AGT

Algebraic & Geometric Topology

Volume 23 (2023)

Connective models for topological modular forms of level n

LENNART MEIER

msp

Algebraic & Geometric Topology **23**:8 (2023) 3553–3586
DOI: 10.2140/agt.2023.23.3553
Published: 5 November 2023

Connective models for topological modular forms of level *n*

LENNART MEIER

We construct and study connective versions of topological modular forms of higher level like $\mathrm{tmf}_1(n)$. In particular, we use them to realize Hirzebruch's level-n genus as a map of ring spectra.

55N34; 55N22, 55P91

1 Introduction

The basic tenet of Waldhausen's philosophy of *brave new algebra* is to replace known notions for commutative rings by corresponding notions for E_∞–ring spectra. These days replacing the integers by the sphere spectrum is no longer so brave and new, but rather a well-established principle. In extension, we might want to find and study E_∞–analogues of other prominent rings as well. The aim of the present paper is to do this for rings of holomorphic modular forms with respect to congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Topological analogues of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ itself were already introduced about twenty years ago. Indeed, Goerss, Hopkins and Miller introduced three spectra TMF, Tmf and tmf of topological modular forms. Recall that the rings $M_*(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z})$

and $\widetilde{M}_*(\mathrm{SL}_2(\mathbb{Z});\mathbb{Z})$ of holomorphic and meromorphic integral modular forms can be defined as the global sections $H^0(\overline{\mathcal{M}}_{\mathrm{ell}};\omega^{\otimes *})$ and $H^0(\mathcal{M}_{\mathrm{ell}};\omega^{\otimes *})$ of powers of a certain line bundle ω on the compactified and uncompactified moduli stack of elliptic curves, respectively.[1] In analogy, TMF is defined as the global sections of a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞–ring spectra on $\mathcal{M}_{\mathrm{ell}}$ with $\pi_{2k}\mathcal{O}^{\mathrm{top}} \cong \omega^{\otimes k}$ and Tmf as the global sections of an analogous sheaf on $\overline{\mathcal{M}}_{\mathrm{ell}}$. The edge maps of the resulting descent spectral sequences take the form of homomorphisms

$$\pi_{2*}\mathrm{TMF} \to \widetilde{M}_*(\mathrm{SL}_2(\mathbb{Z});\mathbb{Z}) \quad \text{and} \quad \pi_{2*}\mathrm{Tmf} \to M_*(\mathrm{SL}_2(\mathbb{Z});\mathbb{Z}).$$

The former morphism is an isomorphism after base change to $\mathbb{Z}\big[\frac{1}{6}\big]$ (while taking higher cohomology of $\omega^{\otimes *}$ into account at the primes 2 and 3) and thus TMF can be really seen as the rightful analogue of $\widetilde{M}(\mathrm{SL}_2(\mathbb{Z});\mathbb{Z})$. In contrast, $\pi_*\mathrm{Tmf}$ has torsion-free summands in negative degree, whereas $M_*(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z})$ is concentrated in nonnegative degrees. The solution is to define tmf simply as the connective cover $\tau_{\geq 0}\mathrm{Tmf}$, and one can show that indeed $\pi_{2*}\mathrm{tmf}\big[\frac{1}{6}\big]$ is isomorphic to $M_*(\mathrm{SL}_2(\mathbb{Z}),\mathbb{Z}\big[\frac{1}{6}\big])$. We mention that one of the motivations for constructing tmf was lifting the Witten genus to a map of E_∞–ring spectra $M\mathrm{String} \to \mathrm{tmf}$ as achieved in Ando, Hopkins and Rezk [1]. For applications to the stable homotopy groups of spheres and exotic spheres, see for instance Hopkins and Mahowald [23], Behrens, Hill, Hopkins and Mahowald [3], Wang and Xu [46] and Isaksen, Wang and Xu [25].

In number theory, it is very common not only to consider modular forms with respect to $\mathrm{SL}_2(\mathbb{Z})$, but also to congruence subgroups of these; the most important being $\Gamma = \Gamma_0(n)$, $\Gamma_1(n)$ or $\Gamma(n)$. Algebrogeometrically, such modular forms can be defined as sections of the pullback of $\omega^{\otimes *}$ to compactifications $\overline{\mathcal{M}}(\Gamma)$ of stacks classifying generalized elliptic curves with certain level structures (see eg Deligne and Rapoport [6], Diamond and Im [7], Conrad [5] and the author's [36]); for example, $\overline{\mathcal{M}}(\Gamma_1(n))$ classifies generalized elliptic curves with a chosen point of order n whose multiples intersect every irreducible component of every geometric fiber. Hill and Lawson [17] defined sheaves of E_∞–ring spectra on these stacks and obtained spectra $\mathrm{Tmf}(\Gamma)$, as their global sections, and $\mathrm{TMF}(\Gamma)$, by restriction to the loci of smooth elliptic curves. The latter spectra are good topological analogues of the rings $\widetilde{M}(\Gamma;\mathbb{Z}[1/n])$ of meromorphic modular forms in

[1]The terms *meromorphic* and *holomorphic* come from the corresponding analytic definitions, where one demands that the given function on the upper half-plane can be continued meromorphically and holomorphically, respectively, to the cusp(s). The former kind of modular form is also sometimes called *weakly holomorphic*.

the sense that $\pi_*\mathrm{TMF}(\Gamma)$ is isomorphic to this ring if Γ is $\Gamma_1(n)$ or $\Gamma(n)$ (with $n \geq 2$) and, if we invert 6, also in the case $\Gamma = \Gamma_0(n)$.

In contrast, neither $\mathrm{Tmf}(\Gamma)$ nor its connective cover $\tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ are in general good analogues of the ring of holomorphic modular forms $M(\Gamma; \mathbb{Z}[1/n])$, even in the nice case of $\Gamma = \Gamma_1(n)$ and $n \geq 2$. Writing $\mathrm{Tmf}_1(n)$ for $\mathrm{Tmf}(\Gamma_1(n))$, the reason is that $H^1(\overline{\mathcal{M}}(\Gamma_1(n)); \omega)$ and thus $\pi_1\mathrm{Tmf}_1(n)$ is nontrivial in general (with $n = 23$ being the first example), while this contribution does not occur in $M(\Gamma; \mathbb{Z}[1/n])$. Following an idea of Lawson, we define a connective version $\mathrm{tmf}_1(n)$ by "artificially" removing π_1, while still retaining the E_∞–structure on $\mathrm{tmf}_1(n)$. The following will be proven as Theorems 2.12 and 2.22.

Theorem 1.1 *There is an essentially unique connective E_∞–ring spectrum $\mathrm{tmf}_1(n)$ with an E_∞–ring map $\mathrm{tmf}_1(n) \to \mathrm{Tmf}_1(n)$ that identifies the homotopy groups of the source with $M(\Gamma_1(n); \mathbb{Z}[1/n])$.*

Moreover, the involution of $\overline{\mathcal{M}}(\Gamma_1(n))$ sending a point of order n on the universal elliptic curve to its negative defines on $\mathrm{tmf}_1(n)$ the structure of a genuine C_2–spectrum. Its slices in the sense of Hill, Hopkins and Ravenel [16] are trivial in odd degrees and can be explicitly identified in even degrees.

The analogous theorem also works to define $\mathrm{tmf}(n)$, but $\mathrm{tmf}_0(n)$ we define only in certain cases since in the general case it is not yet clear what the "correct" definition is. The spectrum $\mathrm{tmf}(n)$ has been further investigated in [21, Theorem 3.14], where a criterion for the nonvanishing of its Tate spectrum is proven.

One of the principal motivations for the consideration of $\mathrm{tmf}_1(n)$ is its connection to the Hirzebruch level-n genera $\mathrm{MU}_* \to M(\Gamma_1(n); \mathbb{Z}[1/n])$. They specialize for $n = 2$ to the classic Ochanine elliptic genus and have similar rigidity properties in general; see Hirzebruch, Berger and Jung [20]. We will prove the following as Theorem 3.6.

Theorem 1.2 *For every $n \geq 2$, there is a ring map $\mathrm{MU} \to \mathrm{tmf}_1(n)$ realizing on homotopy groups the Hirzebruch level-n–genus. Moreover, this map refines to a map $\mathrm{MU}_\mathbb{R} \to \mathrm{tmf}_1(n)$ of C_2–spectra.*

We have two further classes of results on the spectra $\mathrm{tmf}_1(n)$ and their cousins. The first is the following compactness result, contained in Theorem 4.4 and Corollary 4.6.

Theorem 1.3 *The* $\mathrm{tmf}[1/n]$*–modules* $\mathrm{tmf}_0(n)$, $\mathrm{tmf}_1(n)$ *and* $\mathrm{tmf}(n)$ *are perfect*, *ie they are compact objects in the module category*, *in the cases they are defined. In particular*, *their* $\mathbb{F}_p$*–cohomologies are finitely presented over the Steenrod algebra and thus their* p*–completions are fp-spectra in the sense of Mahowald and Rezk* [33].

By a result of Kuhn [28, Theorem 1.7] this implies, for example, that the Hurewicz image of $\pi_*\mathrm{tmf}(\Gamma) \cong \pi_*\Omega^\infty\mathrm{tmf}(\Gamma)$ in $H_*(\Omega^\infty\mathrm{tmf}(\Gamma);\mathbb{F}_p)$ is finite-dimensional, where $\mathrm{tmf}(\Gamma)$ denotes either $\mathrm{tmf}_0(n)$, $\mathrm{tmf}_1(n)$ or $\mathrm{tmf}(n)$. We also note that in contrast to the theorem, $\mathrm{tmf}_1(n)$ will not be a perfect $\mathrm{tmf}_0(n)$–module in general. We also show that $\mathrm{tmf}_0(n)$, $\mathrm{tmf}_1(n)$ and $\mathrm{tmf}(n)$ are faithful as $\mathrm{tmf}[1/n]$–modules, answering a question of Höning and Richter [21, page 21].

The second result is a variant of the decomposition results of the author [37], which we state in this introduction only at the prime 2 and for $\mathrm{tmf}_1(n)$, and which will be proven as Theorem 5.6.

Theorem 1.4 *Let* $n > 1$ *be odd. If one can lift every weight-*1 *modular form for* $\Gamma_1(n)$ *over* $\mathbb{F}_2$ *to a form of the same weight and level over* $\mathbb{Z}_{(2)}$, *we have a* C_2*–equivariant splitting*

$$\mathrm{tmf}_1(n)_{(2)} \simeq \bigoplus_i \Sigma^{n_i\rho}\mathrm{tmf}_1(3)_{(2)},$$

where ρ *denotes the real regular representation of* C_2.

In the author's earlier work [36, Appendix C], it is shown that for $1 < n < 65$ odd indeed every weight-1 modular form for $\Gamma_1(n)$ over $\mathbb{F}_2$ lifts to a form of the same weight and level over $\mathbb{Z}_{(2)}$, while for $n = 65$ it does not. See also [36, Remark 3.14] for a further discussion of this condition.

Conventions and notation

All notions are to be understood suitably derived or ∞–categorical. This means that pushout means either a pushout in the respective ∞–category or a homotopy pushout in the underlying model category. We will use $\otimes$ for the (derived) smash product. Note that this coincides with the coproduct in the ∞–category CAlg of E_∞–ring spectra.

When we use G–spectra, we will always mean genuine G–spectra. The notations $\tau_{\leq k}$ and $\tau_{\geq k}$ denote the k–(co)connective cover of a spectrum and we use the same notation for the slice-(co)connective covers of a G–spectrum. Furthermore, we denote by $\mathbb{S}$

the sphere (G–)spectrum. In some parts of this article, we have the opportunity to use $\mathrm{RO}(C_2)$–graded homotopy groups of C_2–spectra. We will use the notation σ for the sign representation and ρ or $\mathbb{C}$ for the regular representation of C_2.

We will use the notations $\mathrm{TMF}_1(n)$ and $\mathrm{TMF}(\Gamma_1(n))$ interchangeably and similarly in related contexts.

Acknowledgements

I want to thank Tyler Lawson for explaining to me the idea of how to construct a connective model for $\mathrm{TMF}_1(n)$, and for the sketch of an argument that $\mathrm{tmf}_1(3)$ is not perfect as a $\mathrm{tmf}_0(3)$–module. It is also a pleasure to acknowledge the influence and encouragement of Mike Hill. Furthermore I want to thank Eva Höning and Birgit Richter for their interest and remarks on a preliminary version, and the referee for their extensive comments.

Finally, I want to thank the Hausdorff Institute for hospitality in 2015 when part of this work was undertaken. Apologies for the subsequent delay in publication.

2 The construction of connective topological modular forms

The aim of this section is to construct connective spectra $\mathrm{tmf}(\Gamma)$ of topological modular forms and thereby prove Theorem 1.1. Here Γ denotes a congruence subgroup Γ in the following sense, which is a bit more restrictive than the standard definition.

Definition 2.1 We call $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ a *congruence subgroup of level n* if $\Gamma = \Gamma(n)$ or $\Gamma_1(n) \subset \Gamma \subset \Gamma_0(n)$.[2]

As explained in [17; 37, Section 2.1], we can associate with every such Γ a (nonconnective and nonperiodic) E_∞–ring spectrum $\mathrm{Tmf}(\Gamma)$. (See also [44, Theorem 5.2] for the case of $\Gamma(n)$.) These arise as global sections of sheaves of E_∞–ring spectra $\mathcal{O}^{\mathrm{top}}$ on stacks $\overline{\mathcal{M}}(\Gamma)$ classifying generalized elliptic curves with certain level structures; the details will not be important for the purposes of this article, but see for instance [6; 5; 45; 36]. Our goal in this section is to construct a nice connective version $\mathrm{tmf}(\Gamma)$ for $\mathrm{Tmf}(\Gamma)$. For this, we will fix a localization $\mathbb{Z}_S$ of the integers and restrict mostly to tame congruence subgroups.

[2]We refer to [8] for background on the congruence subgroups $\Gamma_1(n)$, $\Gamma(n)$ and $\Gamma_0(n)$ and their relationship to moduli of elliptic curves. This material though is barely necessary for the present paper, as we use the congruence subgroups primarily as notation.

