from vertices at height k. We compare $d_l^h P_k(T)$ with $P_k d_l^h(T)$. We consider different cases depending on the difference between k and l.

- If $l > k$, d_l^h is computed by gluing together a pair (e, f) of adjacent edges of height bigger than k (and its mirror pair) and performing a shuffle at the new target vertex. These operations only change a connected subtree whose vertices all have height bigger than k, and, by construction, k–prunings commute with such operations. Hence $d_l^h P_k = P_k d_l^h$.

- If $l = k$, then we can write $d_k^h P_k(T) = \sum_{(T',T'')} \sum_{(e,f)} d_{e,f}(T' \otimes S_k(T''))$, where the sum is over symmetric k–prunings (T', T'') of T and pairs of adjacent edges (e, f) in the positive half-plane with a common source vertex of height k in T' or $S_k(T'')$. We also note that $S_k(T'')$ has a unique vertex w of height k. There is an obvious bijection

$$\bigsqcup_{\substack{v\in V(T)\\ h(v)=k}} E(v) \leftrightarrow \bigsqcup_{\substack{u\in V(T')\\ h(v)=k}} E(u) \sqcup (E(w) \setminus \{e_0\})$$

that maps an edge to its image in T' (if it is not pruned away) or in $S_k(T'')$ (if it is), and that arises from the fact that, for elementary prunings, $T = T' \cup T'' \cup r(T'')$. The edge e_0 is the unique edge belonging to the central vertical stem whose source vertex is w. Moreover, this bijection preserves the properties of belonging to the positive and negative half-plane. Therefore, we can write the summation above expressing $d^h_k P_k(T)$ as the sum of three pieces:

- The first piece is the sum of the terms corresponding to (e, f) such that (e, f) come from adjacent edges in T. These terms correspond to symmetric k–prunings of $d^S_{e,f,\sigma}(T)$, for shuffles σ at the common vertex of e and f. Hence, their sum yields $P_k d^h_k(T)$.
- The second piece is the sum of the terms corresponding to (e, f) in $S_k(T'')$ such that $e \neq e_0$ and (e, f) do not come from adjacent vertices of T. Under this condition, the symmetric vertex permutation $\sigma(S_k(T''))$ of $S_k(T'')$ at w that switched the positions of e and f still produces a shuffle of the scraps of the elementary prunings involved in (T', T''). Every tree in $d_{(e,f)}(S_k(T''))$ cancel out with a tree in $d_{(f,e)}(\sigma(S_k(T'')))$. Hence, this second piece is 0.
- The third piece is given by the terms corresponding to (e, f) with $e = e_0$. These terms yield $(\mathrm{id} \otimes d_0) P_k(T)$.

Finally, we deduce that $d^h_k P_k(T) = (\mathrm{id} \otimes d_0) P_k(T) + P_k d^h_k(T)$.

- If $l = k-1$, $P_k d^h_{k-1}(T) = \sum_{(e,f)} \sum_{(T',T'')} T' \otimes S_k(T'')$, where the sum is taken over couples (e, f) of adjacent edges in T whose common source v has height $k-1$, and symmetric k–prunings (T', T'') of trees in $d_{(e,f)}(T)$. Let v_e and v_f be the targets of e and f, respectively. By construction, $d_{(e,f)}(T)$ glues v_e and v_f to a single vertex $\bar{v}$, such that $E(\bar{v}) = E(v_e) \sqcup E(v_f)$, suitably shuffled. Let A be the set of edges removed by the corresponding elementary symmetric prunings at $\bar{v}$ and at $r(\bar{v})$, the mirror vertex of $\bar{v}$ (which might coincide). We retrieve symmetric k–prunings for which $E(v_e) \nsubseteq A$ and $E(v_f) \nsubseteq A$ from symmetric k–prunings (T', T'') of T by applying $d_{(e,f)}$ to T'. Now assume that v is not on the central stem of the tree. If $e \neq \min(E(v))$, it is the successor of an edge $g \in E(v)$, and the terms of $P_k d_{e,f}(T)$ for which $E(v_e) \subseteq A$ cancel out with the terms of $P_k d_{g,e}(T)$ for which $E(v_e) \subseteq A$. Similarly, all the terms for which $E(v_f) \subseteq A$ and $f \neq \max(E(v))$ cancel out. The only remaining terms are those in which we remove an entire subtree corresponding to $\min(E(v))$ — which is the mirror image of $\max(E(r(v))$. If v belongs to the central axis, we must slightly modify the argument to take into account only edges in the positive half-plane and shows that the surviving terms are those in which an entire subtree stemming from $\max(E(v))$ is removed. The sum of all these elements is exactly equal to the correcting term $(\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes \mathrm{id})P^{\min}_{k-1}(T)$. We deduce that

$$d^h_{k-1}P_k(T) = P_k d^h_{k-1}(T) + (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes \mathrm{id})P^{\min}_{k-1}(T).$$

- If $l < k-1$, since k–prunings only depend on the part of the tree above height k and d^h_l does not change it, the same argument used for $l > k$ shows that $d^h_l P_k = P_k d^h_l$.

Combining the equalities obtained in these cases yields (1). $\square$

We are now ready to construct a cochain representative of the cohomological coproduct map $H^*(W_{B_n}) \to \bigoplus_{i=0}^n H^*(W_{B_i})\otimes H^*(W_{B_{n-i}})$.

Proposition 4.9 *Let $\Delta_k : \mathrm{FN}^*_{W_{B_n}}\otimes\mathbb{F}_2 \to \bigoplus_{i=0}^n \mathrm{FN}^*_{W_{B_i}}\otimes \mathrm{FN}^*_{W_{B_{n-i}}}\otimes\mathbb{F}_2$ be the linear maps defined recursively by the formulas*

- $\Delta_0 = P_0$,
- *for* $k>0$, $\Delta_k = (\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})\Delta_{k-1}$.

Then

1. *the limit $\Delta = \varinjlim \Delta_k$ exists,*
2. *Δ is a cochain map,*
3. *Δ represents the cohomology coproduct map at the cochain level.*

Proof (1) Let $\boldsymbol{a}\in \mathrm{FN}^*_{W_{B_n}}$ and let $m=\max\{a_0,\dots,a_{n-1}\}$. Statement (2) of Lemma 4.8 guarantees that $\Delta_k(\boldsymbol{a}) = \Delta_m(\boldsymbol{a})$ for all $k>m$. Thus, the sequence $\{\Delta_k\}_{k=0}^\infty$ stabilizes and consequently has a limit.

(2) We first observe that Lemma 4.8(4) and (5) imply that

$$(\mathrm{id}\otimes C)(P^{\min}_k\otimes\mathrm{id})\Delta_k = \Delta_k - \Delta_{k-1}$$

for all $k\ge 0$, with the convention that $\Delta_{-1}(T) = T\otimes I$. Combining this remark with Lemma 4.8(3) and (5), we obtain that, for all $k\ge 1$,

$$\begin{aligned}
&(\mathrm{id}\otimes C)(\mathrm{id}\otimes d_{m_{k-1}}\otimes\mathrm{id})(\mathrm{id}\otimes C\otimes\mathrm{id})(P_k\otimes\mathrm{id}\otimes\mathrm{id})(P^{\min}_{k-1}\otimes\mathrm{id})\Delta_{k-1}\\
&\qquad = (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(\mathrm{id}\otimes C\otimes\mathrm{id})(P_k\otimes\mathrm{id}\otimes\mathrm{id})(P^{\min}_{k-1}\otimes\mathrm{id})\Delta_{k-1}\\
&\qquad = (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes C)(P^{\min}_{k-1}\otimes\mathrm{id})\Delta_{k-1}\\
&\qquad = (\mathrm{id}\otimes d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes\mathrm{id})(\Delta_{k-1}-\Delta_{k-2}).
\end{aligned}$$

We use this to prove by induction on k that $\Delta_k d = d\Delta_k + (\mathrm{id}\otimes d_0)(\Delta_k - \Delta_{k-1})$. For $k=0$ this identity is the content of the first statement of Lemma 4.8. For $k>0$, we deduce from the identity above and the previous lemma that

$$
\begin{aligned}
\Delta_k d &= (\mathrm{id}\otimes C)(P_k\otimes \mathrm{id})\Delta_{k-1} d \\
&= (\mathrm{id}\otimes C)(P_k\otimes \mathrm{id}) d\Delta_{k-1} + (\mathrm{id}\otimes C)(P_k\otimes d_0)(\Delta_{k-1}-\Delta_{k-2}) \\
&= (\mathrm{id}\otimes C)(P_k\otimes \mathrm{id}) d\Delta_{k-1} + (\mathrm{id}\otimes C) d(P_k\otimes \mathrm{id})(\Delta_{k-1}-\Delta_{k-2}) \\
&\qquad + (d - d_{m_{k-1}})(\mathrm{id}\otimes C)(P_k\otimes \mathrm{id})(\Delta_{k-1}-\Delta_{k-2}) \\
&= (\mathrm{id}\otimes C) d(P_k\otimes \mathrm{id})\Delta_{k-1} + (\mathrm{id}\otimes C)(\mathrm{id}\otimes d_0\otimes \mathrm{id})[(P_k - \mathrm{id}\otimes I)\otimes \mathrm{id}]\Delta_{k-1} \\
&\qquad + (\mathrm{id}\otimes C) d(P_k\otimes \mathrm{id})(\Delta_{k-1}-\Delta_{k-2}) + d(\mathrm{id}\otimes C)(P_k\otimes \mathrm{id})(\Delta_{k-1}-\Delta_{k-2}) \\
&= (\mathrm{id}\otimes C) d(P_k\otimes \mathrm{id})\Delta_{k-2} + (\mathrm{id}\otimes d_0)(\mathrm{id}\otimes C)[(P_k - \mathrm{id}\otimes I)\otimes \mathrm{id}]\Delta_{k-1} \\
&\qquad + d\Delta_k - d(1\otimes C)(P_k\otimes \mathrm{id})\Delta_{k-2} \\
&= d\Delta_k - (\mathrm{id}\otimes d_0)(\Delta_k - \Delta_{k-1}) + (\mathrm{id}\otimes C) d(P_k\otimes \mathrm{id})\Delta_{k-2} \\
&\qquad - d(\mathrm{id}\otimes C)(P_k\otimes \mathrm{id})\Delta_{k-2} \\
&= d\Delta_k - (\mathrm{id}\otimes d_0)(\Delta_k - \Delta_{k-1}).
\end{aligned}
$$

To justify the last equality, we observe that $(P_k\otimes \mathrm{id})\Delta_{k-2}$ is a sum of terms of the form $\boldsymbol{c}\otimes\boldsymbol{a}\otimes\boldsymbol{b}$ with $b_0 < \min\{a_i\} - 1$, and we apply the stronger clause of Lemma 4.8(3).

Now the identity $d\Delta = \Delta d$ follows by passing to the limit, and using that the sequence $\{\Delta_k\}_{k=0}^{\infty}$ stabilizes.

(3) Consider the dg-module U over $\mathbb{F}_2$ with basis given by symmetric planar level trees with antisymmetric labels in any finite subset $I\subseteq\mathbb{N}$, not necessarily $\{-n,\dots,n\}$, with the symmetric tree differential. Note that $\bigoplus_{n\geq 0}\widetilde{\mathrm{FN}}^*_{W_{B_n}}\otimes\mathbb{F}_2$ embeds in U in the obvious way. We observe that the linear maps P_k, $P_k^{\min}$ and C lift to linear maps $\widetilde{P}_k, \widetilde{P}_k^{\min}\colon U\to U\otimes U$ and $\widetilde{C}\colon U\otimes U\to U$. $\widetilde{P}_k$ and $\widetilde{P}_k^{\min}$ are still defined via prunings, but we additionally keep track of the labels of the subtrees involved. We compute $\widetilde{C}$ on $T'\otimes T''$ by splitting T'' symmetrically along the vertical axis, keeping labels, and symmetrically attach the two parts to T' to obtain a new basis element of U. Lemma 4.8 still holds for this labeled version of the morphisms by the same proof. Consequently, there is a labeled version $\widetilde{\Delta}\colon U\to U\otimes U$ of Δ, constructed recursively via finite approximations $\widetilde{\Delta}_k$, that still commutes with the differential. Note that we can also embed

$$\widetilde{\mathrm{FN}}^*_{W_{B_n}}\otimes\widetilde{\mathrm{FN}}^*_{W_{B_m}}\otimes\mathbb{F}_2$$

into $U \otimes U$ by keeping the labels of trees in $\widetilde{\mathrm{FN}}^*_{W_{B_n}}$ and relabeling trees in $\widetilde{\mathrm{FN}}^*_{W_{B_m}}$ via the bijection $\{0, \dots, m-1\} \to \{n, \dots, n+m-1\}$ that raises numbers by n. There is also a projection $U \times U \to \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \widetilde{\mathrm{FN}}^*_{W_{B_m}} \otimes \mathbb{F}_2$ that maps every basis element of $U \otimes U$ that does not belong to $\widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \widetilde{\mathrm{FN}}^*_{W_{B_m}} \otimes \mathbb{F}_2$ to 0. By induction, we easily see that restricting each $\widetilde{\Delta}_k$ for all k (and, consequently, $\widetilde{\Delta}$) to $\widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \widetilde{\mathrm{FN}}^*_{W_{B_m}} \otimes \mathbb{F}_2$ and composing with this projection we obtain linear maps $\bigoplus_{n\geq 0} \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \mathbb{F}_2 \to \bigoplus_{n\geq 0} \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \bigoplus_{n \geq 0} \widetilde{\mathrm{FN}}^*_{W_{B_n}} \otimes \mathbb{F}_2$ that are equivariant with respect to the monomorphisms $W_{B_n} \times W_{B_m} \to W_{B_{n+m}}$ and satisfy the same formal relation with respect to the differential. By identifying $\mathrm{FN}^*_{W_{B_n}}$ with the invariant subspace $(\widetilde{\mathrm{FN}}^*_{W_{B_n}})^{W_{B_n}}$, the limit map $\widetilde{\Delta}$ restricts to Δ, which is thus a cochain-level realization of the coproduct map. $\square$

We now turn our attention to the transfer map. We need a preliminary definition.

Definition 4.10 (partially from [8]) Let $\boldsymbol{a} = [a_0 : \dots : a_{n-1}] \in \mathrm{FN}^*_{W_{B_n}}$ be as defined above. In what follows, we assume, by convention, that $a_{-1} = a_n = 0$. We say that the chain $[a_i : \dots : a_j]$ is a *k–block* of $\boldsymbol{a}$ if $a_l > k$ for all $i \leq l \leq j$ and $\max\{a_{i-1}, a_{j+1}\} \leq k$. We say that a k–block $[a_i : \dots : a_j]$ of $\boldsymbol{a}$ is *principal* if, in addition, $\min_{0 \leq r < i} a_r = k$. We denote by $\mathrm{PBl}_k(\boldsymbol{a})$ the tuple of the principal k–blocks of $\boldsymbol{a}$, ordered from left to right.

For example, the basis element $\boldsymbol{a} = [3:2:3:1:2:1:3:2:0:3]$ has four 1–blocks: $B_{1,1} = [3:2:3]$, $B_{1,2} = [2]$, $B_{1,3} = [3:2]$, and $B_{1,4} = [3]$. $\mathrm{PBl}_1(\boldsymbol{a}) = (B_{1,2}, B_{1,3})$.

Note that a basis element $\boldsymbol{a}$ is uniquely determined by $\{\mathrm{PBl}_k(\boldsymbol{a})\}_{k=0}^{\infty}$, the collection of its principal blocks. To retrieve $\boldsymbol{a}$ from these data, we can use the following procedure. First, for all $k \geq 0$, add an entry equal to k before each principal k–block and concatenate all such tuples to obtain $\boldsymbol{a}_k$. Then, we obtain $\boldsymbol{a}$ as the concatenation of $\dots, \boldsymbol{a}_k, \boldsymbol{a}_{k-1}, \dots, \boldsymbol{a}_0$. This sequence is necessarily finite because for $k > \max_{i=0}^{n-1} a_i$, $\mathrm{PBl}(\boldsymbol{a})$ is the empty 0–tuple. With this method, we can construct a basis element $\boldsymbol{a}$ from an eventually empty collection of tuples $\{B_k\}$, where the entries of B_k are tuples of natural numbers strictly bigger than k.

We also observe that k–blocks can be retrieved from the corresponding symmetric level tree T. They are given by the connected components of $T \cap \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y > k\}$. Interpreted this way, a k–block is principal if and only if it does not intersect the central vertical axis but is contained in the $(k-1)$–block intersecting it.

Proposition 4.11 *Given $\boldsymbol{a} \in \mathrm{FN}^*_{W_{B_n}}$, $\boldsymbol{b} \in \mathrm{FN}^*_{W_{B_m}}$ and $k \geq 0$, let $n_{a,k}$ and $n_{b,k}$ be the lengths of $\mathrm{PBl}_k(\boldsymbol{a})$ and $\mathrm{PBl}_k(\boldsymbol{b})$, respectively. Given a sequence $\underline{\sigma} = \{\sigma_k\}_{k=0}^{\infty}$ of permutations $\sigma_k \in \Sigma_{n_{a,k}+n_{b,k}}$, define $\underline{\sigma}(\boldsymbol{a},\boldsymbol{b})$ as the unique basis elements of $\mathrm{FN}^*_{W_{B_{n+m}}}$ such that, for all $k \geq 0$, $\mathrm{PBl}_k(\underline{\sigma}(\boldsymbol{a},\boldsymbol{b})) = \sigma_k(\mathrm{PBl}_k(\boldsymbol{a}), \mathrm{PBl}_k(\boldsymbol{b}))$, where $(\mathrm{PBl}_k(\boldsymbol{a}), \mathrm{PBl}_k(\boldsymbol{b}))$ is the concatenated $(n_{a,k}+n_{b,k})$–tuple and σ_k acts on $(n_{a,k}+n_{b,k})$–tuples by permuting the entries. Let $\odot\colon \mathrm{FN}^*_{W_{B_n}} \otimes \mathrm{FN}^*_{W_{B_m}} \otimes \mathbb{F}_2 \to \mathrm{FN}^*_{W_{B_{n+m}}} \otimes \mathbb{F}_2$ be the homomorphism that maps $\boldsymbol{a} \otimes \boldsymbol{b}$ to the sum $\sum_{\underline{\sigma}} \underline{\sigma}(\boldsymbol{a},\boldsymbol{b})$ over sequences of permutations $\underline{\sigma} = \{\sigma_k\}_{k=0}^{\infty}$ such that σ_k is a $(n_{a,k}, n_{b,k})$–shuffle for all $k \geq 0$. Informally, $\boldsymbol{a} \odot \boldsymbol{b}$ is the sum of basis elements whose principal k–blocks are obtained by shuffling the principal k–blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$. This defines a morphism of complexes that induces the transfer product in cohomology.*

Proof The reflection arrangement of $W_{B_n} \times W_{B_m}$, with its product reflection action on $\mathbb{R}^n \times \mathbb{R}^m$, is $\mathcal{A}_{B_n} \times \mathcal{A}_{B_m} = \{H \times \mathbb{R}^m\}_{H \in \mathcal{A}_{B_n}} \cup \{\mathbb{R}^n \times H'\}_{H' \in \mathcal{A}_{B_m}}$. Being it a subarrangement of $\mathcal{A}_{B_{n+m}}$, we have a natural inclusion

$$Y^{(\infty)}_{W_{B_{n+m}}} \to Y^{(\infty)}_{\mathcal{A}_{B_n} \times \mathcal{A}_{B_m}} \cong Y^{(\infty)}_{W_{B_n}} \times Y^{(\infty)}_{W_{B_m}}.$$

We can explicitly obtain such inclusion by splitting a configuration of $n+m$ points into the two subconfigurations consisting of its first n points and its last m points, respectively, and relabeling the indices of the second one. This map is a $(W_{B_n} \times W_{B_m})$–equivariant homotopy equivalence.

Therefore, passing to quotients, this yields a map

$$\pi\colon \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_n} \times W_{B_m}} \to \frac{Y^{(\infty)}_{W_{B_n}}}{W_{B_n}} \times \frac{Y^{(\infty)}_{W_{B_m}}}{W_{B_m}}$$

that models the standard homotopy equivalence $B(W_{B_n} \times W_{B_m}) \simeq B(W_{B_n}) \times B(W_{B_m})$. Moreover, the obvious quotient map

$$\pi'\colon \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_n} \times W_{B_m}} \to \frac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_{n+m}}}$$

is a covering model for $B(W_{B_n} \times W_{B_m}) \to B(W_{B_{n+m}})$.

Let $x = [a_0 : \cdots : a_{n-1}] \otimes [b_0 : \cdots : b_{m-1}]$ be a basis element for the Fox–Neuwirth complex $\mathrm{FN}^*_{B_n} \otimes \mathrm{FN}^*_{B_m} \otimes \mathbb{F}_2$. Let σ be a smooth singular simplex transverse to our strata. By construction, the evaluation of $[a_0 : \cdots : a_{n-1}] \odot [b_0 : \cdots : b_{m-1}]$ on σ is the sum of the evaluations of x on $\pi(\tilde{\sigma})$, as $\tilde{\sigma}$ varies among all liftings of σ. A direct calculation shows that some $\pi(\tilde{\sigma})$ intersects the stratum corresponding to x if and only if σ intersects some stratum $e(\boldsymbol{c})$, where the k–principal blocks of $\boldsymbol{c}$ are obtained by shuffling the k–principal blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$. □

We conclude the treatment of the structural maps on the cohomology of W_{B_n} with some potentially helpful remarks. Since we will not use these facts in this paper, we will not provide complete statements nor proofs of these last claims. Nevertheless, it should be straightforward, although notationally heavy, to fill in the details.

Remark 4.12 (1) The transfer and coproduct maps commute already at the cochain level. To see this, you can observe that, by construction, $\Delta(\boldsymbol{a})$ is a sum of tensors $\boldsymbol{a}' \otimes \boldsymbol{a}''$, where $\mathrm{PBl}_k(\boldsymbol{a}'')$ is given by the leftovers of symmetric k–prunings of $\boldsymbol{a}$, suitably shuffled, and that the pruning map P_k itself commute with $\odot$.

(2) The same constructions of the coproduct map in terms of prunings and the transfer map in terms of principal block shuffles can be generalized to the cohomology with integral coefficients. In these cases, additional signs that we can compute from those appearing in Theorem 3.1 are required.

4.3 Structural morphisms: A_D

The coproduct and the transfer product for W_{D_n} are described geometrically, similarly to what we did for W_{B_n}. However, some complications arise. For example, we cannot repeat the proof of Proposition 4.11 as it is for $\mathrm{FN}^*_{W_{D_n}}$, because, in this case, a product of strata $S \times S' \subseteq Y^{(\infty)}_{W_{D_n}} \times Y^{(\infty)}_{W_{D_m}}$ is not necessarily the closure of a union of strata in $Y^{(\infty)}_{W_{D_{n+m}}}$. However, these ideas adapt well to the cochain complex $\mathrm{FN}'^*_{W_{D_n}}$, which we will use in the following as a cochain model. We can retrieve the identities we need in $\mathrm{FN}^*_{W_{D_n}}$ by using the equivalence φ of Lemma 4.1.

We can now state the formulas parallel to Propositions 4.9 and 4.11 for W_{D_n}. First, we consider the following oriented versions of the pruning and concatenation maps. Given a symmetric k–pruning (T', T'') of a symmetric planar level tree T, let $\mathcal{O}$ and $\mathcal{O}'$ be orientations of T and T' respectively. Fix an antisymmetric labeling λ' of T' inducing $\mathcal{O}'$, and an antisymmetric labeling λ of T inducing $\mathcal{O}$ such that its restriction to T', seen as a subtree of T, is λ'. By keeping track of the labels of scraps, λ induces an antisymmetric labeling λ'' on $S_k(T'')$ and, consequently, an orientation $\mathcal{O}''$. Unless the k–pruning is trivial, it is always possible to find such labelings λ and λ', and the resulting orientation $\mathcal{O}''$ only depends on $\mathcal{O}$ and $\mathcal{O}'$.

Definition 4.13 Let $k \in \mathbb{N}$ and let $(T, \mathcal{O})$ be an oriented symmetric planar level tree. An *oriented k–pruning* of $(T, \mathcal{O})$ is a quadruple $(T', \mathcal{O}', T'', \mathcal{O}'')$ where

- (T', T'') is a k–pruning of T,
- $\mathcal{O}'$ is an orientation of T',
- $\mathcal{O}''$ is the orientation of $S_k(T'')$ determined from $\mathcal{O}$ and $\mathcal{O}'$ via the procedure above.

An oriented k–pruning of $(T', \mathcal{O}', T'', \mathcal{O}'')$ of T is called *positive* (resp. *negative*) if $\mathcal{O}'$ is the positive (resp. negative) orientation of T'.

Given a nontrivial k–pruning (T', T'') of T, there are precisely two ways to extend it to an oriented k–pruning $(T', \mathcal{O}, T'', \mathcal{O}'')$, one positive and one negative.

We now mimic the construction we produced for W_{B_n} to describe the coproduct. We thus consider the following maps:

- the positive and negative k–pruning maps
$$P_k^+, P_k^- : \bigoplus_{n\geq 0} \mathrm{FN}'^*_{W_{D_n}} \otimes \mathbb{F}_2 \to \bigoplus_{n\geq 0} \mathrm{FN}'^*_{W_{D_n}} \otimes \bigoplus_{m\geq 0} \mathrm{FN}'^*_{W_{D_m}} \otimes \mathbb{F}_2$$
given by the formula
$$P_k^{\pm}(T) = \sum_{(T',\mathcal{O}',T'',\mathcal{O}'')} (T', \mathcal{O}') \otimes (S_k(T''), \mathcal{O}''),$$
where the sum runs over all positive and negative oriented k–prunings of T, respectively;
- $\widehat{C} : \mathrm{FN}'^*_{W_{D_n}} \otimes \mathrm{FN}'^*_{W_{D_m}} \otimes \mathbb{F}_2 \to \mathrm{FN}'^*_{W_{D_{n+m}}} \otimes \mathbb{F}_2$, the oriented concatenation map, given by the formulas
$$\widehat{C}((\boldsymbol{a}, +) \otimes (\boldsymbol{b}, +)) = (C(\boldsymbol{a} \otimes \boldsymbol{b}), +), \quad \widehat{C}((\boldsymbol{a}, +) \otimes (\boldsymbol{b}, -)) = (C(\boldsymbol{a} \otimes \boldsymbol{b}), -),$$
$$\widehat{C}((\boldsymbol{a}, -) \otimes (\boldsymbol{b}, +)) = (C(\boldsymbol{a} \otimes \boldsymbol{b}), -), \quad \widehat{C}((\boldsymbol{a}, -) \otimes (\boldsymbol{b}, -)) = (C(\boldsymbol{a} \otimes \boldsymbol{b}), +).$$

We can also define $\Delta_k^+, \Delta_k^- : \mathrm{FN}'^*_{W_{D_n}} \otimes \mathbb{F}_2 \to \bigoplus_{i=0}^n \mathrm{FN}'^*_{W_{D_i}} \otimes \mathrm{FN}'^*_{W_{D_{n-i}}} \otimes \mathbb{F}_2$ by the recursive formulas

- $\Delta_0^{\pm} = P_0^{\pm}$,
- $\Delta_k^{\pm} = (\mathrm{id} \otimes \widehat{C})(P_k^{\pm} \otimes \mathrm{id})\Delta_{k-1}^{\pm}$ if $k \geq 1$.

Let Δ be the direct limit $\varinjlim_k (\Delta_k^+ + \Delta_k^-)$.

Proposition 4.14 *The oriented pruning coproduct Δ is a cochain map and induces the coproduct $\Delta : A_D \to A_D \otimes A_D$ in cohomology.*

Proof It is enough to observe that, looking at the proof of Proposition 4.9, we can obtain the map $\Delta\colon \mathrm{FN}'^{*}_{W_{D_n}} \otimes \mathbb{F}_2 \to \bigoplus_{i=0}^{n} \mathrm{FN}'^{*}_{W_{D_i}} \otimes \mathrm{FN}'^{*}_{W_{D_{n-i}}} \otimes \mathbb{F}_2$ from

$$\widetilde{\Delta}\colon \widetilde{\mathrm{FN}}^{*}_{W_{B_n}} \otimes \mathbb{F}_2 \to \bigoplus_{i=0}^{n} \widetilde{\mathrm{FN}}^{*}_{W_{B_i}} \otimes \widetilde{\mathrm{FN}}^{*}_{W_{B_{n-i}}} \otimes \mathbb{F}_2$$

by restricting to W_{D_n}–invariants. □

Proposition 4.15 *Let $\boldsymbol{a}^{\pm}$ and $\boldsymbol{b}^{\pm}$ be generic basis elements of $\mathrm{FN}'^{*}_{W_{D_n}}$ and $\mathrm{FN}'^{*}_{W_{D_m}}$ respectively, where $\boldsymbol{a}$ (resp. $\boldsymbol{b}$) is defined by an n–tuple $\underline{a}$ (resp. an m–tuple $\underline{b}$) of nonnegative integers. Let $\odot\colon \mathrm{FN}'^{*}_{W_{D_n}} \otimes \mathrm{FN}'^{*}_{D_m} \otimes \mathbb{F}_2 \to \mathrm{FN}'^{*}_{W_{D_{n+m}}} \otimes \mathbb{F}_2$ be the homomorphism that maps $\boldsymbol{a}^{\pm} \otimes \boldsymbol{b}^{\pm}$ to the sum of all elements $\boldsymbol{c}^{\pm}$, such that the principal k–blocks of $\boldsymbol{c}$ are obtained by shuffling the principal k–blocks of $\boldsymbol{a}$ and $\boldsymbol{b}$ for all $k \geq 0$, and the sign of $\boldsymbol{c}$ is deduced from the signs of $\boldsymbol{a}$ and $\boldsymbol{b}$ by applying the multiplication sign rule $(+,+) \mapsto +$, $(+,-) \mapsto -$, $(-,+) \mapsto -$, and $(-,-) \mapsto +$. This map is a morphism of complexes and induces the transfer product in cohomology.*

Proof The proof is essentially the same as that of Proposition 4.11. □

5 The almost-Hopf ring presentations

This section contains the statements of the Hopf ring presentation for A_B and the almost-Hopf ring presentation for A_D. We thus state our main theorems, whose proof will be postponed until Section 7 because we still need to develop some necessary algebraic machinery. In the first subsection, we construct our generators, providing cochain representatives and a geometric interpretation. In the second one, we explain our relations and state Theorems 5.9 and 5.15. We then apply these results to extract combinatorially accessible additive bases for A_B and A_D in Section 5.2. Finally, the last subsection is devoted to the link between all these almost-Hopf ring structures.

5.1 Generators

We define certain cohomology classes that we will later prove to generate our (almost) Hopf rings. We begin with A_B.

Definition 5.1 In $\mathrm{FN}^{*}_{W_{B_n}}$, the following cochains are defined for $k \geq 0$, $m > 0$, and $n > 0$:

- $\gamma_{k,m} = [0:\underbrace{\underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}}:0:\underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}}:0:\cdots:0:\underbrace{1:1:\cdots:1}_{2^k-1 \text{ times}}}_{m \text{ times}}]$,
- $\delta_n = [\underbrace{1:1:\cdots:1}_{n \text{ times}}]$.

A direct calculation shows that both $\gamma_{k,m}$ and δ_n have trivial differential, and thus define cohomology classes $\gamma_{k,n} \in H^{m(2^k-1)}(W_{B_{m2^k}}, \mathbb{F}_2)$ and $\delta_n \in H^n(W_{B_n}; \mathbb{F}_2)$, that we still denote, with a slight abuse of notation, by the same symbols. While the proof of this fact is entirely straightforward, we provide a proof for the sake of completeness.

Lemma 5.2 *$\gamma_{k,m}$ and δ_n are cocycles in $\mathrm{FN}^*_{W_{B_{m2^k}}} \otimes \mathbb{F}_2$ and $\mathrm{FN}^*_{W_{B_n}} \otimes \mathbb{F}_2$, respectively.*

Proof $\gamma_{k,m}$ is represented by the symmetric planar level tree in Figure 4. We prove that $d_i(\gamma_{k,m}) = 0$ for all $0 \le i < m2^k$ by considering different cases:

- If $i \neq l2^k$ for $0 \le l < m$, the addend d_i of the differential identifies two edges adjacent in a vertex v_j for $1 \le j \le m$, and performs a vertex shuffle at the new vertex. Exactly two possible vertex shuffles yield the same tree. Hence $d_i(\gamma_{k,m}) = 0$.
- If $i = l2^k$ for some $1 \le l < m$, then $d_i(\gamma_{k,m})$ is obtained by gluing together v_l and v_{l+1} and shuffling the outgoing edges of these two vertices. Since all these shuffles yield the same tree, and there is an even number of them — precisely $\binom{2^{k+1}}{2^k}$ — we have again that $d_i(\gamma_{k,m}) = 0$.
- If $i = 0$, v_1 and its mirror vertex are glued to the central axis of the tree, and the corresponding outgoing edges are permuted with a symmetric shuffle. Again, there is an even number of them (precisely 2^{2^k}), and thus $d_0(\gamma_{k,m}) = 0$.

δ_n is represented by a symmetric planar level tree with $2n+1$ leaves and a single internal vertex of height 1. The same proof used in the second case of $\gamma_{k,m}$ shows that $d_i(\delta_n) = 0$ for all $0 \le i < n$. □

Another possible point of confusion is that the symbol $\gamma_{k,m}$ is used in [8] to indicate a class in $H^{m(2^k-1)}(\Sigma_{m2^k}; \mathbb{F}_2)$. The class we define is the image of this cohomology class of the symmetric group in $H^{m(2^k-1)}(W_{B_{m2^k}}; \mathbb{F}_2)$ via the map induced by the projection $\pi: W_{B_{m2^k}} \to \Sigma_{m2^k}$, as we will prove later (Proposition 5.26).

We can interpret all the cohomology classes that we defined above geometrically.

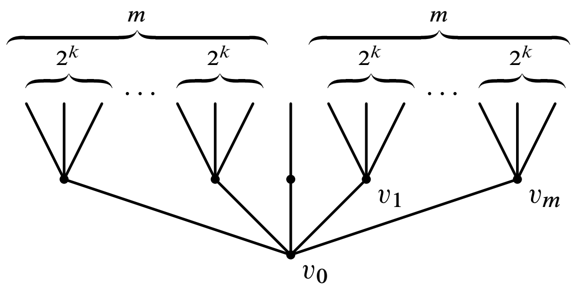

Figure 4: The planar symmetric level tree representing $\gamma_{k,m}$.

Proposition 5.3 *The following statements are true*:

(1) *Consider the proper submanifold $\Gamma_{k,m}$ of $Y^{(\infty)}_{W_{B_{m2^k}}}/W_{B_{m2^k}}$ consisting of 2^m points that can be partitioned into m sets of 2^k points, where all the points in the same subset share the first coordinate. Then $\gamma_{k,m}$ is the Thom class of $\Gamma_{k,m}$ in $Y^{(\infty)}_{W_{B_{m2^k}}}$, in the sense of* [7, Definition 4.6].

(2) *Consider the vector bundle $\eta\colon E(W_{B_n})\times_{W_{B_n}}\mathbb{R}^n\to B(W_{B_n})$, where W_{B_n} acts on $\mathbb{R}^n$ via its irreducible reflection representation. Then δ_n is the n–dimensional Stiefel–Whitney class of η (the nonoriented version of the Euler class).*

Proof The description of $\gamma_{k,n}$ is a direct consequence of the conclusions of the geometric arguments of the previous section.

Regarding the second point, consider the vector bundle $\eta\colon E(\eta)\to B(\eta)$ above, with zero section $\sigma_0\colon B(\eta)\to E(\eta)$, and let $T(\eta)\in H^n(E(\eta),E(\eta)\setminus\sigma_0(B(\eta)))$ be its Thom class. Define

$$\widetilde{X}_n=\{(\underline{x}_1,\dots,\underline{x}_n)\in Y^{(\infty)}_{B_n}\mid(x_1)_1=\cdots=(x_n)_1=0\}.$$

We observe that $\widetilde{X}_n$ is a proper submanifold of $Y^{(\infty)}_{W_{B_n}}$ and that the Thom class of the image X_n of $\widetilde{X}_n$ in $Y^{(\infty)}/W_{B_n}$ is δ_n. We observe that the normal bundle of X_n in $Y^{(\infty)}/W_{B_n}$ is isomorphic to $\eta|_{X_n}$. Since restriction of vector bundles to subspaces preserve Thom classes, we deduce that, if we take $Y^{(\infty)}/W_{B_n}$ as a model for $B(W_{B_n})$, then $j^*(l^*)^{-1}k^*(T(\eta))=\delta_n$, where

- $k\colon(E(\eta|_{X_n}),E(\eta|_{X_n})\setminus\sigma_0(X_n))\to(E(\eta),E(\eta)\setminus\sigma_0(B(\eta)))$,
- $l\colon(E(\eta|_{X_n}),E(\eta|_{X_n})\setminus\sigma_0(X_n))\to(B(W_{B_n}),B(W_{B_n})\setminus X_n)$ is a tubular neighborhood of X_n in $B(\eta)$, and
- $j\colon(B(W_{B_n}),\varnothing)\to(B(W_{B_n}),B(W_{B_n})\setminus X_n)$.

Note that the induced map l^* in cohomology is invertible by excision.

Let $\Phi\colon H^*(B(\eta);\mathbb{F}_2) \to H^*(E(\eta), E(\eta)\setminus\sigma_0(B(\eta));\mathbb{F}_2)$ be the Thom isomorphism. We recall that $\Phi(\alpha) = p^*(\alpha)\cup T(\eta)$, where $p\colon E(\eta)\to B(\eta)$ is the projection. We know, for instance from Milnor and Stasheff's book [13, page 91], that the Thom class $T(\eta)$ and $w_n(\eta)$ are linked by the formula

$$w_n(\eta) = \Phi^{-1}(\mathrm{Sq}^n(T(\eta))) = \Phi^{-1}(T(\eta)^2).$$

Therefore, in order to prove that $w_n(\eta) = \delta_n$, it is sufficient to show that

$$i^*(T(\eta)) = p^* j^* (l^*)^{-1} k^*(T(\eta)),$$

where i is the obvious inclusion map between pairs of spaces,

$$i\colon (E(\eta),\varnothing) \to \big(E(\eta), E(\eta)\setminus\sigma_0(B(\eta))\big).$$

To prove this claim, we first observe that we can use a slightly different model for $B(\eta)$. We recall that there is a tubular neighborhood $\widetilde{N}$ of $\widetilde{X}_n$ in $Y^{(\infty)}_{W_{B_n}}$ determined by an embedding of the total space of the normal bundle. Explicitly, we can define the embedding by the formula

$$(\underline{x}_1,\dots,\underline{x}_n)\times(\lambda_1,\dots,\lambda_n) \in \widetilde{X}_n\times\mathbb{R}^n \mapsto (\underline{x}_1+\lambda_1\boldsymbol{e}_1,\dots,\underline{x}_n+\lambda_n\boldsymbol{e}_1),$$

where $\boldsymbol{e}_1$ is the first element of the canonical basis of $\mathbb{R}^\infty$. Hence

$$\widetilde{N} = \big\{(\underline{x}_1,\dots,\underline{x}_n) \mid (\underline{x}_i - (x_i)_1\boldsymbol{e}_1) \neq \pm(\underline{x}_j - (x_j)_1\boldsymbol{e}_1) \text{ for all } 1\le i<j\le n,\\ (\underline{x}_i - (x_i)_1\boldsymbol{e}_1)\neq 0 \text{ for all } 1\le i\le n\big\}.$$

Note that the action of W_{B_n} preserves $\widetilde{N}$, and $\widetilde{N}$ is provided with a stratification induced from that on $Y^{(\infty)}_{W_{B_n}}$ by restriction. Further, every stratum of N is obtained from a stratum of $Y^{(\infty)}_{W_{B_n}}$ by removing an infinite-codimensional affine subspace. Thus, $\widetilde{N}\to Y^{(\infty)}_{W_{B_n}}$ is a homotopy equivalence. $\widetilde{N}$ is still contractible, and therefore we can use its quotient $N = \widetilde{N}/W_{B_n}$ as an alternative model for $B(W_{B_n})$. In this model, the inclusion l is an isomorphism. Thus we do not need to worry about excision maps, and this simplifies the argument. The claim now follows by observing that i and kjp are homotopic. An explicit homotopy is

$$((\underline{x}_1,\dots,\underline{x}_n),\underline{\lambda},t)\in\widetilde{N}\times_{W_{B_n}}\mathbb{R}^n\times[0,1]\\ \mapsto \big((\underline{x}_1-(1-t)(x_1)_1\boldsymbol{e}_1,\dots,\underline{x}_n-(1-t)(x_n)_1\boldsymbol{e}_1),((1-t)\underline{\lambda}+t((x_1)_1,\dots,(x_n)_1))\big). \quad\square$$

We now turn our attention to W_{D_n}. First, we give the following definition.

Definition 5.4 Let $n \geq 1$ and $m \geq 0$. We define $\delta^0_{n;m} \in H^n(W_{D_{n+m}};\mathbb{F}_2)$ as the restriction of $\delta_n \odot 1_m \in H^*(W_{B_{n+m}};\mathbb{F}_2)$ to the cohomology of $W_{D_{n+m}}$. We also let $\delta^0_{0:m}$ be the unique nonzero class in $H^0(W_{D_m};\mathbb{F}_2)$ for all $m \geq 0$.

We will require some other generators that do not arise as restrictions of cohomology classes of W_{B_n}.

Definition 5.5 Given $k, m \geq 1$, we define two cochains in $\mathrm{FN}^{m2^k}_{W_{D_n}}$:

- $\gamma^+_{k,m} = [0 : \underbrace{\underbrace{1:\cdots:1}_{2^k-1\text{ times}} :0: \underbrace{1:\cdots:1}_{2^k-1\text{ times}} :0:\cdots: \underbrace{1:\cdots:1}_{2^k-1\text{ times}}}_{m\text{ times}}]$;
- $\gamma^-_{k,m} = [1:0: \underbrace{\underbrace{1:\cdots:1}_{2^k-2\text{ times}} :0: \underbrace{1:\cdots:1}_{2^k-1\text{ times}} :0:\cdots: \underbrace{1:\cdots:1}_{2^k-1\text{ times}}}_{m\text{ times}}]$.

Lemma 5.6 *$\gamma^+_{k,m}$ and $\gamma^-_{k,m}$ are cocycles.*

Proof The cochain equivalence φ^* of Lemma 4.1 maps $\gamma^{\pm}_{k,m}$ to

$$[0: \underbrace{\underbrace{1:1:\cdots:1}_{2^k-1\text{ times}} :0: \underbrace{1:1:\cdots:1}_{2^k-1\text{ times}} :0:\cdots:0: \underbrace{1:1:\cdots:1}_{2^k-1\text{ times}}}_{m\text{ times}}]^{\pm}.$$

The same proof used for Lemma 5.2, with the additional requirement of keeping track with orientations, shows that these cochains in $\mathrm{FN}'^{m2^k}_{W_{D_n}}$ are cocycles. As φ^* is injective, $\gamma^{\pm}_{k,m}$ must also be a cocycle.

An alternative proof can be obtained by directly using the De Concini and Salvetti description of the boundary in $C^{W_{D_n}}_*$ and dualizing. $\square$

A consequence of the previous lemma is that $\gamma^+_{k,m}$ and $\gamma^-_{k,m}$ represent cohomology classes, that, once again, we denote by the same symbols with a slight abuse of notation.

To adapt our notation to Giusti and Sinha's for the alternating groups, we will refer to $\gamma^+_{k,m}$ (resp. $\gamma^-_{k,m}$) for some k and m as positively (resp. negatively) charged generators, and to $\delta^0_{n:m}$ for some n and m as neutral generators.

5.2 Relations

This subsection is devoted to deriving algebraic relations between the generators defined above. We will mainly obtain the relations as a consequence of the results in Section 4.

We first focus on A_B. We can retrieve in A_B the same relations among the classes $\gamma_{k,m}$ that appear in Giusti, Salvatore and Sinha's Theorem 2.3.

Proposition 5.7 *The following formulas hold in A_B:*

$$\Delta(\gamma_{k,m}) = \sum_{i+j=m} \gamma_{k,i} \otimes \gamma_{k,j}, \quad \gamma_{k,n} \odot \gamma_{k,m} = \binom{n+m}{n} \gamma_{k,n+m}.$$

Proof We use the chain-level formulas computed in Propositions 4.9 and 4.11.

To compute the coproduct, we represent $\gamma_{k,m}$ by the symmetric planar level tree depicted in Figure 4. Note that $P_l(\gamma_{k,m})$ is trivial for $l \geq 2$ and that the 0–pruning map gives $P_0(\gamma_{k,m}) = \sum_{i+j=m} \gamma_{k,i} \otimes \gamma_{k,j}$. Therefore it is enough to prove that $P_1(\gamma_{k,n}) = \gamma_{k,n} \otimes I$, where I is the trivial symmetric level tree, for all $k \geq 0$ and $n > 0$. Consider a 1–pruning (T', T'') of $\gamma_{k,m}$. Every vertex v_i, for $1 \leq i \leq m$, as depicted in Figure 4 corresponds to a vertex u_i of height 1 in T'. Let $2^k - n_i$ be the number of outgoing edges of u_i for some integer $0 \leq n_i < 2^k$. We can obtain the pruning (T', T'') from $\gamma_{k,m}$ by applying a sequence of elementary 1–prunings at each vertex v_i and their mirror vertices $r(v_i)$ that prunes away a_i outgoing edges from v_i and $n_i - a_i$ outgoing edges from $r(v_i)$, for some $0 \leq a_i \leq n_i$. Therefore, summing over all the possible shuffles of leftovers, whose number is

$$\frac{\left(\sum_{i=1}^m n_i\right)!}{\prod_{i=1}^m a_i! \prod_{i=1}^m (n_i - a_i)!},$$

we deduce that $(T', S_1(T''))$ appears in $P_1(\gamma_{k,m})$ with coefficient

$$\sum_{0 \leq a_1 \leq n_1, \dots, 0 \leq a_m \leq n_m} \frac{\left(\sum_{i=1}^m n_i\right)!}{\prod_{i=1}^m a_i! \prod_{i=1}^m (n_i - a_i)!} = \frac{\left(\sum_{i=1}^m n_i\right)!}{\prod_{i=1}^m n_i!} \prod_{i=1}^m 2^{n_i}.$$

This number is even unless $n_i = 0$ for all $1 \leq i \leq m$, yielding the trivial 1–pruning.

The transfer product formula follows directly from the application of the cochain-level map of Proposition 4.11, by observing that $\gamma_{k,m}$ has m principal 0–blocks all equal to $[1, \dots, 1]$, where the entry 1 is repeated $2^k - 1$ times, and that it has no principal l–block for $l \geq 1$. Thus $\gamma_{k,n} \odot \gamma_{k,m}$ is given by a single basis element in $\mathrm{FN}^*_{W_{B2^k}(n+m)}$ (representing $\gamma_{k,n+m}$) counted as many times as the number of (n, m)–shuffles, that is the binomial coefficient appearing in the equation. □

We can obtain coproduct formulas for δ_n via the same geometric description. The following is again a consequence of the formulas in Lemmas 4.9 and 4.11.

Proposition 5.8 $$\Delta(\delta_n) = \sum_{k+l=n} \delta_k \otimes \delta_l, \quad \delta_n \odot \delta_m = \binom{n+m}{n} \delta_{n+m}.$$

Proof Since all the entries of δ_n are equal to 1, the cochain-level coproduct map on δ_n reduces to the 1–pruning map P_1 and provides the desired formula. We compute the transfer product as in Proposition 5.7, by observing that δ_n has no principal l–blocks for $l \neq 1$, and has n principal 1–blocks all empty. □

These relations will suffice to describe A_B completely. We restate here our main result, which we will prove in Section 7.

Theorem 5.9 (main theorem for type B) *The Hopf ring A_B is generated by classes $\gamma_{k,n}$ (with $k \geq 0$ and $n > 0$) and δ_n (with $n > 0$) with the relations described in Propositions 5.7 and 5.8, together with the following additional relations*:

- *the product $\cdot$ of generators from different components is* 0;
- *$\gamma_{0,n}$ is the $\cdot$–product unit of the n^{th} component.*

We now turn our attention to A_D. A trick borrowed from [9, page 9] can be used to simplify the presentation of this almost-Hopf ring. We recall that there is an involution $\iota\colon A_D \to A_D$. We can define A'_D to be the bigraded $\mathbb{F}_2$–vector space defined by $(A'_D)_{n,d} = H^d(W_{D_n}; \mathbb{F}_2)$ if $(n,d) \neq (0,0)$ and $(A'_D)_{0,0} = \mathbb{F}_2\{1^+, 1^-\}$. We can embed A_D as a vector space in A'_D by identifying the nonzero class in $H^0(W_{D_0}; \mathbb{F}_2)$ with $1^+ + 1^-$.

Lemma 5.10 *The following statements are true in A_D*:

(1) $\iota(x \odot y) = \iota(x) \odot y = x \odot \iota(y)$,

(2) $(\iota \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \iota)\Delta(x) = \Delta\iota(x)$,

(3) $\iota(x \cdot y) = \iota(x) \cdot \iota(y)$.

Proof (1) $\odot$ is commutative, and the following diagram induces a pullback of finite coverings at the level of classifying spaces:

$$\begin{tikzcd} W_{D_n} \times W_{D_m} \arrow[r] \arrow[d, "c_{s_0}\times\mathrm{id}"'] & W_{D_{n+m}} \arrow[d, "c_{s_0}"] \\ W_{D_n} \times W_{D_m} \arrow[r] & W_{D_{n+m}} \end{tikzcd}$$

(2) This follows from the cocommutativity of Δ and the commutativity of the diagram above.

(3) It is the cohomological consequence of the diagonal map being equivariant with respect to the conjugation c_{s_0}. □

Proposition 5.11 *Write the coproduct of every element $x \in A_D$ in A_D as*

$$x \otimes 1_0 + \bar{\Delta}(x) + 1_0 \otimes x,$$

so $\bar{\Delta}$ is the reduced coproduct. By letting $1^- \cdot 1^+ = 0$, $1^- \cdot 1^- = 1^-$, $1^+ \cdot 1^+ = 1^+$, $1^- \odot 1^- = 1^+$ and $\Delta(1^\pm) = 1^+ \otimes 1^\pm + 1^- \otimes 1^\mp$, the almost-Hopf ring structure on A_D extends to an almost-Hopf ring structure on A'_D such that $1^- \odot x = \iota(x)$, $1^- \cdot x = 0$ and $\Delta(x) = 1^+ \otimes x + 1^- \otimes \iota(x) + \bar{\Delta}(x) + x \otimes 1^+ + \iota(x) \otimes 1^-$ for every $x \in A'_D$ of positive degree.

Proof Using the formulas in the statement of this proposition, we can extend $\odot$ and $\cdot$ uniquely to two commutative products on A'_D and Δ to a unique cocommutative coproduct on A'_D. The coassociativity of Δ follows from Lemma 5.10(3). The associativity of $\cdot$ on A'_D is obvious. The bialgebra structure of A'_D with $\cdot$ and Δ follows from the bialgebra structure on A_D and (2) of the previous lemma. Moreover, the fact that the transfer product with 1^- is associative follows from (1). Hopf ring distributivity with classes involving a transfer product with 1^- follows again from (3) of the result referenced above. □

Instead of determining a presentation for A_D, we calculate a presentation for A'_D because we can write it more concisely. For example, $\gamma^-_{k,m} = 1^- \odot \gamma^+_{k,m}$ in A'_D; thus the formulas for $\gamma^-_{k,m}$ arise as a direct consequence of the formulas for $\gamma^+_{k,m}$ and the almost-Hopf ring structure of A'_D. The two approaches are equivalent.

Proposition 5.12 *Let $k, m \geq 1$ and $n \geq 0$. The following coproduct formulas hold in A'_D, where $\gamma^-_{k,m-l} = 1^- \odot \gamma^+_{k,m-l}$:*

$$\Delta(\gamma^+_{k,m}) = \sum_{l=0}^{m} \gamma^+_{k,l} \otimes \gamma^+_{k,m-l} + \gamma^-_{k,l} \otimes \gamma^-_{k,m-l},$$

$$\Delta(\delta^0_{n:m}) = \sum_{i=0}^{n} \sum_{j=0}^{m} \delta^0_{i:j} \otimes \delta^0_{n-i:m-j}.$$

Moreover, the transfer product in A'_D satisfies the following formulas for every choice of indexes:

$$\gamma^+_{k,a} \odot \gamma^+_{k,b} = \binom{a+b}{a}\gamma^+_{k,a+b},$$

$$b \odot b' = 0 \text{ if } b \text{ and } b' \text{ are cup products of neutral generators (ie } \delta^0_{n:m}\text{)},$$

$$\delta^0_{n:m} \odot 1^- = \delta^0_{n:m}.$$

Proof Note that $\gamma^-_{k,m} = 1_- \odot \gamma^+_{k,m} = \iota(\gamma^+_{k,m})$ as a direct consequence of Lemma 4.2 and the definition of the cochain representatives of these classes. The coproduct formulas for $\gamma^+_{k,m}$ follow from Lemma 4.1 and Proposition 4.14. More precisely, we observe that mapping $\gamma^+_{k,m}$ into $\mathrm{FN}'^*_{W_{D_{m2^k}}} \otimes \mathbb{F}_2$ via φ^* yields a cohomology class represented by the same symmetric planar level tree of Figure 4, with positive orientation. The same proof of Proposition 5.7 holds in this case by keeping track of orientations.

The coproduct formula for $\delta^0_{n:m}$ is a consequence of Proposition 5.8, the Hopf ring properties of A_B, and the fact that the restriction map $\rho\colon A_B \to A_D$ preserves coproducts.

Regarding transfer product, we prove the first identity using Proposition 4.15 precisely in the same way as the second part of Proposition 5.7.

Let $\rho\colon A_B \to A_D$ be the restriction map. For every $x \in H^*(W_{B_n};\mathbb{F}_2)$ and $y \in H^*(W_{B_m};\mathbb{F}_2)$, we can prove that $\rho(x) \odot \rho(y) = 0$ in $H^*(W_{D_{n+m}};\mathbb{F}_2)$ with the same argument used in [9, Proposition 3.14]. Essentially, it is sufficient to observe that both the restriction $H^*(W_{B_n} \times W_{B_m};\mathbb{F}_2) \to H^*(W_{D_n} \times W_{D_m};\mathbb{F}_2)$ and the transfer $H^*(W_{D_n} \times W_{D_m};\mathbb{F}_2) \to H^*(W_{D_{n+m}};\mathbb{F}_2)$ factor through the cohomology of the subgroup $G = W_{D_{n+m}} \cap (W_{B_n} \times W_{B_m})$, and that the composition

$$H^*(G;\mathbb{F}_2) \xrightarrow{\text{res}} H^*(W_{D_n} \times W_{D_m};\mathbb{F}_2) \xrightarrow{\text{tr}} H^*(G;\mathbb{F}_2)$$

is 0 for mod 2 coefficients because $W_{D_n} \times W_{D_m}$ has even index in G. In particular, nontrivial transfer products of blocks obtained by cup-multiplying neutral generators must be 0. The last relation also follows from the invariance of $\delta^0_{n:m}$ with respect to the involution ι. □

After these coproduct and transfer product formulas, we will also need some cup product relations. Since the Fox–Neuwirth type cell complex does not behave well with cup products, we found that it is simpler to obtain these formulas via restriction to elementary abelian subgroups. This approach is fruitful because of a detection theorem for these subgroups. We postpone the proof of the following proposition to Section 6, where we will explain this in detail.

Proposition 5.13 *Let $\gamma^-_{k,m} = 1^- \odot \gamma^+_{k,m}$ as an element of A'_D. Then the following formulas hold in A_D:*

(1) $\gamma^+_{k,n} \cdot \gamma^-_{h,m} = 0$ *for all* $n, m, k \geq 1$ *and* $h \geq 2$*;*

(2) $\gamma^+_{1,m}\gamma^-_{1,m} = (\gamma^+_{1,m-1})^2 \odot \delta^0_{2:0}$ *for all* $m \geq 1$*;*

(3) *the $\cdot$ product of generators belonging to different components is* 0 *and* $\delta^0_{0:m}$ *is the $\cdot$–product unit of the m^{th} component*

(4) $\delta^0_{1:m} = 0$ *for all* $m \geq 0$*;*

(5) $\delta^0_{n:m} \cdot \gamma^+_{k,\frac{n+m}{2^k}} = \delta^0_{n:0} \cdot \gamma^+_{k,\frac{n}{2^k}} \odot \gamma^+_{k,\frac{m}{2^k}}$ *for all* $k > 0$ *and* $m, n \geq 0$*, where we understand that* $\gamma^+_{k,r} = 0$ *if* r *is not an integer.*

The last relation we require involves the behavior of the coproduct with the transfer product. We need a preliminary remark. Let $b \in A'_D$ be an element obtained as a cup-product of positively and neutrally charged generators (ie $\gamma^+_{k,m}$ or $\delta^0_{n:m}$), with at least one positively charged generator. Note that, by Propositions 5.12 and 5.13, $\Delta(b)$ can be written as a sum $\sum_i b'_i \otimes b''_i$ where b'_i and b''_i are elements obtained as iterated transfer products of elements of the same form, or the images of such elements via the involution $\iota = 1^- \odot \_$. We let $\Delta'(b)$ be that sum restricted only to addends $b'_i \otimes b''_i$ in which the involution is not performed to obtain b'_i or b'_i is fixed by ι and the involution is not performed to obtained b''_i. As Δ is $(\iota \otimes \iota)$–invariant, this intuitively amounts to keeping half of the addends of the coproduct in A'_D.

Proposition 5.14 (cf [9, Theorem 3.21]) *Let $\tau : \alpha \otimes \beta \in A'_D \otimes A'_D \mapsto \beta \otimes \alpha \in A'_D \otimes A'_D$ be the map that exchanges the two factors. For all $b \in A'_D$ the cup-product of positively and neutrally charged generators, with at least a positively charged generator appearing, and for all $x \in A'_D$, we have that*

$$\Delta(b \odot x) = (\odot \otimes \odot) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta'(b) \otimes \Delta(x)),$$

where Δ' is the expression described above.

The proof of the analog of this proposition is done in [9] by a careful examination of certain spectral sequences. It can be done this way also for A_D. Still, we decide to argue here using detection by elementary abelian subgroups that for finite Coxeter groups comes for free and leads to a shorter proof. Therefore, we postpone the proof of this proposition until the next section.

We restate our presentation theorem for A'_D, whose proof we postpone.

Theorem 5.15 (main theorem for type D) *A'_D is generated, as an almost-Hopf ring, by the classes $\delta^0_{n:m}$ for $n \geq 0$ and $m \geq 0$, $\gamma^+_{k,m}$ for $k, m \geq 1$, and 1^- defined above, under the relations described in Propositions 5.12, 5.13 and 5.14 and the relations $1^- \odot 1^- = 1^+$, $1^- \cdot 1^- = 1^-$, $1^+ \cdot 1^- = 0$ and $\Delta(1^-) = 1^+ \otimes 1^- + 1^- \otimes 1^+$ coming from Proposition 5.11.*

5.3 Additive bases

We describe here additive bases for A_B and A_D. In this subsection, we assume that the statements of Theorems 5.9 and 5.15 are true. They do not rely logically on the existence of such bases in A_B and A_D. Thus this choice does not invalidate their proof.

We begin with A_B.

Definition 5.16 (cf [7]) A *gathered block* in A_B is an element of the form

$$b = \delta^{t_0}_m \prod_{k=1}^{n} \gamma^{t_k}_{k,\frac{m}{2^k}},$$

where m is a positive integer, 2^n divides m, and n is the maximal index such that $\gamma_{n,\frac{m}{2^n}}$ appears in b with a nonzero exponent. The *profile* of b is the $(n+1)$–tuple $(t_0, \dots, t_n)$. We also allow $n = 0$: in this case, $b = \delta^{t_0}_m$ for some $t_0 \geq 0$.

A *Hopf monomial* is a transfer product of gathered blocks $x = b_1 \odot \cdots \odot b_r$. We denote by $\mathcal{M}_B$ the set of Hopf monomials whose constituent gathered blocks have pairwise different profiles.

Note that, given a possible profile $(t_0, \dots, t_n)$, for all $l \geq 1$, there is a unique gathered block b in the $(l2^n)^{\text{th}}$ component having that profile. As a notational convention, we denote it $b_{l,\underline{t}}$.

We can describe elements of $\mathcal{M}_B$ graphically. We represent $\gamma_{k,n}$ as a rectangle of width $n2^k$ and height $1 - 2^{-k}$ and δ_n as a rectangle of width n and height 1. The width of a box is the number of the component to which the class belongs. Its area is its cohomological dimension. We understand the cup product of two generators as stacking the corresponding boxes on top of the other. In contrast, their transfer product corresponds graphically to placing them next to each other horizontally. The profile of a gathered block is described by the height of the rectangles of the corresponding column. Thus, we can represent every gathered block as a column made of boxes with the same width. Hence, an element of $\mathcal{M}_B$ is a diagram consisting of columns placed next to each other, such that there are not two columns that consist of rectangles of the

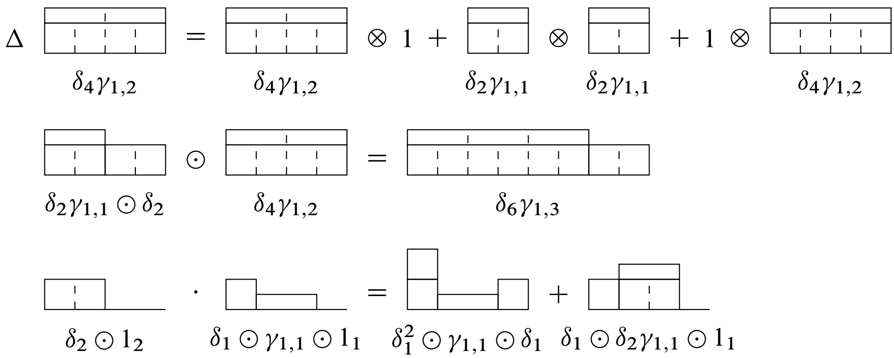

Figure 5: Computations via skyline diagrams.

same height. Following the notation of Giusti, Salvatore and Sinha [7], we call these objects *B–skyline diagrams* or, more concisely, *skyline diagrams* where it is clear that we are considering the Hopf ring A_B.

As in [7], the coproduct and the two products in $\mathcal{M}_B$ have a graphical description, derived from our relations:

- We divide rectangles corresponding to δ_n or $\gamma_{k,n}$ in n equal parts via vertical dashed lines. The coproduct is then given by dividing along all vertical lines (dashed or not) of full height and then partitioning the new columns into two to make two new skyline diagrams.

- The transfer product of two skyline diagrams is given by placing them next to each other and merging every two columns with constituent boxes of the same heights, with a coefficient of 0 if the widths of these columns share a 1 in their binary expansion.

- To compute the cup product of two diagrams, we consider all possible ways to split each into columns, along vertical lines (dashed or not) of full height. We then match columns of each in all possible ways up to automorphism and stack the resulting matched columns to build a new diagram.

We depict some examples of calculations with skyline diagrams in Figure 5.

Proposition 5.17 (cf [7, Proposition 6.4]) *$\mathcal{M}_B$ is an additive basis for A_B and Δ, and $\odot$ and $\cdot$ of basis elements are computed graphically via the algorithmic procedures described above.*

Proof We first prove the correctness of the graphical interpretation of the structural morphisms. The coproduct of a gathered block $b_{m,\underline{t}}$ with profile $(t_0, \dots, t_n)$ is of the form

$$\Delta(b_{m,\underline{t}}) = \sum_{i+j=m} b_{i,\underline{t}} \otimes b_{j,\underline{t}}.$$

We prove this formula by induction on the number of cup-product generators constituting $b_{i,\underline{t}}$: for single generators δ_m or $\gamma_{k,\frac{m}{2^k}}$ the formula appears in the set of relations for A_B, and the induction step is a consequence of the bialgebra structure formed by $\cdot$ and Δ. Thus, the graphical procedure for the calculation of the coproduct is correct on single-column skyline diagrams. As a general skyline diagram represents the transfer product of its columns, the general algorithm is justified because Δ and $\odot$ form a bialgebra.

