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We complement the previous computation of the Chow–Witt rings of classifying spaces of special linear groups by an analogous computation for the general linear groups. This case involves discussion of nontrivial dualities. The computation proceeds along the lines of the classical computation of the integral cohomology of $BO(n)$ with local coefficients, as done by Čadek. The computations of Chow–Witt rings of classifying spaces of GL_n are then used to compute the Chow–Witt rings of the finite Grassmannians. As before, the formulas are close parallels of the formulas describing integral cohomology rings of real Grassmannians.

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1 Introduction

The computation of the Chow ring of Grassmannians is fundamental in algebraic geometry. The computation for finite Grassmannians provides the basis of Schubert calculus and its applications in enumerative geometry, while the computation for the infinite Grassmannians (ie the classifying space $B\mathrm{GL}_n$ of vector bundles) describes the characteristic classes of vector bundles in terms of Chern classes. Given any cohomology theory, one can ask for similar computations and how these computations provide information on characteristic classes of vector bundles (in the case of infinite Grassmannians) or provide variations of Schubert calculus with relevance for combinatorics and representation theory (in the case of finite Grassmannians). Indeed, many such investigations have been done in recent years for algebraic versions of complex-oriented cohomology theories. On the other hand, several cohomology theories have recently been considered which detect aspects related to real algebraic geometry and the theory of quadratic forms; for example the Chow–Witt rings $\widetilde{\mathrm{CH}}^\bullet(X)$ introduced by Barge and Morel in [7] and studied in depth by Fasel — see eg [12; 13]. Other strongly related examples are given by the cohomology $H^\bullet(X, \mathbf{W})$ of the sheaf of Witt rings $\boldsymbol{W}$ on the small (Zariski or Nisnevich) site of a smooth scheme X — appearing, for example, in the study of the Gersten–Witt spectral sequence converging to Witt groups of X by Balmer and Walter [6] — and the cohomology $H^\bullet(X, \boldsymbol{I}^\bullet)$ of the sheaves $\boldsymbol{I}^n$ of powers of the fundamental ideal in the Witt ring (which by work of Jacobson [18] is strongly related to singular cohomology of the real realization). The goal of the present paper is to compute these cohomology theories for the (finite and infinite) Grassmannians. As mentioned above, the case of infinite Grassmannians serves to better understand the relevant characteristic classes of vector bundles which may be relevant for splitting questions — as in the original work of Barge and Morel [7] on Euler class obstructions for splitting, or more recently in the work of Asok and Fasel, eg [3; 4]. The case of finite Grassmannian allows to set up a variant of Schubert calculus combining features from complex and real Grassmannians, see [28], which could be useful for refined or $\mathbb{A}^1$–enumerative questions over general fields as considered recently by Kass and Wickelgren in [19], Levine in [20], and others.

The present paper is a sequel to work with Hornbostel [16], which computed the Chow–Witt rings of classifying spaces of the symplectic and special linear groups. We provide a similar computation of the total Chow–Witt ring of $B\mathrm{GL}_n$, essentially by a combination and extension of the techniques developed in [16] and Čadek's computation of cohomology of $BO(n)$ with twisted coefficients in [26]. Once the characteristic classes for vector bundles and their relations are known, we are also able to compute the Chow–Witt rings of the finite Grassmannians. It turns out that the formulas describing the $\boldsymbol{I}$–cohomology of Grassmannians are direct analogues of the classical formulas for integral cohomology of real Grassmannians, so the Chow–Witt rings capture information from the singular cohomology of both real and complex Grassmannians (via the $\boldsymbol{I}$–cohomology and Chow ring components, respectively).

1.1 Chow–Witt rings of infinite Grassmannians

We first formulate the results for the infinite Grassmannians $B\mathrm{GL}_n$. As these are not smooth schemes, there are two choices for talking about their cohomology — either using finite-dimensional approximations by smooth schemes (as in Totaro's approach to Chow groups of classifying spaces [24]) or in the framework of the motivic homotopy category of Morel and Voevodsky [22] (in which all the cohomology theories considered here happen to be representable). Both approaches yield equivalent results, but we adopt the former point of view for the present paper; see also the beginning of Section 3 for a slightly more detailed discussion.

Before going into a detailed description of the Chow–Witt ring of $B\mathrm{GL}_n$, we need to introduce some notation; see also the more detailed discussions around the relevant cohomology theories in Section 2. The ultimate goal is the computation of the Chow–Witt ring $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n)$ which is a graded algebra over the Grothendieck–Witt ring $\mathrm{GW}(F)$ of quadratic forms over the base field F. The Chow–Witt ring combines two pieces of information, via a cartesian square in point (1) of Theorem 1.1: one piece, described in point (2) of the theorem, comes from the Chow ring and is related to complex Grassmannians; the other piece, described in point (3), is the $\boldsymbol{I}$–cohomology ring which is related to the real Grassmannian. Finally, the two pieces are glued together by means of maps from both pieces to the mod 2 Chow ring $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$ which are described in point (4) of the theorem.

The main new computations in this paper concern the $\boldsymbol{I}$–cohomology ring as a graded algebra over the Witt ring $W(F)$ of quadratic forms (obtained from $\mathrm{GW}(F)$ as a quotient by the ideal generated by the hyperbolic form). Both the Chow–Witt ring and the $\boldsymbol{I}$–cohomology ring involve possible twists by line bundles. These are related to orientability questions and are in some ways similar to the cohomology with local coefficients for real manifolds. There is consequently an additional grading by the mod 2 Picard group for the Chow–Witt and $\boldsymbol{I}$–cohomology ring, since tensor squares of line bundles do not change isomorphism classes of the cohomology groups. In the particular case of the Grassmannians (both finite and infinite), there are essentially only two line bundles, or dualities, to consider: the trivial duality and the nontrivial one given by the (dual of the) determinant $\det\gamma_n^\vee$ of the universal rank n bundle γ_n. Thus, the Chow–Witt and $\boldsymbol{I}$–cohomology rings of $B\mathrm{GL}_n$ are graded algebras with grading by $\mathbb{Z}\oplus\mathrm{Pic}(B\mathrm{GL}_n)/2\cong\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. They are in fact graded-commutative (in the sense that the correction factor for switching cup product factors is determined by the cohomological degrees of the cohomology classes). For a detailed discussion of subtleties in the graded-commutativity of total $\boldsymbol{I}$–cohomology or Chow–Witt rings, see Remark 2.3.

As a last piece of notation, it turns out to be convenient for the description of the Chow–Witt rings to also upgrade the (integral and mod 2) Chow ring to a $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ algebra by defining a product on

$$\mathrm{CH}^\bullet(B\mathrm{GL}_n)^{\oplus 2}=\{(\alpha,\mathscr{L})\mid \alpha\in\mathrm{CH}^\bullet(B\mathrm{GL}_n),\mathscr{L}\in\mathrm{Pic}(B\mathrm{GL}_n)/2\}$$

by $(\alpha,\mathscr{L})\cdot(\beta,\mathscr{L}'):=(\alpha\cup\beta,\mathscr{L}\otimes\mathscr{L}')$, ie by taking intersection products from the Chow ring combined with the group structure of $\mathrm{Pic}(B\mathrm{GL}_n)/2$. With this definition, the reduction map

$$\rho\colon H^q(B\mathrm{GL}_n,\boldsymbol{I}^q)\oplus H^q(B\mathrm{GL}_n,\boldsymbol{I}^q(\det\gamma_n^\vee))\to\mathrm{Ch}^q(B\mathrm{GL}_n)^{\oplus 2}$$

becomes a map of $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded algebras. Since this map is compatible with the direct sum decomposition, we will also denote the summands by $\rho\colon H^q(B\mathrm{GL}_n,\boldsymbol{I}^q(\mathscr{L}))\to\mathrm{Ch}^q(B\mathrm{GL}_n)$.

With these preparations, the following result now describes the total Chow–Witt ring of $B\mathrm{GL}_n$; see Theorems 3.24 and 3.27 and Proposition 3.26.

Theorem 1.1 *Let F be a perfect field of characteristic $\neq 2$.*

(1) *The following square, induced from the pullback description of the Milnor–Witt K–theory sheaf, is a cartesian square of $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded $\mathrm{GW}(F)$–algebras:*

$$\begin{array}{ccccc}
\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n,\mathscr{O}\oplus\det\gamma_n^\vee) & \longrightarrow & \ker\partial_{\mathscr{O}}\oplus\ker\partial_{\det\gamma_n^\vee} & \hookrightarrow & \mathrm{CH}^\bullet(B\mathrm{GL}_n)^{\oplus 2}\\
\downarrow & & \downarrow{\scriptstyle \bmod 2} & & \\
H^\bullet_{\mathrm{Nis}}(B\mathrm{GL}_n,\boldsymbol{I}^\bullet\oplus\boldsymbol{I}^\bullet(\det\gamma_n^\vee)) & \xrightarrow[\rho]{} & \mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2} & &
\end{array}$$

In the upper-right corner of this diagram, we have the kernels of the (twisted) integral Bockstein maps

$$\partial_{\mathscr{L}}\colon\mathrm{CH}^\bullet(B\mathrm{GL}_n)\to\mathrm{Ch}^\bullet(B\mathrm{GL}_n)\xrightarrow{\beta_{\mathscr{L}}}H^{\bullet+1}(B\mathrm{GL}_n,\boldsymbol{I}^{\bullet+1}(\mathscr{L})).$$

(2) *The kernels of the twisted integral Bockstein operations inside the Chow ring*

$$\mathrm{CH}^\bullet(B\mathrm{GL}_n) \cong \mathbb{Z}[c_1, \dots, c_n]$$

can be described as

$$\ker \partial_{\mathcal{O}} = \mathbb{Z}[c_i^2, c_1 c_{2i} + c_{2i+1}, c_1 c_n, (2)], \quad \ker \partial_{\det \gamma_n^\vee} = \langle c_{2i+1}, c_n, (2)\rangle_{\ker \partial_{\mathcal{O}}}.$$

The first is a subring of $\mathrm{CH}^\bullet(B\mathrm{GL}_n)$, *the second is a sub–*$\ker \partial_{\mathcal{O}}$*–module, and* (2) *denotes the ideal generated by* 2 *in* $\mathrm{CH}^\bullet(B\mathrm{GL}_n)$.

(3) *The cohomology ring* $H^\bullet_{\mathrm{Nis}}(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \gamma_n^\vee))$ *is generated by the following characteristic classes: the even Pontryagin classes* p_{2i} *in degree* $(4i, 0)$, *the Euler class* e_n *in degree* $(n, 1)$ *and the (twisted) Bocksteins of products of Stiefel–Whitney classes. The latter classes are defined as*

$$\beta_J = \beta_{\mathcal{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \quad \text{and} \quad \tau_J = \beta_{\det \gamma_n^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}),$$

with the index set J *running through the (possibly empty) sets* $\{j_1, \dots, j_l\}$ *of positive natural numbers such that* $0 < j_1 < \cdots < j_l \le \left[\frac{1}{2}(n-1)\right]$; *in the special case of* $J = \varnothing$, *the corresponding classes are* $\beta_\varnothing = \beta_{\mathcal{O}}(1)$ *and* $\tau_\varnothing = \beta_{\det \gamma_n^\vee}(1)$. *For an index set* $J = \{j_1, \dots, j_l\}$, *the degree of* β_J *is* $\left(1 + 2\sum_{i=1}^l j_i, 0\right)$ *and the degree of* τ_J *is* $\left(1 + 2\sum_{i=1}^l j_i, 1\right)$. *The* $\boldsymbol{I}$*–cohomology ring can then be explicitly identified as the* $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$*–graded-commutative* $W(F)$*–algebra*

$$W(F)[p_2, p_4, \dots, p_{[(n-1)/2]}, e_n, \{\beta_J\}_J, \{\tau_J\}_J, \tau_\varnothing]$$

modulo the relations

(a) $I(F)\beta_J = I(F)\tau_J = I(F)\tau_\varnothing = 0$;

(b) *if* $n = 2k+1$ *is odd and* $k \ge 1$, *we have* $e_{2k+1} = \tau_{\{k\}}$, *and for* $n = 1$ *we have* $e_1 = \tau_\varnothing$;

(c) *for two index sets* J *and* J', *where* J' *can be empty,*

$$\beta_J \cdot \beta_{J'} = \sum_{k \in J} \beta_{\{k\}} \cdot p_{(J \setminus \{k\}) \cap J'} \cdot \beta_{\Delta(J \setminus \{k\}, J')},$$

$$\beta_J \cdot \tau_{J'} = \sum_{k \in J} \beta_{\{k\}} \cdot p_{(J \setminus \{k\}) \cap J'} \cdot \tau_{\Delta(J \setminus \{k\}, J')},$$

$$\tau_J \cdot \beta_{J'} = \beta_J \cdot \tau_{J'} + \tau_\varnothing \cdot p_{J \cap J'} \cdot \beta_{\Delta(J, J')},$$

$$\tau_J \cdot \tau_{J'} = \beta_J \cdot \beta_{J'} + \tau_\varnothing \cdot p_{J \cap J'} \cdot \tau_{\Delta(J, J')},$$

where we set $p_A = \prod_{i=1}^l p_{a_i}$ *for an index set* $A = \{a_1, \dots, a_l\}$.

(4) *The reduction morphism* ρ *is given by*

$$p_{2i} \mapsto \bar{c}_{2i}^2, \quad e_n \mapsto \bar{c}_n, \quad \beta_{\mathcal{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \mapsto \mathrm{Sq}^2_{\mathcal{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}).$$

Under the homomorphism $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n, \mathcal{O}) \to \mathrm{CH}^\bullet(B\mathrm{GL}_n)$, *the Chow–Witt-theoretic Pontryagin class is mapped as*

$$p_i \mapsto (-1)^i c_i^2 + 2 \sum_{j=\max\{0, 2i-n\}}^{i-1} (-1)^j c_j c_{2i-j}.$$

It can be shown that formulas similar to the above description of $\boldsymbol{I}^\bullet$–cohomology are true for real-étale cohomology, but as algebra over $H^0_{\text{rét}}(F,\mathbb{Z}) \cong \operatorname{colim}_n I^n(F)$. For $F = \mathbb{R}$, the real cycle class map induces an isomorphism

$$H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det\gamma_n^\vee)) \xrightarrow{\cong} H^\bullet(BO(n), \mathbb{Z}\oplus\mathbb{Z}^{\mathrm{t}}),$$

where the target was computed in [26], and it sends algebraic characteristic classes to their topological counterparts. For this result and a discussion of the required compatibilities, eg between localization sequences and the real cycle class maps, see Hornbostel, Xie, Zibrowius and the author [17].

1.2 Chow–Witt rings of finite Grassmannians

The second point of the paper is to provide a computation of the Chow–Witt rings of the finite Grassmannians $\mathrm{Gr}(k,n)$. The full description is even longer than the description of the Chow–Witt ring of $B\mathrm{GL}_n$ above, so we will only give pointers to the main results in the text. First, the Chow–Witt ring is again given in terms of a cartesian square combining the kernels of integral Bockstein maps with $\boldsymbol{I}^\bullet$–cohomology; see Theorem 5.10. The $\boldsymbol{I}^\bullet$–cohomology of $\mathrm{Gr}(k,n)$ can be described as follows: the characteristic classes of the tautological rank k subbundle $\mathscr{S}_k$ and the tautological rank $n-k$ quotient bundle $\mathscr{Q}_{n-k}$ generate the $\boldsymbol{I}^\bullet$–cohomology, except in the case where $k(n-k)$ is odd, in which we have a new class R in degree $n-1$. They naturally satisfy the relations in the $\boldsymbol{I}^\bullet$–cohomology of $B\mathrm{GL}_k$ and $B\mathrm{GL}_{n-k}$, and they also satisfy the relations which are consequences of the Whitney sum formula for the extension

$$0 \to \mathscr{S}_k \to \mathscr{O}^{\oplus n} \to \mathscr{Q}_{n-k} \to 0.$$

There are a few further relations involving the potential class R. All these statements are established in Theorem 5.7. The reduction morphisms

$$H^\bullet(\mathrm{Gr}(k,n), \boldsymbol{I}^\bullet(\mathscr{L})) \to \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$$

are described in Proposition 5.8. Except for the new fact that $R \mapsto \bar{c}_{k-1}\bar{c}^{\perp}_{n-k}$, the description of the reduction morphisms follows directly from the ones for $B\mathrm{GL}_k$ and $B\mathrm{GL}_{n-k}$. This also provides a description of the kernel of the integral Bockstein maps; cf Theorem 5.10. Again, similar formulas would be true in real-étale cohomology, and for $F = \mathbb{R}$ the above description recovers exactly the integral cohomology of the real Grassmannians $\mathrm{Gr}_k(\mathbb{R}^n)$ (with local coefficients); see [17].

1.3 Decomposition of $\boldsymbol{I}$–cohomology

The present paper is a significantly revised version of its predecessor. While the previous version established the results mostly following the proof strategy of [16] fairly closely, the revised proofs follow a different strategy. The key new insight arises from a decomposition of $\boldsymbol{I}$–cohomology, described in Section 2.4,

$$0 \to \operatorname{Im}\beta_{\mathscr{L}}(X) \to H^q(X, \boldsymbol{I}^q(\mathscr{L})) \to H^q(X, \boldsymbol{W}(\mathscr{L})) \to 0.$$

This decomposition arises from the twisted Bär sequence, an algebraic analogue of the long exact Bockstein sequence in topology, which is discussed in more detail in Section 2.1. The cohomology with coefficients in the sheaf $\boldsymbol{W}$ of Witt rings is a theory in which η is invertible, and much more amenable to long exact sequence calculations than $\boldsymbol{I}$–cohomology; see work of Ananyevskiy [1]. Moreover, if $\boldsymbol{W}$–cohomology is free, the above sequence splits. In that case, the reduction morphism $\rho\colon H^q(X, \boldsymbol{I}^q(\mathscr{L})) \to \mathrm{Ch}^q(X)$ is injective on the image of $\beta_{\mathscr{L}}$; hence the torsion classes in $\mathrm{Im}\,\beta_{\mathscr{L}}$ can be computed from the knowledge of the Steenrod squares $\mathrm{Sq}^2_{\mathscr{L}}$ on the mod 2 Chow ring. This way, the computation of $\boldsymbol{I}$–cohomology splits into two significantly easier parts, the computation of $\boldsymbol{W}$–cohomology which can be done by the same methods as calculations of rational cohomology of real Grassmannians — see Milnor and Stasheff [21] and Sadykov [23] — and the computation of $\mathrm{Im}\,\beta$ which only requires knowledge of the mod 2 Chow theory. The freeness of $\boldsymbol{W}$–cohomology, which implies the above decomposition of $\boldsymbol{I}$–cohomology, can therefore be seen as the algebraic analogue of the statement that "all torsion in the cohomology of real Grassmannians is 2–torsion". This gets rid of problems as in Remark 2.2 or Remark 7.2 of [16]. Moreover, the proof of freeness of $\boldsymbol{W}$–cohomology, and therefore the torsion statement, is significantly easier than in topology (where it is not clear that integral cohomology modulo the image of the integral Bockstein maps is even a cohomology theory). The $\mathrm{Im}\,\beta$–$\boldsymbol{W}$–decomposition of $\boldsymbol{I}$–cohomology will be a useful tool for a number of upcoming computations (where the real topological counterparts have only 2–torsion), such as classifying spaces of orthogonal groups and flag varieties.