Definition 2.2 We say that a congruence subgroup Γ of level n is *tame* with respect to $\mathbb{Z}_S$ if $n \geq 2$ and n is invertible in $\mathbb{Z}_S$; in the case $\Gamma_1(n) \subset \Gamma \subset \Gamma_0(n)$ we demand additionally that $\gcd(6, [\Gamma : \Gamma_1(n)])$ is invertible in $\mathbb{Z}_S$.[3]

The definition ensures that the order of every automorphism of a point in $\overline{\mathcal{M}}(\Gamma)$ is invertible and thus the stack is of cohomological dimension one. As explained in [37, Section 2.1], in this case $\pi_*\tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ is concentrated in even degrees except for $\pi_1\mathrm{Tmf}(\Gamma)$, which might be nonzero. (The smallest n for which this happens is 23.) Moreover, the even homotopy groups of $\mathrm{Tmf}(\Gamma)$ are precisely isomorphic to the ring of holomorphic modular forms $M(\Gamma; \mathbb{Z}[1/n])$.

Following the lead of [29, Proposition 11.1] (and additional explanations by its author), we will first describe a general procedure to kill π_1 for E_∞–rings that applies to $\tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ for Γ tame. We will then present a C_2–equivariant refinement that helps to define a nice version of $\mathrm{tmf}(\Gamma)$ also in some nontame cases; see Construction 2.24. We note that our techniques are only necessary if $\pi_1\mathrm{Tmf}(\Gamma)$ is nontrivial as otherwise the usual connective cover defines a perfectly good version of $\mathrm{tmf}(\Gamma)$.

2.1 The nonequivariant argument

Let R be a connective E_∞–ring spectrum with $\pi_0 R$ an étale extension of $\mathbb{Z}_S$, a localization of $\mathbb{Z}$, and $\eta \cdot 1 = 0$; here, $\eta \in \pi_1\mathbb{S}$ is the Hopf element and $1 \in \pi_0 R$ the unit. (The relevant example for us is $R = \tau_{\geq 0}\mathrm{Tmf}(\Gamma)_S$ with $\pi_0 R = \mathbb{Z}_S$ if $\Gamma_1(n) \subset \Gamma \subset \Gamma_0(n)$ and $\pi_0 R = \mathbb{Z}_S[\zeta_n]$ if $\Gamma = \Gamma(n)$.) We want to construct a map $R' \to R$ of E_∞–ring spectra which is injective on π_* and with cokernel $\pi_1 R$. In the following, we localize everything implicitly at the set S — so $\mathbb{Z}$ really means $\mathbb{Z}_S$, etc.

Let A first be a general E_∞–ring spectrum. For an A–module M, we denote by

$$\mathbb{P}_A(M) \simeq A \oplus M \oplus (M^{\otimes_A 2})_{h\Sigma_2} \oplus \cdots$$

the free unital E_∞–A–algebra on M; cf [32, Example 3.1.3.14].

Definition 2.3 Let $x\colon \Sigma^k A \to A$ be an A–linear map. We define its *E_∞–cone* $C^A(x)$ as the pushout $A \otimes_{\mathbb{P}_A(\Sigma^k A)} A$ of E_∞–ring spectra. Here, the first map $\mathbb{P}_A(\Sigma^k A) \to A$ is the free E_∞–map on x, while the second arises from applying $\mathbb{P}_A$ to the unique map $\Sigma^k A \to 0$.

[3] As the quotient $\Gamma_0(n)/\Gamma_1(n)$ is $(\mathbb{Z}/n)^\times$, the latter condition reduces to $\gcd(6, \varphi(n))$ being invertible in the case $\Gamma = \Gamma_0(n)$. Thus we require that 2 is invertible and also 3 if n is divisible by a prime of the form $3k+1$ or by 9.

Note that if B is an E_∞–A–algebra, we have $C^A(x)\otimes_A B\simeq C^B(x)$. Writing the usual cone $C(x)$ as the pushout $A\sqcup_{\Sigma^k A\oplus A} A$ in A–modules produces a map $C(x)\to C^A(x)$ via the inclusion $A\oplus\Sigma^k A\to\mathbb{P}^A(\Sigma^k A)$ of the first two summands and the identity id_A.

Lemma 2.4 *If $x=0$, the canonical map $C(x)\to C^A(x)$ is split as a map of A–modules.*

Proof The pushout square

$$\begin{tikzcd} A\oplus\Sigma^k A \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & C(0)\simeq A\oplus\Sigma^{k+1}A \end{tikzcd} \tag{2.5}$$

arises from the pushout square

$$\begin{tikzcd} \Sigma^k A \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma^{k+1}A \end{tikzcd} \tag{2.6}$$

via the functor $\mathrm{Mod}_A\to\mathrm{CAlg}_A$ of square-zero extension. In particular, it is a diagram of E_∞–A–algebras. As the E_∞–pushout square (P) defining $C^A(0)$ arises from (2.6) as well, but via $\mathbb{P}_A$, we see that the square (2.5) receives a map from the square (P). The resulting map $C^A(0)\to C(0)$ defines a splitting of $C(0)\to C^A(0)$ by the universal property of the pushout square (2.5). □

We will apply our general consideration to the connective E_∞–ring spectrum R we have fixed. As η is zero in π_*R, we obtain an E_∞–map $C^{\mathbb{S}}(\eta)\to R$. This induces an E_∞–map $\tau_{\le 1}C^{\mathbb{S}}(\eta)\to\tau_{\le 1}R$; see [16, Proposition 4.35].

Lemma 2.7 *The 1–coconnective cover $\tau_{\le 1}C^{\mathbb{S}}(\eta)$ is equivalent to $H\mathbb{Z}$.*

Proof We claim that the canonical map $C(\eta)\to C^{\mathbb{S}}(\eta)$ is 2–connected. By the Hurewicz theorem, we can test this after tensoring with $H\mathbb{Z}$ and thus it suffices to show that the resulting map $C(\eta\otimes H\mathbb{Z})\to C^{H\mathbb{Z}}(\eta\otimes H\mathbb{Z})$ is 2–connected. But $\eta\otimes H\mathbb{Z}$ agrees with the 0–map $\Sigma H\mathbb{Z}\to H\mathbb{Z}$. Thus, we have to show that

$$H\mathbb{Z}\oplus\Sigma^2 H\mathbb{Z}\to C^{H\mathbb{Z}}(\eta\otimes H\mathbb{Z})\simeq\mathbb{P}^{H\mathbb{Z}}\Sigma^2 H\mathbb{Z}\simeq H\mathbb{Z}\oplus\Sigma^2 H\mathbb{Z}\oplus(\Sigma^4 H\mathbb{Z})_{hC_2}\oplus\cdots$$

is 2–connected. As noted above, the map is split injective and thus must be indeed an isomorphism on π_i even for $i\le 3$. □

The inclusion of 1–truncated connective E_∞–ring spectra into all connective E_∞–ring spectra admits a left adjoint by [31, Proposition 5.5.6.18; 32, Proposition 7.1.3.14]. By [32, Proposition 7.1.3.15(3)], it agrees with $\tau_{\le 1}$ on underlying spectra.

By [32, Theorem 7.5.0.6], we can extend the E_∞–map $H\mathbb{Z} = \tau_{\le 1}C^{\mathbb{S}}(\eta) \to \tau_{\le 1}R$ to an E_∞–map $H\pi_0 R \to \tau_{\le 1}R$ since the map $\mathbb{Z} \to \pi_0 R$ is étale. Define now R' via the homotopy pullback square

$$\begin{tikzcd} R' \arrow[r] \arrow[d] & H\pi_0 R \arrow[d] \\ R \arrow[r] & \tau_{\le 1} R \end{tikzcd} \tag{2.8}$$

This construction provides the existence part of the following proposition.

Proposition 2.9 *Let R be a connective E_∞–ring spectrum such that $\pi_0 R$ is an étale extension of a localization $\mathbb{Z}_S$ of the integers and $\eta \cdot 1 = 0$ in $\pi_1 R$. Then there exists a morphism $R' \to R$ of E_∞–ring spectra inducing an isomorphism on π_i for $i \neq 1$ and satisfying $\pi_1 R' = 0$. Moreover, for every other $R'' \to R$ with these properties, there is an equivalence $R'' \to R'$ of E_∞–ring spectra over R.*

Proof It remains to show uniqueness. We localize again everything implicitly at S. We first note that the map $H\mathbb{Z} \to \tau_{\le 1}R$ constructed above is actually the unique E_∞–map with this source and target. Indeed, for connectivity reasons, we have an equivalence of mapping spaces $\mathrm{Map}_{\mathrm{CAlg}}(H\mathbb{Z}, \tau_{\le 1}R) \simeq \mathrm{Map}_{\mathrm{CAlg}}(C^{\mathbb{S}}(\eta), \tau_{\le 1}R)$. The latter is equivalent to the space of nullhomotopies of η in $\tau_{\le 1}R$, ie $\mathrm{Map}_{\mathrm{Sp}}(\Sigma^2\mathbb{S}, \tau_{\le 1}R) \simeq *$. Using that thus $\tau_{\le 1}R$ has an essentially unique structure of an $H\mathbb{Z}$–E_∞–algebra, we deduce again from [32, Theorem 7.5.0.6] that the space of E_∞–maps from $H\pi_0 R$ to $\tau_{\le 1}R$ is equivalent to the set of ring homomorphisms $\pi_0 R \to \pi_0 R$.

Given now $R'' \to R$ as in the proposition, we obtain a map $R'' \to \tau_{\le 1}R'' \simeq H\pi_0 R \to \tau_{\le 1}R$. We see that R'' arises as a pullback of a diagram of the same shape as (2.8), but possibly with a map $H\pi_0 R \to \tau_{\le 1}R$ inducing a different isomorphism f on π_0 than the identity. The paragraph above implies that using the map f on $H\pi_0 R$ we obtain an equivalence between the cospans constructing R' and R'' and thus between R' and R'' over R. $\square$

To apply this to topological modular forms, we need the following two lemmas.

Lemma 2.10 *Let Γ be a tame congruence subgroup with respect to a localization $\mathbb{Z}_S$. Then η is zero in $\pi_1 \mathrm{Tmf}(\Gamma)_S$.*

Proof According to [37, Proposition 2.5], the descent spectral sequence

$$H^s(\overline{\mathcal{M}}(\Gamma); \omega^{\otimes t}) \Rightarrow \pi_{2t-s}\mathrm{Tmf}(\Gamma)$$

for $\mathrm{Tmf}(\Gamma)_S$ is concentrated in lines 0 and 1. Thus $\pi_1\mathrm{Tmf}(\Gamma)_S \cong H^1(\overline{\mathcal{M}}(\Gamma)_S; \omega)$ and it suffices to show that the image of η in $H^1(\overline{\mathcal{M}}(\Gamma)_S; \omega)$ is trivial. This is the content of [36, Proposition 2.16] unless $\Gamma_1(n) \subsetneq \Gamma \subsetneq \Gamma_0(n)$. For the remainder of the proof, assume that we are in this case and set $G = \Gamma/\Gamma_1(n)$.

We will argue that the map

$$H^1(\overline{\mathcal{M}}(\Gamma)_{(2)}; \omega) \to H^1(\overline{\mathcal{M}}(\Gamma_1(n)_{(2)}; \omega)$$

is isomorphic to the inclusion of G–invariants. As η vanishes in the target, this will imply the vanishing of η in the source.

The map $\overline{\mathcal{M}}(\Gamma_1(n))_{(2)} \to \overline{\mathcal{M}}(\Gamma)_{(2)}$ induces a map

$$c\colon \mathcal{X} = [\overline{\mathcal{M}}(\Gamma_1(n))_{(2)}/G] \to \overline{\mathcal{M}}(\Gamma)_{(2)}$$

from the stack quotient. Denote the pullback of ω to $\mathcal{X}$ also by ω. By [37, Lemma A.2], the induced map $H^1(\overline{\mathcal{M}}(\Gamma)_{(2)}; \omega) \to H^1(\mathcal{X}; \omega)$ is an isomorphism. Moreover, the descent spectral sequence

$$H^i(G; H^j(\overline{\mathcal{M}}(\Gamma_1(n))_{(2)}; \omega)) \Rightarrow H^{i+j}(\mathcal{X}; \omega)$$

is concentrated in the zero-line since the order of G is invertible in $\mathbb{Z}_{(2)}$ by the tameness of Γ. Thus,

$$H^1(\overline{\mathcal{M}}(\Gamma)_{(2)}; \omega) \cong H^1(\mathcal{X}; \omega) \to H^1(\overline{\mathcal{M}}(\Gamma_1(n)_{(2)}; \omega)$$

is indeed the inclusion of G–invariants. $\square$

Lemma 2.11 *Let Γ be a tame congruence subgroup with respect to a localization $\mathbb{Z}_S$. Then $\pi_0\mathrm{Tmf}(\Gamma) \cong \mathbb{Z}_S$ if $\Gamma_1(n) \subset \Gamma \subset \Gamma_0(n)$ and $\pi_0\mathrm{Tmf}(\Gamma) \cong \mathbb{Z}_S[\zeta_n]$ if $\Gamma = \Gamma(n)$.*

Proof As recalled above, we have $\pi_0\mathrm{Tmf}(\Gamma) \cong H^0(\overline{\mathcal{M}}(\Gamma); \mathcal{O}_{\overline{\mathcal{M}}(\Gamma)})$. In the cases that $\Gamma = \Gamma_0(n)$, $\Gamma_1(n)$ or $\Gamma(n)$ the computation of this group is classical and can be found for instance in [36, Proposition 2.13]. The case of $\Gamma_1(n) \subsetneq \Gamma \subsetneq \Gamma_0(n)$ follows by identifying $H^0(\overline{\mathcal{M}}(\Gamma); \mathcal{O}_{\overline{\mathcal{M}}(\Gamma)})$ with $H^0(\overline{\mathcal{M}}(\Gamma); \mathcal{O}_{\overline{\mathcal{M}}(\Gamma)})^{\Gamma/\Gamma_1(n)}$ again using [37, Lemma A.2]. $\square$

This allows us to use Proposition 2.9 to define $\mathrm{tmf}(\Gamma)_S$ in the tame case by killing π_1 from $\tau_{\geq 0}\mathrm{Tmf}(\Gamma)_S$. Summarizing we obtain:

Theorem 2.12 *For every set of primes S and every congruence subgroup Γ that is tame with respect to $\mathbb{Z}_S$, there is up to equivalence a unique connective E_∞–ring spectrum $\mathrm{tmf}(\Gamma)_S$ with an E_∞–ring map $\mathrm{tmf}(\Gamma)_S \to \mathrm{Tmf}(\Gamma)_S$ that identifies the homotopy groups of the source with the ring of holomorphic modular forms $M(\Gamma; \mathbb{Z}_S)$.*

Formally, we could also apply this procedure in some nontame cases (for instance if we localize away from 2), but the author knows of no reason to regard these constructions in these cases as "correct".