Regarding $\odot$, the transfer product of two Hopf monomials corresponds to the horizontal juxtaposition of the corresponding skyline diagrams. Thus, we only need to justify the merging of columns. In formulas, this reads as follows. Fix a profile $\underline{t} = (t_0, \dots, t_n)$, with $t_k \geq 0$ for $0 \leq k < n$ and $t_n > 0$. Then

$$b_{i,\underline{t}} \odot b_{j,\underline{t}} = \binom{i+j}{i} b_{i+j,\underline{t}}.$$

Again, we prove this by induction on $r = t_0 + \cdots + t_n$. For $r = 1$, gathered blocks with profile $\underline{t}$ are single generators, and the formula above is exactly our transfer product relation among them. For $r > 1$, the induction step is proved by combining the coproduct formula for $\gamma_{l,(i+j)2^{n-l}}$, Hopf ring distributivity, and the fact that cup products of elements in different components is 0 to deduce that

$$\gamma_{l,(i+j)2^{n-l}}(b_{i,\underline{t}} \odot b_{j,\underline{t}}) = (b_{i,\underline{t}} \cdot \gamma_{l,i2^{n-l}}) \odot (b_{j,\underline{t}} \cdot \gamma_{l,j2^{n-l}}),$$

or the analogous formula with δ_{i+j} in place of $\gamma_{n,(i+j)2^{n-l}}$ if $n = 0$.

The $\cdot$–product algorithm above graphically encodes Hopf ring distributivity.

Finally, we prove that $\mathcal{M}_B$ is an additive basis for A_B. We consider the bigraded vector space V over $\mathbb{F}_2$ with skyline diagrams or, equivalently, $\mathcal{M}_B$ as a basis. Define linear maps $\odot, \cdot : V \otimes V \to V$ and $\Delta : V \to V \otimes V$ by computing their values on basis elements via the algorithm above. Note that these maps define a Hopf ring structure on V. There is a map $V \to A_B$ that realizes every Hopf monomial as the corresponding element of A_B. Since the procedures to compute the structural morphisms on $\mathcal{M}_B$ are deduced from the Hopf ring structure of A_B and the relations of Theorem 5.9, this map is a morphism of Hopf rings. We also note that V is generated as a Hopf ring by single rectangles, corresponding to $\gamma_{k,n}$ and δ_n, and that the relations of Theorem 5.9 are satisfied in V. Since A_B is presented by such generators and relations, it follows that the map $V \to A_B$ is an isomorphism. $\square$

We now construct an additive basis for A_D, assuming Theorem 5.15. The first step is to identify the subalgebra of A_D under the cup product generated by neutral generators. Let $\widetilde{\mathcal{B}}^0$ be the set of Hopf monomials $x \in A_B$ of the form $x = \delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}$, ordered with $a_1 > \cdots > a_r$ and $k_1 \geq 2$. These correspond to skyline diagrams in which only boxes of height 1 appear and in which the highest column has width strictly bigger than 1.

Lemma 5.18 *Every element of $\widetilde{\mathcal{B}}^0 \cap H^*(W_{B_n};\mathbb{F}_2)$ lies in the cup product subalgebra generated by $\delta_n, \delta_{n-1} \odot 1_1, \dots, \delta_1 \odot 1_{n-1}$. Moreover, the images in A_D of elements of $\widetilde{\mathcal{B}}^0$ are a vector space basis for the subalmost-Hopf ring generated by elements of the form $\delta^0_{n:m}$ for $n, m \geq 0$.*

Proof Let $\overline{\mathcal{B}}^0$ be the set of Hopf monomials $x \in A_B$ of the form $x = \delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}$ ordered with $a_1 > \cdots > a_r$, without the condition $k_1 \geq 2$. We can define an injective function $\varepsilon_n \colon \overline{\mathcal{B}}^0 \cap H^*(W_{B_n};\mathbb{F}_2) \to \mathbb{N}^n$ given by

$$\varepsilon_n(\delta_{k_1}^{a_1} \odot \cdots \odot \delta_{k_r}^{a_r}) = (\underbrace{a_1, \dots, a_1}_{k_1 \text{ times}}, a_2, \dots, a_{r-1}, \underbrace{a_r, \dots, a_r}_{k_r \text{ times}})$$

By identifying $\overline{\mathcal{B}}^0 \cap H^*(W_{B_n};\mathbb{F}_2)$ with a subset of $\mathbb{N}^n$ this way, the lexicographic ordering on $\mathbb{N}^n$ induces a total order on $\overline{\mathcal{B}}^0$. We observe that $\prod_{i=1}^n (\delta_i \odot 1_{n-i})^{a_i}$ is a linear combination of elements of $\overline{\mathcal{B}}^0$. In this linear combination, the maximal nonzero Hopf monomial corresponds to $\bigl(\sum_{i=1}^n a_i, \sum_{i=2}^n a_i, \dots, a_{n-1} + a_n, a_n\bigr)$. Moreover, this belongs to $\widetilde{\mathcal{B}}^0$ if and only if $a_1 = 0$, ie if and only if $\delta_1 \odot 1_{n-1}$ does not appear as a factor. Since these are all different, $\delta_n, \delta_{n-1} \odot 1_1, \dots, \delta_1 \odot 1_{n-1}$ generate, under the cup product, a polynomial subalgebra with basis $\overline{\mathcal{B}}^0 \cap H^*(W_{B_n};\mathbb{F}_2)$. By Proposition 5.13, the kernel of the restriction map to $H^*(W_{D_n};\mathbb{F}_2)$ on this subalgebra is the ideal generated by $\delta_1 \odot 1_{n-1}$. Consequently, the images of elements of $\widetilde{\mathcal{B}}^0$ in A'_D are a basis for the cup product subalgebra generated by the elements $\delta^0_{n:m}$. Since the transfer products of these elements are trivial and this subalgebra is closed under coproduct by Proposition 5.12, this is a subalmost-Hopf ring. $\square$

Definition 5.19 We call a *neutral gathered block* in A_D an element $b \in A'_D$ obtained as the image in A'_D of an element of the set $\widetilde{\mathcal{B}}^0$ considered in the previous lemma. A *positively charged gathered block*, or simply *positive gathered block*, is an element of the form $b=(\delta^0_{2^n m:0})^{t_0}\prod_{i=1}^n(\gamma^+_{k,m2^{n-k}})^{t_k}$, for some $n,m\geq 1$, $t_k\geq 0$ for $0\leq k<n$ and $t_n>0$. The *profile* of b is $(t_0,\dots,t_n)$. A *negatively charged gathered block*, or simply *negative gathered block*, is an element of the form $b=(\delta^0_{2^n m:0})^{t_0}\prod_{i=1}^n(\gamma^+_{k,m2^{n-k}})^{t_k}$, for some $n,m\geq 1$, $t_k\geq 0$ for $0\leq k<n$ and $t_n>0$. The *profile* of b is $(t_0,\dots,t_n)$. A *Hopf monomial* is a transfer product of gathered blocks.

Note that, given a possible profile $\underline{t}=(t_0,\dots,t_n)$, for all $l\geq 1$, there is a unique positively (resp. negatively) charged gathered block in the $(l2^n)^{\text{th}}$ component having that profile. As a notational convention, we denote it by $b^+_{l,\underline{t}}$ (resp. $b^-_{l,\underline{t}}$). Moreover, we stress that we require that a positively charged generator and a negatively charged one do not appear in the same gathered block. This is not a restriction since, due to Proposition 5.13, a cup product of two such generators is 0, or we can write it as a transfer product of gathered blocks. Therefore Hopf monomials generate A'_D as an $\mathbb{F}_2$–vector space.

We also define a filtration of A'_D that we will use to extract an additive basis from this set of (linear) generators.

Definition 5.20 Define the *weight* of a neutral gathered block b as $\mathrm{w}(b)=0$. Define the weight of a positively or negatively charged gathered block $b^\pm_{l,\underline{t}}$, with profile $\underline{t}=(t_0,\dots,t_n)$, $n\geq 1$, as $\mathrm{w}(b^\pm_{l,\underline{t}})=l2^{n-1}t_1$. Define the weight of a Hopf monomial $x=b_1\odot\cdots\odot b_r$ as the sum $\mathrm{w}(x)=\mathrm{w}(b_1)+\cdots+\mathrm{w}(b_r)$ of the weights of its constituent gathered blocks. Define the *weight filtration* as the increasing filtration $F(A'_D)=\{F_n(A'_D)\}_{n=0}^\infty$ of A'_D such that $F_n(A'_D)$ is the linear span of Hopf monomials in A'_D of weight at most n.

We first compute formulas for the coproduct and transfer product of gathered blocks in A'_D. These are essentially the charged versions of the corresponding identities in A_B, except for gathered blocks involving the generators $\gamma^\pm_{1,n}$, for which this is true only in the graded space $\mathrm{gr}_F(A'_D)$ associated with the weight filtration. Complete formulas in A'_D are complicated and can be retrieved recursively on the filtration F.

Lemma 5.21 *Let $n\geq 1$. Let $\underline{t}=(t_0,\dots,t_n)$ with $t_k\geq 0$ for all $0\leq k<n$ and $t_n>0$. In any almost-Hopf ring satisfying the relations of Theorem 5.15 the following statements are true for all $i,j>0$:*

1. $1^-\odot b^+_{i,\underline{t}}=b^-_{i,\underline{t}}$ *and* $1^-\odot b^-_{i,\underline{t}}=b^+_{i,\underline{t}}$;
2. *if $n>1$, the coproduct satisfies* $\Delta(b^+_{m,\underline{t}})=\sum_{i+j=n}(b^+_{i,\underline{t}}\otimes b^+_{j,\underline{t}}+b^-_{i,\underline{t}}\otimes b^-_{j,\underline{t}})$ *and* $\Delta(b^-_{m,\underline{t}})=\sum_{i+j=n}(b^+_{i,\underline{t}}\otimes b^-_{j,\underline{t}}+b^-_{i,\underline{t}}\otimes b^+_{j,\underline{t}})$;

(3) *if* $n > 1$, $\odot$ *satisfies* $b^+_{i,\underline{t}} \odot b^+_{j,\underline{t}} = b^-_{i,\underline{t}} \odot b^-_{j,\underline{t}} = \binom{i+j}{i} b^+_{i+j,\underline{t}}$ *and* $b^+_{i,\underline{t}} \odot b^-_{j,\underline{t}} = b^-_{i,\underline{t}} \odot b^+_{j,\underline{t}} = \binom{i+j}{i} b^-_{i+j,\underline{t}}$;

(4) *for all neutral gathered block* b^0, $b^+_{i,\underline{t}} \odot b^0 = b^-_{i,\underline{t}} \odot b^0$;

(5) *for all profiles* $\underline{u}$, *possibly different from* $\underline{t}$, $b^+_{i,\underline{t}} \odot b^+_{j,\underline{u}} = b^-_{i,\underline{t}} \odot b^-_{j,\underline{u}}$, $b^+_{i,\underline{t}} \odot b^-_{j,\underline{u}} = b^-_{i,\underline{t}} \odot b^+_{j,\underline{u}}$;

(6) *if* $n > 1$, *for all profiles* $\underline{u}$, $b^+_{i,\underline{t}} \cdot b^-_{j,\underline{u}} = b^-_{i,\underline{t}} \cdot b^+_{j,\underline{u}} = 0$;

(7) *for all neutral gathered block* b', $b^-_{i,\underline{t}} \cdot b' = 1^- \odot (b^+_{i,\underline{t}} \cdot b')$.

Moreover, (2), (3) *and* (6) *are true in* $\mathrm{gr}_F(A'_D)$ *even if* $n = 1$.

Proof (1) Recall that, by definition, $\gamma^-_{k,m} = 1^- \odot \gamma^+_{k,m}$. Combining the link between transfer product and coproduct provided by Proposition 5.14 with the coproduct formula for 1^- and $\gamma^+_{k,m}$, we deduce that

$$\Delta(\gamma^-_{k,m}) = \sum_{i+j=m} (\gamma^+_{k,i} \otimes \gamma^-_{k,j} + \gamma^-_{k,i} \otimes \gamma^+_{k,j}),$$

with the convention that $\gamma^{\pm}_{k,0} = 1^{\pm}$. Then, we can prove that $1^- \odot b^+_{i,\underline{t}} = b^-_{i,\underline{t}}$ by induction on the number of cup-product factors of the involved gathered block. If $b^+_{i,\underline{t}}$ is a single generator $\gamma^+_{k,m}$, the statement holds by definition. The induction step

$$(b^-_{i,\underline{t}} \cdot \gamma^-_{k,\frac{i2^n}{2^k}}) = (1^- \odot b^+_{i,\underline{t}}) \cdot \gamma^-_{k,\frac{i2^n}{2^k}} = 1^- \odot (b^+_{i,\underline{t}} \cdot \gamma^+_{k,\frac{i2^n}{2^k}})$$

is deduced from Hopf ring distributivity and the coproduct formula derived above for $\gamma^-_{k,m}$, using that $1^- \cdot 1^- = 1^-$, $1^- \cdot 1^+ = 0$ and that the cup product of elements in different components is zero. The statement for negatively charged gathered blocks is obtained from its analog for positively charged ones by taking the transfer product of both members of the identity with 1^- and using the relation $1^- \odot 1^- = 1^+$.

(2) We begin with the case of positively charged gathered blocks $b^+_{m,\underline{t}}$. We proceed, again, by induction on the number of $\cdot$ generators appearing in the expression of $b^+_{m,\underline{t}}$. If $b^+_{m,\underline{t}}$ is a single generator, then the statement holds by the coproduct identities of Proposition 5.12. The induction step follows from the fact that $\cdot$ and Δ form a bialgebra, and relations 1,2,3 of Proposition 5.13. For instance, for $k \geq 2$, we explicitly have

$$\Delta(b^+_{m,\underline{t}} \cdot \gamma^+_{k,\frac{m2^n}{2^k}}) = (\cdot \otimes \cdot)(\mathrm{id} \otimes \tau \otimes \mathrm{id}) \Bigg[\sum_{i+j=m} (b^+_{i,\underline{t}} \otimes b^+_{j,\underline{t}} + b^-_{i,\underline{t}} \otimes b^-_{j,\underline{t}}) \otimes \sum_{r+s=\frac{m2^n}{2^k}} (\gamma^+_{k,r} \otimes \gamma^+_{k,s} + \gamma^-_{k,r} \otimes \gamma^-_{k,s}) \Bigg]$$

$$= \sum_{i+j=m} \big(b^+_{i,\underline{t}} \cdot \gamma^+_{k,\frac{i2^n}{2^k}} \otimes b^+_{j,\underline{t}} \cdot \gamma^+_{k,\frac{j2^n}{2^k}} + b^-_{i,\underline{t}} \cdot \gamma^-_{k,\frac{i2^n}{2^k}} \otimes b^-_{j,\underline{t}} \cdot \gamma^-_{k,\frac{j2^n}{2^k}} + b^+_{i,\underline{t}} \cdot \gamma^-_{k,\frac{i2^n}{2^k}} \otimes b^+_{j,\underline{t}} \cdot \gamma^-_{k,\frac{j2^n}{2^k}} + b^-_{i,\underline{t}} \cdot \gamma^+_{k,\frac{i2^n}{2^k}} \otimes b^-_{j,\underline{t}} \cdot \gamma^+_{k,\frac{j2^n}{2^k}}\big)$$

$$= \sum_{i+j=m} \big(b^+_{i,\underline{t}} \cdot \gamma^+_{k,\frac{i2^n}{2^k}} \otimes b^+_{j,\underline{t}} \cdot \gamma^+_{k,\frac{j2^n}{2^k}} + b^-_{i,\underline{t}} \cdot \gamma^-_{k,\frac{i2^n}{2^k}} \otimes b^-_{j,\underline{t}} \cdot \gamma^-_{k,\frac{j2^n}{2^k}}\big).$$

We only need to be careful when $k_1 > 1$ because $\gamma^+_{1,l}\gamma^-_{1,l}$ is not necessarily 0. Note that for $k \geq 2$ we have by Hopf ring distributivity

$$\gamma^\pm_{k,r}\gamma^+_{1,2^{k-1}r}\gamma^-_{1,2^{k-1}r} = \gamma^\pm_{k,r}((\gamma^+_{1,2^{k-1}r-1})^2 \odot \delta^0_{2:0}) = 0,$$

because the coproduct of $\gamma^\pm_{k,r}$ does not have an addend $x' \otimes x''$ with the component of x'' equal to 2. This observation guarantees that, if $n > 1$, the mixed-charge terms vanish. Even if $n = 1$, we obtain the additional terms by applying relation 2 of Proposition 5.13 to expressions of this form, and this procedure lowers weights. Thus, the desired formula holds in $\mathrm{gr}_F(A'_D)$ in this case.

The formulas for negatively charged gathered blocks are, once again, obtained by applying the transfer product with 1^-.

(3) The formula is easily deduced from the coproduct formulas (2) by induction on the number of $\cdot$–product generators appearing in $b^\pm_{i,\underline{t}}$. In the case $n = 1$, we use the obvious fact that $\odot$ preserves the weight filtration to deduce that the desired formula holds in the graded space.

(4) This is a combination of (1) and the relations $1^- \odot \delta^0_{n:m} = \delta^0_{n:m}$.

(5) This is a combination of (1) and the relations $1^\pm \odot 1^\pm = 1^+$ and $1^\pm \odot 1^\mp = 1^-$.

(6) If $n > 1$, it follows directly from relation (1) of Proposition 5.13. If $n = 1$, assume that $b^+_{i,\underline{t}} \in F_a$ and $b^-_{j,\underline{u}} \in F_b$. Relation (2) of Proposition 5.13 provides a way to write $b^+_{i,\underline{t}} \cdot b^-_{j,\underline{u}}$ as a product of the form $((\gamma^+_{1,l-1})^2 \odot \delta^0_{2:0}) \cdot b^+_{i,\underline{t}'} \cdot b^-_{j,\underline{u}'}$ for some $l \geq 1$, where $b^+_{i,\underline{t}'} \in F_{a-l}$ and $b_{i,\underline{u}'} \in F_{b-l}$. By relation (5) of the same proposition, these $\cdot$ products preserve the weight filtration. Therefore the statement is true in $\mathrm{gr}_F(A'_D)$.

(7) We argue as we did for (1), combining the formula given in (1) with the relation $1^- \odot \delta_{n:m} = \delta_{n:m}$, which implies that neutral gathered blocks are invariant by the action of $1^- \odot \_$. □

Using this lemma, we can use Hopf monomials in the additive basis for A_B to construct basis elements of A_D by adding charges.

Proposition 5.22 *Let $\mathcal{M}_B$ be the Hopf monomial basis for A_B of Proposition 5.17. Let $\widetilde{\mathcal{B}}^0 \subseteq \mathcal{M}_B$ be as in Lemma 5.18. Let $\widetilde{\mathcal{B}}^c$ be the subset of $\mathcal{M}_B$ consisting of nontrivial Hopf monomials in which every constituent gathered block has profile $\underline{t} = (t_0, \dots, t_n)$ with $n \geq 1$. Let $\mathcal{M}_D = \mathcal{B}^0 \sqcup \mathcal{B}^+ \sqcup \mathcal{B}^- \subseteq A'_D$, where*

- $\mathcal{B}^+ = \{x^+ = \bigodot_{i=1}^k b^+_{l_i,\underline{t}_i}\}_{x=\bigodot_{i=1}^k b_{l_i,\underline{t}_i} \in \widetilde{\mathcal{B}}^c} \cup \{1^+\}$,
- $\mathcal{B}^- = \{x^- = \bigodot_{i=1}^{k-1} b^+_{l_i,\underline{t}_i} \odot b^-_{l_k,\underline{t}_k}\}_{x=\bigodot_{i=1}^k b_{l_i,\underline{t}_i} \in \widetilde{\mathcal{B}}^c} \cup \{1^-\}$,
- $\mathcal{B}^0 = \{x^0 = \rho(y) \odot z^+\}_{x=y\odot z, y \in \widetilde{\mathcal{B}}^0 \setminus \{1_0\}, z \in \widetilde{\mathcal{B}}^c \cup \{1_0\}}$.

Then $\mathcal{M}_D$ is an additive basis for A'_D.

Before providing a proof of this statement, we make a remark that clarifies the cumbersome identity of Proposition 5.14.

Remark 5.23 Proposition 5.22 provides a direct sum decomposition of A'_D as an $\mathbb{F}_2$–vector space with three addends, V^+, V^- and V^0, defined as the linear span of $\mathcal{B}^+$, $\mathcal{B}^-$ and $\mathcal{B}^0$, respectively. Note that the involution $\iota = 1^- \odot \_$ switches V^+ and V^- and fixes all elements of V^0 by Lemma 5.21. We can consider the linear projection $\pi \colon V \to V^+$ defined as the identity on V^+ and as 0 on V^- and V^0. With this notation, we can rewrite Proposition 5.14 as

$$\Delta(b \odot x) = (\odot \otimes \odot)(\pi \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta)(b \otimes x)$$

for all $x \in A'_D$ and b charged gathered block.

A further reduction can be performed. We can consider the free $\mathbb{F}_4$–module $\widetilde{V}$ with basis $\mathcal{M}_D$ and define a $\mathbb{Z}$–linear map $\widetilde{\Delta} \colon A'_D \otimes A'_D \to \widetilde{V}$ as follows. Given $x, y \in \mathcal{M}_D$, first compute the expansions of the coproducts $\Delta(x) = \sum_i x'_i \otimes x''_i$ and $\Delta(y) = \sum_j y'_j \otimes y''_j$ on the basis $\mathcal{M}_D \otimes \mathcal{M}_D$ of $A'_D \otimes A'_D$. Then, let

$$\widetilde{\Delta}(x \otimes y) = \sum_i \sum_j (x'_i \odot y'_j) \otimes (x''_i \odot y''_j),$$

where, this time, the sum is computed in $\widetilde{V}$. Recall that both $\odot$ and Δ are $(\iota \otimes \iota)$–invariant. Hence, each addend appears twice, except possibly the elements of the form $(x'_i \odot y'_j) \otimes (x''_i \odot y''_j)$, where x'_i, x''_i, y'_j and y''_j are all fixed by ι. But this implies that these classes belong to $\mathcal{B}^0$, and thus their transfer product is zero. Consequently, such addends do not appear in the summation. This implies that $\widetilde{\Delta}(x \otimes y)$ is killed by the multiplication by 2, and thus $\widetilde{\Delta}$ extends linearly to a map as desired. The image of $\widetilde{\Delta}$ is contained in the image of the embedding $\xi \colon A'_D \hookrightarrow \widetilde{V}$ that maps every $x \in \mathcal{M}_D$ to

$2x \in \widetilde{V}$. We can rephrase Proposition 5.14 by saying that

$$(\Delta\odot)\colon x \otimes y \in A'_D \otimes A'_D \mapsto \Delta(x \odot y) \in A'_D$$

is the unique linear map satisfying $\xi \circ (\Delta\odot) = \widetilde{\Delta}$. We immediately see that this statement is equivalent to the formulation above when x or y is a charged gathered block. If both x and y belong to $\mathcal{B}^0$, then $\widetilde{\Delta}(x \otimes y) = 0$ because the transfer product of two neutral gathered blocks is always zero. The general case follows by induction on the number of $\odot$–factors in the Hopf monomials involved.

Proof of Proposition 5.22 We can write every element in an almost-Hopf ring with generators $\gamma^+_{k,n}$, 1^- and $\delta^0_{n:m}$ satisfying the relations of Theorem 5.15 as a linear combination of addends in $\mathcal{M}_D$ due to Lemmas 5.18 and 5.21. Therefore $\mathcal{M}_D$ is a set of linear generators for A'_D.

The fact that Hopf monomials in $\mathcal{M}_D$ are linearly independent is a byproduct of the proof of Theorem 5.15. It is nevertheless possible to provide a fully independent proof that a basis for the almost-Hopf ring with the presentation of Theorem 5.15 has an additive basis given by $\mathcal{M}_D$, but we will not provide it, as it would be uselessly long. □

5.4 Comparison between A_Σ, A_B and A_D

In this subsection, we compute the action of the connecting homomorphisms on the elements of the additive bases determined in the previous subsection.

We first start with the link between A_Σ and A_B. We recall that there are a natural injection $j\colon \Sigma_n \to W_{B_n}$ and a natural projection $\pi\colon W_{B_n} \to \Sigma_n$, providing linear maps linking A_B and A_Σ. We begin by analyzing the relationship between A_Σ and A_B.

Proposition 5.24 *Let $j\colon \Sigma_n \to W_{B_n}$ and $\pi\colon W_{B_n} \to \Sigma_n$ be the natural homomorphisms. The induced maps $j^*\colon A_B \to A_\Sigma$ and $\pi^*\colon A_\Sigma \to A_B$ are Hopf-ring homomorphisms.*

Proof It is obvious from the fact that the diagrams

$$\begin{tikzcd} \dfrac{\mathrm{Conf}_{n+m}((0,+\infty)^\infty)}{\Sigma_n \times \Sigma_m} \arrow[r, "j"] \arrow[d, "p"] & \dfrac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_n} \times W_{B_m}} \arrow[d, "p"] \\ \dfrac{\mathrm{Conf}_{n+m}((0,+\infty)^\infty)}{\Sigma_{n+m}} \arrow[r, "j"] & \dfrac{Y^{(\infty)}_{W_{B_{n+m}}}}{W_{B_{n+m}}} \end{tikzcd} \qquad \begin{tikzcd} \dfrac{E(\mathbb{F}_2)^{n+m} \times E(\Sigma_{n+m})}{(\mathbb{F}_2 \wr \Sigma_n) \times (\mathbb{F}_2 \wr \Sigma_m)} \arrow[r, "\pi"] \arrow[d, "p"] & \dfrac{E(\Sigma_{n+m})}{\Sigma_n \times \Sigma_m} \arrow[d, "p"] \\ \dfrac{E(\mathbb{F}_2)^{n+m} \times E(\Sigma_{n+m})}{\mathbb{F}_2 \wr \Sigma_{n+m}} \arrow[r, "\pi"] & \dfrac{E(\Sigma_{n+m})}{\Sigma_{n+m}} \end{tikzcd}$$

are pullbacks of finite coverings, where p indicates covering maps. □

The following proposition is a direct consequence of Corollary 4.6 and Proposition 5.24.

Proposition 5.25 *With reference to the notation of Theorem 2.3, $j^*(\gamma_{k,n}) = \gamma_{k,n}$ and $j^*(\delta_n) = 0$. More generally, given a B–skyline diagram $x \in \mathcal{M}_B$, $j^*(x)$ is zero if x contains a rectangle of height 1. Otherwise, it is obtained by interpreting x as a skyline diagram in A_Σ.*

We can now use our algebraic description to compute the action of π^* on generators.

Proposition 5.26 *$\pi^*(\gamma_{k,n}) = \gamma_{k,n}$. For a skyline diagram $x \in A_\Sigma$, $\pi^*(x)$ is obtained by interpreting x as a B–skyline diagram without rectangles of height 1.*

Proof We proceed by induction on n. If $n = 1$, since $\pi \circ j = \mathrm{id}$, π^* is injective. Hence $\pi^*(\gamma_{k,1})$ is a nonzero class in $H^{2^k-1}(W_{B_{2k}}; \mathbb{F}_2)$. Thanks to Proposition 5.24, $\pi^*(\gamma_{k,1})$ is primitive. From our description of A_B in terms of skyline diagrams, formalized with the statement of Proposition 5.17, we see that the only nontrivial primitive of A_B in the right component and cohomological degree is $\gamma_{k,1}$. For $n > 1$, Proposition 5.24 guarantees that π^* preserves coproducts. Hence we inductively have that $\pi^*(\gamma_{k,n}) + \gamma_{k,n}$ is primitive. However, there are no nonzero primitive in that bidegree, thus $\pi^*(\gamma_{k,n}) = \gamma_{k,n}$. □

We now turn to A_D. There is a restriction map $\rho\colon A_B \to A_D$ induced by the inclusions $W_{D_n} \hookrightarrow W_{B_n}$. Moreover, we recall that we have natural injections $i_+, i_-\colon \Sigma_n \to W_{D_n}$ determining maps $A_D \to A_\Sigma$ and an involution $\iota\colon A_D \to A_D$ induced on $H^*(W_{D_n}; \mathbb{F}_2)$ by the conjugation with $s_0 \in W_{B_n}$. We analyzed these maps in Section 4.1.

First, we explain the relation between $\gamma^+_{k,m}$ and $\gamma^-_{k,m}$ and the natural maps between W_{D_n}, W_{B_n} and Σ_n.

Proposition 5.27 *For all $n, k \geq 1$ and $m \geq 0$,*

$$\begin{array}{ll} i_+^*(\gamma^+_{k,n}) = \gamma_{k,n}, & i_+^*(\gamma^-_{k,n}) = 0, \\ i_-^*(\gamma^-_{k,n}) = \gamma_{k,n}, & i_-^*(\gamma^+_{k,n}) = 0, \end{array} \quad i_+^*(\delta^0_{n:m}) = i_-^*(\delta^0_{n:m}) = 0.$$

More generally, with reference to Proposition 5.22, i_+^* (resp. i_-^*) is zero on all neutral or negatively (resp. neutral or positively) charged Hopf monomials. We obtain the value of positively (resp. negatively) charged Hopf monomials under i_+^* (resp. i_-^*) by forgetting the charge to get a Hopf monomial in $\mathcal{M}_B$ and then applying j^* as described in Proposition 5.25.

Proof The formulas involving $\gamma^{\pm}_{k,n}$ are a direct consequence of Corollary 4.6 and the form of the chain representative of $\gamma_{k,n} \in \mathrm{FN}^*_{\Sigma_n} \otimes \mathbb{F}_2$ retrieved in [8, Definition 4.9]. To deduce that $i_+^*(\delta^0_{n:m}) = 0$, we recall that $\delta^0_{n:m} = \rho(\delta_n \odot 1_m)$ and that the composition $\Sigma_n \xhookrightarrow{i_+} W_{D_n} \hookrightarrow W_{B_n}$ is equal to j. By Proposition 5.25 $j^*(\delta_n \odot 1_m) = 0$; therefore $i_+^*(\delta^0_{n:m}) = 0$. The same is also true for $i_-^*(\delta^0_{n:m})$ because i_- is obtained by composing i_+ with the conjugation with an element of W_{B_n}, whose action is trivial on elements coming from A_B. $\square$

Since we identify the involution ι with the transfer product with 1^-, the following proposition is essentially a restatement of the description of the previous subsection.

Proposition 5.28 *If x^0 is a neutral Hopf monomial in $\mathcal{M}_D$, then $\iota(x^0) = x^0$. If $x^{\pm}$ is a charged Hopf monomial in $\mathcal{M}_D$, we get $\iota(x^{\pm})$ by inverting the charge.*

To complete the description of the homomorphisms connecting our structures, we need to compute the restriction $\rho\colon A_B \to A_D$ and transfer $\mathrm{tr}\colon A_D \to A_B$ maps. To do this, we need to establish preliminary identities.

Lemma 5.29 *For all $x, x' \in A_D$ and for all $y \in A_B$, the following identities are satisfied*:

(1) $\rho(\mathrm{tr}(x) \odot y) = x \odot \rho(y)$,

(2) $\mathrm{tr}(x) \cdot \mathrm{tr}(x') = \mathrm{tr}(x \cdot x' + \iota(x) \cdot x')$,

(3) $\mathrm{tr}(x \cdot \rho(y)) = \mathrm{tr}(x) \cdot y$,

(4) $\mathrm{tr}(x \odot x') = \mathrm{tr}(x) \odot \mathrm{tr}(x')$.

Proof The first statement follows from the fact that this commutative diagram induces a pullback of covering spaces at the level of classifying spaces:

$$\begin{tikzcd} W_{D_n} \times W_{D_m} \arrow[r] \arrow[d] & W_{D_n} \times W_{B_m} \arrow[d] \\ W_{D_{n+m}} \arrow[r] & W_{B_{n+m}} \end{tikzcd}$$

Regarding the second statement, since the conjugation by s_0 is an endomorphism of the covering $B(W_{D_n} \times W_{D_n}) \to B(W_{D_{2n}} \cap (W_{B_n} \times W_{B_n}))$,

$$\mathrm{tr}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})}_{W_{D_n} \times W_{D_n}} c^*_{s_0} = c^*_{s_0} \,\mathrm{tr}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})}_{W_{D_n} \times W_{D_n}} .$$

Moreover, the classifying space functor applied to the following square produces a diagram homotopy equivalent to a pullback of covering, where d and d' are diagonal maps:

$$\begin{tikzcd} W_{D_n} \arrow[r, "d"] \arrow[d, "j"'] & W_{D_n} \times W_{D_n} \arrow[d] \\ W_{B_n} \arrow[r, "d'"'] & W_{D_{2n}} \cap (W_{B_n} \times W_{B_n}) \end{tikzcd}$$

Hence $\operatorname{tr} d^* = d'^* \operatorname{tr}_{W_{D_n} \times W_{D_n}}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})}$. These facts imply that, denoting by d the diagonal subgroups,

$$\begin{aligned}
&\operatorname{tr}(x) \cdot \operatorname{tr}(x') \\
&= \rho_{d(W_{B_n}}^{W_{B_n} \times W_{B_n}} \operatorname{tr}_{W_{D_n} \times W_{D_n}}^{W_{B_n} \times W_{B_n}} (x \otimes x') \\
&= \rho_{d(W_{B_n})}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})} \rho_{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})}^{W_{B_n} \times W_{B_n}} \operatorname{tr}_{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})}^{W_{B_n} \times W_{B_n}} \operatorname{tr}_{W_{D_n} \times W_{D_n}}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})} (x \otimes x') \\
&= d'^* (\mathrm{id} + c_{s_0}^*) \operatorname{tr}_{W_{D_n} \times W_{D_n}}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})} (x \otimes x') \\
&= d'^* \operatorname{tr}_{W_{D_n} \times W_{D_n}}^{W_{D_{2n}} \cap (W_{B_n} \times W_{B_n})} (\mathrm{id} + c_{s_0}^*)(x \otimes x') \\
&= \operatorname{tr} d^* (\mathrm{id} + c_{s_0}^*)(x \otimes x') \\
&= \operatorname{tr}(x \cdot x' + \iota(x) \cdot x').
\end{aligned}$$

Similarly, the last two statements follow from the diagrams below, where the vertical maps of the first one are the diagonal morphisms:

$$\begin{tikzcd} W_{D_n} \arrow[r] \arrow[d] & W_{B_n} \arrow[d] \\ W_{D_n} \times W_{B_n} \arrow[r] & W_{B_n} \times W_{B_n} \end{tikzcd} \qquad \begin{tikzcd} W_{D_n} \times W_{D_m} \arrow[r] \arrow[d] & W_{D_{n+m}} \arrow[d] \\ W_{B_n} \times W_{B_m} \arrow[r] & W_{B_{n+m}} \end{tikzcd}$$

□

Proposition 5.30 *The transfer map* $\operatorname{tr}\colon A_D \to A_B$ *is such that* $\operatorname{tr}(\gamma_{k,n}^{\pm}) = \gamma_{k,n}$ *and* $\operatorname{tr}(\delta_{n:m}^0) = 0$*. More generally, if* $b_{l,\underline{t}}^{\pm}$ *is a charged gathered block with profile* $\underline{t} = (t_0, t_1, \dots, t_n)$*, then* $\operatorname{tr}(b_{l,\underline{t}}^{\pm}) = b_{l,\underline{t}}$ *if* $n \geq 2$*, while if* $n = 1$ *the transfer of gathered blocks is computed inductively by the formula*

$$b_{l,\underline{t}} = \sum_{a=0}^{\lfloor t_1/2 \rfloor} \operatorname{tr}(b_{l,(t_0,t_1-a)}^{\pm} (\gamma_{1,l}^{\mp})^a).$$

The transfer of every neutral gathered block is 0*, and we realize the transfer of a Hopf monomial as the transfer product of the transfer of its constituent gathered blocks.*

Proof The statement for generators is a direct consequence of their definition at the cochain level. The general claim for Hopf monomials in A_D follows directly from Lemma 5.29. $\square$

Proposition 5.31 *$\rho(\gamma_{k,n}) = \gamma^+_{k,n} + \gamma^-_{k,n}$ for all $n, k \geq 1$. Moreover, $\rho(\delta_m) = \delta^0_{m:0}$ for $n \geq 2$ and $\rho(\delta_1) = 0$. More generally, for every Hopf monomial $x \in \mathcal{M}_B$, $\rho(x)$ can be computed as follows. If $x = b_{l,\underline{t}}$ is a gathered block with profile $\underline{t} = (t_0, \dots, t_n)$, we have that*

$$\rho(x) = \begin{cases} x^0 & \text{if } n = 0, \\ \sum_{a=0}^{t_1} \binom{t_1}{a} b^+_{l,(t_0,a)} (\gamma^-_{1,l})^{t_1 - a} & \text{if } n = 1, \\ b^+_{l,\underline{t}} + b^-_{l,\underline{t}} & \text{if } n \geq 2. \end{cases}$$

The restriction of a Hopf monomial x with a constituent gathered block in $\widetilde{\mathcal{B}}^0$ is x^0. We calculate the restriction of a Hopf monomial $x \in \widetilde{\mathcal{B}}^c$ as follows. First, replace every constituent gathered block in x with the sum of the positively or neutrally charged elements of its restriction. Then, write the resulting linear combination as a sum of Hopf monomials in A_D. Finally, add to that the negatively charged counterpart of every positively charged Hopf monomials appearing in the sum.

Proof Using the cochain-level representative of $\gamma_{k,n}$ introduced in Definition 5.1, we immediately see that its restriction is represented in $\mathrm{FN}'^*_{W_{D_{n2^k}}}$ by the sum of two elements obtained by providing this cochain with the two possible orientations. These elements correspond to cochain representatives of $\gamma^+_{k,n}$ and $\gamma^-_{k,n}$ via the cochain equivalence φ of Lemma 4.1. The formulas for δ_m are a consequence of the generators' definition in A_D and relation (4) of Proposition 5.13. $\square$

We conclude this section with a short description of the Gysin sequence of the double cover $W_{D_n} \to W_{B_n}$. In [9], Giusti and Sinha adopt the analysis of a similar Gysin exact sequence as the starting point to compute the cohomology of the alternating groups as an almost-Hopf ring. While we retrieve that as a byproduct of our algebraic description, we stress that Giusti and Sinha's approach could be used in our framework as an alternative method to deduce relations in A'_D. Indeed, a direct consequence of the following proposition is that $\mathcal{M}_D = \mathcal{B}^0 \sqcup \mathcal{B}^+ \sqcup \mathcal{B}^-$ is the polarized basis arising from a Gysin decomposition in the sense of [9].

Proposition 5.32 (cf [9, Section 3]) *The restriction $\rho\colon A_B \to A'_D$ and the transfer $\mathrm{tr}\colon A'_D \to A_D$ fit into the Gysin sequence*

$$\cdots \xrightarrow{\partial_{k-1}} H^k(W_{B_n}; \mathbb{F}_2) \xrightarrow{\rho_k} H^k(W_{D_n}; \mathbb{F}_2) \xrightarrow{\mathrm{tr}_k} H^k(W_{B_n}; \mathbb{F}_2) \xrightarrow{\partial_k} H^{k+1}(W_{B_n}; \mathbb{F}_2) \xrightarrow{\rho_{k+1}} \cdots,$$

where ∂ is the multiplication with $\delta_1 \odot 1_{n-1}$. It can be described on skyline diagrams by the operation of replacing each column corresponding to δ_k^m with the diagram corresponding to $\delta_1^{m+1} \odot \delta_{k-1}^m$.

Proof By a general fact, the connecting homomorphism ∂ is the multiplication with the Euler class e of the double covering. In the case $n = 1$, this covering is isomorphic to the universal double covering $S^\infty \to \mathbb{P}^\infty(\mathbb{R})$, and its Euler class is δ_1. For bigger n, the Euler class is $\delta_1 \odot 1_{n-1}$ because it is the only class in the right degree that restricts to δ_1.

$\mathrm{tr} \circ \rho = 0$ because we are working modulo 2. Therefore the transfer of a neutral gathered block is 0. If $b = b_{l,\underline{t}}^{\pm}$ is a charged gathered block, then the restriction of $\mathrm{tr}(b)$ must be $b + \iota(b)$, and the multiplication with $\delta_1 \odot 1_{n-1}$ must kill $\mathrm{tr}(b)$. These two conditions force $\mathrm{tr}(b) = b_{l,\underline{t}}$. Since tr preserves the transfer product $\odot$, the formula for a general Hopf monomial follows. □

6 Restriction to elementary abelian subgroups

We recall here some theorems from Swenson's thesis [18], which constitute the formal framework in which we will calculate the cohomology of W_{B_n} and W_{D_n}. We will then exploit these theorems to determine the restriction of our generators in A_B and A_D to elementary abelian 2–subgroups. This yields the restriction of all the cohomology of the groups W_{B_n} and W_{D_n} to maximal elementary abelian subgroups, because the structural morphisms of our almost-Hopf rings behave in a predictable way: cup products and coproducts are preserved by such restriction, while the relation with transfer product is determined via double cosets formulas, as stated in Adem and Milgram's book [1, Section II.6].

6.1 Quillen's theorem for finite reflection groups

The relevance of these restriction maps is encompassed by a result of Quillen [15; 16], which we state here. Let G be a finite group and $\mathcal{F}$ a family of subgroups. Let $\theta_g \colon H^*(K;\mathbb{F}_p) \to H^*(gKg^{-1};\mathbb{F}_p)$ be the conjugation homomorphism. Define

$$\mathcal{F}^n = \{\{f_K\}_{K\in\mathcal{F}},\ f_K \in H^n(K;\mathbb{F}_p) \mid g^{-1}Kg \subseteq K' \Longrightarrow f_K = \theta_g^*(f_{K'})|_K \text{ for all } K, K'\}.$$

Alternatively, we can consider $\mathcal{F}$ as a category in which

$$\mathrm{Hom}(K, K') = \{g \mid g^{-1}Kg \subseteq K'\}.$$

Then $\mathcal{F}^n$ is the inverse limit of the functor H^n from $\mathcal{F}$ into the category of $\mathbb{F}_p$–vector spaces. In other words, $\mathcal{F}^*$ consists of collections of cohomology classes of groups in $\mathcal{F}$ that are stable under restrictions and conjugation by elements of G. Observe that $\mathcal{F}^* = \bigoplus_n \mathcal{F}^n$ has a natural ring structure.

Theorem 6.1 [15, Theorem 6.2, page 564] *Let G be a finite group. Let $\mathcal{F}^*$ be as before. The map $q_G\colon H^*(G;\mathbb{F}_p) \to \mathcal{F}^*$ given by $q_G(f) = \{f|_K\}_K$ is a well-defined ring homomorphism. Moreover, if $\mathcal{F}$ is the family of elementary abelian p–subgroups, then the kernel and cokernel of q_G are nilpotent.*

Hence elementary abelian p–subgroups give much information on the $\mathbb{F}_p$–cohomology of a group. In the case of a finite reflection group, an even stronger property holds.

Theorem 6.2 [18, Theorem 11, page 2] *If G is a finite reflection group and $\mathcal{F}$ is the family of elementary abelian p–subgroups of G, then q_G is an isomorphism.*

6.2 Restriction from A_B

For the reasons explained in the previous subsection, Swenson has calculated the elementary abelian 2–subgroups of W_{B_n}. Before stating his result, we need to recall the structure of elementary abelian 2–groups of the symmetric group Σ_n on n objects. The relevant calculations are reviewed in [1]. Σ_n admits a transitive elementary abelian 2–subgroup if and only if $n = 2^k$. In this case, all these subgroups are conjugated in Σ_n to the image V_k of the homomorphism $\rho_k\colon \mathbb{F}_2^k \hookrightarrow \Sigma_{2^k}$ given by the regular action of $\mathbb{F}_2^k$ on itself. More generally, a maximal elementary abelian 2–subgroup of Σ_n is conjugated to a direct product

$$V_{k_1} \times \cdots \times V_{k_r} \hookrightarrow \Sigma_{2^{k_1}} \times \cdots \times \Sigma_{2^{k_r}} \hookrightarrow \Sigma_{2^{k_1}+\cdots+2^{k_r}}.$$

Hence, conjugacy classes of maximal elementary abelian 2–subgroups in Σ_n are parametrized by partitions π of n such that every element of π is an integral power of 2 and the multiplicity of $1 = 2^0$ in π is at most 1.

To further simplify notation, we borrow from Swenson's thesis the following definition.

Definition 6.3 [18] Let $n \in \mathbb{N}$. We say that a partition π of n is *admissible* if it consists only of parts that are integral powers of 2.

The main results about elementary abelian 2–subgroups in W_{B_n} is the following:

Proposition 6.4 [18, page 22] *Let $A_1, A_2 \leq W_{B_n}$ be maximal elementary abelian 2–subgroups. Then*

- $\widetilde{A}_i = A_i \cap \Sigma_n \leq \Sigma_n$ *is conjugated to a subgroup of the form $V_{k_1} \times \cdots \times V_{k_r}$, with $k_i \geq 0$ for all i;*
- A_1 *and* A_2 *are conjugated in W_{B_n} if and only if $\widetilde{A}_1$ and $\widetilde{A}_2$ are conjugated in Σ_n.*

In particular, conjugacy classes of maximal elementary abelian 2–subgroups in W_{B_n} are parametrized by admissible partitions π. Moreover, if we denote by A_π the subgroup corresponding to a partition π, we have that $A_{(2^k)} = V_k \times C_k$, where $C_k \cong \mathbb{F}_2$ is the center of $W_{B_{2^k}}$ and, more generally, if m_i is the multiplicity of 2^i in a partition π, then A_π is isomorphic to the direct product $\prod_i A_{(2^i)}^{m_i}$. Let $d_{2^i-1}, \dots, d_{2^i-2^{i-1}}$ be the Dickson invariants in $H^*(V_i;\mathbb{F}_2) \hookrightarrow H^*(A_{(2^i)};\mathbb{F}_2)$ and define

$$f_{2^i} = \prod_{y \in H^1(V_i;\mathbb{F}_2)} (x+y),$$

where $x \in H^1(A_{2^i};\mathbb{F}_2)$ is the linear dual to the nontrivial element in the C_i–factor of A_{2^i}. There is a natural isomorphism

$$[H^*(A_\pi;\mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)} \cong \bigotimes_i (\mathbb{F}_2[f_{2^i}, d_{2^i-1}, \dots, d_{2^i-2^{i-1}}]^{\otimes m_i})^{\Sigma_{m_i}}.$$

We can calculate the restriction of our generating classes $\gamma_{k,n}$ and δ_n to these abelian subgroups. The calculation for $\gamma_{k,n}$ has been essentially carried out by Giusti, Salvatore and Sinha [7]. We state here the result.

Proposition 6.5 [7, Corollary 7.6, page 189] *Let $l, n \geq 1$. Let π be a partition of $n2^l$ consisting of powers of 2, $\pi = (2^{k_1}, \dots, 2^{k_r})$. Then*

$$\gamma_{l,n}|_{A_\pi} = \begin{cases} \bigotimes_{i=1}^r d_{2^{k_i}-2^{k_i-l}} & \text{if } k_i \geq l \text{ for all } 1 \leq i \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 6.6 *Let $n \geq 0$. Let $\pi = (2^{k_1}, \dots, 2^{k_r})$ be an admissible partition. The restriction of δ_n to the cohomology of the maximal elementary abelian 2–subgroup A_π is equal to $\bigotimes_{i=1}^r f_{2^{k_i}}$. Moreover, δ_n is the unique class in $H^n(W_{B_n};\mathbb{F}_2)$ that has this property for every π.*

Proof We observe that the restrictions of a cohomology class to A_π for all the partitions π of n determine its restriction to every elementary abelian 2–subgroup (not necessarily maximal). Hence, by Theorem 6.2, a class that satisfies the condition in the statement for every π is necessarily unique.

Let $U_n = \mathbb{R}^n$ be the reflection representation of W_{B_n}. Recall that, if $n = 2^k$ and $\pi = (2^k)$, then $A_\pi = V_k \times C_k$, where $C_k = \langle t \rangle$ is a cyclic group of order 2, the center of W_{B_n}, and $V_k = \langle v_1, \dots, v_k \rangle \leq \Sigma_{2^k}$ is the subgroup defined above. $H^*(A_\pi; \mathbb{F}_2)$ is polynomial on degree 1 elements $x, y_1, \dots, y_k$, the linear duals to $t, v_1, \dots, v_k$, respectively. Given $a \in A_\pi \setminus \{1\}$, let ε_a, sgn_a, and $\mathbb{R}\langle a \rangle$ be the 1–dimensional trivial representation, the signum representation, and the regular representation of $\langle a \rangle \cong \mathbb{F}_2$, respectively. We first observe that, since t acts on U_n as the multiplication by -1, $U_n|_{A_\pi} \cong \mathrm{sgn}_t \otimes U_n|_{V_k}$. Moreover, the inclusion of V_k in Σ_{2^k} is given by the regular representation; hence

$$U_n|_{V_k} \cong \bigotimes_{i=1}^{k} \mathbb{R}\langle v_i \rangle \cong \bigoplus_{S \subseteq \{1,\dots,k\}} \bigotimes_{i=1}^{k} U_{S,i},$$

where $U_{S,i}$ is equal to sgn_{v_i} if $i \in S$, and to ε_{v_i} if $i \notin S$. Thus, with the notation used before in this document, the Stiefel–Whitney class of $U_n|_{A_\pi}$ is

$$\prod_{S \subseteq \{1,\dots,n\}} \left(1 + x + \sum_{i \in S} y_i\right).$$

Its n–dimensional part is exactly f_{2^k}. Hence, the thesis for $\pi = (2^k)$ follows from the naturality of the characteristic classes and Proposition 5.3

In the case of a general admissible partition $\pi = (2^{k_1}, \dots, 2^{k_r})$, the proposition follows from the fact that $A_\pi \cong \prod_{i=1}^{r} A_{(2^{k_i})}$ and $U_n|_{A_\pi} \cong \bigoplus_{i=1}^{r} U_{2^{k_i}}|_{A_{(2^{k_i})}}$. □

To complete the calculation of the restriction morphisms from A_B to maximal elementary abelian 2–subgroups, we need to describe how such maps behave with the structural morphisms of A_B. Restrictions preserve cup products, and, regarding the coproduct, there is nothing to say because every maximal elementary abelian subgroup of $W_{B_n} \times W_{B_m}$ is itself a maximal elementary abelian subgroup of $W_{B_{n+m}}$. The only nontrivial behavior occurs with the transfer product. We describe it in the following proposition.

Proposition 6.7 *Let $x, y \in A_B$ be in positive components n and m respectively. Let $\pi = (2^{k_1}, \dots, 2^{k_r})$ be an admissible partition of $n+m$. For all $I \subseteq \{1, \dots, r\}$, write $I = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$ and let $\pi_I = (2^{k_{i_1}}, \dots, 2^{k_{i_s}})$. Then*

$$(x \odot y)|_{A_\pi} = \sum_{I,J} \tau_{I,J}(x|_{A_{\pi_I}} \otimes y|_{A_{\pi_J}}),$$

where the sum runs over all partitions $\{1,\dots,r\} = I \sqcup J$ *of* $\{1,\dots,r\}$ *into two subsets such that* $\sum_{i\in I} 2^{k_i} = n$ *(and, consequently,* $\sum_{j\in J} 2^{k_j} = m$*), and*

$$\tau_{I,J}\colon H^*(A_{\pi_I};\mathbb{F}_2)\otimes H^*(A_{\pi_J};\mathbb{F}_2)\to H^*(A_\pi;\mathbb{F}_2)$$

is the obvious permutation of tensor factors.

Proof We begin by assuming that $r=1$; thus $\pi=(2^k)$ for some k and $n+m=2^k$. Then, since A_π acts transitively on $\{1,\dots,2^k\}$, no conjugate of A_π in $W_{B_{n+m}}$ is contained in $W_{B_n}\times W_{B_m}$. Given that A_π is abelian, the classically known property stated in [1, Proposition 5.6, page 69] implies that the transfer map

$$H^*(A_\pi\cap\sigma(W_{B_n}\times W_{B_m})\sigma^{-1};\mathbb{F}_2)\to H^*(A_\pi;\mathbb{F}_2)$$

is identically zero. Eilenberg's double coset formula then guarantees that the composition of the restriction with the transfer product

$$H^*(W_{B_n};\mathbb{F}_2)\otimes H^*(W_{B_m};\mathbb{F}_2)\xrightarrow{\odot} H^*(W_{B_{n+m}};\mathbb{F}_2)\to H^*(A_\pi;\mathbb{F}_2)$$

is zero. Thus $(x\odot y)|_{A_{(2^k)}}=0$.

In the general case, the restriction of $x\odot y$ to this subgroup factors through the r–fold coproduct. By the calculations above, addends in this coproduct for which a factor is a nontrivial transfer product restrict to 0. Since $\odot$ and Δ form a bialgebra structure on A_B, the other addends have the desired form. □

6.3 Restriction from A_D and proof of relations

We can adapt the argument to calculate the restriction to elementary abelian subgroups of generators also in the D_n case. First, we state the analog of Proposition 6.4. Recall that a partition π of n is admissible if and only if it consists of parts that are powers of 2.

Theorem 6.8 [18, Theorem 5.4.3, page 40] *Let* π *be an admissible partition of* n*. Let* m_1 *and* m_2 *be the multiplicities of* 1 *and* 2 *in* π*. We write* $\pi=(1)^{m_1}\cup(2)^{m_2}\cup\pi'$*. Let* $A_\pi\le W_{B_n}$ *the maximal elementary abelian* 2*–subgroup corresponding to* π *and let* $\widehat{A}_\pi = A_\pi\cap W_{D_n}$*. Then* $\widehat{A}_\pi$ *is maximal as an elementary abelian subgroup of* W_{D_n} *if and only if* $m_1\neq 2$*. Moreover:*

- *If* $m_1>0$*, then* $\widehat{A}_\pi=\ker\bigl(\sum\colon\mathbb{F}_2^{m_1}\to\mathbb{F}_2\bigr)\times A_{(2)^{m_2}\cup\pi'}$*. If* $e_1,\dots,e_{m_1}$ are the *elementary symmetric functions in* $H^*(\mathbb{F}_2^{m_1};\mathbb{F}_2)=H^*(A_{(1)^{m_1}};\mathbb{F}_2)$*, we define* $\bar{e}_i=$

$e_i + e_1 e_{i-1}$ if $2 \le i < m$ and $\bar{e}_m = e_1 e_{m-1}$. There is an isomorphism between the invariant subalgebra $[H^*(\widehat{A}_\pi;\mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)}$ and

$$\mathbb{F}_2[\bar{e}_2,\dots,\bar{e}_m] \otimes [H^*(A_{(2)^{m_2}\sqcup\pi'};\mathbb{F}_2)]^{N_{W_{B_{n-m_1}}}(A_{(2)^{m_2}\sqcup\pi'})}.$$

Moreover, the cohomological restriction from $A_{(1)^m}$ to $\widehat{A}_{(1)^m}$ is given by $e_1 \mapsto 0$ and $e_i \mapsto \bar{e}_i$ if $2 \le i \le m$.

- *If $m_1 = 0$ and $m_2 > 0$, then $\widehat{A}_\pi = A_\pi$. Identifying $H^*(A_{(2)};\mathbb{F}_2)^{\otimes m_2}$ with $\bigotimes_{i=1}^{m_2} \mathbb{F}_2[x_i, y_i]$, we can define*

$$h_{m_2} = \sum_{\substack{S\subseteq\{1,\dots,m_2\}\\ |S|=2l}} \prod_{i\in S}(x_i+y_i)\prod_{j\notin S} x_j.$$

Then $[H^*(\widehat{A}_\pi;\mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)}$ is the free $[H^*(A_\pi;\mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)}$–module with basis $\{1, h_{m_2}\otimes 1_{H^*(A_{\pi'};\mathbb{F}_2)}\}$.

- *If $m_1 = m_2 = 0$, then $\widehat{A}_\pi = A_\pi$ and $N_{W_{D_n}}(A_\pi) = N_{W_{B_n}}(A_\pi)$; hence*

$$[H^*(\widehat{A}_\pi;\mathbb{F}_2)]^{N_{W_{D_n}}(\widehat{A}_\pi)} = [H^*(A_\pi;\mathbb{F}_2)]^{N_{W_{B_n}}(A_\pi)}.$$

Moreover, if $m_1 \ne 0$ or $m_2 \ne 0$, then A_π is W_{B_n}–conjugate to A' if and only if $\widehat{A}_\pi$ is W_{D_n}–conjugate to $A' \cap W_{D_n}$. Conversely, if $m_1 = m_2 = 0$, then the W_{B_n}–conjugacy class of A_π contains exactly two W_{D_n}–conjugacy classes of elementary abelian 2–subgroups.

We now determine the restriction of our generators to the elementary abelian subgroups.

Proposition 6.9 *Let $n = 2^k m$, for some $k, m \ge 1$. Let π be an admissible partition of n. Let m_1 and m_2 be the multiplicities of 1 and 2 in π. Then*

1. *for every $k \ge 1$, if $m_1 = m_2 = 0$, then $\gamma^+_{k,m}|_{A_\pi} = \gamma_{k,m}|_{A_\pi}$, $\gamma^+_{k,m}|_{A_\pi^{s_0}} = 0$, $\gamma^-_{k,m}|_{A_\pi} = 0$, and $\gamma^-_{k,m}|_{A_\pi^{s_0}} = \gamma_{k,m}|_{A_\pi^{s_0}}$;*
2. *for every $k \ge 2$, if $m_1 \ne 0$ or $m_2 \ne 0$, or for $k=1$ if $m_1 \ne 0$, then $\gamma^\pm_{k,m}|_{\widehat{A}_\pi} = 0$;*
3. *if $m_1 = 0$ but $m_2 \ne 0$, ie $\pi = (2)^{m_2} \sqcup \pi'$, then the restriction of $\gamma^+_{1,m}$ (resp. $\gamma^-_{1,m}$) to $\widehat{A}_\pi = A_{(2)^{m_2}} \times A_{\pi'}$ is $h_{m_2} \otimes \gamma_{1,m-m_2}|_{A_{\pi'}}$ (resp. $(d_1^{\otimes m_2} + h_{m_2}) \otimes \gamma_{1,m-m_2}|_{A_{\pi'}}$);*
4. *if $\pi = (1)^{m_1} \sqcup \pi'$, then the restriction of $\delta^0_{k:m}$ to $\widehat{A}_\pi = \widehat{A}_{(1)^{m_1}} \times A_{\pi'}$ is*

$$1 \otimes (\delta_k \odot 1_{W_{B_{m-m_1}}})|_{A_{\pi'}} + \sum_{i=2}^{k} \bar{e}_i \otimes (\delta_{k-i} \odot 1_{W_{B_{m-m_1+i}}})|_{A_{\pi'}},$$

with the convention that $1_{W_{B_r}} = 0$ when $r < 0$.

Proof If π has more than 1 element and is different from $(1,1,\dots,1)$, then the restriction to $\widehat{A}_\pi$ or $\widehat{A}_\pi^{s_0}$ factors through the coproduct. Thus, by applying the coproduct formulas of Proposition 5.12, we can inductively reduce to these two cases.

We begin by assuming that $\pi=(2^n)$ has only one element, and we prove the first statement. If $k\geq 1$ and $n\geq 2$, the restriction of $\gamma^\pm_{k,2^l}$ to A_π $(n=k+l)$ must be $N_{W_{D_{2^n}}}(A_\pi)$–invariant. Hence, for degree reasons, it can be 0 or $d_{2^n-2^l}$. Since $i_+^*(\gamma^+_{k,2^l})=\gamma_{k,2^l}$ (resp. $i_+^*(\gamma^-_{k,2^l})=0$) by Proposition 5.27, its restriction to $A_\pi\cap\Sigma_{2^n}$ must be the Dickson invariant of degree 2^n-2^l (resp. 0). This forces

$$\gamma^+_{k,2^l}|_{A_\pi}=d_{2^n-2^l}=\gamma_{k,2^l}|_{A_\pi}$$

(resp. $\gamma^-_{k,2^l}|_{A_\pi}=0$). By essentially the same argument, considering i_- instead of i_+, we determine the restrictions to $A_\pi^{s_0}$, proving the first point.

Claim (2) is immediate from the fact that, if $k\geq 1$, there are no nonzero elements in $H^*(\widehat{A}_{(1)^{2^k n}};\mathbb{F}_2)^{N_{W_{D_{2^k n}}}(\widehat{A}_{(1)^{2^k n}})}$ in the same degree of $\gamma^\pm_{k,n}$, and that if $k\geq 2$ the coproduct of $\gamma^\pm_{k,n}$ has no element in component 2.

To prove (3) when $\pi=(2)$, we notice that $A_{(2)}=W_{D_2}$ and $\gamma^+_{1,1}$ can be identified with h_1, while $\gamma^-_{1,1}$, with d_1+h_1.

By the coproduct formula for $\delta^0_{k:m}$, we have $\delta^0_{k:m}|_{A_{(1)}^{k+m}}=e_k$. Thus, the last statement for $\pi=(1,\dots,1)$ follows directly by combining Proposition 6.6 and Theorem 6.8. □

As in A_B, the behavior of the restriction to maximal elementary abelian 2–subgroups with the cup product and coproduct is straightforward. We describe the relation between such restriction maps and the transfer product in the following proposition, which is the counterpart of Proposition 6.7.

Proposition 6.10 *Let $x,y\in A_D$ be elements in positive component n and m respectively. Let $\pi=(2^{k_1},\dots,2^{k_r})$ be an admissible partition of $n+m$. For all $I\subseteq\{1,\dots,r\}$, write $I=\{i_1,\dots,i_s\}$ with $i_1<\dots<i_s$ and let $\pi_I=(2^{k_{i_1}},\dots,2^{k_{i_s}})$. Then*

$$(x\odot y)|_{\widehat{A}_\pi}=\sum_{I,J}\tau_{I,J}(x|_{\widehat{A}_{\pi_I}}\otimes y|_{\widehat{A}_{\pi_J}}+\iota(x)|_{\widehat{A}_{\pi_I}}\otimes\iota(y)|_{\widehat{A}_{\pi_J}}),$$

where the sum runs over all partitions $\{1,\dots,r\}=I\sqcup J$ of $\{1,\dots,r\}$ into two subsets such that $\sum_{i\in I}2^{k_i}=n$ (and, consequently, $\sum_{j\in J}2^{k_j}=m$) and at least one between I and J does not contains any $l\in\{1,\dots,r\}$ such that $k_l=0$, and where

$\tau_{I,J}\colon H^*(\widehat{A}_{\pi_I};\mathbb{F}_2)\otimes H^*(\widehat{A}_{\pi_J};\mathbb{F}_2)\to H^*(\widehat{A}_\pi;\mathbb{F}_2)$ is the obvious permutation of tensor factors. Moreover, if $1\notin\pi$ and $2\notin\pi$,

$$(x\odot y)|_{\widehat{A}_\pi^{s_0}}=c_{s_0}^*\sum_{I,J}\tau_{I,J}(x|_{\widehat{A}_{\pi_I}}\otimes\iota(y)|_{\widehat{A}_{\pi_J}}+\iota(x)|_{\widehat{A}_{\pi_I}}\otimes y|_{\widehat{A}_{\pi_J}}),$$

where I, J, $\tau_{I,J}$ are as above, and $c_{s_0}^*\colon H^*(\widehat{A}_\pi;\mathbb{F}_2)\to H^*(\widehat{A}_\pi^{s_0};\mathbb{F}_2)$ is induced by the conjugation with s_0.

Proof We cannot repeat the proof of Proposition 6.7 because in A'_D the transfer product and the coproduct do not form a bialgebra. Therefore, we argue by considering Eilenberg's double coset formula associated with the two subgroups $W_{D_n}\times W_{D_m}$ and $\widehat{A}_\pi$ of $W_{D_{n+m}}$. We preliminarily fix some notation. Let P_π be the partition of the set $\{1,\dots,n+m\}$ given by

$$P_\pi=\left\{\{1,\dots,2^{k_1}\},\{2^{k_1}+1,\dots,2^{k_1}+2^{k_2}\},\dots,\left\{\sum_{l=1}^{r-1}2^{k_l}+1,\dots,n+m\right\}\right\}.$$

Moreover, let $P_{j,\pi}=\{\sum_{l=1}^{j-1}2^{k_l}+1,\dots,\sum_{l=1}^{j}2^{k_l}\}$.

