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Asymptotic dimension of graphs of groups and one-relator groups

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Asymptotic dimension of graphs of groups and one-relator groups

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We prove a new inequality for the asymptotic dimension of HNN-extensions. We deduce that the asymptotic dimension of every finitely generated one-relator group is at most two, confirming a conjecture of A Dranishnikov. As corollaries we calculate the exact asymptotic dimension of right-angled Artin groups and we give a new upper bound for the asymptotic dimension of fundamental groups of graphs of groups.

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1.	Introduction	3587
2.	Asymptotic dimension of HNN-extensions	3591
3.	Asymptotic dimension of one-relator groups	3604
4.	Graphs of groups	3610
	References	3612

1 Introduction

In 1993, M Gromov introduced the notion of the asymptotic dimension of metric spaces (see [12]) as an invariant of finitely generated groups. It can be shown that if two metric spaces are quasi-isometric then they have the same asymptotic dimension. The asymptotic dimension $\operatorname{asdim} X$ of a metric space X is defined by: $\operatorname{asdim} X \leq n$ if and only if, for every $R > 0$, there exists a uniformly bounded covering $\mathcal{U}$ of X such that the R–multiplicity of $\mathcal{U}$ is smaller than or equal to $n+1$ (ie every R–ball in X intersects at most $n+1$ elements of $\mathcal{U}$). There are many equivalent ways to define the asymptotic dimension of a metric space. It turns out that the asymptotic dimension of an infinite tree is 1 and the asymptotic dimension of $\mathbb{E}^n$ is n.

In 1998, the asymptotic dimension achieved particular prominence in geometric group theory after the publication of a paper of Guoliang Yu (see [24]) which proved the Novikov higher signature conjecture for manifolds whose fundamental group has finite asymptotic dimension. Unfortunately, not all finitely presented groups have finite asymptotic dimension. For example, Thompson's group F has infinite asymptotic dimension since it contains $\mathbb{Z}^n$ for all n. However, we know for many classes of groups that they have finite asymptotic dimension. For instance, hyperbolic, relative hyperbolic, mapping class groups of surfaces and one-relator groups have finite asymptotic dimension (see G Bell and A Dranishnikov [3], Bestvina, Bromberg and Fujiwara [6], Osin [18] and D Matsnev [17]). The exact computation of the asymptotic dimension of groups or finding the optimal upper bound is more delicate. Another remarkable result is that of Buyalo and Lebedeva (see [7]), where in 2006 they established the equality, for hyperbolic groups,

$$\operatorname{asdim} G = \dim \partial_\infty G + 1.$$

The inequalities of Bell and Dranishnikov (see [2; 9]) play a key role in finding an upper bound for the asymptotic dimension of groups. However, in some cases the upper bounds that the inequalities of Bell and Dranishnikov provide us are quite far from being optimal. An example is the asymptotic dimension of one-relator groups.

We prove some new inequalities that can be a useful tool for the computation of the asymptotic dimension of groups. As an application we give the optimal upper bound for the asymptotic dimension of one-relator groups which was conjectured by Dranishnikov. As a further corollary we calculate the exact asymptotic dimension of any right-angled Artin group (Theorem 1.2) — this has been proven earlier by N Wright [23] by different methods.

The first inequality and one of the main results we prove is the following:

Theorem 1.1 *Let $G*_N$ be an HNN-extension of the finitely generated group G over N. Then*

$$\operatorname{asdim} G*_N \leq \max\{\operatorname{asdim} G, \operatorname{asdim} N + 1\}.$$

Next, we calculate the asymptotic dimension of the right-angled Artin groups. To be more precise, let Γ be a finite simplicial graph. We denote by $A(\Gamma)$ the *right-angled Artin group* (RAAG) associated to the graph Γ. We set

$$\operatorname{Sim}(\Gamma) = \max\{n : \Gamma \text{ contains the 1–skeleton of the standard } (n-1)\text{–simplex } \Delta^{n-1}\}.$$

Then by applying Theorem 1.1 we obtain the following:

Theorem 1.2 *Let Γ be a finite simplicial graph. Then*

$$\operatorname{asdim} A(\Gamma) = \operatorname{Sim}(\Gamma).$$

In 2005, Bell and Dranishnikov (see [4]) gave a proof that the asymptotic dimension of one-relator groups is finite and also gave an upper bound, namely the length of the relator plus one. Let $G = \langle S \mid r \rangle$ be a finitely generated one-relator group such that $|r| = n$. Then

$$\operatorname{asdim} G \leq n + 1.$$

To prove this upper bound, Bell and Dranishnikov used an inequality for the asymptotic dimension of HNN-extensions; see [2]. In particular, let G be a finitely generated group and let N be a subgroup of G. Then

$$\operatorname{asdim} G *_N \leq \operatorname{asdim} G + 1.$$

In 2006, Matsnev (see [17]) proved a sharper upper bound for the asymptotic dimension of one-relator groups: if $G = \langle S \mid r \rangle$ is a one-relator group, then

$$\operatorname{asdim} G \leq \left\lceil \tfrac{1}{2} \operatorname{length}(r) \right\rceil.$$

Here by $\lceil a \rceil$ ($a \in \mathbb{R}$) we denote the minimal integer greater than or equal to a.

Applying Theorem 1.1, we answer a conjecture of Dranishnikov (see [8]) giving the optimal upper bound for the asymptotic dimension of one-relator groups.

Theorem 1.3 *Let G be a finitely generated one-relator group. Then*

$$\operatorname{asdim} G \leq 2.$$

We note that R C Lyndon (see [14]) has shown that the *cohomological dimension* of a torsion-free one-relator group is smaller than or equal to 2. Our result can be seen as a large-scale analog of this. We note that the large-scale geometry of one-relator groups can be quite complicated; for example, one-relator groups can have very large isoperimetric functions; see eg Platonov [19].

It is worth noting that L Sledd showed that the Assouad–Nagata dimension of any finitely generated $C'\left(\frac{1}{6}\right)$ group is at most two; see [20].

Theorem 1.3 combined with the results of M Kapovich and B Kleiner (see [13]) leads us to a description of the boundary of hyperbolic one-relator groups.

We determine also the one-relator groups that have asymptotic dimension exactly two. We prove that every infinite finitely generated one-relator group G that is not a free group or a free product of a free group and a finite cyclic group has asymptotic dimension equal to 2 (Proposition 3.5). We obtain the following:

Corollary *Let G be finitely generated freely indecomposable one-relator group which is not cyclic. Then*

$$\operatorname{asdim} G = 2.$$

Moreover, we describe the finitely generated one-relator groups:

Corollary *Let G be a finitely generated one-relator group. Then one of the following is true*:

(i) *G is finite cyclic*, *and* $\operatorname{asdim} G = 0$.

(ii) *G is a nontrivial free group or a free product of a nontrivial free group and a finite cyclic group*, *and* $\operatorname{asdim} G = 1$.

(iii) *G is an infinite freely indecomposable not cyclic group or a free product of a nontrivial free group and an infinite freely indecomposable not cyclic group*, *and* $\operatorname{asdim} G = 2$.

Using Theorem 1.1 and an inequality of Dranishnikov about the asymptotic dimension of amalgamated products (see [9]) we obtain a more general theorem for the asymptotic dimension of fundamental groups of graphs of groups:

Theorem 1.4 *Let $(\mathbb{G}, Y)$ be a finite graph of groups with vertex groups $\{G_v : v \in Y^0\}$ and edge groups $\{G_e : e \in Y^1_+\}$. Then*

$$\operatorname{asdim} \pi_1(\mathbb{G}, Y, \mathbb{T}) \leq \max_{v \in Y^0, e \in Y^1_+} \{\operatorname{asdim} G_v, \operatorname{asdim} G_e + 1\}.$$

Using the previous theorem, we can obtain, for example, that the asymptotic dimension of a graph of surface groups (with genus ≥ 2) with free edge groups is two. Theorem 1.4 says that the asymptotic dimension doesn't *jump* as long as there exists a vertex group with asymptotic dimension greater than the asymptotic dimension of any edge group.

The paper is organized as follows. In Section 2 we prove the inequality for the asymptotic dimension of HNN-extensions. In Section 2.1 we compute the asymptotic dimension of RAAGs. Next, in Section 3 we give the optimal upper bound for the asymptotic dimension of one-relator groups. In Section 3.1 we describe the one-relator groups with asymptotic dimension 0, 1 and 2. In Section 4 a new upper bound for the asymptotic dimension of graphs of groups is obtained.

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2 Asymptotic dimension of HNN-extensions

Let X be a metric space and $\mathcal{U}$ a covering of X. We say that the covering $\mathcal{U}$ is *d–bounded* or *d–uniformly bounded* if $\sup_{U\in\mathcal{U}}\{\operatorname{diam} U\} \le d$. The *Lebesgue number* $L(\mathcal{U})$ of the covering $\mathcal{U}$ is

$$L(\mathcal{U}) = \sup\{\lambda : \text{if } A \subseteq X \text{ with } \operatorname{diam} A \le \lambda \text{ then there exists } U \in \mathcal{U} \text{ such that } A \subseteq U\}.$$

We recall that the order $\operatorname{ord}(\mathcal{U})$ of the cover $\mathcal{U}$ is the smallest number n (if it exists) such that each point of the space belongs to at most n sets in the cover. For a metric space X, we say that (r,d)–$\dim X \le n$ if, for $r > 0$, there exists a d–bounded cover $\mathcal{U}$ of X with $\operatorname{ord}(\mathcal{U}) \le n+1$ and with Lebesgue number $L(\mathcal{U}) > r$. We refer to such a cover as an *(r,d)–cover* of X. The following proposition is due to Bell and Dranishnikov (see [2]):

Proposition 2.1 *For a metric space X, $\operatorname{asdim} X \le n$ if and only if there exists a function $d(r)$ such that $(r,d(r))$–$\dim X \le n$ for all $r > 0$.*

We recall that the family X_i of subsets of X satisfies the inequality $\operatorname{asdim} X_i \le n$ *uniformly* if, for every $R > 0$, there exists a D–bounded covering $\mathcal{U}_i$ of X_i with R–$\operatorname{mult}(\mathcal{U}_i) \le n+1$ for every i. For the proofs of Theorems 2.2 and 2.3 see [1].