Notation 2.13 We will use the abbreviations

$$\mathrm{tmf}_1(n) = \mathrm{tmf}(\Gamma_1(n)), \quad \mathrm{tmf}_0(n) = \mathrm{tmf}(\Gamma_0(n)), \quad \mathrm{tmf}(n) = \mathrm{tmf}(\Gamma(n)),$$

when these make sense.

Remark 2.14 For every ring spectrum R, we can consider the stack $\mathcal{X}_R$ associated to the graded Hopf algebroid $(\mathrm{MU}_{2*}(R), (\mathrm{MU} \otimes \mathrm{MU})_{2*}(R))$. If R is complex orientable, this coincides with the stack quotient $[\operatorname{Spec} \pi_{2*}R/\mathbb{G}_m]$. In [38, Definition 5.5] we introduced cubic versions $\mathcal{M}_1(n)_{\mathrm{cub}}$ and $\mathcal{M}_0(n)_{\mathrm{cub}}$ of the moduli stacks $\mathcal{M}(\Gamma_1(n))$ and $\mathcal{M}(\Gamma_0(n))$. These come with a finite morphism to the moduli stack $\mathcal{M}_{\mathrm{cub}}$ of cubic curves, where we allow arbitrary Weierstraß equations. In [38, Theorem 5.19] we showed that $\mathcal{M}_1(n)_{\mathrm{cub}} \simeq [M(\Gamma_1(n); \mathbb{Z}[1/n])/\mathbb{G}_m]$ for $n \geq 2$. In combination, we see that $\mathcal{X}_{\mathrm{tmf}_1(n)} \simeq \mathcal{M}_1(n)_{\mathrm{cub}}$ for $n \geq 2$. In the case $n = 1$, the corresponding equivalence $\mathcal{X}_{\mathrm{tmf}} \simeq \mathcal{M}_{\mathrm{cub}}$ has a quite different character and was shown in [34]. Whether there are equivalences $\mathcal{X}_{\mathrm{tmf}_0(n)} \simeq \mathcal{M}_0(n)_{\mathrm{cub}}$ for a suitable definition of $\mathrm{tmf}_0(n)$ remains open, to the knowledge of the author, even for $n = 3$.

2.2 The C_2–equivariant argument

All the stacks $\mathcal{M}(\Gamma)$ come with an involution induced from postcomposing the level structure with the $[-1]$–automorphism of the elliptic curve. As explained in Remark 2.15, this induces a C_2–action on $\mathrm{Tmf}(\Gamma)$. Our goal in this subsection is to define suitable C_2–spectra $\mathrm{tmf}(\Gamma)$ in the tame case. This will allow us to construct an E_∞–ring spectrum $\mathrm{tmf}(\Gamma)$ also if there is just a tame subgroup $\Gamma' \subset \Gamma$ of index 2; see Construction 2.24.

Remark 2.15 The goal of this remark is to clarify the construction of the C_2–action on $\mathrm{Tmf}(\Gamma)$ sketched above.

Denote the automorphism of $\mathcal{M}(\Gamma)$ described above by t. As t commutes with the forgetful map $\mathrm{pr}\colon \mathcal{M}(\Gamma) \to \mathcal{M}_{\mathrm{ell}}$, this defines a C_2–action inside the slice category $(\mathrm{Stacks}/\mathcal{M}_{\mathrm{ell}})^{\text{ét},\mathrm{op}}$ of stacks étale over $\mathcal{M}_{\mathrm{ell}}$. We will use a lax commutative triangle

$$\begin{tikzcd}
(\mathrm{Stacks}/\mathcal{M}_{\mathrm{ell}})^{\text{ét},\mathrm{op}} \arrow[rr, "N"] \arrow[dr, "\mathcal{O}^{\mathrm{top}}"'] & {} \arrow[d, phantom, "\Leftarrow"] & (\mathrm{Stacks}/\overline{\mathcal{M}}_{\mathrm{ell}})^{\text{log-ét},\mathrm{op}} \arrow[dl, "\mathcal{O}^{\mathrm{top}}"] \\
& \mathrm{CAlg}(\mathrm{Sp}) &
\end{tikzcd}$$

Here, the diagonal arrows are the Goerss–Hopkins–Miller and Hill–Lawson sheaves of ring spectra. The horizontal arrow N is a normalization construction; see for example [22, Proposition 2.27]. The canonical map $\mathcal{O}^{\mathrm{top}}(N(U)) \to \mathcal{O}^{\mathrm{top}}(U)$ for $U \to \mathcal{M}_{\mathrm{ell}}$ étale comes from the fact that $U \subset N(U)$ is an open substack and the Hill–Lawson sheaf restricts to the Goerss–Hopkins–Miller sheaf.

Applying the left diagonal arrow to $(\mathcal{M}(\Gamma), t)$ gives a C_2–action on $\mathrm{TMF}(\Gamma)$. Doing the same with the composite of the right diagonal arrow and the horizontal arrow produces the C_2–action on $\mathrm{Tmf}(\Gamma)$. Moreover, we obtain a C_2–map $\mathrm{Tmf}(\Gamma) \to \mathrm{TMF}(\Gamma)$.

As explained in [37, Example 6.12], the C_2–action induced by t on $\mathrm{TMF}(\Gamma)$ is equivalent to the one induced by the C_2–action in $(\mathrm{Stacks}/\mathcal{M}_{\mathrm{ell}})^{\text{ét},\mathrm{op}}$ given by $\mathrm{id}_{\mathcal{M}(\Gamma)}$ on $\mathcal{M}(\Gamma)$, but choosing the $[-1]$–isomorphism between the elliptic curves classified by pr and $\mathrm{pr}\,\mathrm{id}_{\mathcal{M}(\Gamma)}$. This C_2–action induces multiplication by -1 on the pullback of ω to $\mathcal{M}(\Gamma)$: indeed, the $[-1]$–automorphism of an elliptic curve induces multiplication by -1 on the sheaf of differentials. Moreover, pr classifies precisely the pullback of the universal elliptic curve $\mathcal{E}^{\mathrm{uni}}$ and ω is the restriction of $\Omega^1_{\mathcal{E}^{\mathrm{uni}}/\mathcal{M}_{\mathrm{ell}}}$ to $\mathcal{M}_{\mathrm{ell}}$ along the zero section.

Thus, if Γ is tame, it implies that C_2 acts by $(-1)^k$ on $\pi_{2k}\mathrm{TMF}(\Gamma) \cong H^0(\mathcal{M}(\Gamma); \omega^{\otimes k})$. Since $\pi_{2k}\mathrm{Tmf}(\Gamma)$ injects in the tame case into $\pi_{2k}\mathrm{TMF}(\Gamma)$, the same is true for $\pi_{2k}\mathrm{Tmf}(\Gamma)$.

The action t can be trivial, eg for $\Gamma = \Gamma_0(n)$ or $\Gamma(2)$. This forces $\pi_{2k}\mathrm{Tmf}(\Gamma) = 0$ for k odd in these cases (as t acts both by 1 and -1 and the groups are torsion-free). This corresponds to the classical fact that there are no modular forms of odd weight if $-\mathrm{id}$ is in Γ.

In the following we will use standard notation from equivariant homotopy theory. In particular, for an inner product space V with G–action, we denote by $S(V)$ the unit sphere and by S^V the 1–point compactification as G–spaces. We denote by $a = a_\sigma\colon S^0 \to S^\sigma$ the inclusion for σ the real sign representation of C_2.

The Hopf map defines a C_2–map $\overline{\eta}\colon S(\mathbb{C}^2) \to S^{\mathbb{C}}$, where C_2 acts on $\mathbb{C}$ via complex conjugation. This stabilizes to an element in $\pi_\sigma^{C_2}\mathbb{S}$, which restricts to $\eta \in \pi_1^e\mathbb{S}$.

Lemma 2.16 *The homotopy groups $\pi_\sigma^{C_2}(\mathbb{S})$ and $\pi_{-\sigma}^{C_2}\mathbb{S}$ are infinite cyclic and generated by $\overline{\eta}$ and a, respectively.*

Proof For $\pi_\sigma^{C_2}(\mathbb{S})$, this is proven as formula (8.1) in [2]. (Note that they use the notation $\pi_{p,q}^s$ for our $\pi_{p\sigma+q}^{C_2}(\mathbb{S})$.) Proposition 7.0 in op. cit. implies that the homomorphism $\pi_{-\sigma}^{C_2}\mathbb{S} \to \pi_0\mathbb{S}$, taking a map $\mathbb{S} \to \Sigma^\sigma\mathbb{S}$ to its geometric fixed points is an isomorphism. Taking fixed points of the map a clearly gives the identity map $S^0 \to S^0$, which yields the result. □

In the following, we denote by $\tau_{\leq i}$ the slice coconnective cover, by $\tau_{\geq i}$ the slice connective cover and by $\tau_i = \tau_{\geq i}\tau_{\leq i}$ the i^{th} slice for C_2–spectra. We refer to [16] for background about the slice filtration. We denote by $H\underline{\mathbb{Z}}$ the C_2–Eilenberg–Mac Lane spectrum for the constant Mackey functor $\underline{\mathbb{Z}}$.

Lemma 2.17 *We have an equivalence $\tau_{\leq 1}C\overline{\eta} \simeq H\underline{\mathbb{Z}}$.*

Proof It suffices to show that the first slice of $C\overline{\eta}$ is null and the zeroth slice is $H\underline{\mathbb{Z}}$. As shown in [16] and summarized in [18, Section 2.4], this is implied by the calculations $\underline{\pi}_0C\overline{\eta} \cong \underline{\mathbb{Z}}$ and $\underline{\pi}_\sigma C\overline{\eta} = 0$. These follows easily by the long exact sequence arising from the cofiber sequence

$$S^\sigma \xrightarrow{\overline{\eta}} S^0 \to C\overline{\eta}$$

and the computations of $\pi_{-\sigma}^{C_2}\mathbb{S}$, $\pi_0^{C_2}\mathbb{S}$ and $\pi_\sigma^{C_2}(\mathbb{S})$ above, using also that $\pi_{-1}^{C_2}S^\sigma = 0$. □

The following lemma is a C_2–slice analogue of Lewis's equivariant Hurewicz theorem [30, Theorem 2.1]. Recall that a C_2–spectrum is *k–slice connected* if and only if $\tau_{\leq k}X = 0$.

Lemma 2.18 *A connective C_2–spectrum X is k–slice connected if and only if $H\underline{\mathbb{Z}} \otimes X$ is k–slice connected.*

Spelled out, the latter condition is equivalent to $\underline{H}_V(X;\underline{\mathbb{Z}}) = \underline{\pi}_V(H\underline{\mathbb{Z}} \otimes X) = 0$ for all C_2–representations V of the form $i\rho$ or $i\rho - 1$ with $|V| \leq k$.

Proof If X is k–slice connected, the same is true for $H\underline{\mathbb{Z}} \otimes X$. For the converse, assume that $\underline{H}_V^{C_2}(X;\underline{\mathbb{Z}}) = 0$ for all C_2–representations V of the form $i\rho$ or $i\rho - 1$ with $|V| \leq k$. By induction on k, we can assume that X is $(k-1)$–slice connected and we need to show that $\tau_k X = 0$ to deduce that X is indeed k–slice connected. Let W be $\frac{1}{2}k\rho$ if k is even and $\frac{1}{2}(k+1)\rho - 1$ if k is odd. As $\tau_{\geq k+1}X$ and its suspension are k–slice connected, the direction discussed above shows $\underline{H}_W(\tau_{\geq k+1}X;\underline{\mathbb{Z}}) = \underline{H}_W(\Sigma\tau_{\geq k+1}X;\underline{\mathbb{Z}}) = 0$. Thus,

$$0 = \underline{H}_W(X;\underline{\mathbb{Z}}) \to \underline{H}_W(\tau_k X;\underline{\mathbb{Z}})$$

is an isomorphism. As summarized in [18, Section 2.4], the slice $\tau_k X$ is of the form $S^W \otimes HM$ for some Mackey functor M and we deduce that $\underline{H}_0(HM;\underline{\mathbb{Z}}) \cong \underline{H}_W(\tau_k X;\underline{\mathbb{Z}}) = 0$.

We know that $\tau_0\mathbb{S} = H\underline{\mathbb{Z}}$. As HM is (slice) connective, a similar argument to before shows that

$$M \cong \underline{\pi}_0(\mathbb{S} \otimes HM) \cong \underline{\pi}_0(H\underline{\mathbb{Z}} \otimes HM) = \underline{H}_0(HM;\underline{\mathbb{Z}}) = 0.$$

Thus, $\tau_k X = 0$, as was to be shown. □

For an element $x \in \pi_k^{C_2}\mathbb{S}$, we can define a (naive) C_2–equivariant E_∞–cone $C^{\mathbb{S}}(x)$ as in the nonequivariant situation in the preceding subsection. The arguments for the following two results are quite analogous to those of the preceding section, so we allow ourselves to be brief.

Lemma 2.19 *The map $C(\overline{\eta}) \to C^{\mathbb{S}}(\overline{\eta})$ is slice-2–connected.*

Proof By Lemma 2.18 it suffices to check that $C(\overline{\eta}) \otimes H\underline{\mathbb{Z}} \to C^{\mathbb{S}}(\overline{\eta}) \otimes H\underline{\mathbb{Z}}$ is slice-2–connected. Since $\underline{\pi}_\sigma H\underline{\mathbb{Z}} = 0$ and thus $\overline{\eta}$ becomes zero in $H\underline{\mathbb{Z}}$, this agrees with

$$\begin{aligned} H\underline{\mathbb{Z}} \oplus \Sigma^\rho H\underline{\mathbb{Z}} \to C^{H\underline{\mathbb{Z}}}(\Sigma H\underline{\mathbb{Z}}) &\simeq \mathbb{P}^{H\underline{\mathbb{Z}}}\Sigma^\rho H\underline{\mathbb{Z}} \\ &\simeq H\underline{\mathbb{Z}} \oplus \Sigma^\rho H\underline{\mathbb{Z}} \oplus (\Sigma^{2\rho} H\underline{\mathbb{Z}})_{hC_2} \oplus \cdots. \end{aligned}$$

Analogously to Lemma 2.4, the map is split injective and thus indeed slice-2–connected (even slice-3–connected). □

Together with Lemma 2.17 this implies that $\tau_{\leq 1}C^{\mathbb{S}}(\overline{\eta}) \simeq H\underline{\mathbb{Z}}$. To deduce the analogue of Proposition 2.9, we will need one more categorical result.