Assume that $1\notin\pi$. A set of representatives for $W_{D_{n+m}}/(W_{D_n}\times W_{D_m})$ is the set $\mathrm{Sh}(n,m)\cdot\{1,t\}$, where $\mathrm{Sh}(n.m)\subseteq\Sigma_{n+m}\hookrightarrow W_{D_{n+m}}$ is the set of (n,m)–shuffles, and $t=s_0\times s_0\in W_{B_n}\times W_{B_m}$. Note that $\widehat{A}_\pi\subseteq(\sigma t^\varepsilon)(W_{D_n}\times W_{D_m})(\sigma t^\varepsilon)^{-1}$ if and only if $\sigma(\{1,\dots,n\})$ is a union of parts of P_π. Since $\widehat{A}_\pi$ is abelian, these provide the only nonzero terms in the summation of the double coset formula. Moreover, by inspecting the image of $\{1,\dots,n\}\subseteq\{1,\dots,n+m\}$ under the signed permutation action of $W_{D_{n+m}}\subseteq W_{B_{n+m}}$, we see that if σt^ε and $\sigma' t^{\varepsilon'}$ are two coset representatives satisfying this condition, then $\widehat{A}_\pi\sigma t^\varepsilon(W_{D_n}\times W_{D_m})=\widehat{A}_\pi\sigma' t^{\varepsilon'}(W_{D_n}\times W_{D_m})$ if and only if $\sigma=\sigma'$ and $\varepsilon=\varepsilon'$.

Consequently, the double coset formula allows us to rewrite $\rho_{\widehat{A}_\pi}^{W_{D_{n+m}}}(x\odot y)$ as the sum

$$\sum_{\substack{I\subseteq\{1,\dots,r\}\\ \sum_{i\in I}2^{k_i}=n}}(c_{\sigma_I}^*\rho_{\widehat{A}_{\pi_I}\times\widehat{A}_{\pi_J}}^{W_{D_n}\times W_{D_m}}(x\otimes y)\otimes+c_{\sigma_I}^*(c_{s_0}^*\otimes c_{s_0}^*)\rho_{\widehat{A}_{\pi_I}^{s_0}\otimes\widehat{A}_{\pi_J}^{s_0}}^{W_{D_n}\times W_{D_m}}),$$

where σ_I is the unique (n,m)–shuffle satisfying $\sigma(\{1,\dots,n\})=\bigcup_{i\in I}P_{i,\pi}$ and $J=\{1,\dots,r\}\setminus I$. The statement follows by observing that $c_{\sigma_I}^*=\tau_{I,J}$ and

$$c_{s_0}^*\rho_{\widehat{A}_{\pi'}^{s_0}}^{W_{D_l}}=\rho_{\widehat{A}_{\pi'}}^{W_{D_l}}\iota$$

for all $l\geq 1$ and an π' admissible partition of l.

The case of $\widehat{A}_\pi^{s_0}$ where $1,2 \notin \pi$ is done similarly. If $1 \in \pi$, the same argument holds, but if there exists $i \in I$ and $j \in J$ such that $k_i = k_j = 0$, then, interpreting the elements of $W_{D_{n+m}}$ as signed permutations, $(p_i, -p_i)(p_j, -p_j)$ belongs to $\widehat{A}_\pi$ but not to $(\sigma_I t^\varepsilon)(W_{D_n} \times W_{D_m})(\sigma_I t^\varepsilon)^{-1}$, where $P_{i,\pi} = \{p_i\}$ and $P_{j,\pi} = \{p_j\}$. Thus, we need to restrict the summation only to partitions $\{1, \dots, r\} = I \sqcup J$ in which all the occurrences of 1 in π belong to the same part. $\square$

This result provides a way to detect the charge of a Hopf monomial via restriction to maximal elementary abelian 2–subgroups. We first fix preliminary notation.

Definition 6.11 With the notation of Theorem 6.8, write $H^*(\widehat{A}_{(2)}; \mathbb{F}_2) = \mathbb{F}_2[x, y]$ and let $z = x + y$. Let $H^+_{A_{(2)}}$ (resp. $H^-_{A_{(2)}}$) be the vector subspace generated by elements of the form $x^a z^b$ where $a > b$ (resp. $b > a$). If π is an admissible partition of n, write $\pi = (1)^{m_1} \cup (2)^{m_2} \cup \pi'$ where $1 \notin \pi'$ and $2 \notin \pi'$. For $S \subseteq \{1, \dots, m_2\}$, we define $H_{i,S} = H^+_{A_{(2)}}$ if $i \notin S$ and $H^-_{A_{(2)}}$ if $i \in S$. Then we define

$$
\begin{aligned}
H^+_{\widehat{A}_\pi} &= \begin{cases} 0 & \text{if } m_1 > 0, \\ \bigoplus_{S \subset \{1,\dots,m_2\}, |S|=2k} \bigotimes_{i=1}^{m_2} H_{i,S} \otimes H^*(A_{\pi'}; \mathbb{F}_2) & \text{if } m_1 = 0, \end{cases} \\
H^-_{\widehat{A}_\pi} &= \begin{cases} 0 & \text{if } m_1 > 0, \\ \bigoplus_{S \subset \{1,\dots,m_2\}, |S|=2k+1} \bigotimes_{i=1}^{m_2} H_{i,S} \otimes H^*(A_{\pi'}; \mathbb{F}_2) & \text{if } m_1 = 0. \end{cases}
\end{aligned}
$$

Moreover, if $m_1 = m_2 = 0$, we define $H^+_{A_\pi^{s_0}} = 0$ and $H^-_{A_\pi^{s_0}} = H^*(A_\pi^{s_0}; \mathbb{F}_2)$.

Proposition 6.12 *Referring to Definition 6.11, for every maximal elementary abelian 2–subgroup $A = \widehat{A}_\pi$ or $A = A_\pi^{s_0}$ of W_{D_n}, the restriction of a positively (resp. negatively) charged Hopf monomial in $\mathcal{M}_D \cap H^*(W_{D_n}; \mathbb{F}_2)$ to the cohomology of $A_{(\pi)}$ belongs to H^+_A (resp. H^-_A).*

Proof Every positively charged gathered block b restricts to an element of H^+_A. Nontrivial computations arise only if $b = (\delta^0_{2m:0})^r (\gamma^+_{1,m})^s$ with $r \geq 0$ and $s > 0$ and $A = A_{(2)^m}$. In this case, with the notation of Theorem 6.8, we observe that

$$
h_m^{2^k} = \sum_{\substack{S \subseteq \{1,\dots,m\} \\ |S|=2l}} \prod_{i \in S} z_i^{2^k} \prod_{j \notin S} x_j^{2^k},
$$

where $z_i = x_i + y_i$. Thus $h_m^{2^k} \in H^+_{(2)^m}$. Let 2^k be the biggest power of 2 smaller than s. Then $h_m^{s-2^k}$ is a sum of pure tensors of the form $w_1 \otimes \cdots \otimes w_m$, where w_i is a monomial in x_i and z_i with total degree smaller than 2^k. Therefore, $h_m^s = h_m^{2^k} h_m^{s-2^k}$

still belongs to $H^+_{(2)^m}$. The restriction of b to $A_{(2)^m}$ is equal to $\bigotimes_{i=1}^m (x_i z_i)^r h_m^s$, which belongs to $H^+_{(2)^m}$ because multiplication by $\bigotimes_{i=1}^m x_i z_i$ preserves $H^+_{(2)^m}$.

We see the corresponding statement for negatively charged gathered blocks by noting that conjugation with s_0 exchanges H^+_A and $H^-_{A^{s_0}}$.

In general, a positively (resp. negatively) charged Hopf monomial x is a transfer product of gathered blocks, all positively charged (resp. all positively charged except one). Consequently, Proposition 6.10 yields the statement for x. □

We can finally complete our relations for A'_D by providing the proofs of the two leftover propositions of Section 5.2.

Proof of Proposition 5.14 Let b be a positively charged gathered block in A_D and $x \in A_D$. From Lemma 5.10 and the definition of Δ' we deduce that $\Delta(b) = \Delta'(b) + (\iota \otimes \iota)\Delta'(b)$, and that $\Delta(\iota(b)) = (\mathrm{id} \otimes \iota + \iota \otimes \mathrm{id})\Delta'(b)$. During this proof, we assume, by convention, that $x|_{A_\pi} = 0$ when $x \in H^*(W_{D_n}; \mathbb{F}_2)$ and π is not an admissible partition of n. Let $\pi = (2^{k_1}, \dots, 2^{k_r})$ and $\pi' = (2^{h_1}, \dots, 2^{h_s})$ be admissible partitions of some integers. From Proposition 6.10, we deduce that

$$
\begin{aligned}
&[(\odot\otimes\odot)(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\Delta'\otimes\Delta)(b\otimes x)]|_{\widehat{A}_\pi\times\widehat{A}_{\pi'}}\\
&= \sum_{\substack{I\sqcup J=\{1,\dots,r\}\\ I'\sqcup J'=\{1,\dots,s\}}} \tau_{I,I',J,J'}\big(\Delta'(b)\otimes\Delta(x)+(\mathrm{id}\otimes\iota)\Delta'(b)\otimes(\mathrm{id}\otimes\iota)\Delta(x)\\
&\qquad\qquad +(\iota\otimes\mathrm{id})\Delta'(b)\otimes(\iota\otimes\mathrm{id})\Delta(x)\\
&\qquad\qquad\qquad +(\iota\otimes\iota)\Delta'(b)\otimes(\iota\otimes\iota)\Delta(x)\big)|_{\widehat{A}_{\pi_I}\times\widehat{A}_{\pi'_{I'}}\times\widehat{A}_{\pi_J}\times\widehat{A}_{\pi'_{J'}}}\\
&= \sum_{\substack{I\sqcup J=\{1,\dots,r\}\\ I'\sqcup J'=\{1,\dots,s\}}} \tau_{I,I',J,J'}\big((\mathrm{id}+\iota\otimes\iota)\Delta'(b)\otimes\Delta(x)\\
&\qquad\qquad +(\mathrm{id}+\iota\otimes\iota)\Delta'(\iota(b))\otimes\Delta(\iota(x))\big)|_{\widehat{A}_{\pi_I}\times\widehat{A}_{\pi'_{I'}}\times\widehat{A}_{\pi_J}\times\widehat{A}_{\pi'_{J'}}}\\
&= \sum_{\bar{I}\sqcup\bar{J}=\{1,\dots,r+s\}} (b\otimes x+\iota(b)\otimes\iota(x))|_{\widehat{A}_{(\pi\sqcup\pi')_{\bar{I}}}\times\widehat{A}_{(\pi\sqcup\pi')_{\bar{J}}}}\\
&= [\Delta(b\odot x)]|_{\widehat{A}_\pi\times\widehat{A}_{\pi'}}.
\end{aligned}
$$

In these equalities we used the identities of Lemma 5.10 to perform the substitutions $(\iota\otimes\iota)\Delta(x) = \Delta(x)$ and $(\mathrm{id}\otimes\iota)\Delta(x) = (\iota\otimes\mathrm{id})\Delta(x) = \Delta(\iota(x))$; $\pi\sqcup\pi'$ is assumed to be $(2^{k_1}, \dots, 2^{k_r}, 2^{h_1}, \dots, 2^{h_s})$; $I = \bar{I}\cap\{1,\dots,r\}$ and $J = \bar{J}\cap\{1,\dots,r\}$, while I' and J' are $\bar{I}\cap\{r+1,\dots,r+s\}$ and $\bar{J}\cap\{r+1,\dots,r+s\}$ suitably shifted. The sum should be over all I, J, I' and J' such that at least one between I and J does not contain an l such that $k_l=0$ and at least one between I' and J' does not contain an l such that $h_l=0$. However, since the restriction of positively charged gathered blocks is zero on elementary abelian 2–subgroups corresponding to admissible partitions containing 1, we can restrict the sum only to the terms for which $k_i\neq 0$ for all $i\in I$ and $h_i\neq 0$ for all $i'\in I'$. This condition is equivalent to $\bar{I}$ not containing 1, and we can, once again, restrict the last sum only to these terms and get the last equality. □

Proof of Proposition 5.13 Using Proposition 6.9, the newly proved Proposition 5.14, Proposition 6.10, and the fact that cup products commute with restrictions, we check that the desired identity hold when restricted to maximal elementary abelian 2–subgroups. Then Theorem 6.2 yields the relations in A_D. □

7 Proofs of the main theorems

We devote this section to the proofs of the presentation theorems for A_B and A_D. They will be proved by comparing restrictions to elementary abelian 2–subgroups and exploiting Theorem 6.2. We will separate two technical lemmas from the proofs for the sake of clarity of exposition.

We first provide a proof for our structure theorem for A_B.

Lemma 7.1 *Let $k>0$. The kernel of the restriction map*

$$H^*(A_{(2^k)};\mathbb{F}_2)^{N_{W_{B_{2k}}}(A_{(2^k)})}\to H^*(A_{(2^k)}\cap A_{(2^{k-1},2^{k-1})};\mathbb{F}_2)$$

is the ideal generated by d_{2^k-1}.

Proof From Swenson's description of A_π, stated as in Proposition 6.4, we can identify $A_{(2^k)}$ with the image of the diagonal embedding $\mathrm{id}\times d:\Sigma_2\times V_{k-1}\to\Sigma_2\wr V_{k-1}\to W_{B_{2k}}$. Its intersection with the product $A_{(2^{k-1},2^{k-1})}=V_{k-1}\times V_{k-1}$ is identified with the subgroup $V_{k-1}\subseteq\Sigma_2\times V_{k-1}$, embedded diagonally in $W_{B_{2k}}$.

The restriction to this subgroup maps f_{2^k} to $(f_{2^{k-1}})^2$, $d_{2^k-2^l}$ to $(d_{2^{k-1}-2^{l-1}})^2$ if $l>0$, and d_{2^k-1} to 0. This is known, but we sketch a proof for completeness. If we chose bases $\{x,y_1,\dots,y_k\}$ of $H^1(A_{(2^k)};\mathbb{F}_2)$ and $\{x,y_1,\dots,y_{k-1}\}$ of $H^1(A_{(2^{k-1})};\mathbb{F}_2)$ as in Section 6.2, the restriction is given by $x\mapsto x$, $y_i\mapsto y_i$ if $1\le i<k$ and $y_k\mapsto 0$. The polynomial $F_k(t)=\prod_{v\in H^1(V_k;\mathbb{F}_2)}(t+v)$ in $H^*(V_k;\mathbb{F}_2)[t]$ restricts to $(F_{k-1}(t))^2$. Since $f_{2^k}=F_k(x)$, we deduce the formula for f_{2^k}. The identities for $d_{2^k-2^l}$ are obtained from this by using the classical identity $F_k(t)=\sum_{i=0}^k t^{2^i}d_{2^k-2^i}$. □

Proof of Theorem 5.9 Let A'_B be the Hopf ring generated by $\gamma_{k,m}$ and δ_m with the desired relations. Since the relations mentioned above hold in A_B, there exists an obvious morphism $\varphi\colon A'_B \to A_B$.

We need to fix a total ordering $\leq$ on the set $\mathcal{P}_n$ of admissible partitions of n such that, for all $\pi, \pi' \in \mathcal{P}_n$, $\pi' > \pi$ if π' is a refinement of π. In other words, $\leq$ extends the partial ordering given by refinement. Let b be a nontrivial gathered block in A_B. There exist unique nonnegative integers n and m such that $b = \prod_{i=1}^n \gamma_{i,2^{n-i}m}^{a_i} \delta_{2^n m}^{a_0}$ with $a_n \neq 0$. We consider the partition of $2^n m$ $\pi_b = (2^n, \dots, 2^n)$. Given $x = b_1 \odot \cdots \odot b_r \in \mathcal{M}_B$, let $\pi_x = \bigsqcup_{i=1}^r \pi_{b_i}$. As a consequence of Propositions 6.5, 6.6 and 6.7, $x|_{A_\pi} \neq 0$ implies that $\pi_x > \pi$. Explicitly, if $b = \prod_{i=1}^n \gamma_{i,2^{n-i}m}^{a_i} \delta_{2^n m}^{a_0}$,

$$b|_{A_{\pi_b}} = \left(f_{2^n}^{a_0} \prod_{i=1}^n d_{2^n - 2^{n-i}}^{a_i} \right)^{\otimes m}.$$

For any $x = b_1 \odot \cdots \odot b_r \in \mathcal{M}_B$, $x|_{A_{\pi_n}}$ is the symmetrization of $\bigotimes_{i=1}^r b_i|_{A_{\pi_{b_i}}}$. Given a partition π, let $\mathcal{M}_\pi$ be the set of elements $x \in \mathcal{M}$ such that $\pi_x = \pi$.

We first prove that φ is injective. We proceed by contradiction, and we assume that there exists a nontrivial sum $\sum_i x_i$ of elements of $\mathcal{M}_B$ that is 0 when restricted to every elementary abelian 2–subgroup. Let π be maximal among the set of partitions $\{\pi_{x_i}\}_i$. Since, by the explicit calculation above, the restrictions of the elements of $\mathcal{M}_\pi$ to A_π are linearly independent, this gives a contradiction.

To prove surjectivity, it is sufficient, by Theorem 6.2, to prove that an element α of the Quillen group $\mathcal{F}^*_{W_{B_n}}$ can be written as the image via $q_{W_{B_n}}$ of a linear combination of elements of $\mathcal{M}_B$. Note that such an α is determined by its values α_π on the maximal abelian 2–subgroups A_π. Let $\bar{\pi}_\alpha = \max\{\pi \in \mathcal{P}_n \mid \alpha_\pi \neq 0\}$ with respect to the chosen linear ordering. We write $\bar{\pi}_\alpha = (2^{k_1}, \dots, 2^{k_r})$. We proceed by induction on $\bar{\pi}_\alpha$. $\alpha_{\bar{\pi}_\alpha}$ must be invariant with respect to the action of the normalizer $N_{W_{B_n}}(A_{\bar{\pi}_\alpha})$. By Swenson's description of these invariant subalgebras stated in Proposition 6.4, it is a sum of elements $\sum_i c_{i,1} \otimes \cdots \otimes c_{i,r}$, with

$$c_{i,j} = \prod_{l=1}^{k_j - 1} d_{2^{k_j} - 2^{k_j - l}}^{a_{i,j,l}} f_{2^{k_j}}^{a_{i,j,0}} \in H^*(A_{(2^{k_j})}; \mathbb{F}_2).$$

We must have $a_{i,j,k_j} \neq 0$ for all i and j. Otherwise, we can define a partition π' obtained from $\bar{\pi}_\alpha$ by substituting 2^{k_j} with two parts both equal to $2^{k_j - 1}$ and observe that, by Lemma 7.1, we must have $\alpha_{\bar{\pi}_\alpha}|_{A_{\bar{\pi}_\alpha} \cap A_{\pi'}} \neq 0$. Thus $\alpha_{\pi'} \neq 0$ and this would contradict the maximality of $\bar{\pi}_\alpha$. By our calculations above, since $\alpha_{\bar{\pi}_\alpha}$ must be invariant by permutations of tensor factors, this condition guarantees the existence of an element x in the linear span of $\mathcal{M}_{\bar{\pi}_\alpha}$ such that $x|_{A_{\bar{\pi}_\alpha}} = \alpha_{\bar{\pi}_\alpha}$. This reduces the statement to $\alpha' = \alpha + q_{W_{B_n}}(x)$ for which, by construction, $\bar{\pi}_{\alpha'} < \bar{\pi}_\alpha$, and completes the induction argument. □

We now focus on the presentation of A_D.

Lemma 7.2 *Let $\mathcal{M}_2 \subseteq \mathcal{M}_D$ be the set of Hopf monomials in A'_D whose constituent gathered blocks are all of the form $(\delta^0_{2k:0})^r(\gamma^{\pm}_{1,k})^s$ with $r \geq 0$ and $s > 0$, or of the form $(\delta^0_{2:0})^a$ with $a \geq 0$. Then, for all $m \geq 0$, $\mathcal{M}_2 \cap H^*(W_{D_{2m}}; \mathbb{F}_2)$ restricts to a linearly independent set in $H^*(\hat{A}_{(2)^m}; \mathbb{F}_2)$. Moreover, the image of $\mathcal{M}_2$ in the cohomology of $\hat{A}_{(2)^m}$ generates the kernel of the restriction*

$$\rho_{2,1} \colon H^*(\hat{A}_{(2)^m}; \mathbb{F}_2)^{N_{W_{D_{2m}}}(\hat{A}_{(2)^m})} \to H^*(\hat{A}_{(1)^4 \cup (2)^{m-2}} \cap \hat{A}_{(2,2)}; \mathbb{F}_2).$$

Proof Note that, due to Theorem 6.8 and Proposition 6.9, the Hopf monomials in $\mathcal{M}_2 \cap H^*(W_{D_2}; \mathbb{F}_2)$ restrict to linearly independent elements in $H^*(\hat{A}_{(2)}; \mathbb{F}_2)$. Therefore, to prove the linear independence claim for $m > 1$, it is enough to check that the restrictions of the elements of $\mathcal{M}_2 \cap H^*(W_{D_{2m}}; \mathbb{F}_2)$ to $H^*(W^m_{D_2}; \mathbb{F}_2)$ (which is a component $\Delta_{(2)^m}$ of the coproduct) are linearly independent. Let $\mathcal{F}$ be the weight filtration on A'_D provided by Definition 5.20. It is enough to prove that this set is linearly independent when working in the associated graded spaces $\mathrm{gr}_{\mathcal{F}}(A'_D)$ and $\mathrm{gr}_{\mathcal{F}}(H^*(W^m_{D_2}; \mathbb{F}_2))$. In this setting, the image of a gathered block $b^+_{l,\underline{t}} \in \mathcal{M}_2$ (resp. $b^-_{l,\underline{t}} \in \mathcal{M}_2$) under $\mathrm{gr}_{\mathcal{F}}(\Delta_{(2)^m})$ is $\sum_{\varepsilon_1,\dots,\varepsilon_l} \bigotimes_{i=1}^m b^{\varepsilon_i}_{1,\underline{t}}$, where the sum is over all l–tuples $(\varepsilon_1, \dots, \varepsilon_l)$ with $\varepsilon_i \in \{+,-\}$ and the cardinality of the set $\{i : 1 \leq i \leq l, \varepsilon_i = -\}$ is even (resp. odd). Combining this with Proposition 5.14, we check the claim directly.

By Propositions 6.9 and 6.10, every element of $\mathcal{M}_2$ restricts to 0 on $\hat{A}_\pi$ whenever $1 \in \pi$. Therefore, it is contained in the kernel of $\rho_{2,1}$. We now prove the opposite inclusion. With the notation of Theorem 6.8, we write

$$H^*(A_{(2)^m}; \mathbb{F}_2)^{N_{W_{D_{2m}}}(A_{(2)^m})} = (\mathbb{F}_2[f_2, d_1]^{\otimes m})^{\Sigma_m}\{1, h_m\}.$$

We note that $A_{(2)^m} \cap \hat{A}_{(1)^4 \cup (2)^{m-2}} = A_{(2)^m} \cap A_{(1)^4 \cup (2)^{m-2}}$. Moreover, $h_m = \gamma^+_{1,m}|_{A_{(2)^m}}$ is 0 when restricted to $A_{(2)^m} \cap \hat{A}_{(1)^4 \cup (2)^{m-2}}$. Therefore, Lemma 7.1 implies that $\ker(\rho_{2,1})$ is the ideal generated by h_m and $d_1^{\otimes m} = \rho_{2,1}(\gamma^+_{1,m} + \gamma^-_{1,m})|_{A_{(2)^m}}$. Finally, the generators belong to the image of $\mathcal{M}_2$, the linear subspace generated by $\mathcal{M}_2$ is a $\cdot$–subalgebra by our formulas in A'_D, and restriction maps preserve cup products. □

Proof of Theorem 5.15 Let A''_D be the almost Hopf ring generated by elements of the form $\delta^0_{n:m}$, $\gamma^{\pm}_{k,m}$, and 1^- with the desired relations. Let $\varphi\colon A''_D \to A'_D$ be the obvious morphism. We also consider the $\mathbb{F}_2$–vector space A'''_D with basis $\mathcal{M}_D$. By our relations for A'_D, $\mathcal{M}_D$ generates A''_D. Thus, there is a surjective linear map $\varphi'\colon A'''_D \to A''_D$. To prove that φ' is an isomorphism it is enough to prove that $\varphi'' = \varphi'\varphi$ is. Since in component 0 this is obvious, we can consider only positive components and replace A'_D with A_D. For technical reasons, we consider the set $\mathcal{M}'_D$, which differ from $\mathcal{M}_D$ by replacing neutral gathered blocks with elements of the form $\rho\bigl(\prod_{i=2}^n(\delta_i \odot 1_{n-i})^{k_i}\bigr)$ for $k_2,\dots,k_n \geq 0$. As shown in Lemma 5.18, this corresponds, at the level of A'''_D, to performing a change of basis. Hence, it does not affect the argument.

We adapt the argument used in the proof of Theorem 5.9. We define π_x for $x \in \mathcal{M}'_D$ as we did for A_B, with the only difference that gathered blocks of the form $b = (\delta^0_{2:0})^m$ have $\pi_b = (2)$, because $(1,1)$ does not define a maximal elementary abelian subgroup in W_{D_2}. It is still true that $x|_{\widehat{A}_\pi} = 0$ unless π_x is a refinement of π. We extend refinement of admissible partitions to a total ordering $\leq$, and we use the same argument by induction on $\leq$ adopted for A_B. Our choice of the new basis $\mathcal{M}'_D$ makes evident that for all admissible partitions π the set $\mathcal{M}'_\pi = \{x \in \mathcal{M}'_D \mid \pi_x = \pi\}$ restricts to a linearly independent set in the cohomology of $\widehat{A}_\pi$ when $1 \in \pi$, and Lemma 7.2 guarantees that this is true if $1 \notin \pi$ and $2 \in \pi$. Hence, the injectivity part works verbatim. We need to adapt the surjectivity argument for admissible partitions π such that $1 \notin \pi$ and $2 \in \pi$ (in all other cases, nothing changes). In these cases, we use Lemma 7.2 instead of Lemma 7.1 to carry on the proof. □

8 Steenrod algebra action

This section is devoted to the calculation of the Steenrod algebra action on A_B and A_D. We first observe that, since the coproducts and transfer products are induced by (stable) maps, they satisfy a Cartan formula with respect to Steenrod squares. In other words, A_B and A_D are almost-Hopf rings over the Steenrod algebra. Thus it is sufficient to determine the action of the Steenrod squares on the generators δ_{2^n}, $\gamma_{k,2^n}$, $\delta^0_{n:m}$ and $\gamma^{\pm}_{k,2^n}$.

Definition 8.1 [7] We define the following notions:

- The *height* (ht) of a gathered block in A_B or A_D is the number of generators that are cup-multiplied to obtain it, and the height of a Hopf monomial $x = b_1 \odot \cdots \odot b_r$ is $\max_{i=1}^r \operatorname{ht}(b_i)$.

- The *effective scale* (effsc) of a gathered block in the cohomology of W_{B_n} (resp. W_{D_n}) is the least l such that $n/2^l$ is an integer and its restriction to $W_{B_{2^l}}^{n/2^l}$ (resp. $W_{D_{2^l}}^{n/2^l}$) is nonzero, and the effective scale of a Hopf monomial $x = b_1 \odot \cdots \odot b_r$ as $\min_{i=1}^r \operatorname{effsc}(b_i)$.
- A *full-width* monomial is a Hopf monomial in A_B (resp. A_D) of which no constituent block is of the form $1_{W_{B_n}}$ (resp. $1_{W_{D_n}}$).

Theorem 8.2 (cf [7, Theorem 8.3, page 191]) *Let $k, n \geq 1$ and $i \geq 0$. Then, in A_B, the following formulas hold:*

- *$\mathrm{Sq}^i(\gamma_{k,2^n})$ is the sum of all the full-width monomials $x \in \mathcal{M}_B$ of degree $2^{n+k} - 2^n + i$ with $\mathrm{ht}(x) \leq 2$ and $\operatorname{effsc}(x) \geq k$ in which generators of the form δ_k do not appear.*
- *$\mathrm{Sq}^i(\delta_{2^n})$ is the sum of all the full-width monomials $x \in \mathcal{M}_B$ of degree $2^n + i$ with $\mathrm{ht}(x) \leq 2$ and $\operatorname{effsc}(x) \geq 1$ such that a generator of the form δ_k appears in every constituent gathered block of x.*

Proof The calculation for $\mathrm{Sq}^i(\gamma_{k,2^n})$ is an obvious consequence of [7, Theorem 8.3, page 191]. Regarding $\mathrm{Sq}^i(\delta_{2^n})$, since δ_{2^n} is the top-dimensional Stiefel–Whitney class of the reflection representation U_{2^n} by Proposition 5.3, by Wu's formula $\mathrm{Sq}^i(\delta_{2^n}) = w_i(U_{2^n})\delta_{2^n}$. Defining, by convention, $\gamma_{k,0} = 1$, let

$$u_i = \sum_{\substack{j_0,\dots,j_n \geq 0, \sum_{r=1}^{n-1} 2^r j_r + j_n + j_0 = 2^n \\ \sum_{r=1}^{n-1}(2^r-1) j_r + j_n = i}} \bigodot_{r=1}^{n-1} \gamma_{r,j_r} \odot \delta_{r_n} \odot 1_{W_{B_{j_0}}}.$$

We computed the restriction of $w_i(U_{2^n})$ to the maximal elementary abelian subgroups A_π in the proof of Proposition 6.6. It coincides with the restriction of u_i by our previous calculations based on Proposition 6.5. Thus,

$$\mathrm{Sq}^i(\delta_{2^n}) = w_i(U_{2^n})\delta_{2^n} = u_i \delta_{2^n},$$

and this class is exactly the sum of all the desired Hopf monomials x. □

Regarding the calculation of the Steenrod squares on the generators of A_D, We observe that the calculation for $\mathrm{Sq}^i(\delta_{n:m})$ is implicit in Theorem 8.2 since $\delta_{n:m} = \rho(\delta_n \odot 1_m)$ and ρ commute with Steenrod operations. Thus we only need to consider generators of the form $\gamma_{k,n}^{\pm}$.

Theorem 8.3 *Let $k, n \geq 1$ and $i \geq 0$. Then, in A_D, $\mathrm{Sq}^i(\gamma^+_{k,n})$ (resp. $\mathrm{Sq}^i(\gamma^-_{k,n})$) is the sum of all the full-width monomials $x \in \mathcal{B}^+$ (resp. $x \in \mathcal{B}^-$) of degree $2^{n+k} - 2^n + i$ with $\mathrm{ht}(x) \leq 2$ and $\mathrm{effsc}(x) \geq l$ in which generators of the form $\delta_{n:m}$ do not appear.*

Proof We recall that Definition 6.11 provides, for all maximal elementary abelian 2–subgroup $A \subseteq W_{D_{2n}}$, subspaces H^+_A and H^-_A of the cohomology of A. A direct calculation shows that $\mathrm{Sq}^i(h_n) \in H^+_{(2)^n}$. Since restrictions preserve the Steenrod squares, $\mathrm{Sq}^i(\gamma^+_{1,n})$ is mapped to an element of H^+_A for all maximal elementary abelian 2–subgroups $A \subseteq W_{D_{2n}}$ and all choices of i and n. Similarly, the restriction of $\mathrm{Sq}^i(\gamma^-_{1,n})$ to every such subgroup A lies in H^-_A. Let x^+ (resp. x^-) be the sum of all the positively (resp. negatively) charged Hopf monomials considered in the statement. By Proposition 6.12, the restriction of x^+ (resp. x^-) belongs to H^+_A (resp. H^-_A).

Moreover $\mathrm{Sq}^i(\gamma^+_{k,n}) + \mathrm{Sq}^i(\gamma^-_{k,n}) = \rho(\mathrm{Sq}^i(\gamma_{k,n}))$. Consequently, Theorem 8.2 implies that $\mathrm{Sq}^i(\gamma^+_{k,n}) + \mathrm{Sq}^i(\gamma^-_{k,n}) = x^+ + x^-$.

Since $H^+_A \cap H^-_A = 0$ for all A, the two facts above guarantee that $\mathrm{Sq}^i(\gamma^+_{k,n}) = x^+$ and $\mathrm{Sq}^i(\gamma^-_{k,n}) = x^-$. □

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Operads in unstable global homotopy theory

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We study operads in unstable global homotopy theory, which is the homotopy theory of spaces with compatible actions by all compact Lie groups. We show that the theory of these operads works remarkably well, as for example it is possible to give a model structure for the category of algebras over any such operad. We define global E_∞–operads, a good generalization of E_∞–operads to the global setting, and we give a rectification result for algebras over them.

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1 Introduction

Operads were first introduced by May [17] to study infinite loop spaces. Since then they have found uses in many areas of mathematics, including algebra, higher category theory, geometry, and mathematical physics. In general, an operad codifies a collection of operations of varying arity in a symmetric monoidal category.

An algebra over an operad $\mathcal{O}$ is a representation of the abstract operations that the operad encodes as actual operations in some object. For example, an algebra over the commutative operad Comm is a commutative monoid in the given symmetric monoidal category. Another important example is that of an E_∞*–operad*, which encodes a binary operation that is unital, associative and commutative but only up to all higher homotopies.

One area that has seen increased interest in the last decade is equivariant homotopy theory. It is dedicated to studying the homotopy theory of spaces with an action by a topological group G. One can construct operads in the category of G–spaces, and this yields a theory that is remarkably different to the nonequivariant case. Unlike in the nonequivariant case, where all E_∞–operads are equivalent, there are multiple possible nonequivalent notions of what an E_∞–operad in G–spaces could be, all of which are nonequivariantly E_∞–operads. For example, there is the naive one, an E_∞–operad in spaces given the trivial G–action. This is however not the best choice when one wants to study objects like equivariant infinite loop spaces or equivariant spectra with some multiplicative structure.

Instead the better choice is to look at both the G–action and the Σ_n–action on each $\mathcal{O}_n$ at the same time. An E_∞*–*G*–operad* is an operad in G–spaces where each $\mathcal{O}_n$ is a universal space for the family of *graph subgroups* of $G \times \Sigma_n$. Algebras over an E_∞–G–operad have more structure than algebras over a naive E_∞–operad in G–spaces.

In this paper we look at operads in the setting of *unstable global equivariant homotopy theory*. This is the homotopy theory of spaces which have simultaneous and compatible actions by all compact Lie groups. There are important equivariant constructions, like equivariant K–theory spectra and equivariant Thom spectra, that can be understood as a single globally equivariant object. We work with the model for unstable global homotopy theory based on *orthogonal spaces*, introduced by Schwede [22]. An orthogonal space can be thought of as the unstable analog of an orthogonal spectrum. There are some similarities between the theories of operads in the global equivariant setting and the G–equivariant setting for a single group G, but operads in the global equivariant setting are technically better behaved.

An orthogonal space has an *underlying* K*–space* for each compact Lie group K. We study orthogonal spaces through these compatible K–actions for each K. A morphism of orthogonal spaces is said to be a *global equivalence* if it is an equivalence of underlying K–spaces for each compact Lie group K. There is a model structure in the category of orthogonal spaces with these global equivalences as the weak equivalences, called the *global model structure*.

A natural question to consider is whether one can construct a model structure on $\underline{\mathcal{A}lg}(\mathcal{O})$ the category of algebras over a given operad $\mathcal{O}$ in orthogonal spaces using the global model structure on the underlying category.

Question Does the forgetful functor create the weak equivalences and fibrations of some model structure on the category of algebras over a given operad $\mathcal{O}$?

The first place where this question was examined for a general category was in Spitzweck's PhD thesis [25], which provided some conditions under which this is true. The main technical point there was a factorization for pushout diagrams in the category of algebras over an operad. A different approach was used by Berger and Moerdijk [3]. Pavlov and Scholbach [19] studied this question most extensively with full generality, and White and Yau [26] studied an analogous question for semimodel structures. We use a different factorization for pushout diagrams given by Sagave and Schlichtkrull [21], originally from Elmendorf and Mandell [7].

The first main result that we obtain is that the desired model structure exists for any operad in orthogonal spaces.

Theorem 4.11 *Let $\mathcal{O}$ be any operad in $(\underline{\mathcal{S}pc}, \boxtimes)$ the category of orthogonal spaces, with the positive global model structure and the symmetric monoidal structure given by the box product. Then there is a cofibrantly generated model category structure on $\underline{\mathcal{A}lg}(\mathcal{O})$, the category of algebras over $\mathcal{O}$, where the forgetful functor $U_{\underline{\mathcal{A}lg}(\mathcal{O})}$ creates the weak equivalences and fibrations, and sends cofibrations in $\underline{\mathcal{A}lg}(\mathcal{O})$ to h–cofibrations in $\underline{\mathcal{S}pc}$.*

This result is surprising, in that it holds for all operads. Such a result generally holds for all operads if the category is nice enough, for example symmetric spectra based on simplicial sets; see the work of Harper [12]. One relevant property there is that all simplicial sets are cofibrant. Since not all orthogonal spaces are cofibrant, the approach of [12] does not apply to the present case.

Instead we use that the box product of orthogonal spaces is fully homotopical. By definition, this means that the box product of two global equivalences is a global equivalence, without any cofibrancy assumptions. This in turn removes any cofibrancy assumptions on the operad in Theorem 4.11.

Theorem 4.11 was proven by Schwede [22] for the specific case of the commutative operad Comm. Algebras over Comm are the commutative monoids in orthogonal spaces with respect to the box product, which are usually called *ultracommutative* *monoids*, and they have a very rich structure. We generalize the result in [22] to any operad. To accomplish this we need to use several different technical results and tools.

Some of these technical results deal with the Σ_n–objects in the category of orthogonal spaces, and so we study them in detail. We consider more generally orthogonal spaces which have an additional action by a fixed compact Lie group G, which we call *G–orthogonal spaces*. Thus, the underlying K–space of a G–orthogonal space is a $(K\times G)$–space.

We define the notion of a G–global equivalence between G–orthogonal spaces, which takes into account both the G–action, and the action by each compact Lie group K. We also study various properties of these G–orthogonal spaces. In the appendix we give a model structure for G–orthogonal spaces which has the G–global equivalences as weak equivalences. Since G is any compact Lie group, the results of the appendix are new in this generality.

This notion of "globally equivariant objects" with an additional action by a fixed group G was studied extensively by Lenz in the context of algebraic K–theory [15]. There, various model structures were given for a discrete group G not necessarily finite. Orthogonal spaces and orthogonal spectra with a G–action for a compact Lie group G were also studied from the global point of view by Schwede [23]; however no model structure was defined there.

Our second main result is a characterization of morphisms of operads in orthogonal spaces that induce a Quillen equivalence between the respective categories of algebras.

Theorem 4.14 *Let $g\colon \mathcal{O}\to\mathcal{P}$ be a morphism of operads in $(\underline{\mathcal{Spc}},\boxtimes)$, the category of orthogonal spaces, with the positive global model structure and the symmetric monoidal structure given by the box product. Then the extension and restriction adjunction $(g_!, g^*)$ is a Quillen equivalence between the respective categories of algebras if and only if for each $n\geq 0$ the morphism $g_n\colon \mathcal{O}_n\to\mathcal{P}_n$ is a Σ_n–global equivalence.*

As was the case with Theorem 4.11, this result applies in full generality, to any morphism between any two operads. For a morphism g between "nice" operads, it is enough to require that the morphisms g_n are weak equivalences in the underlying category to obtain a Quillen equivalence, as shown by Spitzweck [25]. However, for arbitrary operads in orthogonal spaces Theorem 4.14 does *not* hold if each g_n is merely a global equivalence, it additionally needs to be a Σ_n–global equivalence. In particular, if $\mathcal{O}$ is a topological E_∞–operad given the trivial global structure, $\underline{\mathcal{Alg}}(\mathcal{O})$ is not Quillen equivalent to the category of ultracommutative monoids. Thus, as in the G–equivariant setting mentioned at the beginning, this naive global E_∞–operad is not the best one to consider.

Instead, Theorem 4.14 suggest a good notion of what an E_∞–operad in the global equivariant sense should be. We define a *global E_∞–operad* to be an operad $\mathcal{O}$ in $(\underline{\mathcal{Spc}}, \boxtimes)$ such that each $\mathcal{O}_n$ is Σ_n–globally equivalent to $*$, the one-point orthogonal space. Then the naive global E_∞–operads of the previous paragraph are not actually global E_∞–operads. For any global E_∞–operad $\mathcal{O}$, Theorem 4.14 implies that the category of algebras over $\mathcal{O}$ is Quillen equivalent to the category of ultracommutative monoids. Thus, any algebra over a global E_∞–operad can be rectified to an ultracommutative monoid, and so these algebras also encode the highest possible level of commutativity.

In this article we provide several examples of global E_∞–operads. Some of these are global analogs of classical operads in (equivariant) homotopy theory. These include a global version of the little disks operad and the Steiner operad, which are constructed in a similar way to the little disks and Steiner G–operads associated to a G–universe for a compact Lie group G.

In the G–equivariant case, there is a whole hierarchy of nonequivalent operads between a naive E_∞–operad in G–spaces and an E_∞–G–operad. These in-between operads are called *N_∞–operads*, and were introduced by Blumberg and Hill [4]. They codify various levels of commutativity, by imposing the existence of certain additive transfers/multiplicative norms. In the global setting, there is also a hierarchy of operads between the naive global E_∞–operads and the global E_∞–operads. These operads in orthogonal spaces are the global analogs of N_∞–operads. We provide a classification of them in [1].

Structure of this paper

In Section 2 we begin by recalling the basic properties of operads as defined in any symmetric monoidal category. We then introduce unstable global homotopy theory, to put in context the questions that we examine. We also give plenty of examples of operads in orthogonal spaces, to build some intuition.

In Section 3, we study G–orthogonal spaces. We begin by defining the G–global equivalences, and checking their basic properties. We then look at how G–global equivalences interact with taking G–orbits and with the box product. Lastly we introduce the h–cofibrations of G–orthogonal spaces, which are used in the proofs of our main results, Theorems 4.11 and 4.14, presented in Section 4.

In Section 5 we introduce global E_∞–operads, and check that several of the examples of global operads given in Section 2 are global E_∞–operads.

There is a model structure on G–orthogonal spaces with the G–global equivalences as the weak equivalences. For completeness, we present the construction of this model structure in the appendix. We do not need this model structure to prove our main theorems.

Notation and conventions

We introduce here various mathematical and notational conventions that are used throughout this article.

Whenever we talk about a space we are referring to a compactly generated weak Hausdorff topological space. We use $\underline{\mathrm{Top}}$ to denote the category of such spaces. In the rare cases where we refer to a general topological space, we do so explicitly. We underline the names used for specific categories, like $\underline{\mathrm{Set}}$ or $\underline{\mathrm{Top}}$, but not the variables like $\mathscr{C}$. In particular, $\underline{G}$ denotes the one-object groupoid associated to a group G.

We often use i_l to refer to the boundary map $i_l \colon \partial D^l \to D^l$ in $\underline{\mathrm{Top}}$ for each $l \geq 0$. Similarly we use j_l for the inclusion $j_l \colon D^l \cong D^l \times \{0\} \to D^l \times [0,1]$ for $l \geq 0$.

We use $\times$ for the categorical product, $\boxtimes$ for the box product of orthogonal spaces introduced in Remark 2.6, and $\otimes$ for the tensor product in a generic symmetric monoidal category.

In this article we only consider compact Lie groups, and closed subgroups of them. By default, an inner product space refers to a real inner product space, finite-dimensional unless stated otherwise, and for a compact Lie group G, a G–representation means an orthogonal G–representation in an inner product space, also finite-dimensional unless stated otherwise.

A *complete G–universe* is a countably infinite-dimensional orthogonal G–representation with nonzero fixed points, and such that for each finite-dimensional G–representation V, a countably infinite direct sum of copies of V embeds G–equivariantly into $\mathcal{U}_G$. We denote a complete G–universe by $\mathcal{U}_G$.

We write Σ_n for the symmetric group on n elements. By default, group actions are left group actions. Sometimes we turn a right action into a left action and vice versa by acting via the inverse, without saying so explicitly.

Let G be a compact Lie group. For any set $\mathcal{F}$ of closed subgroups of G, we say that a morphism $f: X \to Y$ of G–spaces is an *$\mathcal{F}$–equivalence* (resp. an *$\mathcal{F}$–fibration*) if for any $H \in \mathcal{F}$ the restriction of f to the H–fixed points $f^H: X^H \to Y^H$ is a weak homotopy equivalence (resp. a Serre fibration). For each set $\mathcal{F}$ of closed subgroups of G there is a cofibrantly generated model structure on $\underline{\mathrm{GTop}}$ the category of G–spaces, with the $\mathcal{F}$–equivalences as weak equivalences and the $\mathcal{F}$–fibrations as fibrations; see [22, Proposition B.7]. We refer to the cofibrations of this model structure as *$\mathcal{F}$–cofibrations*. If the set $\mathcal{F}$ is the set of all closed subgroups of G, we instead use G–equivalences, G–fibrations, and G–cofibrations to refer to these classes of morphisms.

Given two compact Lie groups K and G we refer often to the set of *graph subgroups* of $K \times G$, denoted by $\mathscr{F}(K, G)$ and defined in Definition 3.1. We generally use Γ to denote a graph subgroup. Whenever we also need to refer to the continuous homomorphism ϕ associated to the graph subgroup, we use Γ_ϕ to denote the graph subgroup.

Finally, when we talk about small objects in a category with respect to a class of morphisms, we follow the conventions of [13, Section 2.1.1]. We use the letters I, J and K to denote sets of generating (acyclic) cofibrations of various model categories.

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2 Background

2.1 Operads

Let $\mathscr{C}$ be a cocomplete symmetric monoidal category, where the tensor product preserves all small colimits in both variables. We follow the exposition of [9] to define operads in $\mathscr{C}$. Let Σ_*–$\mathscr{C}$ denote the category of *symmetric sequences* in $\mathscr{C}$, these are sequences $\{X(n)\}_{n\in\mathbb{N}}$ of objects of $\mathscr{C}$ where each $X(n)$ has a right Σ_n–action. So explicitly, Σ_*–$\mathscr{C}$ is the functor category $\mathrm{Fun}\big(\coprod_{n\in\mathbb{N}} \underline{\Sigma_n}, \mathscr{C}\big)$.

One can define a *composition monoidal structure* on Σ_*–$\mathscr{C}$, denoted by $\circ$; see [9, 2.2.1 and 2.2.2]. Then an *operad* in $\mathscr{C}$ is just a monoid in $(\Sigma_*\text{–}\mathscr{C}, \circ)$. An operad $\mathcal{O}$ in $\mathscr{C}$ gives a monad $\mathcal{F}(\mathcal{O})$ on $\mathscr{C}$; see [9, 2.1.1 and 2.2.1]. An *algebra* over this operad is defined as an algebra over the monad $\mathcal{F}(\mathcal{O})$. We use $\underline{\mathcal{A}lg}(\mathcal{O})$ to denote the category of algebras over $\mathcal{O}$, and write $F_{\underline{\mathcal{A}lg}(\mathcal{O})}$ and $U_{\underline{\mathcal{A}lg}(\mathcal{O})}$ for the adjoint free and forgetful functors between $\mathscr{C}$ and $\underline{\mathcal{A}lg}(\mathcal{O})$.

From now on, let $\mathscr{C}$ additionally be a cofibrantly generated model category. Given an operad $\mathcal{O}$ in $\mathscr{C}$ we want to lift the model structure of $\mathscr{C}$ through the forgetful functor $U_{\underline{\mathcal{A}lg}(\mathcal{O})}\colon \underline{\mathcal{A}lg}(\mathcal{O}) \to \mathscr{C}$. That is, we want to consider the class of those morphisms which $U_{\underline{\mathcal{A}lg}(\mathcal{O})}$ sends to weak equivalences, and the class of those sent to fibrations, and ask the question of whether these two classes determine a model structure on $\underline{\mathcal{A}lg}(\mathcal{O})$. If they do, we say that the operad $\mathcal{O}$ is *admissible*.

The result [24, Lemma 2.3] gives conditions under which one can lift a model structure to the category of algebras over a monad. Let I and J denote sets of generating cofibrations and acyclic cofibrations of $\mathscr{C}$, respectively. Set $\mathsf{I}_{\mathcal{O}} = F_{\underline{\mathcal{A}lg}(\mathcal{O})}(\mathsf{I})$ and $\mathsf{J}_{\mathcal{O}} = F_{\underline{\mathcal{A}lg}(\mathcal{O})}(\mathsf{J})$, and let $\mathsf{I}_{\mathcal{O}}$–reg and $\mathsf{J}_{\mathcal{O}}$–reg denote the regular $\mathsf{I}_{\mathcal{O}}$–cofibrations and regular $\mathsf{J}_{\mathcal{O}}$–cofibrations in $\underline{\mathcal{A}lg}(\mathcal{O})$. Those are the transfinite compositions of cobase changes in $\underline{\mathcal{A}lg}(\mathcal{O})$ of morphisms in $\mathsf{I}_{\mathcal{O}}$ and $\mathsf{J}_{\mathcal{O}}$. Applying [24, Lemma 2.3] to the monad $\mathcal{F}(\mathcal{O})$ associated to an operad $\mathcal{O}$, and using that $\mathcal{F}(\mathcal{O})$ always preserves filtered colimits [9, Proposition 2.4.1], one obtains the following result.

Lemma 2.1 ([24, Lemma 2.3] applied to operads) *Let $\mathcal{O}$ be an operad in $\mathscr{C}$. Assume that the sources of morphisms in $\mathsf{I}_{\mathcal{O}}$ and $\mathsf{J}_{\mathcal{O}}$ are small with respect to $\mathsf{I}_{\mathcal{O}}$–reg and $\mathsf{J}_{\mathcal{O}}$–reg, respectively, and that every morphism in $\mathsf{J}_{\mathcal{O}}$–reg is a weak equivalence in $\mathscr{C}$. Then $\underline{\mathcal{A}lg}(\mathcal{O})$ is a cofibrantly generated model category where $U_{\underline{\mathcal{A}lg}(\mathcal{O})}$ creates the weak equivalences and fibrations and $\mathsf{I}_{\mathcal{O}}$ and $\mathsf{J}_{\mathcal{O}}$ are generating sets of cofibrations and acyclic cofibrations.*

We have the following refinement of the result above, inspired by and similar to [9, Proposition 11.1.14].

Theorem 2.2 *Let $\mathscr{C}$ be a symmetric monoidal category which is also a cofibrantly generated model category with sets of generating cofibrations and acyclic cofibrations I*

and J, *respectively*, *and such that the monoidal product preserves all small colimits in each variable. Let* $\mathcal{H}cof$ *be a class of morphisms in* $\mathscr{C}$ *which satisfies the following*:

(a) $\mathcal{H}cof$ *is closed under retracts and transfinite compositions.*

(b) *The sources of morphisms in* I *and* J *are small with respect to* $\mathcal{H}cof$.

(c) *A map which is a transfinite composition of morphisms that are both in* $\mathcal{H}cof$ *and are weak equivalences*, *is a weak equivalence.*

Fix any operad $\mathcal{O}$ *in* $\mathscr{C}$, *and assume that for each pushout in* $\underline{\mathcal{Alg}}(\mathcal{O})$ *of the form*

$$\begin{tikzcd} F_{\underline{\mathcal{Alg}}(\mathcal{O})}(X) \arrow[r, "F_{\underline{\mathcal{Alg}}(\mathcal{O})}(i)"] \arrow[d] & F_{\underline{\mathcal{Alg}}(\mathcal{O})}(Y) \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{1}$$

the following hold:

(1) *If* $i \in \mathsf{I}$ *then* $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(f)$ *is in* $\mathcal{H}cof$.

(2) *If* $i \in \mathsf{J}$ *then* $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(f)$ *is a weak equivalence.*

Then the conditions of Lemma 2.1 are satisfied, *so* $\underline{\mathcal{Alg}}(\mathcal{O})$ *is a cofibrantly generated model category*, *where* $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ *creates the weak equivalences and fibrations*, *and* $\mathsf{I}_{\mathcal{O}}$ *and* $\mathsf{J}_{\mathcal{O}}$ *are generating sets of cofibrations and acyclic cofibrations of* $\underline{\mathcal{Alg}}(\mathcal{O})$. *Furthermore*, $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ *sends cofibrations to morphisms in* $\mathcal{H}cof$.

Note that in the conditions of Theorem 2.2 the class of morphisms $\mathcal{H}cof$ is not required to contain all the cofibrations of $\mathscr{C}$. In our application of Theorem 2.2 in Section 4.1, we take $\mathcal{H}cof$ to be the class of h–cofibrations, the morphisms with the homotopy extension property, hence the notation. In most settings, including the model of unstable global homotopy theory we use, the class of h–cofibrations does contain all cofibrations, but as mentioned this is not necessary.

Proof of Theorem 2.2 We have to check that the sources of morphisms in $\mathsf{I}_{\mathcal{O}}$ and $\mathsf{J}_{\mathcal{O}}$ are small with respect to $\mathsf{I}_{\mathcal{O}}$–reg and $\mathsf{J}_{\mathcal{O}}$–reg, respectively, and that every morphism in $\mathsf{J}_{\mathcal{O}}$–reg is a weak equivalence in $\mathscr{C}$. Then by Lemma 2.1 the claim follows.

By [5, Proposition 4.3.2], $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ preserves filtered colimits. Morphisms in $\mathsf{I}_{\mathcal{O}}$–reg are transfinite compositions of cobase changes of morphisms with the form $F_{\underline{\mathcal{Alg}}(\mathcal{O})}(i)$ for $i \in \mathsf{I}$, as in diagram (1). Therefore, by our assumptions, $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ sends morphisms in $\mathsf{I}_{\mathcal{O}}$–reg to $\mathcal{H}cof$.

Let X be the source of a morphisms in I. By (b), X is κ–small with respect to $\mathcal{H}cof$ for some cardinal κ. Let λ be a κ–filtered ordinal, and $V\colon \lambda \to \underline{\mathcal{Alg}}(\mathcal{O})$ a λ–sequence which lands in $\mathsf{I}_{\mathcal{O}}$–reg. Then

$$\begin{aligned}\operatorname*{colim}_{\lambda} \operatorname{Hom}_{\underline{\mathcal{Alg}}(\mathcal{O})}(F_{\underline{\mathcal{Alg}}(\mathcal{O})}(X), V) &\cong \operatorname*{colim}_{\lambda} \operatorname{Hom}_{\mathscr{C}}(X, U_{\underline{\mathcal{Alg}}(\mathcal{O})} \circ V) \\ &\cong \operatorname{Hom}_{\mathscr{C}}\bigl(X, \operatorname*{colim}_{\lambda} U_{\underline{\mathcal{Alg}}(\mathcal{O})} \circ V\bigr) \\ &\cong \operatorname{Hom}_{\mathscr{C}}\bigl(X, U_{\underline{\mathcal{Alg}}(\mathcal{O})}\bigl(\operatorname*{colim}_{\lambda} V\bigr)\bigr) \\ &\cong \operatorname{Hom}_{\underline{\mathcal{Alg}}(\mathcal{O})}\bigl(F_{\underline{\mathcal{Alg}}(\mathcal{O})}(X), \operatorname*{colim}_{\lambda} V\bigr)\end{aligned}$$

and $F_{\underline{\mathcal{Alg}}(\mathcal{O})}(X)$ is κ–small with respect to $\mathsf{I}_{\mathcal{O}}$–reg. In the second isomorphism we are using that $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ sends morphisms in $\mathsf{I}_{\mathcal{O}}$–reg to $\mathcal{H}cof$ and that X is κ–small with respect to $\mathcal{H}cof$.

Let T denote the class of morphisms with the right lifting property with respect to $\mathsf{I}_{\mathcal{O}}$. By adjointness these are precisely those morphisms which $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ sends to acyclic fibrations. By adjointness again $\mathsf{J}_{\mathcal{O}}$ has the left lifting property with respect to T, and then so does $\mathsf{J}_{\mathcal{O}}$–reg.

Let f be a morphism of $\underline{\mathcal{Alg}}(\mathcal{O})$ which has the left lifting property with respect to T. Use the small object argument for $\mathsf{I}_{\mathcal{O}}$ on f to obtain that f is a retract of $h \in \mathsf{I}_{\mathcal{O}}$–reg, and therefore $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(f)$ is a retract of $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(h) \in \mathcal{H}cof$. Therefore $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ sends $\mathsf{J}_{\mathcal{O}}$–reg to $\mathcal{H}cof$, and so we can repeat the previous argument to obtain that sources of morphisms in $\mathsf{J}_{\mathcal{O}}$ are small with respect to $\mathsf{J}_{\mathcal{O}}$–reg.

For the second condition, let $f \in \mathsf{J}_{\mathcal{O}}$–reg be the transfinite composition of morphisms f_α, such that each f_α is a cobase change (in $\underline{\mathcal{Alg}}(\mathcal{O})$) of $F_{\underline{\mathcal{Alg}}(\mathcal{O})}(j_\alpha)$ for $j_\alpha \in \mathsf{J}$. The morphism $F_{\underline{\mathcal{Alg}}(\mathcal{O})}(j_\alpha)$ has the left lifting property with respect to T, and then so does each f_α. Then by the previous discussion $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(f_\alpha)$ is in $\mathcal{H}cof$, and it is a weak equivalence by the hypothesis of the theorem. Since $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ preserves transfinite compositions, $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(f)$ is a transfinite composition of morphisms that are both in $\mathcal{H}cof$ and are weak equivalences, and so $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(f)$ is a weak equivalence. □

Remark 2.3 Usually in a category with both a model structure and a monoidal structure, two compatibility conditions are required; see for example [24, Definition 3.1]. These are the pushout product axiom, and the requirement that the unit is cofibrant. These are not necessary to prove Theorem 2.2, but something similar to the pushout product axiom is usually needed to actually check that condition (2) of Theorem 2.2 holds in practice.

Remark 2.4 The question of under which conditions on a category and on an operad one can lift the model structure to the category of algebras over said operad has been studied in a few different places and with diverse methods. As far as we can tell, the first place where this was examined in generality was in Spitzweck's PhD thesis [25]. It was proven there that a model structure can be lifted assuming that the category $\mathscr{C}$ satisfies the monoid axiom and that the operad is cofibrant [25, Theorem 4].

The condition that an operad is cofibrant in the model structure on operads constructed in [25] is quite restrictive. It is stronger than asking that each $\mathcal{O}_n$ is cofibrant, or even Σ_n–cofibrant. In addition in the setting of global homotopy theory, it is not enough to look at the category of algebras over a cofibrant replacement of an operad (in the usual sense), as this gives the wrong homotopy theory, which we show in Remark 4.16.

2.2 Unstable global homotopy theory

Unstable global homotopy theory is the homotopy theory of spaces which have simultaneous and compatible actions by all compact Lie groups. A model for this is the category of orthogonal spaces, studied in detail in [22, Chapter 1].

Definition 2.5 [22, Definition 1.1.1] Let $\underline{L}$ be the $\underline{\mathrm{Top}}$–enriched category where the objects are finite-dimensional real inner product spaces, and the morphisms are the linear isometric embeddings between them.

An *orthogonal space* is a $\underline{\mathrm{Top}}$–enriched functor $\underline{L} \to \underline{\mathrm{Top}}$. We use $\underline{\mathcal{S}pc}$ to denote the $\underline{\mathrm{Top}}$–enriched category of orthogonal spaces. Note the similarity of this definition to the definition of orthogonal spectra as enriched functors.

If we have a compact Lie group K, and V is a K–representation, then $X(V)$ inherits a K–action, where $k \in K$ acts via $X(k)$. In this sense, orthogonal spaces have actions by all compact Lie groups. For each compact Lie group K we fix a complete K–universe $\mathcal{U}_K$ for the rest of this article. Let $s(\mathcal{U}_K)$ denote the poset of finite-dimensional subrepresentations of $\mathcal{U}_K$. Then we can associate to any orthogonal space the K–space

$$X(\mathcal{U}_K) = \operatorname*{colim}_{V \in s(\mathcal{U}_K)} X(V),$$

which we call the *underlying K–space*. This yields a functor

$$(-)(\mathcal{U}_K)\colon \underline{\mathcal{S}pc} \to \underline{\mathrm{KTop}}.$$

A *global equivalence* of orthogonal spaces is, roughly speaking, a morphism which for each compact Lie group K, induces K–equivalences on suitable homotopy colimits of finite-dimensional K–representations and equivariant embeddings between them. The precise definition can be found in [22, Definition 1.1.2]. It is also a special case of the definition of a G–global equivalence that we give in Definition 3.2 of this article, with $G = e$.

An orthogonal space X is said to be *closed* if for each linear isometric embedding ψ, the map $X(\psi)$ is a closed embedding. A morphism between closed orthogonal spaces is a global equivalence if and only if for each compact Lie group K the induced map on the underlying K–spaces is a K–equivalence; see [22, Proposition 1.1.17], or Proposition 3.5 below for the analogous result for G–global equivalences.

Remark 2.6 There are two symmetric monoidal structures on $\underline{\mathcal{S}pc}$ which are relevant for us. The first one is the categorical product, denoted by $\times$, which is computed levelwise. The second is the box product, denoted by $\boxtimes$. It is constructed as a Day convolution product in [22, Section 1.3]. The unit of both is the terminal one-point constant orthogonal space $*$.

The box product can also be defined via a universal property. For each $X, Y \in \underline{\mathcal{S}pc}$, consider the orthogonal spaces Z with a bimorphism $(X, Y) \to Z$ from X and Y. Then the box product of X and Y is an orthogonal space $X \boxtimes Y$ and a bimorphism $i: (X, Y) \to X \boxtimes Y$ of orthogonal spaces which is initial among such bimorphisms with source (X, Y).

Remark 2.7 On $\underline{\mathcal{S}pc}$ there are two cofibrantly generated model structures whose weak equivalences are precisely the global equivalences: the global model structure [22, Theorem 1.2.21] and the positive global model structure [22, Theorem 1.2.23]. In the rest of this article we only consider the positive global model structure.

2.3 Examples of operads in unstable global homotopy theory

In this paper, we study operads in orthogonal spaces with respect to the box product. This subsection is mostly devoted to showcasing several examples, which we study in more detail in Section 5.

Remark 2.8 For orthogonal spaces $X, Y \in \underline{\mathcal{S}pc}$, we can construct a bimorphism $(X, Y) \to X \times Y$ via

$$(2) \qquad X(V) \times Y(W) \xrightarrow{X(\iota_1)\times Y(\iota_2)} X(V \oplus W) \times Y(V \oplus W) = (X \times Y)(V \oplus W).$$

This bimorphism yields a morphism of orthogonal spaces

$$\rho_{X,Y}\colon X \boxtimes Y \to X \times Y$$

which is natural in X and Y. This means that the identity functor is a lax symmetric monoidal functor from $(\underline{\mathcal{S}pc}, \times)$ to $(\underline{\mathcal{S}pc}, \boxtimes)$, or equivalently an oplax symmetric monoidal functor from $(\underline{\mathcal{S}pc}, \boxtimes)$ to $(\underline{\mathcal{S}pc}, \times)$.

Therefore given an operad $\mathcal{O}$ in $(\underline{\mathcal{S}pc}, \times)$, the natural transformation ρ gives an operad in $(\underline{\mathcal{S}pc}, \boxtimes)$, with the same $\mathcal{O}_n$ for all $n \geq 0$. We denote this resulting operad in $(\underline{\mathcal{S}pc}, \boxtimes)$ by $\mathcal{O}^{\boxtimes}$.

Example 2.9 (constant operads in $\underline{\mathcal{S}pc}$ obtained from topological operads) For any $X \in \underline{\mathrm{Top}}$, we can consider the constant orthogonal space $\overline{X}$, that is the constant functor $\underline{L} \to \underline{\mathrm{Top}}$ with value X. This means that for any group K the underlying K–space of $\overline{X}$ is just X with the trivial K–action.

Any operad in spaces $\mathcal{O}$ induces a constant operad $\overline{\mathcal{O}}$ in $\underline{\mathcal{S}pc}$, such that $(\overline{\mathcal{O}})_n = \overline{\mathcal{O}_n}$.

Construction 2.10 (operad from a functor to topological operads) Given a continuous functor F from $\underline{L}$ to the category of operads in spaces, $\mathcal{OP}$–$\underline{\mathrm{Top}}$, we can obtain from it an operad in $(\underline{\mathcal{S}pc}, \times)$ by permuting the functoriality on $\underline{L}$ with the functoriality on $\coprod_{n\in\mathbb{N}} \underline{\Sigma_n}$. Thus we obtain objects

$$\mathcal{O}_{F,n} = (-)_n \circ F \in \underline{\Sigma_n \mathcal{S}pc}$$

The operadic structure on each $F(V)$ gives rise to an operadic structure on these $\mathcal{O}_{F,n}$ with respect to the categorical product of orthogonal spaces. Then the natural morphism ρ from $\boxtimes$ to $\times$ turns $\mathcal{O}_F$ into an operad with respect to the box product, $\mathcal{O}_F^{\boxtimes}$.