Theorem 2.2 (infinite union theorem) *Let $X = \bigcup_a X_a$ be a metric space where the family $\{X_a\}$ satisfies the inequality $\operatorname{asdim} X_a \le n$ uniformly. Suppose further that, for every $r > 0$, there is a subset $Y_r \subseteq X$ with $\operatorname{asdim} Y_r \le n$, so that $d(X_a \setminus Y_r, X_b \setminus Y_r) \ge r$ whenever $X_a \ne X_b$. Then $\operatorname{asdim} X \le n$.*

Theorem 2.3 (finite union theorem) *For every metric space presented as a finite union $X = \bigcup_i X_i$,*

$$\operatorname{asdim} X = \max\{\operatorname{asdim} X_i\}.$$

A partition of a metric space X is a presentation as a union $X = \bigcup_i W_i$ such that $\operatorname{Int}(W_i) \cap \operatorname{Int}(W_j) = \varnothing$ whenever $i \ne j$. We denote by ∂W_i the topological boundary of W_i and by $\operatorname{Int}(W_i)$ the topological interior. We have that $\partial W \cap \operatorname{Int}(W) = \varnothing$. The boundary can be written as

$$\partial W_i = \{x \in X : d(x, W_i) = d(x, X \setminus W_i) = 0\}.$$

For the proof of the following theorem see [9]:

Theorem 2.4 (partition theorem) *Let X be a geodesic metric space. Suppose that for every $R > 0$ there is $d > 0$ and a partition $X = \bigcup_i W_i$ with* $\operatorname{asdim} W_i \le n$ *uniformly in i and such that (R,d)–*$\dim\bigl(\bigcup_i \partial W_i\bigr) \le n-1$*, where ∂W_i is taken with the metric restricted from X. Then* $\operatorname{asdim} X \le n$.

Let G be a finitely generated group, N a subgroup of G and $\phi\colon N \to G$ a monomorphism. We set $\overline{G} = G*_N$, the HNN-extension of G over the subgroup N with respect to the monomorphism ϕ. We fix a finite generating set S for the group G. Then the set $\overline{S} = S \cup \{t, t^{-1}\}$ is a finite generating set for the group $\overline{G}$ and we set $C(\overline{G}) = \operatorname{Cay}(\overline{G}, \overline{S})$, its Cayley graph.

Normal forms for HNN-extensions There are two types of *normal forms* for HNN-extensions: the right normal form and the left normal form. We use both.

Right normal form Let S_N and $S_{\phi(N)}$ be sets of representatives of *right cosets* of G/N and of $G/\phi(N)$, respectively. Then every $w \in \overline{G}$ has a unique normal form $w = g t^{\epsilon_1} s_1 t^{\epsilon_2} s_2 \cdots t^{\epsilon_k} s_k$ where $g \in G$, $\epsilon_i \in \{-1, 1\}$, if $\epsilon_i = 1$ then $s_i \in S_N$, if $\epsilon_i = -1$ then $s_i \in S_{\phi(N)}$, and if $s_i = 1$ then $\epsilon_i \epsilon_{i+1} > 0$. We say that the *length* of the right normal form of w is k.

Left normal form Let ${}_N S$ and ${}_{\phi(N)} S$ be sets of representatives of *left cosets* of G/N and of $G/\phi(N)$, respectively. Then every $w \in \overline{G}$ has a unique normal form $w = s_1 t^{\epsilon_1} s_2 t^{\epsilon_2} \cdots s_k t^{\epsilon_k} g$ where $g \in G$, $\epsilon_i \in \{-1, 1\}$, if $\epsilon_i = 1$ then $s_i \in_{\phi(N)} S$, if $\epsilon_i = -1$ then $s_i \in_N S$, and if $s_i = 1$ then $\epsilon_{i-1}\epsilon_i > 0$. We say that the *length* of the left normal form of w is k.

We observe that the *lengths* of the right and the left normal form of an element coincide, and denote this length by $l(w)$.

Convention When we write a normal form we mean the right normal form, unless otherwise stated.

The group $\overline{G} = G*_N$ acts on its Bass–Serre tree T. There is a natural projection $\pi\colon G*_N \to T$ defined by the action: $\pi(g) = gG$.

Lemma 2.5 *The map $\pi\colon \overline{G} \to T$ extends to a simplicial map from the Cayley graph, $\pi\colon C(\overline{G}, S) \to T$, which is* 1*–Lipschitz.*

Proof Let $g \in \overline{G}$ and $s \in \overline{S}$. Then the vertex g is mapped to the vertex $\pi(g) = \pi(gs) = gG$. If $s \in S$, then the edge $[g, gs]$ is mapped to the vertex $\pi(g) = \pi(gs) = gG$.

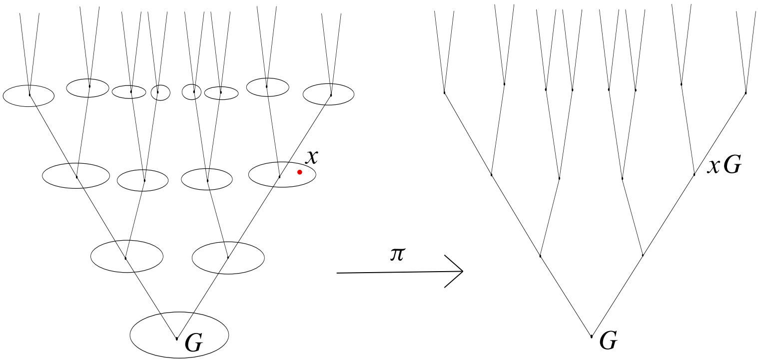

Figure 1: An illustration of the projection $\pi : C(\overline{G}, S) \to T$.

If $s \in \{t, t^{-1}\}$, without loss of generality we may assume that $s = t$, and so the edge $[g, gs]$ is mapped to the edge $[\pi(g), \pi(gs)] = [gG, gtG]$ of T.

We observe that the simplicial map $\pi : C(\overline{G}) \to T$ is 1–Lipschitz. □

The base vertex G separates T into two parts, $T_- \setminus G$ and $T_+ \setminus G$, where

$$\pi^{-1}(T_+) = \{w \in \overline{G} : \text{if } w = gt^{\epsilon_1}s_1t^{\epsilon_2}s_2\cdots t^{\epsilon_k}s_k \text{ is the normal form of } w \text{ then } \epsilon_1 = 1\}$$

and similarly

$$\pi^{-1}(T_-) = \{w \in \overline{G} : \text{if } w = gt^{\epsilon_1}s_1t^{\epsilon_2}s_2\cdots t^{\epsilon_k}s_k \text{ is the normal form of } w \text{ then } \epsilon_1 = -1\}.$$

We note that both $T_+ \setminus G$ and $T_- \setminus G$ are unions of connected components of T and $\pi^{-1}(T_+)$ and $\pi^{-1}(T_-)$ are unions of connected components of $C(\overline{G})$. See Figure 2 for an illustration of T_- and T_+.

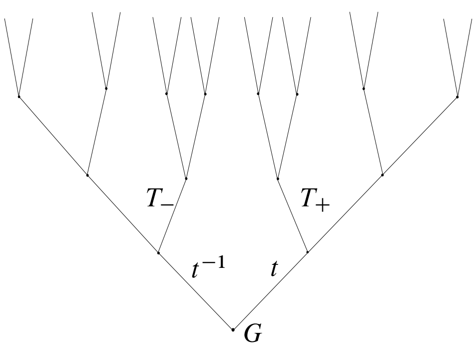

Figure 2: An illustration of T_+ and T_-.

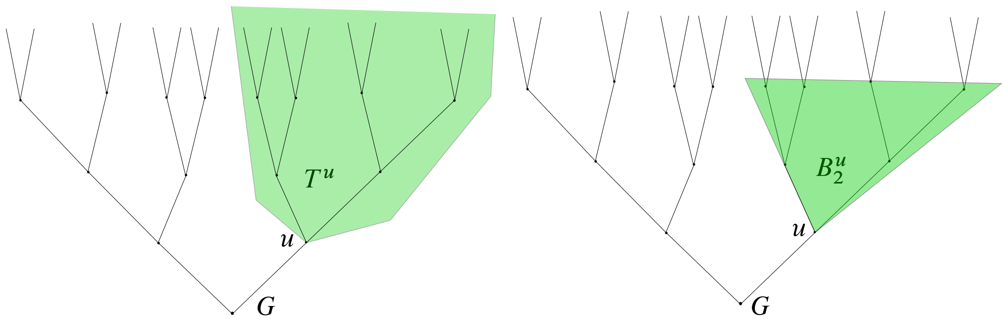

Figure 3: Left: an illustration of T^u. Right: an illustration of B^u_r, where $r = 2$.