The shorter, alternative way to describe the structure of the $\boldsymbol{I}$–cohomology of $B\mathrm{GL}_n$ or $\mathrm{Gr}(k,n)$ (at least additively) is then the following. The $\boldsymbol{I}$–cohomology splits as a direct sum of the image of $\beta_{\mathscr{L}}$, which is a 2–torsion group with the same structure as in the integral cohomology of the real Grassmannians, and the $\boldsymbol{W}$–cohomology, which is a free $W(F)$–algebra having the same presentation as the rational cohomology of the real Grassmannians. The multiplication on the torsion part can be described completely by reduction to mod 2 Chow theory where we have the classical formulas from Schubert calculus. Conceptual descriptions for the multiplication can be found in Casian and Stanton [10] (with an interesting link to representation theory of real Lie groups) and Casian and Kodama [9] (explicitly in terms of signed Young diagrams); see also the discussion of checkerboard fillings for Young diagrams to compute $\mathrm{Sq}^2_{\mathscr{L}}$ in [28]. The description of Chow–Witt rings of finite Grassmannians is used in a sequel [28] to develop an oriented Schubert calculus which allows us to establish arithmetic refinements of classical Schubert calculus.

Structure of the paper We provide a recollection on relevant statements from Chow–Witt theory, in particular the twisted Steenrod squares, in Section 2. The relevant characteristic classes for vector bundles are recalled in Section 3, where we also formulate the main structural results on the Chow–Witt ring of $B\mathrm{GL}_n$. The inductive computation of the $\boldsymbol{I}^\bullet$–cohomology is done in Section 4. The results on Chow–Witt rings of finite Grassmannians are formulated in Section 5 and the proofs are given in Section 6.

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Jens Hornbostel, Thomas Hudson, Lorenzo Mantovani, Toan Manh Nguyen and Konrad Voelkel clarified that the use of the $\operatorname{Im}\beta$–$\boldsymbol{W}$–decomposition of $\boldsymbol{I}$–cohomology could improve structural statements and streamline proofs. I am grateful to Marc Levine and Ákos Matszangosz for various corrections and to an anonymous referee at AGT for many and detailed comments that helped eliminate a number of mistakes and greatly improve the presentation of the paper.

In some of the results describing kernels of Steenrod squares Sq^2 and Bockstein maps β, the obvious and necessary generators in the image of the Steenrod square have been omitted: in Theorem 1.1(2), Corollary 3.13, and Proposition 3.26, Steenrod squares of products of even Stiefel–Whitney classes $\mathrm{Sq}^2_{\mathscr{L}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$ have to be added to the list of generators. Formulations are correct for Corollary 3.14 and Theorem 5.10. This doesn't affect either proofs or main results. I would like to thank Jan Hennig and Marc Levine for pointing this out.

2 Recollection on Chow–Witt rings

Throughout this article, we consider a perfect base field F of characteristic $\mathrm{char}(F)\neq 2$. All the relevant cohomology groups will be Nisnevich cohomology groups, ie Ext-groups between Nisnevich sheaves on the small site of a smooth scheme. Note, however, that for all the sheaves we consider, Nisnevich and Zariski cohomology are isomorphic. The relevant sheaves will be denoted by boldface letters, such as the sheaves $\boldsymbol{K}_n^{\mathrm{M}}$ and $\boldsymbol{K}_n^{\mathrm{MW}}$ of Milnor and Milnor–Witt K–groups, respectively, the sheaf $\boldsymbol{W}$ of Witt rings and the sheaves $\boldsymbol{I}^n$ of powers of the fundamental ideal in $\boldsymbol{W}$. Sections of these sheaves are mostly taken over fields and are denoted by the more usual letters K_n^{M}, K_n^{MW}, W and I^n, respectively.

Since this is a sequel to [16], most of the general facts concerning Chow–Witt rings relevant for the computation in the present paper can already be found in the discussion of [16, Section 2] (or, of course, in the original literature; see [loc. cit.] for references). The same applies to the general discussion of Chow–Witt rings of classifying spaces; all the statements relevant for the present paper can be found in [16, Section 3]. We freely use the definitions, facts and notation from [16].

2.1 Twisted coefficients and cohomology operations

What has to be discussed in slightly more detail is the use of twisted coefficients in Chow–Witt groups and $\boldsymbol{I}^\bullet$–cohomology which were only mentioned in passing in [16]. If $\mathscr{L}$ is a line bundle on a smooth scheme X, then there are twisted sheaves $\boldsymbol{I}^n(\mathscr{L})$ and $\boldsymbol{K}_n^{\mathrm{MW}}(\mathscr{L})$. For the construction as well as a description of Gersten-type complexes computing $H^n_{\mathrm{Nis}}(X,\boldsymbol{I}^n(\mathscr{L}))$ and $H^n_{\mathrm{Nis}}(X,\boldsymbol{K}_n^{\mathrm{MW}}(\mathscr{L}))\cong\widetilde{\mathrm{CH}}^n(X,\mathscr{L})$, see [13, Section 10] and [4, Section 2]. In particular, Theorem 2.3.4 of [4] provides an identification of the definition of twisted Chow–Witt groups in [13] with the Nisnevich cohomology of the twisted Milnor–Witt K–theory sheaves. If $\mathscr{L}$ and $\mathscr{N}$ are two line bundles on X, then there are canonical isomorphisms

$$\widetilde{\mathrm{CH}}^\bullet(X,\mathscr{N})\cong\widetilde{\mathrm{CH}}^\bullet(X,\mathscr{L}^2\otimes\mathscr{N}).$$

The twisted versions of Chow–Witt groups and $\boldsymbol{I}^n$–cohomology have functorial pullbacks, pushforwards and a localization sequence (where the cohomology of the closed subscheme appears with twist by the normal bundle of the inclusion). Formulations and references to the relevant literature can all be found in [16, Section 2.1].

We also need to discuss twisted analogues of the facts on cohomology operations discussed in [16, Section 2.3]. If X is a smooth scheme and $\mathscr{L}$ is a line bundle on X, we can twist the exact sequence of fundamental ideals by $\mathscr{L}$ to get an exact sequence of strictly $\mathbb{A}^1$–invariant Nisnevich sheaves of abelian groups

$$0 \to \boldsymbol{I}^{n+1}(\mathscr{L}) \to \boldsymbol{I}^n(\mathscr{L}) \to \boldsymbol{K}_n^{\mathrm{M}}/2 \to 0.$$

This is analogous to the topological exact sequence $0 \to \mathbb{Z}^{\mathrm{t}} \to \mathbb{Z}^{\mathrm{t}} \to \mathbb{Z}/2\mathbb{Z} \to 0$ for a local system $\mathbb{Z}^{\mathrm{t}}$ with fiber $\mathbb{Z}$. Associated to the previous exact sequence of Nisnevich sheaves, we get a twisted analogue of the Bär sequence used in [16],

$$\cdots \to H^n(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \xrightarrow{\eta} H^n(X, \boldsymbol{I}^n(\mathscr{L})) \xrightarrow{\rho} \mathrm{Ch}^n(X) \xrightarrow{\beta_{\mathscr{L}}} H^{n+1}(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \to \cdots .$$

This is an analogue of the long exact Bockstein sequence in topology. The maps in this sequence are

(1) the connecting map $\beta_{\mathscr{L}}$, a Chow–Witt analogue of the Bockstein operation twisted by a local system in classical algebraic topology,
(2) the map η induced by the inclusion $\boldsymbol{I}^{n+1} \hookrightarrow \boldsymbol{I}^n$ and
(3) the reduction map ρ induced by the quotient map $\boldsymbol{I}^n \to \boldsymbol{K}_n^{\mathrm{M}}/2$.

A discussion of the twisted Bockstein maps in the topological context of cohomology of $BO(n)$ can be found in [26].

Remark 2.1 A funny side remark on some of the differences between $\mathrm{Ch}^n(X)$ and mod 2 singular cohomology: By the universal coefficient formula, mod 2 singular cohomology H^n in general contains mod 2 reductions of integral classes in H^n *as well as* classes related to 2–torsion classes in H^{n-1}. This is not true for $\mathrm{Ch}^n(X)$, viewed as mod 2 reduction of $\mathrm{CH}^n(X)$ — by definition all classes in $\mathrm{Ch}^n(X)$ are simply mod 2 reductions of $\mathrm{CH}^n(X)$. However, the Bär sequence encodes a behavior of $\mathrm{Ch}^n(X)$ exactly analogous to mod 2 singular cohomology: there are some classes which lift to integral cohomology $H^n(X, \boldsymbol{I}^n)$, and some classes which don't (because they have nontrivial images under the Bockstein operation).

For a line bundle $\mathscr{L}$ on a smooth scheme X, the twisted Bockstein map

$$\beta_{\mathscr{L}} \colon \mathrm{Ch}^n(X) \to H^{n+1}(X, \boldsymbol{I}^{n+1}(\mathscr{L}))$$

can be used to define twisted versions of integral Stiefel–Whitney classes analogous to those defined in [14]. The composition with the reduction morphism $\rho \colon H^{n+1}(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \to \mathrm{Ch}^{n+1}(X)$ has been identified in [2, Theorem 3.4.1]. This is a twisted version of Totaro's identification [25, Theorem 1.1] and a Chow–Witt version of [26, Lemma 2].

Proposition 2.2 *Let X be a smooth scheme and $\mathscr{L}$ be a line bundle over X. Denote by*

$$\beta_{\mathscr{L}}\colon \mathrm{Ch}^i(X) \to H^{i+1}(X, \boldsymbol{I}^{i+1}(\mathscr{L}))$$

the twisted Bockstein map. Then for all $x \in \mathrm{Ch}^i(X)$,

$$\rho \circ \beta_{\mathscr{L}}(x) = \bar{c}_1(\mathscr{L}) \cdot x + \mathrm{Sq}^2(x) =: \mathrm{Sq}^2_{\mathscr{L}}(x).$$

2.2 Oriented intersection product and total Chow–Witt ring

The oriented intersection product for the Chow–Witt ring has the form

$$\widetilde{\mathrm{CH}}^i(X, \mathscr{L}_1) \times \widetilde{\mathrm{CH}}^j(X, \mathscr{L}_2) \to \widetilde{\mathrm{CH}}^{i+j}(X, \mathscr{L}_1 \otimes \mathscr{L}_2);$$

there is a similar product on twisted $\boldsymbol{I}^\bullet$–cohomology rings. With these products, the Chow–Witt ring is a $\langle -1 \rangle$–graded commutative algebra over the Grothendieck–Witt ring $\mathrm{GW}(F)$, and $\bigoplus_n H^n(X, \boldsymbol{I}^n)$ is a (-1)–graded commutative algebra over the Witt ring $W(F)$; see eg [16, Section 2.2].

The total Chow–Witt ring of a smooth scheme X is defined by

$$\bigoplus_{\mathscr{L} \in \mathrm{Pic}(X)/2} \widetilde{\mathrm{CH}}^\bullet(X, \mathscr{L});$$

see eg [12, Definition 6.10]. Strictly speaking, a total Chow–Witt ring doesn't exist because identifications $\widetilde{\mathrm{CH}}^\bullet(X, \mathscr{L}) \cong \widetilde{\mathrm{CH}}^\bullet(X, \mathscr{N})$ for isomorphic line bundles $\mathscr{L}$ and $\mathscr{N}$ depend on the choice of isomorphism between the line bundles. However, the technical inaccuracy of neglecting such choices of isomorphisms between different representatives of isomorphism classes of line bundles can be fixed by the methods in [5]. The same goes for the total $\boldsymbol{I}$–cohomology ring

$$\bigoplus_{\mathscr{L} \in \mathrm{Pic}(X)/2, q \in \mathbb{N}} H^q(X, \boldsymbol{I}^q(\mathscr{L})).$$

Note that $\mathrm{Pic}(B\mathrm{GL}_n) \cong \mathbb{Z}$ and $\mathrm{Pic}(\mathrm{Gr}(k,n)) \cong \mathbb{Z}$; in particular, there are only two nontrivial dualities to consider for the total Chow–Witt rings of $B\mathrm{GL}_n$ and $\mathrm{Gr}(k,n)$. For $B\mathrm{GL}_n$, the nontrivial element of $\mathrm{Pic}(B\mathrm{GL}_n)/2$ is given by $\det \gamma_n^\vee$, the dual of the determinant of the universal rank n bundle. Note that this corresponds precisely to the well-known topological fact that there are exactly two isomorphism classes of local systems on $BO(n)$, the trivial one and the one for the sign representation of $\pi_1(BO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ on the coefficient ring $\mathbb{Z}$.

Remark 2.3 There is a serious subtlety concerning the graded commutativity of the total $\boldsymbol{I}$–cohomology ring which we want to discuss at this point. For the correct formulation of graded commutativity of twisted Chow–Witt/$\boldsymbol{I}$–cohomology/$\boldsymbol{W}$–cohomology groups, one has to use graded line bundles, as explained, for example, in [15]. Specializing to the $\boldsymbol{I}$–cohomology situation, the correction factor for graded commutativity of the cup product

$$H^i(X, \boldsymbol{I}^i(\mathscr{L}_1, a_1)) \times H^j(X, \boldsymbol{I}^j(\mathscr{L}_2, a_2)) \to H^{i+j}(X, \boldsymbol{I}^{i+j}(\mathscr{L}_1 \otimes \mathscr{L}_2, a_1 + a_2))$$

would be $(-1)^{(a_1+i)(a_2+j)}$; see [15, Section 3.4].[1] In particular, the degrees of the graded line bundles play an important role. However, for the specific situation of the present paper, this problem is rather invisible: for classifying spaces $B\mathrm{GL}_n$ and Grassmannians $\mathrm{Gr}(k,n)$, the correct graded line bundle for the nontrivial twists would be $(\det \mathscr{V}, \mathrm{rk}\,\mathscr{V})$ for $\mathscr{V} = \gamma_n$ the universal rank n bundle on $B\mathrm{GL}_n$ or $\mathscr{V} = \mathscr{S}_k, \mathscr{Q}_{n-k}$ the tautological subbundle and quotient bundle on the Grassmannian $\mathrm{Gr}(k,n)$. Since the classes in twisted $\boldsymbol{I}$–cohomology of these spaces only appear as multiples of Euler classes of *even rank bundles*, the contributions of the degrees of the twist bundles play no role in the correction term $(-1)^{(a_1+i)(a_2+j)}$. In particular, graded commutativity for the total $\boldsymbol{I}$–cohomology ring in the present paper always means that the correction factor is $(-1)^{ij}$, ie it only depends on the cohomological degrees of the cohomology classes involved. In fact, a posteriori, the $\boldsymbol{I}$–cohomology ring turns out to be commutative after all: all the nontorsion classes in the $\boldsymbol{I}$–cohomology (Pontryagin classes and Euler classes) have even degrees, and all torsion classes are in fact 2–torsion, so signs don't matter.

2.3 The fundamental square

After having discussed all the relevant preliminaries, there are now twisted analogues of the key diagram from [16], for any line bundle $\mathscr{L}$ on X:

$$\begin{array}{ccccccc}
 & & \mathrm{CH}^n(X) & = & \mathrm{CH}^n(X) & & \\
 & & \downarrow & & \downarrow{\scriptstyle 2} & & \\
H^n(X,\boldsymbol{I}^{n+1}(\mathscr{L})) & \longrightarrow & \widetilde{\mathrm{CH}}^n(X,\mathscr{L}) & \longrightarrow & \mathrm{CH}^n(X) & \xrightarrow{\partial_{\mathscr{L}}} & H^{n+1}(X,\boldsymbol{I}^{n+1}(\mathscr{L})) \\
\| & & \downarrow & & \downarrow{\scriptstyle \mathrm{mod}\ 2} & & \| \\
H^n(X,\boldsymbol{I}^{n+1}(\mathscr{L})) & \xrightarrow[\eta]{} & H^n(X,\boldsymbol{I}^n(\mathscr{L})) & \xrightarrow[\rho]{} & \mathrm{Ch}^n(X) & \xrightarrow{\beta_{\mathscr{L}}} & H^{n+1}(X,\boldsymbol{I}^{n+1}(\mathscr{L})) \\
 & & \downarrow & & \downarrow & \searrow^{\mathrm{Sq}^2_{\mathscr{L}}} & \downarrow{\scriptstyle \rho} \\
 & & 0 & \longrightarrow & 0 & & \mathrm{Ch}^{n+1}(X)
\end{array}$$

As already mentioned in [16], there is a twisted analogue of [16, Proposition 2.11], which states that for F a perfect field of characteristic unequal to 2 and a smooth scheme X over F, the canonical map

$$c\colon \widetilde{\mathrm{CH}}^\bullet(X,\mathscr{L}) \to H^\bullet(X,\boldsymbol{I}^\bullet(\mathscr{L})) \times_{\mathrm{Ch}^\bullet(X)} \ker \partial_{\mathscr{L}}$$

induced from the above key square is always surjective, and is injective if $\mathrm{CH}^\bullet(X)$ has no nontrivial 2–torsion. This way we can determine the additive structure of twisted Chow–Witt groups; if we consider the total Chow–Witt ring (ie the direct sum of twisted Chow–Witt groups over $\mathrm{Pic}(X)/2$), the fiber square also describes the oriented intersection product. The result applies, in particular, to $B\mathrm{GL}_n$ and the Grassmannians $\mathrm{Gr}(k,n)$ (or more generally flag varieties G/P for reductive groups) because these are

[1] The formulation in [15] is for Chow–Witt rings and consequently the correction factor there is $\langle(-1)^{(a_1+i)(a_2+j)}\rangle$. For $\boldsymbol{I}$–cohomology, this reduces to $(-1)^{(a_1+i)(a_2+j)}$; see also the discussion of graded commutativity of the various cohomology theories in [16, Definition 2.4 and Proposition 2.5].

known to have 2–torsion-free Chow groups. This implies that we only need to determine the individual terms of the fiber product to get a description of the Chow–Witt ring.