If the functor F is constant, then $\mathcal{O}_F$ is just the constant operad of Example 2.9. However this process becomes interesting when the actions of the linear isometric embeddings are nontrivial, as we discuss below.

Example 2.11 (little disks) For each inner product space V consider the topological operad $\mathcal{LD}(V)$ of little disks in V. These assemble into a continuous functor

$$\mathcal{LD}\colon \underline{L} \to \mathcal{OP}\text{–}\underline{\mathrm{Top}}.$$

We understand an element of $\mathcal{LD}(V)_n$ as a set of n center points $v_i \in D(V)$ in the open unit disk of V and n radii r_i that parametrize a rectilinear embedding of n copies of $D(V)$ into itself. Then for a linear isometric embedding $\psi\colon V \to W$, the map $\mathcal{LD}(\psi)\colon \mathcal{LD}(V) \to \mathcal{LD}(W)$ acts by sending each v_i to $\psi(v_i)$.

By Construction 2.10 we obtain an operad $\mathcal{LD}^{\times}$ in $(\underline{\mathcal{S}pc}, \times)$, and $\mathcal{LD}^{\boxtimes}$, shortened to $\mathcal{LD}$, in $(\underline{\mathcal{S}pc}, \boxtimes)$. The operad given by the underlying spaces of each $\mathcal{LD}_n$ is precisely the E_∞–operad of spaces obtained as the colimit of the little disks operads for $\mathbb{R}^m$. Similarly, for a compact Lie group K, the underlying K–space of $\mathcal{LD}_n$ is exactly $\mathcal{LD}(\mathcal{U}_K)_n$, the n^{th} space of the K–equivariant little disks operad for the complete K–universe $\mathcal{U}_K$, described for example in [4, Definition 3.11(ii)].

Analogously, there is a global version of the Steiner operad.

Example 2.12 (Steiner operad) For each inner product space V let $R(V)$ be the space of distance reducing topological embeddings $f\colon V \to V$, where distance reducing means that $\|f(x) - f(y)\| \leq \|x - y\|$. This is a continuous functor $\underline{L} \to \underline{\mathrm{Top}}$, where for each linear isometric embedding $\psi\colon V \to W$ and $f \in R(V)$, the embedding $R(\psi)(f)\colon W \to W$ is given by

$$(\psi \circ f \circ \psi^{-1}) \oplus \mathrm{Im}(\psi)^{\perp}\colon \mathrm{Im}(\psi) \oplus \mathrm{Im}(\psi)^{\perp} \to \mathrm{Im}(\psi) \oplus \mathrm{Im}(\psi)^{\perp}.$$

A Steiner path for V is a map $h\colon [0,1] \to R(V)$ such that $h(1) = \mathrm{id}_V$. Let $\mathcal{K}(V)_n$ be the space of tuples $(h_1, \ldots, h_n)$ of n Steiner paths such that the images of the $h_i(0)$ are disjoint. These form $\mathcal{K}(V)$, the Steiner operad for V, and these assemble into a continuous functor $F\colon \underline{L} \to \mathcal{OP}$–$\underline{\mathrm{Top}}$, which gives the Steiner operad $\mathcal{K}$ in $(\underline{\mathcal{S}pc}, \boxtimes)$. As in the case of the little disks, for a compact Lie group K the underlying K–space of $\mathcal{K}_n$ is exactly $\mathcal{K}(\mathcal{U}_K)_n$, the n^{th} space of the K–equivariant Steiner operad for the complete K–universe $\mathcal{U}_K$, described for example in [4, Definition 3.11(iv)]. The underlying nonequivariant operad of $\mathcal{K}$ is an E_∞–operad in spaces.

Example 2.13 (endomorphism operads) The symmetric monoidal category $(\underline{\mathcal{S}pc}, \boxtimes)$ is closed; see [22, Remark C.12]. Let Hom denote the internal Hom functor of $\underline{\mathcal{S}pc}$. For each $X \in \underline{\mathcal{S}pc}$ we can consider the endomorphism operad $\mathrm{End}(X)$ in $(\underline{\mathcal{S}pc}, \boxtimes)$, where $\mathrm{End}(X)_n = \mathrm{Hom}(X^{\boxtimes n}, X)$.

Example 2.14 For each compact Lie group K the underlying K–space functor

$$(-)(\mathcal{U}_K)\colon \underline{\mathcal{S}pc} \to \underline{\mathrm{KTop}}$$

has a right adjoint R_K, constructed in [22, Construction 1.2.25]. Being a right adjoint, R_K is strong monoidal with respect to the categorical products in $\underline{\mathrm{KTop}}$ and $\underline{\mathcal{S}pc}$. Therefore if $\mathcal{O}$ is any operad in $(\underline{\mathrm{KTop}}, \times)$ then $R_K(\mathcal{O})$ is an operad in $(\underline{\mathcal{S}pc}, \times)$, and by Remark 2.8 we also obtain an operad in $(\underline{\mathcal{S}pc}, \boxtimes)$.

3 G–orthogonal spaces

In order to talk about operads in $\underline{\mathcal{S}pc}$, we need to know more about the structure of orthogonal spaces which have an action by the symmetric group Σ_n. In this section we study orthogonal spaces with an additional action by a general compact Lie group G. We denote by $\underline{G\mathcal{S}pc}$ the category of continuous functors from $\underline{G}$ to $\underline{\mathcal{S}pc}$, which we call *$G$–orthogonal spaces*.

Definition 3.1 Let K and G be compact Lie groups. A closed subgroup $\Gamma \leq K \times G$ is a *graph subgroup* if $\Gamma \cap (\{e_K\} \times G) = \{e_{K\times G}\}$. We denote the set of graph subgroups of $K \times G$ by $\mathcal{F}(K,G)$. They are called graph subgroups because for any $\Gamma \in \mathcal{F}(K,G)$ there is a closed subgroup $H \leq K$ and a continuous homomorphism $\phi\colon H \to G$ such that Γ is precisely the graph of ϕ.

Let i_l denote the boundary map $i_l\colon \partial D^l \to D^l$ in $\underline{\mathrm{Top}}$, for each $l \geq 0$. We use this notation throughout the paper. Given a G–orthogonal space X, a compact Lie group K, and a K–representation V, $X(V)$ has a $(K\times G)$–action.

Definition 3.2 (G–global equivalence) For a compact Lie group G, a morphism f of $\underline{G\mathcal{S}pc}$ is a *G–global equivalence* if for each compact Lie group K, each graph subgroup $\Gamma \in \mathcal{F}(K,G)$, each K–representation V and $l \geq 0$, the following statement holds. For any lifting problem

$$\begin{tikzcd} \partial D^l \arrow[r, "\alpha"] \arrow[d, hook, "i_l"] & X(V)^\Gamma \arrow[d, "f(V)^\Gamma"] \\ D^l \arrow[r, "\beta"] & Y(V)^\Gamma \end{tikzcd}$$

there is a K–equivariant linear isometric embedding $\psi: V \to W$ into a K–representation W such that there exists a morphism $\lambda: D^l \to X(W)^\Gamma$ which satisfies that in the diagram

$$\begin{tikzcd} \partial D^l \arrow[r, "\alpha"] \arrow[d, hook, "i_l"] & X(V)^\Gamma \arrow[r, "X(\psi)^\Gamma"] & X(W)^\Gamma \arrow[d, "f(W)^\Gamma"] \\ D^l \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^\Gamma \arrow[r, "Y(\psi)^\Gamma"] & Y(W)^\Gamma \end{tikzcd}$$

the upper left triangle commutes, and the lower right triangle commutes up to homotopy relative to ∂D^l.

Note that for $G = e$ this is just the definition of global equivalence from [22, Definition 1.1.2] mentioned in Section 2.2. As it was the case for global equivalences, this definition is meant to capture that for each compact Lie group K taking some suitable homotopy colimit of K–representations yields an $\mathscr{F}(K,G)$–equivalence. We can make this more explicit with the following proposition, analogous to [22, Proposition 1.1.7].

Definition 3.3 For a compact Lie group K, we say that a nested sequence $\{V_i\}_{i\in\mathbb{N}}$ of K–representations

$$V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots$$

is *exhaustive* if for each K–representation V there is an equivariant linear isometric embedding of V into some V_i.

Proposition 3.4 *A morphism $f\colon X \to Y$ in $\underline{G\mathcal{S}pc}$ is a G–global equivalence if and only if for each compact Lie group K and each exhaustive sequence of K–representations $\{V_i\}_{i\in\mathbb{N}}$, the map*

$$\mathrm{tel}_i\, f(V_i)\colon \mathrm{tel}_i\, X(V_i) \to \mathrm{tel}_i\, Y(V_i)$$

induced on the mapping telescopes of the sequences $X(V_i)$ and $Y(V_i)$ of $(K{\times}G)$–spaces and $(K{\times}G)$–equivariant maps is an $\mathscr{F}(K,G)$–equivalence.

Proof First we assume that for each compact Lie group K and each exhaustive sequence of orthogonal K–representations, $\mathrm{tel}_i\, f(V_i)$ is an $\mathscr{F}(K,G)$–equivalence of $(K{\times}G)$–spaces. Any compact Lie group K has an exhaustive sequence of representations $\{V_i\}_{i\in\mathbb{N}}$, so for any K–representation V, any graph subgroup $\Gamma\in\mathscr{F}(K,G)$ and any lifting problem (α,β) for $f(V)^\Gamma$, since $\{V_i\}_{i\in\mathbb{N}}$ is exhaustive, we can embed V into some V_n, and so we assume that $V=V_n$.

Now we fix some notation. Let

$$c_{X,n}\colon X(V_n) \to \mathrm{tel}_i\, X(V_i)$$

be the canonical $(K{\times}G)$–equivariant map. Let $\mathrm{tel}_{[0,n]}\, X(V_i)$ denote the truncated mapping telescope. Let

$$\pi_{X,n}\colon \mathrm{tel}_{[0,n]}\, X(V_i) \to X(V_n)$$

be the $(K{\times}G)$–equivariant canonical projection. Slightly abusing notation we also use $c_{X,n}$ for the canonical map

$$c_{X,n}\colon X(V_n) \to \mathrm{tel}_{[0,n]}\, X(V_i).$$

For $n \le m$, let

$$c_{X,n,m}\colon \mathrm{tel}_{[0,n]}\, X(V_i) \to \mathrm{tel}_{[0,m]}\, X(V_i)$$

denote the inclusion of truncated mapping telescopes, and

$$c_{X,n,\infty}\colon \mathrm{tel}_{[0,n]} X(V_i) \to \mathrm{tel}_i X(V_i)$$

the canonical map.

Taking fixed points commutes with the construction of the mapping telescopes by [22, Proposition B.1], so $(\mathrm{tel}_i X(V_i))^\Gamma \cong \mathrm{tel}_i X(V_i)^\Gamma$ for each graph subgroup $\Gamma \in \mathcal{F}(K,G)$. Since we assumed that $\mathrm{tel}_i f(V_i)^\Gamma$ is a weak homotopy equivalence, by [18, Lemma 9.6] there is a map λ associated to the lifting problem $(c^\Gamma_{X,n} \circ \alpha, c^\Gamma_{Y,n} \circ \beta)$ such that the upper-left triangle commutes and the lower-right one commutes up to homotopy relative ∂D^l, witnessed by a homotopy H:

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha"] \arrow[d, hook, "i_l"] & X(V_n)^\Gamma \arrow[r, "c^\Gamma_{X,n}"] & \mathrm{tel}_i X(V_i)^\Gamma \arrow[d, "\mathrm{tel}_i f(V_i)^\Gamma"] \\
D^l \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V_n)^\Gamma \arrow[r, "c^\Gamma_{Y,n}"'] & \mathrm{tel}_i Y(V_i)^\Gamma
\end{tikzcd}$$

Both λ and H have compact domains, and since the Γ–fixed points of the mapping telescopes are colimits along the closed embeddings $c^\Gamma_{X,n,m}$, both λ and H factor through some stage $m \geq n$ with $\psi\colon V_n \to V_m$, giving

$$\lambda'\colon D^l \to \mathrm{tel}_{[0,m]} X(V_i)^\Gamma \quad \text{and} \quad H'\colon D^l \times [0,1] \to \mathrm{tel}_{[0,m]} Y(V_i)^\Gamma.$$

Then $\pi^\Gamma_{X,m} \circ \lambda'$ and $\pi^\Gamma_{X,m} \circ H'$ satisfy the requirements for the lifting problem

$$(X(\psi)^\Gamma \circ \alpha,\ X(\psi)^\Gamma \circ \beta),$$

so f is a G–global equivalence.

Now assume that f is a G–global equivalence. Fix a compact Lie group K, a graph subgroup $\Gamma \in \mathcal{F}(K,G)$, and an exhaustive sequence of K–representations $\{V_i\}_{i\in\mathbb{N}}$. We have to check that $\mathrm{tel}_i f(V_i)^\Gamma$ is a weak homotopy equivalence.

For a lifting problem (α,β) for $\mathrm{tel}_i f(V_i)^\Gamma$, since ∂D^l and D^l are compact, α and β factor through some stage n, as

$$\alpha'\colon \partial D^l \to \mathrm{tel}_{[0,n]} X(V_i)^\Gamma \quad \text{and} \quad \beta'\colon D^l \to \mathrm{tel}_{[0,n]} Y(V_i)^\Gamma.$$

For each n, there is a homotopy from the identity on $\mathrm{tel}_{[0,n]} X(V_i)$ to $c_{X,n} \circ \pi_{X,n}$, which is $(K\times G)$–equivariant and natural in X, given by retracting the truncated mapping telescope. By [22, Lemma 1.1.5] this means that if there is a solution of the lifting problem

$$(c^\Gamma_{X,n} \circ \pi^\Gamma_{X,n} \circ \alpha',\ c^\Gamma_{Y,n} \circ \pi^\Gamma_{Y,n} \circ \beta')$$

then there is a solution of the lifting problem (α', β'). In this proof we use the terminology *solution of a lifting problem*[1] in the sense of [22, Lemma 1.1.5] and Definition 3.2, ie a map that makes the upper-left triangle commute and the lower-right triangle commute up to homotopy relative ∂D^l:

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha'"] \arrow[d, hook, "i_l"] & \mathrm{tel}_{[0,n]} X(V_i)^\Gamma \arrow[r, "\pi^\Gamma_{X,n}"] \arrow[d, "\mathrm{tel}_{[0,n]} f(V_i)^\Gamma"] & X(V_n)^\Gamma \arrow[r, "c^\Gamma_{X,n}"] \arrow[d, "f(V_m)^\Gamma"] & \mathrm{tel}_{[0,n]} X(V_i)^\Gamma \arrow[d, "\mathrm{tel}_{[0,n]} f(V_i)^\Gamma"] \\
D^l \arrow[r, "\beta'"'] & \mathrm{tel}_{[0,n]} Y(V_i)^\Gamma \arrow[r, "\pi^\Gamma_{Y,n}"'] & Y(V_n)^\Gamma \arrow[r, "c^\Gamma_{Y,n}"'] & \mathrm{tel}_{[0,n]} Y(V_i)^\Gamma
\end{tikzcd}$$

The new lifting problem $(\pi^\Gamma_{X,n} \circ \alpha', \pi^\Gamma_{Y,n} \circ \beta')$ has as solution λ after evaluating at some larger $m \geq n$ with embedding $\psi\colon V_n \to V_m$, because f is a G–global equivalence and $\{V_i\}_{i \in \mathbb{N}}$ is an exhaustive sequence of K–representations. Then $c^\Gamma_{X,m} \circ \lambda$ is a solution of the lifting problem

$$(c^\Gamma_{X,m} \circ X(\psi)^\Gamma \circ \pi^\Gamma_{X,n} \circ \alpha',\ c^\Gamma_{Y,m} \circ Y(\psi)^\Gamma \circ \pi^\Gamma_{Y,n} \circ \beta'),$$

and since $\pi^\Gamma_{X,m} \circ c^\Gamma_{X,n,m} = X(\psi)^\Gamma \circ \pi^\Gamma_{X,n}$, the map $c^\Gamma_{X,m} \circ \lambda$ is also a solution of

$$(c^\Gamma_{X,m} \circ \pi^\Gamma_{X,m} \circ c^\Gamma_{X,n,m} \circ \alpha',\ c^\Gamma_{Y,m} \circ \pi^\Gamma_{Y,m} \circ c^\Gamma_{Y,n,m} \circ \beta').$$

By [22, Lemma 1.1.5] and the previously mentioned homotopy from the identity on $\mathrm{tel}_{[0,m]} X(V_i)$ to $c_{X,m} \circ \pi_{X,m}$, the lifting problem $(c^\Gamma_{X,n,m} \circ \alpha', c^\Gamma_{Y,n,m} \circ \beta')$ has a solution λ'. Note that we did not obtain a solution of (α', β'), but since

$$c^\Gamma_{X,m,\infty} \circ c^\Gamma_{X,n,m} = c^\Gamma_{X,n,\infty},$$

the map $c^\Gamma_{X,m,\infty} \circ \lambda'$ is a solution of the original lifting problem

$$(\alpha, \beta) = (c^\Gamma_{X,m,\infty} \circ c^\Gamma_{X,n,m} \circ \alpha',\ c^\Gamma_{Y,m,\infty} \circ c^\Gamma_{Y,n,m} \circ \beta').$$

Since any lifting problem for $\mathrm{tel}_i\ f(V_i)^\Gamma$ has a solution, by [18, Lemma 9.6] the map $\mathrm{tel}_i\ f(V_i)^\Gamma$ is a weak homotopy equivalence. □

Recall that an orthogonal space X is said to be *closed* if for each linear isometric embedding ψ the map $X(\psi)$ is a closed embedding. We similarly define a *closed G–orthogonal space* to be a G–orthogonal space X such that $X(\psi)$ is a closed embedding

[1] To avoid confusion with the more common meaning of the terminology *solution of a lifting problem*, we do not use it outside of this proof.

for each linear isometric embedding ψ. We have fixed a complete K–universe $\mathcal{U}_K$ for each compact Lie group K. The *underlying* $(K\times G)$*–space* of a G–orthogonal space X is the underlying K–space of X as an orthogonal space with the induced G–action. This is precisely the colimit $X(\mathcal{U}_K) = \mathrm{colim}_{V\in s(\mathcal{U}_K)} X(V)$ over the finite-dimensional subrepresentations of $\mathcal{U}_K$. Therefore analogously to [22, Proposition 1.1.17] we have the following simpler characterization of G–global equivalences.

Proposition 3.5 *A morphism $f\colon X\to Y$ in $\underline{G\mathcal{S}pc}$ between closed G–orthogonal spaces is a G–global equivalence if and only if for each compact Lie group K the map induced on the underlying $(K\times G)$–spaces*

$$f(\mathcal{U}_K)\colon X(\mathcal{U}_K)\to Y(\mathcal{U}_K)$$

is an $\mathcal{F}(K,G)$–equivalence of $(K\times G)$–spaces.

Proof The colimits that define $X(\mathcal{U}_K)$ and $Y(\mathcal{U}_K)$ can be written as sequential colimits

$$\operatorname*{colim}_{V\in s(\mathcal{U}_K)} X(V)\cong \operatorname*{colim}_{i\in\mathbb{N}} X(V_i),$$

for a nested sequence of finite-dimensional subrepresentations $\{V_i\}_{i\in\mathbb{N}}$ of $\mathcal{U}_K$ which cover all of $\mathcal{U}_K$. These are colimits of $(K\times G)$–spaces along closed embeddings because X and Y are closed. Then for each $\Gamma\in\mathcal{F}(K,G)$, taking Γ–fixed points commutes with this colimit along closed embeddings; see [22, Proposition B.1(ii)]. Since additionally ∂D^l and D^l are compact, a lifting problem for $(\mathrm{colim}_{i\in\mathbb{N}} f(V_i))^\Gamma$ factors through some stage n of the sequential colimit. By [18, Lemma 9.6] we obtain that if f is a G–global equivalence the map $(\mathrm{colim}_{i\in\mathbb{N}} f(V_i))^\Gamma$ is a weak homotopy equivalence.

For the other implication, assume that $f(\mathcal{U}_K)^\Gamma$ is a weak homotopy equivalence for each compact Lie group K and each $\Gamma\in\mathcal{F}(K,G)$. Let V be a K–representation. Then V embeds into $\mathcal{U}_K$, so we may fix an embedding $V\to\mathcal{U}_K$ and call it ψ. Given any lifting problem

$$(\alpha\colon \partial D^l\to X(V)^\Gamma,\ \beta\colon D^l\to Y(V)^\Gamma)$$

for $f(V)^\Gamma$, consider the lifting problem

$$(X(\psi)^\Gamma\circ\alpha,\ Y(\psi)^\Gamma\circ\beta)$$

for $f(\mathcal{U}_K)^\Gamma$. By [18, Lemma 9.6], since $f(\mathcal{U}_K)^\Gamma$ is a weak homotopy equivalence there exists some $\lambda\colon D^l\to X(\mathcal{U}_K)^\Gamma$ such that $\lambda\circ i_l = X(\psi)^\Gamma\circ\alpha$ and $f(\mathcal{U}_K)^\Gamma\circ\lambda$ is homotopic relative ∂D^l to $Y(\psi)^\Gamma\circ\beta$, and we denote this homotopy by H.

Both λ and H factor through some stage of the colimits as

$$\lambda' \colon D^l \to X(V_j)^\Gamma \quad \text{and} \quad H' \colon D^l \times [0,1] \to Y(V_j)^\Gamma.$$

We can choose j so that V_j contains the image of ψ since $\{V_i\}_{i \in \mathbb{N}}$ covers $\mathcal{U}_K$. Then λ' and the homotopy H' witness that f is a G–global equivalence. □

Remark 3.6 When defining G–global equivalences, we could have decided to look at all subgroups instead of only the graph subgroups of $K \times G$. This would give a strictly smaller class of G–global equivalences.

We consider only the graph subgroups because between G–free orthogonal spaces, the graph subgroups tell the whole story, since the fixed points of any nongraph subgroup are empty. This means that looking at this bigger class of G–global equivalences is enough for the proof of Theorem 4.11. It also leads to Theorem 4.14, which states that a map f of operads in $(\underline{\mathcal{S}pc}, \boxtimes)$ gives a Quillen equivalence if and only if each f_n is a Σ_n–global equivalence in the sense of the graph subgroups, even if the operads themselves are not Σ_n–free.

The G–global equivalences are in fact a part of a model structure on $\underline{G\mathcal{S}pc}$, which we call the G–global model structure. We include in this section the basic facts about G–global equivalences, as well as the results about G–global equivalences that are most relevant to the proofs of Section 4. We relegate the construction of this G–global model structure to the appendix.

The characterization of G–global equivalences given by Proposition 3.4 makes it simple to check the following general properties of G–global equivalences.

Lemma 3.7 *For compact Lie groups G and H we have the following properties*:

(i) ***2-out-of-6*** *Consider three composable morphisms of G–orthogonal spaces f, g and h. If $g \circ f$ and $h \circ g$ are G–global equivalences, then f, g, h and $h \circ g \circ f$ are G–global equivalences.*

(ii) *A retract of a G–global equivalence is a G–global equivalence.*

(iii) *If $f \colon X \to Y$ is a G–global equivalence and g is homotopic to f through morphisms of G–orthogonal spaces, then g is a G–global equivalence.*

(iv) *For a G–orthogonal space X, and an H–global equivalence $f \colon Y \to Z$ between H–orthogonal spaces, the morphism $X \times f$ is a $(G \times H)$–global equivalence.*

(v) *For a G–global equivalence $f \colon X \to Y$ and an H–global equivalence $f' \colon X' \to Y'$, the morphism $f \times f'$ is a $(G \times H)$–global equivalence.*

(vi) *For a G–global equivalence $f\colon X \to Y$ and a continuous homomorphism $\eta\colon H \to G$ the restriction $\eta^* f$ is an H–global equivalence.*

Proof Let K be a compact Lie group and $\{V_i\}_{i\in\mathbb{N}}$ an exhaustive sequence of K–representations.

(i) By Proposition 3.4 the maps

$$\mathrm{tel}_i (g\circ f)(V_i) = \mathrm{tel}_i\, g(V_i)\circ \mathrm{tel}_i\, f(V_i) \quad \text{and} \quad \mathrm{tel}_i (h\circ g)(V_i) = \mathrm{tel}_i\, h(V_i)\circ \mathrm{tel}_i\, g(V_i)$$

are $\mathscr{F}(K,G)$–equivalences. Since the class of $\mathscr{F}(K,G)$–equivalences satisfies the 2-out-of-6 property, by Proposition 3.4 again we obtain that f, g, h and $h\circ g\circ f$ are G–global equivalences.

(ii) As before, if g is a retract of f then $\mathrm{tel}_i\, g(V_i)$ is a retract of $\mathrm{tel}_i\, f(V_i)$, and $\mathscr{F}(K,G)$–equivalences are closed under retracts.

(iii) If $H\colon X\times[0,1]\to Y$ is a homotopy through morphisms of G–orthogonal spaces then it induces a homotopy through $(K\times G)$–equivariant maps on mapping telescopes. Since a map $(K\times G)$–homotopic to an $\mathscr{F}(K,G)$–equivalence is an $\mathscr{F}(K,G)$–equivalence, we see that g is also a G–global equivalence.

(iv) The canonical map

$$\mathrm{tel}_i (X\times Y)(V_i) \to \mathrm{tel}_i\, X(V_i)\times \mathrm{tel}_i\, Y(V_i)$$

is a $(K\times G\times H)$–equivalence. Consider a graph subgroup $\Gamma_\phi\in\mathscr{F}(K,G\times H)$ associated to a homomorphism ϕ. Let $\pi_G: G\times H\to G$ and $\pi_H: G\times H\to H$ denote the respective projections. Then

$$(\mathrm{tel}_i\, X(V_i)\times \mathrm{tel}_i\, f(V_i))^{\Gamma_\phi} \cong \mathrm{tel}_i\, X(V_i)^{\Gamma_{\pi_G\circ\phi}}\times \mathrm{tel}_i\, f(V_i)^{\Gamma_{\pi_H\circ\phi}}$$

where $\Gamma_{\pi_H\circ\phi}$ is the graph subgroup associated to $\pi_H\circ\phi$. Since $\mathrm{tel}_i\, f(V_i)$ is an $\mathscr{F}(K,H)$–equivalence, $\mathrm{tel}_i\, X(V_i)\times\mathrm{tel}_i\, f(V_i)$ is an $\mathscr{F}(K,G\times H)$–equivalence and by the 2-out-of-6 property so is $\mathrm{tel}_i(X\times f)(V_i)$.

(v) We have $f\times f' = (Y\times f')\circ(f\times X')$ and each of these is a $(G\times H)$–global equivalence by (iv).

(vi) Consider a graph subgroup $\Gamma_\phi\in\mathscr{F}(K,H)$ associated to a homomorphism ϕ. Then

$$(\mathrm{tel}_i\, \eta^* f(V_i))^{\Gamma_\phi} = (\eta^*(\mathrm{tel}_i\, f(V_i)))^{\Gamma_\phi} = (\mathrm{tel}_i\, f(V_i))^{\Gamma_{\eta\circ\phi}}$$

which is a weak homotopy equivalence. □

We now turn to the box product of G–orthogonal spaces, to check that it preserves G–global equivalences. The box product of orthogonal spaces is fully homotopical with respect to the global equivalences; that is, the box product of two global equivalences is a global equivalence, and this does not require any cofibrancy assumptions on the morphisms or the orthogonal spaces involved. Our goal is to check that the box product is also fully homotopical with respect to the G–global equivalences.

Given compact Lie groups G and H, a G–orthogonal space X and an H–orthogonal space Y, $X \boxtimes Y$ has a canonical action by $G \times H$, and so does $X \times Y$. The natural morphism $\rho_{X,Y}$ of Remark 2.8 is $(G\times H)$–equivariant. By [22, Theorem 1.3.2(i)] this $\rho_{X,Y}$ is a global equivalence of underlying orthogonal spaces. We now adapt that proof to show that it is additionally a $(G\times H)$–global equivalence. We start with a technical lemma.

Lemma 3.8 *Given $F\colon \underline{L} \to \underline{L}$ a continuous endofunctor, a natural transformation $\eta\colon \mathrm{Id} \Rightarrow F$, and a G–orthogonal space X, the morphism $X \circ \eta\colon X \to X \circ F$ is a G–global equivalence.*

Proof We use the fact that for each compact Lie group K and each K–representation V, the two embeddings

$$F(\eta_V), \eta_{F(V)}\colon F(V) \to F(F(V))$$

are homotopic relative to $\eta_V\colon V \to F(V)$ through K–equivariant linear isometric embeddings. This is shown in the proof of the equivalent result where X is just an orthogonal space in [22, Theorem 1.1.10].

Given a compact Lie group K, a K–representation V, a graph subgroup $\Gamma \in \mathscr{F}(K, G)$ and a lifting problem (α, β) as in the following diagram, the linear isometric embedding η_V and the map β together witness that $X \circ \eta$ is a G–global equivalence:

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha"] \arrow[d, hook, "i_l"] & X(V)^\Gamma \arrow[r, "X(\eta_V)^\Gamma"] \arrow[d, "X(\eta_V)^\Gamma"'] & X(F(V))^\Gamma \arrow[d, "X(\eta_{F(V)})^\Gamma"] \arrow[dl, equal] \\
D^l \arrow[r, "\beta"] & X(F(V))^\Gamma \arrow[r, "X(F(\eta_V))^\Gamma"'] & X(F(F(V)))^\Gamma
\end{tikzcd}$$

The upper left trapezoid commutes by construction. For the lower right triangle, $F(\eta_V)$ and $\eta_{F(V)}$ are homotopic through K–equivariant linear isometric embeddings, therefore $X(F(\eta_V))$ and $X(\eta_{F(V)})$ are homotopic through $(K\times G)$–equivariant maps, and $X(F(\eta_V))^\Gamma$ and $X(\eta_{F(V)})^\Gamma$ are homotopic. Since the original homotopy was relative to η_V, and $\beta \circ i_l = X(\eta_V)^\Gamma \circ \alpha$, the obtained homotopy between $X(F(\eta_V))^\Gamma \circ \beta$ and $X(\eta_{F(V)})^\Gamma \circ \beta$ is relative i_l. Thus $X \circ \eta$ is a G–global equivalence. □

Proposition 3.9 *Given a G–orthogonal space X and an H–orthogonal space Y, the morphism of $(G \times H)$–orthogonal spaces $\rho_{X,Y}$ is a $(G \times H)$–global equivalence.*

Proof Consider the endofunctor $\mathrm{sh}\colon \underline{L} \to \underline{L}$ that sends V to $V \oplus V$. We have two natural transformations, $\iota_1, \iota_2\colon \mathrm{Id} \Rightarrow \mathrm{sh}$, given by the embeddings into the first and second factor respectively. We also denote by sh the functor of orthogonal spaces given by precomposing with sh, $\mathrm{sh}(X) = X \circ \mathrm{sh}$.

The universal bimorphism i that exhibits $X \boxtimes Y$ as the box product of X and Y gives a morphism of orthogonal spaces $\lambda\colon X \times Y \to \mathrm{sh}(X \boxtimes Y)$ through the maps

$$i_{V,V}\colon X(V) \times Y(V) \to (X \boxtimes Y)(V \oplus V) = (\mathrm{sh}(X \boxtimes Y))(V).$$

We need to check that $\lambda \circ \rho_{X,Y}$ and $\mathrm{sh}(\rho_{X,Y}) \circ \lambda$ in the diagram

$$X \boxtimes Y \xrightarrow{\rho_{X,Y}} X \times Y \xrightarrow{\lambda} \mathrm{sh}(X \boxtimes Y) \xrightarrow{\mathrm{sh}(\rho_{X,Y})} \mathrm{sh}(X \times Y)$$

are $(G \times H)$–global equivalences, and then we can use Lemma 3.7(i), the 2-out-of-6 property, to obtain that $\rho_{X,Y}$ is a $(G \times H)$–global equivalence.

We have that $\mathrm{sh}(\rho_{X,Y}) \circ \lambda$ evaluated at V is the same as the map associated to $\rho_{X,Y}$ at level (V, V) given in (2) of Remark 2.8, by the constructions of λ and $\rho_{X,Y}$. This means that

$$\mathrm{sh}(\rho_{X,Y}) \circ \lambda = X(\iota_1) \times Y(\iota_2),$$

where each morphism on the right is a G–global equivalence or an H–global equivalence respectively by Lemma 3.8, and so their product is a $(G \times H)$–global equivalence by Lemma 3.7(v).

Next we use that $\lambda \circ \rho_{X,Y}$ is homotopic through $(G \times H)$–equivariant morphisms to $(X \boxtimes Y)(\iota_1)$, since the homotopy between them given in the proof of [22, Theorem 1.3.2(i)] is through $(G \times H)$–equivariant morphisms. Additionally $(X \boxtimes Y)(\iota_1)$ is a $(G \times H)$–global equivalence by Lemma 3.8, so by Lemma 3.7(iii) $\lambda \circ \rho_{X,Y}$ is a $(G \times H)$–global equivalence. □

Corollary 3.10 *For a G–global equivalence $f\colon X \to Y$ and an H–global equivalence $f'\colon X' \to Y'$, the morphism $f \boxtimes f'$ is a $(G \times H)$–global equivalence. If $H = G$ then $f \boxtimes f'$ is a G–global equivalence. Therefore for any $X \in \underline{G\mathcal{S}pc}$, the functor $X \boxtimes -$ preserves G–global equivalences.*

Proof First,
$$\rho_{Y,Y'} \circ (f \boxtimes f') = (f \times f') \circ \rho_{X,X'},$$
and $\rho_{Y,Y'}$ and $\rho_{X,X'}$ are $(G\times H)$–global equivalences by Proposition 3.9. Since $f \times f'$ is also a $(G\times H)$–global equivalence by Lemma 3.7(v), by the 2-out-of-6 property so is $f \boxtimes f'$.

If $H = G$, by restricting along the diagonal homomorphism $\Delta\colon G \to G \times G$ and using Lemma 3.7(vi), we obtain that $f \boxtimes f'$ is a G–global equivalence and therefore $X \boxtimes -$ preserves G–global equivalences. □

Now we proceed with a technical lemma which we use to prove the two subsequent propositions. The first one discusses what happens to G–global equivalences between G–free orthogonal spaces when taking orbits, if G is finite. The second one shows that G–global equivalences are preserved by inducing from a finite subgroup.

Lemma 3.11 *Let H be a finite group and K and G compact Lie groups. Assume that we have equivariant maps of $(K\times G\times H)$–spaces $f\colon X \to Y$ and $g\colon Y \to Z$ such that Z is Hausdorff and H–free. Then the map on orbits $f/H\colon X/H \to Y/H$ is an $\mathcal{F}(K,G)$–equivalence if and only if f is an $\mathcal{F}(K,G\times H)$–equivalence.*

Proof First note that since Z is H–free, so are X and Y. For any graph subgroup $\Gamma_\phi \in \mathcal{F}(K,G)$ given by a continuous homomorphism $\phi\colon L \to G$, [22, Proposition B.17] gives a natural homeomorphism for X, Y and Z,
$$\coprod_{[\psi]} X^{\Gamma_\psi}/C(\psi) \to (X/H)^{\Gamma_\phi}.$$
The disjoint union on the left is indexed by the conjugacy classes of continuous homomorphisms $\psi\colon \Gamma_\phi \to H$. Here $C(\psi)$ denotes the centralizer of the image of ψ in H.

Fix a graph subgroup $\Gamma_\phi \in \mathcal{F}(K,G)$. A homomorphism $\psi\colon \Gamma_\phi \to H$, as a subgroup of $K \times G \times H$, has elements $(k,\phi(k),\psi(k,\phi(k)))$ for $k \in L$, so $\Gamma_\psi \in \mathcal{F}(K,G\times H)$. Conversely, for a graph subgroup $\Gamma_\psi \in \mathcal{F}(K,G\times H)$, let ϕ be the homomorphism $\pi_G \circ \psi\colon L \to G$ where $\pi_G\colon G \times H \to G$ is the projection. Then Γ_ψ is a graph subgroup of $\Gamma_\phi \times H$, so that ψ can be seen as a homomorphism $\Gamma_\phi \to H$.

Therefore the map on orbits f/H is an $\mathcal{F}(K,G)$–equivalence if and only if for each $\Gamma_\psi \in \mathcal{F}(K,G\times H)$ the map $f^{\Gamma_\psi}/C(\psi)$ is a weak homotopy equivalence.

For each $\Gamma_\psi \in \mathcal{F}(K, G \times H)$, the centralizer of the image of ψ, $C(\psi) \leq H$, is finite. Additionally, Z^{Γ_ψ} is $C(\psi)$–free and a subspace of Z, so Hausdorff. Therefore the $C(\psi)$–action on Z^{Γ_ψ} is properly discontinuous, and since f^{Γ_ψ} and g^{Γ_ψ} are $C(\psi)$–equivariant, the $C(\psi)$–actions on X^{Γ_ψ} and Y^{Γ_ψ} are also properly discontinuous.

This means that

$$X^{\Gamma_\psi} \to X^{\Gamma_\psi}/C(\psi) \quad \text{and} \quad Y^{\Gamma_\psi} \to Y^{\Gamma_\psi}/C(\psi)$$

are covering maps, and since f^{Γ_ψ} is $C(\psi)$–equivariant, it induces a map of coverings. Then we consider the long exact sequence of homotopy groups for these covering maps. We obtain that $f^{\Gamma_\psi}/C(\psi)$ is a weak homotopy equivalence if and only if f^{Γ_ψ} is a weak homotopy equivalence. For π_n for $n \geq 2$ this can be seen by using the five lemma and for π_0 and π_1 it can be checked explicitly.

Thus we finally obtain that f/H is an $\mathcal{F}(K, G)$–equivalence if and only if f is an $\mathcal{F}(K, G\times H)$–equivalence. □

This next proposition is similar to [21, Lemma 8.1].

Proposition 3.12 *Let H be a finite group and G a compact Lie group. Consider two morphisms of $(G\times H)$–orthogonal spaces $f : X \to Y$ and $g : Y \to Z$, where for Z we know that for each inner product space V the space $Z(V)$ is Hausdorff and H–free. Then $f/H : X/H \to Y/H$ is a G–global equivalence if and only if f is a $(G\times H)$–global equivalence.*

Proof By Proposition 3.4 we know that $f/H : X/H \to Y/H$ is a G–global equivalence if and only if for each compact Lie group K and exhaustive sequence of K–representations $\{V_i\}_{i\in\mathbb{N}}$ the map

$$\mathrm{tel}_i\ f/H(V_i) : \mathrm{tel}_i\ X/H(V_i) \to \mathrm{tel}_i\ Y/H(V_i)$$

is an $\mathcal{F}(K, G)$–equivalence.

Taking H–orbits commutes with colimits and product with [0, 1], so it commutes with taking mapping telescopes. Therefore $\mathrm{tel}_i\ f/H(V_i) \cong \mathrm{tel}_i\ f(V_i)/H$. Now f and g induce $(K\times G\times H)$–equivariant maps on mapping telescopes

$$\mathrm{tel}_i\ X(V_i) \xrightarrow{\mathrm{tel}_i\ f(V_i)} \mathrm{tel}_i\ Y(V_i) \xrightarrow{\mathrm{tel}_i\ g(V_i)} \mathrm{tel}_i\ Z(V_i).$$

Since each $Z(V)$ is Hausdorff and H–free, so is $\mathrm{tel}_i\ Z(V_i)$. By Proposition 3.4 again f is a $(G\times H)$–global equivalence if and only if $\mathrm{tel}_i\ f(V_i)$ is an $\mathcal{F}(K, G\times H)$–equivalence for each K and $\{V_i\}_{i\in\mathbb{N}}$. By Lemma 3.11 $\mathrm{tel}_i\, f(V_i)/H$ is an $\mathcal{F}(K,G)$–equivalence if and only if $\mathrm{tel}_i\, f(V_i)$ is an $\mathcal{F}(K,G\times H)$–equivalence, which yields the result. □

Proposition 3.13 *For a compact Lie group G, a finite subgroup $H\le G$, and an H–global equivalence $f\colon X\to Y$, the morphism $G\times_H f$ is a G–global equivalence.*

Proof We first need to check that $G\times f$ is a $(G\times H)$–global equivalence, for the action where G acts on the left on the G factor, and H acts both on the right on the G factor and on the left on the f factor.

Consider a compact Lie group K and an exhaustive sequence of K–representations $\{V_i\}_{i\in\mathbb{N}}$. The functor $G\times -$ commutes with colimits and the functor $-\times[0,1]$, so it commutes with taking mapping telescopes. Therefore it suffices to check that $G\times\mathrm{tel}_i\, f(V_i)$ is an $\mathcal{F}(K,G\times H)$–equivalence.

For any graph subgroup $\Gamma_\phi\in\mathcal{F}(K,G\times H)$, the image of Γ_ϕ under the projection

$$\pi_{K\times H}\colon K\times G\times H\to K\times H$$

is the graph subgroup $\Gamma_{\pi_H\circ\phi}$. Therefore

$$(\mathrm{tel}_i\, f(V_i))^{\Gamma_\phi}=(\mathrm{tel}_i\, f(V_i))^{\Gamma_{\pi_H\circ\phi}},$$

and the latter is a weak homotopy equivalence since $\mathrm{tel}_i\, f(V_i)$ is an $\mathcal{F}(K,H)$–equivalence. Then

$$(G\times\mathrm{tel}_i\, f(V_i))^{\Gamma_\phi}=G^{\Gamma_\phi}\times\mathrm{tel}_i\, f(V_i)^{\Gamma_\phi}$$

is also a weak homotopy equivalence.

Lastly, the projection $G\times Y\to G$ is a $(G\times H)$–equivariant map, where again G acts on G on the left and H acts on the right on G and on the left on Y. With this action G is H–free and Hausdorff, so by Proposition 3.12, $G\times_H f$ is a G–global equivalence. □

In the appendix we further explore some more technical aspects of G–orthogonal spaces. In particular, we construct the G–global model structure on $\underline{G\mathcal{S}pc}$. The G–flat cofibrations are the cofibrations of this model structure. However for our admissibility results on operads in $\underline{\mathcal{S}pc}$ we need to work with a bigger class of morphisms than that of the G–flat cofibrations. This is why we now introduce the class of G–h–cofibrations of $\underline{G\mathcal{S}pc}$. In the appendix we also study the compatibility of G–global equivalences and G–h–cofibrations.

The category $\underline{G\mathcal{S}pc}$ is tensored over $\underline{\mathrm{Top}}$. Thus we can define what a homotopy of morphisms of G–orthogonal spaces is in the usual way using the interval. We can also similarly define what a G–homotopy equivalence of G–orthogonal spaces is.

Definition 3.14 (G–h–cofibration) A morphism in $\underline{G\mathcal{S}pc}$ is an *h–cofibration* if it has the homotopy extension property. A morphism $f\colon X \to Y$ has the homotopy extension property if and only if there is a retraction in $\underline{G\mathcal{S}pc}$ for the induced morphism

$$X \times [0,1] \cup_X Y \to Y \times [0,1].$$

We call these morphisms the *G–h–cofibrations*.

Lemma 3.15 *The class of G–h–cofibrations is closed under coproducts, transfinite compositions, cobase changes and retracts. Additionally each G–flat cofibration is a G–h–cofibration.*

Proof On a category tensored and cotensored over $\underline{\mathrm{Top}}$ the h–cofibrations can be equivalently defined as those morphisms that have the left lifting property with respect to $\mathrm{ev}_0\colon X^{[0,1]} \to X$ for all objects X; see [22, Definition A.28]. This shows the first part.

The G–level model structure for $\underline{G\mathcal{S}pc}$ that we construct in Theorem A.2 is topological, and all objects are fibrant so by [22, Corollary A.30(iii)] each G–flat cofibration is a G–h–cofibration. □

Lemma 3.16 *Let G be a compact Lie group.*

(i) *Consider a closed normal subgroup $H \leq G$. For a G–h–cofibration of G–orthogonal spaces $f\colon X \to Y$, the morphism on orbits $f/H\colon X/H \to Y/H$ is a (G/H)–h–cofibration.*

(ii) *Consider a continuous homomorphism $\alpha\colon H \to G$ between compact Lie groups. For a G–h–cofibration of G–orthogonal spaces $f\colon X \to Y$, the morphism*

$$\alpha^* f\colon \alpha^*(X) \to \alpha^*(Y)$$

is an H–h–cofibration.

(iii) *Consider a compact Lie group H and an H–orthogonal space Z. For a G–h–cofibration of G–orthogonal spaces $f\colon X \to Y$, the morphisms $Z \boxtimes f$ and $Z \times f$ are $(H\times G)$–h–cofibrations.*

(iv) *Consider a closed subgroup $H \leq G$. For an H–h–cofibration of H–orthogonal spaces $f\colon X \to Y$, the morphism $G \times_H f\colon G \times_H X \to G \times_H Y$ is a G–h–cofibration.*

Proof (i) Suppose that we have a retraction in $\underline{G\mathcal{S}pc}$

$$r\colon Y \times [0,1] \to X \times [0,1] \cup_X Y.$$

Taking orbits commutes with pushouts and the product with $[0,1]$.

Thus the morphism

$$r/H\colon Y/H\times[0,1]\to(X\times[0,1]\cup_X Y)/H\cong X/H\times[0,1]\cup_{X/H}Y/H$$

is the retraction that witnesses that f/H is a G/H–h–cofibration.

(ii) As before, the functor α^* commutes with pushouts and the product with [0, 1], and the morphism α^*r is the retraction that witnesses that α^*f is an H–h–cofibration.

(iii) The functors $Z\boxtimes -$ and $Z\times -$ commute with pushouts and the product with [0, 1]. The $(H\times G)$–equivariant morphisms $Z\boxtimes r$ and $Z\times r$ witness that $Z\boxtimes f$ and $Z\times f$ are $(H\times G)$–h–cofibrations respectively.

(iv) This follows from (i), (ii) and (iii). □

4 Main results for operads in $(\underline{\mathcal{S}pc},\boxtimes)$

4.1 Lifting the positive global model structure to $\underline{\mathcal{A}lg}(\mathcal{O})$

In this subsection, our goal is to prove Theorem 4.11, that states that any operad in $(\underline{\mathcal{S}pc},\boxtimes)$ is admissible. By this we mean that for any operad $\mathcal{O}$ in $(\underline{\mathcal{S}pc},\boxtimes)$, the positive global model structure on $\underline{\mathcal{S}pc}$ lifts through $U_{\underline{\mathcal{A}lg}(\mathcal{O})}\colon\underline{\mathcal{A}lg}(\mathcal{O})\to\underline{\mathcal{S}pc}$ to give a model structure on $\underline{\mathcal{A}lg}(\mathcal{O})$.

The condition that we need to check to obtain that any operad is admissible is the following.

Condition 4.1 For any $Z\in\underline{\Sigma_n\mathcal{S}pc}$ and any generating cofibration i of $\underline{\mathcal{S}pc}$, the morphism $Z\boxtimes_{\Sigma_n}i^{\Box n}$ is an h–cofibration. For any $Z\in\underline{\Sigma_n\mathcal{S}pc}$ and any generating acyclic cofibration j of $\underline{\mathcal{S}pc}$, the morphism $Z\boxtimes_{\Sigma_n}j^{\Box n}$ is an h–cofibration and a global equivalence.

Note that the fact that we consider any possible Z here is crucial in removing any cofibrancy assumptions on the operad. The symbol $\Box$ denotes the pushout product of two morphisms and $i^{\Box n}$ denotes the n^{th} iterated pushout product of i with itself.

Remark 4.2 Condition 4.1 is strongly related to the property named *symmetric h–monoidality* defined in [20, Definition 4.2.4]. Note that there are two different definitions of *h–cofibrations* in the literature. The one used in [20] and [19] was first given in [2, Definition 1.1], and it is weaker than the definition we used for $\underline{\mathcal{S}pc}$ and $\underline{G\mathcal{S}pc}$.

The property of $\underline{\mathcal{S}pc}$ being symmetric h–monoidal is not directly related to Condition 4.1; however the spirit of it is the same. In [19, Theorem 5.11] it is proven that in a category which satisfies certain technical assumptions and is symmetric h–monoidal each operad is admissible. Using Condition 4.1 instead of symmetric h–monoidality simplifies some arguments in the case of orthogonal spaces. Most of this subsection is dedicated to checking Condition 4.1.

First of all, in order to check Condition 4.1 we should give an explicit description of the generating (acyclic) cofibrations of the positive global model structure on $\underline{\mathcal{S}pc}$. They can also be obtained from the generating (acyclic) cofibrations of the G–global model structure described in Theorem A.2 and Construction A.14 by setting $G = e$ and adding everywhere the requirement that $V \neq 0$ (this $V \neq 0$ requirement is the difference between the *positive* global model structure and the global model structure).

Remark 4.3 (generating (acyclic) cofibrations of the positive global model structure) In $\underline{\mathcal{S}pc}$ we have a semifree orthogonal space for each compact Lie group G and each G–representation V, given by $L_{G,V} = \underline{L}(V,-)/G$. This semifree orthogonal space is the representing object for the functor $(-)(V)^G$ given by evaluating at V and then taking G–fixed points.

Recall that i_l denotes the boundary map $i_l \colon \partial D^l \to D^l$ in $\underline{\mathrm{Top}}$, for each $l \geq 0$. Similarly, let j_l denote the inclusion $j_l \colon D^l \cong D^l \times \{0\} \to D^l \times [0,1]$ for $l \geq 0$. We use this notation throughout the paper.

The morphisms in I, the generating cofibrations of the positive global model structure, are of the form $L_{G,V} \times i_l$ for a compact Lie group G, a faithful G–representation $V \neq 0$, and $l \geq 0$.

The generating acyclic cofibrations are $\mathsf{J} \cup \mathsf{K}$, where morphisms in J are of the form $L_{G,V} \times j_l$ for a compact Lie group G, a faithful G–representation $V \neq 0$, and $l \geq 0$. Morphisms in K are of the form $\iota_{\rho_{G,V,W}} \mathbin{\square} i_l$ for a compact Lie group G, a faithful G–representation $V \neq 0$, a G–representation W, and $l \geq 0$. The morphism $\rho_{G,V,W} \colon L_{G,V\oplus W} \to L_{G,V}$ is given by restriction to V, and $\iota_{\rho_{G,V,W}}$ is the mapping cylinder inclusion of $\rho_{G,V,W}$.

The generating acyclic cofibrations in the set K are more complex. Before checking Condition 4.1 for them, we need to prove several auxiliary lemmas. We first deal with the case of the morphisms in I and J.

Proposition 4.4 *Let K be a compact Lie group, $n \geq 1$, and let Z be a $(K\times\Sigma_n)$–orthogonal space. For a generating cofibration $i \in \mathsf{I}$, the morphism $Z \boxtimes_{\Sigma_n} i^{\square n}$ is a*

K–h–cofibration. For a generating acyclic cofibration j in the set J, the morphism $Z \boxtimes_{\Sigma_n} j^{\square n}$ is a K–h–cofibration and a K–global equivalence.

Remark 4.5 This proposition is stated in more generality than Condition 4.1 so that we can also use it later in the proof of Theorem 4.14.

Proof Let $i = L_{G,V} \times i_l \in \mathsf{I}$. Then

$$Z \boxtimes i^{\square n} = Z \boxtimes L_{G,V}^{\boxtimes n} \times i_l^{\square n},$$

which is a $(K \times \Sigma_n)$–h–cofibration because $i_l^{\square n}$ is a Σ_n–h–cofibration of Σ_n–spaces. Then by Lemma 3.16(i) $Z \boxtimes_{\Sigma_n} i^{\square n}$ is a K–h–cofibration.

Let $j = L_{G,V} \times j_l \in \mathsf{J}$. By the same argument as before we obtain that $Z \boxtimes_{\Sigma_n} j^{\square n}$ is a K–h–cofibration. Since $j_l^{\square n}$ is a Σ_n–homotopy equivalence of Σ_n–spaces, we also obtain that $Z \boxtimes L_{G,V}^{\boxtimes n} \times j_l^{\square n}$ is a $(K \times \Sigma_n)$–homotopy equivalence of orthogonal spaces, so $Z \boxtimes_{\Sigma_n} j^{\square n}$ is a K–homotopy equivalence. Therefore it is a K–level equivalence, and thus a K–global equivalence. $\square$

Proposition 4.6 *Let $f\colon X \to Y$ be a morphism of orthogonal spaces such that for each $n \geq 1$ the morphism $f^{\boxtimes n}$ is a Σ_n–global equivalence, and such that for each $n \geq 1$ the morphism $f^{\square n}$ is a Σ_n–h–cofibration. Then for each $n \geq 1$ the morphism $f^{\square n}$ is a Σ_n–global equivalence.*

Proof We use strong induction. For the base case, $f^{\square 1} = f^{\boxtimes 1} = f$ is a global equivalence.

Assume that the result holds for each $i < n$. We decompose $f^{\boxtimes n}$ by applying [21, Lemma A.8] to the pushout diagram given by $X = X \to Y$, obtaining

$$X^{\boxtimes n} = Q_0^n(f) \to Q_1^n(f) \to \cdots \to Q_{n-1}^n(f) \xrightarrow{f^{\square n}} Q_n^n(f) = Y^{\boxtimes n}.$$

Note that the last step of this decomposition is precisely $f^{\square n}$. In the rest of this article we also use $Q_{n-1}^n(f)$ to denote the source of the n–fold pushout product of f, following the notation of [21], originally introduced in [7, Section 12].

For each step $1 \leq i < n$ there is a Σ_n–equivariant pushout diagram of orthogonal spaces

$$\begin{tikzcd}
\Sigma_n \times_{\Sigma_{n-i} \times \Sigma_i} X^{\boxtimes n-i} \boxtimes Q_{i-1}^i(f) \arrow[r, "\Sigma_n \times_{\Sigma_{n-i} \times \Sigma_i} X^{\boxtimes n-i} \boxtimes f^{\square i}"] \arrow[d] & \Sigma_n \times_{\Sigma_{n-i} \times \Sigma_i} X^{\boxtimes n-i} \boxtimes Y^{\boxtimes i} \arrow[d] \\
Q_{i-1}^n(f) \arrow[r] & Q_i^n(f) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By Corollary 3.10 and the induction hypothesis, $X^{\boxtimes n-i} \boxtimes f^{\square i}$ is a $(\Sigma_{n-i}\times\Sigma_i)$–global equivalence. Then by Proposition 3.13,

$$\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} X^{\boxtimes n-i} \boxtimes f^{\square i}$$

is a Σ_n–global equivalence. Additionally by Lemma 3.16 it is a Σ_n–h–cofibration.

By Corollary A.9 this means that $Q^n_{i-1}(f) \to Q^n_i(f)$ is a Σ_n–global equivalence for each $1 \le i < n$. Since so is $f^{\boxtimes n}$, by the 2-out-of-6 property for Σ_n–global equivalences $f^{\square n}$ is a Σ_n–global equivalence. □

Lemma 4.7 *For $f\colon X \to Y$ a homotopy equivalence between orthogonal spaces and $n \ge 1$, the morphism $f^{\boxtimes n}$ is a Σ_n–homotopy equivalence of orthogonal spaces, and therefore a Σ_n–global equivalence.*

Proof Let $g\colon Y \to X$ be a homotopy inverse to f and H a homotopy between $f\circ g$ and Id_X. Then for each $n \ge 1$,

$$H^{\boxtimes n}\circ (X^{\boxtimes n}\times \Delta)\colon X^{\boxtimes n}\times [0,1] \to Y^{\boxtimes n}$$

is a Σ_n–equivariant homotopy between $(f\circ g)^{\boxtimes n}$ and $\mathrm{Id}_X^{\boxtimes n}$, where $\Delta\colon [0,1]\to[0,1]^n$ is the diagonal. The same can be done for $g\circ f$.

Then we obtain that $f^{\boxtimes n}$ is a Σ_n–homotopy equivalence. Therefore it is a Σ_n–level equivalence, and thus a Σ_n–global equivalence. □

Proposition 4.8 *For each generating acyclic cofibration $k \in \mathsf{K}$, the morphism $k^{\square n}$ is a Σ_n–h–cofibration. Concretely, let G be a compact Lie group, consider a faithful G–representation $V \ne 0$, a G–representation W, $n \ge 1$ and $l \ge 0$. Let $\iota_{\rho_{G,V,W}}$ be the morphism given in Remark 4.3. Then $k^{\square n} = (\iota_{\rho_{G,V,W}} \square i_l)^{\square n}$ is a Σ_n–h–cofibration. In particular, since $\iota_{\rho_{G,V,W}} \square i_0 = \iota_{\rho_{G,V,W}}$, we get that $\iota^{\square n}_{\rho_{G,V,W}}$ is a Σ_n–h–cofibration.*

Proof Consider the decomposition of $\iota_{\rho_{G,V,W}}$ given by $g\circ i_{L_{G,V\oplus W}}$ in the diagram

$$\begin{tikzcd}
L_{G,V\oplus W} \arrow[r, "i_{L_{G,V\oplus W}}"] & L_{G,V\oplus W} \amalg L_{G,V} \arrow[r, "g"] & M_{\rho_{G,V,W}} \\
& L_{G,V\oplus W} \amalg L_{G,V\oplus W} \arrow[u] \arrow[r, "L_{G,V\oplus W}\times i_1"] & L_{G,V\oplus W}\times[0,1] \arrow[u] \arrow[ul, phantom, "\lrcorner"]
\end{tikzcd}$$

We use results from [10] that deal with the interaction between the pushout product and operations on morphisms like composition and cobase change. The structure of this proof is convoluted because in general we cannot prove that the pushout product of two Σ_n–h–cofibrations is a Σ_n–h–cofibration. This forces us to carry the $-\square\, i_l^{\square n}$

term around. We also have to decompose $(\iota_{\rho_{G,V,W}} \square i_l)^{\square n}$ into simpler morphisms which we can prove to be Σ_n–h–cofibrations, using cobase changes and compositions, because these operations preserve the class of Σ_n–h–cofibrations.

By [10, Lemma 15] we can decompose $\iota^{\square n}_{\rho_{G,V,W}}$ into $f_0 \circ f_1 \circ \cdots \circ f_n$. Here for each $0 \le j \le n$, the morphism f_j is a Σ_n–equivariant cobase change of

$$\Sigma_n \times_{\Sigma_{n-j} \times \Sigma_j} g^{\square n-j} \square i^{\square j}_{L_{G,V \oplus W}}.$$

By [10, Lemma 17] we can write $\iota^{\square n}_{\rho_{G,V,W}} \square i_l^{\square n}$ as $f'_0 \circ f'_1 \circ \cdots \circ f'_n$, where each f'_j is a cobase change of $f_j \square i_l^{\square n}$.

We also have [10, Lemma 13], which states that for any h_0 if a morphism h_1 is a cobase change of h_2, then $h_1 \square h_0$ is a cobase change of $h_2 \square h_0$. By iterating this result, and using the associativity of the pushout product, we obtain that $g^{\square n-j}$ is a cobase change of $(L_{G,V \oplus W} \times i_1)^{\square n-j}$. Similarly $i^{\square j}_{L_{G,V\oplus W}}$ is a cobase change of $\varnothing \to L^{\boxtimes j}_{G,V}$. Furthermore these cobase changes can be checked to be through equivariant maps.

Note that $i_1^{\square n-j} \cong i_{n-j}$. For each $0 \le j \le n$ we can apply [10, Lemma 13] again to obtain that $g^{\square n-j} \square i^{\square j}_{L_{G,V\oplus W}}$ is a $(\Sigma_{n-j} \times \Sigma_j)$–equivariant cobase change of

$$(L^{\boxtimes n-j}_{G,V\oplus W} \times i_1^{\square n-j}) \square (\varnothing \to L^{\boxtimes j}_{G,V}) \cong L^{\boxtimes n-j}_{G,V\oplus W} \boxtimes L^{\boxtimes j}_{G,V} \times i_{n-j}.$$

We want to check that for each $0 \le j \le n$,

$$(\Sigma_n \times_{\Sigma_{n-j} \times \Sigma_j} (L^{\boxtimes n-j}_{G,V\oplus W} \boxtimes L^{\boxtimes j}_{G,V} \times i_{n-j})) \square i_l^{\square n} \tag{3}$$

is a Σ_n–h–cofibration. As a morphism of $(\Sigma_n \times \Sigma_n)$–orthogonal spaces, this is isomorphic to

$$\Sigma_n \times_{\Sigma_{n-j} \times \Sigma_j} ((L^{\boxtimes n-j}_{G,V\oplus W} \boxtimes L^{\boxtimes j}_{G,V} \times i_{n-j}) \square i_l^{\square n}).$$

The map $i_{n-j} \square i_l^{\square n} \cong i_{n-j+ln}$ is a $(\Sigma_{n-j} \times \Sigma_n)$–$h$–cofibration of spaces. Therefore by Lemma 3.16,

$$L^{\boxtimes n-j}_{G,V\oplus W} \boxtimes L^{\boxtimes j}_{G,V} \times (i_{n-j} \square i_l^{\square n})$$

is a $(\Sigma_{n-j} \times \Sigma_j \times \Sigma_n)$–$h$–cofibration and the morphism (3) is a Σ_n–h–cofibration.

Recall that for each $0 \le j \le n$ the morphism $g^{\square n-j} \square i^{\square j}_{L_{G,V\oplus W}}$ is a cobase change of

$$L^{\boxtimes n-j}_{G,V\oplus W} \boxtimes L^{\boxtimes j}_{G,V} \times i_{n-j}.$$

Thus applying [10, Lemma 13] again we obtain that $f_j \square i_l^{\square n}$ is a cobase change of (3) so it is also a Σ_n–h–cofibration. We are using the fact that the induction functor $\Sigma_n \times_{\Sigma_{n-j} \times \Sigma_j} -$ preserves pushouts.

Finally, each f'_j was a cobase change of $f_j \mathbin{\square} i_l^{\square n}$, so it is also a Σ_n–h–cofibration. Thus their composition

$$k^{\square n} = (\iota_{\rho_{G,V,W}} \mathbin{\square} i_l)^{\square n} = \iota_{\rho_{G,V,W}}^{\square n} \mathbin{\square} i_l^{\square n}$$

is a Σ_n–h–cofibration. ☐

Proposition 4.9 *For each generating acyclic cofibration $k \in \mathsf{K}$, the morphism $k^{\square n}$ is a Σ_n–global equivalence.*

Proof The generating acyclic cofibration k is of the form $\iota_{\rho_{G,V,W}} \mathbin{\square} i_l$ for a compact Lie group G, a faithful G–representation $V \neq 0$, a G–representation W, and $l \geq 0$.

We first check that

$$\rho_{G,V,W}^{\boxtimes n} \colon L_{G,V\oplus W}^{\boxtimes n} \to L_{G,V}^{\boxtimes n}$$

is a Σ_n–global equivalence. By [22, Example 1.3.3] the orthogonal space $L_{G,V}^{\boxtimes n}$ is isomorphic to L_{G^n,V^n} and thus closed. The $(\Sigma_n \wr G)$–representation V^n is faithful, so for each compact Lie group K by [22, Proposition 1.1.26(ii)] the restriction map

$$\rho_{V^n,W^n}(\mathcal{U}_K) \colon \underline{L}(V^n \oplus W^n, \mathcal{U}_K) \to \underline{L}(V^n, \mathcal{U}_K)$$

is a $(K \times (\Sigma_n \wr G))$–homotopy equivalence. Using that

$$\underline{L}(V^n, \mathcal{U}_K) \cong \operatorname*{colim}_{V' \in s(\mathcal{U}_K)} \underline{L}(V^n, V')$$

and the fact that $-/G^n$ preserves colimits, we can obtain that

$$\rho_{G,V,W}^{\boxtimes n}(\mathcal{U}_K) \cong \rho_{V^n,W^n}(\mathcal{U}_K)/G^n$$

is a $(K\times\Sigma_n)$–homotopy equivalence. Therefore $\rho_{G,V,W}^{\boxtimes n}(\mathcal{U}_K)$ is an $\mathscr{F}(K, \Sigma_n)$–equivalence, and so $\rho_{G,V,W}^{\boxtimes n}$ is a Σ_n–global equivalence.