We consider the Bass–Serre tree T as a metric space with the simplicial metric $\bar{d}$. If Y is a graph, we denote by Y^0 or $V(Y)$ the vertices of Y. For $u \in T^0$ we denote by $|u|$ the distance to the vertex with label G. We note that the distance of the vertex wG from G in the Bass–Serre tree T equals the length $l(w)$ of the normal form of w, $|wG| = l(w)$.

We recall that a *full subgraph* of a graph Γ is a subgraph formed from a subset of vertices V and from all of the edges that have both endpoints in the subset V. If A is a subgraph of Γ we define the *edge closure* $E(A)$ of A to be the full subgraph of Γ formed from $V(A)$. Obviously, $V(E(A)) = V(A)$.

We fix some notation on the Bass–Serre tree T and on the Cayley graph.

In the tree T We denote by B^T_r the r–ball in T centered at G ($r \in \mathbb{N}$). There is a partial order on vertices of T defined by setting $v \leq u$ if and only if v lies in the geodesic segment $[G, u]$ joining the base vertex G with u. For $u \in T^0$ of nonzero level (ie $u \neq G$) and $r > 0$, we set

$$T^u = E(\{v \in T^0 : u \leq v\}), \quad B^u_r = E(\{v \in T^u : |v| \leq |u| + r\}).$$

For every vertex $u \in T^0$ represented by a coset $g_u G$, we have $B^u_r = g_u B^T_r \cap T^u$. We also observe that $B^u_r = E(\{v \in T^u : \bar{d}(v, u) \leq r\})$. See Figure 3 for an illustration of the sets T^u and B^u_r

In the Cayley graph For $R \in \mathbb{N}$, let

$$M_R = \{g \in \overline{G} : \operatorname{dist}(g, N \cup \phi(N)) = R\}.$$

Letting $u = g_u G$, we set $M^u_R = g_u M_R \cap \pi^{-1}(T^u)$. We observe that $\pi(M^u_R) \subseteq B^u_R$ since π is 1–Lipschitz.

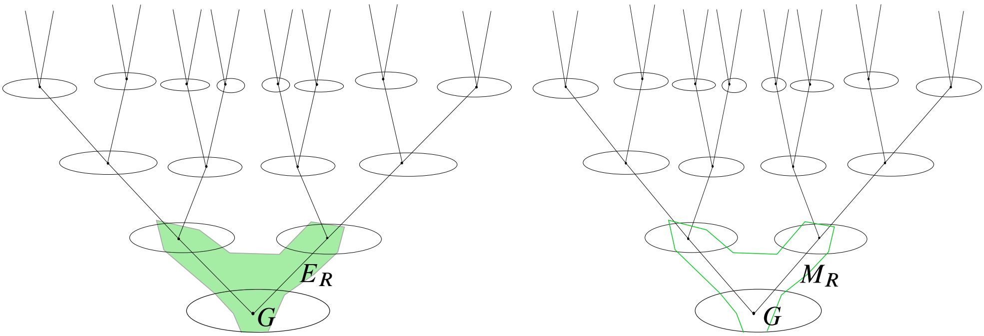

Figure 4: Left: an illustration of E_R. Right: an illustration of M_R.

Letting $u = g_u G$, we set $E_R = E\big(N_R(N \cup \phi(N))\big)$ and

$$E_R^u = g_u E_R \cap \pi^{-1}(T^u).$$

Obviously, $M_R^u \subseteq E_R^u \subseteq \pi^{-1}(B_R^u)$.

Convention We associate every $u \in T^0$ to an element $g_u \in \overline{G}$ such that

(i) $u = g_u G$, and

(ii) if the left normal form of g_u is $s_1 t^{\epsilon_1} s_2 t^{\epsilon_2} \cdots s_k t^{\epsilon_k} g$ then $g = 1_{\overline{G}}$.

We see that in this way we may define a bijective map from T^0 to the set $\mathcal{G}_T$ which consists of the elements of $\overline{G}$ such that conditions (i) and (ii) hold.

Proposition 2.6 *If $4 < 4R \le r$ and the distinct vertices $u, u' \in T^0$, satisfy $|u|, |u'| \in \{nr : n \in \mathbb{N}\}$, then*

$$d(M_R^u, M_R^{u'}) \ge 2R.$$

Proof We distinguish two cases. See the left and right parts of Figure 5 for cases 1 and 2, respectively.

Case 1 $(|u| \ne |u'|)$ Recall that every path γ in $C(\overline{G})$ projects to a path $\pi(\gamma)$ in the tree T. Then, since

$$M_R^u = g_u M_R \cap \pi^{-1}(T^u) \subseteq \pi^{-1}(B_R^u), \quad M_R^{u'} = g_{u'} M_R \cap \pi^{-1}(T^{u'}) \subseteq \pi^{-1}(B_R^{u'})$$

and π is 1–Lipschitz,

$$d(M_R^u, M_R^{u'}) \ge \bar{d}(B_R^u, B_R^{u'}) \ge r - R \ge 3R.$$

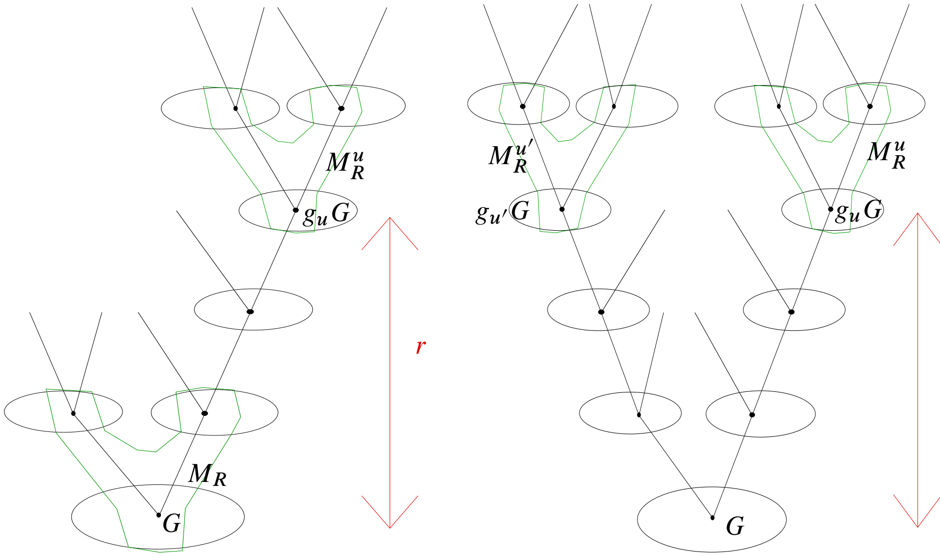

Figure 5: Left: an illustration of Case 1 of Proposition 2.6, where $u' = G$. Right: an illustration of Case 2 of Proposition 2.6.

Case 2 $(|u| = |u'|$ with $u \neq u')$ Denote by ζ_0 the last vertex of the common geodesic segment $[G, \zeta_0]$ of the geodesics $[G, u]$ and $[G, u']$. Observe that $\bar{d}(u, \zeta_0), \bar{d}(u', \zeta_0) \geq 1$. Let $x \in M_R^u$, $y \in M_R^{u'}$ and let γ be a geodesic from x to y. Then the path $\pi(\gamma)$ passes

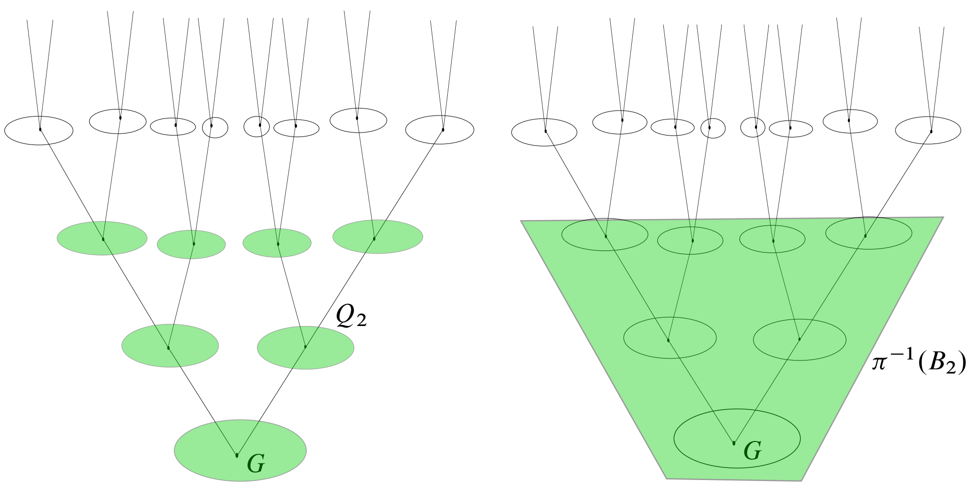

Figure 6: Left: an illustration of Q_m, for $m = 2$ (Proposition 2.8). We note that $Q_m = V(\pi^{-1}(B_r))$. Right: an illustration of $\pi^{-1}(B_r)$, where $r = 2$.

through the vertices u, u' and ζ_0, so the geodesic γ intersects both $g_u(N \cup \phi(N))$ and $g_{u'}(N \cup \phi(N))$. Hence

$$\begin{aligned} d(x,y) &\geq \operatorname{dist}\bigl(x, g_u(N \cup \phi(N))\bigr) + \operatorname{dist}\bigl(y, g_{u'}(N \cup \phi(N))\bigr) + \operatorname{length}([\zeta_0, u']) \\ &\qquad\qquad + \operatorname{length}([\zeta_0, u]) \\ &\geq R + R + 2 = 2(R+1). \end{aligned}$$ □

For $w \in G*_N$, we denote by $\|w\|$ the distance from w to $1_{\overline{G}}$ in the Cayley graph $\operatorname{Cay}(\overline{G}, \overline{S})$.