2.4 Decomposing $\boldsymbol{I}$–cohomology into $\boldsymbol{W}$–cohomology and the image of $\boldsymbol{\beta}$

One of the features which is new and has not been used in either [16] or the previous version of the present paper is the use of $\boldsymbol{W}$–cohomology. For a smooth F–scheme X, we can consider the restriction of the Nisnevich sheaf $\boldsymbol{W}$ of Witt groups to the small Nisnevich site of X, and then take its Nisnevich cohomology $H^\bullet(X,\boldsymbol{W})$. This cohomology theory has been considered before, as it appears in the context of the Gersten–Witt spectral sequence converging to the Witt groups of a smooth scheme X; see [6]. Some of its properties discussed below make it also very suitable as a stepping stone in computations of $\boldsymbol{I}$–cohomology and Chow–Witt rings.

As before, if $\mathscr{L}$ is a line bundle on X, we can consider the twisted $\boldsymbol{W}$–cohomology groups $H^\bullet(X,\boldsymbol{W}(\mathscr{L}))$. The product structure on the Witt rings induces an intersection product

$$H^i(X,\boldsymbol{W}(\mathscr{L}_1))\times H^j(X,\boldsymbol{W}(\mathscr{L}_2))\to H^{i+j}(X,\boldsymbol{W}(\mathscr{L}_1\otimes\mathscr{L}_2)),$$

and we can consider the total $\boldsymbol{W}$–cohomology ring $\bigoplus_{q,\mathscr{L}\in\mathrm{Pic}(X)/2} H^q(X,\boldsymbol{W}(\mathscr{L}))$ (again using [5] to make sense of this). Similar to the $\boldsymbol{I}$–cohomology ring, the total $\boldsymbol{W}$–cohomology ring is a (-1)–graded commutative algebra over the Witt ring $W(F)$; see Remark 2.3.

There is a morphism $(\boldsymbol{I}^n)_{n\in\mathbb{Z}}\to(\boldsymbol{W})_{n\in\mathbb{Z}}$ which in degree n is given by the natural inclusion $\boldsymbol{I}^n\hookrightarrow\boldsymbol{W}$, with the usual convention of $\boldsymbol{I}^n=\boldsymbol{W}$ for $n\le 0$. This morphism induces a $W(F)$–algebra homomorphism

$$\bigoplus_{q,\mathscr{L}} H^q(X,\boldsymbol{I}^q(\mathscr{L}))\to\bigoplus_{q,\mathscr{L}} H^q(X,\boldsymbol{W}(\mathscr{L}))$$

from the total $\boldsymbol{I}$–cohomology ring to the total $\boldsymbol{W}$–cohomology ring.

The relation with $\boldsymbol{I}$–cohomology can be made more precise. The pieces

$$H^{i-1}(X,\boldsymbol{K}^{\mathrm{M}}_{n-1}/2)\to H^i(X,\boldsymbol{I}^n(\mathscr{L}))\to H^i(X,\boldsymbol{I}^{n-1}(\mathscr{L}))\to H^i(X,\boldsymbol{K}^{\mathrm{M}}_{n-1}/2)$$

of the Bär sequence provide isomorphisms $H^i(X,\boldsymbol{I}^n(\mathscr{L}))\to H^i(X,\boldsymbol{I}^{n-1}(\mathscr{L}))$ for $i>n$, because the outer terms vanish. This can be seen from the Gersten resolution for mod 2 Milnor K–theory together with the fact that $(\boldsymbol{K}^{\mathrm{M}}_{n-1}/2)_{-c}=0$ for $c>n-1$. Moreover, from the Gersten resolution for $\boldsymbol{I}^n$, we also see that the natural morphisms $H^i(X,\boldsymbol{I}^n(\mathscr{L}))\to H^i(X,\boldsymbol{W}(\mathscr{L}))$ are isomorphisms for $i>n$. Now with this reinterpretation, we can consider the piece of the Bär sequence for the boundary case $i=n$,

$$\mathrm{Ch}^{n-1}(X)\cong H^{n-1}(X,\boldsymbol{K}^{\mathrm{M}}_{n-1}/2)\xrightarrow{\beta_{\mathscr{L}}} H^n(X,\boldsymbol{I}^n(\mathscr{L}))\to H^n(X,\boldsymbol{W}(\mathscr{L}))\to 0.$$

In particular, $\boldsymbol{I}$–cohomology is a combination of $\boldsymbol{W}$–cohomology with the image of the Bockstein morphism β. We get a stronger splitting result if the $\boldsymbol{W}$–cohomology is free:

Lemma 2.4 *Let X be a smooth scheme over a field F of characteristic $\neq 2$, and let $\mathscr{L}$ be a line bundle on X. If $H^n(X, \boldsymbol{W}(\mathscr{L}))$ is free as a $W(F)$–module, then we have a splitting*

$$H^n(X, \boldsymbol{I}^n(\mathscr{L})) \cong \operatorname{Im} \beta_{\mathscr{L}} \oplus H^n(X, \boldsymbol{W}(\mathscr{L})).$$

In this case, the reduction morphism $\rho\colon H^n(X, \boldsymbol{I}^n(\mathscr{L})) \to \mathrm{Ch}^n(X)$ is injective on the image of $\beta_{\mathscr{L}}$.

Proof The Bär sequence is a long exact sequence of $W(F)$–modules. The first claim follows from the piece

$$\mathrm{Ch}^{n-1}(X) \xrightarrow{\beta_{\mathscr{L}}} H^n(X, \boldsymbol{I}^n(\mathscr{L})) \to H^n(X, \boldsymbol{W}(\mathscr{L})) \to 0.$$

If the last group is free as $W(F)$–module, then the sequence splits as claimed.

For the second claim, we first note that the existence of a splitting implies

$$\big(I(F) \cdot H^n(X, \boldsymbol{I}^n(\mathscr{L}))\big) \cap \operatorname{Im} \beta_{\mathscr{L}} = 0.$$

This follows since the splitting map $H^n(X, \boldsymbol{I}^n(\mathscr{L})) \to \operatorname{Im} \beta_{\mathscr{L}}$ is a $W(F)$–module map, and the $W(F)$–module structure on $\mathrm{Ch}^{n-1}(X)$ and hence $\operatorname{Im} \beta_{\mathscr{L}}$ is a direct sum of copies of $W(F)/I(F)$. Thus the splitting map necessarily sends every element in $I(F) \cdot H^n(X, \boldsymbol{I}^n(\mathscr{L}))$ to zero. But the splitting map is the identity on $\operatorname{Im} \beta_{\mathscr{L}}$, which implies the claim concerning the intersection.

Now to prove the second claim of the lemma, we want to use the exact piece of the Bär sequence

$$H^n(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \xrightarrow{\eta} H^n(X, \boldsymbol{I}^n(\mathscr{L})) \xrightarrow{\rho} \mathrm{Ch}^n(X).$$

If we can show $\operatorname{Im} \beta_{\mathscr{L}} \cap \operatorname{Im} \eta = 0$, then the injectivity claim of the lemma follows. Suppose we have a nonzero element $\alpha \in \operatorname{Im} \beta_{\mathscr{L}} \cap \operatorname{Im} \eta$. Then we can factor the inclusion of the corresponding $W(F)/I(F)$–summand as

$$W(F)/I(F) \to H^n(X, \boldsymbol{I}^{n+1}(\mathscr{L})) \xrightarrow{\eta} H^n(X, \boldsymbol{I}^n(\mathscr{L})).$$

This map is now the inclusion of a direct summand (as $W(F)$–modules), but it is also multiplication by η from the factorization. Therefore, it is the zero map, contradicting the assumption $0 \neq \alpha \in \operatorname{Im} \beta_{\mathscr{L}} \cap \operatorname{Im} \eta$, which proves injectivity. □

Corollary 2.5 *Let X be a smooth scheme over a field F of characteristic $\neq 2$, and let $\mathscr{L}$ be a line bundle on X. If the total $\boldsymbol{W}$–cohomology ring is free as a $W(F)$–module, then the image of the maps $\beta_{\mathscr{L}}$ for $\mathscr{L} \in \mathrm{Pic}(X)/2$ coincides exactly with the $W(F)$–torsion in $\boldsymbol{I}^\bullet$–cohomology. In particular, the image of the maps $\beta_{\mathscr{L}}$ for $\mathscr{L} \in \mathrm{Pic}(X)/2$ is an ideal in the total $\boldsymbol{I}^\bullet$–cohomology ring.*

Remark 2.6 The freeness of $\boldsymbol{W}$–cohomology in this lemma will play an important role in our computations. It is an algebraic replacement of the classical statement that "all torsion in the cohomology of the Grassmannians is 2–torsion", as formulated in eg [8, Lemma 2.2]. Using the splitting in Lemma 2.4 is a different strategy than the cumbersome proofs in [16] which were needed to establish that ρ is injective on the image of β; see Remark 7.2 and the discussion before Proposition 8.6 in [16].

There are two reasons why the decomposition of $\boldsymbol{I}$–cohomology as a direct sum of $\boldsymbol{W}$–cohomology and the image of β is so effective as a computational tool. On the one hand, the image of β is basically known in the relevant cases — all it requires is knowledge of the Chow ring together with the action of Sq^2. On the other hand, computations in $\boldsymbol{W}$–cohomology are simpler than for $\boldsymbol{I}$–cohomology because the localization sequence takes the following simplified form: Assume X is a smooth scheme, $Z \subseteq X$ a smooth closed subscheme of pure codimension c with open complement $U = X \setminus Z$, and $\mathscr{L}$ is a line bundle on X. Denote the inclusions by $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$, and denote by $\mathscr{N}$ the determinant of the normal bundle for Z in X. Then we have a localization sequence for $\boldsymbol{W}$–cohomology,

$$\cdots \to H^i(U, \boldsymbol{W}(\mathscr{L})) \xrightarrow{\partial} H^{i-c+1}(Z, \boldsymbol{W}(\mathscr{L} \otimes \mathscr{N}_Z)) \xrightarrow{i_*} H^{i+1}(X, \boldsymbol{W}(\mathscr{L})) \xrightarrow{j^*} H^{i+1}(U, \boldsymbol{W}(\mathscr{L})) \to \cdots .$$

This has the distinct advantage that there are no index shifts in the coefficients (such as what happens for $\boldsymbol{I}$–cohomology) and we really get an honest long exact sequence (as opposed to only a piece of a long exact sequence containing the "geometric bidegrees"). This way, computations of $\boldsymbol{W}$–cohomology can follow their classical topology counterparts much more closely than is possible for $\boldsymbol{I}$–cohomology.

Remark 2.7 One explanation of the simplified form of the localization sequence is that the $\boldsymbol{W}$–cohomology ring $\bigoplus_n H^\bullet(X, \boldsymbol{W})$ considered above is part of the η–inverted Witt group theory considered, for example, in [1]. Essentially, it is the quotient of the η–inverted Witt ring of X modulo $\eta - 1$. Some of the formulas for $\boldsymbol{W}$–cohomology of Grassmannians we develop in this paper already appear in [loc. cit.]. On the other hand, some of the computations for $\boldsymbol{W}$–cohomology in Section 6 could surely be done more generally for other cohomology theories in which η is invertible.

3 Characteristic classes for vector bundles

The next two sections will provide a computation of the Chow–Witt ring of $B\mathrm{GL}_n$. The global structure of the argument is similar to the computation of integral cohomology with local coefficients of $BO(n)$; see [26]. Some of the relevant adaptations to the Chow–Witt setting have already been made in [16]. Additionally, the decomposition of $\boldsymbol{I}$–cohomology into the image of β and $\boldsymbol{W}$–cohomology will significantly simplify the approach of [16], rendering the arguments even closer to their topological counterparts.

In this section, we begin by setting up the localization sequence and defining the relevant characteristic classes for vector bundles. We formulate the main structure results concerning the Chow–Witt and $\boldsymbol{I}^\bullet$–cohomology ring of $B\mathrm{GL}_n$ and establish the basic relations between the characteristic classes. The inductive proof of the structure theorem will be done in the next section.

Before embarking on the computation of cohomology of $B\mathrm{GL}_n$, we need to briefly discuss the issues arising from the classifying spaces not being smooth (in particular finite-dimensional) schemes; see also similar discussions in [16]. The cohomology theories discussed in Section 2 are usually applied to smooth schemes, and some techniques like Gersten-style complexes only work for smooth schemes.

There are then two approaches to extend the definition and computational tools like localization sequences to classifying spaces:

(i) One possibility is to use finite-dimensional approximations to the classifying space, built from representations of the group in question, as in Totaro's definition of Chow groups of classifying spaces [24]. In this approach, only finite-dimensional schemes are considered. Any particular such finite-dimensional approximation of the classifying space only captures the cohomology in a limited range of degrees. On the other hand, stabilization results imply that for any degree, one can always find a suitably high-dimensional approximation which correctly computes cohomology in this degree. This hinges on the fact that the cohomology theories we consider are based on cycles which implies that the degree q cohomology reflects the structure of codimension q subvarieties in a smooth scheme (as opposed to what would happen for algebraic or hermitian K–theory, for example).

(ii) The other possibility is to extend the cohomology theories discussed in Section 2 to all spaces in the Morel–Voevodsky motivic homotopy category [22]. All these cohomology theories are representable in motivic homotopy, because they satisfy Nisnevich descent and homotopy invariance. The classifying spaces can be constructed as spaces in the motivic homotopy category. This provides a definition of cohomology of classifying spaces which correctly computes all degrees at the same time.

For the present paper, we will work with the first viewpoint, using finite-dimensional approximations, as discussed below. In particular, all cohomology groups will in fact be cohomology groups of smooth schemes. Referring to $H^\bullet(B\mathrm{GL}_n)$ means that whenever we are interested in a particular cohomological degree q, we are actually considering a suitably high-dimensional smooth scheme X and compute $H^q(X)$. All the discussions (in particular ones using localization sequences or intersection products) will always only involve a finite number of degrees, so that this is indeed possible.

3.1 Setup of localization sequence

We begin by setting up the localization sequence for the inductive computation of the cohomology of $B\mathrm{GL}_n$, following the procedure for SL_n in [16, Section 5.1].

Let V be a finite-dimensional representation of GL_n such that outside a closed GL_n–stable subset Y of codimension s, the action of GL_n is free and the quotient $X(V) := (V \setminus Y)/\mathrm{GL}_n$ is a GL_n–torsor (ie a GL_n–principal bundle). For any s, there is a GL_n–representation satisfying this requirement; see the discussion [24, Section 1, Remark 1.4]. Then the Chow–Witt group $\widetilde{\mathrm{CH}}^q(X(V), \mathscr{L})$ is up to isomorphism independent of the choice of representation V for $q \leq s-2$, ie computes $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n, \mathscr{L})$ in degrees $\leq s-2$. Moreover, a finite-dimensional model for the universal GL_n–torsor is given by the projection $p\colon V \setminus Y \to X(V)$. The tautological GL_n–representation on $\mathbb{A}^n$ gives rise to a vector bundle $\gamma_V\colon E_n(V) \to X(V)$ associated to the GL_n–torsor $p\colon V \setminus Y \to X(V)$.

Denote by $S_n(V)$ the complement of the zero-section of $\gamma_V\colon E_n(V) \to X(V)$. As in the case of SL_n, the complement $S_n(V)$ can be identified as an approximation of the classifying space $B\mathrm{GL}_{n-1}$. Moreover, the quotient map $q\colon (V\setminus Y)/\mathrm{GL}_{n-1}\to X(V)$ induces a morphism

$$\widetilde{\mathrm{CH}}^\bullet(X(V),\mathscr{L})\xrightarrow{\gamma_V^*}\widetilde{\mathrm{CH}}^\bullet(S_n(V),\gamma_V^*(\mathscr{L}))\cong\widetilde{\mathrm{CH}}^\bullet((V\setminus Y)/\mathrm{GL}_{n-1},q^*(\mathscr{L}))$$

which models the stabilization map $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n,\mathscr{L})\to\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_{n-1},\iota^*\mathscr{L})$ for the standard inclusion $\iota\colon B\mathrm{GL}_{n-1}\to B\mathrm{GL}_n$. Consequently, we get the following localization sequence:

Proposition 3.1 *There is a long exact sequence of Chow–Witt groups of classifying spaces*

$$\cdots\to\widetilde{\mathrm{CH}}^{q-n}(B\mathrm{GL}_n,\mathscr{L}\otimes\det\gamma_n)\to\widetilde{\mathrm{CH}}^{q}(B\mathrm{GL}_n,\mathscr{L})\to\widetilde{\mathrm{CH}}^{q}(B\mathrm{GL}_{n-1},\iota^*(\mathscr{L}))$$
$$\to H^{q+1-n}(B\mathrm{GL}_n,\boldsymbol{K}^{\mathrm{MW}}_{q-n}(\mathscr{L}\otimes\det\gamma_n))\to H^{q+1}(B\mathrm{GL}_n,\boldsymbol{K}^{\mathrm{MW}}_{q}(\mathscr{L}))\to\cdots.$$

The first map is the composition of the dévissage isomorphism with the forgetting of support — alternatively "multiplication with the Euler class of the universal bundle γ_n". The second map is the restriction along the stabilization inclusion $\iota\colon \mathrm{GL}_{n-1}\to\mathrm{GL}_n$.

There are similar exact sequences for the other coefficients, $\boldsymbol{I}^\bullet(\mathscr{L})$, $\boldsymbol{K}^{\mathrm{M}}_\bullet$ and $\boldsymbol{W}(\mathscr{L})$, and the change-of-coefficients maps induce commutative ladders of exact sequences. Notably, the localization sequence for $\boldsymbol{W}$–cohomology is

$$\cdots\to H^{q-n}(B\mathrm{GL}_n,\boldsymbol{W}(\mathscr{L}\otimes\det\gamma_n))\xrightarrow{e_n}H^{q}(B\mathrm{GL}_n,\boldsymbol{W}(\mathscr{L}))$$
$$\xrightarrow{\iota^*}H^{q}(B\mathrm{GL}_{n-1},\boldsymbol{W}(\iota^*\mathscr{L}))\xrightarrow{\partial}H^{q-n+1}(B\mathrm{GL}_n,\boldsymbol{W}(\mathscr{L}\otimes\det\gamma_n))\to\cdots.$$

The proof is the same line of argument as for the case SL_n in [16, Proposition 5.1].