Now we use the mapping cylinder to decompose $\rho_{G,V,W}$ as $\pi_{\rho_{G,V,W}} \circ \iota_{\rho_{G,V,W}}$. Since $\pi_{\rho_{G,V,W}}$ is a homotopy equivalence, by Lemma 4.7 $\pi_{\rho_{G,V,W}}^{\boxtimes n}$ is a Σ_n–global equivalence, and then so is $\iota_{\rho_{G,V,W}}^{\boxtimes n}$.

We use Propositions 4.6 and 4.8 to obtain that for each $n \geq 1$ the morphism $\iota_{\rho_{G,V,W}}^{\square n}$ is a Σ_n–global equivalence. Finally, by Corollary A.10 and Proposition 4.8 again, we get that $k^{\square n} = \iota_{\rho_{G,V,W}}^{\square n} \mathbin{\square} i_l^{\square n}$ is a Σ_n–global equivalence. ☐

Proposition 4.10 *Let $n \geq 1$ and let Z be a Σ_n–orthogonal space. For each generating acyclic cofibration $k \in \mathsf{K}$, the morphism $Z \boxtimes_{\Sigma_n} k^{\square n}$ is an h–cofibration and a global equivalence.*

Proof First,
$$Z \boxtimes k^{\square n} = Z \boxtimes (\iota_{\rho_{G,V,W}} \square i_I)^{\square n}$$
is a Σ_n–h–cofibration by Proposition 4.8, and a Σ_n–global equivalence by Proposition 4.9 and Corollary 3.10.

Consider the Σ_n–orthogonal space $L_{G,V}^{\boxtimes n} \cong L_{G^n,V^n}$. For each inner product space U the group G^n acts freely (since V is faithful), smoothly and properly (since G^n is compact) on $\underline{L}(V^n, U)$, as long as $|U| \geq |V|^n$. Therefore
$$L_{G^n,V^n}(U) = \underline{L}(V^n, U)/G^n$$
is Hausdorff; and since V^n is a faithful Σ_n–representation, $L_{G,V}^{\boxtimes n}(U) \cong L_{G^n,V^n}(U)$ is also Σ_n–free.

If $|U| < |V|^n$ then $\underline{L}(V^n, U)$ is empty, so in particular $L_{G^n,V^n}(U)$ is still Hausdorff and Σ_n–free.

The morphism $\pi_{\rho_{G,V,W}}$ induces a Σ_n–equivariant map of orthogonal spaces from the target of $Z \boxtimes k^{\square n}$ to $* \boxtimes L_{G,V}^{\boxtimes n} \times *$, and so by Proposition 3.12 $Z \boxtimes_{\Sigma_n} k^{\square n}$ is a global equivalence. It is an h–cofibration by Lemma 3.16(i). □

Note that the fact that $L_{G,V}^{\boxtimes n}(U)$ is Σ_n–free in this last proof is important. It lets us avoid the assumption that the components $\mathcal{O}_n$ of the operad $\mathcal{O}$ are Σ_n–free in the following theorem.

Theorem 4.11 *Let $\mathcal{O}$ be any operad in $(\underline{\mathcal{Spc}}, \boxtimes)$ the category of orthogonal spaces, with the positive global model structure and the symmetric monoidal structure given by the box product. Then there is a cofibrantly generated model category structure on $\underline{\mathcal{Alg}}(\mathcal{O})$, the category of algebras over $\mathcal{O}$, where the forgetful functor $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ creates the weak equivalences and fibrations, and sends cofibrations in $\underline{\mathcal{Alg}}(\mathcal{O})$ to h–cofibrations in $\underline{\mathcal{Spc}}$.*

Proof Let $\mathcal{Hcof}$ be the class of h–cofibrations in $\underline{\mathcal{Spc}}$. It satisfies conditions (a), (b) and (c) of Theorem 2.2 by Lemma 3.15, Lemma A.16, and Corollary A.12 respectively, with $G = e$ in all of them.

Consider a morphism $i : X \to Y$ in $\underline{\mathcal{Spc}}$ and a pushout in $\underline{\mathcal{Alg}}(\mathcal{O})$ of the form
$$\begin{tikzcd}
F_{\underline{\mathcal{Alg}}(\mathcal{O})}(X) \arrow[r, "F_{\underline{\mathcal{Alg}}(\mathcal{O})}(i)"] \arrow[d] & F_{\underline{\mathcal{Alg}}(\mathcal{O})}(Y) \arrow[d] \\
A \arrow[r, "f"] & B \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

We use the filtration of [21, Proposition A.16], originally introduced in the proof of [7, Theorem 12.4], with $k = 0$, where $U_0^{\mathcal{O}} = U_{\underline{\mathcal{Alg}}(\mathcal{O})}$. We obtain a decomposition of $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(f)$ as the infinite composition of morphisms

$$f_n \colon P_{n-1}U_{\underline{\mathcal{Alg}}(\mathcal{O})}(B) \to P_n U_{\underline{\mathcal{Alg}}(\mathcal{O})}(B)$$

for $n \geq 1$, with $P_0 U_{\underline{\mathcal{Alg}}(\mathcal{O})}(B) = U_{\underline{\mathcal{Alg}}(\mathcal{O})}(A)$. For each $n \geq 1$, [21, Proposition A.16] gives the following pushout in $\underline{\mathcal{Spc}}$:

$$\begin{tikzcd}
U_n^{\mathcal{O}}(A) \boxtimes_{\Sigma_n} Q_{n-1}^n(i) \arrow[r, "U_n^{\mathcal{O}}(A)\boxtimes_{\Sigma_n} i^{\square n}"] \arrow[d] & U_n^{\mathcal{O}}(A) \boxtimes_{\Sigma_n} (Y)^{\boxtimes n} \arrow[d] \\
P_{n-1}U_0^{\mathcal{O}}(B) \arrow[r, "f_n"] \arrow[ur, phantom, very near end, "\ulcorner"] & P_n U_0^{\mathcal{O}}(B)
\end{tikzcd}$$

Both the class $\mathcal{Hcof}$, and the class of morphisms in $\underline{\mathcal{Spc}}$ which are both h–cofibrations and global equivalences, are closed under infinite composition and cobase change (see the results of the appendix). Propositions 4.4 and 4.10 imply that if i is a generating cofibration $U_n^{\mathcal{O}}(A) \boxtimes_{\Sigma_n} i^{\square n}$ is an h–cofibration, and if i is a generating acyclic cofibration then $U_n^{\mathcal{O}}(A) \boxtimes_{\Sigma_n} i^{\square n}$ is an h–cofibration and global equivalence. Therefore all the conditions of Theorem 2.2 hold, and $\underline{\mathcal{Alg}}(\mathcal{O})$ is a cofibrantly generated model category where $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ creates the weak equivalences and fibrations. Furthermore $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ sends cofibrations in $\underline{\mathcal{Alg}}(\mathcal{O})$ to h–cofibrations in $\underline{\mathcal{Spc}}$. □

4.2 Characterizing which morphisms of operads induce Quillen equivalences

We study now morphisms of operads and the associated functors between their respective categories of algebras, with the goal of classifying which morphisms of operads in orthogonal spaces induce Quillen equivalences between the respective categories of algebras.

Consider for now a general symmetric monoidal category $(\mathcal{C}, \otimes, *)$, where the tensor product preserves all colimits in both variables. Let $g \colon \mathcal{O} \to \mathcal{P}$ be a morphism of operads, understood as a morphism of monoids in $(\Sigma_*\text{–}\mathcal{C}, \circ)$. The morphism g induces an adjoint pair of functors

$$g! \colon \underline{\mathcal{Alg}}(\mathcal{O}) \rightleftarrows \underline{\mathcal{Alg}}(\mathcal{P}) : g^*,$$

called the extension functor and the restriction functor respectively. The specific details can be found in [9, Section 3.3.5] for example.

We use $\theta\colon \mathcal{F}(\mathcal{O}) \Rightarrow \mathcal{F}(\mathcal{P})$ to denote the natural transformation induced by g between the monads $\mathcal{F}(\mathcal{O})$ and $\mathcal{F}(\mathcal{P})$, associated to the operads $\mathcal{O}$ and $\mathcal{P}$ respectively. For X an algebra over $\mathcal{P}$, we use $\zeta_X\colon \mathcal{F}(\mathcal{P})(X) \to X$ to denote its structure map. Then $g^*(X)$ is just X with structure map $\zeta_X \circ \theta_X$. Additionally, since $U_{\underline{\mathcal{A}lg}(\mathcal{O})} \circ g^* = U_{\underline{\mathcal{A}lg}(\mathcal{P})}$, we have that $g_! \circ F_{\underline{\mathcal{A}lg}(\mathcal{O})}$ is left adjoint to $U_{\underline{\mathcal{A}lg}(\mathcal{P})}$, so it is naturally isomorphic to $F_{\underline{\mathcal{A}lg}(\mathcal{P})}$. This is the only information about the extension functor that we need.

For the proof of Theorem 4.14 we need to consider again the functors $U_k^{\mathcal{O}}$ for $k \geq 0$ from [21, Proposition 10.1], originally introduced in the proof of [7, Theorem 12.4]. The functor $U_k^{\mathcal{O}}$ goes from $\underline{\mathcal{A}lg}(\mathcal{O})$ to $\underline{\Sigma_k \mathscr{C}}$, the category of Σ_k–objects in $\mathscr{C}$, and $U_0^{\mathcal{O}} = U_{\underline{\mathcal{A}lg}(\mathcal{O})}$.

Construction 4.12 Let $\mathcal{O}$ and $\mathcal{P}$ be two operads, and let g be a morphism of operads $g\colon \mathcal{O} \to \mathcal{P}$. For a general $\mathcal{O}$–algebra X, a $\mathcal{P}$–algebra Y, and a map of $\mathcal{O}$–algebras $\gamma\colon X \to g^*(Y)$, we construct certain maps

$$g_{k,\gamma}\colon U_k^{\mathcal{O}}(X) \to U_k^{\mathcal{P}}(Y)$$

in $\underline{\Sigma_k \mathscr{C}}$ for each $k \geq 0$, in a way that is natural in γ and preserves filtered colimits. It is important to note that the morphism $g_{k,\gamma}$ is not $U_k^{\mathcal{O}}(\gamma)$ unless $k = 0$. In fact, $U_k^{\mathcal{O}} \circ g^*$ is not $U_k^{\mathcal{P}}$ for $k \neq 0$, so $g_{k,\gamma}$ and $U_k^{\mathcal{O}}(\gamma)$ do not have the same target for $k \neq 0$.

Consider the functors

$$\mathcal{O}(-,k)\colon \mathscr{C} \to \underline{\Sigma_k \mathscr{C}}$$

constructed in [21, Section A.9], which for an operad $\mathcal{O}$ and $k \geq 0$ are given by

$$\mathcal{O}(X,k) = \coprod_{n \in \mathbb{N}} \mathcal{O}(n+k) \otimes_{\Sigma_n} X^{\otimes n}.$$

Note that $\mathcal{O}(-,0) = \mathcal{F}(\mathcal{O})$. The construction of the functors $U_k^{\mathcal{O}}$ in [21, Definition A.10] is given by the coequalizer

$$\mathcal{O}(\mathcal{O}(X,0),k) \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \mathcal{O}(X,k) \to U_k^{\mathcal{O}}(X).$$

The morphism of operads g and the map of $\mathcal{O}$–algebras γ together induce a Σ_k–equivariant morphism of coequalizer diagrams. The induced morphism between the coequalizers $U_k^{\mathcal{O}}(X)$ and $U_k^{\mathcal{P}}(Y)$ is our desired $g_{k,\gamma}$.

This construction preserves filtered colimits because tensor powers preserve them, and thus so do the functors $\mathcal{O}(-,k)$.

Now we restrict ourselves to the case of operads in $(\underline{\mathcal{S}pc}, \boxtimes)$, where we have the model structures on $\underline{\mathcal{A}lg}(\mathcal{O})$ obtained in Theorem 4.11.

Proposition 4.13 *For any morphism $g\colon \mathcal{O} \to \mathcal{P}$ of operads in $(\underline{\mathcal{S}pc}, \boxtimes)$, the restriction functor g^* preserves and reflects fibrations and weak equivalences. Thus the pair $(g_!, g^*)$ is a Quillen adjunction.*

Proof The functors $U_{\underline{\mathcal{A}lg}(\mathcal{O})}$ and $U_{\underline{\mathcal{A}lg}(\mathcal{P})}$ preserve and reflect fibrations and weak equivalences, and $U_{\underline{\mathcal{A}lg}(\mathcal{O})} \circ g^* = U_{\underline{\mathcal{A}lg}(\mathcal{P})}$. □

Theorem 4.14 *Let $g\colon \mathcal{O} \to \mathcal{P}$ be a morphism of operads in $(\underline{\mathcal{S}pc}, \boxtimes)$, the category of orthogonal spaces, with the positive global model structure and the symmetric monoidal structure given by the box product. Then the induced adjunction $(g_!, g^*)$ is a Quillen equivalence between the respective categories of algebras if and only if for each $n \geq 0$ the morphism $g_n\colon \mathcal{O}_n \to \mathcal{P}_n$ is a Σ_n–global equivalence.*

Proof The right adjoint g^* preserves and reflects weak equivalences. Therefore the pair $(g_!, g^*)$ is a Quillen equivalence if and only if for each cofibrant $A \in \underline{\mathcal{A}lg}(\mathcal{O})$ the unit $\eta_A\colon A \to g^*(g_!(A))$ is a weak equivalence in $\underline{\mathcal{A}lg}(\mathcal{O})$, that is, a global equivalence of underlying orthogonal spaces (see for example [8, Lemma 3.3] for a proof).

We first assume that each g_n is a Σ_n–global equivalence, and check that for each cofibrant $A \in \underline{\mathcal{A}lg}(\mathcal{O})$ the unit $\eta_A\colon A \to g^*(g_!(A))$ is a global equivalence.

First assume that the cofibrant algebra A is the colimit of a λ–sequence of morphisms $\{f_\beta\}_{\beta\in\lambda}$ beginning at $A_0 = \mathcal{O}_0$, for a limit ordinal λ. Note that the initial object of $\underline{\mathcal{A}lg}(\mathcal{O})$ is $\mathcal{O}_0$, since it is $F_{\underline{\mathcal{A}lg}(\mathcal{O})}(\varnothing)$. Assume that each f_β is a cobase change of a morphism of the form $F_{\underline{\mathcal{A}lg}(\mathcal{O})}(i_\beta)$, for $i_\beta \in \mathrm{I}$ $i_\beta\colon X_\beta \to Y_\beta$ a generating positive flat cofibration of orthogonal spaces. We want to check that $U_{\underline{\mathcal{A}lg}(\mathcal{O})}(\eta_A)$ is a global equivalence.

By evaluating the unit of the adjunction η on the λ–sequence that gives rise to A, we obtain the diagram

$$\begin{tikzcd}
A_0 = \mathcal{O}_0 \arrow[r, "f_0"] \arrow[d, "\eta_{A_0}"] & A_1 \arrow[r, "f_1"] \arrow[d, "\eta_{A_1}"] & \cdots \arrow[r] & A_\beta \arrow[r, "f_\beta"] \arrow[d, "\eta_{A_\beta}"] & \cdots \\
g^*(g_!(A_0)) \arrow[r, "g^*(g_!(f_0))"] & g^*(g_!(A_1)) \arrow[r, "g^*(g_!(f_1)))"] & \cdots \arrow[r] & g^*(g_!(A_\beta)) \arrow[r, "g^*(g_!(f_\beta))"] & \cdots
\end{tikzcd}$$

We apply $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ to the whole diagram. By Theorem 4.11, $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ sends cofibrations to h–cofibrations, so $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(f_\beta)$ is an h–cofibration. The morphism $g_!(f_\beta)$ is a cofibration, so

$$U_{\underline{\mathcal{Alg}}(\mathcal{O})}(g^*(g_!(f_\beta))) = U_{\underline{\mathcal{Alg}}(\mathcal{P})}(g_!(f_\beta))$$

is also an h–cofibration. Since $U_{\underline{\mathcal{Alg}}(\mathcal{O})}$ preserves filtered colimits, $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(\eta_A)$ is

$$\operatorname*{colim}_{\beta\in\lambda} U_{\underline{\mathcal{Alg}}(\mathcal{O})}(\eta_{A_\beta}).$$

We have to check that each $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(\eta_{A_\beta})$ is a global equivalence, and for this we follow the proof of the similar statement in [21, Lemma 9.13]. We prove this by induction, but we in fact need to work with a stronger property. For each β and each $k \geq 0$, let $g_{k,\beta}$ be the morphism $g_{k,\eta_{A_\beta}}$ given in Construction 4.12. We check by transfinite induction on β that for each $k \geq 0$ the morphism

$$g_{k,\beta}\colon U_k^{\mathcal{O}}(A_\beta) \to U_k^{\mathcal{P}}(g_!(A_\beta))$$

is a Σ_k–global equivalence. For $k = 0$ this reduces to our desired result.

The base case concerns $A_0 = \mathcal{O}_0 = F_{\underline{\mathcal{Alg}}(\mathcal{O})}(\varnothing)$. By [21, Lemma A.13] the Σ_k–orthogonal space $U_k^{\mathcal{O}}(F_{\underline{\mathcal{Alg}}(\mathcal{O})}(\varnothing))$ is isomorphic to $\mathcal{O}(\varnothing, k)$, and $\mathcal{O}(\varnothing, k)$ equals $\mathcal{O}_k$. Similarly $g_!(F_{\underline{\mathcal{Alg}}(\mathcal{O})}(\varnothing))$ is isomorphic to $F_{\underline{\mathcal{Alg}}(\mathcal{P})}(\varnothing)$, and then

$$U_k^{\mathcal{P}}(F_{\underline{\mathcal{Alg}}(\mathcal{P})}(\varnothing)) \cong \mathcal{P}(\varnothing, k) = \mathcal{P}_k,$$

and under these identifications, the morphism $g_{k,0}$ corresponds to g_k, which is a Σ_k–global equivalence by the condition of the theorem. Remarkably, this is the only part of the proof where this condition is used.

Then we check the induction step for a successor ordinal $\beta+1$. For this we use the filtration of [21, Proposition A.16], originally introduced in the proof of [7, Theorem 12.4], in the same way that it is used in the proof of [21, Lemma 9.13]:

$$\begin{tikzcd}
U_k^{\mathcal{O}}(A_\beta) = F_0 U_k^{\mathcal{O}}(A_{\beta+1}) \arrow[r] \arrow[d, "g_{k,\beta}"] & F_1 U_k^{\mathcal{O}}(A_{\beta+1}) \arrow[r] \arrow[d] & \cdots \\
U_k^{\mathcal{P}}(g_!(A_\beta)) = F_0 U_k^{\mathcal{P}}(g_!(A_{\beta+1})) \arrow[r] & F_1 U_k^{\mathcal{P}}(g_!(A_{\beta+1})) \arrow[r] & \cdots
\end{tikzcd} \tag{4}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname*{colim}_{j\in\mathbb{N}} F_j U_k^{\mathcal{O}}(A_{\beta+1}) = U_k^{\mathcal{O}}(A_{\beta+1}) \arrow[d, "g_{k,\beta+1}"] \\
\cdots \arrow[r] & \operatorname*{colim}_{j\in\mathbb{N}} F_j U_k^{\mathcal{P}}(g_!(A_{\beta+1})) = U_k^{\mathcal{P}}(g_!(A_{\beta+1}))
\end{tikzcd}$$

Assume that for each $k \geq 0$ the morphism $g_{k,\beta}$ is a Σ_k–global equivalence. Each horizontal map is a cobase change of

$$U^{\mathcal{O}}_{j+k}(A_\beta) \boxtimes_{\Sigma_j} i_\beta^{\Box j} \quad \text{or} \quad U^{\mathcal{P}}_{j+k}(g_!(A_\beta)) \boxtimes_{\Sigma_j} i_\beta^{\Box j},$$

which are Σ_k–h–cofibrations by Proposition 4.4.

Each vertical map is obtained from the previous by the following morphism of pushout diagrams:

$$\begin{tikzcd}
F_{j-1}U^{\mathcal{O}}_k(A_{\beta+1}) \arrow[rr] & & F_{j-1}U^{\mathcal{P}}_k(g_!(A_{\beta+1})) \\
U^{\mathcal{O}}_{j+k}(A_\beta)\boxtimes_{\Sigma_j} Q^j_{j-1}(i_\beta) \arrow[u] \arrow[rr, "g_{j+k,\beta}\boxtimes_{\Sigma_j} Q^j_{j-1}(i_\beta)"] \arrow[d, "U^{\mathcal{O}}_{j+k}(A_\beta)\boxtimes_{\Sigma_j} i_\beta^{\Box j}"'] & & U^{\mathcal{P}}_{j+k}(g_!(A_\beta))\boxtimes_{\Sigma_j} Q^j_{j-1}(i_\beta) \arrow[u] \arrow[d, "U^{\mathcal{P}}_{j+k}(g_!(A_\beta))\boxtimes_{\Sigma_j} i_\beta^{\Box j}"] \\
U^{\mathcal{O}}_{j+k}(A_\beta)\boxtimes_{\Sigma_j} (Y_\beta)^{\boxtimes j} \arrow[rr, "g_{j+k,\beta}\boxtimes_{\Sigma_j} (Y_\beta)^{\boxtimes j}"] & & U^{\mathcal{P}}_{j+k}(g_!(A_\beta))\boxtimes_{\Sigma_j} (Y_\beta)^{\boxtimes j}
\end{tikzcd}$$

By the induction hypothesis, the morphism

$$g_{j+k,\beta} \colon U^{\mathcal{O}}_{j+k}(A_\beta) \to U^{\mathcal{P}}_{j+k}(g_!(A_\beta))$$

is a Σ_{j+k}–global equivalence. Here $Y_\beta = L_{G,V} \times i_l$, so we can project to $L_{G,V}^{\boxtimes j}$ and use Corollary 3.10 and Proposition 3.12 as in the proof of Proposition 4.10 to check that the two rightmost vertical maps are Σ_k–global equivalences.

Then we can use induction on j and the gluing lemma, Lemma A.8, to obtain that each vertical map of (4) is also a Σ_k–global equivalence. Finally, by Lemma A.11, $g_{k,\beta+1}$ is a Σ_k–global equivalence.

If β is a limit ordinal, we just need to use Lemma A.11, and the fact that the construction of $g_{k,\beta}$ preserves filtered colimits.

We have proven that $g_{k,\beta}$ is a Σ_k–global equivalence for each k and β. Setting $k = 0$ we have our original intended result that $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(\eta_{A_\beta})$ is a global equivalence for each β. By Lemma A.11 with $G = e$ the morphism $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(\eta_A)$ is a global equivalence.

If $A \in \underline{\mathcal{Alg}}(\mathcal{O})$ is cofibrant, then it is a retract of an algebra A' of the kind we were considering at the beginning of this proof, and the unit η_A is a retract of $\eta_{A'}$. Since retracts preserve weak equivalences, η_A is a weak equivalence in $\underline{\mathcal{Alg}}(\mathcal{O})$. Therefore $(g_!, g^*)$ is a Quillen equivalence.

We prove the other implication now. Assume that $(g_!, g^*)$ is a Quillen equivalence. We want to prove that for each $n \geq 0$ the morphism g_n is a Σ_n–global equivalence.

Consider the free orthogonal space $L_{\mathbb{R}} = \underline{L}(\mathbb{R}, -)$, which is positively flat. Then $F_{\underline{\mathcal{Alg}}(\mathcal{O})}(L_{\mathbb{R}})$ is cofibrant in $\underline{\mathcal{Alg}}(\mathcal{O})$. Since $(g_!, g^*)$ is a Quillen equivalence the unit

$$\eta_{F_{\underline{\mathcal{Alg}}(\mathcal{O})}(L_{\mathbb{R}})} \colon F_{\underline{\mathcal{Alg}}(\mathcal{O})}(L_{\mathbb{R}}) \to g^*(g_!(F_{\underline{\mathcal{Alg}}(\mathcal{O})}(L_{\mathbb{R}})))$$

is a weak equivalence, so its underlying morphism of orthogonal spaces is a global equivalence.

The $\mathcal{P}$–algebra $g_!(F_{\underline{\mathcal{Alg}}(\mathcal{O})}(L_{\mathbb{R}}))$ is naturally isomorphic to $F_{\underline{\mathcal{Alg}}(\mathcal{P})}(L_{\mathbb{R}})$. After post-composing $\eta_{F_{\underline{\mathcal{Alg}}(\mathcal{O})}(L_{\mathbb{R}})}$ with g^* of this isomorphism, we obtain a morphism

$$F_{\underline{\mathcal{Alg}}(\mathcal{O})}(L_{\mathbb{R}}) \to g^*(F_{\underline{\mathcal{Alg}}(\mathcal{P})}(L_{\mathbb{R}})),$$

whose underlying morphism of orthogonal spaces is precisely

$$\theta_{L_{\mathbb{R}}} = \coprod_{n \in \mathbb{N}} g_n \boxtimes_{\Sigma_n} L_{\mathbb{R}}^{\boxtimes n}.$$

Since $\theta_{L_{\mathbb{R}}}$ is a global equivalence, each $g_n \boxtimes_{\Sigma_n} L_{\mathbb{R}}^{\boxtimes n}$ is a global equivalence. If $n = 0$, we obtain that g_0 is a global equivalence. For each $n \geq 1$, $L_{\mathbb{R}}^{\boxtimes n} \cong L_{\mathbb{R}^n}$, and the orthogonal space $L_{\mathbb{R}^n}$ is Σ_n–free and Hausdorff at each inner product space V. Thus by Proposition 3.12 the morphism $g_n \boxtimes L_{\mathbb{R}^n}$ is a Σ_n–global equivalence for each $n \geq 1$.

The morphisms $\rho_{\mathcal{O}_n, L_{\mathbb{R}^n}}$ and $\rho_{\mathcal{P}_n, L_{\mathbb{R}^n}}$ are Σ_n–global equivalences by Proposition 3.9 and Lemma 3.7(vi). By the 2-out-of-6 property of Σ_n–global equivalences we obtain that $g_n \times L_{\mathbb{R}^n}$ is a Σ_n–global equivalence:

$$\begin{tikzcd}
\mathcal{O}_n \boxtimes L_{\mathbb{R}^n} \arrow[r, "g_n \boxtimes L_{\mathbb{R}^n}"] \arrow[d, "\rho_{\mathcal{O}_n, L_{\mathbb{R}^n}}"'] & \mathcal{P}_n \boxtimes L_{\mathbb{R}^n} \arrow[d, "\rho_{\mathcal{P}_n, L_{\mathbb{R}^n}}"] \\
\mathcal{O}_n \times L_{\mathbb{R}^n} \arrow[r, "g_n \times L_{\mathbb{R}^n}"] & \mathcal{P}_n \times L_{\mathbb{R}^n}
\end{tikzcd}$$

By Proposition 3.4, for each compact Lie group K and each exhaustive sequence of K–representations $\{V_i\}_{i \in \mathbb{N}}$, the map

$$\mathrm{tel}_i(g(V_i) \times L_{\mathbb{R}^n}(V_i)) \colon \mathrm{tel}_i(\mathcal{O}_n(V_i) \times L_{\mathbb{R}^n}(V_i)) \to \mathrm{tel}_i(\mathcal{P}_n(V_i) \times L_{\mathbb{R}^n}(V_i))$$

is an $\mathscr{F}(K, \Sigma_n)$–equivalence. The canonical map

$$\mathrm{tel}_i(\mathcal{O}_n(V_i) \times L_{\mathbb{R}^n}(V_i)) \to (\mathrm{tel}_i\, \mathcal{O}_n(V_i)) \times (\mathrm{tel}_j\, L_{\mathbb{R}^n}(V_j))$$

is also an $\mathscr{F}(K, \Sigma_n)$–equivalence, and the same holds for $\mathcal{P}_n$. Therefore

$$(\mathrm{tel}_i\, g_n(V_i)) \times (\mathrm{tel}_j\, L_{\mathbb{R}^n}(V_j))$$

is an $\mathscr{F}(K, \Sigma_n)$–equivalence.

For each $\Gamma_\phi \in \mathcal{F}(K, \Sigma_n)$, we can pull the Σ_n–action on $\mathbb{R}_n$ through $\phi\colon H \to \Sigma_n$ to get an H–action. Then the H–representation $\mathbb{R}^n$ embeds into some K–representation (see [6, Theorem III.4.5]), which in turn embeds into some V_i, so $(\mathrm{tel}_j\, L_{\mathbb{R}^n}(V_j))^{\Gamma_\phi}$ is nonempty. Thus $\mathrm{tel}_i\, g_n(V_i)$ is an $\mathcal{F}(K, \Sigma_n)$–equivalence, and g_n is a Σ_n–global equivalence for each $n \geq 0$. $\square$

Remark 4.15 The previous theorem generalizes, in the setting of algebras over operads in $(\underline{\mathcal{S}pc}, \boxtimes)$, the classical result that a morphism g between cofibrant operads induces a Quillen equivalence if the underlying morphism of each g_n is a weak equivalence (see [9, 12.5.A] for example). For orthogonal spaces, and a morphism g between operads which are *not necessarily cofibrant*, by the previous theorem we require the stronger condition that each g_n is not just a global equivalence, but also a Σ_n–global equivalence.

The question of which morphisms between more general operads induce Quillen equivalences was also answered in [19, Theorem 7.5]. The key property there is whether the morphisms g_n are *symmetric flat* weak equivalences as defined in [19, Definition 2.1(vii)]. However, Σ_n–global equivalences are not necessarily symmetric flat.

Remark 4.16 Given $\mathcal{O}$, an operad in $(\underline{\mathcal{S}pc}, \boxtimes)$, we could take a cofibrant replacement of it in the J–semimodel category $\mathcal{OP}\text{–}\underline{\mathcal{S}pc}$ of operads in $(\underline{\mathcal{S}pc}, \boxtimes)$, constructed in [25, Theorem 3]. This would be a cofibrant operad $\mathcal{O}'$ and a morphism of operads $g\colon \mathcal{O}' \to \mathcal{O}$ such that each g_n is a global equivalence. But as we just saw in Theorem 4.14, this g does not induce a Quillen equivalence between the categories of algebras of $\mathcal{O}$ and $\mathcal{O}'$ unless each g_n is additionally a Σ_n–global equivalence. This means that simply taking a cofibrant replacement $\mathcal{O}'$ in $\mathcal{OP}\text{–}\underline{\mathcal{S}pc}$ of an operad $\mathcal{O}$, and looking at the model structure on $\mathcal{A}lg(\mathcal{O}')$ does not give the correct homotopy theory of the algebras over $\mathcal{O}$.

Additionally, we cannot have a functor $F^c\colon \mathcal{OP}\text{–}\underline{\mathcal{S}pc} \to \mathcal{OP}\text{–}\underline{\mathcal{S}pc}$, with a natural transformation $\eta\colon F^c \Rightarrow \mathrm{Id}_{\mathcal{OP}\text{–}\underline{\mathcal{S}pc}}$ such that each $\eta(\mathcal{O})_n$ is a Σ_n–global equivalence, and $F^c(\mathcal{O})$ is cofibrant in the J–semimodel structure of [25, Theorem 3]. Assume that we had such a functor F^c, then consider a morphism of operads $g\colon \mathcal{O} \to \mathcal{O}'$ which satisfies that each g_n is a global equivalence, but does not satisfy that each g_n is a Σ_n–global equivalence. An example of such a morphism is given by the unique morphism from one of the naive global E_∞–operads of Remark 5.9 to the terminal operad Comm.

In that case each $F^c(g)_n$ would be a global equivalence by the 2-out-of-6 property, so $F^c(g)$ induces a Quillen equivalence between $\mathcal{A}lg(F^c(\mathcal{O}))$ and $\mathcal{A}lg(F^c(\mathcal{O}'))$ because

$F^c(\mathcal{O})$ and $F^c(\mathcal{O}')$ are cofibrant operads. The morphisms of operads $\eta(\mathcal{O})$ and $\eta(\mathcal{O}')$ would also induce Quillen equivalences by Theorem 4.14, But this would imply that g induces a Quillen equivalence between the categories of algebras, which contradicts the only if part of Theorem 4.14.

This means that in order to study the genuine homotopy theory of algebras over operads in $(\underline{\mathcal{S}pc}, \boxtimes)$, we cannot restrict ourselves to looking only at cofibrant operads.

5 Global E_∞–operads

Let Comm be the terminal operad in $(\underline{\mathcal{S}pc}, \boxtimes)$, where each Comm_n is the constant one-point orthogonal space. Algebras over Comm are precisely the commutative monoids in $\underline{\mathcal{S}pc}$ with respect to the box product, which are called commutative orthogonal monoid spaces or ultracommutative monoids in [22, Definition 1.4.14]. The unit and multiplication maps imply that a commutative monoid in $(\underline{\mathcal{S}pc}, \boxtimes)$ is precisely a lax symmetric monoidal functor $(\underline{L}, \oplus) \to (\underline{\mathrm{Top}}, \times)$.

Definition 5.1 A *global E_∞–operad* in $(\underline{\mathcal{S}pc}, \boxtimes)$ is an operad $\mathcal{O}$ in $(\underline{\mathcal{S}pc}, \boxtimes)$ such that each $\mathcal{O}_n$ is Σ_n–globally equivalent to $*$ with the trivial Σ_n–action.

Remark 5.2 By Theorem 4.14, if $\mathcal{O}$ is a global E_∞–operad in $(\underline{\mathcal{S}pc}, \boxtimes)$ and g is the unique morphism of operads $g\colon \mathcal{O} \to \mathrm{Comm}$, then the induced Quillen adjunction $(g_!, g^*)$ is a Quillen equivalence between $\underline{\mathcal{A}lg}(\mathcal{O})$ and $\underline{\mathcal{A}lg}(\mathrm{Comm})$, the category of ultracommutative monoids. This justifies why we gave the previous definition of a global E_∞–operad.

Furthermore, the algebras over a global E_∞–operad are endowed with plenty of additional structure, just like ultracommutative monoids. It is also relatively simple to characterize when a given operad in $\underline{\mathcal{S}pc}$ (like the ones constructed in Section 2.3) is a global E_∞–operad.

Proposition 5.3 *Let $\mathcal{O}$ be a global E_∞–operad in $(\underline{\mathcal{S}pc}, \boxtimes)$, and let $g\colon \mathcal{O} \to$ Comm be the unique morphism of operads. There is a homotopical functor*

$$R\colon \underline{\mathcal{A}lg}(\mathcal{O}) \to \underline{\mathcal{A}lg}(\mathrm{Comm})$$

and a zigzag of natural weak equivalences between $g^* \circ R$ and the identity on $\underline{\mathcal{A}lg}(\mathcal{O})$.

For $A \in \underline{\mathcal{A}lg}(\mathcal{O})$, $R(A)$ is an ultracommutative monoid, thus R is a functor that rectifies algebras over global E_∞–operads into ultracommutative monoids.

Proof Let $C\colon \underline{\mathcal{Alg}}(\mathcal{O}) \to \underline{\mathcal{Alg}}(\mathcal{O})$ be a cofibrant replacement functor in $\underline{\mathcal{Alg}}(\mathcal{O})$ constructed via the small object argument, and let $\alpha\colon C \Rightarrow \mathrm{id}_{\underline{\mathcal{Alg}}(\mathcal{O})}$ be the associated natural weak equivalence. Then $U_{\underline{\mathcal{Alg}}(\mathcal{O})}(\alpha_A)$ is a global equivalence for each $A \in \underline{\mathcal{Alg}}(\mathcal{O})$. Furthermore, the adjunction unit for $C(A)$, the morphism $\eta_{C(A)}\colon C(A) \to g^*(g_!(C(A)))$, is a global equivalence in $\underline{\mathcal{Spc}}$ because the right adjoint g^* preserves and reflects weak equivalences; see [8, Lemma 3.3]. Then $R = g_! \circ C$ is the desired functor, and α and η form the desired zigzag of natural weak equivalences. □

Lemma 5.4 *The operads $\mathcal{LD}$ and $\mathcal{K}$ constructed in Examples 2.11 and 2.12 respectively are reduced ($\mathcal{LD}_0 = \mathcal{K}_0 = *$). For each $n \geq 0$, the orthogonal spaces $\mathcal{LD}_n$ and $\mathcal{K}_n$ are closed, and for each $V \in \underline{L}$, the Σ_n–spaces $\mathcal{LD}_n(V)$ and $\mathcal{K}_n(V)$ are Σ_n–free and Hausdorff.*

Proof This follows from the properties of the little disks operads $\mathcal{LD}(V)$ and Steiner operads $\mathcal{K}(V)$ for an inner product space V. By construction they are reduced, and for each $n \geq 0$ they are Σ_n–free and Hausdorff, so the same is true for $\mathcal{LD}$ and $\mathcal{K}$. For a linear isometric embedding $\psi\colon V \to W$, the maps $\mathcal{LD}_n(\psi)$ and $\mathcal{K}_n(\psi)$ are closed embeddings, so the operads $\mathcal{LD}$ and $\mathcal{K}$ are closed. □

We now give several examples of global E_∞–operads. To check that $\mathcal{LD}$ and $\mathcal{K}$ are global E_∞–operads we first need the following technical lemma.

Lemma 5.5 *Let K be a compact Lie group, $\mathcal{U}_K$ a K–universe (not necessarily complete), $L \leq K$, and T an L–set. Let $\mathrm{Conf}^L_T(\mathcal{U}_K)$ denote the space of L–equivariant T–configurations in $\mathcal{U}_K$, that is, L–equivariant embeddings of T in $\mathcal{U}_K$. Then $\mathrm{Conf}^L_T(\mathcal{U}_K)$ is either empty or contractible.*

Proof Decompose $\mathcal{U}_K$ as

$$\mathcal{U}_K \cong \bigoplus_{\lambda\in\Lambda}\bigoplus_{n\in\mathbb{N}} \lambda \cong \bigoplus_{n\in\mathbb{N}}\bigoplus_{\lambda\in\Lambda} \lambda = \bigoplus_{n\in\mathbb{N}} U_n,$$

where Λ is a set of finite-dimensional irreducible K–representations.

Let P be the linear isometric embedding

$$\bigoplus_{n\in\mathbb{N}} U_n \to \bigoplus_{n\in\mathbb{N}} U_n, \quad (u_0, u_1, \dots) \mapsto (0, u_0, u_1, \dots).$$

Then P is a K–equivariant nonsurjective linear isometric embedding.

We give a homotopy H between the identity and

$$P\circ -\colon \mathrm{Conf}^L_T(\mathcal{U}_K)\to \mathrm{Conf}^L_T(\mathcal{U}_K),$$

the map given by postcomposition with P. For each $s\in[0,1]$, $f\in \mathrm{Conf}^L_T(\mathcal{U}_K)$ and $t\in T$ this homotopy H is given by

$$H_s(f)(t)=(1-s)(f(t))+s(P(f(t))).$$

It is easy to see that each $H_s(f)$ is L–equivariant. If $H_s(f)(t)=H_s(f)(t')$, arguing coordinatewise we see that $t=t'$.

Now assume that $\mathrm{Conf}^L_T(\mathcal{U}_K)$ is nonempty, so that $f_0\in \mathrm{Conf}^L_T(\mathcal{U}_K)$. There is an $m\geq 0$ such that the image of f_0 is contained in $\bigoplus_{n\leq m} U_n$. Then

$$H'_s(f)(t)=s(P^{\circ m+1}(f(t)))+(1-s)(f_0(t))$$

gives a homotopy between the constant map with value f_0 and the map

$$P^{\circ m+1}\circ -\colon \mathrm{Conf}^L_T(\mathcal{U}_K)\to \mathrm{Conf}^L_T(\mathcal{U}_K).$$

We can easily see that $H'_s(f)$ is L–equivariant, and we can check that it is an embedding by looking at the projection to $\bigoplus_{n\leq m} U_n$ and to its orthogonal complement separately.

With H we can obtain a homotopy from the identity to $P^{\circ m+1}\circ -$, and combining that homotopy with H' we obtain that $\mathrm{Conf}^L_T(\mathcal{U}_K)$ is contractible. $\square$

Proposition 5.6 *The operads $\mathcal{LD}$ and $\mathcal{K}$ in $(\underline{\mathcal{S}pc},\boxtimes)$ are global E_∞–operads.*

Proof Both operads $\mathcal{LD}$ and $\mathcal{K}$ are closed by Lemma 5.4. For each compact Lie group K and each K–representation V, the $(K\times\Sigma_n)$–spaces $\mathcal{LD}(V)_n$ and $\mathcal{K}(V)_n$ are $(K\times\Sigma_n)$–homotopy equivalent to $\mathrm{Conf}_n(V)$, the configuration space of n points in V, where K acts on V and Σ_n acts by permuting the points. This is [11, Lemma 1.2] and [11, Proposition 1.5] respectively.[2]

We have that

$$\mathrm{Conf}_n(\mathcal{U}_K)\cong \operatorname*{colim}_{V\in s(\mathcal{U}_K)} \mathrm{Conf}_n(V).$$

Consider any graph subgroup $\Gamma_\phi\in\mathcal{F}(K,\Sigma_n)$, with $\phi\colon H\to\Sigma_n$ and $H\leq K$. Let T_ϕ be the set with n elements and an H–action given by ϕ. Since taking fixed points

[2] In the proof of [11, Lemma 1.2], one has to add a small condition to ensure that the little disks are contained in the unit disk and that the constructed maps are well defined.

commutes with filtered colimits along closed embeddings, and the poset $s(\mathcal{U}_K)$ has a cofinal subsequence, $\mathcal{LD}_n(\mathcal{U}_K)^{\Gamma_\phi}$ and $\mathcal{K}_n(\mathcal{U}_K)^{\Gamma_\phi}$ are weakly homotopy equivalent to

$$\mathrm{Conf}_n(\mathcal{U}_K)^{\Gamma_\phi} \cong \mathrm{Conf}^H_{T_\phi}(\mathcal{U}_K),$$

where $\mathrm{Conf}^H_{T_\phi}(\mathcal{U}_K)$ is the space of H–equivariant T_ϕ–configurations in $\mathcal{U}_K$, that is, H–equivariant embeddings of T_ϕ in $\mathcal{U}_K$. Since $\mathcal{U}_K$ is a complete universe, $\mathrm{Conf}^H_{T_\phi}(\mathcal{U}_K)$ is nonempty, so by Lemma 5.5 it is contractible. Thus $\mathcal{LD}_n$ and $\mathcal{K}_n$ are Σ_n–globally contractible, and $\mathcal{LD}$ and $\mathcal{K}$ are global E_∞–operads. □

Recall that $L_V = \underline{L}(V,-)$ is the orthogonal space represented by V.

Proposition 5.7 *For any $V \in \underline{L}$ with $V \neq 0$, the endomorphism operad* $\mathrm{End}(L_V)$ *in* $(\underline{\mathcal{Spc}}, \boxtimes)$ *is a global E_∞–operad.*

Proof We have to check that $\mathrm{Hom}(L_V^{\boxtimes n}, L_V) \to *$ is a Σ_n–global equivalence.

Let $\mathcal{U}_K$ be a complete K–universe for K a compact Lie group. Then the underlying K–space of $\mathrm{End}(L_V)_n$ is

$$\begin{aligned}\mathrm{Hom}(L_V^{\boxtimes n}, L_V)(\mathcal{U}_K) &= \operatorname*{colim}_{W\in s(\mathcal{U}_K)} \mathrm{Hom}(L_V^{\boxtimes n}, L_V)(W)\\ &\cong \operatorname*{colim}_{W\in s(\mathcal{U}_K)} \underline{\mathcal{Spc}}(L_W, \mathrm{Hom}(L_V^{\boxtimes n}, L_V))\\ &\cong \operatorname*{colim}_{W\in s(\mathcal{U}_K)} \underline{\mathcal{Spc}}(L_W \boxtimes L_{V^n}, L_V)\\ &\cong \operatorname*{colim}_{W\in s(\mathcal{U}_K)} \underline{L}(V, W\oplus V^n)\\ &\cong \underline{L}(V, \mathcal{U}_K \oplus V^n).\end{aligned}$$

The first three isomorphisms are induced by a chain of isomorphisms for each W, and we have to check that they are natural in W. For the second isomorphism this holds by the naturality of the $\boxtimes$–Hom adjunction, and for the first and third because of the naturality of the enriched Yoneda lemma. By the same reason they are $(K\times\Sigma_n)$–equivariant.

For any $\Gamma \in \mathcal{F}(K,\Sigma_n)$ we consider the Γ–fixed points

$$\mathrm{Hom}(L_V^{\boxtimes n}, L_V)(\mathcal{U}_K)^\Gamma \cong \underline{L}(V, \mathcal{U}_K\oplus V^n)^\Gamma \cong \underline{L}(V, (\mathcal{U}_K\oplus V^n)^\Gamma).$$

Since $\mathcal{U}_K$ is a complete K–universe, $(\mathcal{U}_K\oplus V^n)^{K\times\Sigma_n}$ is infinite-dimensional, and thus so is $(\mathcal{U}_K\oplus V^n)^\Gamma$, so

$$\underline{L}(V, (\mathcal{U}_K\oplus V^n)^\Gamma) \cong \underline{L}(V, \mathbb{R}^\infty) \simeq *. \qquad \square$$

Remark 5.8 For any $V \in \underline{L}$ with $V \neq 0$, the orthogonal space L_V is an algebra over the global E_∞–operad $\mathrm{End}(L_V)$. However, L_V cannot be given the structure of a commutative monoid over the box product (ultracommutative monoid). In particular, $L_V(0) = \underline{L}(V, 0) = \varnothing$, so there are no morphisms of orthogonal spaces from $*$ to L_V, and thus we cannot give it a unit.

Remark 5.9 Let $\mathcal{O}$ be an E_∞–operad in $\underline{\mathrm{Top}}$. This is an operad such that $\mathcal{O}_n$ is Σ_n–free and weakly contractible for each $n \geq 0$. Let $\overline{\mathcal{O}}$ be the constant operad in orthogonal spaces associated to $\mathcal{O}$, which is closed. The space $\overline{\mathcal{O}}_n(\mathcal{U}_K)$ is just $\mathcal{O}_n$ with the trivial K–action, which means that for $n \geq 2$ and a graph subgroup $\Gamma \in \mathcal{F}(K, \Sigma_n)$ not contained in $K \times \{e\}$, we have that $(\overline{\mathcal{O}}_n(\mathcal{U}_K))^\Gamma = \varnothing$. Therefore $\overline{\mathcal{O}}_n$ is not Σ_n–globally equivalent to $*$ for $n \geq 2$, and so the constant operad $\overline{\mathcal{O}}$ is not a global E_∞–operad.

A similar situation occurs in the classical world of equivariant operads. A nonequivariant E_∞–operad given the trivial G–action is not a good example of an E_∞–operad in G–spaces. Instead one wants to look at E_∞–G–operads, the ones for which $\mathcal{O}_n$ is $\mathcal{F}(G, \Sigma_n)$–equivalent to a point; first defined in [16, Definition VII.1.2].

Remark 5.10 In the G–equivariant setting, there is a hierarchy of nonequivalent operads between a nonequivariant operad given the trivial G–action and an E_∞–G–operad. These in-between operads are called *N_∞–operads*, and were introduced in [4]. They codify various levels of commutativity, by imposing the existence of certain additive transfers/multiplicative norms, which exist for commutative monoids in G–spaces and commutative G–ring spectra respectively.

In the global setting, there is also a hierarchy of operads between the naive global E_∞–operads of Remark 5.9 and the global E_∞–operads. These operads in orthogonal spaces are the global analogs of N_∞–operads. A classification of them will appear in a separate article [1].

Appendix More about G–orthogonal spaces

We now construct the G–global model structure on the category of G–orthogonal spaces, for G a compact Lie group. The process is the same as the one used in [22, Section 1.2] to construct the global model structure on $\underline{\mathcal{S}pc}$, and most of the proofs are almost identical. We first construct a level model structure on $\underline{G\mathcal{S}pc}$ applying the results from [22, Appendix C], and then we consider the left Bousfield localization with the

G–global equivalences as the weak equivalences. We include here several technical results needed to construct this G–global model structure, or used in the main part of this paper.

In this appendix we assume the definitions and results of Section 3.

A.1 G–level model structure

For each compact Lie group G there is an isomorphism of $\underline{\mathrm{Top}}$–enriched categories

$$\underline{\mathrm{Fun}}(\underline{L} \times \underline{G}, \underline{\mathrm{Top}}) \cong \underline{\mathrm{Fun}}(\underline{G}, \underline{\mathrm{Fun}}(\underline{L}, \underline{\mathrm{Top}})) = \underline{G}\text{–}\underline{\mathcal{S}pc}.$$

We have that $\mathfrak{D} = \underline{L} \times \underline{G}$ is a skeletally small $\underline{\mathrm{Top}}$–enriched category. On $\mathfrak{D}$ we have a dimension function on the objects $|-|$ given by the dimension of the inner product space of $\underline{L}$. This function satisfies that if $|V| > |W|$ then $\mathfrak{D}(V, W) = \varnothing$ and if $|V| = |W|$ then V and W are isomorphic in $\mathfrak{D}$. We need to choose for each $m \geq 0$ an object of $\mathfrak{D}$ of dimension m, and we pick $\mathbb{R}^m$.

As input to obtain the G–level model structure, we have to consider for each $m \geq 0$ a model structure on the category of spaces with an action by $\mathfrak{D}(\mathbb{R}^m, \mathbb{R}^m) = O(m) \times G$. We take the model structure given by the graph subgroups, the $\mathcal{F}(O(m), G)$–projective model structure, where a morphism f of $\underline{(O(m) \times G)\mathrm{Top}}$ is a weak equivalence (resp. a fibration) if and only if f^Γ is a weak homotopy equivalence (resp. a Serre fibration) for each $\Gamma \in \mathcal{F}(O(m), G)$. Recall that $\mathcal{F}(O(m), G)$ is the set of graph subgroups of $O(m) \times G$, those $\Gamma \in O(m) \times G$ such that $\Gamma \cap (\{e_{O(m)}\} \times G) = \{e_{O(m) \times G}\}$

This $\mathcal{F}(O(m), G)$–projective model structure is proper, cofibrantly generated, and topological. See for example [22, Proposition B.7] for the construction. We call the weak equivalences of this model structure the $\mathcal{F}(O(m), G)$–equivalences, and we do the same for the fibrations, acyclic fibrations, cofibrations and acyclic cofibrations. We can use the results from [22, Appendix C] to construct a level model structure on $\underline{G\mathcal{S}pc}$ based on the graph subgroups. For this, the following consistency condition, described in [22, Definition C.22], has to be satisfied.

Lemma A.1 (consistency condition) *For each $m, n \geq 0$ and each $\mathcal{F}(O(m), G)$–acyclic cofibration i, the morphism*

$$(\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G) \times_{O(m) \times G} i$$

is an $\mathcal{F}(O(m+n), G)$–acyclic cofibration.

Proof The functor

$$(\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G) \times_{O(m)\times G} -$$

is a left adjoint to the functor given by

$$\mathrm{Map}(\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G, -)^{O(m+n)\times G}.$$

Therefore we only need to check that it sends the generating acyclic cofibrations to acyclic cofibrations.

The generating acyclic cofibrations of the $\mathscr{F}(O(m), G)$–projective model structure are of the form $((O(m) \times G)/\Gamma) \times j_l$, for $\Gamma \in \mathscr{F}(O(m), G)$ and $l \geq 0$. Then the functor mentioned at the beginning takes this generating acyclic cofibration to

$$((\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G)/\Gamma) \times j_l.$$

The left G–action on G is free, and because Γ is a graph subgroup and the $O(m)$–action on $\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n})$ is free, the left G–action on $(\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G)/\Gamma$ is also free. We consider now $\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G$ as an $(O(m+n)\times G\times O(m)\times G)$–space, where the component $O(m+n) \times G$ acts on the left, and $O(m) \times G$ originally acts on the right so we precompose with $(-)^{-1}$ to obtain a left action. The space $\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G$ is a smooth $(O(m+n)\times G\times O(m)\times G)$–manifold. Illman's theorem [14, 7.2] provides an $(O(m+n)\times G\times O(m)\times G)$–equivariant triangulation, so $\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G$ is cofibrant in the projective model structure with respect to all subgroups of $O(m+n) \times G \times O(m) \times G$.

By [22, B.14(i), (iii)], $(\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n})\times G)/\Gamma$ is cofibrant in the projective model structure with respect to all subgroups of $O(m+n)\times G$. This in particular means that it is a retract of a generalized $(O(m+n)\times G)$–CW–complex. Each cell $(O(m+n)\times G)/\Delta\times D^{l'}$ for a subgroup $\Delta \leq O(m+n) \times G$ and $l' \geq 0$ that appears in this equivariant CW–structure induces a $(O(m+n)\times G)$–equivariant map

$$f\colon (O(m+n) \times G)/\Delta \times D^{l'} \to (\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G)/\Gamma.$$

Since the target of f is G–free, so is the source; thus Δ is a graph subgroup. As only graph subgroups can appear in this CW–structure, $(\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G)/\Gamma$ is also cofibrant in the $\mathscr{F}(O(m+n), G)$–projective model structure. Recall that the $\mathscr{F}(O(m+n), G)$–projective model structure is topological. Therefore, the morphism $((\underline{L}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G)/\Gamma) \times j_l$ is the product of a cofibrant object with an acyclic cofibration of $\underline{\mathrm{Top}}$, so it is an acyclic cofibration. $\square$

Theorem A.2 (G–level model structure) *There is a topological cofibrantly generated model structure on the category $\underline{G\mathcal{S}pc}$ of G–orthogonal spaces, which we call the G–level model structure. The weak equivalences (resp. the fibrations) are those morphisms f such that for each $m \geq 0$ and each graph subgroup $\Gamma \in \mathcal{F}(O(m), G)$, the map $f(\mathbb{R}^m)^\Gamma$ is a weak homotopy equivalence (resp. a Serre fibration). We call these respectively the **G–level equivalences** and the **G–level fibrations**.*

A set of generating cofibrations is

$$\mathsf{I}_G = \{((\underline{L}(\mathbb{R}^m, -) \times G)/\,\Gamma) \times i_l \mid m, l \geq 0,\ \Gamma \in \mathcal{F}(O(m), G)\}.$$

A set of generating acyclic cofibrations is

$$\mathsf{J}_G = \{((\underline{L}(\mathbb{R}^m, -) \times G)/\,\Gamma) \times j_l \mid m, l \geq 0, \Gamma \in \mathcal{F}(O(m), G)\}.$$

*We call the cofibrations of this model structure the **G–flat cofibrations**.*

Proof Such a model structure exists by [22, Proposition C.23(i)]. It is cofibrantly generated by [22, Proposition C.23(iii)] because each of the chosen model structures on $\underline{(O(m) \times G)\mathrm{Top}}$ is cofibrantly generated.

The functor

$$(-)(\mathbb{R}^m) \colon \underline{G\mathcal{S}pc} \to \underline{(O(m) \times G)\mathrm{Top}}$$

given by evaluation at $\mathbb{R}^m$ has a left adjoint, which we denote by F_m, and it is given by

$$F_m(A) = (\underline{L}(\mathbb{R}^m, -) \times G) \times_{O(m)\times G} A.$$

The generating cofibrations obtained from [22, Proposition C.23(iii)] are those of the form $F_m(i)$ where i is a generating cofibration of $\underline{(O(m) \times G)\mathrm{Top}}$, which are of the form $((O(m) \times G)/\,\Gamma) \times i_l$ for $\Gamma \in \mathcal{F}(O(m), G)$ and $l \geq 0$. Similarly the generating acyclic cofibrations are of the form $F_m(j)$ for j a generating acyclic cofibration of $\underline{(O(m) \times G)\mathrm{Top}}$.

Each G–orthogonal space of the form $(\underline{L}(\mathbb{R}^m, -) \times G)/\,\Gamma$ is cofibrant in this G–level model structure, because $F_m(((O(m) \times G)/\,\Gamma) \times i_0)$ is a generating cofibration. Using [22, Proposition B.5] we obtain that this model structure is topological, taking $\mathcal{G}$ and $\mathcal{Z}$ in the statement of that proposition to be

$$\mathcal{G} = \{(\underline{L}(\mathbb{R}^m, -) \times G)/\,\Gamma \mid m \geq 0,\ \Gamma \in \mathcal{F}(O(m), G)\} \quad \text{and} \quad \mathcal{Z} = \varnothing. \qquad \square$$

Note that we should call this model structure on $\underline{G\mathcal{S}pc}$ the "G–graph level model structure" to distinguish it from other possible model structures on $\underline{G\mathcal{S}pc}$. In particular, there is at least the level model structure that we would obtain by considering all subgroups of $O(m) \times G$, and not just the graph subgroups. There is also a projective model structure on $\underline{\mathrm{Fun}}(\underline{G}, \underline{\mathcal{S}pc})$. However since we never talk about these two other model structures on $\underline{G\mathcal{S}pc}$, we omit the adjective "graph" everywhere.

Lemma A.3 *If $f : X \to Y$ is a G–level equivalence then for any compact Lie group K and any faithful K–representation V, the map $f(V)$ is an $\mathcal{F}(K, G)$–equivalence. In particular, f is also a G–global equivalence.*

Proof As a finite-dimensional inner product space, V is isomorphic to $\mathbb{R}^l$ for some $l \geq 0$. Let $\Gamma \in \mathcal{F}(K, G)$ be a graph subgroup. Then the image of Γ under the homomorphism $K \times G \to O(\mathbb{R}^l) \times G$ induced by said isomorphism is a graph subgroup Γ'. Then $X(V)^\Gamma$ is naturally (on X) homeomorphic to $X(\mathbb{R}^l)^{\Gamma'}$. Since $f(\mathbb{R}^l)^{\Gamma'}$ is a weak homotopy equivalence, so is $f(V)^\Gamma$. □

Remark A.4 For an inner product space V and a closed subgroup $H \leq O(V) \times G$, the G–orthogonal space

$$\mathcal{D}(V, -)/H = (\underline{L}(V, -) \times G)/H,$$

which we denote by $L_{H,V;G}$, is special. It has a certain freeness condition, namely it is the representing object for the functor $(-)(V)^H$ given by evaluating at V and then taking H–fixed points. We refer to these as the *semifree G–orthogonal spaces*, since they have the same property as the semifree orthogonal spaces $L_{H,V}$.

Explicitly the natural isomorphism between the functors

$$\underline{G\mathcal{S}pc}(L_{H,V;G}, -), (-)(V)^H : \underline{G\mathcal{S}pc} \to \underline{\mathrm{Top}}$$

is given by $f \mapsto f(V)([\mathrm{id}_V, e])$. The inverse isomorphism is given on $Y \in \underline{G\mathcal{S}pc}$ by sending a point $y_0 \in Y(V)^H$ to the morphism of G–orthogonal spaces f given by

$$(\underline{L}(V, W) \times G)/H \to Y(W), \quad [\psi, g] \mapsto Y(\psi)(gy_0).$$

Analogously to the case of the semifree orthogonal spaces, the box product of a semifree G–orthogonal space and a semifree K–orthogonal space has a nice structure. As a $(G\times K)$–orthogonal space it is isomorphic to a semifree $(G\times K)$–orthogonal space, and this can be deduced from the result for orthogonal spaces. Note however that the box product of two semifree G–orthogonal spaces with the G–action given by restriction along the diagonal is not a semifree G–orthogonal space in general.

Lemma A.5 *For compact Lie groups G and K, inner product spaces V and V', and closed subgroups $H \leq O(V) \times G$ and $H' \leq O(V') \times K$, $L_{H,V;G} \boxtimes L_{H',V';K}$ is isomorphic as a $(G\times K)$–orthogonal space to $L_{H\times H',V\oplus V';G\times K}$.*

Proof Since the box product preserves colimits in both variables, we have that

$$\begin{aligned}(\underline{L}(V,-)\times G)/H\boxtimes(\underline{L}(V',-)\times K)/H' &\cong ((\underline{L}(V,-)\times G)\boxtimes(\underline{L}(V',-)\times K))/(H\times H') \\ &\cong (\underline{L}(V\oplus V',-)\times G\times K)/(H\times H').\end{aligned}$$

Here we also used the isomorphism $\underline{L}(V,-)\boxtimes \underline{L}(V',-)\cong \underline{L}(V\oplus V',-)$ from [22, Example 1.3.3] and its naturality on V and V'. □

Lemma A.6 *The pushout product of a G–flat cofibration (recall that these are the cofibrations of the G–level model structure) and a K–flat cofibration is a $(G\times K)$–flat cofibration.*

Proof Given a generating G–flat cofibration $f = L_{\Gamma,\mathbb{R}^m;G}\times i_l$ and a generating K–flat cofibration $g = L_{\Gamma',\mathbb{R}^n;K}\times i_k$, their pushout product is by Lemma A.5 isomorphic to

$$L_{\Gamma\times\Gamma',\mathbb{R}^{m+n};G\times K}\times(i_l \mathbin{\square} i_k)$$

as a morphism of $(G\times K)$–orthogonal spaces. The subgroup

$$\Gamma\times\Gamma' \leq O(m)\times O(n)\times G\times K \leq O(m+n)\times G\times K$$

is a graph subgroup because both Γ and Γ' are graph subgroups. Additionally $i_l \mathbin{\square} i_k$ is homeomorphic to i_{l+k}, and so $f \mathbin{\square} g$ is a generating $(G\times K)$–flat cofibration.

Since the box product of orthogonal spaces is closed, [13, Lemma 4.2.4] implies that the pushout product of a G–flat cofibration and a K–flat cofibration is a $(G\times K)$–flat cofibration. □

Lemma A.7 *The G–level model structure is proper.*

Proof First we check right properness. Consider the pullback diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] & Y \end{tikzcd}$$

where f is a G–level fibration and h is a G–level equivalence. Let $m \leq 0$. Evaluating at $\mathbb{R}^m$ yields a diagram of $(O(m)\times G)$–spaces, which is a pullback diagram because limits of G–orthogonal spaces and $(O(m)\times G)$–spaces are computed in $\underline{\text{Top}}$. Then

$f(\mathbb{R}^m)$ is an $\mathcal{F}(O(m),G)$–fibration and $h(\mathbb{R}^m)$ is an $\mathcal{F}(O(m),G)$–equivalence, and since the $\mathcal{F}(O(m),G)$–projective model structure is right proper by [22, B.7], $g(\mathbb{R}^m)$ is also an $\mathcal{F}(O(m),G)$–equivalence. Thus g is a G–level equivalence.

To check left properness one can use the dual argument. We additionally need to use that if a morphism f of G–orthogonal spaces is a G–flat cofibration, then it is a G–h–cofibration (see Lemma 3.15), which means that each $f(\mathbb{R}^m)$ is an h–cofibration of $(O(m)\times G)$–spaces, and then we need to use the gluing lemma [22, B.6]. □

A.2 G–h–cofibrations and G–global equivalences

We now check that G–global equivalences are preserved by various constructions along G–h–cofibrations. We use these results to finish the construction of the G–global model structure, and in the main part of this paper.

Lemma A.8 (gluing lemma) *Given a commutative diagram of G–orthogonal spaces*

$$\begin{tikzcd} Y \arrow[d, "\beta"] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\alpha"] & Z \arrow[d, "\gamma"] \\ Y' & X' \arrow[l, "f'"'] \arrow[r, "g'"] & Z' \end{tikzcd}$$

where α, β and γ are G–global equivalences, and f and f' are G–h–cofibrations, the morphism induced on the pushouts $Y\cup_X Z\to Y'\cup_{X'} Z'$ is a G–global equivalence.

Proof Consider a compact Lie group K and an exhaustive sequence of K–representations $\{V_i\}_{i\in\mathbb{N}}$. We have the following diagram of equivariant morphisms of $(K\times G)$–spaces:

$$\begin{tikzcd} \mathrm{tel}_i\, Y(V_i) \arrow[d, "\mathrm{tel}_i\, \beta(V_i)"] & \mathrm{tel}_i\, X(V_i) \arrow[l, "\mathrm{tel}_i\, f(V_i)"'] \arrow[r, "\mathrm{tel}_i\, g(V_i)"] \arrow[d, "\mathrm{tel}_i\, \alpha(V_i)"] & \mathrm{tel}_i\, Z(V_i) \arrow[d, "\mathrm{tel}_i\, \gamma(V_i)"] \\ \mathrm{tel}_i\, Y'(V_i) & \mathrm{tel}_i\, X'(V_i) \arrow[l, "\mathrm{tel}_i\, f'(V_i)"'] \arrow[r, "\mathrm{tel}_i\, g'(V_i)"] & \mathrm{tel}_i\, Z'(V_i) \end{tikzcd}$$

Here by Proposition 3.4 $\mathrm{tel}_i\,\alpha(V_i)$, $\mathrm{tel}_i\,\beta(V_i)$ and $\mathrm{tel}_i\,\gamma(V_i)$ are $\mathcal{F}(K,G)$–equivalences, and the formation of mapping telescopes commutes with pushouts, retracts and $-\times[0,1]$, so $\mathrm{tel}_i\, f(V_i)$ and $\mathrm{tel}_i\, f'(V_i)$ are h–cofibrations of $(K\times G)$–spaces. Therefore by the gluing lemma for $\mathcal{F}(K,G)$–equivalences (see for example [22, Proposition B.6]) the induced map on the pushouts of the mapping telescopes is also an $\mathcal{F}(K,G)$–equivalence. Since again taking mapping telescopes commutes with pushouts, this means that $Y\cup_X Z\to Y'\cup_{X'} Z'$ is a G–global equivalence. □

Corollary A.9 *For a pushout diagram of G–orthogonal spaces*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d] \\ X' \arrow[r, "f'"] \arrow[ur, phantom, "\ulcorner"] & Y' \end{tikzcd}$$

where f is a G–global equivalence and either f or g is a G–h–cofibration, f' is a G–global equivalence.