Lemma 2.20 *Let G be a finite group and Sp_G be the ∞–category of G–spectra. Denote by $\mathrm{Sp}_G^{\geq 0}$ the full subcategory of connective G–spectra and by $\mathrm{Sp}_G^{[0,k]}$ that of connective and slice-k–truncated G–spectra. Then the inclusion*

$$\mathrm{CAlg}(\mathrm{Sp}_G^{[0,k]}) \to \mathrm{CAlg}(\mathrm{Sp}_G)$$

admits for every $k \geq 0$ a left adjoint, which agrees on the level of underlying G–spectra with the slice truncation $\tau_{\leq k}$.

Proof Connective G–spectra form a presentable ∞–category with compact generators the $\Sigma^\infty G/H_+$. We obtain $\mathrm{Sp}_G^{[0,k]}$ by localizing $\mathrm{Sp}_G^{\geq 0}$ at the collection S of maps $C \to 0$ for C a slice cell of dimension greater than k. By [31, Proposition 5.5.4.15], $\mathrm{Sp}_G^{[0,k]}$ is presentable again.

If X is connective and $Y \geq k+1$ in the slice filtration, then by [16, Proposition 4.26] $X \otimes Y \geq k+1$. Thus, $\tau_{\leq k}$ is compatible with $\otimes$ in the following sense: if $X \to Y$ in $\mathrm{Sp}_G^{\geq 0}$ induces an equivalence $\tau_{\leq k}X \to \tau_{\leq k}Y$, then $\tau_{\leq k}(X \otimes Z) \to \tau_{\leq k}(Y \otimes Z)$ is an equivalence for every $Z \in \mathrm{Sp}_G^{\geq 0}$. By [32, Proposition 2.2.1.9], $\mathrm{Sp}_G^{[0,k]}$ inherits the structure of a symmetric monoidal ∞–category from $\mathrm{Sp}_G^{\geq 0}$ and $\tau_{\leq k}$ is strong symmetric monoidal, while the inclusion $\mathrm{Sp}_G^{[0,k]} \to \mathrm{Sp}_G^{\geq 0}$ is lax symmetric monoidal. The same proposition gives that the resulting maps

$$(\mathrm{Sp}_G^{[0,k]})^\otimes \to (\mathrm{Sp}_G^{\geq 0})^\otimes \quad \text{and} \quad (\mathrm{Sp}_G^{\geq 0})^\otimes \to (\mathrm{Sp}_G^{[0,k]})^\otimes$$

of ∞–operads are adjoint. Since commutative algebras in such an ∞–operad $\mathcal{C}^\otimes$ are defined as sections of $\mathcal{C}^\otimes \to \mathrm{NFin}_*$ as maps of operads, we see that the resulting maps between $\mathrm{CAlg}(\mathrm{Sp}_G^{[0,k]})$ and $\mathrm{CAlg}(\mathrm{Sp}_G^{\geq 0})$ are indeed adjoint. Here, we use the characterization of an adjunction given by [42], namely the existence of a unit and counit, satisfying the triangle identities up to homotopy. □

Proposition 2.21 *Let R be a connective E_∞–ring C_2–spectrum with $\underline{\pi}_0^{C_2} = \mathbb{Z}_S$ being a localization of $\mathbb{Z}$ and $\overline{\eta} = 0 \in \pi_\sigma^{C_2} R$. Then there is an E_∞–ring C_2–spectrum R' with an E_∞–map $R' \to R$ inducing an equivalence on slices in degree 0 and degrees at least 2 and such that $\tau_1 R' = 0$. Moreover, for every other $R'' \to R$ with these properties, there is an equivalence $R'' \to R'$ of E_∞–ring C_2–spectra over R.*

Proof Since $\overline{\eta}$ is zero in R, we obtain a map $C^{\mathbb{S}}(\overline{\eta})_S \to R \to \tau_{\leq 1} R$ of E_∞–ring C_2–spectra, which factors over $H\underline{\mathbb{Z}}_S = \tau_{\leq 1} C^{\mathbb{S}}(\overline{\eta})_S$. Define now R' via the homotopy pullback square

$$\begin{tikzcd} R' \arrow[r] \arrow[d] & H\underline{\mathbb{Z}}_S \arrow[d] \\ R \arrow[r] & \tau_{\le 1} R \end{tikzcd}$$

The proof of uniqueness is analogous to Proposition 2.9. □

To formulate the consequences for $\mathrm{tmf}(\Gamma)$, we want to recall from [18] that a C_2–spectrum E is strongly even if its odd slices vanish and its even slices are of the form $S^{k\rho} \otimes H\underline{A}$ or, equivalently, if $\underline{\pi}_{k\rho} E$ is constant and $\underline{\pi}_{k\rho-1} E = 0$.

Theorem 2.22 *For every set of primes S and every congruence subgroup $\Gamma_1(n) \subset \Gamma \subset \Gamma_0(n)$ that is tame with respect to $\mathbb{Z}_S$, we can define a strongly even connective E_∞–ring C_2–spectrum $\mathrm{tmf}(\Gamma)_S$ with an E_∞–ring C_2–map $\mathrm{tmf}(\Gamma)_S \to \mathrm{Tmf}(\Gamma)_S$ that identifies the underlying homotopy groups of the source with $M(\Gamma; \mathbb{Z}_S)$.*

Proof Equip $\mathrm{Tmf}(\Gamma) \simeq \mathrm{Tmf}(\Gamma)^{E(C_2)_+}$ with the cofree structure of a C_2–spectrum. We claim that

(1) $\overline{\eta} \in \pi_\sigma^{C_2} \tau_{\ge 0}\mathrm{Tmf}(\Gamma)$ is zero,

(2) the only odd slice of $\tau_{\ge 0}\mathrm{Tmf}(\Gamma)$ is τ_1, and

(3) the even slices of $\mathrm{Tmf}(\Gamma)$ are of the form $S^{k\rho} \otimes H\underline{A}$.

Given these claims, applying Proposition 2.21 to $R = \tau_{\ge 0}\mathrm{Tmf}(\Gamma)$ yields the C_2–spectrum $R' = \mathrm{tmf}(\Gamma)$ with the required properties: the first claim implies that we can apply Proposition 2.21, while the other two ensure that $\mathrm{tmf}(\Gamma)$ is strongly even.

For proving the claims, we will distinguish the (overlapping) cases that $\frac{1}{2} \in \mathbb{Z}_S$ and that Γ is tame for $\mathbb{Z}_{(2)}$.

For the first claim, note that $\pi_\sigma^{C_2} \tau_{\ge 0}\mathrm{Tmf}(\Gamma) \cong \pi_\sigma^{C_2}\mathrm{Tmf}(\Gamma)$ (eg since S^σ is a slice cell). The restriction map $\pi_\sigma^{C_2}\mathrm{Tmf}(\Gamma) \to \pi_1\mathrm{Tmf}(\Gamma)$ is an injection: if $\frac{1}{2} \in \mathbb{Z}_S$, this follows from the homotopy fixed points spectral sequence; else, use the line after (6.15) in [37]. Since $\overline{\eta}$ restricts to $\eta \in \pi_1\mathrm{Tmf}(\Gamma)$, Lemma 2.10 implies thus the vanishing of $\overline{\eta}$.

If Γ is tame for $\mathbb{Z}_{(2)}$, Theorem 6.16 of [37] yields the last two claims. If $\frac{1}{2} \in S$, we obtain $\pi_{k\rho-1}^{C_2}\mathrm{Tmf}(\Gamma) = 0$ by the homotopy fixed point spectral sequence since

$\pi_{2k-1}\mathrm{Tmf}(\Gamma) = 0$ for $k > 1$ by [37, Section 2.1]. This yields the second claim by [18, Proposition 2.9]. For the third claim it is enough to show that $\underline{\pi}_{k\rho}\mathrm{Tmf}(\Gamma)$ are constant Mackey functors; see [18, Proposition 2.13]. This follows again from the homotopy fixed point spectral sequence and the fact that the C_2–action on $\pi_{k\rho}\mathrm{Tmf}(\Gamma)$ is trivial: indeed, C_2 acts by $(-1)^k$ on $\pi_{2k}\mathrm{Tmf}(\Gamma)$ (see Remark 2.15) and the presence of $k\sigma$ twists the action by the same sign. $\square$

Remark 2.23 The case that $\Gamma = \Gamma_0(n)$ is not excluded in the previous theorem, but one easily checks that $\Gamma_0(n)$ can only be tame if $\frac{1}{2} \in \mathbb{Z}_S$. In this case, we obtain simply the cofree C_2–spectrum of $\mathrm{tmf}_0(n)_S$ with the trivial action.

Construction 2.24 Given $\Gamma' \subset \Gamma \subset \Gamma_0(n)$ with Γ' tame with respect to $\mathbb{Z}_S$ and $\Gamma/\Gamma' \cong C_2$, we can extend our previous definition by defining $\mathrm{tmf}(\Gamma)_S$ as $\mathrm{tmf}(\Gamma')_S^{C_2}$ (so for example $\mathrm{tmf}_0(3) = \mathrm{tmf}_1(3)^{C_2}$ as in [18]). If Γ itself is already tame, then $\frac{1}{2} \in \mathbb{Z}_S$. One then easily computes (eg with the slice spectral sequence) that $\pi_*\mathrm{tmf}(\Gamma')_S^{C_2} \cong \pi_*\mathrm{tmf}(\Gamma)_S$ and one can use the uniqueness part of Theorem 2.12 to identify our new definition with the previous one.

Remark 2.25 In the setting of Construction 2.24, the map $\mathrm{tmf}(\Gamma) \to \tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ is an isomorphism in π_* for $* \geq 2$ even if Γ is not tame. Indeed, the cofiber of $\mathrm{tmf}(\Gamma') \to \tau_{\geq 0}\mathrm{Tmf}(\Gamma')$ is the target's first slice and thus by [37, Theorem 6.16] equivalent to $\Sigma^\sigma HM$, where M is the constant Mackey functor on $H^1(\overline{\mathcal{M}}(\Gamma')_S; \omega) \cong \pi_1\mathrm{Tmf}(\Gamma')$. We directly observe that the nonequivariant homotopy groups of $\Sigma^\sigma HM$ vanish in degrees at least 2. Moreover the cofiber sequence $(C_2)_+ \to S^0 \to S^\sigma$ induces a long exact sequence

$$\pi_k^e HM \to \pi_k^{C_2} HM \to \pi_{k-\sigma}^{C_2} HM \to \pi_{k-1}^e HM \xrightarrow{\mathrm{tr}} \pi_{k-1}^{C_2} HM,$$

which implies that $\pi_k^{C_2}\Sigma^\sigma HM = \pi_{k-\sigma}^{C_2} HM = 0$ for $k \geq 2$ and actually also for $k = 1$ if tr is injective, ie if $\pi_1\mathrm{Tmf}(\Gamma')$ has no 2–torsion. Thus, $\mathrm{tmf}(\Gamma) \to \tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ is indeed an isomorphism in π_* for $* \geq 2$, and even for $* = 1$ if $\pi_1\mathrm{Tmf}(\Gamma')$ has no 2–torsion.

3 Realization of Hirzebruch's level-*n* genus

In the previous section we defined ring spectra $\mathrm{tmf}_1(n) = \mathrm{tmf}(\Gamma_1(n))$. The spectra $\mathrm{tmf}_1(n)$ are even for $n \geq 2$ and thus complex orientable. We want to show that there is a complex orientation for $\mathrm{tmf}_1(n)$ such that the corresponding map

$$\mathrm{MU}_{2*} \to \mathrm{tmf}_1(n)_{2*} \cong M(\Gamma_1(n); \mathbb{Z}[1/n])$$

agrees with the level-n genus introduced by Hirzebruch [19] and Witten [48] and studied for instance in [27; 11; 15; 47]. We recall its definition below. For this purpose it will be convenient to use algebrogeometric language, for which we recall first the following set of definitions.

Definition 3.1 A *formal group* over a base scheme S is a Zariski sheaf $F\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Ab}$ that Zariski locally on an affine open $U = \operatorname{Spec} R \subset S$ is isomorphic to $\operatorname{Spf} R[\![t]\!]$. The R–modules $R[\![t]\!]$ glue to the *structure sheaf* $\mathcal{O}_F$ on S and the R–modules $(R[\![t]\!]/t)\cdot dt$ glue to the line bundle $\omega_{F/S}$.[4] An *invariant differential* of a formal group F is a trivialization of $\omega_{F/S}$. A *coordinate* is a section s of $\mathcal{O}_F$ that is of the form $a_0 t + a_1 t^2 + \cdots$ with $a_0 \in R^\times$ for every local trivialization $F|_{\operatorname{Spec} R} \cong \operatorname{Spf} R[\![x]\!]$.

Remark 3.2 There are different ways to state the definition of a formal group, for example as an abelian group object in one-dimensional formal Lie varieties; see [12, Definitions 1.29 and 2.2]. To compare them, note that our formal groups are automatically fpqc sheaves since $\operatorname{Spf} R[\![t]\!]$ is an fpqc sheaf. On the other hand, a trivialization of the sheaf of differentials of a one-dimensional formal Lie variety over $\operatorname{Spec} R$ determines an equivalence to $\operatorname{Spf} R[\![t]\!]$, and such trivializations exist Zariski locally.

We note that the differential ds of a coordinate s of a formal group F is an invariant differential of F, sending $a_0 t + a_1 t^2 + \cdots$ to $a_0\, dt$ locally. If $S = \operatorname{Spec} R$, a coordinate of F is equivalent datum to an isomorphism $F \cong \operatorname{Spf} R[\![s]\!]$.

Recall that given an arbitrary even ring spectrum E, a complex orientation is an element in $\widetilde{E}^2(\mathbb{CP}^\infty)$ restricting to $1 \in \widetilde{E}^2(\mathbb{CP}^1)$ after a homeomorphism $\mathbb{CP}^1 \cong S^2$ is chosen. The formal spectrum $\operatorname{Spf} E^{2*}(\mathbb{CP}^\infty)$ is a formal group over $\operatorname{Spec} E^{2*}(\mathrm{pt})$ and the line bundle ω corresponds to $\widetilde{E}^*(\mathbb{CP}^1)$; it thus comes with a canonical invariant differential corresponding to $1 \in \widetilde{E}^2(\mathbb{CP}^1)$. A complex orientation is thus a coordinate of $\operatorname{Spf} E^{2*}(\mathbb{CP}^\infty)$ in degree $* = 1$ whose differential is the canonical invariant differential.