Lemma 2.7 *Let $w = gt^{\epsilon_1}s_1t^{\epsilon_2}s_2\cdots t^{\epsilon_k}s_k$ be the normal form of w. Then*

$$\|w\| \geq d(s_k, N) \quad \text{if } \epsilon_k = 1 \qquad \text{and} \qquad \|w\| \geq d(s_k, \phi(N)) \quad \text{if } \epsilon_k = -1.$$

Proof Without loss of generality we assume that $\epsilon_k = 1$. Let

$$w = \Bigl(\prod_{i_0=1}^{m_0} s_{i_0}\Bigr)t^{\epsilon_1}\Bigl(\prod_{i_1=1}^{m_1} s_{i_1}\Bigr)t^{\epsilon_2}\cdots t\Bigl(\prod_{i_k=1}^{m_k} s_{i_k}\Bigr)$$

be a shortest presentation of w in the alphabet $\overline{S}$ (we note that $s_{i_j} \notin \{t, t^{-1}\}$). We set $\prod_{i_j=1}^{m_j} s_{i_j} = g_j$ for every $j \in \{1, \ldots, k\}$. Then $w = gt^{\epsilon_1}g_1t^{\epsilon_2}s_2\cdots tg_k = w_0tg_k$. The first step when we rewrite w in normal form starting from the previous presentation is to write $g_k = ns_k$ (where $n \in N$). Then

$$\|w\| \geq \|g_k\| = \|ns_k\| = d(ns_k, 1) = d(s_k, n^{-1}) \geq d(s_k, N).$$ □

We note that there exists an amalgamated product analog of the following proposition, proved by Dranishnikov in [9]:

Proposition 2.8 *Suppose that* asdim $G \leq n$. *Let*

$$Q_m = \{w \in \overline{G} : l(w) \leq m\}.$$

Then asdim $Q_m \leq n$, *for every $m \in \mathbb{N}$.*

Proof We set $P_\lambda = \{w \in \overline{G} : l(w) = \lambda\}$. It is enough to show that asdim $P_\lambda \leq n$, for every $\lambda \in \mathbb{N}$. Indeed, since

$$Q_m = \bigcup_{i=0}^{m} P_i,$$

by the finite union theorem we obtain that asdim $Q_m \leq n$.

Claim *For $\lambda \in \mathbb{N}$ we have* asdim $P_\lambda \leq n$.

Proof We use induction on λ. We have $P_0 = G$, so asdim $P_0 \le n$. We observe that $P_\lambda \subseteq P_{\lambda-1}tG \cup P_{\lambda-1}t^{-1}G$. Using the finite union theorem it suffices to show that $\operatorname{asdim}(P_\lambda \cap P_{\lambda-1}tG) \le n$ and $\operatorname{asdim}(P_\lambda \cap P_{\lambda-1}t^{-1}G) \le n$; we show the first.

To show that asdim $P_\lambda \cap P_{\lambda-1}tG \le n$, we use the infinite union theorem. For $r > 0$ we set $Y_r = P_{\lambda-1}tN_r(N)$. We claim that

$$Y_r \subseteq N_{r+1}(P_{\lambda-1}).$$

Indeed, if $z \in Y_r$ then $z = z_0 t z_1$, where $z_0 \in P_{\lambda-1}$ and $z_1 \in N_r(N)$. Since $z_1 \in N_r(N)$, there exists $n \in N$ with $d(n, z_1) \le$ r. So $z = z_0 t n n^{-1} z_1 = z_0 \phi(n) t n^{-1} z_1$, and

$$d(z, P_{\lambda-1}) \le d(z, z_0\phi(n)) = \|tn^{-1}z_1\| \le \|t\| + \|t^{-1}z_1\| \le 1 + r.$$

Hence Y_r and $P_{\lambda-1}$ are quasi-isomorphic, so asdim $Y_r \le n$. □

We consider the family xtG where $x \in P_{\lambda-1}$. For $xtG \ne ytG$, we have

$$d(xtG \setminus Y_r, ytG \setminus Y_r) = d(xtg, yth) = \|g^{-1}t^{-1}x^{-1}yth\|,$$

where $g, h \in G \setminus N_r(N)$. The first step when we rewrite $g^{-1}t^{-1}x^{-1}yth$ in normal form is to make the substitution $h = ns_k$, where $n \in N$ and $s_k \in S_N$, so $g^{-1}t^{-1}x^{-1}yth = g^{-1}t^{-1}x^{-1}y\phi(n)ts_k$. Since $h \in G \setminus N_r(N)$, we have $\|s_k\| = \|n^{-1}h\| \ge d(h, N) \ge r$.

By Lemma 2.7 we obtain that $\|g^{-1}t^{-1}x^{-1}y\phi(n)ts_k\| \ge \|s_k\| \ge r$.

Finally, by observing that xtG and G are isometric, we deduce that $\operatorname{asdim}(xtG) \le n$ uniformly. Since all the conditions of the infinite union theorem hold,

$$\operatorname{asdim}(P_\lambda \cap P_{\lambda-1}tG) \le n$$

for every $\lambda \in \mathbb{N}$. □

We observe that $E(Q_m) = \pi^{-1}(B_m^T)$ and $Q_m = \overline{G} \cap \pi^{-1}(B_m^T)$.

For $w \in \overline{G}$, we set $T^w = T^{\pi(w)}$, where $\pi(w) = wG$.

We note that there was an attempt to prove the following theorem in [16], however, there is a gap in that proof. We give a few details about this gap right after the proof of Theorem 2.9.

Theorem 2.9 *Let $G*_N$ be an HNN-extension of the finitely generated group G over N. Then*

$$\operatorname{asdim} G*_N \le \max\{\operatorname{asdim} G, \operatorname{asdim} N + 1\}.$$

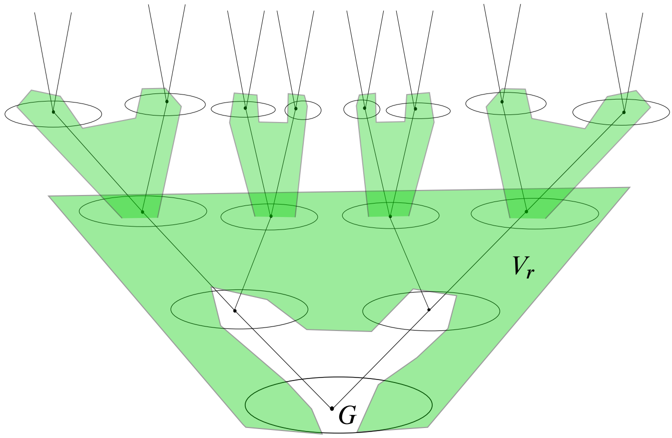

Figure 7: An illustration of V_r.

Proof Let $n = \max\{\operatorname{asdim} G, \operatorname{asdim} N + 1\}$. We denote by $\pi : C(\overline{G}, S) \to T$ the map of Lemma 2.5. We recall that we denote by $l(g)$ the length of the normal form of g.

We use the partition theorem (Theorem 2.4). Let $R, r \in \mathbb{N}$ be such that $R > 1$ and $r > 4R$.

We set

$$U_r = E\Big[\big(\pi^{-1}(B^T_{r-1}) \cap E\big(\{g \in \overline{G} : d(g, N \cup \phi(N)) \geq R\}\big)\big) \cup \Big(\bigcup_{u \in \partial B^T_r} E^u_R\Big)\Big],$$

where $E^u_R = g_u E\big(N_R(N \cup \phi(N))\big) \cap \pi^{-1}(T^u)$. We recall that

$$M_R = \{g \in \overline{G} : d(g, N \cup \phi(N)) = R\}.$$

Let A_R be the collection of the edges between the elements of $M_R \subseteq U_r$. We have that $A_R \subseteq U_r$. We define V_r to be the set obtained by removing the interior of the edges of A_R from U_r. Formally,

$$V_r = U_r \setminus \{\text{interior}(e) : e \in A_R\}.$$

See Figure 7 for an illustration of V_r. We observe that U_r and V_r are subgraphs of $C(\overline{G})$, $\partial U_r = \partial V_r$ and $V_r \cap \overline{G} = U_r \cap \overline{G}$. Obviously, $\bigcup_{u \in \partial B^T_r} E^u_R \subseteq V_r$. We also have

$$V_r \cap \overline{G} = \big(\overline{G} \cap \pi^{-1}(B^T_{r-1}) \cap E\big(\{g \in \overline{G} : d(g, N \cup \phi(N)) \geq R\}\big)\big) \cup \Big(\overline{G} \cap \bigcup_{u \in \partial B^T_r} E^u_R\Big).$$

To be more precise, $V_r \cap \overline{G}$ consists of those $wx \in \overline{G}$ such that $d(w, N \cup \phi(N)) \geq R$ and, if $w = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_k} g_k$ is the normal form of w, then either $k \leq r-1$, or $g_k = 1$ and $k = r$, while, if $x \neq 1$, then $k = r$, $g_k = 1$ and $d(x, N \cup \phi(N)) \leq R$.