Proof Apply the previous lemma to the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, "g"] \arrow[d, equal] & X' \arrow[d, equal] \\ Y & X \arrow[l, "f"'] \arrow[r, "g"] & X' \end{tikzcd}$$

□

Corollary A.10 *For morphisms of G–orthogonal spaces $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ such that f is a G–global equivalence and either f or g is a G–h–cofibration, their pushout product $f \square g$ is a G–global equivalence.*

Similarly, assume instead that $f\colon X_1 \to Y_1$ is a morphism of G–orthogonal spaces and $g\colon X_2 \to Y_2$ is a map of G–spaces. If either f is a G–global equivalence or g is a G–equivalence, and either f or g is a G–h–cofibration, their pushout product $f \square g$ is a G–global equivalence.

Proof By Lemma 3.7(iv) $f \boxtimes X_2$ and $f \boxtimes Y_2$ are G–global equivalences. Depending on the hypothesis, either $f \boxtimes X_2$ or $X_1 \boxtimes g$ is a G–h–cofibration, so by Corollary A.9 the morphism α is a G–global equivalence, and by the 2-out-of-6 property so is $f \square g$:

$$\begin{tikzcd} X_1 \boxtimes X_2 \arrow[r, "X_1 \boxtimes g"] \arrow[d, "f \boxtimes X_2"'] & X_1 \boxtimes Y_2 \arrow[d, "\alpha"] \arrow[ddr, bend left, "f \boxtimes Y_2"] & \\ Y_1 \boxtimes X_2 \arrow[r] \arrow[drr, bend right, "Y_1 \boxtimes g"'] \arrow[ur, phantom, "\ulcorner"] & P \arrow[dr, dashed, "f \square g"] & \\ & & Y_1 \boxtimes Y_2 \end{tikzcd}$$

The same is true if g is a map of G–spaces, since the product of an orthogonal space with a space is the same as the box product with the associated constant orthogonal space. A G–equivalence between constant orthogonal spaces is a G–global equivalence, and similarly a G–h–cofibration of spaces is a G–h–cofibration between constant orthogonal spaces. □

Lemma A.11 *For a limit ordinal λ, consider two λ–sequences in $\underline{G\mathcal{S}pc}$, which are colimit preserving functors $X\colon \lambda \to \underline{G\mathcal{S}pc}$ and $Y\colon \lambda \to \underline{G\mathcal{S}pc}$, and a natural transformation f between them. If for each $\beta \in \lambda$ the morphisms $g_\beta\colon X_\beta \to X_{\beta+1}$ and $h_\beta\colon Y_\beta \to Y_{\beta+1}$ are G–h–cofibrations and the morphism $f_\beta\colon X_\beta \to Y_\beta$ is a G–global equivalence, the morphism induced on the colimits,*

$$\operatorname*{colim}_{\beta\in\lambda} f_\beta\colon \operatorname*{colim}_{\beta\in\lambda} X_\beta \to \operatorname*{colim}_{\beta\in\lambda} Y_\beta,$$

is a G–global equivalence.

Proof By Proposition 3.4 it is enough to check that for each compact Lie group K and exhaustive sequence of K–representations $\{V_i\}_{i\in I}$, the map $\mathrm{tel}_i(\mathrm{colim}_{\beta\in\lambda} f_\beta)(V_i)$ is an $\mathcal{F}(K,G)$–equivalence. The construction of the mapping telescopes commutes with taking colimits, so this map is isomorphic to $\mathrm{colim}_{\beta\in\lambda}(\mathrm{tel}_i\ f_\beta(V_i))$.

For each $\beta \in \lambda$ the map $\mathrm{tel}_i\ f_\beta(V_i)$ is an $\mathcal{F}(K,G)$–equivalence, and the maps $\mathrm{tel}_i\ g_\beta(V_i)$ and $\mathrm{tel}_i\ h_\beta(V_i)$ are h–cofibrations of $(K\times G)$–spaces, and so in particular h–cofibrations of underlying compactly generated weak Hausdorff spaces, and therefore closed embeddings.

For each $\Gamma \in \mathcal{F}(K,G)$ taking Γ–fixed points commutes with filtered colimits along closed embeddings; see [22, Proposition B.1(ii)]. Colimits with the shape of a filtered poset and built out of closed embeddings of compactly generated weak Hausdorff spaces can be computed in the category of all topological spaces; see [22, Proposition A.14(ii)]. Weak Hausdorff spaces are T_1, so by [13, Proposition 2.4.2] maps from compact spaces (∂D^l and D^l in this case) into the colimit of a λ–sequence of closed embeddings (for λ a limit ordinal) factor through some stage $\beta \in \lambda$. Therefore compact spaces are finite in $\underline{\mathrm{Top}}$ relative closed embeddings.

This implies that, for the λ–sequences given by $(\mathrm{tel}_i\ g_\beta(V_i))^\Gamma$ and $(\mathrm{tel}_i\ h_\beta(V_i))^\Gamma$, which consist of closed embeddings, and the natural transformation between them given by the maps $(\mathrm{tel}_i\ f_\beta(V_i))^\Gamma$ which are weak homotopy equivalences, the map induced on the colimits

$$\operatorname*{colim}_{\beta\in\lambda}(\mathrm{tel}_i\ f_\beta(V_i))^\Gamma \cong \big(\operatorname*{colim}_{\beta\in\lambda}(\mathrm{tel}_i\ f_\beta(V_i))\big)^\Gamma$$

is a weak homotopy equivalence. Therefore $\mathrm{tel}_i(\mathrm{colim}_{\beta\in\lambda} f_\beta)(V_i)$ is an $\mathcal{F}(K,G)$–equivalence. □

Corollary A.12 *A transfinite composition of morphisms in $\underline{G\mathcal{S}pc}$ that are G–h–cofibrations and G–global equivalences is a G–global equivalence.*

Proof We check this via transfinite induction on the ordinal λ. Let $Y: \lambda \to \underline{G\mathcal{S}pc}$ be a λ–sequence such that for each $\beta \in \lambda$ the morphism $h_\beta: Y_\beta \to Y_{\beta+1}$ is a G–h–cofibration and a G–global equivalence. The base case and the case where λ is a successor ordinal hold because composition of two G–global equivalences is a G–global equivalence.

If λ is a limit ordinal, set $X: \lambda \to \underline{G\mathcal{S}pc}$ as the constant functor $X_\beta = Y_0$. Define a natural transformation $f: X \Rightarrow Y$ by letting f_β be the morphism $Y_0 \to Y_\beta$. This is the transfinite composition of Y restricted to $\beta + 1$. Then by the induction hypothesis f_β is a G–global equivalence for each $\beta \in \lambda$. Then we use Lemma A.11 to obtain that $\operatorname{colim}_{\beta\in\lambda} f_\beta$ is a G–global equivalence, but this morphism is precisely the transfinite composition of Y. □

A.3 G–global model structure

We now go back to constructing the G–global model structure, starting with the fibrations.

Definition A.13 (G–global fibration) A morphism of G–orthogonal spaces $f: X \to Y$ is a *G–global fibration* if it is a G–level fibration, and for each compact Lie group K, every graph subgroup $\Gamma \in \mathcal{F}(K,G)$, and every linear isometric embedding of K–representations $\psi: V \to W$ with V faithful, the induced map

$$X(V)^\Gamma \to Y(V)^\Gamma \times_{Y(W)^\Gamma} X(W)^\Gamma$$

is a weak homotopy equivalence. Since f is a G–level fibration, and so $f(V)^\Gamma$ and $f(W)^\Gamma$ are Serre fibrations, this is equivalent to the following square being homotopy cartesian:

$$\begin{tikzcd} X(V)^\Gamma \arrow[r, "X(\psi)^\Gamma"] \arrow[d, "f(V)^\Gamma"'] & X(W)^\Gamma \arrow[d, "f(W)^\Gamma"] \\ Y(V)^\Gamma \arrow[r, "Y(\psi)^\Gamma"] & Y(W)^\Gamma \end{tikzcd}$$

Construction A.14 Fix a compact Lie group G. We now construct the set K_G, where $\mathsf{J}_G \cup \mathsf{K}_G$ is a set of generating acyclic cofibrations for the G–global model structure of Theorem A.20. Recall that J_G is the set of generating acyclic cofibrations of the G–level model structure given in Theorem A.2. Let K be a compact Lie group, let V be a faithful K–representation, let W be a K–representation and let $\Gamma \in \mathcal{F}(K,G)$ be a graph subgroup. We consider the restriction morphism of G–orthogonal spaces,

$$\rho_{\Gamma,V,W;G}: L_{\Gamma,V\oplus W;G} = (\underline{L}(V \oplus W, -) \times G)/\Gamma \to (\underline{L}(V,-) \times G)/\Gamma = L_{\Gamma,V;G}.$$

The morphism $\rho_{\Gamma,V,W;G}$ is a G–global equivalence because the semifree G–orthogonal spaces are closed and given a compact Lie group L, the map

$$\rho_{V,W}\colon \underline{L}(V\oplus W,\mathcal{U}_L)\to \underline{L}(V,\mathcal{U}_L)$$

is a $(K\times L)$–homotopy equivalence by [22, 1.1.26(ii)] (recall that $\mathcal{U}_L$ here is a complete L–universe).

Now let κ be a set of representatives of isomorphism classes of triples (K,Γ,V,W) consisting of a compact Lie group K, a faithful K–representation V, a K–representation W, and a graph subgroup $\Gamma\in\mathcal{F}(K,G)$. Let K_G be the set

$$\mathsf{K}_G=\bigcup_{(K,\Gamma,V,W)\in\kappa}\{\iota_{\rho_{\Gamma,V,W;G}}\,\square\, i_l\mid l\geq 0\}.$$

Recall that $\iota_{\rho_{\Gamma,V,W;G}}$ denotes the inclusion of the mapping cylinder

$$L_{\Gamma,V\oplus W;G}\to M_{\rho_{\Gamma,V,W;G}}.$$

Note that here we allow V to be 0. For the generating acyclic cofibrations of the positive global model structure on $\underline{\mathcal{S}pc}$, we do require that $V\neq 0$. If we did that here, in Definition A.13, and in Theorem A.2, we would obtain the *positive G–global model structure*.

Lemma A.15 *Any morphism in $\mathsf{J}_G\cup\mathsf{K}_G$ is a G–global equivalence and a G–flat cofibration. Any morphism obtained from $\mathsf{J}_G\cup\mathsf{K}_G$ by transfinite composition and cobase changes is also a G–global equivalence and a G–flat cofibration.*

Proof Any morphism $f\in\mathsf{J}_G$ is an acyclic cofibration in the G–level model structure, so it is a G–flat cofibration and by Lemma A.3 a G–global equivalence.

Fix a compact Lie group K, a faithful K–representation V, a K–representation W, a graph subgroup $\Gamma\in\mathcal{F}(K,G)$, and $l\geq 0$. Consider $f=\iota_{\rho_{\Gamma,V,W;G}}\,\square\, i_l$ in K_G. We saw in Construction A.14 that $\rho_{\Gamma,V,W;G}$ is a G–global equivalence. The projection $M_{\rho_{\Gamma,V,W;G}}\to L_{\Gamma,V;G}$ from the mapping cylinder of $\rho_{\Gamma,V,W;G}$ to its target is a homotopy equivalence in $G\underline{\mathcal{S}pc}$. Therefore it is a G–level equivalence, and thus a G–global equivalence. By the 2-out-of-6 property $\iota_{\rho_{\Gamma,V,W;G}}$ is also a G–global equivalence.

The G–orthogonal spaces $L_{\Gamma,V\oplus W;G}$ and $L_{\Gamma,V;G}$ are G–flat orthogonal spaces because they are isomorphic to $L_{\Gamma',\mathbb{R}^{n+m};G}$ and $L_{\Gamma'',\mathbb{R}^n;G}$ respectively, for some $n,m\geq 0$, $\Gamma'\in\mathcal{F}(O(n+m),G)$ and $\Gamma''\in\mathcal{F}(O(n),G)$. Then we obtain that

$$L_{\Gamma,V\oplus W;G}\to L_{\Gamma,V\oplus W;G}\amalg L_{\Gamma,V;G}$$

is a G–flat cofibration. Also since the G–level model structure of Theorem A.2 is topological, $L_{\Gamma,V\oplus W;G} \times i_1$ is a G–flat cofibration. Putting this together we obtain that $\iota_{\rho_{\Gamma,V,W;G}}$ is a G–flat cofibration, and again because the G–level model structure is topological so is f. By Corollary A.10, $f = \iota_{\rho_{\Gamma,V,W;G}} \,\square\, i_l$ is a G–global equivalence.

Using the closure properties of Corollaries A.9 and A.12, we obtain the second part of the lemma. $\square$

Lemma A.16 *The sources of all morphisms in I_G, J_G and K_G are finite (and thus small) with respect to the class of maps that are levelwise closed embeddings. Since G–h–cofibrations are levelwise closed embeddings, they are also finite with respect to the class of G–h–cofibrations.*

Proof We first check that for any compact Lie group K, faithful K–representation V, graph subgroup $\Gamma \in \mathscr{F}(K,G)$, and compact space A, the G–orthogonal space $L_{\Gamma,V;G} \times A$ is finite with respect to morphisms which are levelwise closed embeddings.

We recalled in the proof of Lemma A.11 that compact spaces are finite in $\underline{\mathrm{Top}}$ relative closed embeddings. Taking Γ–fixed points commutes with filtered colimits along closed embeddings. Consider a limit ordinal λ, and a λ–sequence $X\colon \lambda \to \underline{G\mathcal{S}pc}$ of levelwise closed embeddings. By the semifreeness property of $L_{\Gamma,V;G} \times A$, and since colimits in $\underline{G\mathcal{S}pc}$ are computed levelwise, we have that

$$\begin{aligned}\underline{G\mathcal{S}pc}\big(L_{\Gamma,V;G} \times A, \operatorname*{colim}_{\beta\in\lambda} X_\beta\big) &\cong \underline{\mathrm{Top}}\big(A, \big(\operatorname*{colim}_{\beta\in\lambda} X_\beta\big)(V)^\Gamma\big)\\ &\cong \underline{\mathrm{Top}}\big(A, \operatorname*{colim}_{\beta\in\lambda}(X_\beta(V)^\Gamma)\big)\\ &\cong \operatorname*{colim}_{\beta\in\lambda} \underline{\mathrm{Top}}(A, (X_\beta(V)^\Gamma))\\ &\cong \operatorname*{colim}_{\beta\in\lambda} \underline{G\mathcal{S}pc}(L_{\Gamma,V;G} \times A, X_\beta).\end{aligned}$$

So for a generating cofibration $i \in \mathsf{I}_G$, its source is of the form $L_{\Gamma,\mathbb{R}^m;G} \times \partial D^l$, so it is finite relative levelwise closed embeddings. Similarly the source of a generating acyclic cofibration $j \in \mathsf{J}_G$ is $L_{\Gamma,\mathbb{R}^m;G} \times D^l$, so it is also finite relative levelwise closed embeddings.

For a generating acyclic cofibration $k = \iota_{\rho_{\Gamma,V,W;G}} \,\square\, i_l$ in K_G, its source is

$$L_{\Gamma,V\oplus W;G} \times D^l \cup_{L_{\Gamma,V\oplus W;G}\times\partial D^l} M_{\rho_{\Gamma,V,W;G}} \times \partial D^l.$$

The G–orthogonal space $M_{\rho_{\Gamma,V,W;G}} \times \partial D^l$ is a finite colimit of objects of the form $L_{\Gamma,V;G} \times A$. Therefore it is also finite relative levelwise closed embeddings, because in $\underline{\mathrm{Set}}$ finite limits commute with filtered colimits. By the same argument, the source of k is also finite relative levelwise closed embeddings.

G–h–cofibrations are levelwise h–cofibrations of spaces, which are closed embeddings in the category of compactly generated weak Hausdorff spaces. Therefore G–h–cofibrations are levelwise closed embeddings. $\square$

Lemma A.17 *A morphism in $\underline{G\mathcal{S}pc}$ is a G–global fibration if and only if it has the right lifting property with respect to $\mathsf{J}_G \cup \mathsf{K}_G$.*

Proof Every linear isometric embedding of K–representations is isomorphic to an embedding of the form $i_V\colon V \to V \oplus W$. Thus Definition A.13 can be altered slightly to say that a morphism f is a G–global fibration if and only if it is a G–level fibration and for each compact Lie group K, graph subgroup $\Gamma \in \mathcal{F}(K, G)$, and K–representations V and W, the square

$$\begin{array}{ccc} X(V)^\Gamma & \xrightarrow{X(i_V)^\Gamma} & X(V\oplus W)^\Gamma \\ {\scriptstyle f(V)^\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle f(V\oplus W)^\Gamma} \\ Y(V)^\Gamma & \xrightarrow{Y(i_V)^\Gamma} & Y(V\oplus W)^\Gamma \end{array}$$

is homotopy cartesian. By Remark A.4, the morphism $\rho_{\Gamma,V,W;G}$ represents the natural transformation

$$(-)(i_V)^\Gamma\colon (-)(V)^\Gamma \Rightarrow (-)(V\oplus W)^\Gamma.$$

By applying [22, Proposition 1.2.16] to the G–level model structure we obtain that the previous square is homotopy cartesian if and only if f has the right lifting property with respect to $\iota_{\rho_{\Gamma,V,W;G}} \mathbin{\square} i_l$ for all $l \geq 0$. The set J_G is a set of generating acyclic cofibrations of the G–level model structure, so a morphism is a G–level fibration if and only if it has the right lifting property with respect to J_G. Therefore a morphism in $\underline{G\mathcal{S}pc}$ is a G–global fibration if and only if it has the right lifting property with respect to $\mathsf{J}_G \cup \mathsf{K}_G$. $\square$

Lemma A.18 *A pullback of a G–global equivalence along a G–level fibration is also a G–global equivalence.*

Proof Consider the pullback square

$$\begin{array}{ccc} P & \xrightarrow{k} & X \\ {\scriptstyle g}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow[h]{} & Y \end{array} \tag{5}$$

where f is a G–global equivalence and h is a G–level fibration. Consider a compact Lie group K, a K–representation V, a graph subgroup $\Gamma \in \mathcal{F}(K,G)$, and a lifting problem given by $\alpha\colon \partial D^l \to P(V)^\Gamma$ and $\beta\colon D^l \to Z(V)^\Gamma$ with $g(V)^\Gamma \circ \alpha = \beta \circ i_l$. Since f is a G–global equivalence, there is a K–representation W, a K–equivariant linear isometric embedding $\psi\colon V \to W$, and a morphism $\lambda\colon D^l \to X(W)^\Gamma$ such that

$$\lambda \circ i_l = X(\psi)^\Gamma \circ k(V)^\Gamma \circ \alpha$$

and there is a ∂D^l–relative homotopy H from $Y(\psi)^\Gamma \circ h(V)^\Gamma \circ \beta$ to $f(W)^\Gamma \circ \lambda$ relative ∂D^l. Since $h(W)^\Gamma$ is a Serre fibration, there is a lift H' in the diagram

$$\begin{tikzcd}
D^l \times \{0\} \cup \partial D^l \times [0,1] \arrow[r, "(Z(\psi)^\Gamma \circ \beta) \cup ((g(W)^\Gamma \circ P(\psi)^\Gamma \circ \alpha) \times [0,1])"] \arrow[d, hook] & Z(W)^\Gamma \arrow[d, "h(W)^\Gamma"] \\
D^l \times [0,1] \arrow[r, "H"] \arrow[ur, dashed, "H'"] & Y(W)^\Gamma
\end{tikzcd}$$

Since the square (5) is a pullback there is a unique $\lambda'\colon D^l \to P(W)^\Gamma$ such that $g(W)^\Gamma \circ \lambda' = H'(-,1)$ and $k(W)^\Gamma \circ \lambda' = \lambda$. Also by the universal property of the pullback (5) we obtain that $\lambda' \circ i_l = P(\psi)^\Gamma \circ \alpha$. Since H' is a homotopy relative ∂D^l between $g(W)^\Gamma \circ \lambda'$ and $Z(\psi)^\Gamma \circ \beta$, we obtain that g is a G–global equivalence. □

Lemma A.19 *If $f\colon X \to Y$ is a G–global equivalence and a G–global fibration then it is also a G–level equivalence.*

Proof Consider $m \geq 0$, a graph subgroup $\Gamma \in \mathcal{F}(O(m), G)$ given by a homomorphism $H \to G$ with $H \leq O(m)$, and a lifting problem of the form

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha"] \arrow[d, hook, "i_l"] & X(\mathbb{R}^m)^\Gamma \arrow[d, "f(\mathbb{R}^m)^\Gamma"] \\
D^l \arrow[r, "\beta"] & Y(\mathbb{R}^m)^\Gamma
\end{tikzcd}$$

Since f is a G–global equivalence, there is an embedding of H–representations $\psi\colon \mathbb{R}^m \to V$ and a map $\lambda\colon D^l \to X(V)^\Gamma$ such that $\lambda \circ i_l = X(\psi)^\Gamma \circ \alpha$ and $f(V)^\Gamma \circ \lambda$ is homotopic relative ∂D^l to $Y(\psi)^\Gamma \circ \beta$.

Since f is a G–level fibration, $f(V)^\Gamma$ is a Serre fibration. By lifting against

$$D^l \times \{0\} \cup \partial D^l \times [0,1] \to D^l \times [0,1],$$

which is a cofibration of spaces, we can replace λ with a λ' such that $\lambda' \circ i_l = X(\psi)^\Gamma \circ \alpha$ and $f(V)^\Gamma \circ \lambda' = Y(\psi)^\Gamma \circ \beta$.

Since f is a G–global fibration,

$$(f(\mathbb{R}^m)^\Gamma, X(\psi)^\Gamma)\colon X(\mathbb{R}^m)^\Gamma \to Y(\mathbb{R}^m)^\Gamma \times_{Y(V)^\Gamma} X(V)^\Gamma$$

is a weak homotopy equivalence. This means that by [18, Lemma 9.6] there is a map λ'' in the diagram

$$\begin{tikzcd} \partial D^l \arrow[r, "\alpha"] \arrow[d, hook, "i_l"] & X(\mathbb{R}^m)^\Gamma \arrow[d, "(f(\mathbb{R}^m)^\Gamma, X(\psi)^\Gamma)"] \\ D^l \arrow[r, "(\beta,\lambda')"'] \arrow[ur, dashed, "\lambda''"] & Y(\mathbb{R}^m)^\Gamma \times_{Y(V)^\Gamma} X(V)^\Gamma \end{tikzcd}$$

such that the upper-left triangle commutes and the lower-right triangle commutes up to homotopy relative ∂D^l. Thus by [18, Lemma 9.6] again $f(\mathbb{R}^m)^\Gamma$ is a weak homotopy equivalence, and so f is a G–level equivalence. □

Theorem A.20 (G–global model structure) *There is a proper topological cofibrantly generated model structure on the category $\underline{G\mathcal{S}pc}$ of G–orthogonal spaces, with the G–global equivalences as the weak equivalences, the G–global fibrations as the fibrations, and the G–flat cofibrations of the G–level model structure as the cofibrations. We call this model structure the G–global model structure.*

I_G is a set of generating cofibrations of this model structure. The set $\mathsf{J}_G \cup \mathsf{K}_G$ is a set of generating acyclic cofibrations. Recall that I_G, J_G and K_G were given in Theorem A.2 and Construction A.14.

Proof $\underline{G\mathcal{S}pc}$ is complete and cocomplete. The G–global equivalences satisfy the 2-out-of-6 property and are closed under retracts by Lemma 3.7(i) and (ii) respectively. The G–global fibrations and G–flat cofibrations are closed under retracts because they can be defined via lifting properties, see Lemma A.17 and Theorem A.2 respectively. Now we have to check the lifting and factorization axioms.

Given a morphism in $\underline{G\mathcal{S}pc}$, we can use the G–level model structure of Theorem A.2 to decompose it into $f \circ i$ where i is a G–flat cofibration and f is a G–level fibration and a G–level equivalence, so it is also a G–global equivalence by Lemma A.3. Given $\psi\colon V \to W$ a linear isometric embedding of faithful K–representations, in the square

$$\begin{tikzcd} X(V)^\Gamma \arrow[r, "X(\psi)^\Gamma"] \arrow[d, "f(V)^\Gamma"'] & X(W)^\Gamma \arrow[d, "f(W)^\Gamma"] \\ Y(V)^\Gamma \arrow[r, "Y(\psi)^\Gamma"] & Y(W)^\Gamma \end{tikzcd}$$

the two vertical morphisms are weak equivalences by Lemma A.3. Therefore this square is homotopy cartesian and f is a G–global fibration. This gives one of the factorization axioms.

For the second factorization axiom, we apply Quillen's small object argument to the set $\mathsf{J}_G \cup \mathsf{K}_G$, which we can do by Lemma A.16. This factors any morphism into $f \circ j$, where by Lemma A.15 we know that j is a G–flat cofibration and a G–global equivalence, and f has the right lifting property with respect to $\mathsf{J}_G \cup \mathsf{K}_G$, so by Lemma A.17 it is a G–global fibration. This gives the second factorization axiom. Note for later that this j by construction has the left lifting property with respect to G–global fibrations.

One of the lifting axioms can be obtained from the G–level model structure. By Lemma A.19, a morphism which is both a G–global fibration and a G–global equivalence is a G–level equivalence, so it has the right lifting property with respect to the G–flat cofibrations.

Lastly, consider a morphism g which is both a G–flat cofibration and a G–global equivalence. We can use Quillen's small object argument on the set $\mathsf{J}_G \cup \mathsf{K}_G$ again to decompose g into $f \circ j$, where f is a G–global fibration and j is a G–global equivalence which has the left lifting property with respect to G–global fibrations. By the 2-out-of-6 property f is also a G–global equivalence. Then by the previously proven lifting axiom g is a retract of j, so it also has the left lifting property with respect to G–global fibrations.

This model structure is right proper by Lemma A.18 (G–global fibrations are G–level fibrations) and left proper by Corollary A.9. Using [22, Proposition B.5] we obtain that this model structure is topological, taking $\mathcal{G}$ and $\mathcal{Z}$ in that statement to be

$$\mathcal{G} = \{L_{\Gamma,\mathbb{R}^m;G} \mid m \geq 0,\ \Gamma \in \mathscr{F}(O(m), G)\}, \quad \mathcal{Z} = \{\iota_{\rho_{\Gamma,V,W;G}} \mid (K,\Gamma,V,W) \in \kappa\}. \ \square$$

Remark A.21 As mentioned in Remark 3.6, we can define a different class of G–global equivalences by checking the condition from Definition 3.2 on all subgroups of $K \times G$ instead of only on the graph subgroups. We can do the same for all the results of this appendix, replacing $\mathscr{F}(K,G)$ everywhere by the set of all closed subgroups of $K \times G$. We can take the G–level model structure given by all subgroups briefly mentioned right after Theorem A.2, and localize it at this smaller class of G–global equivalences. This gives us a model structure with this smaller class of G–global equivalences as the weak equivalences, as well as fibrations and cofibrations that are similarly defined by looking at all subgroups instead of just the graph subgroups.

However, as shown by the various results of this article, the G–global model structure constructed in this appendix is more relevant when looking at operads in $\underline{\mathcal{S}pc}$.

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On some p–differential graded link homologies, II

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In a previous article, we constructed a link invariant categorifying the Jones polynomial at a $2p^{\text{th}}$ root of unity, where p is an odd prime. This categorification utilized an $N=2$ specialization of a differential introduced by Cautis in an $\mathfrak{sl}_N$–link homology theory. Here we give a family of link homologies where the Cautis differential is specialized to a positive integer of the form $N=kp+2$. When k is even, all these link homologies categorify the Jones polynomial evaluated at a $2p^{\text{th}}$ root of unity, but they are distinct link invariants.

57K18; 18G99

1 Introduction

Given any link L, Khovanov and Rozansky [5] constructed a triply graded link homology theory $\mathrm{HHH}(L)$ whose graded Euler characteristic is the HOMFLYPT polynomial of L using the theory of matrix factorizations. Khovanov reformulated this construction using categories of Soergel bimodules [3]. The connection between Soergel bimodules and link homology began with Rouquier's categorification of the braid group [14]. He also extended this categorification to a link homology [15].

Link homology theories are important examples of categorification. In 1994, Crane and Frenkel [2] introduced their categorification program with the purpose of constructing (3+1)–dimensional TQFTs by lifting the (2+1)–dimensional TQFTs coming from quantum groups. The (2+1)–dimensional TQFTs utilize quantum groups at roots of unity. Motivated by this goal, Khovanov [4] introduced the subject of hopfological algebra, which was further developed in [7]. The basic idea is to take a categorification of a quantum group (for a generic quantum parameter) or its representations, defined over a field of characteristic p and look for differentials ∂ such that $\partial^p = 0$. Searching for such p–differentials is equivalent to constructing an action of the Hopf algebra $H = \Bbbk[\partial]/(\partial^p)$ (hence the word "hopfological"). We refer the reader to [10] for a survey of some recent progress in this direction.

Cautis [1] defined an additional differential, depending upon a natural number N, on the chain groups for the triply graded theory, which produced a categorification of the quantum $\mathfrak{sl}_N$–link invariant (also known as the symmetric $\mathfrak{gl}_N$ homology). Independently, Robert and Wagner [13] and Queffelec, Rose and Sartori [12] constructed the same $\mathfrak{sl}_N$–link homology from different perspectives.

In a more recent work [6], Khovanov and Rozansky equipped the triply graded link homology with an action of the positive half of the Witt algebra. One of the Witt algebra generators (denoted by $L_1 = x^2 \frac{\partial}{\partial x}$) in [6] acts as a p–differential over a field of characteristic p on $\mathrm{HHH}(L)$. For degree reasons, this is the only Witt algebra generator that can play the role of a p–differential. In [11], we utilized this p–differential along with the Cautis differential for $N = 2$, to construct a categorification of the Jones polynomial evaluated at a $2p^{\text{th}}$ root of unity. The Cautis differential has the effect of applying $L_1 \wedge (\cdot)$ to $\mathrm{HHH}(L)$. A key property that facilitated the construction in [11] is that the two actions of L_1, as the p–differential and the Cautis differential, commute with each other.

In this work, we generalize the previous results by considering the Cautis differential for $N = kp+2$ where p is an odd prime — this condition could be removed but was used in the braid group action in the prequel [11] — and k is a nonnegative integer. The essential reason that this generalization works is that, in characteristic p, the polynomial algebra generated by x^p lies in the center of the Witt algebra. Therefore the p–differential L_1 still commutes with $L_{kp+1} = x^{kp+2} \frac{\partial}{\partial x}$, the latter now serving as the Cautis differential. Thus, for each $N = kp+2$ and braid β, we obtain a finite-dimensional object $p\mathrm{H}(\beta, kp+2)$ which is well defined in the homotopy category of p–complexes. Our main result is the following.

Theorem 4.9 *Let L be a link presented as the closure of a braid β and p be an odd prime. The object $p\mathrm{H}(\beta, kp+2)$ is a finite-dimensional framed link invariant. When $k \in 2\mathbb{Z}$, its Euler characteristic is the Jones polynomial evaluated at a $2p^{th}$ root of unity.*

Varying the Cautis differential categorifies $\mathfrak{sl}_N$–link invariants for different ranks. But when q is a $2p^{\text{th}}$ root of unity, and k is even, $q^{kp+2} = q^2$ so the $\mathfrak{sl}_{kp+2}$–link invariant is just the Jones polynomial. While this is true on the decategorified level, we show in Section 5 that on the level of homology, the invariant for the Hopf link depends upon k. Thus we obtain a family of distinct link homologies categorifying the Jones polynomial at $2p^{\text{th}}$ roots of unity.

In a parallel direction [8], we show that the root of unity categorification of [11] can be extended to the colored case. Combining the approach of [8] with the current work, one can construct certain colored $\mathfrak{sl}_N$–link homologies, which we plan to explore.

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2 Background

In this section, we recall some background material from [11]. We assume the reader has some familiarity with the constructions in [11].

2.1 p–DG algebras and their relative homotopy categories

Let $\Bbbk$ be a field of characteristic $p > 2$. For any graded or ungraded algebra B over $\Bbbk$, denote by d_0 the zero superdifferential ($d_0^2 = 0$) and by ∂_0 the zero p–differential ($\partial_0^p = 0$) on B, while letting B sit in homological degree zero. When B is graded, the homological grading is independent of the internal grading of B. We will usually refer to the internal grading as the *q–degree* in what follows.

We will let $\mathcal{C}(B, d_0)$ and $\mathcal{C}(B, \partial_0)$ stand for the homotopy categories and p–homotopy categories of B respectively. For more details on hopfological algebra of p–homotopy categories, see [4; 7].

For a graded module M over a graded algebra B, we let $M\{n\}$ denote the module M, where the internal grading has been shifted up by n. When convenient, we sometimes call this shifted module $q^n M$.

We will need the following functor introduced in [11, Section 2.1] which is called the *p–extension functor*. Let B be a $\Bbbk$–algebra. Given a chain complex of B–modules, we repeat every term sitting in odd homological degrees $(p-1)$ times while keeping even degree terms unchanged. More explicitly, for a given complex

$$\cdots \xrightarrow{d_{2k+2}} M_{2k+1} \xrightarrow{d_{2k+1}} M_{2k} \xrightarrow{d_{2k}} M_{2k-1} \xrightarrow{d_{2k-1}} M_{2k-2} \xrightarrow{d_{2k-2}} \cdots,$$

the p–extended complex looks like

$$\cdots \xrightarrow{d_{2k+2}} M_{2k+1} = \cdots = M_{2k+1} \xrightarrow{d_{2k+1}} M_{2k} \xrightarrow{d_{2k}} M_{2k-1} = \cdots = M_{2k-1} \xrightarrow{d_{2k-1}} M_{2k-2} \xrightarrow{d_{2k-2}} \cdots$$

Similarly, for chain maps of B–modules, the odd degree maps are repeated $p-1$ times while the even ones are kept unchanged. In [9, Proposition 2.3], it is shown that this construction leads to an exact functor between homotopy categories

$$\mathcal{P}\colon \mathcal{C}(B, d_0) \to \mathcal{C}(B, \partial_0). \tag{2-1}$$

The exactness of $\mathcal{P}$ means that it commutes with homological shifts, denoted by $[\pm 1]_d$ and $[\pm 1]_\partial$, respectively, on $\mathcal{C}(B, d_0)$ and $\mathcal{C}(B, \partial_0)$, and preserves the class of distinguished triangles.

Suppose (A, ∂_A) is a p–DG algebra, ie a graded algebra equipped with a differential ∂_A of degree two, satisfying

$$\partial_A^p(a) \equiv 0, \quad \partial_A(ab) = \partial_A(a)b + a\partial_A(b) \tag{2-2}$$

for all $a, b \in A$. In other words, A is an algebra object in the module category of the graded Hopf algebra $H_q = \Bbbk[\partial_q]/(\partial_q^p)$, where the primitive degree-two generator $\partial_q \in H_q$ acts on A by the differential ∂_A. Below we will usually take B to be a certain smash product algebra associated with (A, ∂_A), which we next recall.

Given a p–DG algebra A, we may form the *smash product algebra $A \# H_q$* in this case. As a $\Bbbk$–vector space, $A \# H_q$ is isomorphic to $A \otimes H_q$. The multiplication on the smash product, given in pure tensor elements, is determined by

$$(a \otimes \partial_q)(b \otimes \partial_q) = ab \otimes \partial_q^2 + a\partial_A(b) \otimes \partial_q. \tag{2-3}$$

Notice that, by construction, $A \otimes 1$ and $1 \otimes H_q$ sit in $A \# H_q$ as subalgebras.

For later use, let us record a family of *balanced* H_q–modules

$$V_i := \big(\overset{-i}{\Bbbk} = \overset{-i+2}{\Bbbk} = \cdots = \overset{i-2}{\Bbbk} = \overset{i}{\Bbbk} \big) \tag{2-4}$$

for each q–degree i in $\{0, \dots, p-1\}$. Here the module sits in a single homological degree, while the labels on top indicate the various q–degrees that the module lives in. As graded modules over H_q, we have $V_i \cong q^{-i} H_q/(\partial_q^{i+1})$.

We will also need a relative version of certain homotopy categories that play an essential role in [11]. There is an exact forgetful functor between the usual homotopy categories of chain complexes of graded $A\#H_q$–modules

$$\mathcal{F}_d : \mathcal{C}(A \# H_q, d_0) \to \mathcal{C}(A, d_0).$$

An object $K_\bullet$ in $\mathcal{C}(A \# H_q, d_0)$ lies inside the kernel of the functor if and only if, when forgetting the H_q–module structure on each term of $K_\bullet$, the complex of graded A–modules $\mathcal{F}_d(K_\bullet)$ is nullhomotopic. The nullhomotopy map on $\mathcal{F}_d(K_\bullet)$, though, is not required to intertwine H_q–actions.

Likewise, there is an exact forgetful functor

$$\mathcal{F}_\partial : \mathcal{C}(A \# H_q, \partial_0) \to \mathcal{C}(A, \partial_0).$$

Similarly, an object $K_\bullet$ in $\mathcal{C}(A \# H_q, \partial_0)$ lies inside the kernel of the functor if and only if, when forgetting the H_q–module structure on each term of $K_\bullet$, the p–complex of A–modules $\mathcal{F}(K_\bullet)$ is nullhomotopic. The nullhomotopy map on $\mathcal{F}(K_\bullet)$, though, is not required to intertwine H_q–actions.

Definition 2.1 Given a p–DG algebra (A, ∂_A), the *relative homotopy category* is the Verdier quotient

$$\mathcal{C}^{\partial_q}(A, d_0) := \frac{\mathcal{C}(A \# H_q, d_0)}{\mathrm{Ker}(\mathcal{F}_d)}.$$

Likewise, the *relative p–homotopy category* is the Verdier quotient

$$\mathcal{C}^{\partial_q}(A, \partial_0) := \frac{\mathcal{C}(A \# H_q, \partial_0)}{\mathrm{Ker}(\mathcal{F}_\partial)}.$$

The superscripts in the definitions are to remind the reader of the H_q–module structures on the objects.

The categories $\mathcal{C}^{\partial_q}(A, d_0)$ and $\mathcal{C}^{\partial_q}(A, \partial_0)$ are triangulated. By construction, there are factorizations of the forgetful functors

$$\begin{tikzcd} \mathcal{C}(A \# H_q, d_0) \arrow[rr, "\mathcal{F}_d"] \arrow[dr] & & \mathcal{C}(A, d_0) \\ & \mathcal{C}^{\partial_q}(A, d_0) \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(A \# H_q, \partial_0) \arrow[rr, "\mathcal{F}_\partial"] \arrow[dr] & & \mathcal{C}(A, \partial_0) \\ & \mathcal{C}^{\partial_q}(A, \partial_0) \arrow[ur] & \end{tikzcd}$$

Proposition 2.2 [11, Proposition 2.13] *The p–extension functor*

$$\mathcal{P}\colon \mathcal{C}(A \# H_q, d_0) \to \mathcal{C}(A \# H_q, \partial_0)$$

descends to an exact functor, still denoted by $\mathcal{P}$, between the relative homotopy categories,

$$\mathcal{P}\colon \mathcal{C}^{\partial_q}(A, d_0) \to \mathcal{C}^{\partial_q}(A, \partial_0).$$

2.2 p–DG bimodules over the polynomial algebra

The polynomial algebra $R_n = \Bbbk[x_1, \dots, x_n]$ has a natural graded algebra structure by setting the degree of each x_i to be two. We can equip R_n with a p–DG algebra structure, where the generator $\partial_q \in H_q$ acts as a derivation determined by $\partial_q(x_i) = x_i^2$ for $i = 1, \dots, n$. As before, the internal grading on R_n will be referred to as the q–degree. When n is clear from the context, we will abbreviate R_n by just R.

The differential is invariant under the permutation action of the symmetric group S_n on the indices of the variables. Therefore let the subalgebra of polynomials symmetric in variables x_i and x_{i+1} with its inherited H_q–module structure be denoted by

$$R_n^i = \Bbbk[x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \dots, x_n].$$

More generally, given a (Young) subgroup $G \subset S_n$, the invariant subalgebra R_n^G inherits an H_q–algebra structure from R_n (and is thus a p–DG algebra). In particular, we will also use the H_q–subalgebra $R_n^{i,i+1} := R_n^{S_3}$, where S_3 is identified with the subgroup generated by permuting the indices i, $i+1$ and $i+2$.

The (R, R)–bimodule $B_i = R \otimes_{R^i} R$ has the structure of an H_q–module (and is thus a p–DG bimodule) where the differential acts via the Leibniz rule: for any $h \otimes g \in R \otimes_{R^i} R$,

$$\partial_q(h \otimes g) = \partial_q(h) \otimes g + h \otimes \partial_q(g).$$

With respect to $\otimes_R$, the monoidal category of (R,R)–bimodules generated by the B_i has an H_q–module structure, where the ∂_q action is given by the Leibniz rule. We denote this category by $(R,R)\#H_q$–mod.

Let $f=\sum_{i=1}^n a_i x_i \in \mathbb{F}_p[x_1,\dots,x_n]\subset R$ be a linear function. We twist the H_q–action on the bimodule B_i to obtain a bimodule B_i^f defined as follows. As an (R,R)–bimodule, it is the same as B_i but the action of H_q is twisted by defining

$$\partial_q(1\otimes 1)=(1\otimes 1)f. \tag{2-5a}$$

Similarly, we define ${}^fB_i$ where now

$$\partial_q(1\otimes 1)=f(1\otimes 1). \tag{2-5b}$$

For R_n as a bimodule over itself, it is clear that ${}^fR_n\cong R_n^f$ as p–DG bimodules. It follows that there are p^n ways to put an H_q–module structure on a rank-one free module over R_n. Each such H_q–module is quasi-isomorphic to a finite-dimensional p–complex. Choose numbers $b_i\in\{2,\dots,p,p+1\}$ such that $b_i\equiv a_i \pmod p$, $i=1,\dots,n$, and define the H_q–ideal of R,

$$I=(x_1^{p+1-b_1},\dots,x_n^{p+1-b_n}). \tag{2-6}$$

Then the natural quotient map

$$\pi\colon R^f \twoheadrightarrow R^f/(I\cdot R^f) \tag{2-7}$$

is readily seen to be a quasi-isomorphism. The right hand side of (2-7) computes the *slash homology* (see [11, Section 2.1] for more details), denoted by $\mathrm{H}^{/}_{\bullet}$, of R^f.

Lemma 2.3 [11, Lemma 3.1] *For each $f=\sum_i a_i x_i$, the rank-one p–DG module R^f has slash homology*

$$\mathrm{H}^{/}_{\bullet}(R^f)\cong \bigotimes_{i=1}^n V_{p-a_i}\{p-a_i\}.$$

In particular, the slash homology is finite-dimensional, and vanishes if any a_i of $f=\sum_i a_i x_i$ is equal to one.

Corollary 2.4 [11, Corollary 3.2] *Let M be a p–DG module over R which is equipped with a finite filtration, whose subquotients are isomorphic to R^f for various f. Then M has finite-dimensional slash homology.*

2.3 Relative p–Hochschild homology

In [11, Section 2.3], we introduced an absolute version of the p–Hochschild (co)homology functor. In what follows, we will instead need a relative version of p–Hochshild homology for a p–DG algebra, which we recall now. An important reason for introducing the relative homotopy category is that the relative p–Hochschild homology functor descends to this category.

Let (A, ∂_A) be a p–DG algebra. Equip A with the zero differential d_0 and zero p–differential ∂_0, and denote the resulting trivial (p–)DG algebras by (A_0, d_0) and (A_0, ∂_0) respectively. Likewise, for a (p–)DG bimodule M over A, we temporarily denote by M_0 the A–bimodule equipped with zero (p–)differentials.

The usual Hochschild homology of M_0 over (A_0, d_0) in this case carries a natural H_q–action, since the H_q–action commutes with all differentials in the usual simplicial bar complex for A_0.

Definition 2.5 The *relative Hochschild homology* of a p–DG bimodule (M, ∂_M) over (A, ∂_A) is the usual Hochschild homology of M_0 over (A_0, d_0) equipped with the induced H_q–action from ∂_M and ∂_A, and denoted by

$$\mathrm{HH}_\bullet^{\partial_q}(M) := \mathrm{HH}_\bullet(A_0, M_0).$$

Replacing the usual simplicial bar complex by Mayer's p–simplicial bar complex (see [11, Definition 2.10]), we make the next definition (see [11, Section 2.3] for details). Mayer's p–simplicial bar complex is obtained by removing the alternating signs in the usual simplicial bar complex of an algebra. In turn this results in a p–complex bimodule resolution of an algebra.

Definition 2.6 The *relative p–Hochschild homology* of M is the p–complex

$$p\mathrm{HH}_\bullet^{\partial_q}(M) := \mathrm{H}_\bullet^{/}(A_0 \otimes^{\boldsymbol{L}}_{A_0 \otimes A_0^{\mathrm{op}}} M_0) = \mathrm{H}_\bullet^{/}(\boldsymbol{p}(A_0) \otimes_{A_0 \otimes A_0^{\mathrm{op}}} M_0),$$

where the notation $\otimes^{\boldsymbol{L}}$ is the derived tensor functor. Here, the usual simplicial bar resolution of M_0 over A_0 is replaced by *Mayer's p–simplicial bar complex* $\boldsymbol{p}(A_0)$.

Similar to the usual Hochschild homology, the relative p–Hochschild homology is also covariant functor: if $f : M \to N$ is a morphism of p–DG bimodules over A, it induces

$$p\mathrm{HH}_\bullet^{\partial_q}(f) := \mathrm{H}_\bullet^{/}(\mathrm{Id}_{A_0} \otimes f) \colon \mathrm{H}_\bullet^{/}(A_0 \otimes^{\boldsymbol{L}}_{A_0 \otimes A_0^{\mathrm{op}}} M_0) \to \mathrm{H}_\bullet^{/}(A_0 \otimes^{\boldsymbol{L}}_{A_0 \otimes A_0^{\mathrm{op}}} N_0).$$

Proposition 2.7 [11, Proposition 2.20] *The relative p–Hochschild homology descends to a functor defined on the relative homotopy category $\mathcal{C}^{\partial_q}(A, \partial_0)$ of p–DG bimodules over A.*

We also have the trace-like property for relative p–Hochschild homology.

Proposition 2.8 [11, Proposition 2.21] *Given two p–DG bimodules M and N over A, there is an isomorphism of p–complexes of H_q–modules*

$$p\mathrm{HH}^{\partial_q}_{\bullet}(M \otimes^{\boldsymbol{L}}_A N) \cong p\mathrm{HH}^{\partial_q}_{\bullet}(N \otimes^{\boldsymbol{L}}_A M).$$

We next recall a technical tool that allows us to use a simpler bimodule resolution to compute the relative Hochschild homology than the usual simplicial bar resolution.

Theorem 2.9 [11, Theorem 2.22] *Let M be a p–DG bimodule over A. Suppose $f\colon Q_\bullet \to M$ is a p–complex resolution of M over (A_0, ∂_0) which is H_q–equivariant, and each term of $Q_\bullet$ is projective as an $A_0 \otimes A_0^{\mathrm{op}}$–module. Then f induces an isomorphism of H_q–modules*

$$\mathrm{H}^{/}_{\bullet}(A_0 \otimes_{A_0 \otimes A_0^{\mathrm{op}}} Q_\bullet) \cong p\mathrm{HH}^{\partial_q}_{\bullet}(M).$$

2.4 Elementary braiding complexes

Here and below, for ease of notation, we will abbreviate $t^n = [n]_d$ for homological shifts, where $n \in \mathbb{Z}$. Recall that in [11], we show that there are $(R,R)\#H_q$–module homomorphisms

(i) $rb_i\colon R \to q^{-2} B_i^{-(x_i+x_{i+1})}$, where $1 \mapsto (x_{i+1} \otimes 1 - 1 \otimes x_i)$;

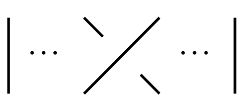
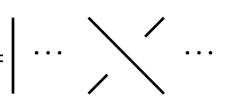
(ii) $br_i\colon B_i \to R$, where $1 \otimes 1 \mapsto 1$.

Thus we have complexes of $(R,R)\#H_q$–modules

(2-8) $$T_i := \big(tB_i \xrightarrow{br_i} R\big), \quad T_i' := \big(R \xrightarrow{rb_i} q^{-2}t^{-1} B_i^{-(x_i+x_{i+1})}\big).$$

In the coming sections we will, for presentation reasons, often omit the various shifts built into the definitions of T_i and T_i'.

We associate respectively to the left and right crossings σ_i and σ_i' between the i^{th} and $(i+1)^{\text{st}}$ strands in (2-9) the chain complexes of $(R,R)\#H_q$–bimodules T_i and T_i',

(2-9) $\sigma_i :=$  $\sigma_i' :=$ 

More generally, if $\beta \in \mathrm{Br}_n$ is a braid group element written as a product in the elementary generators $\sigma_{i_i}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$, where $\epsilon_i \in \{\varnothing, \prime\}$, we assign the chain complex of $(R, R)\#H_q$–bimodules

$$T_\beta := T_{i_1}^{\epsilon_1} \otimes_R \cdots \otimes_R T_{i_k}^{\epsilon_k}. \tag{2-10}$$

The complex is well defined in the relative homotopy category thanks to the following result.

Theorem 2.10 *The complexes of T_i and T_i' are mutually inverse complexes in the relative homotopy category $\mathcal{C}^{\partial_q}(R, R, d_0)$. They satisfy the braid relations*

- $T_i T_j \cong T_j T_i$ *if* $|i-j| > 1$,
- $T_i T_{i+1} T_i \cong T_{i+1} T_i T_{i+1}$ *for all* $i = 1, \dots, n-1$.

Consequently, given any braid group element $\beta \in \mathrm{Br}_n$, the chain complex of T_β associated to it is a well-defined element of the relative homotopy category $\mathcal{C}^{\partial_q}(R, R, d_0)$.

Proof This is proven in [11, Section 3]. □

3 Specialized HOMFLYPT theories

3.1 HOMFLYPT homologies

In this section we categorify the HOMFLYPT polynomial of any link using analogous arguments from [1], [13] and [15] adapted to the p–DG setting.

For the first construction, we will allow complexes of Soergel bimodules to sit in half-integer degrees in the Hochschild (a) and the homological, sometimes called the topological, (t) degrees when considering the usual complexes of vector spaces.

We modify the elementary braiding complexes of (2-8) to be

$$T_i := (at)^{-\frac{1}{2}} q^{-2} \big(t B_i \xrightarrow{br_i} R\big), \quad T_i' := (at)^{\frac{1}{2}} q^2 \big(R \xrightarrow{rb_i} q^{-2} t^{-1} B_i^{-(x_i + x_{i+1})}\big). \tag{3-1}$$

Here we have extended the degree shift convention for q–degrees (see the beginning of Section 2) to a– and t–degrees.

Let $\beta \in \mathrm{Br}_n$ be a braid group element in n strands. By Theorem 2.10, there is a chain complex of $(R_n, R_n)\#H_q$–bimodules T_β, well defined up to homotopy, associated with β. Then set

$$T_\beta = \big(\cdots \xrightarrow{d_0} T_\beta^{i+1} \xrightarrow{d_0} T_\beta^i \xrightarrow{d_0} T_\beta^{i-1} \xrightarrow{d_0} \cdots\big). \tag{3-2}$$

Definition 3.1 The *untwisted H_q–HOMFLYPT homology* of β is the object

$$\widehat{\mathrm{HHH}}^{\partial_q}(\beta) := a^{-\frac{n}{2}} t^{\frac{n}{2}} \mathrm{H}_\bullet\big(\cdots \to \mathrm{HH}_\bullet^{\partial_q}(T_\beta^{i+1}) \xrightarrow{d_t} \mathrm{HH}_\bullet^{\partial_q}(T_\beta^{i}) \xrightarrow{d_t} \mathrm{HH}_\bullet^{\partial_q}(T_\beta^{i-1}) \to \cdots\big)$$

in the category of triply graded H_q–modules, where $d_t := \mathrm{HH}_\bullet^{\partial_q}(d_0)$ is the induced map of d_0 on relative Hochschild homology. Here, the relative Hochschild homology is defined in Definition 2.5, and $\mathrm{H}_\bullet$ means the usual homology of a chain complex.

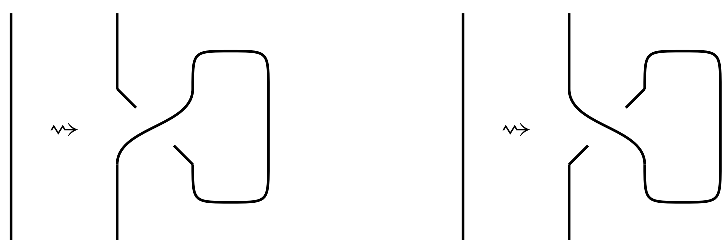
By construction, the space $\widehat{\mathrm{HHH}}^{\partial_q}(\beta)$ is triply graded by topological (t) degree, Hochschild (a) degree as well as quantum (q) degree. When necessary to emphasize each graded piece of the space, we will write $\widehat{\mathrm{HHH}}^{\partial_q}_{i,j,k}(\beta)$ to denote the homogeneous component concentrated in t–degree i, a–degree j and q–degree k.

The following theorem is a particular case of the main result of [6], where we have only kept track of the degree two p–nilpotent differential — which is denoted by L_1 in [6] — in finite characteristic p. The detailed verification given in Section 3.2, however, uses the main ideas of [15] and differs from that of [6]. This proof serves as the model for the other link homology theories in this paper.

Theorem 3.2 *The untwisted H_q–HOMFLYPT homology of β depends only on the braid closure of β as a framed link in $\mathbb{R}^3$.*

As a convention for the framing number of braid closure, if a strand for a component of link is altered as in the left of (3-3), then we say that the framing of the component is increased by 1 (with respect to the blackboard framing). If a strand for a component of link is altered as in the right of (3-3), then we say that the framing of the component is decreased by 1.

(3-3)



Denote by $\mathtt{f}_i(L)$ the framing number of the i^{th} component of a link L. Then, under the Reidemeister moves of (3-3), $\mathtt{f}_i(L)$ is increased or decreased by one when changing from the corresponding left local picture to the right local picture.

We next seek to define a triply graded analogue with a–, t– and q–degrees in the homotopy category of p–complexes. Let us first discuss what degrees of freedom we have in the constructions.

First, we may adapt (3-1) into

$$pT_i := a^u t^v q^w [n]^a_\partial [m]^t_\partial \big(B_i[1]^t_\partial \xrightarrow{br_i} R\big), \tag{3-4a}$$

$$pT_i' := a^{-u} t^{-v} q^{-w} [-n]^a_\partial [-m]^t_\partial \big(R \xrightarrow{rb_i} q^{-2} B_i^{-x_i - x_{i+1}}[-1]^t_\partial\big). \tag{3-4b}$$

Here, the superscripts in homological shifts indicate in which of the three gradings they are occurring. See the discussion around (2-1) for the meaning of the subscripts in the notation. We let $u, v, w, m, n \in \mathbb{Z}$ denote possible grading shifts to be determined, which will be made into the simplest possible form at the end of the next subsection.

Definition 3.3 Let $\beta \in \mathrm{Br}_n$ be a braid group element written as a product in the elementary generators $\sigma_{i_i}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$, where $\epsilon_i \in \{\varnothing, \prime\}$. We assign to β the p–chain complex of $(R_n, R_n)\# H_q$–bimodules

$$pT_\beta := pT_{i_1}^{\epsilon_1} \otimes_R \cdots \otimes_R pT_{i_k}^{\epsilon_k}. \tag{3-5}$$

We will denote the boundary maps in the p–complex pT_β by ∂_0 ($\partial_0^p = 0$), in contrast to the usual, also called the topological, differential d_0 satisfying $d_0^2 = 0$.

Definition 3.4 The *untwisted H_q–HOMFLYPT p–homology* of β is the object

$$p\widehat{\mathrm{HHH}}^{\partial_q}(\beta) := q^{f(n)} \mathrm{H}^{/}_\bullet\big(\cdots \to p\mathrm{HH}^{\partial_q}_\bullet(pT_\beta^{i+1}) \xrightarrow{\partial_t} p\mathrm{HH}^{\partial_q}_\bullet(pT_\beta^{i}) \xrightarrow{\partial_t} p\mathrm{HH}^{\partial_q}_\bullet(pT_\beta^{i-1}) \to \cdots\big)$$

in the homotopy category of bigraded H_q–modules, where $f(n)$ is a function on $\mathbb{N}$ which is determined below in (3-16). Here ∂_t stands for the induced map of the topological differentials on p–Hochschild homology groups $\partial_t := p\mathrm{HH}^{\partial_q}_\bullet(\partial_0)$.

In the definition of the H_q–HOMFLYPT p–homology, we have applied the p–extensions in both the topological and the Hochschild direction so that they can be collapsed into a single degree. The reason will become clearer later when categorifying certain $\mathfrak{sl}_N$ polynomials at prime roots of unity. Therefore, in contrast to $\widehat{\mathrm{HHH}}(\beta)$, $p\widehat{\mathrm{HHH}}(\beta)$ is only doubly graded, and we will adopt the notation $p\widehat{\mathrm{HHH}}_{i,j}(\beta)$ as above to stand for its homogeneous components in topological degree i and q–degree j. Further, the overall grading shift in the definition will be utilized in the invariance under the Markov II move below.

Theorem 3.5 *The untwisted H_q–HOMFLYPT p–homology of β depends only on the braid closure of β as a framed link in $\mathbb{R}^3$.*

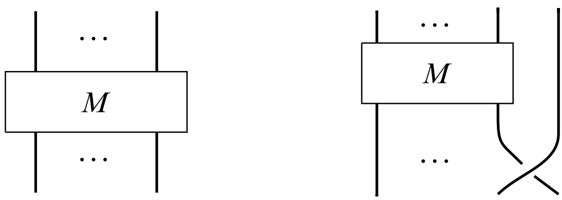
The proof of Theorems 3.2 and 3.5 will occupy the next few subsections, after we introduce the H_q–equivariant (p–)Koszul resolutions.

3.2 Examining Markov II invariance

In this subsection, let us examine the invariance under the Markov II move for $p\widehat{\mathrm{HHH}}$.

In order to satisfy the second Markov move, one needs to show that for a Soergel bimodule M (or a complex of Soergel bimodules) over the polynomial p–DG algebra R_n, that the H_q–HOMFLYPT (p–)homologies of the bimodules (3-6) are isomorphic (up to grading shifts and twists in H_q actions),

(3-6)



By definition, the one-variable p–extended Koszul complex is given by

$$pC_1 = q^2\Bbbk[x]^x \otimes \Bbbk[x]^x[1]^a_\partial \xrightarrow{x\otimes 1 - 1\otimes x} \Bbbk[x]\otimes\Bbbk[x]. \tag{3-7}$$

Set $pC_{n+1} := pC_1^{\otimes n+1}$. For the ease of notation, we will write pC_1' for the p–extended Koszul complex pC_1 in the variable x_{n+1}. Using the isomorphism of p–DG bimodules,

$$\begin{aligned} pC_{n+1} &\otimes_{(R_{n+1},R_{n+1})} ((M\otimes\Bbbk[x_{n+1}])\otimes_{R_{n+1}} pT_n) \\ &= (pC_n\otimes pC_1')\otimes_{(R_{n+1},R_{n+1})} ((M\otimes\Bbbk[x_{n+1}])\otimes_{R_{n+1}} pT_n) \\ &\cong pC_n\otimes_{(R_n,R_n)} (M\otimes_{R_n} (pC_1'\otimes_{(\Bbbk[x_{n+1}],\Bbbk[x_{n+1}])} pT_n)), \end{aligned} \tag{3-8}$$

we are reduced to analyzing the p–homology of the "square" $pC_1'\otimes_{\Bbbk[x_{n+1}],\Bbbk[x_{n+1}]} pT_n$:

$$\begin{tikzcd} a^ut^vq^w({}^{x_{n+1}}B_i^{x_{n+1}})[k+1]^a_\partial[m+1]^t_\partial \arrow[r] \arrow[d] & a^ut^vq^w R^{2x_{n+1}}[k+1]^a_\partial[m]^t_\partial \arrow[d] \\ a^ut^vq^w B_n[k]^a_\partial[m+1]^t_\partial \arrow[r] & a^ut^vq^w R[k]^a_\partial[m]^t_\partial \end{tikzcd} \tag{3-9a}$$

$$
\begin{tikzcd}
q^4 R_{n+1}^{x_n+3x_{n+1}}[1]^a_\partial[1]^t_\partial \arrow[r, "2(x_{n+1}-x_n)"] \arrow[d] \arrow[dd, bend right, "\phi"'] & q^2 R_{n+1}^{2x_{n+1}}[1]^a_\partial \arrow[d] \arrow[dd, bend left, "\mathrm{Id}"] & & 0 \arrow[d] \\
0 \arrow[r] & 0 & & {:=}\, pY_1 \arrow[dd] \\
q^2({}^{x_{n+1}}B_n^{x_{n+1}})[1]^a_\partial[1]^t_\partial \arrow[r, "br"] \arrow[d, "x_{n+1}\otimes 1-1\otimes x_{n+1}"] \arrow[dd, bend right, "\widetilde{br}"'] & q^2 R_{n+1}^{2x_{n+1}}[1]^a_\partial \arrow[d, "0"] & & \\
B_n[1]^t_\partial \arrow[r, "br"] \arrow[dd, bend right, "2\mathrm{Id}"'] & R_{n+1} \arrow[dd, bend left, "2\mathrm{Id}"] & & = pC_1' \otimes_{(\Bbbk[x_{n+1}],\Bbbk[x_{n+1}])} pT_n \arrow[d] \\
q^2 \widetilde{R}_{n+1}^{x_n+x_{n+1}}[1]^t_\partial \arrow[r] \arrow[d, "(x_{n+1}-x_n)\otimes 1-1\otimes(x_{n+1}-x_n)"] & 0 \arrow[d] & & {:=}\, pY_2 \arrow[d] \\
B_n[1]^t_\partial \arrow[r, "br"] & R_{n+1} & & 0
\end{tikzcd}
$$

Figure 1

and $pC_1' \otimes_{\Bbbk[x_{n+1}],\Bbbk[x_{n+1}]} pT_n'$:

$$
\begin{tikzcd}
a^{-u}t^{-v}q^{2-w}R^{2x_{n+1}}[1-k]^a_\partial[-m]^t_\partial \arrow[r] \arrow[d] & a^{-u}t^{-v}q^{2-w}({}^{x_{n+1}}B_n^{-x_n})[1-k]^a_\partial[-1-m]^t_\partial \arrow[d] \\
a^{-u}t^{-v}q^{-w}[-k]^a_\partial[-m]^t_\partial \arrow[r] & a^{-u}t^{-v}q^{-w-2}B_n^{-x_i-x_{i-1}}[-k]^a_\partial[-1-m]^t_\partial
\end{tikzcd}
\tag{3-9b}
$$

Let us begin by studying the first p–complex square (3-9a). We will exhibit a sub–p–complex pY_1 of (3-9a), whose quotient will be denoted by pY_2. Ignoring for the moment the overall grading shift $a^u t^v q^w [k]^a_\partial [m]^t_\partial$ for simplicity, we have a filtration of the square given by a short exact sequence of (p–complexes) of bimodules as in Figure 1, where by definition the first square is pY_1 and the third square is pY_2. Here ϕ is the map that sends 1 to $(x_{n+1}-x_n)\otimes 1 + 1\otimes(x_{n+1}-x_n)$.