Remark 3.2 Note also that for $\mathscr{L}=\det\gamma_n^\vee$, with $\gamma_n^\vee$ the dual of the universal rank n bundle on $B\mathrm{GL}_n$, we have $\iota^*\mathscr{L}\cong\det\gamma_{n-1}^\vee$. Multiplication with the Euler class changes the dualities.

3.2 Euler class

Recall from [16, Definition 5.9] how the Chow–Witt-theoretic Euler class of [4] gives rise to an Euler class in $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n,\det\gamma_n^\vee)$. For a smooth scheme X, the Chow–Witt-theoretic Euler class of a vector bundle $p\colon\mathscr{E}\to X$ of rank n is defined via the formula

$$e_n(p\colon\mathscr{E}\to X):=(p^*)^{-1}s_{0*}(1)\in\widetilde{\mathrm{CH}}^n(X,\det(p)^\vee),$$

where $s_0\colon X\to\mathscr{E}$ is the zero section. Using smooth finite-dimensional approximations to the classifying space $B\mathrm{GL}_n$ provides a well-defined Euler class

$$e_n\in\widetilde{\mathrm{CH}}^n(B\mathrm{GL}_n,\det(\gamma_n)^\vee).$$

In the localization sequence of Proposition 3.1, the Euler class corresponds under the dévissage isomorphism to the Thom class for the universal rank n vector bundle γ_n on $B\mathrm{GL}_n$. This justifies calling the composition

$$\widetilde{\mathrm{CH}}^{q-n}(B\mathrm{GL}_n,\mathscr{L}\otimes\det\gamma_n)\cong\widetilde{\mathrm{CH}}^{q}_{B\mathrm{GL}_n}(E_n,\mathscr{L})\to\widetilde{\mathrm{CH}}^{q}(E_n,\mathscr{L})\cong\widetilde{\mathrm{CH}}^{q}(B\mathrm{GL}_n,\mathscr{L})$$

"multiplication with the Euler class". There are corresponding notions of Euler classes in $\boldsymbol{I}^\bullet$–cohomology, $\boldsymbol{W}$–cohomology, as well as Chow theory; these are compatible with the change of coefficients. The Euler classes are compatible with pullbacks of morphisms between smooth schemes; see [4, Proposition 3.1.1].

3.3 Chern classes

A direct consequence of the above localization sequence for Chow theory is the computation of the Chow ring (with integral and mod 2 coefficients) of the classifying space $B\mathrm{GL}_n$. The formulas are the standard ones found in any intersection theory handbook; see also [16, Proposition 5.2]. As in [loc. cit.], the Chern classes are uniquely determined by their compatibility with stabilization and the identification of the top Chern class with the Euler class of the universal bundle.

Proposition 3.3 *There are unique classes $c_i(\mathrm{GL}_n) \in \mathrm{CH}^i(B\mathrm{GL}_n)$ for $1 \le i \le n$, such that the natural stabilization morphism $\iota\colon B\mathrm{GL}_{n-1} \to B\mathrm{GL}_n$ satisfies $\iota^* c_i(\mathrm{GL}_n) = c_i(\mathrm{GL}_{n-1})$ for $i < n$ and $c_n(\mathrm{GL}_n) = e_n(\mathrm{GL}_n)$. In particular, the Chow–Witt-theoretic Euler class reduces to the top Chern class in the Chow theory. There is a natural isomorphism*

$$\mathrm{CH}^\bullet(B\mathrm{GL}_n) \cong \mathbb{Z}[c_1, c_2, \ldots, c_n].$$

The restriction along the Whitney sum $B\mathrm{GL}_m \times B\mathrm{GL}_{n-m} \to B\mathrm{GL}_n$ maps the Chern classes as

$$c_i \mapsto \sum_{j=i+m-n}^{m} c_j \boxtimes c_{i-j}.$$

Remark 3.4 From the above computations of the Chow ring of $B\mathrm{GL}_n$ we also see the standard fact that $\mathrm{Pic}(B\mathrm{GL}_n) \cong \mathbb{Z}$. Note that for any smooth scheme X and any two line bundles $\mathscr{L}$ and $\mathscr{N}$ over X such that the class of $\mathscr{L}$ in $\mathrm{Pic}(X)$ is divisible by 2,

$$\widetilde{\mathrm{CH}}^\bullet(X, \mathscr{L} \otimes \mathscr{N}) \cong \widetilde{\mathrm{CH}}^\bullet(X, \mathscr{N}).$$

In particular, there are only two relevant dualities to consider for $B\mathrm{GL}_n$: the trivial duality corresponding to the trivial line bundle on $B\mathrm{GL}_n$, and the nontrivial duality corresponding to the determinant of the universal bundle. This closely resembles the classical situation where $\pi_1(BO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ and so there are only two isomorphism classes of local systems on $BO(n)$.

3.4 Pontryagin classes

Recall from [16, Definition 5.6] that the Pontryagin classes of vector bundles are defined as the images of $p_i \in \widetilde{\mathrm{CH}}^\bullet(B\mathrm{Sp}_{2n})$ of [16, Theorem 4.10] under the homomorphism $\widetilde{\mathrm{CH}}^\bullet(B\mathrm{Sp}_{2n}) \to \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n)$, which is induced from the symplectification morphism (ie the standard hyperbolic functor) $B\mathrm{GL}_n \to B\mathrm{Sp}_{2n}$. Note that this means that the Pontryagin classes of vector bundles are elements in the Chow–Witt ring with trivial duality (because they are induced from the symplectic group). As for the special linear groups — see [16, Proposition 5.8] — the Pontryagin classes are compatible with stabilization in the sense that

$$\iota^*(p_i(\mathrm{GL}_n)) = p_i(\mathrm{GL}_{n-1}),$$

where $i < n$ and $\iota^*\colon \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n) \to \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_{n-1})$ is induced from the natural stabilization map $\mathrm{GL}_{n-1} \to \mathrm{GL}_n$. There are corresponding definitions of Pontryagin classes for $\boldsymbol{I}^\bullet$–cohomology and $\boldsymbol{W}$–cohomology, compatible with the natural change-of-coefficient maps

$$\widetilde{\mathrm{CH}}^q(X) \to H^q(X, \boldsymbol{I}^q) \to H^q(X, \boldsymbol{W}).$$

Remark 3.5 (NB concerning odd Pontryagin classes) The above definition produces Pontryagin classes $p_1, \dots, p_n$ for GL_n; in particular, we get *odd* Pontryagin classes p_{2i+1}. These classes turn out to be torsion and are not included explicitly in the presentation of $\boldsymbol{I}$–cohomology — see, for example, in Theorem 3.24 — because they can be expressed as Bockstein classes; see Theorem 3.27. Note that the indexing convention here differs from the one employed in topology, where only the even Chern classes are used in the definition of Pontryagin classes of bundles — the Pontryagin classes in topology correspond to the *even* Pontryagin classes in the present paper. The reason for this choice of indexing here — and also in [16] — is the easier formulation of the Whitney sum formula for Pontryagin classes; see Proposition 3.28 and Remark 3.29.

3.5 Stiefel–Whitney classes and their (twisted) Bocksteins

The localization sequence of Proposition 3.1 immediately implies a theory of Stiefel–Whitney classes which are determined by the compatibility with stabilization and the identification of the top Stiefel–Whitney class with the Euler class of the respective universal bundle; see [16, Proposition 5.4].

Proposition 3.6 *There are unique classes $\bar{c}_i(\mathrm{GL}_n) \in \mathrm{Ch}^i(B\mathrm{GL}_n)$ for $1 \le i \le n$, such that the natural stabilization morphism $\iota\colon B\mathrm{GL}_{n-1} \to B\mathrm{GL}_n$ satisfies $\iota^*\bar{c}_i(\mathrm{GL}_n) = \bar{c}_i(\mathrm{GL}_{n-1})$ for $i < n$ and $\bar{c}_n(\mathrm{GL}_n) = e_n(\mathrm{GL}_n)$. These agree with the Stiefel–Whitney classes in* [14, Definition 4.2]. *There is a natural isomorphism*

$$\mathrm{Ch}^\bullet(B\mathrm{GL}_n) \cong \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \dots, \bar{c}_n].$$

Again, this is a very classical formula. We include it just for the following discussion of the (twisted) Bockstein classes and the action of the respective (twisted) Steenrod squares on $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$.

Recall from Section 2 that for a scheme X and a line bundle $\mathscr{L}$, we have a Bockstein map

$$\beta_{\mathscr{L}}\colon \mathrm{Ch}^n(X) \to H^{n+1}(X, \boldsymbol{I}^{n+1}(\mathscr{L})).$$

For the specific case of $B\mathrm{GL}_n$, there are two relevant line bundles to consider: $\mathscr{O}$ and $\det \gamma_n^\vee$; see Remark 3.4. This leads to two types of Bockstein classes for vector bundles:

Definition 3.7 For a (possibly empty) set $J = \{j_1, \dots, j_l\}$ of positive natural numbers

$$0 < j_1 < \cdots < j_l \leq \left[\tfrac{1}{2}(n-1)\right],$$

there are classes

$$\beta_J := \beta_{\mathcal{O}}(\bar{c}_{2j_1}\bar{c}_{2j_2}\cdots\bar{c}_{2j_l}) \in H^{d+1}(B\mathrm{GL}_n, \boldsymbol{I}^{d+1}),$$
$$\tau_J := \beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\bar{c}_{2j_2}\cdots\bar{c}_{2j_l}) \in H^{d+1}(B\mathrm{GL}_n, \boldsymbol{I}^{d+1}(\det\gamma_n^\vee)),$$

where $d = \sum_{a=1}^{l} 2j_a$.

Remark 3.8 We discuss the special cases with $J = \varnothing$. The Bockstein class $\beta(\varnothing)$ is trivial; see [16, Remark 5.12]. However, the class $\tau(\varnothing)$ is nontrivial; more precisely,

$$\rho(\tau(\varnothing)) = \mathrm{Sq}^2_{\det\gamma_n^\vee}(1) = \bar{c}_1.$$

As a matter of convention, whenever products $c_{2j_1}\cdots c_{2j_l}$ of Chern classes (or their mod 2 reductions, the Stiefel–Whitney classes) appear in the paper, the indices will be positive natural numbers with $0 < j_1 < \cdots < j_l \leq \left[\tfrac{1}{2}(n-1)\right]$.

Lemma 3.9 *For a (possibly empty) set $J = \{j_1, \dots, j_l\}$ of positive natural numbers*

$$0 < j_1 < \cdots < j_l \leq \left[\tfrac{1}{2}(n-1)\right],$$

we have

$$I(F)\beta_{\mathcal{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = 0 \quad \text{and} \quad I(F)\beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = 0$$

in $H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet)$ and $H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet(\det\gamma_n^\vee))$, respectively.

Proof As in [16, Lemma 7.3], this is formal from the $W(F)$–linearity of the maps in the exact Bär sequence. □

Proposition 3.10 *With the notation from Definition 3.7, if $n = 2k+1$,*

$$e_n = \beta_{\det\gamma_n^\vee}(\bar{c}_{n-1}) = \tau_{\{k\}}.$$

Proof This is proved in [14, Theorem 10.1], noting that our Stiefel–Whitney classes in Proposition 3.6 agree with those in [loc. cit.]; see also [16, Proposition 7.5]. □

Combining Lemma 3.9 and Proposition 3.10, we see that the Euler class $e_n \in H^n(B\mathrm{GL}_n, \boldsymbol{I}^n(\det\gamma_n^\vee))$ is $I(F)$–torsion if n is odd.

Remark 3.11 On $B\mathrm{GL}_n$, the Bockstein classes don't contain more information than the Stiefel–Whitney classes; it will follow from Proposition 4.5 combined with Lemma 2.4 that the reduction morphism

$$\rho\colon H^m(B\mathrm{GL}_n, \boldsymbol{I}^m(\mathscr{L})) \to \mathrm{Ch}^m(B\mathrm{GL}_n)$$

is injective on the image of $\beta_{\mathscr{L}}$. However, for a smooth scheme X, it is possible that the Bockstein class is nontrivial while its reduction in the mod 2 Chow ring is trivial. Topologically, this happens if the integral Stiefel–Whitney class is divisible by 2; divisibility results for the integral Stiefel–Whitney classes arise, for example, in Massey's discussion of the obstruction theory for existence of almost complex structures.

3.6 The Wu formula for the Chow ring

We briefly discuss the action of the Steenrod squares $\mathrm{Sq}^2_{\mathscr{L}}$ on $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$. Essentially, this is described by the Wu formula. It's well known, and there are several ways to prove it, for instance deducing it from the Wu formula in singular cohomology via the cycle class map. We give a sketch of argument relying mostly on Fasel's computations with integral Stiefel–Whitney classes in [14].

Proposition 3.12 *The untwisted Steenrod square* $\mathrm{Sq}^2_{\mathbb{O}}$ *is given by*

$$\mathrm{Sq}^2_{\mathbb{O}}\colon \mathrm{Ch}^\bullet(B\mathrm{GL}_n) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n), \quad \bar{c}_j \mapsto \bar{c}_1\bar{c}_j + (j-1)\bar{c}_{j+1}.$$

The twisted Steenrod square $\mathrm{Sq}^2_{\det\gamma_n}$ *is given by*

$$\mathrm{Sq}^2_{\det\gamma_n}\colon \mathrm{Ch}^\bullet(B\mathrm{GL}_n) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n), \quad \bar{c}_j \mapsto (j-1)\bar{c}_{j+1}.$$

The (twisted) Steenrod squares $\mathrm{Sq}^2_{\mathbb{O}}$ *and* $\mathrm{Sq}^2_{\det\gamma_n}$ *of other elements are determined by the above formulas, the derivation property of the Steenrod square* $\mathrm{Sq}^2_{\mathbb{O}}$ *and the relation*

$$\mathrm{Sq}^2_{\det\gamma_n^\vee}(x) = \bar{c}_1\cdot x + \mathrm{Sq}^2_{\mathbb{O}}(x).$$

Proof The first and second statement are equivalent by Proposition 2.2 and noting that $\bar{c}_1(\det\gamma_n) = \bar{c}_1$. So it suffices to prove the claims concerning $\mathrm{Sq}^2_{\det\gamma_n}$.

The second statement about the twisted Steenrod square in case of even Stiefel–Whitney classes is proved in [14, Proposition 10.3, Remark 10.5]. For odd Stiefel–Whitney classes, the vanishing of $\mathrm{Sq}^2_{\det\gamma_n}(\bar{c}_{2n+1})$ is a consequence of the following computation, applied to $x = \bar{c}_{2n}$ and using $\mathrm{Sq}^2_{\det\gamma_n}(\bar{c}_{2n}) = \bar{c}_{2n+1}$:

$$\begin{aligned}\mathrm{Sq}^2_{\det\gamma_n}\circ\mathrm{Sq}^2_{\det\gamma_n}(x) &= \bar{c}_1\cdot\mathrm{Sq}^2_{\det\gamma_n}(x) + \mathrm{Sq}^2_{\mathbb{O}}\circ\mathrm{Sq}^2_{\det\gamma_n}(x)\\ &= \bar{c}_1^2\cdot x + \bar{c}_1\cdot\mathrm{Sq}^2_{\mathbb{O}}(x) + \mathrm{Sq}^2_{\mathbb{O}}(\bar{c}_1\cdot x) + \mathrm{Sq}^2_{\mathbb{O}}\circ\mathrm{Sq}^2_{\mathbb{O}}(x)\\ &= \bar{c}_1^2\cdot x + \bar{c}_1\cdot\mathrm{Sq}^2_{\mathbb{O}}(x) + \bar{c}_1\cdot\mathrm{Sq}^2_{\mathbb{O}}(x) + x\cdot\mathrm{Sq}^2_{\mathbb{O}}(\bar{c}_1)\\ &= 0.\end{aligned}$$

□

Corollary 3.13 *The kernel of the untwisted Steenrod square* $\mathrm{Sq}^2_{\mathbb{O}}$ *is given by the subring*

$$\mathbb{Z}/2\mathbb{Z}[\bar{c}_i^2, \bar{c}_1\bar{c}_{2i} + \bar{c}_{2i+1}, \bar{c}_1\bar{c}_n] \subseteq \mathbb{Z}/2\mathbb{Z}[\bar{c}_1,\dots,\bar{c}_n] = \mathrm{Ch}^\bullet(B\mathrm{GL}_n).$$

The kernel of the twisted Steenrod square $\mathrm{Sq}^2_{\det\gamma_n}$ *is given by the submodule (over the kernel of* $\mathrm{Sq}^2_{\mathbb{O}}$*)*

$$\langle\bar{c}_{2i+1},\bar{c}_n\rangle_{\ker\mathrm{Sq}^2_{\mathbb{O}}} \subseteq \mathbb{Z}/2\mathbb{Z}[\bar{c}_1,\dots,\bar{c}_n] = \mathrm{Ch}^\bullet(B\mathrm{GL}_n).$$

Proof The claims follow from the Wu formula in Proposition 3.12. The odd Chern classes are in the kernel of the twisted Steenrod square, and $\bar{c}_n$ is the image of the Euler class. Even though the twisted Steenrod square for Chern classes is given essentially by the same formula as the Steenrod square in $\mathrm{Ch}^\bullet(B\mathrm{SL}_n)$, the formula differs from [16] since $\mathrm{Sq}^2_{\det\gamma_n^\vee}(\bar{c}_{2i}^2) = \bar{c}_1\bar{c}_{2i}^2$. For the untwisted Steenrod square $\mathrm{Sq}^2_{\mathbb{O}}$, the even classes $\bar{c}_{2i}$ map to $\bar{c}_1\bar{c}_{2i} + \bar{c}_{2i+1}$. Hence the latter classes are in the kernel of the Steenrod square; similarly for $\bar{c}_1\bar{c}_n$. The description of the kernels follow from that; see also [26, page 285]. □

Corollary 3.14 *Consider the mod 2 Chow ring $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$. The images of the Steenrod square maps $\mathrm{Sq}^2_{\mathscr{L}}\colon \mathrm{Ch}^\bullet(B\mathrm{GL}_n) \to \mathrm{Ch}^{\bullet+1}(B\mathrm{GL}_n)$ for $\mathscr{L} = \mathscr{O}, \det\gamma_n^\vee$ are contained in the subring generated by the classes $\bar{c}_1 = \mathrm{Sq}^2_{\det\gamma_n}(1)$, $\bar{c}_{2i}^2$, and $\bar{c}_n$ as well as $\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$ and $\mathrm{Sq}^2_{\det\gamma_n}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$ for (possibly empty) sequences of positive natural numbers $0 < j_1 < j_2 < \cdots < j_l \le \left[\frac{1}{2}(n-1)\right]$.*

Proof The Steenrod squares $\mathrm{Sq}^2_{\mathscr{L}}$ are linear. To determine generators of the image, it thus suffices to consider Steenrod squares of monomials in the Chern classes.