For every vertex $u \in T^0$ satisfying $|u| \in \{nr : n \in \mathbb{N}\}$, we define

$$V_r^u = g_u V_r \cap \pi^{-1}(T^u).$$

Obviously, the sets V_r^u are subgraphs of $C(\overline{G})$ and $V_r^u \nsubseteq \overline{G}$. We observe that $V_r \subseteq \pi^{-1}(B_{r+R}^T)$, so $V_r^u \subseteq \pi^{-1}(B_{r+R}^u)$. Obviously, for every h such that $h = g_1 t^{\epsilon_1} g_2 \cdots t^{\epsilon_r}$ is the *left normal form* of h, $u \leq g_u hG$ and $|u| + r = |g_u hG|$, we have that

$$(g_u M_R \cap \pi^{-1}(T^{g_u G})) \cup (g_u h M_R \cap \pi^{-1}(T^{g_u hG})) \subseteq V_r^u.$$

We also observe that

$$g_u M_R \cap \pi^{-1}(T^{g_u G}) \subseteq \partial V_r^u,$$

which can also be written as

$$M_R^u = M_R^{g_u G} \subseteq \partial V_r^u.$$

We set $V_r^G = V_r$.

Consider the partition

$$C(\overline{G}) = \pi^{-1}(T) = \Biggl(\bigcup_{|u| \in \{nr : n \in \mathbb{N}_+ \cup \{0\}\}} V_r^u \Biggr) \cup E\bigl(N_R(N \cup \phi(N))\bigr). \tag{1}$$

We set

$$Z = \Biggl(\bigcup_{|u| \in \{nr : n \in \mathbb{N}_+\} \cup \{0\}} \partial V_r^u \Biggr) \cup \partial E\bigl(N_R(N \cup \phi(N))\bigr).$$

Observe that if $V_r^u \cap V_r^v \neq \varnothing$, then either $u \leq v$ and $|u| + r = |v|$ or $u \geq v$ and $|v| + r = |u|$. If $V_r^u \cap V_r^v \neq \varnothing$ is such that $u \leq v$ and $|u| + r = |v|$, then

$$V_r^u \cap V_r^v = M_R^v.$$

We deduce that

$$Z = \Biggl(\bigcup_{|u| \in \{nr : n \in \mathbb{N}_+\}} M_R^u \Biggr) \cup M_R.$$

We will show that there exists $d > 0$ such that (R, d)–dim $Z \leq n - 1$. Since M_R is quasi-isometric to $N_R(N \cup \phi(N))$, which is quasi-isometric to $N \cup \phi(N)$, we have that asdim $M_R \leq n - 1$. Then for $R > 0$ there exists an (R, d)–covering $\mathcal{U}$ of M_R with $\operatorname{ord}(\mathcal{U}) \leq n$. In view of Proposition 2.6, the covering

$$\mathcal{V} = \mathcal{U} \cup \bigcup_{|u| \in \{nr : n \in \mathbb{N}_+\}} (g_u \mathcal{U} \cap M_R^u)$$

is an (R, d)–covering of Z with $\operatorname{ord}(\mathcal{V}) \leq n$. We conclude that (R, d)–dim $Z \leq n - 1$.

Next, we will show that $\operatorname{asdim} V_r^u \le n$ and $\operatorname{asdim} N_R(N \cup \phi(N)) \le n$ uniformly. This will complete our proof, since all the conditions of the partition theorem are satisfied. It suffices to show that $\operatorname{asdim} V_r^u \le n$ uniformly and

$$\operatorname{asdim} N_R(N \cup \phi(N)) \le n.$$

We observe that $V_r \subseteq \pi^{-1}(B_{r+R}^T) \subseteq N_1(Q_{r+R})$, so by Proposition 2.8 we have that $\operatorname{asdim} V_r^u \le n$. Since there are at most two isometric classes for the sets V_r^u (for $u \neq G$) of our partition, we conclude that $\operatorname{asdim} V_r^u \le n$ uniformly. Finally, $\operatorname{asdim} N_R(N \cup \phi(N)) \le n-1$ since $N_R(N \cup \phi(N))$ is quasi-isometric to $N \cup \phi(N)$.

By the partition theorem (Theorem 2.4), $\operatorname{asdim} C(\overline{G}) = \operatorname{asdim} \pi^{-1}(T) \le n$. □

We now give a few details about the gap we found in [16]. We use the same notation as on pages 2276–2279 of [16]. We are not going to redefine all the symbols we use. We consider the HNN-extension $A*_C = \langle S_A \mid R_A, ct = tf(c)\rangle$ with respect to a monomorphism f. The idea was to construct a partition for $\pi^{-1}(K_1)$. The building block of this partition is the set $V_r = X_+ \cap \bigl(\bigcap_{|u|=r} X_-^u\bigr)$, where $u = g_u C$ are vertices of K_1.

The problem is that the set V_r is empty when the index $[A:C]$ is at least 2. One can find two vertices u and v of K_1 (where $|u| = |v| = r$) such that $X_-^v \cap X_-^u = \varnothing$. To see that, one must investigate how the dual graph K *behaves* under translations. For example, $K^u = g_u K_1$ when g_u (where u is a vertex of K_1) has a normal form ending with t, while $K^u = g_u K_0$ when g_u (where u is a vertex of K_1) has a normal form ending with t^{-1}.

2.1 Right-angled Artin groups

We use the following theorem of Bell, Dranishnikov and J Keesling; see [5].

Theorem 2.10 *If A and B are finitely generated groups then*

$$\operatorname{asdim} A * B = \max\{\operatorname{asdim} A, \operatorname{asdim} B\}.$$

Let Γ be a finite simplicial graph with n vertices. The *right-angled Artin group* (RAAG) $A(\Gamma)$ associated to the graph Γ has the presentation

$$A(\Gamma) = \langle \{s_u : u \in V(\Gamma)\} \mid \{[s_u, s_v] : [u, v] \in E(\Gamma)\}\rangle.$$

By $[s_u, s_v] = s_u s_v s_u^{-1} s_v^{-1}$ we mean the commutator. We set

$$\operatorname{Val}(\Gamma) = \max\{\operatorname{valency}(u) : u \in V(\Gamma)\}.$$

By valency(u) of a vertex u we denote the number of edges incident to the vertex u. Clearly $\mathrm{Val}(\Gamma) \le \mathrm{rank}(A(\Gamma)) - 1$.

If Γ is a simplicial graph, we denote by 1–skel(Γ) the 1–skeleton of Γ. Recall that a *full subgraph* of a graph Γ is a subgraph formed from a subset of vertices V and from all of the edges that have both endpoints in the subset V.

Conventions Let Γ be simplicial graph, $u \in V(\Gamma)$ and $e \in E(\Gamma)$. We denote by

(i) $\Gamma \setminus \{u\}$ the full subgraph of Γ formed from $V(\Gamma) \setminus \{u\}$,

(ii) $\Gamma \setminus e$ the subgraph of Γ such that $V(\Gamma \setminus e) = V(\Gamma)$ and $E(\Gamma \setminus e) = E(\Gamma) \setminus \{e\}$.

Lemma 2.11 *Let Γ be a finite simplicial graph. Then*

$$\mathrm{asdim}\, A(\Gamma) \le \mathrm{Val}(\Gamma) + 1.$$

Proof By Theorem 2.10, it suffices to prove Lemma 2.11 for connected simplicial graphs, so assume Γ is a connected simplicial graph. We use induction on $\mathrm{rank}(A(\Gamma))$. For $\mathrm{rank}(A(\Gamma)) = 1$ we have that $A(\Gamma)$ is $\mathbb{Z}$, so the statement holds. We assume that the statement holds for every $k \le n$ and we show that it holds for $n+1$ (for $n+1 \ge 2$).

Let Γ be a simplicial graph with $n+1$ vertices. We remove a vertex u from the graph Γ such that valency$(u) = \mathrm{Val}(\Gamma) = m \ge 1$. Let's denote by v_i (for $i \in \{1, \dots, m\}$) the vertices of Γ which are adjacent to u. We set $\Gamma' = \Gamma \setminus \{u\}$. Obviously, $\mathrm{Val}(\Gamma') \le \mathrm{Val}(\Gamma)$. We denote by Y the full subgraph of Γ formed from $\{v_1, \dots, v_m\}$.

We observe that the RAAG $A(\Gamma)$ is an HNN-extension of the RAAG $A(\Gamma')$. To be more precise, we have that

$$A(\Gamma) = A(\Gamma') *_{A(Y)}.$$

By Theorem 2.9 we obtain that

$$\mathrm{asdim}\, A(\Gamma) \le \max\{\mathrm{asdim}\, A(\Gamma'), \mathrm{asdim}\, A(Y) + 1\}.$$

We observe that $\mathrm{Val}(Y) \le \mathrm{Val}(\Gamma) - 1$, so by the inductive hypothesis $(\mathrm{rank}(A(Y)) \le n)$,

$$\mathrm{asdim}\, A(Y) \le \mathrm{Val}(Y) + 1 \le \mathrm{Val}(\Gamma).$$

Since $\mathrm{rank}\, A(\Gamma') = n$, again by the inductive hypothesis, we deduce that

$$\mathrm{asdim}\, A(\Gamma') \le \mathrm{Val}(\Gamma') + 1 \le \mathrm{Val}(\Gamma) + 1.$$

Combining the three previous inequalities, we obtain

$$\mathrm{asdim}\, A(\Gamma) \le \max\{\mathrm{Val}(\Gamma) + 1, \mathrm{Val}(\Gamma) + 1\} = \mathrm{Val}(\Gamma) + 1. \qquad \square$$

Using the previous lemma we can compute the exact asymptotic dimension of $A(\Gamma)$. We note that this has already been computed by Wright [23] using different methods.