[4] If $p\colon C \to S$ is a (generalized) elliptic curve and F is the formal completion of $\mathcal{E}$, this agrees with $\omega_{C/S} = p_* \Omega^1_{C/S}$.

We want to apply this to $E = \mathrm{tmf}_1(n)$ for $n \geq 2$. Essentially by construction, the maps

$$\pi_{2*}\mathrm{tmf}_1(n) \to \pi_{2*}\mathrm{Tmf}_1(n) \to H^0(\overline{\mathcal{M}}_1(n); \omega^{\otimes *}_{\bar{\mathcal{C}}/\overline{\mathcal{M}}_1(n)})$$

are isomorphisms, where $\bar{\mathcal{C}}$ is the universal generalized elliptic curve over $\overline{\mathcal{M}}_1(n)$. For convenience, let $\overline{\mathcal{M}}_1^1(n)$ be the relative spectrum

$$\underline{\mathrm{Spec}}_{\overline{\mathcal{M}}_1(n)}\Big(\bigoplus \omega^{\otimes *}_{\bar{\mathcal{C}}/\overline{\mathcal{M}}_1(n)}\Big),$$

which is the total space of the $\mathbb{G}_m$–torsor associated with $\omega_{\bar{\mathcal{C}}/\overline{\mathcal{M}}_1(n)}$, ie classifies generalized elliptic curves with a point of exact order n and an invariant differential. The resulting morphism

$$\overline{\mathcal{M}}_1^1(n) \to \mathrm{Spec}\, H^0(\overline{\mathcal{M}}_1^1; \mathcal{O}_{\overline{\mathcal{M}}_1^1(n)}) \cong \mathrm{Spec}\, H^0(\overline{\mathcal{M}}_1; \omega^{\otimes *}_{\bar{\mathcal{C}}/\overline{\mathcal{M}}_1(n)}) \cong \mathrm{Spec}\, \pi_{2*}\mathrm{tmf}_1(n)$$

is an open immersion, whose image is covered by the nonvanishing loci of c_4 and Δ; see [38, Proposition 3.5]. We denote by $\mathcal{C}$ the pullback of $\bar{\mathcal{C}}$ to $\overline{\mathcal{M}}_1^1(n)$. Since

$$\mathrm{tmf}_1(n)[c_4]^{-1} \simeq \mathrm{Tmf}_1(n)[c_4^{-1}] \quad \text{and} \quad \mathrm{tmf}_1(n)[\Delta^{-1}] \simeq \mathrm{Tmf}_1(n)[\Delta^{-1}]$$

are elliptic cohomology theories, their formal groups are identified with the restrictions of $\widehat{\mathcal{C}}$ to the nonvanishing loci of c_4 and Δ, respectively, and as a result $\widehat{\mathcal{C}}$ becomes identified with the restriction of $\mathrm{Spf}\,\mathrm{tmf}_1(n)^{2*}(\mathbb{CP}^\infty)$ to $\overline{\mathcal{M}}_1^1(n)$. As $\overline{\mathcal{M}}_1^1(n) \subset \mathrm{Spec}\,\pi_{2*}\mathrm{tmf}_1(n)$ induces an isomorphism on global sections of the structure sheaf, coordinates on $\mathrm{Spf}\,\mathrm{tmf}_1(n)^{2*}(\mathbb{CP}^\infty)$ are in bijection with those on $\widehat{\mathcal{C}}$ and one checks that the canonical invariant differential on the former corresponds to the canonical invariant differential on the latter. Summarizing we obtain:

Lemma 3.3 *Complex orientations* $\mathrm{MU} \to \mathrm{tmf}_1(n)$ *are in bijection with coordinates of* $\widehat{\mathcal{C}}$, *which are homogeneous of degree one and have the canonical invariant differential as differential.*

The Hirzebruch genus relies on a specific such coordinate, which we will construct momentarily. Basically we will follow [20, Chapter 7], but present a more algebro-geometric approach and give an independent treatment. The key point is the existence of a certain meromorphic function on a cover of a given generalized elliptic curve. To the purpose of constructing this function, recall that every section P into the smooth part of a generalized elliptic curve $C \to S$ is an effective Cartier divisor [26, Lemma 1.2.2], ie the kernel $\mathcal{O}_C(-(P))$ of $\mathcal{O}_C \to P_*\mathcal{O}_S$ is a line bundle. Given any linear combination of sections P_i, we denote by $\mathcal{O}_C\big(\sum_i n_i(P_i)\big)$ the corresponding tensor product of line bundles.

Lemma 3.4 *Let $n \geq 2$ and S be a $\mathbb{Z}[1/n]$–scheme. Furthermore let C/S be a generalized elliptic curve with zero-section $e\colon S \to C$ and a chosen point $P\colon S \to C$ of exact order n in the smooth locus.*

(a) *The pullback of $e^*\mathcal{O}_C((P)-(e))$ to S is canonically isomorphic to $\omega_{C/S} = e^*\Omega^1_{C/S}$.*

(b) *Let λ be an invariant differential on C. Then there exists a unique meromorphic function h on C with an n–fold zero at e and an n–fold pole at P as the only pole whose restriction along e coincides with λ^n under the identification of the previous part.*

(c) *There exists a degree-n étale cover $q\colon C' \to C$ by a generalized elliptic curve and a meromorphic function f on C' with $f^n = q^*h$.*

Proof (a) Note that $\mathcal{O}_C(-(e))$ is the ideal sheaf associated to the closed immersion e and the pullback $e^*\mathcal{O}_C((P)-(e))$ coincides with $\mathcal{O}_C(-(e))/\mathcal{O}_C(-(e))^2$ viewed as an $\mathcal{O}_S$–module. Indeed, we can cover S by opens of the form $U \cap S$, where $U \cong \operatorname{Spec} R$ is an affine open in C not intersecting the image of P. The section e corresponds to an element $s \in R$ and $U \cap S \cong \operatorname{Spec} S/s$. Then $e^*\mathcal{O}_C((P)-(e))(U \cap S)$ is the S/s–module $sS \otimes_S S/s$, which is canonically isomorphic to the S/s–module sS/s^2S. For example, by [14, Proposition II.8.12], we obtain a canonical surjective map

$$\mathcal{O}_C(-(e))/\mathcal{O}_C(-(e))^2 \to e^*\Omega^1_{C/S} = \omega_{C/S}$$

between line bundles, which is hence an isomorphism.

(b) Consider the line bundle $\mathcal{O}_C(n(P)-n(e))$. Note that $n \cdot P - n \cdot e = e$ as points on C. By [26, Theorem 2.1.2] in the case that C is an elliptic curve, and by [6, Proposition II.2.7] for generalized elliptic curves, we deduce that $\mathcal{O}_C(n(P)-n(e))$ is the pullback of a line bundle $\mathcal{L}$ on S. By part (a), $\mathcal{L} = e^*p^*\mathcal{L} = \omega^{\otimes n}_{C/S}$. By [6, Proposition II.1.6], we see that the canonical map

$$\omega^{\otimes n}_{C/S} \to p_*p^*\omega^{\otimes n}_{C/S} \cong p_*\mathcal{O}_C(n(P)-n(e))$$

is an isomorphism. Thus

$$\Gamma(\mathcal{O}_C(n(P)-n(e))) \cong \Gamma(\omega^{\otimes n}_{C/S}),$$

where the isomorphism can be identified with the pullback along e. Thus, there is a unique section h of $\mathcal{O}_C(n(P)-n(e))$ whose image is λ^n.

(c) Consider the μ_n–torsor $q\colon C' \to C$ associated with the problem of extracting an n^{th} root out of q^*h as a section of $q^*\mathcal{O}_C((P)-(e))$, in other words the μ_n–torsor associated with the pair $(h, \mathcal{O}_C((P)-(e)))$ in the sense of [39, page 125]. By construction, the required root f exists on C'. By [6, Proposition II.1.17], C' has the structure of a generalized elliptic curve provided that we can lift e to C' and $C' \to S$ has geometrically connected fibers. For the first point, it suffices to provide a section of $C' \times_C S \to S$, ie to provide an n^{th} root of e^*h. Under the identification of part (a), this is provided by λ. For the second point, we assume that $S = \operatorname{Spec} K$ with K algebraically closed of characteristic not dividing n and that C' is not connected. The stabilizer of a component C'_0 must be of the form μ_m with $m < n$, and thus $C' \cong C'_0 \times_{\mu_m} \mu_n$. The μ_m–torsor C'_0 is hence associated with a pair $(g, \mathcal{O}_C((P)-(e)))$ such that $g^{n/m} = h$. The section g provides a trivialization of $\mathcal{O}_C(m(P) - m(e))$. This implies $m \cdot P = e$ on C' [6, Corollaire II.2.4], in contradiction with P being of exact order n. $\square$

Construction 3.5 Let $\mathcal{C}$ be the universal generalized elliptic curve with a point of exact order n over $\overline{\mathcal{M}}_1^1(n)$. It comes, by definition, with a canonical invariant differential λ. From the preceding lemma, we obtain an n–fold étale cover $q\colon \mathcal{C}' \to \mathcal{C}$ together with a meromorphic function f on $\mathcal{C}'$ whose pullback along a lift of e agrees with λ. This function f provides a coordinate for $\widehat{\mathcal{C}}' \cong \widehat{\mathcal{C}}$. Moreover, note that f is uniquely determined by the requirements in the lemma because $\mathcal{C}'$ is irreducible (since $\overline{\mathcal{M}}_1^1(n)$ is irreducible and the locus of smoothness of $\mathcal{C}'$ in it is dense) and thus every other n^{th} root of h would have to differ by a root of unity, resulting in a different pullback to $\overline{\mathcal{M}}_1^1(n)$.

Pulling the orientation induced from f back along a map $\operatorname{Spec} \mathbb{C} \to \overline{\mathcal{M}}_1(n)$ classifying $(\mathbb{C}/\Lambda, 1/n, dz)$ results exactly in the coordinate and orientation chosen in [20].

Theorem 3.6 *For every $n \geq 2$, there is a unique complex orientation of* $\mathrm{MU} \to \mathrm{tmf}_1(n)$ *realizing on homotopy groups the Hirzebruch genus. Moreover, this can be uniquely refined to a morphism* $\mathrm{MU}_{\mathbb{R}} \to \mathrm{tmf}_1(n)$ *of C_2–ring spectra.*

Proof The first part follows from Lemma 3.3 as the Hirzebruch genus is given by a coordinate on the formal group associated with the universal generalized elliptic curve on $\overline{\mathcal{M}}_1^1(n)$. For the second point, we recall from [24, Theorem 2.25] that C_2–ring morphisms $\mathrm{MU}_{\mathbb{R}} \to \mathrm{tmf}_1(n)$ are in bijection with Real orientations of $\mathrm{tmf}_1(n)$, ie a lift of a complex orientation to a class $\mathrm{tmf}_1(n)^{\rho}_{C_2}(\mathbb{CP}^\infty)$. As $\mathbb{CP}^\infty$ can be built by cells in dimensions $k\rho$, the strong-evenness of $\mathrm{tmf}_1(n)$ from Theorem 2.22 implies that the forgetful map

$$\mathrm{tmf}_1(n)^{\rho}_{C_2}(\mathbb{CP}^\infty) \to \mathrm{tmf}_1(n)^2(\mathbb{CP}^\infty)$$

is an isomorphism; thus every complex orientation of $\mathrm{tmf}_1(n)$ refines to a unique Real orientation. □

Remark 3.7 In [11], Franke already gave a related but different algebrogeometric treatment of the Hirzebruch genus.

Remark 3.8 After the first version of this article became available, Senger has shown in [43] that the map $\mathrm{MU} \to \mathrm{tmf}_1(n)$ actually refines to one of E_∞–ring spectra. He also gives a reformulation of our treatment above in terms of Θ^1–structures.

4 Compactness, formality and faithfulness of tmf(Γ)

Given a (tame) congruence subgroup of level n, we will show that $\mathrm{tmf}(\Gamma)$ is a faithful and perfect $\mathrm{tmf}[1/n]$–module. In contrast, for example, $\mathrm{tmf}_1(3)$ will not be a perfect $\mathrm{tmf}_0(3)$–module, not even rationally. The latter result relies on $\mathrm{tmf}_0(3)_{\mathbb{Q}}$ being formal (ie multiplicatively a graded Eilenberg–Mac Lane spectrum), a result we prove in greater generality in a subsection on its own.

4.1 All tmf(Γ) are perfect

Recall that for an A_∞–ring spectrum R, a perfect R–module is a compact object in the ∞–category of left R–modules. Equivalently, the ∞–category of perfect R–modules is the smallest stable ∞–subcategory of all left R–modules that contains R and is closed under retracts. The goal of this section is to show that the spectra $\mathrm{tmf}(\Gamma)$, in the cases we defined them, are perfect $\mathrm{tmf}[1/n]$–modules. The key technical tool is the following proposition.

Proposition 4.1 *Let R be an A_∞–ring spectrum such that*

(1) *$\pi_0 R$ is regular noetherian*,

(2) *all $\pi_n R$ are finitely generated $\pi_0 R$–modules*, *and*

(3) *$H\pi_0 R$ is perfect as a $\tau_{\geq 0} R$–module.*

Let furthermore M be a perfect R–module. Then $\tau_{\geq k} M$ is a perfect $\tau_{\geq 0} R$–module for every $k \in \mathbb{Z}$.