Since $\mathrm{Sq}^2_{\mathscr{O}}$ is a derivation, $\mathrm{Sq}^2_{\mathscr{O}}(x^2) = 2x\mathrm{Sq}^2_{\mathscr{O}}(x) = 0$ and $\mathrm{Sq}^2_{\mathscr{O}}(x^2y) = x^2\mathrm{Sq}^2_{\mathscr{O}}(y)$. In particular, we can always pull out squares. For even Stiefel–Whitney classes, these squares are explicitly included as generators in the statement. For the odd Stiefel–Whitney classes,

$$\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i}\bar{c}_{2i+1}) = \bar{c}_{2i}\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i+1}) + \bar{c}_{2i+1}\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i}) = 2\bar{c}_1\bar{c}_{2i}\bar{c}_{2i+1} + \bar{c}_{2i+1}^2.$$

It thus suffices to show that the Steenrod squares of all products $\bar{c}_{j_1}\cdots\bar{c}_{j_m}$ with no repeating factors are contained in the subring as claimed.

For the odd Stiefel–Whitney classes,

$$\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_{2i+1}x) = \bar{c}_{2i+1}\mathrm{Sq}^2(x) + \bar{c}_1\bar{c}_{2i+1}x = \bar{c}_{2i+1}\mathrm{Sq}^2_{\det\gamma_n}(x).$$

Since $\bar{c}_{2i+1} = \mathrm{Sq}^2_{\det\gamma_n}(\bar{c}_{2i})$ with the special case $\bar{c}_1 = \mathrm{Sq}^2_{\det\gamma_n}(1)$, the odd Stiefel–Whitney classes are also among the generators of the subring listed in the claim. Therefore, we can also pull out all the odd Stiefel–Whitney classes from the products $\bar{c}_{j_1}\cdots\bar{c}_{j_m}$. A similar calculation shows that we can also pull out $\bar{c}_n$, which is also included explicitly among the generators. We have thus established the claim for $\mathrm{Sq}^2_{\mathscr{O}}$.

To show the claim for $\mathrm{Sq}^2_{\det\gamma_n}$, we first have $\mathrm{Sq}^2_{\det\gamma_n}(x^2) = \bar{c}_1x^2$ and

$$\mathrm{Sq}^2_{\det\gamma_n}(x^2y) = \bar{c}_1x^2y + x^2\mathrm{Sq}^2_{\mathscr{O}}(y) = x^2\mathrm{Sq}^2_{\det\gamma_n}(y).$$

This tells us again that we can always pull out squares. For the odd Stiefel–Whitney classes,

$$\mathrm{Sq}^2_{\det\gamma_n}(\bar{c}_{2i+1}x) = \bar{c}_1\bar{c}_{2i+1}x + \bar{c}_{2i+1}\mathrm{Sq}^2_{\det\gamma_n}(x) = \bar{c}_{2i+1}\mathrm{Sq}^2_{\mathscr{O}}(x).$$

Therefore,we can also pull out odd Stiefel–Whitney classes (and by a similar computation also $\bar{c}_n$). □

3.7 The candidate presentation

We define an appropriate graded ring $\mathscr{R}_n/\mathscr{I}_n$ which we will prove to be isomorphic to

$$H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det\gamma_n^\vee)).$$

The ring will be graded by $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, where the degrees $(n, 0)$ are those with $\boldsymbol{I}^\bullet$–coefficients, and the degrees $(n, 1)$ are those with $\boldsymbol{I}^\bullet(\det\gamma_n^\vee)$–coefficients. The ring will in fact be graded-commutative; see the discussion in Remark 2.3. Following [26], we use the notation $\Delta(J, J') = (J \cup J') \setminus (J \cap J')$ for the symmetric difference of two subsets J and J' of a given set.

Definition 3.15 Let F be a field of characteristic $\neq 2$ and denote by $W(F)$ the Witt ring of quadratic forms over F with its fundamental ideal $I(F) \subseteq W(F)$ of even-dimensional forms. For a natural number $n \geq 1$, we define the $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$*–graded-commutative* $W(F)$–algebra

$$\mathcal{R}_n = W(F)[P_1, \ldots, P_{[(n-1)/2]}, X_n, B_J, T_J, T_\varnothing].$$

The classes P_i have degree $(4i, 0)$ and the class X_n has degree $(n, 1)$. For the classes B_J and T_J, the index set J runs through the (possibly empty) sets $\{j_1, \ldots, j_l\}$ of positive natural numbers with $0 < j_1 < \cdots < j_l \leq \left[\frac{1}{2}(n-1)\right]$, and the degrees of B_J and T_J are $(d, 0)$ and $(d, 1)$, respectively, with $d = 1 + 2\sum_{a=1}^{l} j_a$. By convention $B_\varnothing = 0$.

Let $\mathcal{I}_n \subset \mathcal{R}_n$ be the ideal generated by the following relations:

(1) $I(F)B_J = I(F)T_J = I(F)T_\varnothing = 0$.

(2) If $n = 2k+1$ is odd and $k \geq 1$, then $X_{2k+1} = T_{\{k\}}$; for $n = 1$ we have $X_1 = T_\varnothing$.

(3) For any two index sets J and J', where J' can be empty,

$$B_J \cdot B_{J'} = \sum_{k \in J} B_{\{k\}} \cdot P_{(J \setminus \{k\}) \cap J'} \cdot B_{\Delta(J \setminus \{k\}, J')}, \tag{3-1}$$

$$B_J \cdot T_{J'} = \sum_{k \in J} B_{\{k\}} \cdot P_{(J \setminus \{k\}) \cap J'} \cdot T_{\Delta(J \setminus \{k\}, J')}, \tag{3-2}$$

$$T_J \cdot B_{J'} = B_J \cdot T_{J'} + T_\varnothing \cdot P_{J \cap J'} \cdot B_{\Delta(J, J')}, \tag{3-3}$$

$$T_J \cdot T_{J'} = B_J \cdot B_{J'} + T_\varnothing \cdot P_{J \cap J'} \cdot T_{\Delta(J, J')}. \tag{3-4}$$

Here we set $P_A = \prod_{i=1}^{l} P_{a_i}$ for an index set $A = \{a_1, \ldots, a_l\}$, with the usual convention that $P_\varnothing = 1$ (in the degree $(0,0)$ component of $\mathcal{R}_n$).

Example 3.16 We briefly discuss the edge case $n = 1$. In this case, no classes P_i appear, and there are no classes B_J or T_J with J nonempty. The only relevant generators are X_1 and $T_\varnothing$. From relation (2), we get $X_1 = T_\varnothing$. Relation (1) implies that this class is $I(F)$–torsion. From the relations in (3), only (3-4) would be applicable, but that trivializes to $T_\varnothing^2 = T_\varnothing^2$. The resulting ring has $W(F)$ in degree 0, generated by 1, and has a $W(F)/I(F) \cong \mathbb{Z}/2\mathbb{Z}$–summand generated by $T_\varnothing^i$ in degree $(i, i \bmod 2)$ for each $i \geq 1$.

Remark 3.17 We will show in Theorem 3.24 that the ring $\mathcal{R}_n/\mathcal{I}_n$ is isomorphic to the total $\boldsymbol{I}$–cohomology ring of $B\mathrm{GL}_n$. The classes P_i correspond to Pontryagin classes, and the class X_n to the Euler class. The classes B_J and T_J for a (possibly empty) index set $J = \{j_1, \ldots, j_l\}$ of positive natural numbers with $0 < j_1 < \cdots < j_l \leq \left[\frac{1}{2}(n-1)\right]$ correspond to nontwisted and twisted Bockstein classes, respectively,

$$B_J \mapsto \beta_{\mathbb{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}), \quad T_J \mapsto \beta_{\det \gamma_n^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}).$$

The special class $T_\varnothing$ corresponds to $\beta_{\det \gamma_n^\vee}(1)$ whose reduction in $\mathrm{Ch}^1(B\mathrm{GL}_n)$ is the first Stiefel–Whitney class $\bar{c}_1 = \mathrm{Sq}^2_{\det \gamma_n^\vee}(1)$.

Remark 3.18 There are slight differences in the indexing sets between the formulas in [8] and [26]. For the Pontryagin classes, this difference is due to fact that the Euler class squares to the top Pontryagin class. So in Čadek's presentation, there is no need to introduce the top Pontryagin class; on the other hand, Brown only computes cohomology with trivial coefficients and he has to introduce the top Pontryagin class separately. The same thing is true for the Bockstein classes:

$$\beta_{\mathcal{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = \beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_{l-1}})e_{n-1} \quad \text{if } j_l = \tfrac{1}{2}(n-1),$$

and this relation cannot be expressed in cohomology with trivial coefficients. Moreover, the reason why Brown's additional $\bar{c}_1$–factors in the Bockstein classes can be omitted in Čadek's presentation is given by the formula

$$\beta_{\mathcal{O}}(\bar{c}_1\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = \beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_{l-1}})\beta_{\det\gamma_n^\vee}(1).$$

Definition 3.19 Let $n \geq 2$ be a natural number. Define the $W(F)$–algebra homomorphism $\Phi_n : \mathcal{R}_n \to \mathcal{R}_{n-1}$ by

(1) the element P_i maps to P_i if $i < \frac{1}{2}(n-1)$ and maps to X_{n-1}^2 if $i = \frac{1}{2}(n-1)$,

(2) the element X_n maps to 0,

(3) for any index set $J = \{j_1, \dots, j_l\}$,

$$B_J \mapsto \begin{cases} B_J & \text{if } j_l < \frac{1}{2}(n-1), \\ T_{J'} \cdot X_{n-1} & \text{if } j_l = \frac{1}{2}(n-1),\ J = J' \sqcup \{j_l\}, \end{cases}$$
$$T_J \mapsto \begin{cases} T_J & \text{if } j_l < \frac{1}{2}(n-1), \\ B_{J'} \cdot X_{n-1} & \text{if } j_l = \frac{1}{2}(n-1),\ J = J' \sqcup \{j_l\}. \end{cases}$$

Remark 3.20 The above formulas model the restriction of classes from $B\mathrm{GL}_n$ to $B\mathrm{GL}_{n-1}$. On the level of mod 2 Chow rings,

$$\mathrm{Sq}^2_{\mathcal{L}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l}) = \mathrm{Sq}^2_{\mathcal{L}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_{l-1}})\bar{c}_{2j_l} + \bar{c}_1\bar{c}_{2j_1}\cdots\bar{c}_{2j_l} = \mathrm{Sq}^2_{\mathcal{L}\otimes\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_{l-1}})e_{n-1},$$

using Proposition 2.2. Note that the formulas for restriction on the bottom of page 283 in [26] contain some typos, the classes having the wrong degrees.

Proposition 3.21 *With the notation from Definitions 3.15 and 3.19,*

$$\Phi_n(\mathcal{I}_n) \subseteq \mathcal{I}_{n-1}.$$

In particular, the map Φ_n descends to a well-defined ring homomorphism

$$\bar{\Phi}_n : \mathcal{R}_n/\mathcal{I}_n \to \mathcal{R}_{n-1}/\mathcal{I}_{n-1}.$$

Proof We first deal with the relations of type (1). Recall that the map Φ_n is by definition $W(F)$–linear; in particular, it will send $I(F)$ to $I(F)$. Since Φ_n sends B_J to either B_J or $T_{J'} \cdot X_{n-1}$ (and similarly T_J to either T_J or $B_{J'} \cdot X_{n-1}$, with the special case $B_\varnothing = 0$) it is clear the relations of type (1) are preserved.

The relations of type (2) are also preserved since both X_{2k+1} and T_k are mapped to 0 by Φ_n.

It remains to deal with relations of type (3). These relations are trivially preserved if neither J nor J' contains the highest possible index $j_l = \frac{1}{2}(n-1)$. In this case, all the relevant B_J, T_J and P_J will exist both in $\mathcal{R}_n$ and $\mathcal{R}_{n-1}$, and the corresponding relation in $\mathcal{R}_n$ is just mapped to the same relation in $\mathcal{R}_{n-1}$. For the rest of the proof, we will use the numbering (3-1)–(3-4) specified in Definition 3.15.

For relations of type (3-1), assume that $j_l \in J'$ and $j_l \notin J$. On the left-hand side, $B_{J'}$ restricts to $T_{J'\setminus\{j_l\}} \cdot X_{n-1}$ and on the right-hand side, $B_{\Delta(J\setminus\{k\},J')}$ restricts to $B_{\Delta(J\setminus\{k\},J'\setminus\{j_l\})} \cdot X_{n-1}$. The result is the product of a relation of type (3-2) with X_{n-1}. Conversely, if $j_l \in J$ and $j_l \notin J'$, then the left-hand side restricts to $T_{J\setminus\{j_l\}} \cdot X_{n-1} \cdot B_{J'}$. The right-hand side restricts to

$$\sum_{k\in J\setminus\{j_l\}} B_{\{k\}} \cdot P_{(J\setminus\{k\})\cap J'} \cdot T_{\Delta(J\setminus\{k,j_l\},J')} \cdot X_{n-1} + T_\varnothing \cdot X_{n-1} \cdot P_{(J\setminus\{j_l\})\cap J'} \cdot B_{\Delta(J\setminus\{j_l\},J')}.$$

But this is the product of a relation of type (3-3) and X_{n-1}. Finally, when $j_l \in J \cap J'$, the left-hand side restricts to $T_{J\setminus\{j_l\}} \cdot T_{J'\setminus\{j_l\}} \cdot X_{n-1}^2$. The right-hand side restricts to

$$\sum_{k\in J\setminus\{j_l\}} B_{\{k\}} \cdot P_{(J\setminus\{k,j_l\})\cap J'} \cdot X_{n-1}^2 \cdot B_{\Delta(J\setminus\{k\},J')} + T_\varnothing \cdot P_{(J\setminus\{j_l\})\cap J'} \cdot T_{\Delta(J,J')} \cdot X_{n-1}^2.$$

This is a product of a relation of type (3-4) with X_{n-1}^2. The argument for restriction of relations of type (3-2) is completely analogous.

For the restriction of relations of type (3-4), if $j_l \in J$ and $j_l \notin J'$, the left-hand side restricts to $B_J \cdot T_{J'} \cdot X_{n-1}$. The right-hand side restricts to $T_J \cdot B_{J'} \cdot X_{n-1} + T_\varnothing \cdot P_{J\cap J'} \cdot B_{\Delta(J\setminus\{j_l\},J')} \cdot X_{n-1}$. This is the product of a relation of type (3-3) with X_{n-1}, noting that all terms here are 2–torsion. All the other cases are done similarly, and the argument for relations (3-3) is again analogous.

Since $\Phi_n(\mathcal{I}_n) \subset \mathcal{I}_{n-1}$, it follows that the restriction map descends to a $W(F)$–algebra map

$$\overline{\Phi}_n : \mathcal{R}_n/\mathcal{I}_n \to \mathcal{R}_{n-1}/\mathcal{I}_{n-1},$$

as claimed. □

Lemma 3.22 *If n is even, then we have an isomorphism*

$$\mathcal{R}_n/\mathcal{I}_n \cong \mathcal{R}_{n-1}/\mathcal{I}_{n-1}[X_n].$$

In particular, the restriction map $\overline{\Phi}_n : \mathcal{R}_n/\mathcal{I}_n \to \mathcal{R}_{n-1}/\mathcal{I}_{n-1}$ is surjective.

Proof The index sets for the elements P_i are the same for n and $n-1$. In particular, $i \neq \frac{1}{2}(n-1)$ which means that the P_i in $\mathcal{R}_n$ are just mapped to the P_i in $\mathcal{R}_{n-1}$. The same is true for the index sets for B_J and T_J. Moreover, in $\mathcal{R}_{n-1}/\mathcal{I}_{n-1}$ we have $X_{n-1} = T_{\{(n-2)/2\}}$. This proves the surjectivity of Φ_n. The claim about the polynomial ring follows since X_n doesn't appear in any relation in $\mathcal{R}_n$. □

Lemma 3.23 *If n is odd, then there is an exact sequence of graded $W(F)$–algebras*

$$\mathcal{R}_n/\mathcal{I}_n \xrightarrow{\overline{\Phi}_n} \mathcal{R}_{n-1}/\mathcal{I}_{n-1} \to W(F)[X_{n-1}]/(X_{n-1}^2) \to 0.$$

Proof The elements $P_i \in \mathcal{R}_n$ with $i < \frac{1}{2}(n-1)$ are mapped under Φ_n to the elements with the same name in $\mathcal{R}_{n-1}$. The same holds for the elements B_J and T_J where the index set J doesn't contain $\frac{1}{2}(n-1)$. In particular, the subalgebra of $\mathcal{R}_{n-1}/\mathcal{I}_{n-1}$ generated by all P_i, B_J and T_J is in the image. The only elements in $\mathcal{R}_n$ we have not yet considered so far are the new $P_{(n-1)/2}$ and the elements B_J and T_J where J contains $\frac{1}{2}(n-1)$. The element X_{n-1}^2 is in the image of $P_{(n-1)/2}$, the elements $B'_J X_{n-1}$ are in the image of T_J and the elements $T'_J X_{n-1}$ are in the image of B_J. However, the element X_{n-1} itself is not in the image since we noted in Lemma 3.22 that it is a polynomial variable in $\mathcal{R}_{n-1}$. Consequently, defining the morphism $\mathcal{R}_{n-1}/\mathcal{I}_{n-1} \to W(F)[X_{n-1}]/(X_{n-1}^2)$ by sending X_{n-1} to itself and all the other generators to 0 yields the desired exact sequence. □

3.8 Statement of results

Now we are ready to state the main theorem describing the $\boldsymbol{I}^\bullet$–cohomology and Chow–Witt ring of $B\mathrm{GL}_n$. For the $\boldsymbol{I}^\bullet$–cohomology, the result is very close to Čadek's computation of the integral cohomology of $BO(n)$ with twisted coefficients; see [26].