It is not hard to show that $pC_n \otimes_{(R_n,R_n)} (M \otimes_{R_n} pY_2)$ is annihilated by taking first the vertical p–Hochschild homology and then the horizontal topological homology.

$$
\begin{tikzcd}
 & & 0 \arrow[d] \\
q^2 R_{n+1}^{2x_{n+1}}[1]^a_\partial \arrow[r, "\mathrm{Id}"] \arrow[d] \arrow[dd, bend right, "\mathrm{Id}"'] & q^2 R_{n+1}^{2x_{n+1}}[1]^a_\partial[-1]^t_\partial \arrow[d] \arrow[dd, bend left, "rb"] & := pZ_1 \arrow[ddd] \\
0 \arrow[r] & 0 & \\
q^2 R_{n+1}^{2x_{n+1}}[1]^a_\partial \arrow[r, "rb"] \arrow[d, "0"] & ({}^{x_{n+1}}B_n^{-x_n})[1]^a_\partial[-1]^t_\partial \arrow[d, "x_{n+1}\otimes 1-1\otimes x_{n+1}"'] \arrow[dd, bend left, "\widetilde{br}"] & \\
R_{n+1} \arrow[r, "rb"] \arrow[dd, bend right, "\mathrm{Id}"'] & q^{-2} B_n^{-(x_n+x_{n+1})}[-1]^t_\partial \arrow[dd, bend left, "\mathrm{Id}"] & = pC_1' \otimes_{(\Bbbk[x_{n+1}],\Bbbk[x_{n+1}])} pT_n' \arrow[dd] \\
0 \arrow[r] \arrow[d] & \widetilde{R}_{n+1}[1]^a_\partial[-1]^t_\partial \arrow[d, "x_{n+1}\otimes 1-1\otimes x_{n+1}"'] & \\
R_{n+1} \arrow[r, "rb"] & q^{-2} B_n^{-(x_n+x_{n+1})}[-1]^t_\partial & := pZ_2 \arrow[d] \\
 & & 0
\end{tikzcd}
$$

Figure 2

Further, the p–complex pY_1 is quasi-isomorphic to $q^2 R_n^{2x_n}[1]^a_\partial$. Putting back the grading shifts ignored earlier, we obtain the isomorphism

$$
\begin{aligned}
p\widehat{\mathrm{HHH}}^{\partial_q}((M \otimes \Bbbk[x_{n+1}]) \otimes_{R_{n+1}} pT_n) &\cong \mathrm{H}^{/}_{\bullet}(p\mathrm{HH}^{\partial_q}((M \otimes_{R_n} pY_1)) \\
&\cong p\widehat{\mathrm{HHH}}^{\partial_q}(a^u t^v q^{w+2} M[k+1]^a_\partial[m]^t_\partial)^{2x_n}.
\end{aligned} \tag{3-10}
$$

For the second square (3-9b), again there is a short exact sequence of bicomplexes of (R_{n+1}, R_{n+1})–bimodules. Ignoring the overall grading shift $a^{-u}t^{-v}q^{-w}[-k]^a_\partial[-m]^t_\partial$, it is as in Figure 2. Next, consider the morphism of bicomplexes

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \widetilde{R}_{n+1}[1]^a_\partial[-1]^t_\partial \arrow[d, "x_{n+1}\otimes 1-1\otimes x_{n+1}"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
pZ_2 := & R_{n+1} \arrow[r, "rb"] \arrow[rr, bend right, "\mathrm{Id}"'] & B_n^{-(x_n+x_{n+1})}[-1]^t_\partial \arrow[rr, bend right, "br"'] & R_{n+1} \arrow[r, "x_{n+1}-x_n"] & R_{n+1}^{-(x_n+x_{n+1})}[-1]^t_\partial & := pZ_2'
\end{tikzcd}
$$

whose kernel is isomorphic to the contractible p–complex

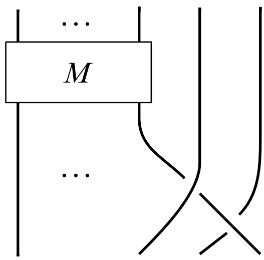
$$pZ_2'' := \begin{array}{c} \widetilde{R}_{n+1}[1]^a_\partial[-1]^t_\partial \\ \Big\downarrow \mathrm{Id} \\ \widetilde{R}_{n+1}[-1]^t_\partial \end{array}$$

Upon taking pHH, the contribution from pZ_2'' vanishes. Taking back into account the overall grading shift, it follows that we have

$$p\widehat{\mathrm{HHH}}^{\partial_q}((M \otimes \Bbbk[x_{n+1}]) \otimes_{R_{n+1}} pT_n') \cong \mathrm{H}^{/}_{\bullet}(p\mathrm{HH}^{\partial_q}((M \otimes_{R_n} pZ_2')) \cong p\widehat{\mathrm{HHH}}^{\partial_q}(a^{-u}t^{-v}q^{-w-2}M[-k]^a_\partial[-m-1]^t_\partial)^{-2x_n}. \tag{3-11}$$

Now, let us observe that taking closure of the following diagram of p–DG bimodules

(3-12)



introduces a canceling pair of Markov II moves. By (3-10) and (3-11), we obtain that

$$p\widehat{\mathrm{HHH}}^{\partial_q}\big((M \otimes \Bbbk[x_{n+1}, x_{n+2}]) \otimes_{R_{n+2}} pT_{n+1} \otimes_{R_{n+2}} pT_{n+2}'\big) \cong q^{f(n+2)}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_{\bullet}(M[1]^a_\partial[-1]^t_\partial)). \tag{3-13}$$

For the last term to be isomorphic to

$$p\widehat{\mathrm{HHH}}^{\partial_q}(M) \cong q^{f(n)}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_{\bullet}(M)), \tag{3-14}$$

we need to require the functor isomorphism

$$[1]^a_\partial[-1]^t_\partial = q^{f(n)-f(n+2)}. \tag{3-15}$$

We are therefore forced to collapse the a grading onto the t grading such that $a = q^r t$, where $r = f(n) - f(n+2) \in \mathbb{Z}$. For simplicity, let us assume that

$$K = f(n) - f(n+1) \tag{3-16}$$

is a constant independent of n. Then $r = 2K \in 2\mathbb{Z}$, and we have $a = q^{2K}t$ such that $[1]^a_\partial = q^{2K}[1]^t_\partial$.

Revisiting (3-4), we now set

$$pT_i := q^{-K-2}[-1]^t_\partial\big(B_i[1]^t_\partial \xrightarrow{br_i} R\big), \tag{3-17a}$$

$$pT_i' := q^{K+2}[1]^t_\partial\big(R \xrightarrow{rb_i} q^{-2}B_i^{-x_i-x_{i+1}}[-1]^t_\partial\big), \tag{3-17b}$$

and

$$p\widehat{\mathrm{HHH}}^{\partial_q}(\beta, K+1) := q^{-Kn}\mathrm{H}^{/}_{\bullet}\big(\cdots \to p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i+1}_\beta) \xrightarrow{\partial_t} p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i}_\beta) \xrightarrow{\partial_t} p\mathrm{HH}^{\partial_q}_{\bullet}(pT^{i-1}_\beta) \to \cdots\big).$$

Recall from Section 2.2 — see the discussion around (2-5a) and (2-5b) — that, for a given linear polynomial $f = \sum_i a_i x_i$, $a_i \in \mathbb{F}_p$, and a p–DG R_n–module M, we can twist the H_q–module structure on M by f. The resulting p–DG module is denoted by M^f.

Theorem 3.6 *Let β_1 and β_2 be two braids whose closures represent the same link L of r components up to framing. Suppose the framing numbers of the closures $\hat\beta_1$ of β_1 and $\hat\beta_2$ of β_2 differ by $\mathtt{f}_i(\hat\beta_1) - \mathtt{f}_i(\hat\beta_2) = a_i$, $i = 1, \dots, r$. Then*

$$\widehat{\mathrm{HHH}}^{\partial_q}(\beta_1) \cong \widehat{\mathrm{HHH}}^{\partial_q}(\beta_2)^{2\sum_{i=1}^r a_i x_i}$$

and

$$p\widehat{\mathrm{HHH}}^{\partial_q}(\beta_1, K+1) \cong p\widehat{\mathrm{HHH}}^{\partial_q}(\beta_2, K+1)^{2\sum_{i=1}^r a_i x_i}$$

where the generator of the polynomial action for the i^{th} component is denoted by x_i and $\widehat{\mathrm{HHH}}^{\partial_q}(\beta_2)^{2\sum_i a_i x_i}$ means that we twist the H_q–module structure on the i^{th} component by $2a_i x_i$.

Proof The topological invariance follows from Theorem 2.10 and the proof above of invariance under the Markov moves. □

3.3 Unlinks and twistings

In this section, we compute $\widehat{\mathrm{HHH}}^{\partial_q}$ and $p\widehat{\mathrm{HHH}}^{\partial_q}$ for the identity element of the braid group Br_n, and define an unframed link invariant in $\mathbb{R}^3$ by correcting the framing factors appearing in Theorem 3.6.

For the unknot, the Koszul resolution C_1 of $\Bbbk[x]$ as bimodules is given by

$$q^2 a\Bbbk[x]^x \otimes \Bbbk[x]^x \xrightarrow{x\otimes 1 - 1\otimes x} \Bbbk[x] \otimes \Bbbk[x].$$

Tensoring this complex with $\Bbbk[x]$ as a bimodule yields

$$q^2 a\Bbbk[x]^{2x} \xrightarrow{0} \Bbbk[x].$$

Thus the homology of the unknot (up to shift) is identified with the bigraded H_q–module

$$\Bbbk[x] \oplus q^2 a \Bbbk[x]^{2x}.$$

More generally, via the Koszul complex $C_n = C_1^{\otimes n}$, we have that the homology of the n–component unlink L_0 is equal to

$$\widehat{\mathrm{HHH}}^{\partial_q}(L_0) \cong a^{-\frac{n}{2}} t^{\frac{n}{2}} \mathrm{HH}_\bullet(R_n) \cong a^{-\frac{n}{2}} t^{\frac{n}{2}} \bigotimes_{i=1}^{n} (\Bbbk[x_i] \oplus q^2 a \Bbbk[x_i]^{2x_i}). \tag{3-18}$$

Alternatively, up to the grading shift $a^{-\frac{n}{2}} t^{\frac{n}{2}}$, we may identify $\widehat{\mathrm{HHH}}^{\partial_q}(L_0)$ with the exterior algebra over R_n generated by the differential forms dx_i of bidegree aq^2 for $i = 1, \dots, n$, subject to the condition that each dx_i accounts for a twisting of the H_q–module structure by $2x_i$.

It follows that, as for the ordinary HOMFLYPT homology, given a framed link L of ℓ components arising as a braid closure $\hat{\beta}$, its untwisted HOMFLYPT H_q–homology $\widehat{\mathrm{HHH}}^{\partial_q}(\beta)$ is a module over

$$\widehat{\mathrm{HHH}}^{\partial_q}_{0,0,\bullet}(L_0) \cong R_\ell,$$

and thus one may consider a twisting of the H_q–module structure on $\widehat{\mathrm{HHH}}^{\partial_q}(\beta)$ by the functor $R_\ell^f \otimes_{R_\ell} (\cdot)$, where f is a linear polynomial in $x_1, \dots, x_\ell$; see Section 2.2.

Definition 3.7 Let L be a framed link arising from the closure of an n–strand braid β. Label the components of L by 1 through ℓ, and set the (*linear*) *framing factor* of β to be the linear polynomial

$$\mathtt{f}_\beta = -\sum_{i=1}^{\ell} 2\mathtt{f}_i x_i.$$

(1) The *H_q–HOMFLYPT homology* of β is the triply graded H_q–module

$$\mathrm{HHH}^{\partial_q}(\beta) := \widehat{\mathrm{HHH}}^{\partial_q}(\beta)^{\mathtt{f}_\beta} \cong R_\ell^{\mathtt{f}_\beta} \otimes_{R_\ell} \widehat{\mathrm{HHH}}^{\partial_q}(\beta).$$

(2) Likewise, the *H_q–HOMFLYPT p–homology* is the doubly graded H_q–module

$$p\mathrm{HHH}^{\partial_q}(\beta, K+1) := p\mathrm{HHH}^{\partial_q}(\beta, K+1)^{\mathtt{f}_\beta} \cong R_\ell^{\mathtt{f}_\beta} \otimes_{R_\ell} p\widehat{\mathrm{HHH}}^{\partial_q}(\beta, K+1).$$

Corollary 3.8 *Given a braid β, both $\mathrm{HHH}^{\partial_q}(\beta)$ and $p\mathrm{HHH}^{\partial_q}(\beta)$ are link invariants that only depend on the closure of β as a link in $\mathbb{R}^3$. Moreover, these invariants satisfy the following properties*:

(i) *The slash homologies of* $\mathrm{HHH}^{\partial_q}(\beta)$ *and* $p\mathrm{HHH}^{\partial_q}(\beta, K+1)$ *are finite-dimensional.*

(ii) *Furthermore, the Euler characteristic of* $\mathrm{HHH}^{\partial_q}(\beta)$ *is equal to the HOMFLYPT polynomial of* $\hat{\beta}$ *in the formal variables* q *and* a*, while the Euler characteristic of* $p\mathrm{HHH}^{\partial_q}(\beta, K+1)$ *is equal to the* $\mathfrak{sl}_{K+1}$*–polynomial of* $\hat{\beta}$ *in a formal* q*–variable.*

(iii) *The Euler characteristic of the slash homology of* $\mathrm{HHH}^{\partial_q}(\beta)$ *is equal to the specialization of the HOMFLYPT polynomial of* $\hat{\beta}$ *at a root of unity* q*, while the Euler characteristic of the slash homology of* $p\mathrm{HHH}^{\partial_q}(\beta, K+1)$ *is the equal to the specialization of the* $\mathfrak{sl}_{K+1}$*–polynomial of* $\hat{\beta}$ *at a root of unity* q*.*

Proof For the first statement, we note that the twisting of the p–DG structure by the framing factor takes care of the Markov II move.

Next, the finite-dimensionality of the homology theories follows, by construction, from the fact that ${}^{f_{i_1}}B_{i_1}^{g_{i_1}} \otimes_R \cdots \otimes_R {}^{f_{i_m}}B_{i_m}^{g_{i_m}}$ is an H_q–module with 2^m–step filtration whose subquotients are isomorphic to R^f as left $R\#H_q$–modules; thus Corollary 2.4 applies. □

Remark 3.9 The previous discussion in Section 3.2 forces us to make a specialization $a = q^r t$ in the homotopy category of t and q–bigraded p–complexes to obtain a framed Markov II invariance. In particular, when $r = K = 0$, this forces the relation, on the Grothendieck group level, that $a = t = -1$. This specialization leads to a categorification of the Alexander skein relation.

4 Specialized homology theories

4.1 A singly graded homology

Fix $k \in \mathbb{N}$. Consider the H_q–Koszul complex in one-variable,

$$C_1 : 0 \to aq^2\Bbbk[x]^x \otimes \Bbbk[x]^x \xrightarrow{d_C} \Bbbk[x] \otimes \Bbbk[x] \to 0, \tag{4-1}$$

where d_C is the map $d_C(f) = (x^{kp+2} \otimes 1 + 1 \otimes x^{kp+2}) f$ and $k \in \mathbb{N}$. We regard the differential on the arrow as an endomorphsim of the Koszul complex, of (a, q)–bidegree $(-1, 2kp+2)$.

Lemma 4.1 *The commutator of the endomorphisms* d_C *and* $\partial_q \in H_q$ *is nullhomotopic on the Koszul complex* C_1*.*

Proof The commutator map $\phi := [d_C, \partial_q]$ is given by

$$\begin{tikzcd}
 & 0 \arrow[r] & \Bbbk[x]^x \otimes \Bbbk[x]^x \arrow[r, "x\otimes 1 - 1\otimes x"] \arrow[d, "\phi"] & \Bbbk[x]\otimes \Bbbk[x] \arrow[r] & 0 \\
0 \arrow[r] & \Bbbk[x]^x \otimes \Bbbk[x]^x \arrow[r, "x\otimes 1 - 1\otimes x"] & \Bbbk[x]\otimes \Bbbk[x] \arrow[r] & 0 &
\end{tikzcd}$$

where ϕ maps the bimodule generator $1\otimes 1 \in \Bbbk[x]^x \otimes \Bbbk[x]^x$ as follows:

$$\begin{aligned}
\phi(1\otimes 1) &= d_C(\partial_q(1\otimes 1)) - \partial_q d_C(1\otimes 1) \\
&= d_C(x\otimes 1 + 1\otimes x) - \partial_q(x^{kp+2}\otimes 1 + 1\otimes x^{kp+2}) \\
&= (x\otimes 1 + 1\otimes x)(x^{kp+2}\otimes 1 + 1\otimes x^{kp+2}) - 2(x^{kp+3}\otimes 1 + 1\otimes x^{kp+3}) \\
&= -x^{kp+3}\otimes 1 + x^{kp+2}\otimes x + x\otimes x^{kp+2} - 1\otimes x^{kp+3} \\
&= x^{kp+2}\otimes 1(1\otimes x - x\otimes 1) + (x\otimes 1 - 1\otimes x)1\otimes x^{kp+2} \\
&= (x\otimes 1 - 1\otimes x)(1\otimes x^{kp+2} - x^{kp+2}\otimes 1).
\end{aligned}$$

We may thus choose a nullhomotopy to be

$$\begin{tikzcd}
 & 0 \arrow[r] & \Bbbk[x]^x \otimes \Bbbk[x]^x \arrow[r, "x\otimes 1 - 1\otimes x"] \arrow[d, "\phi"] \arrow[dl, "h"'] & \Bbbk[x]\otimes \Bbbk[x] \arrow[r] & 0 \\
0 \arrow[r] & \Bbbk[x]^x \otimes \Bbbk[x]^x \arrow[r, "x\otimes 1 - 1\otimes x"] & \Bbbk[x]\otimes \Bbbk[x] \arrow[r] & 0 &
\end{tikzcd}$$

where h is given by multiplication by the element $1\otimes x^{kp+2} - x^{kp+2}\otimes 1$, and acts on the rest of the complex by zero. □

The Koszul complex C_n inherits the endomorphism d_C by forming the n–fold tensor product from the one-variable case. It follows, that for a given p–DG bimodule M over R_n, there is an induced differential, still denoted by d_C, given via the identification

$$\mathrm{HH}^{\partial_q}_\bullet(M) \cong \mathrm{H}_\bullet(M \otimes_{(R_n,R_n)} C_n), \tag{4-2}$$

where the induced differential acts on the right hand side by $\mathrm{Id}_M \otimes d_C$. By construction, d_C has Hochschild degree -1 and q–degree $2kp+2$.

Lemma 4.1 immediately implies the following.

Corollary 4.2 *The induced differential d_C on $\mathrm{HH}^{\partial_q}_\bullet(M)$ commutes with the H_q–action.* □

Remark 4.3 The differential d_C, first observed by Cautis [1], has the following more algebrogeometric meaning. Identifying $\mathrm{HH}^1(R_n)$ as vector fields on $\mathrm{Spec}(R_n) = \mathbb{A}^n$, $\mathrm{HH}^1(R_n)$ acts as differential operators on $\mathrm{HH}_\bullet(M)$ for any (R_n, R_n)–bimodule M, regarded as a coherent sheaf on $\mathbb{A}^n \times \mathbb{A}^n \cong T^*(\mathbb{A}^n)$. Under this identification d_C is given by, up to scaling by a nonzero number, contraction with the vector field

$$\zeta_C := \sum_{i=1}^{n} x_i^{kp+2} \frac{\partial}{\partial x_i}.$$

On the other hand, ∂_q is given by the polynomial derivation by the vector field

$$\zeta_q := \sum_{i=1}^{n} x_i^2 \frac{\partial}{\partial x_i}.$$

Since these two vector fields satisfy

$$[\zeta_C, \zeta_q] = \sum_{i,j} \Big[x_i^{kp+2} \frac{\partial}{\partial x_i}, x_j^2 \frac{\partial}{\partial x_j} \Big] = \sum_i \Big(2x_i^{kp+3} \frac{\partial}{\partial x_i} - (kp+2) x_i^{kp+3} \frac{\partial}{\partial x_i} \Big) = 0,$$

the two actions naturally commute with each other on $\mathrm{HH}_\bullet(M)$ via the Gerstenhaber module structure on $\mathrm{HH}_\bullet(M)$.

In a more general context, Hochschild homology is a Gerstenhaber module over Hochschild cohomology viewed as a Gerstenhaber algebra. We may view d_C and ∂_q as commuting elements in Hochschild cohomology ring but the element d_C acts on homology via cap product $\zeta_C \cap (\cdot)$ and the element ∂_q acts via a Lie algebra action $\mathcal{L}_{\zeta_q}(\cdot)$. The compatibility of these actions is given by the equation

$$\zeta_C \cap \mathcal{L}_{\zeta_q}(x) = [\zeta_C, \zeta_q] \cap x + \mathcal{L}_{\zeta_q}(\zeta_C \cap x).$$

Since $[\zeta_C, \zeta_q] = 0$, these actions commute.

Now we are ready to introduce a further collapsed p–homology theory of a braid closure. Let $\beta \in \mathrm{Br}_n$ be an n–stranded braid. We have associated to β a usual chain complex of H_q–equivariant Soergel bimodules T_β as in (3-2), of which we take $p\mathrm{HH}_\bullet^{\partial_q}$ for each term:

$$\text{(4-3)}\qquad \begin{tikzcd} & \vdots & \vdots & \vdots & \\ \cdots \arrow[r, "\partial_t"] & p\mathrm{HH}_i^{\partial_q}(pT_\beta^{m+1}) \arrow[u, "\partial_C"] \arrow[r, "\partial_t"] & p\mathrm{HH}_i^{\partial_q}(pT_\beta^{m}) \arrow[u, "\partial_C"] \arrow[r, "\partial_t"] & p\mathrm{HH}_i^{\partial_q}(pT_\beta^{m-1}) \arrow[u, "\partial_C"] \arrow[r, "\partial_t"] & \cdots \\ \cdots \arrow[r, "\partial_t"] & p\mathrm{HH}_{i+1}^{\partial_q}(pT_\beta^{m+1}) \arrow[u, "\partial_C"] \arrow[r, "\partial_t"] & p\mathrm{HH}_{i+1}^{\partial_q}(pT_\beta^{m}) \arrow[u, "\partial_C"] \arrow[r, "\partial_t"] & p\mathrm{HH}_{i+1}^{\partial_q}(pT_\beta^{m-1}) \arrow[u, "\partial_C"] \arrow[r, "\partial_t"] & \cdots \\ & \vdots \arrow[u, "\partial_C"] & \vdots \arrow[u, "\partial_C"] & \vdots \arrow[u, "\partial_C"] & \end{tikzcd}$$

Here, ∂_C is a p–differential arising from d_C as follows. By [11, Proposition 4.8], the p–Hochschild homology groups in a column above are identified with the terms in

$$\cdots \xrightarrow{d_C} \mathrm{HH}^{\partial_q}_{2i+1}(pT^m_\beta) = \cdots \\ \cdots = \mathrm{HH}^{\partial_q}_{2i+1}(pT^m_\beta) \xrightarrow{d_C} \mathrm{HH}^{\partial_q}_{2i}(pT^m_\beta) \xrightarrow{d_C} \mathrm{HH}^{\partial_q}_{2i-1}(pT^m_\beta) = \cdots, \tag{4-4}$$

where each term in odd Hochschild degree is repeated $p-1$ times. Here the horizontal differential is the p–Hochschild induced map of the topological differential, which we have denoted by ∂_t to indicate its origin. On the arrows connecting even and odd Hochschild degree terms, we put the map d_C while keeping the repeated terms connected by identity maps. This defines a p–complex structure, denoted by ∂_C, in each column in diagram (4-3). The p–differential ∂_C commutes with the H_q–action on each term by Corollary 4.2. Denote the total p–differential by $\partial_T := \partial_t + \partial_C + \partial_q$, which collapses the double grading into a single q–grading.

Remark 4.4 We would like to emphasize an important point about the vertical grading collapse. In order to p–extend the Koszul complex (4-1) into a p–Koszul complex with ∂_C of degree two, we are forced to make the functor specialization from $[1]^a_d = a$ into $q^{2kp+2}[1]^q_\partial$, so that the p–extended complex looks like

$$pC_1 : 0 \to q^{2kp+4}\Bbbk[x]^x \otimes \Bbbk[x]^x[1]^q_\partial \xrightarrow{d_C} \Bbbk[x] \otimes \Bbbk[x] \to 0. \tag{4-5}$$

Taking tensor products of pC_1, this determines the correct vertical q–degree shifts in each column of diagram (4-3) of the p–Hochschild homology groups.

Notice that, on the level of Grothendieck groups, this has the effect of specializing the formal variable a into $-q^{2kp+2}$.

When $[1]^t_\partial = [1]^q_\partial$ and $a = q^{2kp+2}[1]^q_\partial$, the braiding complexes (3-17) specialize to

$$\begin{aligned} pT_i &:= q^{-kp-3}\big(B_i \xrightarrow{br_i} R[-1]^q_\partial\big), \\ pT'_i &:= q^{kp+3}\big(R[1]^q_\partial \xrightarrow{rb_i} q^{-2}B_i^{-(x_i+x_{i+1})}\big). \end{aligned} \tag{4-6}$$

Comparing (3-17) with (4-6), this forces

$$K = kp + 1. \tag{4-7}$$

This also explains the necessity of p–extension in the collapsed t and a direction in pHHH in the previous section: the homological shift in that direction needs to be p–extended to agree with the homological shift in the q–direction.

Furthermore, the bigrading in diagram (4-3) is now interpreted as a single grading, with both ∂_C and ∂_t raising q–degree by two.

Definition 4.5 Let β be an n stranded braid. The *untwisted* $\mathfrak{sl}_{kp+2}$ *p–homology* of β is the slash homology group

$$p\widehat{\mathrm{H}}(\beta, kp+2) := q^{-n(kp+1)}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}^{\partial_q}_{\bullet}(pT_\beta), \partial_T),$$

viewed as an object in $\mathcal{C}(\Bbbk, \partial_q)$. We will drop the $kp+2$ decoration whenever k is fixed and clear from context.

The homology group $p\widehat{\mathrm{H}}(\beta)$ is only singly graded as an object in $\mathcal{C}(\Bbbk, \partial_q)$. By construction, $p\widehat{\mathrm{H}}(\beta)$ is the slash homology with respect to the ∂_T action on $\bigoplus_{i,j} p\mathrm{HH}^{\partial_q}_i(pT^j_\beta)$; see diagram (4-4). The latter space is doubly graded by the topological degree and q–degree with values in $\mathbb{Z}\times\mathbb{Z}$ (the Hochschild a degree is already forced to be collapsed with the q degree to make the Cautis differential ∂_C homogeneous). However, as in Section 3.2, the Markov II invariance for the homology theory already requires one to collapse the t–grading onto the a–grading, thus also onto the q–grading. We will use $p\widehat{\mathrm{H}}_i(\beta)$ to stand for the homogeneous subspace sitting in some q–degree i.

Remark 4.6 This approach to a categorification of the Jones polynomial, at generic values of q, was first developed by Cautis [1]. We follow the exposition of Robert and Wagner from [13] and the closely related approach of Queffelec, Rose and Sartori [12].

4.2 Topological invariance

In this subsection, we establish the topological invariance of the untwisted homology theory.

Theorem 4.7 *The homology* $p\widehat{\mathrm{H}}(\beta, kp+2)$ *is a finite-dimensional framed link invariant depending only on the braid closure of* β.

Proof The proof of the theorem is similar to [11, Theorem 5.6]. It amounts to showing that taking slash homology of $p\mathrm{HH}^{\partial}_{\bullet}(\beta)$ with respect to ∂_T satisfies the Markov II move.

We start by discussing the normal H_q–equivariant Hochschild homology version. Let L be a link in $\mathbb{R}^3$ obtained as a braid closure $\hat{\beta}$, where $\beta \in \mathrm{Br}_n$ is an n–stranded braid. Recall that the homology groups $\mathrm{HH}^{\partial}_{\bullet}(L)$ are defined by tensoring a complex of Soergel bimodules M determined by β with the Koszul complex C_n and computing its termwise vertical (Hochschild) homology. The differential d_C is defined on the Koszul complex C_n. To emphasize its dependence on n, we will write d_C on C_n as d_n in this proof, and likewise write ∂_n for the p–extended differential on pC_n.

Since

$$C_{n+1} = C_n \otimes C_1' = C_n \otimes \Bbbk[x_{n+1}] \otimes \Lambda\langle dx_{n+1}\rangle \otimes \Bbbk[x_{n+1}],$$

the vertical differential may be inductively defined as

$$d_{n+1} = d_n \otimes \mathrm{Id} + \mathrm{Id} \otimes d_1'. \tag{4-8}$$

Here we have set $C_1' = \Bbbk[x_{n+1}] \otimes \Lambda\langle dx_{n+1}\rangle \otimes \Bbbk[x_{n+1}]$ equipped with part of the Cautis differential

$$d_1' := x_{n+1}^{kp+2} \otimes \iota_{\frac{\partial}{\partial x_{n+1}}} \otimes 1 + 1 \otimes \iota_{\frac{\partial}{\partial x_{n+1}}} \otimes x_{n+1}^{kp+2}.$$

The notation ι denotes the contraction of dx_{n+1} with $\frac{\partial}{\partial x_{n+1}}$. Under p–extension, write ∂_C for the p–extended Cautis differential and ∂_1' as the p–extended differential of d_1'.

We start by reexamining the diagram in Figure 1 with the shifts in (4-6). It will be helpful to keep the a and t gradings separate for the proof, with it understood that $[1]^a_\partial = q^{2kp+2}[1]^q_\partial$ and $[1]^t_\partial = [1]^q_\partial$. Thus we have a short exact sequence as in Figure 3. Further, the sequence splits as bimodules over (R_n, R_n) (see the proof of [11, Proposition 4.12] for an explicit splitting).

We claim that, as modules over $\Bbbk[\partial_T]/(\partial_T^p)$, the p–homology groups

$$p\mathrm{HH}^{\partial_q}_\bullet((M \otimes \Bbbk[x_{n+1}]) \otimes_{R_{n+1}} T_n)$$

fit into a distinguished triangle

$$\mathrm{H}^{/}_\bullet(pC_n \otimes_{(R_n,R_n)} (M \otimes_{R_n} pY_2)) \to \mathrm{HH}^{\partial_q}_\bullet((M \otimes \Bbbk[x_{n+1}]) \otimes_{R_{n+1}} pT_n) \\ \to \mathrm{H}^{/}_\bullet(pC_n \otimes_{(R_n,R_n)} (M \otimes_{R_n} pY_1)) \xrightarrow{[1]} \tag{4-9}$$

after taking vertical slash (p–Hochschild) homology. Note that this p–complex triangle is in reverse order of the filtration in Figure 3.

Indeed, since ∂_C acts on the pY_1 and pY_2 tensor factors via ∂_1', it suffices to check that ∂_1' preserves the submodule arising from pY_2 and presents the part arising from pY_1 as a quotient. To do this, we reexamine the sequence in Figure 1 under vertical slash (p–Hochschild) homology, with the auxiliary a and t–gradings. The part pY_2, under vertical homotopy equivalence, contributes to the horizontal (topological) complex

$$pY_2' := \left(q^{-kp-3} R_{n+1} \xrightarrow{\partial_t = \mathrm{Id}} q^{-kp-3} R_{n+1}[-1]^t_\partial\right) \tag{4-10a}$$

sitting entirely in p–Hochschild degree 0. Likewise, the part pY_1 contributes to the horizontal

$$pY_1' := \left(q^{-kp+1} R_{n+1}^{x_n+3x_{n+1}}[1]^a_\partial \xrightarrow{\partial_t = 2(x_{n+1}-x_n)} q^{-kp-1} R_{n+1}^{2x_{n+1}}[1]^a_\partial[-1]^t_\partial\right) \tag{4-10b}$$

$$
\begin{tikzcd}
q^{-kp+1}R_{n+1}^{x_n+3x_{n+1}}[1]^a_\partial \arrow[r, "2(x_{n+1}-x_n)"] \arrow[d] \arrow[dd, bend right, "\phi"'] & q^{-kp-1}R_{n+1}^{2x_{n+1}}[1]^a_\partial[-1]^t_\partial \arrow[d] \arrow[dd, bend left, "\mathrm{Id}"] \\
0 \arrow[r] & 0 \\
q^{-kp-1}({}^{x_{n+1}}B_n^{x_{n+1}})[1]^a_\partial \arrow[r, "br"] \arrow[d, "x_{n+1}\otimes 1-1\otimes x_{n+1}"] \arrow[dd, bend right, "\widetilde{br}"'] & q^{-kp-1}R_{n+1}^{2x_{n+1}}[1]^a_\partial[-1]^t_\partial \arrow[d, "0"] \\
q^{-kp-3}B_n \arrow[r, "br"] \arrow[dd, bend right, "2\mathrm{Id}"'] & q^{-kp-3}R_{n+1}[-1]^t_\partial \arrow[dd, bend left, "2\mathrm{Id}"] \\
q^{-kp-1}\widetilde{R}_{n+1}^{x_n+x_{n+1}}[1]^a_\partial \arrow[r] \arrow[d, "(x_{n+1}-x_n)\otimes 1-1\otimes(x_{n+1}-x_n)"] & 0 \arrow[d] \\
q^{-kp-3}B_n \arrow[r, "br"] & q^{-kp-3}R_{n+1}[-1]^t_\partial
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[d] \\
:= pY_1 \arrow[d] \\
= pC_1'\otimes_{(\Bbbk[x_{n+1}],\Bbbk[x_{n+1}])} pT_n \arrow[d] \\
:= pY_2 \arrow[d] \\
0
\end{tikzcd}
$$

Figure 3

sitting entirely in p–Hochschild degrees $1,\dots,p-1$. Since ∂_1' decreases the a–degree by one (ie acting vertically downwards), pY_2' must be preserved under ∂_1', acting upon it trivially, and pY_1' is equipped with the quotient action of ∂_1'.

By the above discussion, $\partial_T = \partial_t + \partial_C + \partial_q$ acts on the term containing pY_2' only through $\partial_t + \partial_q$. Since this term is the cone of the identity map, it is nullhomotopic and thus

$$\mathrm{H}^{/}_{\bullet}(pC_n \otimes_{(R_n,R_n)} (M \otimes_{R_n} pY_2)) \cong 0.$$

Consequently, using that $[1]^a_\partial = q^{2kp+2}[1]^t_\partial$, we have an isomorphism

$$\begin{aligned}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_\bullet((M\otimes \Bbbk[x_{n+1}])\otimes_{R_{n+1}} pT_n),\partial_T) &\cong \mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_\bullet(M\otimes_{R_n} pY_1'),\partial_T)\\ &\cong q^{kp+1}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_\bullet(M),\partial_T)^{2x_n}.\end{aligned}$$

The q^{kp+1} factor is canceled out in the overall shift of $p\widehat{\mathrm{H}}$. This finishes the first part of Markov II move.

The other case of the Markov II move is entirely similar, which we omit.

Finally, the finite-dimensionality of $p\widehat{\mathrm{H}}(\beta)$ follows from Corollary 2.4. □

To obtain a categorical link invariant, we need to introduce a p–differential twisting to correct the framing factor occurring in Theorem 4.7, as done in [11, Section 5.3]. For a braid $\beta \in \mathrm{Br}_n$ whose closure is a framed link with ℓ components, choose for each framed component of $\hat{\beta}$ in β a single strand in β that lies in that component after closure, say, the i_r^{th} strand is chosen for the r^{th} component. Then define the polynomial ring $\Bbbk[x_{i_1},\dots,x_{i_\ell}]$ as a subring of $\Bbbk[x_1,\dots,x_n]$ generated by the chosen variables. Set

$$\Bbbk[x_{i_1},\dots,x_{i_\ell}]^{\mathtt{f}_\beta} := \Bbbk[x_{i_1},\dots,x_{i_\ell}]\cdot 1_\beta, \quad \partial(1_\beta) := -\sum_{r=1}^{\ell} 2\mathtt{f}_r x_{i_r} 1_\beta. \tag{4-11}$$

Then we make the twisting of H_q–modules on the $p\mathrm{HH}_\bullet$–level termwise on $p\mathrm{HH}_\bullet(pT_\beta^i)$,

$$p\mathrm{HH}_\bullet^{\mathtt{f}_\beta}(pT_\beta) := p\mathrm{HH}_\bullet(pT_\beta)\otimes_{\Bbbk[x_{i_1},\dots,x_{i_\ell}]}\Bbbk[x_{i_1},\dots,x_{i_\ell}]^{\mathtt{f}_\beta}. \tag{4-12}$$

Definition 4.8 Given $\beta \in \mathrm{Br}_n$ whose closure is a framed link with ℓ components, the $\mathfrak{sl}_{kp+2}$ *p–homology* is the object

$$p\mathrm{H}(\beta, kp+2) := q^{-n(kp+1)}\mathrm{H}^{/}_{\bullet}(p\mathrm{HH}_\bullet^{\mathtt{f}_\beta}(pT_\beta), \partial_T)$$

in the homotopy category $\mathcal{C}(\Bbbk, \partial_q)$.

As done for pHHH, we will often drop $kp+2$ in the notation of the homology.

Theorem 4.9 *The $\mathfrak{sl}_{kp+2}$ p–homology $p\mathrm{H}(\beta, kp+2)$ is a singly graded, finite-dimensional link invariant depending only on the braid closure of β as a link in $\mathbb{R}^3$. Furthermore, when $k \in 2\mathbb{Z}$, its graded Euler characteristic*

$$\chi(p\mathrm{H}(L, kp+2)) := \sum_i q^i \dim_{\Bbbk}(p\mathrm{H}_i(L, kp+2))$$

is equal to the Jones polynomial evaluated at a $2p^{th}$ root of unity.

Proof The above framing twisting compensates for the linear factors appearing in Markov II moves, thus establishing the topological invariance of $p\mathrm{H}(\beta)$.

For the last statement, we will use the fact that the Euler characteristic does not change before or after taking slash homology. This is because, as with the usual chain complexes, taking slash homology only gets rid of acyclic summands whose Euler characteristics are zero.

Let us revisit diagram (4-4). Before collapsing the t and q–gradings, the diagram arises by p–extending $\mathrm{HH}_\bullet(T_\beta)$ in the vertical (t–)direction. Let $P_\beta(v,t)$ be the Poincaré polynomial of the bigraded complex $\mathrm{HH}_\bullet(T_\beta)$ where, for now, v and t are treated as formal variables coming from q and t grading shifts. As shown by Cautis [1], $P_\beta(v,-1)$ is the $\mathfrak{sl}_{kp+2}$ polynomial of the link $\hat{\beta}$ in the variable v.

The p–extension in the topological direction is equivalent to categorically specializing $[1]^t_d$ to $[1]^q_\partial$. It has the effect, on the Euler characteristic level, of specializing $t=-1$. Thus we obtain that the Euler characteristic of $p\mathrm{H}(\beta)$ is equal to $P_\beta(v=q,t=-1)$. This the evaluation of the $\mathfrak{sl}_{kp+2}$ polynomial evaluated at a $2p^{\text{th}}$ root of unity q. When $k\in 2\mathbb{Z}$, we have $q^{kp+2}=q^2$ in

$$\mathbb{O}_p := K_0(\mathcal{C}(\Bbbk,\partial_q)) \cong \frac{\mathbb{Z}[q]}{(1+q^2+\cdots+q^{2(p-1)})},$$

so this evaluation is equal to the value of the Jones polynomial in $\mathbb{O}_p$. $\square$

5 Examples

In this section we compute the various homologies constructed earlier for $(2,n)$ torus links $T_{2,n}$. Note that there are no framing factors to incorporate in this family of examples. The calculations are straightforward modifications of the computations made in [6] and adjusted for p–DG notions in [11, Section 6]. We refer the reader to [11] and just state the modified results here with minimal explanation.

Throughout the remainder of this subsection, let $R=\Bbbk[x_1,x_2]$, $B=B_1$, and $T=T_1$.

5.1 The HOMFLYPT homology of the $(2,n)$ torus link

First note that the homology of the n–component unlink L_0 is

$$p\mathrm{HHH}^{\partial_q}(L_0,K+1) \cong \bigotimes_{i=1}^{n} q^{-K}(\Bbbk[x_i]\oplus q^{2K+2}[1]^t_\partial \Bbbk[x_i]^{2x_i}).$$

The following simplification of $T^{\otimes n}$ is proved in the same way as [11, Lemma 6.1].

Lemma 5.1 *In $\mathcal{C}^{\partial_q}(R,R,\partial_0)$, one has $T^{\otimes n} \cong (q^{-K-1}[-1]^t_\partial)^n$*

$$\big(q^{2(n-1)}B^{(n-1)e_1}[n]^t_\partial \xrightarrow{p_n} q^{2(n-2)}B^{(n-2)e_1}[n-1]^t_\partial \xrightarrow{p_{n-1}} \cdots \\ \cdots \xrightarrow{p_3} q^2B^{e_1}[2]^t_\partial \xrightarrow{p_2} B[1]^t_\partial \xrightarrow{br} R\big),$$

where

$$p_{2i} = 1\otimes(x_2-x_1)-(x_2-x_1)\otimes 1, \quad p_{2i+1} = 1\otimes(x_2-x_1)+(x_2-x_1)\otimes 1.$$

The following result is proved in the same way as [11, Proposition 6.3]

Proposition 5.2 *The bigraded H_q–HOMFLYPT p–homology of a $(2,n)$ torus knot, as an H_q–module depends on the parity of n.*

(i) *If n is odd, it is*

$$q^{-nK-2n-2K}[-n]^t_\partial(q^{2K+2}[1]^t_\partial\Bbbk[x]^{2x}\oplus q^{4K+4}[2]^t_\partial\Bbbk[x]^{4x})$$

$$\oplus \bigoplus_{i\in\{2,4,\dots,n-1\}} q^{-nK-2n-2K}[i-n]^t_\partial\left(q^{2(i-1)}\Bbbk[x]^{2(i-1)x}\oplus q^{2K}[1]^t_\partial\begin{pmatrix} q^{2i}\Bbbk[x]\\ \oplus\\ q^{2i+2}\Bbbk[x]\end{pmatrix} \oplus q^{2i+4+4K}[2]^t_\partial\Bbbk[x]^{2(i+1)x}\right)$$

with the H_q–structure on the middle object

$$\begin{pmatrix}\Bbbk[x]\\ \oplus\\ \Bbbk[x]\end{pmatrix}$$

given by

$$\begin{pmatrix} 2ix & 0\\ -2 & (2i+2)x\end{pmatrix}.$$

(ii) *If n is even, it is*

$$q^{-nK-2n-2K}[-n]^t_\partial(q^{2K+2}[1]^t_\partial\Bbbk[x]^{2x}\oplus q^{4K+4}[2]^t_\partial\Bbbk[x]^{4x})$$

$$\oplus \bigoplus_{i\in\{2,4,\dots,n-2\}} q^{-nK-2n-2K}[i-n]^t_\partial\begin{pmatrix} q^{2(i-1)}\Bbbk[x]^{2(i-1)x}\\ \oplus\\ q^{2K}[1]^t_\partial\begin{pmatrix} q^{2i}\Bbbk[x]\\ \oplus\\ q^{2i+2}\Bbbk[x]\end{pmatrix}\\ \oplus\\ q^{2i+4+4K}[2]^t_\partial\Bbbk[x]^{2(i+1)x}\end{pmatrix}$$

$$\oplus\, q^{-nK-2n-2K}[-n]^t_\partial \begin{pmatrix} q^{2(n-1)}\Bbbk[x_1,x_2]^{(n-1)(x_1+x_2)} \\ \oplus \\ q^{2K}[1]^t_\partial \begin{pmatrix} q^{2n}\Bbbk[x_1,x_2] \\ \oplus \\ q^{2n+2}\Bbbk[x_1,x_2] \end{pmatrix} \\ \oplus \\ q^{2n+4+4K}[2]^t_\partial \Bbbk[x_1,x_2]^{(n+2)(x_1+x_2)} \end{pmatrix} [n]^t_\partial$$

with the H_q–structure on the middle object

$$\begin{pmatrix} q^{2i}\Bbbk[x] \\ \oplus \\ q^{2i+2}\Bbbk[x] \end{pmatrix}$$

given by

$$\begin{pmatrix} 2ix & 0 \\ -2 & (2i+2)x \end{pmatrix}$$

and the H_q–structure on the middle object

$$\begin{pmatrix} q^{2n}\Bbbk[x_1,x_2] \\ \oplus \\ q^{2n+2}\Bbbk[x_1,x_2] \end{pmatrix}$$

given by

$$\begin{pmatrix} (n+1)x_1+(n-1)x_2 & 0 \\ -2 & n(x_1+x_2)+2x_2 \end{pmatrix}.$$

Corollary 5.3 *In the stable category of H_q–modules, the slash homology of the H_q–HOMFLYPT p–homology of a $(2,n)$ torus link $p\mathrm{HHH}^{\partial_q}(T_{2,n},K+1)$ depends on the parity of n.*

(i) *If n is odd, it is*

$$q^{-nK-2n-2K}[-n]^t_\partial(q^{p+2K}V^q_{p-2}[1]^t_\partial \oplus q^{p+4K}V^q_{p-4}[2]^t_\partial)$$
$$\oplus \bigoplus_{i\in\{2,4,\dots,n-1\}} q^{-nK-2n-2K}[-n]^t_\partial\left(q^pV^q_{p-2(i-1)} \oplus \begin{pmatrix} q^{p+2K}V^q_{p-2i} \\ \oplus \\ q^{p+2K}V^q_{p-2i-2}\end{pmatrix}[1]^t_\partial \oplus q^{p+2+4K}V^q_{p-2(i+1)}[2]^t_\partial \right)[i]^t_\partial.$$

(ii) *If n is even, it is*

$$
\begin{aligned}
&q^{-nK-2n-2K}[-n]^t_\partial (q^{p+2K} V^q_{p-2}[1]^t_\partial \oplus q^{p+4K} V^q_{p-4}[2]^t_\partial) \\
&\quad \oplus \bigoplus_{i\in\{2,4,\dots,n-2\}} q^{-nK-2n-2K}[-n]^t_\partial \Bigg(q^p V^q_{p-2(i-1)} \\
&\qquad\qquad \oplus \begin{pmatrix} q^{p+2K} V^q_{p-2i} \\ \oplus \\ q^{p+2K} V^q_{p-2(i+1)} \end{pmatrix} [1]^t_\partial \oplus q^{p+2+4K} V^q_{p-2(i+1)}[2]^t_\partial \Bigg)[i]^t_\partial \\
&\quad \oplus q^{-nK-2n-2K}[-n]^t_\partial \begin{pmatrix} q^{2p} V^q_{p-(n-1)} \otimes V^q_{p-(n-1)} \\ \oplus \\ (q^{2p+2K} V^q_{p-n-1} \otimes V^q_{p-n+1} \oplus q^{2p+2K} V^q_{p-n} \otimes V^q_{p-n-2})[1]^t_\partial \\ \oplus \\ q^{2p+4K} V^q_{p-(n+2)} \otimes V^q_{p-(n+2)}[2]^t_\partial \end{pmatrix} [n]^t_\partial .
\end{aligned}
$$

5.2 The $\mathfrak{sl}_{kp+2}$–homology of the $(2\,,n)$ torus link

To compute this homology, we will use the following tool. If

$$M_\bullet = \left(\cdots \xrightarrow{\partial_t} M_{i+1} \xrightarrow{\partial_t} M_i \xrightarrow{\partial_t} M_{i-1} \xrightarrow{\partial_t} \cdots\right),$$

we write $\mathcal{T}(M_\bullet)$ to be the total complex whose p–differential is the sum $\partial_T := \partial_t + \partial_q$.

Proposition 5.4 [11, Proposition 6.6] *Let $M_\bullet$ be a contractible p–complex of $H_q = \Bbbk[\partial_q]/(\partial_q^p)$–modules. Then the complex $(\mathcal{T}(M_\bullet), \partial_T = \partial_t + \partial_q)$ is acyclic.*

We will be applying Proposition 5.4 in the following situation. Suppose $N_\bullet$ is a p–complex of H_q–modules whose boundary maps preserve the H_q–module structure. Further, let $M_\bullet$ be a sub–p–complex that is closed under the H_q–action, and there is a map σ on $M_\bullet$ as in Proposition 5.4 that preserves the H_q–module structure. Then, when totalizing the p–complexes, we have $\mathcal{T}(M_\bullet) \subset \mathcal{T}(N_\bullet)$ and the natural projection map

$$\mathcal{T}(N_\bullet) \to \mathcal{T}(N_\bullet)/\mathcal{T}(M_\bullet)$$

is a quasi-isomorphism. Similarly, if $M_\bullet$ is instead a quotient complex of $N_\bullet$ that satisfies the condition of Proposition 5.4, and $K_\bullet$ is the kernel of the natural projection map

$$0 \to K_\bullet \to N_\bullet \to M_\bullet \to 0,$$

then the inclusion map of totalized complexes $\mathcal{T}(K_\bullet) \to \mathcal{T}(N_\bullet)$ is a quasi-isomorphism.

We modify the calculation in the previous section of the $(2,n)$ torus link to include the Cautis p–differential ∂_C. Recall that in this singly graded theory, $a = q^{2kp+2}[1]_\partial^q$ and $t = [1]_\partial^q$.

The Hochschild homology $p\mathrm{HH}_\bullet^{\partial_q}(R)$ with the induced Cautis differential ∂_C is given by

$$
\begin{tikzcd}
& R & \\
q^4 R^{2x_1}[1]_\partial^a \arrow[ur, "x_1^{kp+2}"] & & q^4 R^{2x_2}[1]_\partial^a \arrow[ul, "x_2^{kp+2}"'] \\
& q^8 R^{2e_1}[2]_\partial^a \arrow[ul, "-x_2^{kp+2}"] \arrow[ur, "x_1^{kp+2}"'] &
\end{tikzcd}
\tag{5-1}
$$

First we study $p\mathrm{HH}_\bullet^{\partial_q}(br)\colon p\mathrm{HH}_\bullet^{\partial_q}(B)[1]_\partial^q \to p\mathrm{HH}_\bullet^{\partial_q}(R)$,

$$
\begin{tikzcd}
R[1]_\partial^q \arrow[rrr, "1\mapsto 1"] & & & R \\
q^{2kp+4}R[2]_\partial^q \arrow[u, "x_1^{kp+2}+x_2^{kp+2}"] & q^{2kp+6}R[2]_\partial^q \arrow[ul, "x_2^{kp+2}(x_2-x_1)"'] \arrow[r, "\left(\begin{smallmatrix}1 & 0\\ 1 & x_2-x_1\end{smallmatrix}\right)"] & q^{2kp+4}R^{2x_1}[1]_\partial^q \arrow[ur, "x_1^{kp+2}"] & q^{2kp+4}R^{2x_2}[1]_\partial^q \arrow[u, "x_2^{kp+2}"'] \\
q^{4kp+10}R^{3e_1}[3]_\partial^q \arrow[u, "x_2^{kp+2}(x_1-x_2)"] \arrow[ur, "x_1^{kp+2}+x_2^{kp+2}"'] \arrow[rrr, "1\mapsto(x_2-x_1)"] & & & q^{4kp+8}R^{2e_1}[2]_\partial^q \arrow[ul, "-x_2^{kp+2}"] \arrow[u, "x_1^{kp+2}"']
\end{tikzcd}
\tag{5-2}
$$

where the object $q^{2kp+4}R[2]_\partial^q \oplus q^{2kp+6}R[2]_\partial^q$ in the left square is twisted by the matrix

$$
\begin{pmatrix} 2x_1 & 0 \\ 2 & x_1+3x_2 \end{pmatrix}.
\tag{5-3}
$$

Filtering the total complex (5-2) we obtain that it is quasi-isomorphic to

$$\Bbbk\langle x_1^a x_2^b \,|\, 0 \le a \le kp+2,\, 0 \le b \le kp+1\rangle[1]_\partial^q \xrightarrow{1} \Bbbk\langle x_1^a x_2^b \,|\, 0 \le a, b \le kp+1\rangle,$$

which is quasi-isomorphic to

$$\Bbbk\langle x_1^{kp+2}, x_1^{kp+2}x_2, \dots, x_1^{kp+2}x_2^{kp+1}\rangle[1]_\partial^q,$$

where

$$\partial(x_1^{kp+2}x_2^j) = (kp+2+j)x_1^{kp+2}x_2^{j+1}.$$

This is quasi-isomorphic to $q^5 V_1[1]_\partial^q$ if $k=0$. If $k>0$, it's quasi-isomorphic to

$$(q^{3p+2}V_{p-2} \oplus q^{4kp+4}V_2)[1]_\partial^q.$$

Next we analyze

$$p\mathrm{HH}^{\partial_q}_{\bullet}(p_{2i+1})\colon p\mathrm{HH}^{\partial_q}_{\bullet}(q^{4i}B^{2ie_1}[2i+1]^q_\partial)\to \mathrm{HH}^{\partial_q}_{\bullet}(q^{4i-2}B^{(2i-1)e_1}[2i]^q_\partial),$$

$$\begin{tikzcd}[column sep=small]
& & q^{4i}R^{2ie_1}[2i+1]^q_\partial & \\
p\mathrm{HH}^{\partial_q}_{\bullet}(q^{4i}B^{2ie_1}[2i+1]^q_\partial) = & q^{4i+2kp+4}R[2i+1]^q_\partial[1]^q_\partial \arrow[ur, "x_1^{kp+2}+x_2^{kp+2}"] & & q^{4i+2kp+6}R[2i+1]^q_\partial[1]^q_\partial \arrow[ul, "x_2^{kp+2}(x_2-x_1)"'] \\
& & q^{4i+4kp+10}R^{(2i+3)e_1}[2i+1]^q_\partial[2]^q_\partial \arrow[ul, "x_2^{kp+2}(x_1-x_2)"] \arrow[ur, "x_1^{kp+2}+x_2^{kp+2}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& & q^{4i-2}R^{(2i-1)e_1}[2i]^q_\partial & \\
p\mathrm{HH}^{\partial_q}_{\bullet}(q^{4i-2}B^{(2i-1)e_1}[2i]^q_\partial) = & q^{4i+2kp+2}R[2i]^q_\partial[1]^q_\partial \arrow[ur, "x_1^{kp+2}+x_2^{kp+2}"] & & q^{4i+2kp+4}R[2i]^q_\partial[1]^q_\partial \arrow[ul, "x_2^{kp+2}(x_2-x_1)"'] \\
& & q^{4i+4kp+8}R^{(2i+2)e_1}[2i]^q_\partial[2]^q_\partial \arrow[ul, "x_2^{kp+2}(x_2-x_1)"] \arrow[ur, "x_1^{kp+2}+x_2^{kp+2}"'] &
\end{tikzcd}$$

where the differentials for both objects in the middle horizontal rows of the diagrams above are twisted by (5-3) and $p\mathrm{HH}^{\partial_q}_{\bullet}(p_{2i+1}) = 2(x_2-x_1)$ (diagonal multiplication by $2(x_2-x_1)$). Filtering this total complex yields the total complex

$$\begin{tikzcd}
q^{4i}\Bbbk\langle x_1^a x_2^b \mid 0\le a\le kp+2,\, 0\le b\le kp+1\rangle[2i+1]^q_\partial \arrow[d, "2(x_2-x_1)"] \\
q^{4i-2}\Bbbk\langle x_1^a x_2^b \mid 0\le a\le kp+2,\, 0\le b\le kp+1\rangle[2i]^q_\partial
\end{tikzcd} \tag{5-4}$$

This is quasi-isomorphic to

$$q^{4i}\Bbbk\langle x_1^{kp+2}, x_1^{kp+2}x_2,\dots,x_1^{kp+2}x_2^{kp+1}\rangle[2i+1]^q_\partial \oplus q^{4i-2}\Bbbk\langle 1, x_1,\dots,x_1^{kp+1}\rangle[2i]^q_\partial$$

where the differential on the basis elements is given by

$$\begin{tikzcd}
x_1^{kp+2} \arrow[d, "kp+4i+2"'] & & 1 \arrow[d, "4i-2"] \\
x_1^{kp+2}x_2 \arrow[d, "kp+4i+3"'] & \oplus & x_1 \arrow[d, "4i-1"] \\
\vdots \arrow[d, "kp+4i+kp+2"'] & & \vdots \arrow[d, "4i-2+kp"] \\
x_1^{kp+2}x_2^{kp+1} & & x_1^{kp+1}
\end{tikzcd}$$

Thus the total homology of this complex is isomorphic to X_i, which is defined by

$$
\text{(5-5)}\quad
\begin{cases}
\begin{pmatrix} q^{4i}(q^{2(kp+2)+j}V_j \oplus q^{2(kp+2+j+1+(k-1)p)+p-j}V_{p-j})[2i+1]^q_\partial \\ \oplus \\ q^{4i-2}(q^{\bar{j}}V_{\bar{j}} \oplus q^{2((k-1)p+\bar{j}+1)+p-\bar{j}}V_{p-\bar{j}})[2i]^q_\partial \end{pmatrix} & \text{if } j,\bar{j}\neq 0,\\
\begin{pmatrix} q^{4i}(q^{2(kp+2)}V_0 \oplus q^{2(kp+2+kp+1)}V_0)[2i+1]^q_\partial \\ \oplus \\ q^{4i-2}(q^{\bar{j}}V_{\bar{j}} \oplus q^{2((k-1)p+\bar{j}+1)+p-\bar{j}}V_{p-\bar{j}})[2i]^q_\partial \end{pmatrix} & \text{if } j=0,\ \bar{j}\neq 0,\\
\begin{pmatrix} q^{4i}(q^{2(kp+2)+j}V_j \oplus q^{2(kp+2+j+1+(k-1)p)+p-j}V_{p-j})[2i+1]^q_\partial \\ \oplus \\ q^{4i-2}(V_0 \oplus q^{2(kp+1)}V_0)[2i]^q_\partial \end{pmatrix} & \text{if } j\neq 0,\ \bar{j}=0,
\end{cases}
$$

where $j\in\{0,\dots,p\}$ such that $4i+2+j$ is divisible by p and $\bar{j}\in\{0,\dots,p\}$ such that $4i-2+\bar{j}$ is divisible by p.

Once again when n is even, the leftmost term in $T^{\otimes n}$ maps by zero into the rest of the complex so we have to understand the total homology of $p\mathrm{HH}_\bullet(q^{2(n-1)}B^{(n-1)e_1}[n]^q_\partial)$. Filtering

$$
\begin{tikzcd}[column sep=small]
 & q^{2(n-1)}R^{(n-1)e_1}[n]^q_\partial & \\
q^{2(n+1)+2kp}R^{(n-1)e_1}[n]^q_\partial[1]^q_\partial \arrow[ur, "x_1^{kp+2}+x_2^{kp+2}"] & & q^{2(n+2)+2kp}R^{(n-1)e_1}[n]^q_\partial[1]^q_\partial \arrow[ul, "x_2^{kp+2}(x_2-x_1)"'] \\
 & q^{2(n+4)+4kp}R^{(n+2)e_1}[n]^q_\partial[2]^q_\partial \arrow[ul, "x_2^{kp+2}(x_1-x_2)"] \arrow[ur, "x_1^{kp+2}+x_2^{kp+2}"'] &
\end{tikzcd}
$$

where the middle terms $q^{2(n+1)+2kp}R^{(n-1)e_1}[n]^q_\partial[1]^q_\partial \oplus q^{2(n+2)+2kp}R^{(n-1)e_1}[n]^q_\partial[1]^q_\partial$ are further twisted by the matrix (5-3), yields that the diagram above is quasi-isomorphic to

$$
\text{(5-6)}\qquad Y_{\frac{n}{2}} = q^{2(n-1)}\Bbbk\langle x_1^a x_2^b \mid 0\le a\le kp+2,\ 0\le b\le kp+1\rangle[n]^q_\partial
$$

with a differential inherited from the polynomial algebra and twisted by $(n-1)e_1$. All of these computations together with an overall shift of $q^{-(n+2)kp-3n-2}[-n]^q_\partial$ yields the slash homology of the $(2,n)$ torus link for $k>0$,

$$
\text{(5-7)}\quad p\mathrm{H}(T_{2,n},kp+2) \cong
\begin{cases}
q^{-(n+2)kp-3n-2}[-n]^q_\partial\big((q^{3p+2}V_{p-2}\oplus q^{4kp+4}V_2)[1]^q_\partial \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} X_i\big) & \text{if } 2\nmid n,\\
q^{-(n+2)kp-3n-2}[-n]^q_\partial\big((q^{3p+2}V_{p-2}\oplus q^{4kp+4}V_2)[1]^q_\partial \oplus \bigoplus_{i=1}^{\frac{n-2}{2}} X_i \oplus \mathrm{H}^{/}_\bullet(Y_{\frac{n}{2}})\big) & \text{if } 2\mid n,
\end{cases}
$$

where X_i is the p–complex in (5-5) and $Y_{\frac{n}{2}}$ is the p–complex in (5-6).

$$\begin{tikzcd}
1 \arrow[r, "1"] \arrow[d, "1"] & x_2 \arrow[r, "2"] \arrow[d, "1"] & x_2^2 \arrow[r, "3"] \arrow[d, "1"] & \cdots \arrow[r, "p-1"] & x_2^{p-1} \arrow[r, "0"] \arrow[d, "1"] & x_2^p \arrow[r, "1"] \arrow[d, "1"] & x_2^{p+1} \arrow[r, "-2"] \arrow[d, "1"] & {} \\
x_1 \arrow[r, "1"] \arrow[d, "2"] & x_1x_2 \arrow[r, "2"] \arrow[d, "2"] & x_1x_2^2 \arrow[r, "3"] \arrow[d, "2"] & \cdots \arrow[r, "p-1"] & x_1x_2^{p-1} \arrow[r, "0"] \arrow[d, "2"] & x_1x_2^p \arrow[r, "1"] \arrow[d, "2"] & x_1x_2^{p+1} \arrow[r, "-2"] \arrow[d, "2"] & {} \\
\vdots \arrow[d, "p-1"] & \vdots \arrow[d, "p-1"] & \vdots \arrow[d, "p-1"] & \vdots & \vdots \arrow[d, "p-1"] & \vdots \arrow[d, "p-1"] & \vdots \arrow[d, "p-1"] & \\
x_1^{p-1} \arrow[r, "1"] \arrow[d, "0"] & x_1^{p-1}x_2 \arrow[r, "2"] \arrow[d, "0"] & x_1^{p-1}x_2^2 \arrow[r, "3"] \arrow[d, "0"] & \cdots \arrow[r, "p-1"] & x_1^{p-1}x_2^{p-1} \arrow[r, "0"] \arrow[d, "0"] & x_1^{p-1}x_2^p \arrow[r, "1"] \arrow[d, "0"] & x_1^{p-1}x_2^{p+1} \arrow[r, "-2"] \arrow[d, "0"] & {} \\
x_1^p \arrow[r, "1"] \arrow[d, "1"] & x_1^px_2 \arrow[r, "2"] \arrow[d, "1"] & x_1^px_2^2 \arrow[r, "3"] \arrow[d, "1"] & \cdots \arrow[r, "p-1"] & x_1^px_2^{p-1} \arrow[r, "0"] \arrow[d, "1"] & x_1^px_2^p \arrow[r, "1"] \arrow[d, "1"] & x_1^px_2^{p+1} \arrow[r, "-2"] \arrow[d, "1"] & {} \\
x_1^{p+1} \arrow[r, "1"] \arrow[d, "2"] & x_1^{p+1}x_2 \arrow[r, "2"] \arrow[d, "2"] & x_1^{p+1}x_2^2 \arrow[r, "3"] \arrow[d, "2"] & \cdots \arrow[r, "p-1"] & x_1^{p+1}x_2^{p-1} \arrow[r, "0"] \arrow[d, "2"] & x_1^{p+1}x_2^p \arrow[r, "1"] \arrow[d, "2"] & x_1^{p+1}x_2^{p+1} \arrow[r, "-2"] \arrow[d, "2"] & {} \\
x_1^{p+2} \arrow[r, "4"] & x_1^{p+2}x_2 \arrow[r, "5"] & x_1^{p+2}x_2^2 \arrow[r, "6"] & \cdots \arrow[r, "p+2"] & x_1^{p+2}x_2^{p-1} \arrow[r, "p+3"] & x_1^{p+2}x_2^p \arrow[r, "p+4"] & x_1^{p+2}x_2^{p+1} &
\end{tikzcd}$$

Figure 4

Decategorifying the slash homology, for instance on the Hopf link ($n=2$), we obtain that the Euler characteristic of $p\mathrm{H}(T_{2,2}, kp+2)$ is equal to

$$q^{-8}(q^2+q^4+q^6+q^8).$$

Finding the homology of $Y_{\frac{n}{2}}$ is nontrivial. In the example below we take $n=2$ which means we are computing part of the homology for the Hopf link. We also take $k=1$ just for convenience of notation.