Lemma 4.2 *With notation as in the statement of the proposition*, *let X be a $\tau_{\geq 0} R$–module with only finitely many nontrivial homotopy groups*, *all finitely generated over $\pi_0 R$. Then X is a perfect $\tau_{\geq 0} R$–module.*

Proof By induction, we can reduce to the case that $\pi_* X$ is concentrated in a single degree n. Then $X = H\pi_n X$ acquires the structure of a $H\pi_0 R$–module and it is perfect as such because $\pi_0 R$ is regular noetherian and $\pi_n X$ is finitely generated. As $H\pi_0 R$ is perfect over $\tau_{\geq 0} R$, the same is thus true for X. $\square$

Proof of Proposition 4.1 Let M be a perfect R–module. As the truth of the conclusion of the proposition is clearly preserved under retracts in M and also clear for $M = 0$, we can assume by induction that we have a cofiber sequence

$$\Sigma^l R \to N \to M \to \Sigma^{l+1} R$$

where $\tau_{\geq k} N$ is a perfect $\tau_{\geq 0} R$–module for all $k \in \mathbb{Z}$. Taking $\tau_{\geq l}$ on the first two objects gives a diagram

$$\begin{tikzcd}
\Sigma^l \tau_{\geq 0} R \arrow[r] \arrow[d] & \tau_{\geq l} N \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & \Sigma^{l+1} \tau_{\geq 0} R \arrow[d] \\
\Sigma^l R \arrow[r] & N \arrow[r] & M \arrow[r] & \Sigma^{l+1} R
\end{tikzcd}$$

of cofiber sequences. As $\tau_{\geq l} N$ is a perfect $\tau_{\geq 0} R$–module, so is M'. Clearly, we have $\tau_{\geq l+1} M' \simeq \tau_{\geq l+1} M$. As the fiber of $\tau_{\geq l+1} M' \to M'$ fulfills the conditions of the previous lemma, $\tau_{\geq l+1} M$ is perfect as a $\tau_{\geq 0} R$–module.

For a general $k \in \mathbb{Z}$, we make a case distinction: assume first that $k \geq l + 1$. Then the fiber of $\tau_{\geq k} M \to \tau_{\geq l+1} M$ is perfect by the previous lemma; hence $\tau_{\geq k} M$ is perfect as well. If $k \leq l + 1$, consider the fiber of $\tau_{\geq l+1} M \to \tau_{\geq k} M$ instead. $\square$

To apply Proposition 4.1 to topological modular forms, we need the following lemma.

Lemma 4.3 *For every $n \geq 1$, the* $\mathrm{tmf}[1/n]$*–module* $H\pi_0 \mathrm{tmf}[1/n] = H\mathbb{Z}[1/n]$ *is perfect.*

Proof If $2 | n$, there is a 3–cell complex X such that $\mathrm{tmf}[1/n] \otimes X \simeq \mathrm{tmf}_1(2)[1/n]$; see [34, Theorem 4.13]. We have $\pi_* \mathrm{tmf}_1(2)[1/n] = \mathbb{Z}[1/n][b_2, b_4]$. Killing b_2 and b_4 gives $H\mathbb{Z}[1/n]$. Thus, $H\mathbb{Z}[1/n]$ is a perfect $\mathrm{tmf}_1(2)[1/n]$–module and hence also a perfect $\mathrm{tmf}[1/n]$–module.

If $3 | n$, there is an 8–cell complex X such that $\mathrm{tmf}[1/n] \otimes X \simeq \mathrm{tmf}_1(3)[1/n]$; see [34, Theorem 4.10]. We have $\pi_* \mathrm{tmf}_1(3)[1/n] = \mathbb{Z}[1/n][a_1, a_3]$. Killing a_1 and a_3 gives $H\mathbb{Z}[1/n]$ and thus $H\mathbb{Z}[1/n]$ is also a perfect $\mathrm{tmf}[1/n]$–module in this case.

For the general case, let X_i be a collection of $\mathrm{tmf}[1/n]$–modules. Consider

$$\Phi_k\colon \bigoplus_i \mathrm{Hom}_{\mathrm{tmf}[1/n]}\Bigl(H\mathbb{Z}\Bigl[\frac{1}{n}\Bigr], X_i\Bigl[\frac{1}{k}\Bigr]\Bigr) \to \mathrm{Hom}_{\mathrm{tmf}[1/n]}\Bigl(H\mathbb{Z}\Bigl[\frac{1}{n}\Bigr], \bigoplus_i X_i\Bigl[\frac{1}{k}\Bigr]\Bigr).$$

If $k = 2, 3$ or 6, then Φ_k is an equivalence by the previous results. As for every spectrum X, there is a cofiber sequence

$$\Sigma^{-1} X\bigl[\tfrac{1}{6}\bigr] \to X \to X\bigl[\tfrac{1}{2}\bigr] \oplus X\bigl[\tfrac{1}{3}\bigr] \to X\bigl[\tfrac{1}{6}\bigr]$$

and there is a cofiber sequence of maps between mapping spectra

$$\Sigma^{-1}\,\mathrm{fib}(\Phi_6) \to \mathrm{fib}(\Phi_1) \to \mathrm{fib}(\Phi_2 \oplus \Phi_3) \to \mathrm{fib}(\Phi_6).$$

It follows that Φ_1 is an equivalence as well and that $H\mathbb{Z}[1/n]$ is a perfect $\mathrm{tmf}[1/n]$–module. □

Theorem 4.4 *Let Γ be a congruence subgroup of level n, which is tame or has a subgroup $\Gamma' \subset \Gamma$ of index 2 with Γ' tame. Then $\mathrm{tmf}(\Gamma)$ is a perfect $\mathrm{tmf}[1/n]$–module.*

The same conclusion holds without the tameness hypothesis for any $\mathrm{tmf}[1/n]$–module R with a map $R \to \tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ whose fiber has finitely generated homotopy groups over $\mathbb{Z}[1/n]$, concentrated in finitely many degrees.

Proof According to [37, Proposition 2.12] the $\mathrm{Tmf}[1/n]$–module $\mathrm{Tmf}(\Gamma)$ is perfect. All $\pi_k\mathrm{Tmf}[1/n]$ are finitely generated $\mathbb{Z}[1/n]$–modules. Furthermore, $H\pi_0\mathrm{tmf}[1/n] = H\mathbb{Z}[1/n]$ is a perfect $\mathrm{tmf}[1/n]$–module by the previous lemma. This implies that $\tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ is a perfect $\mathrm{tmf}[1/n]$–module by Proposition 4.1.

For any R as in the statement of the theorem, R is thus perfect as well, by Lemma 4.2. To see that $\mathrm{tmf}(\Gamma)$ satisfies the hypotheses on R, note first that every $H^s(\overline{\mathcal{M}}(\Gamma); \omega^{\otimes t})$ is a finitely generated $\mathbb{Z}[1/n]$–module for every s and t since $\overline{\mathcal{M}}(\Gamma)$ is proper over $\mathbb{Z}[1/n]$. If Γ is tame, the cofiber of $\mathrm{tmf}(\Gamma) \to \tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ is by construction $H\pi_1\mathrm{Tmf}(\Gamma)$ and $\pi_1\mathrm{Tmf}(\Gamma) \cong H^1(\overline{\mathcal{M}}(\Gamma); \omega)$. If there is a tame subgroup $\Gamma' \subset \Gamma$ of index 2, the cofiber $\mathrm{tmf}(\Gamma) \to \tau_{\geq 0}\mathrm{Tmf}(\Gamma)$ agrees with $\Sigma^\sigma HM$ for M the constant Mackey functor on $H^1(\overline{\mathcal{M}}(\Gamma'); \omega)$ by Remark 2.25. The exact sequence given in the same remark implies that the homotopy groups of $\Sigma^\sigma HM$ are concentrated in degrees 0 and 1 and are finitely generated $\mathbb{Z}[1/n]$–modules. □

We recall from [33] that a connective p–complete spectrum X is called an *fp-spectrum* if $H_*(X; \mathbb{F}_p)$ is finitely presented as a comodule over the dual Steenrod algebra. They show in [33, Proposition 3.2] that, equivalently, there is a finite spectrum F with nontrivial $\mathbb{F}_p$–homology such that the total group $\pi_*(X \otimes F)$ is finite. The following proposition can be deduced from the known $\mathbb{F}_p$–(co)homology of tmf (see for example [41, Section 21]) and was already noted in [33] for $p = 2$. We prefer to give a less computational proof though.

Proposition 4.5 *The p–completion of* tmf *is an fp-spectrum for all primes p.*

Proof We implicitly p–localize. For $p \neq 3$, [34, Theorem 4.10] implies the existence of a finite spectrum W with nontrivial $\mathbb{F}_p$–homology such that $\mathrm{tmf} \otimes W \simeq \mathrm{tmf}_1(3)$. Choose a complex V such that $BP_*V \cong BP_*/(p^{k_0}, v_1^{k_1}, v_2^{k_2})$ with k_0, k_1 and k_2 positive integers. As $\mathrm{TMF}_1(3)$ is Landweber exact, the sequence p, v_1, v_2 and hence the sequence $p^{k_0}, v_1^{k_1}, v_2^{k_2}$ is regular on $\pi_*\mathrm{TMF}_1(3)$. Since $\pi_*\mathrm{tmf}_1(3) = \mathbb{Z}_{(p)}[a_1, a_3]$ is an integral domain, the sequence is also regular on $\pi_*\mathrm{tmf}_1(3)$. Thus,

$$\pi_*\mathrm{tmf} \otimes W \otimes V \cong \pi_*\mathrm{tmf}_1(3) \otimes V \cong \pi_*\mathrm{tmf}_1(3)/(p^{k_0}, a_1^{k_1}, a_3^{k_2})$$

is a finitely generated $\mathbb{Z}/p^{k_0}$–algebra and of Krull dimension 0. Hence it is of finite length as a $\mathbb{Z}/p^{k_0}$–module, and thus finite.

Essentially the same argument works for $p = 3$ if we choose instead a complex W' with $\mathrm{tmf} \otimes W' \simeq \mathrm{tmf}_1(2)$ as in [34, Theorem 4.13]. □

Corollary 4.6 *The p–completion of* $\mathrm{tmf}(\Gamma)$ *for a congruence subgroup Γ of level n and p not dividing n is an fp-spectrum.*

For implications involving duality we refer to [33] and for an implication for the Hurewicz image in $H_*(\Omega^\infty \mathrm{tmf}(\Gamma); \mathbb{F}_p)$ to [28, Theorem 1.7].

4.2 All $\mathrm{tmf}(\Gamma)_{\mathbb{Q}}$ are formal

The goal of this section is to show that the E_∞–rings $\mathrm{tmf}(\Gamma)_{\mathbb{Q}}$ are formal. While this statement is interesting in its own right, we also need it for further pursuing compactness questions in the following subsection. We begin with the following consequence of Goerss–Hopkins obstruction theory.

Proposition 4.7 *Let A and B be E_∞–$H\mathbb{Q}$–algebras such that π_*A is smooth as a $\mathbb{Q}$–algebra. Then*

$$\pi_i \operatorname{Map}_{\mathrm{CAlg}}(A, B) \cong \begin{cases} \operatorname{Hom}_{\mathrm{grCRings}}(\pi_*A, \pi_*B) & \text{if } i = 0, \\ \operatorname{Hom}_{\pi_*A}(\Omega^1_{\pi_*A/\mathbb{Q}}, \pi_{*+i}B) & \text{if } i > 0, \end{cases}$$

where for π_i with $i > 0$ a basepoint is chosen if a map $A \to B$ exists.

Proof According to [13, Section 4] or [40, Section 6] with $E = H\mathbb{Q}$, there is an obstruction theory for lifting a morphism $\pi_* A \to \pi_* B$ to a morphism $A \to B$, where the obstructions lie in $\mathrm{Ext}^{n+1,n}_{\pi_* A}(\mathbb{L}^{E_\infty}_{\pi_* A/\mathbb{Q}}, \pi_* B)$, where $\mathbb{L}^{E_\infty}$ denotes the E_∞–cotangent complex. As we are working rationally, this coincides with other forms of the cotangent complexes. In particular, we obtain from the smoothness of $\pi_* A$ that $\mathbb{L}^{E_\infty}_{\pi_* A/\mathbb{Q}}$ is isomorphic to $\Omega^1_{\pi_* A/\mathbb{Q}}$ concentrated in degree 0, which again by smoothness is a projective $\pi_* A$–module. Thus the Ext-groups vanish and there is no obstruction to lifting a morphism $\pi_* A \to \pi_* B$ to a morphism $A \to B$. The same sources provide a spectral sequence computing $\pi_* \operatorname{Map}_{\mathrm{CAlg}}(A, B)$, which collapses by a similar Ext-calculation and gives the result. □

Proposition 4.8 *Let $\mathcal{X}$ be a smooth Deligne–Mumford stacks over $\mathbb{Q}$ and $\mathcal{O}$ an even-periodic sheaf of E_∞–ring spectra on $\mathcal{X}$ such that $\pi_0 \mathcal{O} \cong \mathcal{O}_{\mathcal{X}}$ and the $\pi_i \mathcal{O}_{\mathcal{X}}$ are quasicoherent. Assume further that $H^{i+1}(\mathcal{X}; \pi_i \mathcal{O}) = 0$ for all even $i \geq 1$. Then $\mathcal{O}$ is formal, ie equivalent to the (sheafification of the pre)sheaf $H\pi_* \mathcal{O}$ of graded Eilenberg–Mac Lane spectra.*

Proof Note first that $(\mathcal{X}, \mathcal{O})$ actually defines a nonconnective spectral Deligne–Mumford stack and in particular $\mathcal{O}$ is hypercomplete; see eg [37, Lemma B.2]. Set $\mathcal{O}' = H\pi_* \mathcal{O}$. Choosing an étale hypercover $U_\bullet \to \mathcal{X}$ by affines, we can compute $\operatorname{Map}_{\mathrm{CAlg}_{\mathcal{X}}}(\mathcal{O}, \mathcal{O}')$ as the totalization of the cosimplicial diagram

$$M^\bullet = \operatorname{Map}_{\mathrm{CAlg}}(\mathcal{O}(U_\bullet), \mathcal{O}'(U_\bullet)).$$

We observe using Proposition 4.7 that $\pi^0 \pi_0 M^\bullet$ agrees with the set of ring morphisms $\pi_* \mathcal{O} \to \pi_* \mathcal{O}'$, in which we can pick an isomorphism f_0. By [4, Sections 5.2 and 2.4], the vanishing of $\pi^{i+1} \pi_i M^\bullet \cong H^{i+1}(\mathcal{X}, \pi_i \mathcal{O})$ for $i \geq 1$ suffices to lift f_0 to a multiplicative map $\mathcal{O} \to \mathcal{O}'$, which is automatically an equivalence. □

Corollary 4.9 *For all $\overline{\mathcal{M}}(\Gamma)$ the rationalized Goerss–Hopkins–Miller–Hill–Lawson sheaf $\mathcal{O}^{\mathrm{top}}$ is formal.*

Proof We can apply the previous proposition, as $\overline{\mathcal{M}}(\Gamma)_{\mathbb{Q}}$ has cohomological dimension one. (See eg [36, Proposition 2.4(4)].) □

Remark 4.10 In the original account of the construction of $\mathcal{O}^{\mathrm{top}}$ on $\overline{\mathcal{M}}_{\mathrm{ell}}$ in [9], $\mathcal{O}^{\mathrm{top}}_{\mathbb{Q}}$ is actually formal *by construction*. Our argument shows that this choice was necessary, not only for $\overline{\mathcal{M}}_{\mathrm{ell}}$, but also for $\overline{\mathcal{M}}(\Gamma)$. (The former was shown in a different manner already in [17, Proposition 4.47].)