We set

$$\mathrm{Sim}(\Gamma) = \max\{n : \Gamma \text{ contains the 1–skeleton of the standard } (n-1)\text{–simplex } \Delta^{n-1}\}.$$

Obviously if $\Gamma' \subseteq \Gamma$, then $\mathrm{Sim}(\Gamma') \leq \mathrm{Sim}(\Gamma)$.

Theorem 2.12 *Let* Γ *be a finite simplicial graph. Then*

$$\operatorname{asdim} A(\Gamma) = \mathrm{Sim}(\Gamma).$$

Proof By Theorem 2.10, it suffices to prove Theorem 2.12 for connected simplicial graphs, so assume Γ is a connected simplicial graph.

Claim 1 $$\mathrm{Sim}(\Gamma) \leq \operatorname{asdim} A(\Gamma).$$

Proof Let $\mathrm{Sim}(\Gamma) = n$. We observe that $\mathbb{Z}^n = A(S_{n-1}) \leq A(\Gamma)$. It follows that

$$n = \operatorname{asdim} \mathbb{Z}^n \leq \operatorname{asdim} A(\Gamma). \qquad \square$$

Claim 2 $$\operatorname{asdim} A(\Gamma) \leq \mathrm{Sim}(\Gamma).$$

Proof We use induction on $\mathrm{rank}(A(\Gamma))$. For $\mathrm{rank}(A(\Gamma)) = 1$ we have that $A(\Gamma)$ is $\mathbb{Z}$, so the statement holds. We assume that the statement holds for every $r \leq m$, and we show that holds for $m+1$ as well. Let Γ be a connected simplicial graph with $m+1$ vertices.

Let $\mathrm{Sim}(\Gamma) = n$. Then Γ contains the 1–skeleton of the standard $(n-1)$–simplex S_{n-1} (where $S_{n-1} = 1\text{–skel}(\Delta^{n-1})$).

Case 1 $(\Gamma = S_{n-1})$ Then $m+1 = n$, so by Lemma 2.11 we have $\operatorname{asdim} A(S_{n-1}) \leq \mathrm{Val}(S_{n-1}) + 1$. By observing that $\mathrm{Val}(S_{n-1}) = n-1$, we obtain that

$$\operatorname{asdim} A(S_{n-1}) \leq n = \mathrm{Sim}(\Gamma).$$

Case 2 $(S_{n-1} \subsetneq \Gamma)$ We will remove a vertex $u \in V(S_{n-1})$. Let's denote by v_i (for $i \in \{1, \dots, k\}$) the vertices of Γ which are adjacent to u. We set $\Gamma' = \Gamma \setminus \{u\}$. Obviously $\mathrm{Sim}(\Gamma') \leq n$. We denote by Y the full subgraph of Γ formed from $\{v_1, \dots, v_k\}$.

We observe that the RAAG $A(\Gamma)$ is an HNN-extension of the RAAG $A(\Gamma')$. To be more precise,

$$A(\Gamma) = A(\Gamma') *_{A(Y)} .$$

By Theorem 2.9 we obtain that

$$\operatorname{asdim} A(\Gamma) \leq \max\{\operatorname{asdim} A(\Gamma'), \operatorname{asdim} A(Y) + 1\}. \tag{2}$$

Since $\mathrm{Sim}(\Gamma') \le n$ and $\mathrm{rank}(\Gamma') \le m$, by the inductive assumption

$$\text{(3)} \qquad \operatorname{asdim} A(\Gamma') \le \mathrm{Sim}(\Gamma') \le n.$$

We observe that $\mathrm{Sim}(Y) \le n-1$ and $\mathrm{rank}(Y) \le m$. Then by the inductive hypothesis we obtain

$$\text{(4)} \qquad \operatorname{asdim} A(Y) + 1 \le \mathrm{Sim}(Y) + 1 \le n.$$

by (2), (3) and (4) we conclude that

$$\operatorname{asdim} A(\Gamma) \le n = \mathrm{Sim}(\Gamma). \qquad \square$$

3 Asymptotic dimension of one-relator groups

Theorem 3.1 *Let G be a finitely generated one-relator group. Then*

$$\operatorname{asdim} G \le 2.$$

Proof Let $G = \langle S \mid r \rangle$ be a presentation of G where S is finite and r is a cyclically reduced word in $S \cup S^{-1}$. To omit trivial cases, we assume that S contains at least two elements and $|r| > 0$ (we denote by $|r|$ the length of the relator r in the free group $F(S)$).

We may assume that every letter of S appears in r. Otherwise our group G is isomorphic to a free product $H * F$ of a finitely generated one-relator group H with relator r and generating set $S_H \subseteq S$ consisting of all letters which appear in r and a free group F with generating set the remaining letters of S. We recall that the asymptotic dimension of any finitely generated nonabelian free group is equal to one. Then $\operatorname{asdim} G = \max\{\operatorname{asdim} H, \operatorname{asdim} F\} = \max\{\operatorname{asdim} H, 1\}$; see [2].

We denote by $\epsilon_r(s)$ the exponent sum of a letter $s \in S$ in a word r, and by $\mathrm{oc}_r(s)$ the minimum number of the positions of appearance of the elements of the set

$$\{s^k \text{ for some } 0 \ne k \in \mathbb{Z}\}$$

in a cyclically reduced word r. For example, if $r = abcab^{10}a^{-2}c^{-1}$, then $\mathrm{oc}_r(a) = 3$, $\mathrm{oc}_r(b) = 2$, $\mathrm{oc}_r(c) = 2$ and $\epsilon_r(c) = 0$.

We observe that, if there exists $b \in S$ such that $\mathrm{oc}_r(b) = 1$, then the group G is free (see [15, Theorem 5.1, page 198]), so $\operatorname{asdim} G = 1$. From now on we assume that, for every $s \in S$, we have that $\mathrm{oc}_r(s) \ge 2$ (so $|r| \ge 4$).

The proof is by induction on the length of r. We observe that if $|r| = 4$ then the statement of the theorem holds, since by the result of Matsnev [17] we have that

$\operatorname{asdim} G \leq \frac{1}{2}\lfloor |r| \rfloor = \frac{4}{2} = 2$ (where $\lfloor * \rfloor$ is the floor function). We assume that the statement of the theorem holds for all one-relator groups with relator length smaller than or equal to $|r|-1$.

We follow the arguments from the book of Lyndon and Schupp (see [15, Theorem 5.1, page 198]) and the book of Wise (see [22, Construction 18.5]). We distinguish two cases.

Case 1 (there exists a letter $a \in S$ such that $\epsilon_r(a) = 0$) We shall exhibit G as an HNN-extension of a one-relator group G_1 whose defining relator has shorter length than r, over a finitely generated free subgroup F. Let $S = \{a = s_1, s_2, s_3, s_4, \dots, s_k\}$. Set $s_i^{(j)} = a^j s_i a^{-j}$ for $j \in \mathbb{Z}$ and for $k \geq i \geq 2$. Rewrite r, scanning it from left to right and changing any occurrence of $a^j s_i$ to $s_i^{(j)} a^j$, collecting the powers of adjacent a–letters together and continuing with the leftmost occurrence of a or its inverse in the modified word. We denote by r' the modified word in terms of $s_i^{(j)}$. We note that by doing this we make at least one cancellation of a and its inverse. The resulting word r', which represents r in terms of $s_i^{(j)}$ and their inverses, has length at most $|r|-2$. For example, if $r = a s_2 s_3 a s_2^4 a^{-2} s_3$ then $r' = s_2^{(1)} s_3^{(1)} (s_2^{(2)})^4 s_3^{(0)}$.

Let m and M be the minimal and the maximal superscripts, respectively, of all $s_i^{(j)}$ (for $i \geq 2$) occurring in r'. To be more precise,

$$m = \min\{j : s_i^{(j)} \text{ occurs in } r'\} \quad \text{and} \quad M = \max\{j : s_i^{(j)} \text{ occurs in } r'\}.$$

Continuing our example, $m = 0$ and $M = 2$.

Claim 1.1 *In Case 1 we have $M - m > 0$ and $m \leq 0 \leq M$.*

We may assume, replacing r with a suitable permutation if necessary, that r begins with a^k for some $k \neq 0$. Then we can write $r = a^k s w a^n t z$, where $k, n \neq 0$, $a \notin \{s, t\} \subseteq S$ and both a and a^{-1} do not appear in the word z ($\mathrm{oc}_z(a) = 0$). Then we observe that the letter s has as superscript k in the word r' while t has as superscript 0 in the word r'. Since $k \neq 0$, we have that $M - m > 0$. This completes the proof of Claim 1.1.

Claim 1.2 *The group G has a presentation*

$$\langle \{a, s_i^{(j)} : i = 2, \dots, k,\ j = m, \dots, M\} \mid \{r', a s_i^{(j')} a^{-1} (s_i^{(j'+1)})^{-1} : j' = m, \dots, M-1\} \rangle.$$

To verify the claim, let H be the group defined by the presentation given above. The map $\phi\colon G \to H$ defined by

$$a \mapsto a, \quad s_i \mapsto s_i^{(0)}$$

is a homomorphism since $\phi(r) = r'$. On the other hand, the map $\psi\colon H \to G$ defined by
$$a \mapsto a, \quad s_i^{(j)} \mapsto a^j s_i a^{-j}$$
is also a homomorphism since all relators of H are sent to 1_G.