Theorem 3.24 *Let $n \geq 1$ be a natural number.*

(1) *The ring homomorphism*

$$\begin{aligned}\theta_n : \mathcal{R}_n &\to \bigoplus_q H^q(B\mathrm{GL}_n, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_n^\vee)),\\ P_i &\mapsto p_{2i},\\ X_n &\mapsto e_n,\\ B_J &\mapsto \beta_{\mathcal{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \quad \text{for } J = \{j_1, \dots, j_l\},\\ T_J &\mapsto \beta_{\det \gamma_n^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \quad \text{for } J = \{j_1, \dots, j_l\},\\ T_\varnothing &\mapsto \beta_{\det \gamma_n^\vee}(1),\end{aligned}$$

induces a ring isomorphism $\bar{\theta}_n : \mathcal{R}_n/\mathcal{I}_n \xrightarrow{\cong} H^\bullet_{\mathrm{Nis}}(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \gamma_n^\vee))$.

(2) *For any line bundle $\mathcal{L}$ on $B\mathrm{GL}_n$, the reduction morphism*

$$H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet(\mathcal{L})) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$$

induced from the projection $\boldsymbol{I}^n(\mathcal{L}) \to \boldsymbol{K}_n^{\mathrm{M}}/2$ is explicitly given by mapping

$$p_{2i} \mapsto \bar{c}_{2i}^2, \quad \beta_{\mathcal{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}) \mapsto \mathrm{Sq}^2_{\mathcal{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l}), \quad e_n \mapsto \bar{c}_n.$$

(3) *Any class x in the ideal of $H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \gamma_n^\vee))$ generated by β_J and τ_J is trivial if and only if its reduction $\rho(x) \in \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$ is trivial.*

Remark 3.25 This theorem is one of the key components of Theorem 1.1. More concretely, the theorem together with Definition 3.15 provides a generators-and-relations description of the total $\boldsymbol{I}$–cohomology ring

$$\bigoplus_q H^q(B\mathrm{GL}_n, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_n^\vee))$$

as a $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded algebra over the Witt ring $W(F)$. The generators are the Pontryagin classes $p_1,\dots,p_{[(n-1)/2]}$, the Euler class e_n and the (nontwisted and twisted) Bockstein classes

$$\beta_J=\beta_{\mathcal{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})\quad\text{and}\quad\tau_J=\beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$$

for an index set $J=\{j_1,\dots,j_l\}$ of natural numbers $0<j_1<j_2<\cdots<j_l\le\left[\frac{1}{2}(n-1)\right]$, plus the additional $\beta_{\det\gamma_n^\vee}(1)$. Spelling out Definition 3.15, the relations between these classes are as follows:

(1) The Bockstein classes $\beta_{\mathcal{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$, $\beta_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$ and $\beta_{\det\gamma_n^\vee}(1)$ are $I(F)$–torsion.

(2) For $n=2k+1$, the Euler class is a twisted Bockstein class: $e_n=\beta_{\det\gamma_n^\vee}(\bar{c}_{2k})$.

(3) For any two index sets J and J', where J' can be empty, multiplication of Bockstein classes is explicitly given by

$$\beta_J\cdot\beta_{J'}=\sum_{k\in J}\beta_{\{k\}}\cdot p_{(J\setminus\{k\})\cap J'}\cdot\beta_{\Delta(J\setminus\{k\},J')},$$
$$\beta_J\cdot\tau_{J'}=\sum_{k\in J}\beta_{\{k\}}\cdot p_{(J\setminus\{k\})\cap J'}\cdot\tau_{\Delta(J\setminus\{k\},J')},$$
$$\tau_J\cdot\beta_{J'}=\beta_J\cdot\tau_{J'}+\tau_{\varnothing}\cdot p_{J\cap J'}\cdot\beta_{\Delta(J,J')},$$
$$\tau_J\cdot\tau_{J'}=\beta_J\cdot\beta_{J'}+\tau_{\varnothing}\cdot p_{J\cap J'}\cdot\tau_{\Delta(J,J')}.$$

In the above, $p_A=\prod_{i=1}^{l}p_{a_i}$ for an index set $A=\{a_1,\dots,a_l\}$ with the special case $p_{\varnothing}=1$.

Compared to the known integral singular cohomology ring of $BO(n)$, the Bockstein classes generate what is the 2–torsion part of the integral singular cohomology. This part of the $\boldsymbol{I}$–cohomology ring is always the same, independent of the base field F. The part generated by the Pontryagin classes (plus the Euler class for even n) corresponds to the torsion-free part of integral singular cohomology. In $\boldsymbol{I}$–cohomology, it is a free $W(F)$–module. It depends on the base field via $W(F)$, but its rank as a $W(F)$–module is again independent of the base field.

The proof will be given in Section 4. For now we draw some consequences concerning the structure of the Chow–Witt ring of $B\mathrm{GL}_n$.

Proposition 3.26 (1) *The kernel of the composition*

$$\partial_{\mathcal{O}}:\mathrm{CH}^\bullet(B\mathrm{GL}_n)\to\mathrm{Ch}^\bullet(B\mathrm{GL}_n)\xrightarrow{\beta_{\mathcal{O}}}H^{\bullet+1}(B\mathrm{GL}_n,\boldsymbol{I}^{\bullet+1})$$

is the subring

$$\ker\partial_{\mathcal{O}}=\mathbb{Z}\left[\{c_i^2\}_{1\le i\le n},\{c_1c_{2k}+c_{2k+1}\}_{1\le k\le[(n-1)/2]},c_1c_n,(2)\right]\subseteq\mathbb{Z}[c_1,\dots,c_n]\cong\mathrm{CH}^\bullet(B\mathrm{GL}_n).$$

(2) *The kernel of the composition*

$$\partial_{\det\gamma_n^\vee}:\mathrm{CH}^\bullet(B\mathrm{GL}_n)\to\mathrm{Ch}^\bullet(B\mathrm{GL}_n)\xrightarrow{\beta_{\det\gamma_n^\vee}}H^{\bullet+1}(B\mathrm{GL}_n,\boldsymbol{I}^{\bullet+1}(\det\gamma_n^\vee))$$

is the $\ker\partial_{\mathcal{O}}$*–submodule of* $\mathbb{Z}[c_1,\dots,c_n]\cong\mathrm{CH}^\bullet(B\mathrm{GL}_n)$,

$$\ker\partial_{\det\gamma_n^\vee}=\langle\{c_{2k+1}\}_{1\le k\le[(n-1)/2]},c_n,(2)\rangle_{\ker\partial_{\mathcal{O}}}\subseteq\mathrm{CH}^\bullet(B\mathrm{GL}_n).$$

Proof By (3) of Theorem 3.24 and Proposition 2.2, the kernel of $\beta_{\mathcal{L}}$ equals the kernel of $\mathrm{Sq}^2_{\mathcal{L}}$ and the latter is determined by the Wu formula; see Corollary 3.13. Then statements (1) and (2) for the Chow ring follow directly from the corresponding statement for the mod 2 Chow ring in Corollary 3.13, adding as additional generators the elements of the ideal (2). □

The following theorem now establishes the first item of Theorem 1.1. The structure of the $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded algebra on $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2}$ is discussed before the statement of Theorem 1.1.

Theorem 3.27 *There is a cartesian square of $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded $\mathrm{GW}(F)$–algebras*

$$\begin{array}{ccc} \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_n, \mathcal{O}\oplus\det\gamma_n^\vee) & \longrightarrow & \ker\partial_{\mathcal{O}}\oplus\ker\partial_{\det\gamma_n^\vee} \\ \downarrow & & \downarrow{\scriptstyle \text{mod } 2} \\ H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet\oplus\boldsymbol{I}^\bullet(\det\gamma_n^\vee)) & \xrightarrow[\rho]{} & \mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2} \end{array}$$

The right vertical morphism is the natural reduction mod 2 restricted to the kernels of the two boundary maps, and the lower horizontal morphism is the reduction morphism described in Theorem 3.24. The Chow–Witt-theoretic Euler class satisfies $e_n^{\widetilde{\mathrm{CH}}} = (e_n^{\boldsymbol{I}}, c_n)$ with $c_n \in \ker\partial_{\det\gamma_n^\vee}$.[2] *For the Chow–Witt-theoretic Pontryagin classes,*

$$p_i^{\widetilde{\mathrm{CH}}} = \left(p_i^{\boldsymbol{I}}, (-1)^i c_i^2 + 2\sum_{j=\max\{0,2i-n\}}^{i-1} (-1)^j c_j c_{2i-j}\right),$$

where the odd Pontryagin classes[3] *in $\boldsymbol{I}$–cohomology are $I(F)$–torsion and satisfy*

$$p_{2i+1} = (\beta_{\mathcal{O}}(\bar{c}_{2i}))^2 + p_{2i}\beta_{\mathcal{O}}(\bar{c}_1) = \beta_{\mathcal{O}}(\bar{c}_{2i}\bar{c}_{2i+1}).$$

The top Pontryagin class $p_n \in \widetilde{\mathrm{CH}}^{2n}(B\mathrm{Sp}_{2n})$ maps to $e_n^2 \in \widetilde{\mathrm{CH}}^{2n}(B\mathrm{GL}_n, \mathcal{O})$.

Proof The statement about the cartesian square follows directly from [16, Proposition 2.11]. The claims about the reduction from the Chow–Witt ring to $\boldsymbol{I}$–cohomology follows from the definition of the characteristic classes. The statement about e_n and c_n follows from Proposition 3.3. The statement about the p_i has been proved in [16, Theorem 6.10]. For the description of odd Pontryagin classes in terms of Bockstein classes, we first note that the injectivity of restriction to $B\mathrm{SL}_n$ as discussed at the end of the proof of Proposition 4.5 combined with [16, Theorem 6.10 or Proposition 8.16] implies that odd Pontryagin classes are $I(F)$–torsion. The alternative description follows using Theorem 3.24(3) by showing the equality after reduction in $\mathrm{Ch}^\bullet(B\mathrm{GL}_n)$, which is Proposition 4.10; see also [16, Remark 8.17]. The statement about the top Pontryagin class is proved in Proposition 4.5, or [16, Proposition 7.9]. □

[2] Here and in the formula for Pontryagin classes, upper indices have been added to clarify the relevant cohomology theory: Chow–Witt on the left-hand side and $\boldsymbol{I}$–cohomology on the right-hand side.

[3] For a remark on odd Pontryagin classes, see Remark 3.5.

Proposition 3.28 *The restriction along the Whitney sum map* $B(\mathrm{GL}_n \times \mathrm{GL}_m) \to B\mathrm{GL}_{n+m}$ *maps the Pontryagin classes as*

$$p_i \mapsto \sum_{j=\max\{0,i-m\}}^{\min\{i,n\}} p_j \otimes p_{i-j},$$

where the sum is over the indices j *such that* p_j *and* p_{i-j} *are Pontryagin classes for* GL_n *and* GL_m*, respectively.*

Proof The Whitney sum formula follows directly from the Whitney sum formula for the Pontryagin classes of symplectic bundles and the compatibility of Whitney sum and symplectification; see [16]. □

Remark 3.29 The Whitney sum formula above is exactly the classical one from [8]. It is easier to state simply by our conventions — see [16, Remark 5.7] — concerning indexing of the Pontryagin classes.

Example 3.30 To clarify the relation between the cohomology of $B\mathrm{GL}_n$ and $B\mathrm{SL}_n$ — see [16, Example 6.12] — we conclude this section with a detailed description of the cartesian square for $B\mathrm{GL}_3$ with both dualities.

For the trivial duality, we have the cartesian square

$$\begin{array}{ccc} \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_3, \mathbb{O}) & \longrightarrow & \mathbb{Z}[c_i^2, c_1c_2+c_3, c_1c_3, (2)] \\ \downarrow & & \downarrow \\ W(F)[p_2, \beta_{\mathbb{O}}(\bar{c}_1), \beta_{\mathbb{O}}(\bar{c}_2), \beta_{\mathbb{O}}(\bar{c}_1\bar{c}_2)]/\mathcal{I}_{3,\mathbb{O}} & \xrightarrow[\rho]{} & \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \bar{c}_2, \bar{c}_3] \end{array}$$

In the upper-right corner, we have the subring of the Chow ring (in particular containing $1 \in \mathrm{CH}^0(B\mathrm{GL}_3)$) generated by everything 2–divisible, squares of Chern classes and the classes $c_1c_2 + c_3$ and c_1c_3.

The structure of the $W(F)$–algebra in the lower left can be made more explicit using the $\operatorname{Im}\beta$–$\boldsymbol{W}$–decomposition: the $\boldsymbol{W}$–cohomology is

$$H^\bullet(B\mathrm{GL}_3, \boldsymbol{W}) \cong W(F)[p_2].$$

This is a part that depends on the underlying field via $W(F)$, but the presentation as $W(F)$–algebra is independent of the field — it is always a polynomial $W(F)$–algebra in p_2.

The other summand of the $\boldsymbol{I}$–cohomology in the lower-left corner is the image of $\beta_{\mathbb{O}}$, this part is independent of the base, it is the same as the 2–torsion in the integral singular cohomology of $BO(3)$. As generators, we have $\beta_{\mathbb{O}}(\bar{c}_2)$, the odd Pontryagin class $p_1 = \beta_{\mathbb{O}}(\bar{c}_1) = \beta_{\det \gamma_n^\vee}(1)^2$ and the class

$$\beta_{\mathbb{O}}(\bar{c}_3) = \beta_{\mathbb{O}}(\bar{c}_1\bar{c}_2) = \beta_{\det \gamma_n^\vee}(1)\beta_{\det \gamma_n^\vee}(\bar{c}_2).$$

Note that the class $\beta_{\mathbb{O}}(\bar{c}_2)$ is the generator β_J for $J = \{1\}$ and is actually the only generator of this form in this case. All the other classes written above are products of twisted Bockstein classes, as indicated.[4]

[4] See also the discussion of the relation between the presentations of Brown and Čadek in Remark 3.18.

Using that ρ is injective on the image of β, one can use an argument as in the proof of Corollary 3.14 to show that any torsion class is contained in the subring generated by these classes together with p_2. For example, the third Pontryagin class can be expressed as

$$p_3 = \beta_{\mathcal{O}}(\bar{c}_2\bar{c}_3) = \beta_{\mathcal{O}}(\bar{c}_2)^2 + p_2\beta_{\mathcal{O}}(\bar{c}_1).$$

The reductions of the Pontryagin classes are the squares of Chern classes, and the reductions of the other two classes are

$$\rho(\beta_{\mathcal{O}}(\bar{c}_2)) = \bar{c}_1\bar{c}_2 + \bar{c}_3, \quad \rho(\beta_{\mathcal{O}}(\bar{c}_3)) = \bar{c}_1\bar{c}_3.$$

In particular, we recover exactly the generators of $\ker\beta_{\mathcal{O}} \subseteq \mathrm{Ch}^\bullet(B\mathrm{GL}_3)$. As an example, the class $(\beta_{\mathcal{O}}(\bar{c}_2), c_1c_2 + c_3)$ is then a class in the Chow–Witt ring, because both classes have the same mod 2 reductions.

For the nontrivial duality, we have the square

$$\begin{array}{ccc} \widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_3, \det\gamma_n^\vee) & \longrightarrow & \langle c_1, c_3, (2)\rangle_{\ker\partial_{\mathcal{O}}} \\ \downarrow & & \downarrow \\ \langle \beta_{\det\gamma_n^\vee}(1), \beta_{\det\gamma_n^\vee}(\bar{c}_2)\rangle & \longrightarrow & \mathbb{Z}/2\mathbb{Z}[\bar{c}_1, \bar{c}_2, \bar{c}_3] \end{array}$$

Here the upper-right corner is the sub–$\ker_{\mathcal{O}}$–module of $\mathrm{CH}^\bullet(B\mathrm{GL}_3)$ with the indicated generators. For the lower-left corner, the twisted $\boldsymbol{W}$–cohomology is trivial, since the $\boldsymbol{W}$–cohomological Euler class for odd-rank vector bundles is trivial. So the $\boldsymbol{I}$–cohomology is $I(F)$–torsion. As a module over $\bigoplus_q H^q(B\mathrm{GL}_3, \boldsymbol{I}^q)$ — the lower-left corner of the upper diagram for the nontwisted case — it is generated by $\beta_{\det\gamma_n^\vee}(1)$ and the Euler class $e_3 = \beta_{\det\gamma_n^\vee}(\bar{c}_2)$, which in the notation of Remark 3.25 are $\tau_\varnothing$ and $\tau_{\{1\}}$, respectively. The remaining torsion relations, in particular describing further multiplication rules, are not completely spelled out for typesetting reasons.

4 The Chow–Witt ring of $B\mathrm{GL}_n$: proofs

The main goal of this section is to prove Theorem 3.24 which is a Chow–Witt analogue of Čadek's description of integral cohomology of $BO(n)$ with local coefficients. The arguments are based on the decomposition into $\boldsymbol{W}$–cohomology and the image of β.

4.1 Projective spaces

As a first step we need to recall the computations of the $\boldsymbol{I}^\bullet$–cohomology and Chow–Witt rings of projective spaces $\mathbb{P}^n$ from [14]. Since $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, there are only two possible dualities to consider, given by the line bundles $\mathcal{O}_{\mathbb{P}^n}$ and $\mathcal{O}_{\mathbb{P}^n}(1)$.

It is a most classical computation that $\mathrm{Ch}^\bullet(\mathbb{P}^n) \cong \mathbb{Z}/2\mathbb{Z}[\bar{c}_1]/(\bar{c}_1^{n+1})$. The Steenrod squares are given by $\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_1) = \bar{c}_1^2$ and $\mathrm{Sq}^2_{\mathcal{O}(1)}(\bar{c}_1) = 0$. In particular, $\ker\mathrm{Sq}^2_{\mathcal{O}} = \mathbb{Z}/2\mathbb{Z}[\bar{c}_1^2]$, and the kernel of $\mathrm{Sq}^2_{\mathcal{O}(1)}$ is the submodule of $\mathrm{Ch}^\bullet(\mathbb{P}^n)$ generated by odd powers of $\bar{c}_1$.

The following is a direct reformulation of the computations in [14, Section 11].

Proposition 4.1 (1) *If n is odd, then*

$$\bigoplus_q H^q(\mathbb{P}^n, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_1^\vee)) \cong W(F)[e_1, R]/(I(F)\cdot e_1, e_1^{n+1}, e_1 R, R^2).$$

Moreover, $e_1 = \beta_{\mathbb{O}(1)}(1)$ and $R \in H^n(\mathbb{P}^n, \boldsymbol{I}^n)$ is the fundamental class of $\mathbb{P}^n$ (which is orientable in this case). The image of R under the reduction morphism ρ is $\bar{c}_n \in \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$.