We thus need to compute the homology of Z_1, given in Figure 4, where the arrows labeled -2 mean that the differential acts by $x_1^j x_2^{p+1} \mapsto -2x_1^{p+2}x_2^j$.

There is a large contractible summand Z_2 in the upper-left corner. Then there is short exact sequence of complexes

$$Z_2 \to Z_1 \to Z_3$$

where Z_3 is as in Figure 5. The second row from the bottom with the rightmost column, along with the third row from the bottom and second column from the right give a contractible summand Z_4 of Z_3:

$$Z_4 = \Bbbk\langle x_1^{p+1}+x_2^{p+1}, \dots, x_1^{p+1}x_2^{p-1}+x_1^{p-1}x_2^{p+1}\rangle \oplus \Bbbk\langle x_1^p+x_2^p, \dots, x_1^px_2^{p-1}+x_1^{p-1}x_2^p\rangle.$$

$$
\begin{tikzcd}
 & & & & & x_2^p \arrow[r, "1"] \arrow[d, "1"] & x_2^{p+1} \arrow[r, "-2"] \arrow[d, "1"] & {} \\
 & & & & & x_1x_2^p \arrow[r, "1"] \arrow[d, "2"] & x_1x_2^{p+1} \arrow[r, "-2"] \arrow[d, "2"] & {} \\
 & & & & & \vdots \arrow[d, "p-1"] & \vdots \arrow[d, "p-1"] & \\
 & & & & & x_1^{p-1}x_2^p \arrow[r, "1"] \arrow[d, "0"] & x_1^{p-1}x_2^{p+1} \arrow[r, "-2"] \arrow[d, "0"] & {} \\
x_1^p \arrow[r, "1"] \arrow[d, "1"] & x_1^px_2 \arrow[r, "2"] \arrow[d, "1"] & x_1^px_2^2 \arrow[r, "3"] \arrow[d, "1"] & \cdots \arrow[r, "p-1"] & x_1^px_2^{p-1} \arrow[r, "0"] \arrow[d, "1"] & x_1^px_2^p \arrow[r, "1"] \arrow[d, "1"] & x_1^px_2^{p+1} \arrow[r, "-2"] \arrow[d, "1"] & {} \\
x_1^{p+1} \arrow[r, "1"] \arrow[d, "2"] & x_1^{p+1}x_2 \arrow[r, "2"] \arrow[d, "2"] & x_1^{p+1}x_2^2 \arrow[r, "3"] \arrow[d, "2"] & \cdots \arrow[r, "p-1"] & x_1^{p+1}x_2^{p-1} \arrow[r, "0"] \arrow[d, "2"] & x_1^{p+1}x_2^p \arrow[r, "1"] \arrow[d, "2"] & x_1^{p+1}x_2^{p+1} \arrow[r, "-2"] \arrow[d, "2"] & {} \\
x_1^{p+2} \arrow[r, "4"] & x_1^{p+2}x_2 \arrow[r, "5"] & x_1^{p+2}x_2^2 \arrow[r, "6"] & \cdots \arrow[r, "p+2"] & x_1^{p+2}x_2^{p-1} \arrow[r, "p+3"] & x_1^{p+2}x_2^p \arrow[r, "p+4"] & x_1^{p+2}x_2^{p+1} &
\end{tikzcd}
$$

Figure 5

Then there is a short exact sequence of complexes

$$Z_4 \to Z_3 \to Z_5,$$

where Z_5 is

$$
\begin{tikzcd}
x_1^p \arrow[r, "1"] \arrow[d, "1"] & x_1^px_2 \arrow[r, "2"] \arrow[d, "1"] & x_1^px_2^2 \arrow[r, "3"] \arrow[d, "1"] & \cdots \arrow[r, "p-1"] & x_1^px_2^{p-1} \arrow[r, "0"] \arrow[d, "1"] & x_1^px_2^p \arrow[r, "1"] \arrow[d, "1"] & x_1^px_2^{p+1} \arrow[r, "-2"] \arrow[d, "1"] & {} \\
x_1^{p+1} \arrow[r, "1"] \arrow[d, "2"] & x_1^{p+1}x_2 \arrow[r, "2"] \arrow[d, "2"] & x_1^{p+1}x_2^2 \arrow[r, "3"] \arrow[d, "2"] & \cdots \arrow[r, "p-1"] & x_1^{p+1}x_2^{p-1} \arrow[r, "0"] \arrow[d, "2"] & x_1^{p+1}x_2^p \arrow[r, "1"] \arrow[d, "2"] & x_1^{p+1}x_2^{p+1} \arrow[r, "-2"] \arrow[d, "2"] & {} \\
x_1^{p+2} \arrow[r, "4"] & x_1^{p+2}x_2 \arrow[r, "5"] & x_1^{p+2}x_2^2 \arrow[r, "6"] & \cdots \arrow[r, "p+2"] & x_1^{p+2}x_2^{p-1} \arrow[r, "p+3"] & x_1^{p+2}x_2^p \arrow[r, "p+4"] & x_1^{p+2}x_2^{p+1} &
\end{tikzcd}
$$

Now let Z_6 be the contractible subcomplex of Z_5 generated by x_1^{p+1}. That is

$$Z_6 = \Bbbk\langle x_1^{p+1}, 1!x_1^{p+1}x_2 + a_0x_2^{p+1}, \dots, (p-1)!x_1^{p+1}x_2^{p-1} + a_{p-2}x_1^{p+2}x_2^{p-2}\rangle$$

for some coefficients $a_0, \dots, a_{p-2}$. Then there is a short exact sequence of complexes

$$Z_6 \to Z_5 \to Z_7$$

where Z_7 is

$$\begin{tikzcd}
x_1^p \arrow[r, "1"] & x_1^p x_2 \arrow[r, "2"] & x_1^p x_2^2 \arrow[r, "3"] & \cdots \arrow[r, "p-1"] & x_1^p x_2^{p-1} \arrow[r, "0"] & x_1^p x_2^p \arrow[r, "1"] \arrow[d, "1"] & x_1^p x_2^{p+1} \arrow[r, "-2"] \arrow[d, "1"] & {} \\
 & & & & & x_1^{p+1} x_2^p \arrow[r, "1"] \arrow[d, "2"] & x_1^{p+1} x_2^{p+1} \arrow[r, "-2"] \arrow[d, "2"] & {} \\
x_1^{p+2} \arrow[r, "4"] & x_1^{p+2} x_2 \arrow[r, "5"] & x_1^{p+2} x_2^2 \arrow[r, "6"] & \cdots \arrow[r, "p+2"] & x_1^{p+2} x_2^{p-1} \arrow[r, "p+3"] & x_1^{p+2} x_2^p \arrow[r, "p+4"] & x_1^{p+2} x_2^{p+1} &
\end{tikzcd}$$

Consider the contractible summand

$$Z_8 = \Bbbk\langle x_1^p, \dots, x_1^p x_2^{p-1}\rangle.$$

Then there is a short exact sequence

$$Z_8 \to Z_7 \to Z_9,$$

where Z_9 is

$$\begin{tikzcd}
 & & & & & x_1^p x_2^p \arrow[r, "1"] \arrow[d, "1"] & x_1^p x_2^{p+1} \arrow[r, "-2"] \arrow[d, "1"] & {} \\
 & & & & & x_1^{p+1} x_2^p \arrow[r, "1"] \arrow[d, "2"] & x_1^{p+1} x_2^{p+1} \arrow[r, "-2"] \arrow[d, "2"] & {} \\
x_1^{p+2} \arrow[r, "4"] & x_1^{p+2} x_2 \arrow[r, "5"] & x_1^{p+2} x_2^2 \arrow[r, "6"] & \cdots \arrow[r, "p+2"] & x_1^{p+2} x_2^{p-1} \arrow[r, "p+3"] & x_1^{p+2} x_2^p \arrow[r, "p+4"] & x_1^{p+2} x_2^{p+1} &
\end{tikzcd}$$

We now easily decompose Z_9 into a sum of complexes

$$Z_9' \oplus Z_9'' \oplus Z_9''' \oplus Z_9'''',$$

where Z_9' comes from the bottom row. More specifically,

$$\begin{aligned}
Z_9' &= \Bbbk\langle x_1^{p+2}, x_1^{p+2}x_2, \dots, x_1^{p+2}x_2^{p-4}\rangle, \\
Z_9'' &= \Bbbk\langle x_1^{p+2}x_2^{p-3}, x_1^{p+2}x_2^{p-2}, x_1^{p+2}x_2^{p-1}, x_1^{p+2}x_2^{p}, x_1^{p+2}x_2^{p+1}\rangle, \\
Z_9''' &= \Bbbk\bigl\langle x_1^p x_2^p - \tfrac{1}{2}x_1^{p+2}x_2^{p-2}, x_1^{p+1}x_2^p + x_1^p x_2^{p+1} - x_1^{p+2}x_2^{p-1}, -x_1^{p+2}x_2^p + 2x_1^{p+1}x_2^{p+1}\bigr\rangle, \\
Z_9'''' &= \Bbbk\langle 2x_1^{p+2}x_2^{p-1} - 3x_1^{p+1}x_2^p + 3x_1^p x_2^{p+1}\rangle.
\end{aligned}$$

Thus for $k = 1$ and $n = 2$ we get

$$\mathrm{H}^{/}_{\bullet}(Y_{\frac{2}{2}}) \cong q^2[2]_{\partial}^{q}(q^{3p}V_{p-4} \oplus q^{4p+1}V_3 \oplus q^{4p+2}V_2 \oplus q^{4p+2}V_0).$$

Corollary 5.5 *For distinct* $k \in \mathbb{N}$, $p\mathrm{H}(\,\cdot\,, kp+2)$ *are distinct as link homology theories.*

Proof If we repeat the above calculation for $k = 2$ and $n = 2$, everything would proceed in the same way. Other than internal q–grading shifts, the homology $\mathrm{H}^{/}_{\bullet}(Y_{\frac{2}{2}})$ would be the same as above and contain objects V_{p-4}, V_3, V_2 and V_0.

The homology of the Hopf link in [11] does not contain objects of the form V_{p-4} or V_3 (see [11, (6.17)], in particular) in this tail part of the calculation. For more general k, these objects appear with different shifts; see (5-7). Thus we obtain here new categorifications of the Jones polynomial at a $2p^{\mathrm{th}}$ root of unity different from the original one constructed in [11]. □

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Leighton's theorem and regular cube complexes

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Leighton's graph covering theorem states that two finite graphs with common universal cover have a common finite cover. We generalize this to a large family of nonpositively curved special cube complexes that form a natural generalization of regular graphs. This family includes both hyperbolic and nonhyperbolic CAT(0) cube complexes.

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Leighton's graph covering theorem states that two finite graphs with isomorphic universal covers have isomorphic finite covers. First conjectured by Angluin [2] and proven by Leighton [16], whose background was in computer science and the study of networks, the topic has been picked up by topologists and group theorists interested in producing generalizations to graphs with extra structure, including colourings and line patterns; see Bass and Kulkarni [3], Neumann [18], Shepherd [21], and the author [24]. Although it is desirable to generalize such a theorem to higher dimensions, counterexamples are known even when the universal cover is the product of two trees. Standard arithmetic constructions were known to give irreducible lattices acting on the product of trees, and in the 90s nonresidually finite and even simple examples were given; see Burger and Mozes [6] and Wise [22].

A particularly exciting conjecture was made by Haglund in [11] that Leighton's graph covering theorem should generalize to special cube complexes. In the same paper Haglund proved the conjecture for the class of right-angled Fuchsian buildings (commonly referred to as "Bourdon buildings") and more generally for "type-preserving" lattices in the automorphism group of a building associated to a finite graph product of finite groups.

In this paper we will prove Haglund's conjecture for a large family of CAT(0) cube complexes which exhibit symmetry and homogeneity reminiscent of finite regular trees. Let L be a finite simplicial flag complex. An *L–cube-complex* X is a cube complex such that every link is isomorphic to L. Given a flag complex, the Davis complex $D(L)$ of the associated right-angled Coxeter group is a CAT(0) L–cube-complex. In general, $D(L)$ is not the unique CAT(0) L–cube-complex, but in [15] Lazarovich shows that $D(L)$ is unique if and only if L is *superstar-transitive*. Recall that the *star* of simplex σ in L, denoted by $\mathrm{St}(\sigma)$, is the subcomplex given by the union of all simplices containing σ. We say that flag complex L is superstar-transitive if for any two simplices $\sigma, \sigma' \subseteq L$, any isomorphism $\mathrm{St}(\sigma) \to \mathrm{St}(\sigma')$ sending σ to σ' extends to an automorphism of L. Lazarovich also showed that in this case $\mathrm{Aut}(X)$ is virtually simple.

The principal set of examples of superstar transitive flag complexes presented by Lazarovich are *Kneser complexes*. Let Δ be a finite set. The Kneser complex $\mathfrak{K}_n(\Delta)$ is the simplicial flag complex defined with vertex set the n–element subsets of Δ, and edges joining disjoint n–element subsets. In the particular case that $|\Delta| = nd + 1$, the Kneser complex is superstar transitive and its automorphism group is precisely the natural action of the permutation group $\mathrm{Sym}(\Delta)$; see Section 1.2. We prove the following:

Theorem 0.1 *Let $n \geq 2$, $d \geq 1$ and Δ be a finite set of cardinality $nd + 1$. Let L be the Kneser complex $\mathfrak{K}_n(\Delta)$. Suppose that X_1 and X_2 are compact, L–cube-complexes such that all finite-index subgroups of the hyperplane subgroups are separable in $\pi_1 X_1$ and $\pi_1 X_2$, respectively. Then X_1 and X_2 have a common finite cover.*

If the hyperplane subgroups of a compact nonpositively curved cube complex are separable, then there is a finite cover such that the hyperplanes are 2–sided, embed, and do not self-osculate. If no interosculations could be added to this list, then the cube complex would be virtually special. Conversely, specialness implies separable hyperplane subgroups, and it is conjectured that the converse holds as well.

Note that in the case $d = 1$ that L is the set of $n + 1$ disconnected points, so the L–cube-complexes will be $(n+1)$–regular trees. If $n = 2$ and $d = 2$ then L is the famous Petersen graph. In the case when L has no induced squares (as in the case of the Petersen graph), the fundamental groups of X_1 and X_2 will be hyperbolic — see Moussong [17] — and as a consequence of Agol's proof of the virtual Haken conjecture [1], X_1 and X_2 are virtually special. Thus we have:

Corollary 0.2 *Let $n \geq 2$, $d = 1, 2$, $|\Delta| = nd + 1$ and $L = \mathfrak{K}_n(\Delta)$. If X_1 and X_2 are compact L–cube complexes then X_1 and X_2 have common finite covers.*

Proof In the case $d = 1$ the cube complexes are graphs, so it suffices to show that L is square free when $d = 2$. Let $\Delta = \{1, \dots, 2n+1\}$. Suppose that v_1, v_2, v_3 and v_4 are the vertices of an induced square in L. Then without loss of generality we can assume that $v_1 = \{1, \dots, n\}$ and $v_2 = \{n+1, \dots, 2n\}$ since they are disjoint sets. Thus we can further assume that $v_3 = \{2, \dots, n, 2n+1\}$ since it must be an n–element set disjoint from v_2. Then we have a contradiction since v_4 must be an n–element subset disjoint from $v_1 \cup v_3 = \{1, \dots, n, 2n+1\}$, so $v_2 = v_4$. □

0.1 Strategy

The plan is to show (in Proposition 4.1) that each L–cube-complex has a finite cover X admitting a finite orbicovering $X \to X_L$, where X_L is the orbicomplex $W_L \backslash D(L)$. We seek to construct this orbicovering by identifying the link of the 0–cube in X_L with $\mathfrak{K}_n(\Delta)$ and finding a suitable map $\mathrm{lk}(x) \to \mathfrak{K}_n(\Delta)$ for each 0–cube x in X such that the orbicovering is defined. By associating a copy Δ_x of Δ with each 0–cube in X we identify $\mathrm{lk}(x)$ with $\mathfrak{K}_n(\Delta_x)$. The orbicovering is then locally defined by a choice of map $q_x\colon \Delta_x \to \Delta$; see Lemma 1.3.

In order for the set of q_x to define an orbicover we need to ensure that certain conditions are satisfied. If $e = (x, y)$ is a 1–cube, then we need to ensure that e will be mapped to the same half edge in X_L by the maps induced by q_x and q_y. Given a square in X, we also need to ensure that it will be mapped to a quarter-square in X_L.

In Section 3, we formulate the problem in the language of a Δ–*category*, which is a choice of bijection $\phi_e\colon \Delta_x \to \Delta_y$ for each edge $e = (x, y)$, satisfying certain conditions. Most of the action in this paper concerns being able to (virtually) construct a Δ–category. Once we have the Δ–category we obtain a holonomy

$$\Psi\colon \pi_1(X, x) \to \mathrm{Sym}(\Delta_x)$$

and the kernel of this holonomy will give a finite cover for which we can define suitable q_x; see Section 4.

0.2 Previous results and connections to QI–rigidity

A major motivation for proving Haglund's conjecture is the potential applications to Gromov's program of understanding groups up to quasi-isometry [10]. In [11], Haglund proved his conjecture for Bourdon buildings and his result can be combined with a result of Bourdon and Pajot [4] which says that each quasi-isometry of such a building is finite distance from a unique automorphism. Thus we deduce that if G is a group quasi-isometric to the graph product W associated to such a Bourdon building B, then in fact it acts by isometries on B. By Agol's result [1], G will be virtually special, thus acting faithfully on B, and by Haglund G will be weakly commensurable with W. Thus W is quasi-isometrically rigid.

This argument motivates the following problem:

Problem 0.3 *Let $L = \mathfrak{K}_n(\Delta)$, where $|\Delta| = nd + 1$. Is every quasi-isometry of $D(L)$ finite distance from an automorphism?*

A positive answer to Problem 0.3 in the hyperbolic case would immediately give quasi-isometric rigidity for the associated groups W_Γ by a similar argument to the case of Bourdon buildings. That is to say that any group quasi-isometric to W_Γ would be weakly commensurable with W_Γ. In the "higher rank" nonhyperbolic case one might look to Huang's results on the quasi-isometric rigidity of large families of right-angled Artin groups [13]. In this case following would need to be considered:

Problem 0.4 *Suppose that L is a Kneser complex as above, such that W_Γ is not hyperbolic. Are there groups acting geometrically on $D(\Gamma)$ that are not virtually special?*

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1 Preliminaries

1.1 Right-angled Coxeter groups

We refer to Davis [8] for classical background on Coxeter groups and their geometry and to [7] for a recent survey of their large scale geometry.

Let L denote a finite simplicial flag complex. The right-angled Coxeter group W_L associated to L is given by the presentation:

$$W_L = \langle v \in L^{(0)} \mid v^2 = 1 \text{ and } [u, v] = 1 \text{ if } (u, v) \in L^{(1)} \rangle.$$

The Davis complex $D(L)$ is the CAT(0) cube complex obtained from the Cayley 2–complex constructed from the above presentation, after collapsing each v^2 bigon to a single edge, and inserting higher dimensional cubes wherever their 2–skeleta appear. The link of each vertex in $D(L)$ is isomorphic to L, which makes it an *L–cube-complex*. The following theorem tells us when $D(L)$ is the unique CAT(0) L–cube-complex:

Theorem 1.1 [15, Theorem 1.2] *The Davis complex $D(L)$ is the unique CAT(0) cube complex with each link isomorphic to L if and only if L is superstar-transitive.*

If we colour the edges in $D(L)$ according to the corresponding element of L, or alternatively the conjugacy class of the associated generator, we can identify W_L as the subgroup of $\mathrm{Aut}(D(L))$ that preserves the colours. Sometimes this subgroup is referred to as the *type-preserving automorphisms*. The quotient $X_L = W_L \backslash D(L)$ has the structure of an orbicomplex. Each face given by the intersection of k hyperplanes has the associated group $(\mathbb{Z}/2)^k$ with a factor corresponding to a hyperplane.

1.2 Kneser complexes

Let Δ be a finite set. The *Kneser complex* $\mathfrak{K}_n(\Delta)$ is the flag complex with underlying graph with vertex set given by n–elements subsets of Δ, and edges corresponding to disjoint n–element subsets. There is a natural action of $\mathrm{Sym}(\Delta)$ on $\mathfrak{K}_n(\Delta)$.

If $\mathfrak{K} := \mathfrak{K}_n(\Delta)$ is a Kneser complex, then we let $\mathfrak{s}_v = \mathfrak{s}(v) \subseteq \Delta$ denote the subset associated to $v \in \mathfrak{K}^{(0)}$.

Example 1.2 If $|\Delta| = 5$, then $P := \mathfrak{K}_2(\Delta)$ is the *Petersen graph*. It is a simple exercise to verify that P is triangle and square free; see Figure 1.

More generally, if $|\Delta| = nd + 1$, then $\mathfrak{K}_n(\Delta)$ is a $(d-1)$–dimensional flag simplicial complex with a superstar-transitive automorphism group; see [15]. We also note the following:

Lemma 1.3 [9, Corollary 7.8.2] *If $|\Delta| \neq 2n$, then $\mathrm{Aut}(\mathfrak{K}_n(\Delta))$ is equal to $\mathrm{Sym}(\Delta)$.*

Given a subset $\Sigma \subseteq \Delta$, the inclusion induces an embedding $\mathfrak{K}_n(\Sigma) \subseteq \mathfrak{K}_n(\Delta)$, where the vertex in $\mathfrak{K}_n(\Sigma)$ corresponding to $\mathfrak{s} \subseteq \Sigma \subseteq \Delta$ is sent to the corresponding vertex in $\mathfrak{K}_n(\Delta)$. Indeed, an automorphism $(\Delta, \Sigma) \to (\Delta, \Sigma)$ induces an automorphism of $\mathfrak{K}_n(\Delta)$ that restricts to an automorphism on $\mathfrak{K}_n(\Sigma)$. Conversely, by Lemma 1.3, provided $2n$ is not equal to $|\Delta|$ or $|\Sigma|$, an automorphism of $\mathfrak{K}_n(\Delta)$ that preserves $\mathfrak{K}_n(\Sigma)$ gives a self-bijection of Δ that preserves Σ.

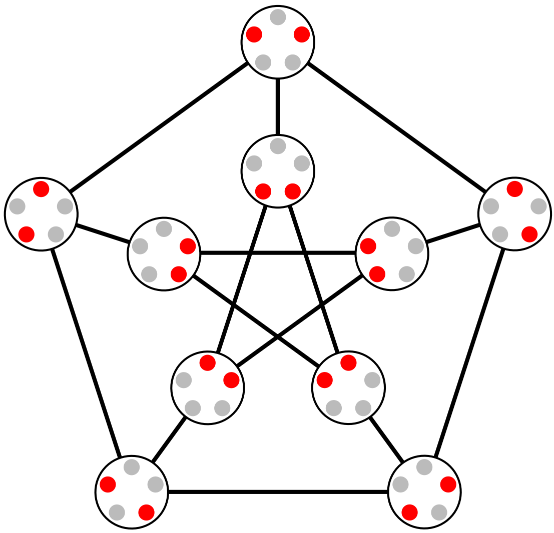

Figure 1: The Petersen graph (courtesy of Tilman Piesk [19]).

Kneser complexes were presented by Lazarovich as a large and readily accessible set of superstar transitive graphs.

Theorem 1.4 [15, Corollary 5.5] *Let $n \geq 2$ and $d \geq 1$. Let $|\Delta| = nd + 1$ and $L := \mathfrak{K}_n(\Delta)$. Then $\mathrm{Aut}(D(L))$ is virtually simple.*

We note that $D(L)$ is Gromov hyperbolic if and only if L does not contain any induced squares [17]. Thus, it is an exercise to verify that $D(L)$ is hyperbolic only if $d \leq 2$.

Remark The most direct means that a result like Theorem 0.1 could be true is if the automorphism group of $D(L)$ were to act properly. In which case any other uniform lattice in $\mathrm{Aut}(D(L))$ would lie inside $\mathrm{Aut}(D(L))$ as a finite-index subgroup. Common covers of the corresponding quotient spaces could be constructed by taking the intersections of the associated lattices. In general the automorphism groups of universal covers will be far too large for this argument to work. Theorem 1.4 is the most extreme example of this: since W_L is residually finite (and indeed virtually special), it cannot lie inside a virtually simple group like $\mathrm{Aut}(D(L))$ as a finite-index subgroup.

2 Special cube complexes

We refer to [5; 12; 14; 20; 23] for more detailed background on nonpositive curvature, cube complexes, and specialness. We outline here the terminology that we will use.

An *n–cube* C is a metric space isometrically identified with $[-1,1]^n$. A 0–cube is a singleton. A *subcube* $S \subseteq C$ of dimension m in an n–cube is the m–cube obtained by restricting $(m-n)$–many coordinates to 1 or -1. The i^{th} *midcube* $M \subset C$, for $1 \leq i \leq n$, is the $(n-1)$–cube obtained by restricting the i^{th} coordinate to 0.

The *reflection* of an n–cube over it's i^{th} midcube $M \subseteq C$ is the map $C \to C$ obtained by multiplying the i^{th} coordinate by -1. Note that all the reflections in a cube commute. The *antipodal* map $C \to C$ is obtained by reflection over all the midcubes in C.

The *link* $\operatorname{lk}(x)$ of a 0–cube x in an n–cube C is the simplex σ given by the ϵ–sphere of x in the ℓ^1–metric (where $1 > \epsilon > 0$). Each subcube of C that contains x has a link at x that gives a corresponding face in σ. If x and y are 0–cubes in C, then x is mapped to y by the composition R of all the reflections over midcubes separating x and y. Thus R induces an isomorphism $\operatorname{lk}(x) \to \operatorname{lk}(y)$.

By a *cube complex* X we will mean a topological space that decomposes into cubes $\mathcal{C}(X)$, such that every subcube of a cube in $\mathcal{C}(X)$ is a cube in $\mathcal{C}(X)$, and such that the intersection of any two cubes $C, C' \in \mathcal{C}(X)$ give subcubes of C and C', or the intersection is empty. The link $\operatorname{lk}(x)$ of a 0–cube x in X is the complex given by the union of all the links of all the cubes containing x, with inclusion of simplices induced by inclusion of subcubes. Alternatively, it can also be thought of as the ϵ neighbourhood of x inside X itself. A cube complex X is *nonpositively curved* if the link of each vertex is a simplicial flag complex. Each n–simplex σ in $\operatorname{lk}(x)$ corresponds to a unique $(n+1)$–cube $C(\sigma)$ in X containing x. Conversely, each $(n+1)$–cube C that contains x corresponds to a simplex $\sigma(C)$ in $\operatorname{lk}(x)$.

Unless otherwise noted, our 1–cubes will be *directed* in the sense that $e = (x, y)$ comes with an initial and terminal 0–cube, denoted by $\iota e = x$ and $\tau e = y$. The *reversed 1–cube* with the opposite direction will be denoted by $\bar{e} = (y, x)$. Let X be a compact nonpositively curved cube complex. A hyperplane Λ in X is an equivalence class of directed 1–cubes generated by the relation $e \sim e'$ if they are opposite faces of a square in X or $\bar{e} = e'$. Associated to the equivalence class is the *realization* of Λ. This is a nonpositively curved cube complex, which we will also denote by Λ, constructed from the midcubes dual to the edges in the equivalence class that immerses by a local isometry $\Lambda \looparrowright X$. Note that this immersion is only a cellular map when both Λ and X have been cubically subdivided. The *hyperplane subgroup* associated to Λ is the image of $\pi_1(\Lambda)$ in $\pi_1(X)$ under the injective homomorphism given by the immersion.

A hyperplane is *embedded* if no two edges in the equivalence class form the corner of a square (that is to say a 2–cube) in X. Equivalently a hyperplane is embedded if the immersion of the realization is an embedding. The *carrier* $N(\Lambda) \subseteq X$ of a hyperplane is the subcomplex obtained by taking all cubes that contain an edge in the associated equivalence class. We say that a hyperplane is *fully clean and 2–sided* if $N(\Lambda) \cong \Lambda \times [-1, 1]$. That is to say that we can extend the embedding of the realization to an embedding $N(\Lambda) = \Lambda \times [-1, 1] \hookrightarrow X$. If the hyperplane subgroups of $\pi_1(X)$ are separable then there is a finite cover of X such that the hyperplanes are fully clean and 2–sided. Indeed, fully clean follows from [12, Lemma 9.14], and with hyperplanes embedded a standard cut-and-paste argument applied to a 1–sided hyperplane yields a degree 2 cover with a two sided hyperplane; see also the proof of [12, Proposition 3.10]. *Thus, we will now assume going forward that all hyperplanes satisfy this condition.* In terms of the definition of specialness, this is equivalent to the hyperplanes being 2–sided, embedded, and without self-osculations. Such a cube complex may fail to be special since interosculations do not contradict this assumption (see Figure 2 for an illustration of the hyperplane pathologies). In terms of the assumptions of Theorem 0.1, if the finite-index subgroups of hyperplane subgroups are separable in $\pi_1 X$, then this remains true of the hyperplane subgroups in a finite cover.

A 0–cube x is *incident* to Λ if it is contained in $N(\Lambda)$. An edge $e = (x, y)$ is *parallel* to Λ if it is contained in $N(\Lambda)$ without being dual to Λ. Under the assumption that the hyperplane Λ is fully clean and 2–sided, the immersion of the realization extends to an embedding $\Lambda \times [-1, 1] \hookrightarrow X$, where the realization is the 0 fiber. The edges parallel to Λ are contained in the -1 and 1 fibers. We will refer to the subcomplexes of X given by the ± 1 fibers as the *sides of the carrier*.

2.1 The adjacency map

Let $e = (x, y)$ be an edge in X dual to Λ and let v be the vertex in $\mathrm{lk}(x)$ corresponding to e, and u be the vertex in $\mathrm{lk}(y)$ corresponding to e. The *star* $\mathrm{Star}(\sigma)$ of a simplex σ in a simplicial complex is the subcomplex spanned by the union of all simplices containing σ. We note that in [15] the star of a simplex is defined by Lazarovich to be the combinatorial 1–neighbourhood. The two notions only coincide in the case when the simplex is the singleton. This alternative notion, which we denote by $\mathrm{St}(\sigma)$ in the introduction, applies to the definition of superstar transitive, but will not be otherwise relevant to the content of this paper.

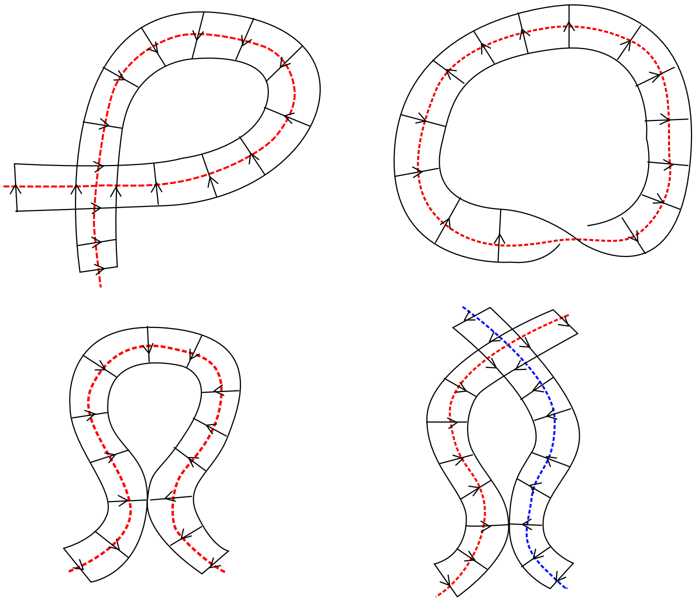

Figure 2: An illustration of the standard hyperplane pathologies. The dotted line depicts the topological realization of the hyperplane. The edges in the equivalence classes are given arrows indicating the direction. The top left depicts a self intersection. The top right depicts a 1–sided hyperplane, and the edges with the arrows reversed also belong to the equivalence class. The bottom left depicts a direct self-osculation. The bottom right depicts an interosculation.

The *adjacency map* for e is the natural isomorphism

$$\mathrm{ad}_e \colon \mathrm{Star}(v) \to \mathrm{Star}(u)$$

such that if $v \in \sigma$ then $\mathrm{ad}_e(\sigma)$ is the unique simplex such that $C(\mathrm{ad}_e(\sigma)) = C(\sigma)$. (This is referred to as the *transfer map* in [15].)

More generally, let x and y be 0–cubes in X that belong to some n–cube. Let C be the minimal such n–cube in X containing x and y. Let $\sigma_x \subseteq \mathrm{lk}(x)$ and $\sigma_y \subseteq \mathrm{lk}(y)$ be the simplices corresponding to C. Then we have a natural adjacency map for C given by the natural isomorphism

$$\mathrm{ad}_C \colon \mathrm{Star}(\sigma_x) \to \mathrm{Star}(\sigma_y)$$

such that if σ is a simplex in $\mathrm{lk}(x)$ containing σ_x, then ad_C on σ is induced by the composition of reflections in $C(\sigma)$ over the midcubes separating x and y. Note that $C(\mathrm{ad}_C(\sigma)) = C(\sigma)$.

Furthermore, suppose that x, y and z are 0–cubes in C such that C is the minimal cube containing x and z. If C_1 and C_2 are the minimal subcubes in C containing x, y and y, z respectively, then $\mathrm{ad}_C = \mathrm{ad}_{C_1} \circ \mathrm{ad}_{C_2}$, where each ad_{C_i} is suitably restricted.

3 Constructing Δ–categories

This section will be devoted to constructing a Δ–category on a compact L–cube-complex X such that all finite-index subgroups of the hyperplane subgroups are separable in $\pi_1 X$, where L is the Kneser graph as specified in the statement of Theorem 0.1. We will assume, as stated in Section 2, that we have passed to a finite-index cover such that the hyperplanes are fully clean and 2–sided.

3.1 A note on notation

In what follows we will be constructing a category over a cube complex. We will be doing this by assigning objects to 0–cubes and assigning morphisms to each 1–cube. For example we might denote the morphism associated to e by ϕ_e. In this case, given an edge path $\gamma = (e_1, \dots, e_n)$, we will let ϕ_γ denote the composition $\phi_{e_n} \circ \cdots \circ \phi_{e_1}$. If all the edges are parallel to a given hyperplane Λ, then we will call γ a *parallel path*.

3.2 Our objects

Let $n \geq 2$, $d \geq 1$ and Δ be a finite set with $|\Delta| = nd + 1$. Let X be a compact, nonpositively curved cube complex with 2–sided hyperplanes such that $\mathrm{lk}(x)$ is isomorphic to $\mathfrak{K}_n(\Delta)$. To each 0–cube x in X let Δ_x be a copy of Δ, and identify $\mathrm{lk}(x)$ with the associated Kneser complex $\mathfrak{K}_n(\Delta_x)$. Let Λ be a hyperplane incident to x. Let e be the 1–cube dual to Λ with $\tau e = x$. Let $v = \sigma(e)$ be the vertex in $\mathrm{lk}(x)$ corresponding to e. The identification of $\mathrm{lk}(x)$ with $\mathfrak{K}(\Delta_x)$ allows us to define $\Lambda_x := \mathfrak{s}(v) \subseteq \Delta_x$. We will also let $\mathfrak{s}(e) := \mathfrak{s}(v)$ when it is clear which 0–cube link we are working with. This is well defined since Λ is fully clean, so the 1–cube e is the only 1–cube dual to Λ incident to x.

3.3 Δ–categories on X

Definition 3.1 A Δ*–category on* X is a collection of bijections

$$\phi_e \colon \Delta_x \to \Delta_y,$$

one for each 1–cube $e = (x, y)$ in X, such that the following conditions are satisfied:

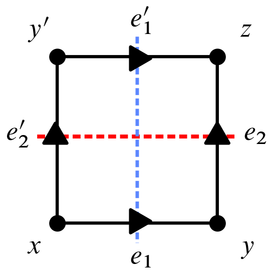

Figure 3: The square. The hyperplane Λ^1 is depicted as the vertical dotted line with the arrows on e_1 and e_1' giving the direction. The hyperplane Λ^2 is the horizontal dotted line with the arrows on e_2 and e_2' giving the direction.

(1) **Invertibility** If e is a directed one cube then $\phi_{\bar{e}} = \phi_e^{-1}$.

(2) Let $e_1 = (x, y)$, $e_2 = (y, z)$, $e_1' = (y', z)$, $e_2' = (x, y')$ be the edges bounding a square S, and Λ^i the hyperplane dual to e_i and e_i' (see Figure 3). Then:

  (a) **Commutativity** $\phi_{e_2} \circ \phi_{e_1} = \phi_{e_1'} \circ \phi_{e_2'}$.

  (b) **Parallel transport** $\phi_{e_1}(\Lambda_x^2) = \Lambda_y^2$ and $\phi_{e_2'}(\Lambda_x^1) = \Lambda_{y'}^1$.

Remark The parallel transport condition applied to all squares containing e allows us to deduce that $\phi_{e_1}(\Lambda_x^1) = \Lambda_y^1$.

Let $\{\phi_e\}$ be a Δ–category on X, and $f\colon \widehat{X} \to X$ a cover. By identifying each link $\operatorname{lk}(\hat{x})$ in $\widehat{X}$ with $\mathfrak{K}_n(\widehat{\Delta}_{\hat{x}})$, where $\widehat{\Delta}_{\hat{x}}$ is the copy of Δ assigned to $\hat{x}$, by Lemma 1.3 the induced isomorphism between the links

$$f_{\hat{x}}\colon \operatorname{lk}(\hat{x}) \to \operatorname{lk}(f(\hat{x}))$$

induces an isomorphism

$$\mathfrak{f}_{\hat{x}}\colon \widehat{\Delta}_{\hat{x}} \to \Delta_{f(\hat{x})}.$$

Thus we can lift the Δ–category $\{\phi_e\}$ on X to a unique Δ–category on $\widehat{X}$ such that the following diagram commutes, for each 1–cube $\hat{e} = (\hat{x}, \hat{y})$ in $\widehat{X}$ mapping to $e = (x, y)$ in X:

$$\begin{tikzcd}
\widehat{\Delta}_{\hat{x}} \arrow[r, "\hat{\phi}_{\hat{e}}"] \arrow[d, "\mathfrak{f}_{\hat{x}}"'] & \widehat{\Delta}_{\hat{y}} \arrow[d, "\mathfrak{f}_{\hat{y}}"] \\
\Delta_x \arrow[r, "\phi_e"] & \Delta_y
\end{tikzcd}$$

It is straightforward to verify that $\{\hat{\phi}_{\hat{e}}\}$ satisfies the invertibility and commutativity conditions, since $\mathfrak{f}_{\hat{x}}$ is invertible, and since the squares in X lift to squares in $\widehat{X}$. Parallel transport holds for $\{\hat{\phi}_{\hat{e}}\}$ by tracing the correspondence of n–element subsets of $\widehat{\Delta}_{\hat{x}}$ to vertices in $\mathrm{lk}(\hat{x})$, which then correspond to hyperplanes incident to $\hat{x}$. Consider a square in $\widehat{X}$ covering the square in Figure 3, labelled with the vertices $\hat{x}$, $\hat{y}$, $\hat{y}'$ and $\hat{z}$, and bounded by edges $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_1'$ and $\hat{e}_2'$. Then for the hyperplane $\widehat{\Lambda}^2$ covering Λ^2, we deduce that

$$\hat{\phi}_{\hat{e}_1}(\widehat{\Lambda}^2_{\hat{x}}) = \hat{\phi}_{\hat{e}_1} \circ \mathfrak{f}_{\hat{x}}^{-1}(\Lambda^2_x) = \mathfrak{f}_{\hat{y}}^{-1} \circ \phi_{e_1}(\Lambda^2_x) = \mathfrak{f}_{\hat{y}}^{-1}(\Lambda^2_y) = \widehat{\Lambda}^2_{\hat{y}}.$$

The second equality follows from commutativity of the above square, and the third from parallel transport for $\{\phi_e\}$ in X. The corresponding conclusion follows similarly for $\widehat{\Lambda}^1$.

3.4 Constructing a Δ–category

We will construct our Δ–category in two stages. In the first stage we will define functions ϕ_e^* that will be defined on subsets of the domain Δ_x. We note that in this section we will be composing functions whose domain and ranges will be subsets of larger sets. In this case the composition will be given by restricting to the intersection of the corresponding domains and ranges.

Lemma 3.2 *There exists a unique family of functions*

$$\{\phi_e^* : (\Delta_x - \Lambda_x) \to (\Delta_y - \Lambda_y) \mid \Lambda \ \textit{ is dual to } \ e = (x, y) \in X^{(1)}\}$$

such that:

(1) $\phi_{\bar{e}}^* = (\phi_e^*)^{-1}$.

(2) *Let* $e_1 = (x, y)$, $e_2 = (y, z)$, $e_1' = (y', z)$ *and* $e_2' = (x, y')$ *be the edges bounding a square* S, *and* Λ^i *the hyperplane dual to* e_i *and* e_i' (see Figure 3). *Then*

   (a) *after suitably restricting domains*,

$$\phi_{e_2}^* \circ \phi_{e_1}^* = \phi_{e_2'}^* \circ \phi_{e_1'}^* : (\Delta_x - \Lambda_x^1 - \Lambda_x^2) \to (\Delta_z - \Lambda_z^1 - \Lambda_z^2),$$

   (b) $\phi_{e_1}^*(\Lambda_x^2) = \Lambda_y^2$ *and* $\phi_{e_2'}^*(\Lambda_x^1) = \Lambda_{y'}^1$.

Proof Let $e = (x, y)$ be a directed 1–cube in X dual to Λ. Let v be the vertex in $\mathrm{lk}(x)$ corresponding to $\bar{e}$, and u be the vertex in $\mathrm{lk}(y)$ corresponding to e. Then $\mathrm{Star}(v)$ decomposes as the simplicial join $v * \mathfrak{K}_n(\Delta_x - \Lambda_x)$ and similarly $\mathrm{Star}(u)$ decomposes as $u * \mathfrak{K}_n(\Delta_y - \Lambda_y)$. Thus the adjacency map ad_e restricts to an isomorphism

$$\mathfrak{K}_n(\Delta_x - \Lambda_x) \to \mathfrak{K}_n(\Delta_y - \Lambda_y).$$

Since $|\Delta_x - \Lambda_x| = |\Delta_y - \Lambda_y| = n(d-1)+1$, by Lemma 1.3 this isomorphism is induced by the bijection

$$\phi_e^*\colon (\Delta_x - \Lambda_x) \to (\Delta_y - \Lambda_y).$$

(This requires checking that $n(d-1)+1 \neq 2n$ for $n \geq 2$ and $d \geq 1$.)

In the case that $d = 1$ there are no squares in X, so conditions (2)(a)–(b) are satisfied automatically. So we assume $d \geq 2$. Suppose that $e_1 = (x, y)$, $e_2 = (y, z)$, $e_1' = (y', z)$ and $e_2' = (x, y')$ are the edges bounding a square S, and Λ^i is the hyperplane dual to e_i and e_i' (see Figure 3). We now check that conditions (2)(a)–(b) are satisfied.

Verifying (2)(b) follows from observing that $\Lambda_x^2 \subseteq \Delta_x - \Lambda_x^1$ corresponds to a vertex $u \in \operatorname{lk}(x) = \mathfrak{K}_n(\Delta_x)$ and $\Lambda_y^2 \subseteq \Delta_y - \Lambda_y^1$ corresponds to a vertex v in $\operatorname{lk}(y) = \mathfrak{K}_n(\Delta_y)$ such that $\mathrm{ad}_{e_1}(u) = v$. (Stare at Figure 3.) Thus $\phi_{e_1}^*(\Lambda_x^2) = \Lambda_y^2$ and similarly $\phi_{e_2'}^*(\Lambda_x^1) = \Lambda_{y'}^1$.

We now consider (2)(a). Observe that (2)(b) implies

$$\phi_{e_2}^* \circ \phi_{e_1}^*((\Delta_x - \Lambda_x^2) - \Lambda_x^1) = \phi_{e_2}^*((\Delta_y - \Lambda_y^1) - \Lambda_y^2) = \Delta_z - \Lambda_z^1 - \Lambda_z^2.$$

Combined with the similar set of equalities for $\phi_{e_2'}^* \circ \phi_{e_1'}^*$ this verifies (2)(b) when $d = 2$ since there is only one possible map between singletons.

In the case that $d > 2$, let $\sigma_x \subseteq \operatorname{lk}(x)$, $\sigma_y \subseteq \operatorname{lk}(y)$, $\sigma_{y'} \subseteq \operatorname{lk}(y')$ and $\sigma_z \subseteq \operatorname{lk}(z)$ denote the 1–simplices corresponding to the square S. We know that

$$\mathrm{ad}_S = \mathrm{ad}_{e_2} \circ \mathrm{ad}_{e_1} = \mathrm{ad}_{e_1'} \circ \mathrm{ad}_{e_2'}\colon \operatorname{Star}(\sigma_x) \to \operatorname{Star}(\sigma_z).$$

We also have the decomposition

$$\operatorname{Star}(\sigma_x) = \sigma_x * \mathfrak{K}_n(\Delta_x - \Lambda_x^1 - \Lambda_x^2)$$

and similar decompositions for the stars of σ_y, $\sigma_{y'}$ and σ_z. The adjacency map ad_S therefore restricts to an isomorphism

$$\mathfrak{K}_n(\Delta_x - \Lambda_x^1 - \Lambda_x^2) \to \mathfrak{K}_n(\Delta_z - \Lambda_z^1 - \Lambda_z^2)$$

which, by Lemma 1.3, is induced by an isomorphism

$$(\Delta_x - \Lambda_x^1 - \Lambda_x^2) \to (\Delta_z - \Lambda_z^1 - \Lambda_z^2)$$

that must coincide with the compositions $\phi_{e_2}^* \circ \phi_{e_1}^*$ and $\phi_{e_1'}^* \circ \phi_{e_2'}^*$, as their restrictions induce the same isomorphism. Thus $\phi_{e_2}^* \circ \phi_{e_1}^* = \phi_{e_1'}^* \circ \phi_{e_2'}^*$.

Finally, we show uniqueness of the family $\{\phi_e^*\}$. Note that $|\Delta_x - \Lambda_x| = (n-1)d + 1$. Therefore, if $n = 1$ then uniqueness is trivial as the maps are between singletons. Otherwise, for $n > 1$ each element of $\Delta_x - \Lambda_x$ is given by the intersection of the n–element subsets Λ_y that contain the given element and are disjoint from Λ_x. Thus, condition (2)(b) applied to each these Λ_y allows us to deduce that ϕ_e^* is uniquely determined on the given element of $\Delta_x - \Lambda_x$. □

We will refer to the maps $\{\phi_e^*\}$ as the *pre–Δ–category*. Note that if $f\colon \widehat{X} \to X$ is a cover, then we can lift the pre–Δ–category to $\widehat{X}$ and check that the conditions are satisfied, in the same way we checked for the Δ–category. Alternatively, since such pre–Δ–categories are unique, we could instead verify that the following square commutes:

$$\begin{tikzcd}
(\widehat{\Delta}_{\hat{x}} - \widehat{\Lambda}_{\hat{x}}) \arrow[r, "\hat{\phi}_{\hat{e}}^*"] \arrow[d, "\mathfrak{f}_{\hat{x}}"'] & (\widehat{\Delta}_{\hat{y}} - \widehat{\Lambda}_{\hat{y}}) \arrow[d, "\mathfrak{f}_{\hat{y}}"] \\
(\Delta_x - \Lambda_x) \arrow[r, "\phi_e^*"] & (\Delta_y - \Lambda_y)
\end{tikzcd}$$

This would follow from the parallel transport conditions and the correspondence between hyperplanes and the corresponding subsets of $\widehat{\Delta}_{\hat{x}}$, in a similar fashion to the argument given for lifting Δ–categories.

3.5 The hyperplane parallel holonomy

As a consequence of Lemma 3.2 we deduce that if an edge $e = (x, y)$ is parallel to Λ then we have a bijection

$$\psi_e\colon \Lambda_x \to \Lambda_y$$

obtained by restricting ϕ_e^* as given by Lemma 3.2. Indeed, if Λ' is the hyperplane dual to e, then $\Lambda_x \subseteq \Delta_x - \Lambda'_x$. We note that this is a category, with the n–element set Λ_x associated to each vertex x that Λ is incident to, and there is a morphism ψ_e associated to each edge e parallel to Λ. In fact, since Λ is 2–sided, there is a category corresponding to each side.

Thus if we fix a choice of side of Λ and a 0–cube p in Λ as a basepoint, we obtain a *parallel holonomy*

$$\Psi_p\colon \pi_1(\Lambda, p) \to \mathrm{Sym}(\Lambda_x).$$

If e' is the edge dual to Λ with midpoint p such that $\tau e' = x$ lies on the given side, this holonomy is given by identifying Λ with the side of the hyperplane carrier containing the basepoint x, and letting the equivalence class of a parallel path $[\gamma] = [e_1, \dots, e_n]$ based at x map to

$$\Psi_p([\gamma]) = \psi_\gamma,$$

where ψ_γ denotes the composition $\psi_{e_n} \circ \cdots \circ \psi_{e_1}$. Conditions (1) and (2)(a) in Lemma 3.2 ensure that this does not depend on the choice of representative.

We note that the triviality of the holonomy does not depend on the choice of basepoint p (but may depend on the side of the carrier that is chosen). Indeed, given another 1–cube e'' dual to Λ, with $\tau e'' = y$ on the same side of Λ, with midpoint p', we can check the following diagram commutes:

$$\begin{tikzcd}
\pi_1(\Lambda, p) \arrow[r, "\Psi_p"] \arrow[d] & \mathrm{Sym}(\Lambda_x) \arrow[d] \\
\pi_1(\Lambda, p') \arrow[r, "\Psi_{p'}"] & \mathrm{Sym}(\Lambda_y)
\end{tikzcd}$$

We have chosen some path γ connecting x to y in $\tau(\Lambda)$. The left vertical map is given by conjugating closed loops by $[\gamma]$, in the standard fashion, and the right vertical map is given by conjugating by ψ_γ.

The kernel of Ψ_p is a finite-index normal subgroup of $\pi_1(\Lambda)$, and by the assumptions of Theorem 0.1 will be separable in $\pi_1 X$.

Lemma 3.3 *There exists a finite cover $\widehat{X} \to X$ such that the parallel holonomies in $\widehat{X}$ are trivial.*

Proof Let Ψ be a parallel holonomy for some hyperplane Λ, and some choice of side and basepoint. The kernel of Ψ is a finite-index normal subgroup of $\pi_1(\Lambda)$, and therefore, by the assumption of Theorem 0.1, will be separable in $\pi_1 X$. Let $\{\mathrm{id}, g_1, \dots, g_\ell\}$ be a minimal set of representatives for the left cosets of $\ker(\Psi)$ in $\pi_1(\Lambda)$. As $g_i \notin \ker(\Psi)$, by separability there exists a finite-index subgroup $N_i \leqslant \pi_1(X)$ such that $\ker(\Psi) \subseteq N_i$ and $g_i \notin N_i$. Thus $\ker(\Psi) = \bigcap_{i=1}^{\ell} N_i \cap \pi_1(\Lambda)$, since we know $\ker(\Psi) \subseteq \bigcap_{i=1}^{\ell} N_i$ and that if $g_i h \in \bigcap_{i=1}^{\ell} N_i \cap \pi_1(\Lambda)$, where $h \in \ker(\Psi)$, then $g_i \in \bigcap_{i=1}^{\ell} N_i$. The normal core, $\mathrm{Core}\big(\bigcap_{i=1}^{\ell} N_i\big)$, is a finite-index normal subgroup of $\pi_1 X$ such that $\pi_1(\Lambda) \cap \mathrm{Core}\big(\bigcap_{i=1}^{\ell} N_i\big)$ is contained in $\ker(\Psi)$.

By repeating this for each side of each hyperplane, and intersecting all the resulting normal cores, we obtain a finite-index normal subgroup $N \leqslant \pi_1(X)$ such that for each hyperplane Λ, the intersection $N \cap \pi_1(\Lambda)$ is contained in the kernel of the parallel holonomies on either side of Λ. Then the desired finite cover $f : \widehat{X} \to X$ is given by N.

Let $\{\hat{\phi}^*_{\hat{e}}\}$ denote the lift of the pre–Δ–category on X to $\widehat{X}$. Then the following diagram commutes, where hyperplane $\widehat{\Lambda}$ covers Λ, and the bottom arrow is the isomorphism induced by conjugation by $\mathfrak{f}_{\hat{x}}$:

$$\begin{tikzcd} \pi_1(\widehat{\Lambda}) \arrow[r, "f_*"] \arrow[d, "\widehat{\Psi}_{\hat{p}}"'] & \pi_1(\Lambda) \arrow[d, "\Psi_p"] \\ \mathrm{Sym}(\widehat{\Delta}_{\hat{x}}) \arrow[r] & \mathrm{Sym}(\Delta_x) \end{tikzcd}$$

Indeed, if we take a combinatorial path $[\hat{\gamma}]$ given by the edge sequence $\hat{e}_1, \dots, \hat{e}_n$ that traversed the vertices $\hat{x} = \hat{x}_0, \hat{x}_1, \dots, \hat{x}_{n-1}$, and let $f([\hat{\gamma}]) = [\gamma]$ with $f(\hat{e}_i) = e_i$ and $f(\hat{x}_i) = x_i$, we deduce that

$$\begin{aligned} \Psi_p \circ f_*([\hat{\gamma}]) = \Psi_p([\gamma]) = \psi_\gamma = \psi_{e_n} \circ \dots \circ \psi_{e_1} \\ &= \mathfrak{f}_{\hat{x}_0} \circ \hat{\psi}_{\hat{e}_n} \circ \mathfrak{f}^{-1}_{\hat{x}_{n-1}} \circ \mathfrak{f}_{\hat{x}_{n-1}} \circ \dots \circ \mathfrak{f}_{\hat{x}_1} \circ \hat{\psi}_{\hat{e}_1} \circ \mathfrak{f}^{-1}_{\hat{x}_0} \\ &= \mathfrak{f}_{\hat{x}_0} \circ \hat{\psi}_{\hat{e}_n} \circ \dots \circ \hat{\psi}_{\hat{e}_1} \circ \mathfrak{f}^{-1}_{\hat{x}_0} \\ &= \mathfrak{f}_{\hat{x}_0} \circ \widehat{\Psi}_{\hat{p}} \circ \mathfrak{f}^{-1}_{\hat{x}_0}. \end{aligned}$$

Thus the square commutes and the parallel holonomies in $\widehat{X}$ are trivial. □

3.6 Extending the maps ϕ^*_e

By Lemma 3.3, we now assume that we have passed to a suitable finite cover such that X has trivial parallel holonomies in its pre–Δ–category. Given an edge e dual to Λ, it remains to extend ϕ^*_e, and this means making a choice of bijection $\Lambda_x \to \Lambda_y$. We can certainly make such choices so that the inversion condition (2)(a) is satisfied, and condition (2)(b) holds as it holds for ϕ^*_e. It therefore remains to ensure we can make our choices so that the commutativity condition (2)(a) is satisfied.

For each hyperplane Λ let $e = (x, y)$ be a choice of edge dual to Λ. We make a choice of map

$$\phi^\circ_e \colon \Lambda_x \to \Lambda_y$$

that extends ϕ^*_e to ϕ_e.

Suppose that e' is some other edge dual to Λ such that $\tau e'$ lies on the same side of Λ as τe. Then let $\gamma = (e_1, \dots, e_p)$ be an edge path parallel to Λ that connects τe to $\tau e'$. We also let $\gamma' = (e'_1, \dots, e'_q)$ be an edge path parallel to Λ that connects ιe to $\iota e'$. Then we define

$$\phi^\circ_{e'} = \psi_{e_p} \circ \dots \circ \psi_{e_1} \circ \phi^\circ_e \circ \psi^{-1}_{e'_1} \circ \dots \circ \psi^{-1}_{e'_q},$$

where ψ_{e_i} and $\psi_{e_i'}$ are the parallel holonomies on either side of Λ. Since the parallel holonomies are trivial, $\phi^\circ_{e'}$ will not depend on the choice of paths γ and γ'. We let $\phi^\circ_{\bar{e}} = (\phi^\circ_e)^{-1}$ and recover that $\phi^\circ_{\bar{e}'} = (\phi^\circ_{e'})^{-1}$.

It remains to check that $\{\phi_e\}$, as defined, satisfy our commutativity relations. Let $e_1 = (x, y)$, $e_2 = (y, z)$, $e_1' = (y', z)$ and $e_2' = (x, y')$ be edges bounding a square, and let Λ^i be the hyperplane dual to e_i and e_i' (see Figure 3). Then we consider the separate cases

$$\phi_{e_2} \circ \phi_{e_1} = \begin{cases} \phi^*_{e_2} \circ \phi^*_{e_1} \colon (\Delta_x - \Lambda^1_x - \Lambda^2_x) \to (\Delta_z - \Lambda^1_z - \Lambda^2_z), \\ \phi^\circ_{e_2} \circ \phi^*_{e_1} \colon \Lambda^2_x \to \Lambda^2_z, \\ \phi^*_{e_2} \circ \phi^\circ_{e_1} \colon \Lambda^1_x \to \Lambda^1_z, \\ \phi^\circ_{e_2} \circ \phi^\circ_{e_1} \colon \varnothing \to \varnothing. \end{cases}$$

It follows from Lemma 3.2 that $\phi^*_{e_2} \circ \phi^*_{e_1} = \phi^*_{e_1'} \circ \phi^*_{e_2'}$. By considering the parallel holonomies with respect to Λ^2 we can see that

$$\phi^\circ_{e_2} \circ \phi^*_{e_1} = \phi^\circ_{e_2} \circ \psi_{e_1} = \psi_{e_1'} \circ \phi^\circ_{e_2'} = \phi^*_{e_1'} \circ \phi^\circ_{e_2'}.$$

A similar sequence of equalities gives that $\phi^*_{e_2} \circ \phi^\circ_{e_1} = \phi^\circ_{e_1'} \circ \phi^*_{e_2'}$. Altogether this allows us to conclude that $\phi_{e_2} \circ \phi_{e_1} = \phi_{e_1'} \circ \phi_{e_2'}$, and that $\{\phi_e\}$ is a Δ–category, and that we have proven the following:

Proposition 3.4 *Let $n \geq 2$, $d \geq 1$ and Δ be a finite set of cardinality $nd + 1$. Let L be the Kneser complex $\mathfrak{K}_n(\Delta)$. Suppose that X is an L–cube-complex such that hyperplane subgroups have separable finite-index subgroups. Then there exists a finite cover $\widehat{X} \to X$, such that there is a Δ–category over X.*

4 The holonomy

Given a Δ–category $\{\phi_e\}$ for X we obtain a holonomy map

$$\Phi_x \colon \pi_1(X, x) \to \mathrm{Sym}(\Delta_x),$$

where the homotopy class $[\gamma] = [e_1, \ldots, e_n]$ of the edge path based at x has image

$$\Phi_x([\gamma]) = \phi_\gamma.$$

The invertibility and commutativity conditions guarantee that this does not depend on the choice of representative of the homotopy class. Note that if Φ_x is trivial, then the holonomy is trivial with respect to any basepoint since the following diagram commutes:

$$\begin{tikzcd} \pi_1(X,x) \arrow[r, "\Phi_x"] \arrow[d] & \mathrm{Sym}(\Delta_x) \arrow[d] \\ \pi_1(X,y) \arrow[r, "\Phi_y"] & \mathrm{Sym}(\Delta_y) \end{tikzcd}$$

If γ is an edge path connecting x to y, then the vertical left arrow is the isomorphism given by conjugating a homotopy class of based loops by $[\gamma]$, and the vertical right arrow is the isomorphism given by conjugating by ϕ_γ.

The kernel of Φ_x is a finite-index normal subgroup of $\pi_1 X$ and corresponds to a finite-sheeted, regular cover $f\colon \widehat{X} \to X$. Lift the Δ–category on X to a Δ–category $\{\hat{\phi}_{\hat{e}}\}$ on $\widehat{X}$. We can check that the following diagram commutes:

$$\begin{tikzcd} \pi_1(\widehat{X},\hat{x}) \arrow[r, "\widehat{\Phi}_{\hat{x}}"] \arrow[d, "f_*"'] & \mathrm{Sym}(\Delta_{\hat{x}}) \arrow[d] \\ \pi_1(X,x) \arrow[r, "\Phi_x"] & \mathrm{Sym}(\Delta_x) \end{tikzcd}$$

The 0–cube $\hat{x}$ is chosen so that $f(\hat{x}) = x$, and the right vertical arrow is the isomorphism given by conjugation by $\mathfrak{f}_{\hat{x}}$. Thus we conclude that the holonomy $\widehat{\Phi}_x$ on $\widehat{X}$ is trivial. If the holonomy on X obtained from a Δ–category is trivial, then we say that the Δ–category itself is *flat*.

4.1 Constructing the orbicover

Proposition 4.1 *Let $L = \mathfrak{K}_n(\Delta)$ where $|\Delta| = nd + 1$. Let X be a compact L–cube-complex that has a flat Δ–category on X. Then there is an orbicomplex cover $X \to X_L$, where $X_L = W_L \backslash D(L)$.*

Proof Let $\{\phi_e\}$ be the flat Δ–category on X. For a basepoint x, fix an identification $q_x\colon \Delta_x \to \Delta$. For any other 0–cube y in X, let $q_y = q_x \circ \phi_\gamma$ where γ is an edge path connecting y to x. Note that q_y does not depend on the choice of γ since the Δ–category is flat.

We will prove the claim by producing an orbicomplex cover $X \to X_L$. First we map all 0–cubes in X to the unique 0–cube in X_L. We can extend X to the 1–skeleton of X by mapping each 1–cube $e = (x, y)$ dual to Λ to the half 1–cube corresponding to $q(\Lambda_x)$. This makes sense since we know that $q_x(\Lambda_x) = q_y \circ \phi_e(\Lambda_x) = q_y(\Lambda_y)$ by the remark following Definition 3.1, so e and $\bar{e}$ are mapped to the same half edge.

Now we want to extend $X^{(1)} \to X_L$ to the 2–skeleton. Let $e_1 = (x, y)$, $e_2 = (y, z)$, $e_1' = (y', z)$ and $e_2' = (x, y')$ be the directed 1–cubes bounding a square S in X such that e_i and e_i' are dual to the hyperplane Λ^i (as in Figure 3). We want to show that e_i and e_i' map to the same half edge, and the e_1 and e_2' map to half edges that bound a quarter-square in X_L. The first fact follows from the parallel transport property since $\phi_{e_1}(\Lambda_x^2) = \Lambda_y^2$ so $q_x(\Lambda_x^2) = q_y \circ \phi_{e_1}(\Lambda_x^2) = q_y(\Lambda_y^2)$. The second follows from the fact that $\Lambda_x^1 \cap \Lambda_x^2 = \varnothing$ since e_1 and e_2' bound the corner of a square, so $q_x(\Lambda_x^1) \cap q_x(\Lambda_x^2) = \varnothing$.

It is immediate that we can extend $X^{(2)} \to X_L$ to the entire skeleton since the higher dimension cubes are entirely determined by the 1–skeleton. In this particular case, we have an orbicovering since the induced maps on the vertex links are isomorphisms. Thus we can lift this orbicovering to an isomorphism $\widetilde{X} \to D(L)$ such that the deck transformation group $\pi_1(X)$ is a subgroup of W_L. □

Proof of Theorem 0.1 Let X_1 and X_2 be our L–cube-complexes. Finite-index subgroups of the hyperplane subgroups are separable, so by Proposition 3.4 there is a finite cover $X_i' \to X_i$ such that there is a Δ–category over X_i'. By considering the holonomy given by the Δ–category, we can pass to a further finite cover $\widehat{X}_i \to X_i'$ such that the induced Δ–category is flat. By Proposition 4.1, there are finite orbicovers $f_i : \widehat{X}_i \to X_L$. The common cover is then obtained by taking the intersection of the corresponding deck transformation groups inside of W_L. □

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