It is easy to verify that $\psi \circ \phi$ is the identity map of G. The homomorphism $\phi \circ \psi\colon H \to H$ maps $a \mapsto a$, $s_i^{(0)} \mapsto s_i \mapsto s_i^{(0)}$ and $s_i^{(j)} \mapsto a^j s_i a^{-j} \mapsto a^j s_i^{(0)} a^{-j}$. Now we show that $s_i^{(j)} = a^j s_i^{(0)} a^{-j}$. We have
$$a^1 s_i^{(0)} a^{-1} = s_i^{(1)}, \quad a^1 s_i^{(1)} a^{-1} = s_i^{(2)}, \dots \quad a^1 s_i^{(j-1)} a^{-1} = s_i^{(j)}.$$
Combining these equations $s_i^{(j)} = a^1 s_i^{(j-1)} a^{-1} = a^2 s_i^{(j-2)} a^{-2} = \cdots = a^j s_i^{(0)} a^{-j}$, so $\phi \circ \psi = \mathrm{id}_H$.

Since $\phi \circ \psi$ and $\psi \circ \phi$ are the identity maps on H and G, respectively, we deduce that ϕ is an isomorphism. This completes the proof of Claim 1.2.

We set
$$G_1 = \langle \{s_i^{(j)} : i = 2, \dots, k,\ j = m, \dots, M\} \mid r' \rangle,$$
and note that there exists a letter $s_{i_m} \in S$ such that $s_{i_m}^{(m)}$ appears in r' and a letter $s_{i_M} \in S$ such that $s_{i_M}^{(M)}$ appears in r'.

Let F and Λ be the subgroups of G_1 generated by
$$X = \{s_i^{(j)} : i = 2, \dots, k,\ j = m, \dots, M-1\}$$
and
$$Y = \{s_i^{(j)} : i = 2, \dots, k,\ j = m+1, \dots, M\},$$
respectively.

Claim 1.3 *The groups F and Λ are free subgroups of G_1.*

This claim follows by the Freiheitssatz (see [15, Theorem 5.1, page 198]); since X omits a generator of G_1 occurring in r' (this is the letter $s_{i_M}^{(M)}$) the subgroup F is free. The same holds for Λ, since Y omits the letter $s_{i_m}^{(m)}$.

Claim 1.4 *We have that $G \simeq G_1 *_F$.*

In particular, the map $s_i^{(j)} \mapsto s_i^{(j+1)}$ from X to Y extends to an isomorphism from F to Λ. Thus H is exhibited as the HNN-extension of G_1 over the finitely generated free group F using a as a stable letter. Since $G \simeq H$ (Claim 1.2),
$$G \simeq G_1 *_F.$$

By the fact that $|r'| < |r|$ and the inductive assumption we have that $\operatorname{asdim} G_1 \leq 2$. To conclude, we apply the inequality for HNN-extensions (Theorem 2.9): $\operatorname{asdim} G \leq \max\{\operatorname{asdim} G_1, \operatorname{asdim} F + 1\} = \max\{\operatorname{asdim} G_1, 2\} = 2$.

Case 2 ($|\epsilon_r(s)| \geq 1$ for every letter $s \in S$) Let $S = \{a = s_1, b = s_2, s_3, s_4, \dots, s_k\}$ and $S_1 = \{t, x, s_i : 3 \leq i \leq k\}$. We consider the homomorphism between the free group $F(S)$ and the free group $F(S_1)$

$$\phi\colon a \mapsto t^{-\epsilon_r(b)}x, \quad b \mapsto t^{\epsilon_r(a)}, \quad s_i \mapsto s_i \quad \text{for } 3 \leq i \leq k. \tag{5}$$

We set

$$\Gamma = \langle S_1 \mid r(t, x, s_3, \dots, s_k) \rangle,$$

where we denote by $r(t, x, s_3, \dots, s_k)$ the modified word in terms of t, x and s_i for $3 \leq i \leq k$ which is obtained from r when we replace a generator s with $\phi(s)$. Then ϕ induces a homomorphism

$$\phi\colon G \to \Gamma.$$

The following claim shows that the homomorphism ϕ is actually a monomorphism into Γ, so we have an embedding of G into Γ via ϕ:

Claim 2.1 *The homomorphism $\phi\colon G \to \Gamma$ is a monomorphism.*

Proof We set $S_2 = \{a, t, s_i : 3 \leq i \leq k\}$ and $S_1 = \{x, t, s_i : 3 \leq i \leq k\}$. We define $g\colon F(S) \to F(S_2)$ and $f\colon F(S_2) \to F(S_1)$ by

$$g\colon a \mapsto a, \quad b \mapsto t^{\epsilon_r(a)}, \quad s_i \mapsto s_i \quad \text{for } 3 \leq i \leq k,$$

and

$$f\colon a \mapsto t^{-\epsilon_r(b)}x, \quad t \mapsto t, \quad s_i \mapsto s_i \quad \text{for } 3 \leq i \leq k.$$

We set $r_2 = g(r)$, $G_2 = \langle S_2 \mid r_2 \rangle$ and $r_1 = f \circ g(r) = r(t, x, s_3, \dots, s_k)$, and we observe that $\Gamma = \langle S_1 \mid r_1 \rangle$. Then g induces a homomorphism $\bar{g}\colon G \mapsto G_2$ and f induces a homomorphism $\bar{f}\colon G_2 \mapsto \Gamma$. Obviously, $\phi = \bar{f} \circ \bar{g}$.

We can easily see that $\bar{f}$ is an isomorphism. Indeed, the homomorphism $\psi\colon \Gamma \to G_2$ given by

$$x \mapsto t^{\epsilon_r(b)}a, \quad t \mapsto t, \quad s_i \mapsto s_i \quad \text{for } 3 \leq i \leq k$$

is the inverse homomorphism of $\bar{f}$.

It is enough to prove that $\bar{g}$ is a monomorphism. This follows by the fact that the group G_2 is the amalgamated product $G *_{\mathbb{Z}} \langle t \rangle$, where $\mathbb{Z} = \langle \lambda \rangle$, and $\psi_1(\lambda) = b$ and $\psi_2(\lambda) = t^{\epsilon_r(a)}$ are the corresponding monomorphisms. We can see that $\bar{g}$ is the inclusion of G into the amalgamated product, so $\bar{g}$ is injective. □

We denote by $r(t,x,s_3,\dots,s_k)$ the modified word in terms of t, x and s_i for $3\le i\le k$ which can be obtained from r when we replace a generator s with $\phi(s)$ and p with the cyclically reduced $r(t,x,s_3,\dots,s_k)$. We observe that $\epsilon_p(t)=0$ and that x occurs in p.

If the letter t occurs in the word p, from Case 1 we have that Γ is an HNN-extension of some group H over a free subgroup F, namely $\Gamma = H*_F$. As in Case 1, by assuming that p starts with t or t^{-1} we introduce new variables $s_i^{(j)} = t^j s_i t^{-j}$. Using these variables, we rewrite p as a word w, eliminating all occurrences of t and its inverse. Then we observe that $|w|\le |r|-1$. By using the inductive assumption for w we obtain

$$\operatorname{asdim} G \le \operatorname{asdim}\Gamma \le 2.$$

If the letter t does not occur in the word p, we observe that

$$|p|\le |r|-1.$$

Then

$$\Gamma = \langle t\rangle * \Gamma',$$

where

$$\Gamma' = \langle \{x, s_i : i=3,\dots,k\} \mid p\rangle.$$

Since $\operatorname{asdim}(G_1 * G_2) = \max\{\operatorname{asdim} G_1, \operatorname{asdim} G_2\}$ holds (see [2]) we have that

$$\operatorname{asdim}\Gamma = \max\{1, \operatorname{asdim}\Gamma'\}.$$

Then, by the inductive assumption for p, $\operatorname{asdim}\Gamma' \le 2$. Finally, we conclude that

$$\operatorname{asdim} G \le \operatorname{asdim}\Gamma \le 2. \qquad \square$$

3.1 One-relator groups with asymptotic dimension two

We recall that a nontrivial group H is *freely indecomposable* if H cannot be expressed as a free product of two nontrivial groups.

A natural question derived from Theorem 3.1 is which one-relator groups have asymptotic dimension two. In this subsection, we will show that the asymptotic dimension of every finitely generated one-relator group that is not a free group or a free product of a free group and a finite cyclic group is exactly two.

We will use Propositions 3.2 and 3.3 from [10] and [21], respectively.

Proposition 3.2 *Let G be an infinite finitely generated one-relator group with torsion. If G has more than one end, then G is a free product of a nontrivial free group and a freely indecomposable one-relator group.*

Proposition 3.3 *Let G be a torsion-free infinite finitely generated group. If G is virtually free, then it is free.*

Lemma 3.4 *Let G be an infinite finitely generated one-relator group that is not a free group or a free product of a nontrivial free group and a freely indecomposable one-relator group. Then G is not virtually free.*

Proof If G has torsion, by Proposition 3.2 G has exactly one end, so G cannot be virtually free. If G is torsion free, by Proposition 3.3 we obtain that G is free and this is a contradiction by the assumption of the lemma. □

We note that every finite one-relator group is cyclic. To see this, it is enough to observe that every one-relator group with at least two generators has infinite abelianization.