(2) *If n is even, then*

$$\bigoplus_q H^q(\mathbb{P}^n, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_1^\vee)) \cong W(F)[e_1, e_n^\perp]/(I(F)\cdot e_1, e_1^{n+1}, e_1 e_n^\perp, (e_n^\perp)^2).$$

Again, $e_1 = \beta_{\mathbb{O}(1)}(1)$, and the class $e_n^\perp \in H^n(\mathbb{P}^n, \boldsymbol{I}^n(\det \gamma_1^\vee))$ is the Euler class of the rank n hyperplane bundle on $\mathbb{P}^n \cong (\mathbb{P}^n)^\vee$.

Proof Note that [14] only establishes the additive structure statements, not quite the full presentation of the ring structure as formulated. Nevertheless, the statements about the ring structure follow from this: since we already know some characteristic classes of vector bundles, we obtain a ring homomorphism from our claimed presentation to the cohomology ring of $\mathbb{P}^n$. Additively, we also know that the Euler class reduces to $\bar{c}_1$; in particular the nontriviality of the powers of the Euler class is then immediate and this already deals with all the torsion classes. The statement for the nontorsion classes R and $e_n^\perp$ follows directly, since these cannot have nontrivial intersections with anything else for dimension reasons. □

Remark 4.2 The classical presentations of the integral cohomology of real projective spaces are recovered exactly for $F = \mathbb{R}$. The algebraic Euler class maps to the topological Euler class under real realization, so the real realization morphism also induces an isomorphism from $\boldsymbol{I}^\bullet$–cohomology to the integral cohomology of real projective space; see [17].

The following is the Chow–Witt version of [26, Lemma 1]. This is a consequence of the above restatement of the computations in [14, Section 11], noting that $B\mathrm{GL}_1 \cong \mathbb{P}^\infty$.

Proposition 4.3 *The Euler class $e_1 \in H^1(\mathbb{P}^\infty, \boldsymbol{I}^1(\det \gamma_1^\vee))$ is nontrivial. Moreover, $e_1 = \beta_{\det \gamma_1^\vee}(1)$. There is an isomorphism*

$$\bigoplus_q H^q(\mathbb{P}^\infty, \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \gamma_1^\vee)) \cong W(F)[e_1]/(I(F)\cdot e_1).$$

The reduction morphism $H^1(\mathbb{P}^\infty, \boldsymbol{I}^1(\det \gamma_1^\vee)) \to \mathrm{Ch}^1(\mathbb{P}^\infty)$ maps e_1 to $\mathrm{Sq}^2_{\det \gamma_1^\vee}(1) = \bar{c}_1$. In particular, Theorem 3.24 is true for $n = 1$.

Remark 4.4 Alternatively, we can formulate the description of the $\boldsymbol{I}$–cohomology of projective space in terms of the decomposition into $\boldsymbol{W}$–cohomology and the image of β. The $\boldsymbol{W}$–cohomology of $\mathbb{P}^n$

is an exterior $W(F)$–algebra on one generator, which is $e_n^\perp \in H^n\big(\mathbb{P}^n, \boldsymbol{W}(\mathcal{O}(1))\big)$ for n even and $R = [\mathrm{pt}] \in H^n(\mathbb{P}^n, \boldsymbol{W})$ for n odd. The image of $\beta_{\mathcal{L}}$ is identified with the image of $\mathrm{Sq}^2_{\mathcal{L}}$ and consists of the appropriate powers of e_1. Multiplication with torsion classes can be computed after reduction in $\mathrm{Ch}^\bullet(\mathbb{P}^n)$.

4.2 Computation of $\boldsymbol{W}$–cohomology

The next step is the computation of the $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_n$.

Proposition 4.5 *The $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_n$ is given by*

$$H^\bullet(B\mathrm{GL}_n, \boldsymbol{W} \oplus \boldsymbol{W}(\det \gamma_n^\vee)) \cong \begin{cases} W(F)[p_2, p_4, \dots, p_{n-2}, e_n] & \text{if } n \equiv 0 \bmod 2, \\ W(F)[p_2, p_4, \dots, p_{n-1}] & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

The morphisms $H^\bullet(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L})) \to H^\bullet(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\mathcal{L}))$, induced by the stabilization morphism $B\mathrm{GL}_{n-1} \to B\mathrm{GL}_n$, are compatible with Pontryagin classes. The restriction along $B\mathrm{GL}_{2n+1} \to B\mathrm{GL}_{2n}$ maps p_{2n} to e_{2n}^2.

Proof We note that the compatibility of the Pontryagin classes with stabilization follows from their definition; see [16, Proposition 5.8].

The result is proved by induction. The base case for the induction is given by $B\mathrm{GL}_1 \cong \mathbb{P}^\infty$. In this case, the claim is that

$$H^q(\mathbb{P}^\infty, \boldsymbol{W}(\mathcal{L})) \cong \begin{cases} W(F) & \text{if } q = 0 \text{ and } \mathcal{L} = \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

This follows from Fasel's computations; see Proposition 4.3.

For the inductive step, we use the localization sequence of Proposition 3.1,

$$\cdots \to H^{q-n}(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L} \otimes \det \gamma_n)) \xrightarrow{e_n} H^q(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L})) \xrightarrow{\iota^*} H^q(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\iota^*\mathcal{L})) \xrightarrow{\partial} H^{q-n+1}(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L} \otimes \det \gamma_n)) \to \cdots.$$

If n is even, then by the induction hypothesis $H^\bullet(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\mathcal{L}))$ is a polynomial $W(F)$–algebra generated by the Pontryagin classes $p_2, \dots, p_{n-2}$. Since the stabilization morphism ι^* is compatible with the Pontryagin classes, it is surjective, hence $\partial = 0$. Thus, e_n is injective. Induction on the cohomological degree proves the claim that e_n is a new polynomial generator; alternatively, we can use the splitting principle of [16, Proposition 7.8] to show independence of e_n from the Pontryagin classes.

If n is odd, we know that $e_n = 0$ in $\boldsymbol{W}$–cohomology, since by Proposition 3.10 it is in the image of β. Therefore, the boundary map

$$\partial\colon H^{n-1}(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\det \gamma_{n-1}^\vee)) \to H^0(B\mathrm{GL}_n, \boldsymbol{W})$$

is surjective. The target is a cyclic $W(F)$–module generated by 1, and by the inductive assumption the image is a cyclic $W(F)$–module generated by ∂e_{n-1}. In particular, $\partial e_{n-1} = 1$, up to a unit in $W(F)$. By the derivation property for ∂, the boundary map is trivial on $H^\bullet(B\mathrm{GL}_{n-1}, \boldsymbol{W})$ and injective on $H^\bullet(B\mathrm{GL}_{n-1}, \boldsymbol{W}(\det \gamma_{n-1}^\vee))$. This implies that the $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_n$ is a polynomial $W(F)$–algebra generated by the Pontryagin classes $p_2, \dots, p_{n-1}$.

Finally, to prove the claim concerning restriction of the top Pontryagin class, consider the morphism

$$o^*\colon H^\bullet(B\mathrm{GL}_{2n(+1)}, \boldsymbol{W}(\det\gamma_n^\vee)) \to H^\bullet(B\mathrm{SL}_{2n(+1)}, \boldsymbol{W})$$

given by pullback to the orientation cover. This maps the Pontryagin classes and Euler class to their respective counterparts for $B\mathrm{SL}_{2n(+1)}$. From the present computation of the $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_{2n(+1)}$ and the computations in [16, Theorem 1.3] for $B\mathrm{Sl}_{2n(+1)}$ we conclude that o^* is injective. Moreover, $p_{2n} - e_{2n}^2$ is mapped to 0 by [16, Theorem 1.3] which proves the claim. □

Remark 4.6 For the case $B\mathrm{SL}_n$, the analogous formulas can be obtained from the general machinery for η–inverted cohomology theories in [1].

4.3 Relations in the mod 2 Chow ring

In this subsection we show that the ideal $\mathcal{I}_n$ of relations between characteristic classes is annihilated by the composition

$$\mathcal{R}_n \xrightarrow{\theta_n} H^\bullet(B\mathrm{GL}_n, \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det\gamma_n^\vee)) \xrightarrow{\rho} \mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2}.$$

Lemma 4.7 *Assume n is odd. With the above notation,*

$$\rho(e_n) = \rho\circ\beta_{\det\gamma_n^\vee}(\bar{c}_{n-1}) = \bar{c}_n.$$

Proof This follows from [14, Proposition 10.3, Remark 10.5], the identification $\mathrm{Sq}^2_{\det\gamma_n^\vee} = \rho\circ\beta_{\det\gamma_n^\vee}$ from Proposition 2.2, and the identification of Stiefel–Whitney classes with reductions of Chern classes in Proposition 3.6. □

Proposition 4.8 *For two index sets J and J', the elements*

$$B_J \cdot B_{J'} - \sum_{k\in J} B_{\{k\}} \cdot P_{(J\setminus\{k\})\cap J'} \cdot B_{\Delta(J\setminus\{k\}, J')},$$
$$B_J \cdot T_{J'} - \sum_{k\in J} B_{\{k\}} \cdot P_{(J\setminus\{k\})\cap J'} \cdot T_{\Delta(J\setminus\{k\}, J')},$$
$$T_J \cdot B_{J'} - B_J \cdot T_{J'} + T_\varnothing \cdot P_{J\cap J'} \cdot B_{\Delta(J,J')},$$
$$T_J \cdot T_{J'} - B_J \cdot B_{J'} + T_\varnothing \cdot P_{J\cap J'} \cdot T_{\Delta(J,J')}$$

have trivial images under the composition $\rho\circ\theta_n\colon \mathcal{R}_n \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$.

Proof The first relation can be established as in [16, Proposition 7.13]. Note that $\rho\circ\theta_n$ maps the elements B_J and T_J to the elements $\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_k})$ and $\mathrm{Sq}^2_{\det\gamma_n^\vee}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_k})$, respectively; see Proposition 2.2. The proofs of the other relations can be done by the same manipulations as detailed in [26, Lemma 4]. □

Corollary 4.9 *The composition $\rho\circ\theta_n\colon \mathcal{R}_n \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)^{\oplus 2}$ factors through the quotient $\mathcal{R}_n/\mathcal{I}_n$.*

Proof This follows directly from Lemma 4.7 and Proposition 4.8. □

Proposition 4.10 *Let $2i+1 \le n$ be an odd natural number.*[5] *Then*

$$\rho(p_{2i+1}) = (\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_{2i}))^2 + \rho(p_{2i})\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_1) = \mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_{2i}\bar{c}_{2i+1}).$$

Proof The claim follows from the computations

$$\begin{aligned}\rho(p_{2i+1})(\mathscr{E}_n) = \bar{c}_{4i+2}(\mathscr{E}_n \oplus \overline{\mathscr{E}}_n) = \bar{c}_{4i+2}(\mathscr{E}_n^{\oplus 2}) = \bar{c}_{2i+1}(\mathscr{E}_n)^2,\\ (\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_{2i}))^2 + \rho(p_{2i})\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_1) = (\bar{c}_{2i+1} + \bar{c}_1\bar{c}_{2i})^2 + \bar{c}_{2i}^2\bar{c}_1^2 = \bar{c}_{2i+1}^2,\\ \mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_{2i}\bar{c}_{2i+1}) = \bar{c}_{2i}\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_{2i+1}) + \bar{c}_{2i+1}\mathrm{Sq}^2_{\mathcal{O}}(\bar{c}_{2i}) = \bar{c}_{2i+1}^2;\end{aligned}$$

see [8, page 288]. □

4.4 Proof of Theorem 3.24

We first note that Proposition 4.5, in combination with Lemma 2.4, a priori implies a splitting of $\boldsymbol{I}$–cohomology into $\boldsymbol{W}$–cohomology and the image of β, and this is the key tool in the proof. This already establishes part (3) of the theorem.

Part (2) of the theorem follows from Proposition 2.2 for the Bockstein classes and [16, Corollary 7.11] for the Pontryagin and Euler classes.

To prove part (1) of the theorem, consider the ring homomorphism

$$\theta_n : \mathcal{R}_n \to \bigoplus_{q,\mathscr{L}} H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathscr{L}))$$

defined in Theorem 3.24. The first step is to show that θ_n factors through the quotient $\mathcal{R}_n/\mathcal{I}_n$, ie that $\theta_n(\mathcal{I}_n) = 0$. We consider the relations generating $\mathcal{I}_n$ given in Definition 3.15. Relations of type (1) hold by Lemma 3.9, relations of type (2) by Proposition 3.10. Relations of type (3) are annihilated by the composition $\rho \circ \theta_n : \mathcal{R}_n \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$ by Proposition 4.8. By Proposition 4.5, the $\boldsymbol{W}$–cohomology of $B\mathrm{GL}_n$ is free, hence Lemma 2.4 implies that the reduction $\rho\colon H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathscr{L})) \to \mathrm{Ch}^q(B\mathrm{GL}_n)$ is injective on the image of $\beta_{\mathscr{L}}$. Since all relations of type (3) are in the image of $\beta_{\mathscr{L}}$, those relations have trivial image under θ_n. Therefore, we get a well-defined ring homomorphism

$$\bar{\theta}_n : \mathcal{R}_n/\mathcal{I}_n \to \bigoplus_{q,\mathscr{L}} H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathscr{L})).$$

We now prove that the ring homomorphism $\bar{\theta}_n$ is surjective. First, we note that $\bar{\theta}_n$ surjects onto $\mathrm{Im}\,\beta_{\mathscr{L}}$ if and only if the composition

$$\rho \circ \bar{\theta}_n : \mathcal{R}_n/\mathcal{I}_n \to \bigoplus_{q,\mathscr{L}} H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathscr{L})) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$$

surjects onto the image of $\mathrm{Sq}^2_{\mathscr{L}}\colon \mathrm{Ch}^{\bullet-1}(B\mathrm{GL}_n) \to \mathrm{Ch}^\bullet(B\mathrm{GL}_n)$. By Corollary 3.14, we know that the image of $\mathrm{Sq}^2_{\mathscr{L}}$ is contained in the subring generated by the classes $\mathrm{Sq}^2_{\mathscr{L}}(\bar{c}_{2j_1}\cdots\bar{c}_{2j_l})$, $\mathrm{Sq}^2_{\det\gamma_n}(1)$, $\bar{c}_{2i}^2$ and $\bar{c}_n$. By part (2) of the theorem, all these classes are reductions of classes in the image of θ_n, proving

[5] Recall that the definition of Pontryagin classes includes a definition of odd Pontryagin classes; see Remark 3.5.

that $\bar{\theta}_n$ surjects onto the image of β. It then suffices to show that the composition

$$\mathcal{R}_n/\mathcal{I}_n \xrightarrow{\bar{\theta}_n} \bigoplus_{q,\mathcal{L}} H^q(B\mathrm{GL}_n, \boldsymbol{I}^q(\mathcal{L})) \to \bigoplus_{q,\mathcal{L}} H^q(B\mathrm{GL}_n, \boldsymbol{W}(\mathcal{L}))$$

is surjective, where the second map is the projection onto $\boldsymbol{W}$–cohomology. But this follows from Proposition 4.5, finishing the surjectivity proof.

Finally, we prove that $\bar{\theta}_n$ is injective. First, we consider the $W(F)$–torsion-free part of $\mathcal{R}_n/\mathcal{I}_n$ which is generated, as a commutative graded $W(F)$–algebra, by the P_i, and X_{2n} if applicable. The restriction of $\bar{\theta}_n$ to that subalgebra is injective by Proposition 4.5. The injectivity on the torsion part, ie the ideal generated by the classes B_J, T_J for $J = \{j_1, \dots, j_l\}$ and $T_\varnothing$ can be checked after composition with ρ, by the decomposition of Lemma 2.4 (and Proposition 4.5) and the resulting fact that ρ is injective on the image of β. The direct translation (replacing w_i by $\bar{c}_i$ and Sq^1 by Sq^2) of the argument on page 285 of [26] takes care of that; see also [16, Proposition 8.15].

5 Chow–Witt rings of finite Grassmannians: statement of results

In the following two sections, we compute the Chow–Witt rings of the finite Grassmannians $\mathrm{Gr}(k,n)$. The results are stated in the present section, and the proofs are deferred to the next section.

5.1 Generators from characteristic classes

The first step is to get enough classes in $\widetilde{\mathrm{CH}}^\bullet(\mathrm{Gr}(k,n), \mathcal{L})$. We realize the Grassmannian $\mathrm{Gr}(k,n)$ over the field F as the variety of k–dimensional F–subspaces of $V = F^n$. Recall that we have an exact sequence of vector bundles on $\mathrm{Gr}(k,n)$,

$$0 \to \mathcal{S}_k \to \mathcal{O}_{\mathrm{Gr}(k,n)}^{\oplus n} \to \mathcal{Q}_{n-k} \to 0.$$

Here, $\mathcal{S}_k$ is the *tautological subbundle*, mapping a point $[W]$ corresponding to a k–dimensional subspace $W \subset V$ to W, and $\mathcal{Q}_{n-k}$ is the *tautological quotient bundle*, mapping a point $[W]$ to the quotient space V/W.

There is a vector bundle torsor $f\colon \mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k}) \to \mathrm{Gr}(k,n)$ over the Grassmannian. This is an $\mathbb{A}^1$–weak equivalence, and the above exact sequence of vector bundles splits over $\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$. Consequently, we obtain an $\mathbb{A}^1$–fiber sequence

$$\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k}) \to B\mathrm{GL}_k \times B\mathrm{GL}_{n-k} \xrightarrow{\oplus} B\mathrm{GL}_n,$$

where the second map is the Whitney sum map and the first map classifies the pair $(f^*\mathcal{S}_k, f^*\mathcal{Q}_{n-k})$. We can also consider the map $c\colon \mathrm{Gr}(k,n) \to B\mathrm{GL}_k \times B\mathrm{GL}_{n-k}$ obtained by composing a homotopy inverse of f with the inclusion of the homotopy fiber, and this map classifies the pair $(\mathcal{S}_k, \mathcal{Q}_{n-k})$.