Proposition 4.11 *Let Γ be a congruence group. Then the E_∞–rings* $\mathrm{tmf}(\Gamma)_{\mathbb{Q}}$ *are formal.*

Proof Set $R = H(H^0(\overline{\mathcal{M}}(\Gamma), \pi_*\mathcal{O}^{\mathrm{top}}_{\mathbb{Q}}))$. We want to construct an equivalence between R and $\mathrm{tmf}(\Gamma)_{\mathbb{Q}}$. By the preceding corollary, we know that $\mathcal{O}^{\mathrm{top}}_{\mathbb{Q}}$ on $\overline{\mathcal{M}}(\Gamma)$ is formal. In particular this provides us with compatible maps $R \to \mathcal{O}^{\mathrm{top}}(U)_{\mathbb{Q}}$ for all affines U étale over $\overline{\mathcal{M}}(\Gamma)$. Taking the homotopy limit, we obtain a map $R \to \mathrm{Tmf}(\Gamma)_{\mathbb{Q}}$. The uniqueness part of Theorem 2.12 identifies R with $\mathrm{tmf}(\Gamma)_{\mathbb{Q}}$. □

4.3 Not all tmf(Γ) are perfect

While we have seen above that $\mathrm{tmf}(\Gamma)$ for a congruence group of level n is always perfect as a $\mathrm{tmf}[1/n]$–module, we will see in this subsection that it is not necessarily compact as a $\mathrm{tmf}(\Gamma')[1/n]$–module for $\Gamma \subset \Gamma'$. The author learned this argument from Tyler Lawson.

Lemma 4.12 *For $R = \mathrm{tmf}(\Gamma)_{\mathbb{Q}}$, the R–module $H\pi_0 R$ can only be perfect if $\pi_* R$ is regular.*

Proof By [10, Theorem 19.1, Corollary 19.5 and Theorem 19.12], $\pi_* R$ is regular if and only if the graded $\mathbb{Q}$–vector space $\mathrm{Tor}^{\pi_* R}_*(\pi_0 R, \pi_0 R)$ is concentrated in finitely many dimensions. Because R is formal by Proposition 4.11, this Tor agrees with $\pi_*(H\pi_0 R \otimes_R H\pi_0 R)$. Clearly, $H\pi_0 R$ being a perfect R–module would imply the finite-dimensionality of this quantity. □

It is actually very rare that $\pi_*\mathrm{tmf}(\Gamma)_{\mathbb{Q}} \cong M_*(\Gamma; \mathbb{Q})$ is regular. One of the few exceptions is $\Gamma = \Gamma_1(3)$, where we obtain the ring $\mathbb{Q}[a_1, a_3]$. In contrast for $\Gamma = \Gamma_0(3)$, we obtain its C_2–fixed points, ie $\mathbb{Q}[a_1^2, a_3^2, a_1 a_3] \cong \mathbb{Q}[x, y, z]/xz - y^2$, which is not regular. Thus, $H\mathbb{Q}$ is a perfect $\mathrm{tmf}_1(3)$–module, but is by the previous lemma not a perfect $\mathrm{tmf}_0(3)_{\mathbb{Q}}$–module. We obtain:

Proposition 4.13 *The* $\mathrm{tmf}_0(3)$*–module* $\mathrm{tmf}_1(3)$ *is not perfect, not even rationally.*

4.4 All tmf(Γ) are faithful

The goal of this section is to show that if Γ is a congruence subgroup of level n, then $\mathrm{tmf}(\Gamma)$ is (if defined) a faithful $\mathrm{tmf}[1/n]$–module, ie tensoring with it is conservative.

Lemma 4.14 *For every congruence subgroup* Γ *of level* n*, the* $\mathrm{Tmf}[1/n]$*–module* $\mathrm{Tmf}(\Gamma)$ *is faithful.*

Proof By [35], the derived stack $(\overline{\mathcal{M}}_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}})$ is 0–affine, ie the global sections functor

$$\Gamma\colon \mathrm{QCoh}(\overline{\mathcal{M}}_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}}) \to \mathrm{Mod}_{\mathrm{Tmf}}$$

is a symmetric monoidal equivalence and the same holds after inverting n. Thus our claim is equivalent to showing that tensoring with $f_*\mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}(\Gamma)}$ for $f\colon \overline{\mathcal{M}}(\Gamma) \to \overline{\mathcal{M}}_{\mathrm{ell},\mathbb{Z}[1/n]}$ is conservative on $\mathrm{QCoh}(\overline{\mathcal{M}}_{\mathrm{ell}}, \mathcal{O}^{\mathrm{top}})$. This can be checked étale locally, where $f_*\mathcal{O}^{\mathrm{top}}_{\overline{\mathcal{M}}(\Gamma)}$ is free of positive rank as f is finite and flat (see eg [36, Proposition 2.4]) and of positive rank everywhere (as $\overline{\mathcal{M}}_{\mathrm{ell},\mathbb{Z}[1/n]}$ is irreducible and $\overline{\mathcal{M}}(\Gamma)$ not empty). □

In the following we fix a congruence subgroup Γ and a multiplicatively closed subset S of $\mathbb{Z}$ such that $\mathrm{tmf}(\Gamma)_S$ is defined (ie Γ is tame or of index 2 in a tame Γ).

Proposition 4.15 *The* tmf_S*–module* $\mathrm{tmf}(\Gamma)_S$ *is faithful for every congruence subgroup* Γ.

Proof Let $M \in \mathrm{Mod}_{\mathrm{tmf}_S}$ with $M \otimes_{\mathrm{tmf}_S} \mathrm{tmf}(\Gamma)_S = 0$. It suffices to show that $M_{(p)} = 0$ for all p not in S. Consider the case $p = 2$ and localize everything implicitly at 2. As $\mathrm{tmf}_1(3)$ is faithful over tmf (see [34, Theorem 4.10]), it suffices to show that $M' = M \otimes_{\mathrm{tmf}} \mathrm{tmf}_1(3)$ vanishes. Our assumption implies

$$(M \otimes_{\mathrm{tmf}} \mathrm{Tmf}) \otimes_{\mathrm{Tmf}} \mathrm{Tmf}(\Gamma) = 0,$$

so by the faithfulness of $\mathrm{Tmf}(\Gamma)$ also $M \otimes_{\mathrm{tmf}} \mathrm{Tmf} = 0$. Thus, $M' \otimes_{\mathrm{tmf}_1(3)} \mathrm{Tmf}_1(3) = 0$. Moreover, $\mathrm{tmf}(\Gamma) \otimes_{\mathrm{tmf}} H\mathbb{Z}$ is a faithful $H\mathbb{Z}$–module as its π_0 is a faithful $\mathbb{Z}$–module. Thus $M' \otimes_{\mathrm{tmf}_1(3)} H\mathbb{Z} \simeq M \otimes_{\mathrm{tmf}} H\mathbb{Z} = 0$.

Recall now that $\pi_*\mathrm{tmf}_1(3) \cong \mathbb{Z}[a_1, a_3]$. The map $\mathrm{tmf}_1(3)[a_i^{-1}] \to \mathrm{Tmf}_1(3)[a_i^{-1}]$ is an equivalence for $i = 1, 3$ since the cofiber of $\mathrm{tmf}_1(3) \to \mathrm{Tmf}_1(3)$ is coconnective. Thus the considerations above imply that $M'[a_1^{-1}]$, $M'[a_3^{-1}]$ and $M'/(a_1, a_3)$ all vanish, which implies the vanishing of M'.

The argument for $p = 3$ is similar with $\mathrm{tmf}_1(2)$ in place of $\mathrm{tmf}_1(3)$ and for $p > 3$ we can use tmf itself as $\pi_*\mathrm{tmf}\left[\frac{1}{6}\right] \cong \mathbb{Z}\left[\frac{1}{6}\right][c_4, c_6]$ is a polynomial ring. □

5 Splittings

Our goal in this setting is to show that $\mathrm{tmf}_1(n)$ often splits p–locally into small pieces.

Fixing a natural number $n \geq 2$ and a prime p not dividing n, we will work throughout this section implicitly p–locally. We demand that $M(\Gamma_1(n), \mathbb{Z}_{(p)}) \to M(\Gamma_1(n); \mathbb{F}_p)$ is surjective. In general, this is a subtle condition, but it is for example always fulfilled if $n \leq 28$; see [36, Remark 3.14]. Equivalently, we can ask that $H^1(\overline{\mathcal{M}}_1(n); \omega) \cong \pi_1 \mathrm{Tmf}_1(n)$ does not have p–torsion. We note that this leaves plenty of cases where $\pi_1 \mathrm{Tmf}_1(n) \neq 0$ and hence $\mathrm{tmf}_1(n)$ is not the naive connective cover of $\mathrm{Tmf}_1(n)$, of which the smallest is $n = 23$.

By Theorem 1.3 of [37], we have a splitting

(5.1)
$$\mathrm{Tmf}_1(n) \simeq \bigoplus_i \Sigma^{2n_i} R$$

of Tmf–modules, where R is $\mathrm{Tmf}_1(3)$, $\mathrm{Tmf}_1(2)$ or Tmf, depending on whether the prime p is 2, 3 or bigger than 3. In this splitting all n_i are nonnegative.

Theorem 5.2 *Under the conditions as above, we have a splitting*

$$\mathrm{tmf}_1(n) \simeq \bigoplus_i \Sigma^{2n_i} r,$$

where $r = \tau_{\geq 0} R$.

Proof Consider the composition

$$f \colon \bigoplus_i \Sigma^{2n_i} r \to \bigoplus_i \tau_{\geq 0} \Sigma^{2n_i} R \to \tau_{\geq 0} \mathrm{Tmf}_1(n).$$

Here, the second map is just the connective cover of (5.1) (using that $\tau_{\geq 0}$ commutes with direct sums) and the first map is the direct sum of the maps

$$\Sigma^{2n_i} r \simeq \tau_{\geq 2n_i} \Sigma^{2n_i} R \to \tau_{\geq 0} \Sigma^{2n_i} R.$$

Since all negative homotopy of R is in odd degrees, we see that f is an isomorphism on even homotopy groups. Moreover, the source has only homotopy groups in even degrees.

Recall that we defined $\mathrm{tmf}_1(n)$ as a pullback

$$\begin{tikzcd}
\mathrm{tmf}_1(n) \arrow[r] \arrow[d] & H\mathbb{Z} \arrow[d] \\
\tau_{\geq 0} \mathrm{Tmf}_1(n) \arrow[r] & \tau_{[0,1]} \mathrm{Tmf}_1(n)
\end{tikzcd}$$

where we still localize implicitly everywhere at p. This implies a fiber sequence

$$\mathrm{tmf}_1(n) \to \tau_{\geq 0}\mathrm{Tmf}_1(n) \to \Sigma H\pi_1\mathrm{Tmf}_1(n).$$

To factor f over $\mathrm{tmf}_1(n)$, it is enough to show that $H^1(\Sigma^{2n_i}r; A) = 0$ with any coefficients A. This is clear anyhow for $n_i \geq 1$, so assume $n_i = 0$. We know that $\tau_{[0,1]}r \simeq H\mathbb{Z}$ and we have $H^1(H\mathbb{Z}; A) \cong H^1(\mathbb{S}; A) = 0$ (as the cofiber of $\mathbb{S} \to H\mathbb{Z}$ is 1–connected).

Now $\pi_*\mathrm{tmf}_1(n)$ is concentrated in even degrees and $\mathrm{tmf}_1(n) \to \tau_{\geq 0}\mathrm{Tmf}_1(n)$ induces a π_*–isomorphism in even degrees. In total, we see that f induces an isomorphism on π_*. □

Remark 5.3 The condition that $\pi_1\mathrm{Tmf}_1(n) \cong H^1(\overline{\mathcal{M}}_1(n); \omega)$ does not have p–torsion is actually necessary in the preceding theorem. One can indeed show that $\mathrm{Tmf}_1(n)$ can be recovered as $\mathrm{tmf}_1(n) \otimes_{\mathrm{tmf}} \mathrm{Tmf}$. Thus a p–local tmf–linear splitting of $\mathrm{tmf}_1(n)$ into shifted copies of r implies a p–local splitting of $\mathrm{Tmf}_1(n)$ into copies of R. As the latter has torsion-free homotopy groups, such a splitting can indeed only occur if the homotopy groups of $\mathrm{Tmf}_1(n)$ are p–torsion-free as well.

We now fix $p = 2$ and are thus assuming that $\pi_1\mathrm{Tmf}_1(n) \cong H^1(\overline{\mathcal{M}}_1(n); \omega)$ does not have 2–torsion — this is true for all odd $2 \leq n < 65$ by [36, Remark 3.14], for example. In this setting we also want to prove connective versions of the C_2–equivariant refinement

$$\mathrm{Tmf}_1(n)_{(2)} \simeq_{C_2} \bigoplus_i \Sigma^{n_i\rho}\mathrm{Tmf}_1(3)_{(2)} \tag{5.4}$$

of (5.1) given in [37, Theorem 6.19], where ρ is the regular representation of C_2. We need the following lemma:

Lemma 5.5 *Let A be an abelian group without 2–torsion, and denote by $\underline{A}$ the corresponding constant C_2–Mackey functor. Then $\pi^{C_2}_{-\sigma}H\underline{A} \cong A \otimes \mathbb{Z}/2$, and the map*

$$[H\underline{\mathbb{Z}}, \Sigma^\sigma H\underline{A}]^{C_2} \xrightarrow{\pi_0^{C_2}} A \otimes \mathbb{Z}/2$$

is an isomorphism.

Proof Smashing the fundamental cofiber sequence

$$(C_2)_+ \to S^0 \to S^\sigma \to \Sigma(C_2)_+$$

with $S^{-\sigma}$ and mapping out of it yields an exact sequence

$$\pi_{-1}^{e} H\underline{A} \leftarrow \pi_{-\sigma}^{C_2} H\underline{A} \leftarrow \pi_{0}^{C_2} H\underline{A} \leftarrow \pi_{0}^{e} H\underline{A}.$$

The rightmost arrow can be identified with the transfer $\mathrm{tr} = 2\colon A \to A$ of the constant Mackey functor, while $\pi_{-1}^{e} H\underline{A} = 0$. We obtain $\pi_{-\sigma}^{C_2} H\underline{A} \cong A \otimes \mathbb{Z}/2$ as claimed.