The following proposition is the main result of this subsection:

Proposition 3.5 *Let G be a finitely generated one-relator group that is not a free group or a free product of a free group and a finite cyclic group. Then*

$$\operatorname{asdim} G = 2.$$

Proof By Theorem 3.1, $\operatorname{asdim} G \le 2$. If G is finite then it is cyclic. If G is infinite, $1 \le \operatorname{asdim} G$. By a theorem of T Gentimis [11], $\operatorname{asdim} G = 1$ if and only if G is virtually free. We assume that G is an infinite virtually free group. So, if G is torsion free, then by Proposition 3.3 we obtain that G is free. If G has torsion, then, by Lemma 3.4, G is a free product of a nontrivial free group and a freely indecomposable one-relator group G_1. Observe that, if G_1 is an infinite noncyclic group, then by the same lemma G_1 is not virtually free, so G is not virtually free either, which is a contradiction. We conclude that $\operatorname{asdim} G = 2$. □

Corollary *Let G be a finitely generated freely indecomposable one-relator group which is not cyclic. Then*

$$\operatorname{asdim} G = 2.$$

Proposition 3.6 [15, Proposition 5.13, page 107] *Let $G = \langle x_1, \dots, x_n \mid r \rangle$ be a finitely generated one-relator group, where r is of minimal length under $\operatorname{Aut}(F(\{x_1, \dots, x_n\}))$ and contains exactly the generators $x_1, \dots, x_k$ for some k with $0 \le k \le n$. Then G is isomorphic to the free product $G_1 * G_2$, where $G_1 = \langle x_1, \dots, x_k \mid r \rangle$ is freely indecomposable and G_2 is free with basis $\{x_{k+1}, \dots, x_n\}$.*

The above results combine to give the following corollary, which describes the finitely generated one-relator groups:

Corollary *Let G be a finitely generated one-relator group. Then one of the following is true*:

(i) *G is finite cyclic*, *and* $\operatorname{asdim} G = 0$.

(ii) *G is a nontrivial free group or a free product of a nontrivial free group and a finite cyclic group*, *and* $\operatorname{asdim} G = 1$.

(iii) *G is an infinite freely indecomposable not cyclic group or a free product of a nontrivial free group and an infinite freely indecomposable noncyclic group*, *and* $\operatorname{asdim} G = 2$.

We can further describe the boundaries of hyperbolic one-relator groups. We recall the following result of Buyalo and Lebedeva (see [7]) for hyperbolic groups:

$$\operatorname{asdim} G = \dim \partial_\infty G + 1.$$

Let G be an infinite finitely generated hyperbolic one-relator group that is not virtually free. By Gentimis [11] we obtain that $\operatorname{asdim} G \neq 1$, so $\operatorname{asdim} G = 2$. Using the previous equality we obtain that G has one-dimensional boundary. Applying a theorem of Kapovich and Kleiner (see [13]) we can describe the boundaries of hyperbolic one-relator groups:

Proposition 3.7 *Let G be a hyperbolic one-relator group. Then* $\operatorname{asdim} G = 0, 1$ *or* 2.

(i) *If* $\operatorname{asdim} G = 0$, *then G is finite.*

(ii) *If* $\operatorname{asdim} G = 1$, *then G is virtually free and the boundary is a Cantor set.*

(iii) *If* $\operatorname{asdim} G = 2$, *providing that G does not split over a virtually cyclic subgroup*, *then one of the following holds*:

   (a) *$\partial_\infty G$ is a Menger curve.*

   (b) *$\partial_\infty G$ is a Sierpinski carpet.*

   (c) *$\partial_\infty G$ is homeomorphic to S^1.*

4 Graphs of groups

We will prove a general theorem for the asymptotic dimension of fundamental groups of finite graphs of groups.

Theorem 4.1 *Let $(\mathbb{G}, Y)$ be a finite graph of groups with vertex groups $\{G_v : v \in Y^0\}$ and edge groups $\{G_e : e \in Y^1_+\}$. Then*

$$\operatorname{asdim} \pi_1(\mathbb{G}, Y, \mathbb{T}) \le \max_{v \in Y^0, e \in Y^1_+} \{\operatorname{asdim} G_v, \operatorname{asdim} G_e + 1\}.$$

Proof We use induction on the number $\#E(Y)$ of edges of the graph Y. For $\#E(Y) = 1$ we distinguish two cases. The first case is when the fundamental group $\pi_1(\mathbb{G}, Y, \mathbb{T})$ is an amalgamated product. Here the theorem follows by the inequality of Dranishnikov (see [9])

$$\operatorname{asdim} A *_C B \le \max\{\operatorname{asdim} A, \operatorname{asdim} B, \operatorname{asdim} C + 1\}.$$

The second case is when the fundamental group $\pi_1(\mathbb{G}, Y, \mathbb{T})$ is an HNN-extension. Here the theorem follows by Theorem 2.9.

We assume that the theorem holds for $E(Y) \le m$. Let $(\mathbb{G}, Y)$ be a finite graph of groups with $\#E(Y) = m + 1$. We denote by $\mathbb{T}$ a maximal tree of Y.

We distinguish two cases:

Case 1 $(Y = \mathbb{T})$ We remove a terminal edge $e' = [v, u]$ from the graph Y so that the full subgraph of Y, denoted by Γ and formed from the vertices $V(Y) \setminus \{u\}$, is connected. We observe that Γ is also a tree, which we denote by $\mathbb{T}'$.

Then $\pi_1(\mathbb{G}, Y, \mathbb{T}) = \pi_1(\mathbb{G}, \Gamma, \mathbb{T}') *_{G_{e'}} G_u$, so by the inequality for amalgamated products of Dranishnikov (see [9]),

$$\operatorname{asdim} \pi_1(\mathbb{G}, Y, \mathbb{T}) \le \max\{\operatorname{asdim} \pi_1(\mathbb{G}, \Gamma, \mathbb{T}'), \operatorname{asdim} G_u, \operatorname{asdim} G_{e'} + 1\}.$$

Since $\#E(\Gamma) = m$, by the inductive assumption we obtain that

$$\operatorname{asdim} \pi_1(\mathbb{G}, \Gamma, \mathbb{T}') \le \max_{v \in Y^0 \setminus \{u\}, e \in Y^1_+ \setminus \{e'\}} \{\operatorname{asdim} G_v, \operatorname{asdim} G_e + 1\},$$

so

$$\operatorname{asdim} \pi_1(\mathbb{G}, Y, \mathbb{T}) \le \max_{v \in Y^0, e \in Y^1_+} \{\operatorname{asdim} G_v, \operatorname{asdim} G_e + 1\}.$$

Case 2 $(\mathbb{T} \subsetneq Y)$ We remove from Y an edge $e' = [v, u]$ which doesn't belong to $\mathbb{T}$. Since the tree $\mathbb{T}$ is a maximal tree of Y and $e' \notin E(\mathbb{T})$, we have that the graph $\Gamma = Y \setminus e'$ is connected and $\mathbb{T} \subseteq \Gamma$. Then $\pi_1(\mathbb{G}, Y, \mathbb{T}) = \pi_1(\mathbb{G}, \Gamma, \mathbb{T}) *_{G_{e'}}$, so by the inequality for HNN-extensions (Theorem 2.9) we have

$$\operatorname{asdim} \pi_1(\mathbb{G}, Y, \mathbb{T}) \le \max\{\operatorname{asdim} \pi_1(\mathbb{G}, \Gamma, \mathbb{T}), \operatorname{asdim} G_{e'} + 1\}.$$

Since $\#E(\Gamma)=m$, by the inductive assumption

$$\operatorname{asdim}\pi_1(\mathbb{G},\Gamma,\mathbb{T})\le \max_{v\in Y^0,e\in Y^1_+\setminus\{e'\}}\{\operatorname{asdim}G_v,\operatorname{asdim}G_e+1\},$$

so

$$\operatorname{asdim}\pi_1(\mathbb{G},Y,\mathbb{T})\le \max_{v\in Y^0,e\in Y^1_+}\{\operatorname{asdim}G_v,\operatorname{asdim}G_e+1\}. \qquad \square$$

We obtain as a corollary the following:

Proposition 4.2 *Let $(\mathbb{G},Y)$ be a finite graph of groups with vertex groups $\{G_v : v\in Y^0\}$ and edge groups $\{G_e : e\in Y^1_+\}$. We assume that*

$$\max_{e\in Y^1_+}\{\operatorname{asdim}G_e\}<\max_{v\in Y^0}\{\operatorname{asdim}G_v\}=n.$$

Then $\operatorname{asdim}\pi_1(\mathbb{G},Y,\mathbb{T})=n$.

As a corollary of Proposition 4.2, the asymptotic dimension of a graph of one-ended hyperbolic groups with n–dimensional boundary with free edge groups is $n+1$.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 23 Issue 8 (pages 3417–3908) 2023

Partial Torelli groups and homological stability 3417

ANDREW PUTMAN

On symplectic fillings of small Seifert 3–manifolds 3497

HAKHO CHOI and JONGIL PARK

Milnor–Witt motivic cohomology of complements of hyperplane arrangements 3531

KEYAO PENG

Connective models for topological modular forms of level n 3553

LENNART MEIER

Asymptotic dimension of graphs of groups and one-relator groups 3587

PANAGIOTIS TSELEKIDIS

Pressure metrics for deformation spaces of quasifuchsian groups with parabolics 3615

HARRISON BRAY, RICHARD CANARY and LIEN-YUNG KAO

The $\mathrm{Sp}_{k,n}$–local stable homotopy category 3655

DREW HEARD

Smooth one-dimensional topological field theories are vector bundles with connection 3707

DANIEL BERWICK-EVANS and DMITRI PAVLOV

Round fold maps on 3–manifolds 3745

NAOKI KITAZAWA and OSAMU SAEKI

The upsilon invariant at 1 of 3–braid knots 3763

PAULA TRUÖL

Cusps and commensurability classes of hyperbolic 4–manifolds 3805

CONNOR SELL

The group of quasi-isometries of the real line cannot act effectively on the line 3835

SHENGKUI YE and YANXIN ZHAO

Simplicial model structures on pro-categories 3849

THOMAS BLOM and IEKE MOERDIJK