To finish the proof, we recall from Section 2.2 that $\tau_{\leq 1} C\overline{\eta} \simeq H\underline{\mathbb{Z}}$. As $\Sigma^{\sigma} H\underline{A} \leq 1$ in the slice filtration, this implies that $[H\underline{\mathbb{Z}}, \Sigma^{\sigma} H\underline{A}]^{C_2} \cong [C\overline{\eta}, \Sigma^{\sigma} H\underline{A}]^{C_2}$. This sits in a long exact sequence

$$0 = \pi_{1}^{C_2} H\underline{A} \to [C\overline{\eta}, \Sigma^{\sigma} H\underline{A}] \to \pi_{-\sigma}^{C_2} H\underline{A} \to \pi_{0}^{C_2} H\underline{A} = A.$$

As A does not have 2–torsion and we have shown above that $\pi_{-\sigma}^{C_2} H\underline{A} \cong A \otimes \mathbb{Z}/2$, the result follows. □

Theorem 5.6 *Assuming that $n \geq 3$ is odd and $H^1(\overline{\mathcal{M}}_1(n); \omega)$ does not have 2–torsion, we have 2–locally a C_2–equivariant splitting*

$$\mathrm{tmf}_1(n) \simeq \bigoplus_i \Sigma^{n_i \rho} \mathrm{tmf}_1(3).$$

Proof We localize everywhere implicitly at 2 and consider the map

$$\bigoplus_i \Sigma^{n_i \rho} \mathrm{tmf}_1(3) \to \bigoplus_i \tau_{\geq 0} \Sigma^{n_i \rho} \mathrm{Tmf}_1(3) \xrightarrow{\tau_{\geq 0}\Phi} \tau_{\geq 0} \mathrm{Tmf}_1(n),$$

for a chosen C_2–equivalence Φ between $\bigoplus_i \Sigma^{n_i \rho} \mathrm{Tmf}_1(3)$ and $\mathrm{Tmf}_1(n)$. We have a fiber sequence

$$\mathrm{tmf}_1(n) \to \tau_{\geq 0} \mathrm{Tmf}_1(n) \to \Sigma^{\sigma} H\underline{A},$$

where $A = H^1(\overline{\mathcal{M}}_1(n); \omega)$ since by [37, Theorem 6.16], $\Sigma^{\sigma} H\underline{A}$ is the 1–slice of $\mathrm{Tmf}_1(n)$. On $\pi_0^{C_2}$ this induces (using Lemma 5.5) a short exact sequence

$$0 \to \mathbb{Z} \to \pi_0^{C_2} \mathrm{Tmf}_1(n) \xrightarrow{r} A \otimes \mathbb{Z}/2 \to 0. \tag{5.7}$$

The composite $\bigoplus \Sigma^{n_i \rho} \mathrm{tmf}_1(3) \to \Sigma^{\sigma} H\underline{A}$ factors over the 1–slice coconnective cover of the source, which agrees with $H\underline{\mathbb{Z}}$ since there is precisely one n_i equaling 0 (by considering nonequivariant homotopy groups). Using Lemma 5.5 again, the resulting map $H\underline{\mathbb{Z}} \to \Sigma^{\sigma} H\underline{A}$ is null if and only if the image $r(\Phi(1))$ of $\Phi(1)$ in $A \otimes \mathbb{Z}/2$ is 0.

We want to show that we can change Φ so that this is true. Using Φ, the C_2–spectrum $\mathrm{Tmf}_1(n)$ gets the structure of a $\mathrm{Tmf}_1(3)$–module. Thus, $\mathrm{Tmf}_1(3)$–module maps

$\bigoplus_{i=0}^{N} \Sigma^{n_i\rho}\mathrm{Tmf}_1(3) \to \mathrm{Tmf}_1(n)$ correspond to a sequence of classes $x_i \in \pi^{C_2}_{n_i\rho}\mathrm{Tmf}_1(n)$ by considering the images of $1 \in \pi^{C_2}_{n_i\rho}\Sigma^{n_i\rho}\mathrm{Tmf}_1(3)$. Denote the sequence corresponding to Φ by $e_0,\dots,e_N$. By possibly reordering, we can assume $n_0 = 0$. We construct a new map $\Phi'\colon \bigoplus_{i=0}^{N} \Sigma^{n_i\rho}\mathrm{Tmf}_1(3) \to \mathrm{Tmf}_1(n)$ corresponding to $x_0, x_1, \dots, x_N$ with $x_i = e_i$ for $i > 0$, and x_0 corresponding to the image of $u \in \mathbb{Z}$ in (5.7), where u maps to $\mathrm{res}^{C_2}_e(e_0)$ along the isomorphism $\mathbb{Z} \cong \pi^e_0\mathrm{tmf}_1(n) \to \pi^e_0\mathrm{Tmf}_1(n)$. As Φ' and Φ induce the same map on underlying homotopy groups, the map Φ' is an equivalence. By construction, $r(x_0) = 0$.

Thus the map

$$\bigoplus_i \Sigma^{n_i\rho}\mathrm{tmf}_1(3) \to \bigoplus_i \tau_{\geq 0}\Sigma^{n_i\rho}\mathrm{tmf}_1(3) \xrightarrow{\tau_{\geq 0}\Phi'} \tau_{\geq 0}\mathrm{Tmf}_1(n)$$

factors indeed over $\mathrm{tmf}_1(n)$. As before, the map $\Sigma^{n_i\rho}\mathrm{tmf}_1(3) \to \mathrm{tmf}_1(n)$ induces an isomorphism on underlying homotopy groups. Both source and target are strongly even and thus the map is a C_2–equivariant equivalence by [18, Lemma 3.4]. □

References

[1] **M Ando**, **M J Hopkins**, **C Rezk**, *Multiplicative orientations of KO–theory and of the spectrum of topological modular forms*, preprint (2010) Available at `https://rezk.web.illinois.edu/koandtmf.pdf`

[2] **S Araki**, **K Iriye**, *Equivariant stable homotopy groups of spheres with involutions, I*, Osaka Math. J. 19 (1982) 1–55 MR Zbl

[3] **M Behrens**, **M Hill**, **M J Hopkins**, **M Mahowald**, *Detecting exotic spheres in low dimensions using* coker J, J. Lond. Math. Soc. 101 (2020) 1173–1218 MR Zbl

[4] **A K Bousfield**, *Homotopy spectral sequences and obstructions*, Israel J. Math. 66 (1989) 54–104 MR Zbl

[5] **B Conrad**, *Arithmetic moduli of generalized elliptic curves*, J. Inst. Math. Jussieu 6 (2007) 209–278 MR Zbl

[6] **P Deligne**, **M Rapoport**, *Les schémas de modules de courbes elliptiques*, from "Modular functions of one variable, II" (P Deligne, W Kuyk, editors), Lecture Notes in Math. 349, Springer (1973) 143–316 MR Zbl

[7] **F Diamond**, **J Im**, *Modular forms and modular curves*, from "Seminar on Fermat's last theorem" (V K Murty, editor), CMS Conf. Proc. 17, Amer. Math. Soc., Providence, RI (1995) 39–133 MR Zbl

[8] **F Diamond**, **J Shurman**, *A first course in modular forms*, Graduate Texts in Math. 228, Springer (2005) MR Zbl

[9] **C L Douglas**, **J Francis**, **A G Henriques**, **M A Hill** (editors), *Topological modular forms*, Mathematical Surveys and Monographs 201, Amer. Math. Soc., Providence, RI (2014) MR Zbl

[10] **D Eisenbud**, *Commutative algebra*, Graduate Texts in Math. 150, Springer (1995) MR Zbl

[11] **J Franke**, *On the construction of elliptic cohomology*, Math. Nachr. 158 (1992) 43–65 MR Zbl

[12] **P G Goerss**, *Quasi-coherent sheaves on the moduli stack of formal groups*, preprint (2008) arXiv 0802.0996

[13] **P G Goerss**, **M J Hopkins**, *Moduli spaces of commutative ring spectra*, from "Structured ring spectra" (A Baker, B Richter, editors), London Math. Soc. Lecture Note Ser. 315, Cambridge Univ. Press (2004) 151–200 MR Zbl

[14] **R Hartshorne**, *Algebraic geometry*, Graduate Texts in Math. 52, Springer (1977) MR Zbl

[15] **R Herrera**, *Elliptic genera of level N on complex π_2–finite manifolds*, C. R. Math. Acad. Sci. Paris 344 (2007) 317–320 MR Zbl

[16] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. 184 (2016) 1–262 MR Zbl

[17] **M Hill**, **T Lawson**, *Topological modular forms with level structure*, Invent. Math. 203 (2016) 359–416 MR Zbl

[18] **M A Hill**, **L Meier**, *The C_2–spectrum $\mathrm{Tmf}_1(3)$ and its invertible modules*, Algebr. Geom. Topol. 17 (2017) 1953–2011 MR Zbl

[19] **F Hirzebruch**, *Elliptic genera of level N for complex manifolds*, from "Differential geometrical methods in theoretical physics" (K Bleuler, M Werner, editors), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 250, Kluwer, Dordrecht (1988) 37–63 MR Zbl

[20] **F Hirzebruch**, **T Berger**, **R Jung**, *Manifolds and modular forms*, Aspects of Mathematics E20, Friedr. Vieweg & Sohn, Braunschweig (1992) MR Zbl

[21] **E Höning**, **B Richter**, *Detecting and describing ramification for structured ring spectra*, preprint (2021) arXiv 2101.12655

[22] **M J Hopkins**, **T Lawson**, *Strictly commutative complex orientation theory*, Math. Z. 290 (2018) 83–101 MR Zbl

[23] **M J Hopkins**, **M Mahowald**, *From elliptic curves to homotopy theory*, preprint (2000) Available at `https://tinyurl.com/HopkinsMahowald2000`

[24] **P Hu**, **I Kriz**, *Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence*, Topology 40 (2001) 317–399 MR Zbl

[25] **D C Isaksen**, **G Wang**, **Z Xu**, *Stable homotopy groups of spheres: from dimension 0 to 90*, preprint (2020) arXiv 2001.04511

[26] **N M Katz**, **B Mazur**, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies 108, Princeton Univ. Press (1985) MR Zbl

[27] **I M Krichever**, *Generalized elliptic genera and Baker–Akhiezer functions*, Mat. Zametki 47 (1990) 34–45 MR Zbl In Russian; translated in Math. Notes 47 (1990) 132–142

[28] **N J Kuhn**, *Adams filtration and generalized Hurewicz maps for infinite loopspaces*, Invent. Math. 214 (2018) 957–998 MR Zbl

[29] **T Lawson**, *The Shimura curve of discriminant 15 and topological automorphic forms*, Forum Math. Sigma 3 (2015) art. id. e3 MR Zbl

[30] **L G Lewis, Jr**, *The equivariant Hurewicz map*, Trans. Amer. Math. Soc. 329 (1992) 433–472 MR Zbl

[31] **J Lurie**, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton Univ. Press (2009) MR Zbl

[32] **J Lurie**, *Higher algebra*, book project (2017) Available at `https://url.msp.org/Lurie-HA`

[33] **M Mahowald**, **C Rezk**, *Brown–Comenetz duality and the Adams spectral sequence*, Amer. J. Math. 121 (1999) 1153–1177 MR Zbl

[34] **A Mathew**, *The homology of tmf*, Homology Homotopy Appl. 18 (2016) 1–29 MR Zbl

[35] **A Mathew**, **L Meier**, *Affineness and chromatic homotopy theory*, J. Topol. 8 (2015) 476–528 MR Zbl

[36] **L Meier**, *Additive decompositions for rings of modular forms*, Doc. Math. 27 (2022) 427–488 MR Zbl

[37] **L Meier**, *Topological modular forms with level structure: decompositions and duality*, Trans. Amer. Math. Soc. 375 (2022) 1305–1355 MR Zbl

[38] **L Meier**, **V Ozornova**, *Rings of modular forms and a splitting of* $\mathrm{TMF}_0(7)$, Selecta Math. 26 (2020) art. id. 7 MR Zbl

[39] **J S Milne**, *Étale cohomology*, Princeton Mathematical Series No. 33, Princeton Univ. Press (1980) MR Zbl

[40] **P Pstrągowski**, **P VanKoughnett**, *Abstract Goerss–Hopkins theory*, Adv. Math. 395 (2022) art. id. 108098 MR Zbl

[41] **C Rezk**, *Supplementary notes for Math* 512 (*version* 0.18), lecture notes (2002) Available at `https://rezk.web.illinois.edu/512-spr2001-notes.pdf`

[42] **E Riehl**, **D Verity**, *Homotopy coherent adjunctions and the formal theory of monads*, Adv. Math. 286 (2016) 802–888 MR Zbl

[43] **A Senger**, *Obstruction theory and the level n elliptic genus*, Compos. Math. 159 (2023) 2000–2021 MR Zbl

[44] **V Stojanoska**, *Duality for topological modular forms*, Doc. Math. 17 (2012) 271–311 MR Zbl

[45] **K Česnavičius**, *A modular description of $\mathcal{X}_0(n)$*, Algebra Number Theory 11 (2017) 2001–2089 MR Zbl

[46] **G Wang**, **Z Xu**, *The triviality of the 61–stem in the stable homotopy groups of spheres*, Ann. of Math. 186 (2017) 501–580 MR Zbl

[47] **J Wang**, **Y Wang**, **Z Yu**, *Elliptic genera of level N for complete intersections*, Front. Math. China 16 (2021) 1043–1062 MR Zbl

[48] **E Witten**, *The index of the Dirac operator in loop space*, from "Elliptic curves and modular forms in algebraic topology" (P S Landweber, editor), Lecture Notes in Math. 1326, Springer (1988) 161–181 MR Zbl

Mathematical Institute, Utrecht University
Utrecht, Netherlands

f.l.m.meier@uu.nl

Received: 13 May 2021 Revised: 4 July 2022

Algebraic & Geometric Topology

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2023 is US $650/year for the electronic version, and $940/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebraic & Geometric Topology

Volume 23 Issue 8 (pages 3417–3908) 2023