Note that there are two possible dualities on $B\mathrm{GL}_k$, corresponding to the line bundles $\mathcal{O}$ and $\det \gamma_k^\vee$; and similarly there are two possible dualities on $B\mathrm{GL}_{n-k}$ corresponding to $\mathcal{O}$ and $\det \gamma_{n-k}^\vee$. Consequently, there are four possible dualities on $B\mathrm{GL}_k \times B\mathrm{GL}_{n-k}$, given by the four possible exterior products of the above line bundles. For any choice of line bundles $\mathscr{L}_k$ and $\mathscr{L}_{n-k}$ on $B\mathrm{GL}_k$ and $B\mathrm{GL}_{n-k}$, respectively, the classifying map c above induces homomorphisms of Chow–Witt groups

$$\widetilde{\mathrm{CH}}^\bullet(B\mathrm{GL}_k \times B\mathrm{GL}_{n-k}, \mathscr{L}_k \boxtimes \mathscr{L}_{n-k}) \to \widetilde{\mathrm{CH}}^\bullet(\mathrm{Gr}(k,n), c^*(\mathscr{L}_k \boxtimes \mathscr{L}_{n-k})).$$

Note that the bundle $c^*(\mathscr{L}_k \boxtimes \mathscr{L}_{n-k})$ is trivial (modulo squares of line bundles) if and only if $\mathscr{L}_k$ and $\mathscr{L}_{n-k}$ are either both trivial or both nontrivial. This follows from the fact that the assignment $(\mathscr{L}_k, \mathscr{L}_{n-k}) \mapsto c^*(\mathscr{L}_k \boxtimes \mathscr{L}_{n-k})$ can be computed by pulling back both line bundles to the Grassmannian and then taking the tensor product; hence it induces the addition

$$\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \cong \mathrm{Ch}^1(B\mathrm{GL}_k \times B\mathrm{GL}_{n-k}) \to \mathrm{Ch}^1(\mathrm{Gr}(k,n)) \cong \mathbb{Z}/2\mathbb{Z}.$$

The induced homomorphisms assemble into a ring homomorphism of the total Chow–Witt rings (to the extent that this makes sense; see the remarks on [5] in Section 2).

This means that the characteristic classes of the tautological bundles $\mathscr{S}_k$ and $\mathscr{Q}_{n-k}$ provide classes in the Chow–Witt ring of $\mathrm{Gr}(k,n)$. For the definition of these classes and relations satisfied by them; see Section 3 and in particular Theorem 3.24, or the main result Theorem 1.1. The characteristic classes for the tautological subbundle $\mathscr{S}_k$ are

(1) the Pontryagin classes $p_1, p_2, \dots, p_{k-1}$,
(2) the Euler class e_k,
(3) the (twisted) Bockstein classes $\beta_{\mathscr{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$ and $\beta_{\det \gamma_k^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$, and
(4) the Chern classes c_i.

Similarly, for the tautological quotient bundle $\mathscr{Q}_{n-k}$, we have the same characteristic classes (with different index sets); these will be denoted by an additional superscript $(-)^\perp$.[6] This provides a number of canonical elements in $\widetilde{\mathrm{CH}}^\bullet(\mathrm{Gr}(k,n), \mathscr{L})$. It turns out that in the cases where $\dim \mathrm{Gr}(k,n) = k(n-k)$ is even, these classes generate the Chow–Witt ring; in the case where the dimension is odd, there is essentially one additional class arising as lift of an Euler class.

Remark 5.1 We follow the convention of [16, Remark 5.7], including all Pontryagin classes without added signs or reindexing. While the odd Pontryagin classes are $I(F)$–torsion and can be expressed in terms of Bockstein classes, this convention makes the Whitney sum formula for Pontryagin classes easier to state; see Proposition 3.28 and the subsequent remark.

5.2 Chow rings of Grassmannians

Before giving the statement concerning the structure of the Chow–Witt rings, we discuss the Chow rings of the Grassmannians. This result is very well-known and can be found in the relevant books on intersection theory, such as [11].

[6] The notation is suggestive that $\mathscr{Q}_{n-k}$ is the complement of $\mathscr{S}_k$ in $\mathscr{O}^{\oplus n}$, after pulling back to $\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$.

Remark 5.2 The key relation in the Chow ring (with integral or mod 2 coefficients) is the Whitney sum formula. For this, in the statements below, we will use the notation $c=\sum_{i=0}^{k} c_i$ and $c^\perp=\sum_{i=0}^{n-k} c_i^\perp$ for the total Chern classes of the tautological subbundles and quotient bundles $\mathcal{S}_k$ and $\mathcal{Q}_{n-k}$, respectively. Similarly, we will use the notation $\bar{c}=\sum_{i=0}^{k}\bar{c}_i$ and $\bar{c}^\perp=\sum_{i=0}^{n-k}\bar{c}_i^\perp$ for the total Stiefel–Whitney classes of the tautological subbundles and quotient bundles, respectively. The Whitney sum formula for the extension

$$0\to\mathcal{S}_k\to\mathcal{O}_{\mathrm{Gr}(k,n)}^{\oplus n}\to\mathcal{Q}_{n-k}\to 0$$

is then simply written as $c\cdot c^\perp=1$.

Proposition 5.3 *Let F be an arbitrary field. With the notation from Remark 5.2,*

$$\mathrm{CH}^\bullet(\mathrm{Gr}(k,n))\cong\mathbb{Z}[c_1,\dots,c_k,c_1^\perp,\dots,c_{n-k}^\perp]/(c\cdot c^\perp=1).$$

Proposition 5.4 *Let F be an arbitrary field. With the notation from Remark 5.2,*

(1) *there is a canonical isomorphism*

$$\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))\cong\mathbb{Z}/2\mathbb{Z}[\bar{c}_1,\dots,\bar{c}_k,\bar{c}_1^\perp,\dots,\bar{c}_{n-k}^\perp]/(\bar{c}\cdot\bar{c}^\perp=1);$$

(2) *the Steenrod square is given by*

$$\mathrm{Sq}^2_{\mathcal{O}}\colon\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))\to\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)),\quad \bar{c}_j^{(\perp)}\mapsto\bar{c}_1^{(\perp)}\bar{c}_j^{(\perp)}+(j-1)\bar{c}_{j+1}^{(\perp)};$$

(3) *the twisted Steenrod square is given by*

$$\mathrm{Sq}^2_{\det\mathcal{S}_k^\vee}\colon\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))\to\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)),\quad \bar{c}_j^{(\perp)}\mapsto(j-1)\bar{c}_{j+1}^{(\perp)}.$$

The following consequence of the description of the Chow ring given in Proposition 5.4 will be relevant later.

Proposition 5.5 *Let $1\le k<n$ and consider the ring*

$$A=\mathbb{Z}/2\mathbb{Z}[\bar{c}_1,\dots,\bar{c}_k,\bar{c}_1^\perp,\dots,\bar{c}_{n-k}^\perp]/(\bar{c}\cdot\bar{c}^\perp=1).$$

(1) *The kernel of multiplication by $\bar{c}_{n-k}^\perp$ is the ideal $\langle\bar{c}_k\rangle\subseteq A$.*

(2) *The cokernel of multiplication by $\bar{c}_{n-k}^\perp$ is*

$$A/\langle\bar{c}_{n-k}^\perp\rangle\cong\mathbb{Z}/2\mathbb{Z}[\bar{c}_1,\dots,\bar{c}_k,\bar{c}_1^\perp,\dots,\bar{c}_{n-k-1}^\perp]/(\bar{c}\cdot\bar{c}^\perp=1).$$

Proof Statement (2) about the cokernel being $A/\langle\bar{c}_{n-k}^\perp\rangle$ is clear. The explicit description of the algebra also follows directly.

For (1), clearly $\langle\bar{c}_k\rangle\subseteq\ker\bar{c}_{n-k}^\perp$ since $\bar{c}_k\bar{c}_{n-k}^\perp=0$ follows from the Whitney sum relation. The reverse inclusion can be seen, for example, by a dimension count in the kernel-cokernel exact sequence for multiplication by $\bar{c}_{n-k}^\perp$. □

Remark 5.6 This is also the formula for the mod 2 cohomology of the real Grassmannians; see eg [21]. The notation for the classes $\bar{c}_i$ and $\bar{c}_i^\perp$ is due to the fact that these are the mod 2 reductions of the Chern classes from the Chow ring. In the real realization these would go exactly to the corresponding Stiefel–Whitney classes.

5.3 Statement of the main results

Now we are ready to state the main results describing the $\boldsymbol{I}^\bullet$–cohomology of the finite Grassmannians. The lengthy formulation boils down to: "the characteristic classes of the tautological bundle and its complement generate the cohomology (except for a new class R in degree $(n-1,0)$ when $k(n-k)$ is odd) and the only new relations come from the Whitney sum formula".

As before in Remark 5.2, we will denote by $p = 1 + p_1 + \cdots + p_k$ and $p^\perp = 1 + p_1^\perp + \cdots + p_{n-k}^\perp$ for the total Pontryagin classes of the tautological subbundles and quotient bundles $\mathscr{S}_k$ and $\mathscr{Q}_{n-k}$, respectively.

Theorem 5.7 *Let F be a perfect field of characteristic $\neq 2$, and let $1 \leq k < n$. The cohomology ring $\bigoplus_q H^q(\mathrm{Gr}(k,n), \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \mathscr{S}_k^\vee))$ is isomorphic to the $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded $W(F)$–algebra generated by*

- (G1) *the Pontryagin classes $p_1, p_2, \ldots, p_k$ of the tautological rank k subbundle and the Pontryagin classes $p_1^\perp, p_2^\perp, \ldots, p_{n-k}^\perp$ of the tautological rank $n-k$ quotient bundle, where the class $p_i^{(\perp)}$ has degree $(2i,0)$;*
- (G2) *the Euler classes e_k and $e_{n-k}^\perp$, having degrees $(k,1)$ and $(n-k,1)$, respectively;*
- (G3) *for every set $J = \{j_1, \ldots, j_l\}$ of natural numbers $0 < j_1 < \cdots < j_l \leq \left[\frac{1}{2}(k-1)\right]$, possibly empty, there are Bockstein classes $\beta_J = \beta_{\mathcal{O}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$ and $\tau_J = \beta_{\det \mathscr{S}_k^\vee}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$ in degrees $(d,0)$ and $(d,1)$, respectively, where $d = 1 + 2\sum_{i=1}^l j_i$;*
- (G4) *for every set $J = \{j_1, \ldots, j_l\}$ of natural numbers $0 < j_1 < \cdots < j_l \leq \left[\frac{1}{2}(n-k-1)\right]$, possibly empty, there are Bockstein classes $\beta_J^\perp = \beta_{\mathcal{O}}(\bar{c}_{2j_1}^\perp \cdots \bar{c}_{2j_l}^\perp)$ and $\tau_J^\perp = \beta_{\det \mathscr{S}_k^\vee}(\bar{c}_{2j_1}^\perp \cdots \bar{c}_{2j_l}^\perp)$ in degrees $(d,0)$ and $(d,1)$, respectively, where $d = 1 + 2\sum_{i=1}^l j_i$;*
- (G5) *if $k(n-k)$ is odd, there is a class R in degree $(n-1,0)$*

subject to the relations

- (R1) *the classes p_i, e_k, β_J and τ_J satisfy the relations holding in the total $\boldsymbol{I}$–cohomology ring of $B\mathrm{GL}_k$, and the classes $p_i^\perp$, $e_{n-k}^\perp$, $\beta_J^\perp$ and $\tau_J^\perp$ satisfy the relations in the total $\boldsymbol{I}$–cohomology ring of $B\mathrm{GL}_{n-k}$ (see Theorem 3.24);*[7]
- (R2) *$p \cdot p^\perp = 1$, ie the product of the total Pontryagin classes is 1;*
- (R3) *$e_k \cdot e_{n-k}^\perp = 0$;*
- (R4) *$\beta_{\mathcal{O}}(\bar{c} \cdot \bar{c}^\perp) = 1$ and $\beta_{\det \mathscr{S}_k^\vee}(\bar{c} \cdot \bar{c}^\perp) = \tau_\varnothing = \tau_\varnothing^\perp$, ie applying the (twisted) Bockstein to the product of the total Stiefel–Whitney classes in $\mathrm{Ch}^\bullet$ is trivial;*
- (R5) *$R^2 = 0$, and the product of R with an $I(F)$–torsion class α is zero if and only if $\bar{c}_{k-1}\bar{c}_{n-k}^\perp \rho(\alpha) = 0$.*

[7] In particular, the classes $\beta_\varnothing = \beta_\varnothing^\perp = 0$.

Proposition 5.8 *Let F be a perfect field of characteristic $\neq 2$, and let $1 \leq k < n$. The reduction homomorphism*

$$\rho\colon \bigoplus_q H^q(\mathrm{Gr}(k,n), \boldsymbol{I}^q \oplus \boldsymbol{I}^q(\det \mathscr{S}_k)) \to \mathrm{Ch}^q(\mathrm{Gr}(k,n))$$

is given by

$$p_{2i}^{(\perp)} \mapsto (\bar{c}_{2i}^{(\perp)})^2, \quad e_k \mapsto \bar{c}_k, \quad e_{n-k}^{\perp} \mapsto \bar{c}_{n-k}^{\perp},$$

$$\beta_{\mathscr{L}}(\bar{c}_{2j_1}^{(\perp)} \cdots \bar{c}_{2j_l}^{(\perp)}) \mapsto \mathrm{Sq}^2_{\mathscr{L}}(\bar{c}_{2j_1}^{(\perp)} \cdots \bar{c}_{2j_l}^{(\perp)}), \quad R \mapsto \bar{c}_{k-1}\bar{c}_{n-k}^{\perp} = \bar{c}_k\bar{c}_{n-k-1}^{\perp}.$$

The reduction homomorphism $\rho_{\mathscr{L}}$ is injective on the image of the Bockstein map $\beta_{\mathscr{L}}$.

Remark 5.9 This presentation gives a complete description of the cup product. To multiply two torsion classes, we first rewrite the complementary classes $\bar{c}_{2i}^{\perp}$ in terms of polynomials in the ordinary classes $\bar{c}_{2j}$. (It follows directly from the well-known presentation of $\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$ that it is generated by the classes $\bar{c}_i$ and the complementary classes $\bar{c}_j^{\perp}$ can be expressed in terms of these.) The product of classes of the form $\beta_{\mathscr{L}}(\bar{c}_{2j_1} \cdots \bar{c}_{2j_l})$ is then given by the relation in $H^\bullet(B\mathrm{GL}_k)$. Note also that the product of R with an even Pontryagin class is independent of the Pontryagin classes. The product of R with a torsion class is a torsion class, and so it can be determined by computation in $\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$. More detailed descriptions of how to work out products can be found in [28].

Theorem 5.10 *Let F be a perfect field of characteristic $\neq 2$, and let $1 \leq k < n$. Then there is a cartesian square of $\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$–graded $\mathrm{GW}(F)$–algebras*

$$\begin{array}{ccc} \widetilde{\mathrm{CH}}^\bullet(\mathrm{Gr}(k,n), \mathcal{O} \oplus \det \mathscr{S}_k^\vee) & \longrightarrow & \ker \partial_{\mathcal{O}} \oplus \ker \partial_{\det \mathscr{S}_k^\vee} \\ \downarrow & & \downarrow {\scriptstyle \mathrm{mod}\ 2} \\ H^\bullet(\mathrm{Gr}(k,n), \boldsymbol{I}^\bullet \oplus \boldsymbol{I}^\bullet(\det \mathscr{S}_k^\vee)) & \xrightarrow[\rho]{} & \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))^{\oplus 2} \end{array}$$

Here $\det \mathscr{S}_k^\vee$ is the determinant of the dual of the tautological rank k subbundle on $\mathrm{Gr}(k,n)$, and

$$\partial_{\mathscr{L}}\colon \mathrm{CH}^\bullet(\mathrm{Gr}(k,n)) \to \mathrm{Ch}^\bullet(\mathrm{Gr}(k,n)) \xrightarrow{\beta_{\mathscr{L}}} H^{\bullet+1}(\mathrm{Gr}(k,n), \boldsymbol{I}^{\bullet+1}(\mathscr{L}))$$

is the (twisted) integral Bockstein map.

The kernel of the integral Bockstein map $\partial_{\mathscr{L}}$ is the preimage under reduction mod 2 of the subalgebra of $\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$ generated by $(c_i{}^{(\perp)})^2$, c_k, $c_{n-k}^{\perp}$, and $c_k c_{n-k-1}^{\perp}$ together with the image of $\mathrm{Sq}^2_{\mathscr{L}}$.

Proof This follows from [16, Proposition 2.11] since the Chow ring of $\mathrm{Gr}(k,n)$ is 2–torsion-free; see Proposition 5.3. The description of $\boldsymbol{I}^\bullet$–cohomology is given in Theorem 5.7, and the description of the reduction morphism ρ is given in Proposition 5.8. The description of the kernel of the boundary map follows directly from the definition and the Bär sequence, ie that the kernel of $\beta_{\mathscr{L}}$ is exactly the image of the reduction map $\rho_{\mathscr{L}}$. □

Remark 5.11 We can determine the images of Euler classes and Pontryagin classes in Chow theory using Theorem 3.27.

5.4 Examples

The following are two examples describing the $\boldsymbol{I}^\bullet$–cohomology of small Grassmannians. For alternative descriptions of the $\boldsymbol{I}^\bullet$–cohomology, using even Young diagrams for the $\boldsymbol{W}$–part and checkerboard fillings for Young diagrams for the image of β; see [28].

Example 5.12 Let us work out the components of the cartesian square of Theorem 5.10 in the example case $\mathrm{Gr}(2,4)$.

First, the mod 2 Chow ring $\mathrm{Ch}^\bullet(\mathrm{Gr}(2,4))$ is generated by the Stiefel–Whitney classes $\bar{c}_1^{(\perp)}$ and $\bar{c}_2^{(\perp)}$ of the tautological bundles, and the relations from the Whitney formula for Stiefel–Whitney classes are

$$\bar{c}_1=\bar{c}_1^{\perp},\quad \bar{c}_2+\bar{c}_1^2+\bar{c}_2^{\perp}=0,\quad \bar{c}_1\bar{c}_2^{\perp}+\bar{c}_2\bar{c}_1^{\perp}=\bar{c}_1^3=0,\quad \bar{c}_2^2+\bar{c}_2\bar{c}_1^2=0.$$

From these relations, the usual well-known description of the mod 2 Chow ring follows. The description of the integral Chow ring $\mathrm{CH}^\bullet(\mathrm{Gr}(2,4))$ is completely analogous, just in terms of Chern classes.

With the relations between the Stiefel–Whitney classes, we can now compute the twisted and untwisted Bocksteins of Stiefel–Whitney classes. This provides information both for the kernel of $\partial_{\mathscr{O}}$ and $\partial_{\det\mathscr{S}_2^\vee}$ in $\mathrm{CH}^\bullet(\mathrm{Gr}(2,4))$, and information on the torsion classes in $\boldsymbol{I}^\bullet$–cohomology. The first relation of Stiefel–Whitney classes above implies $\beta_{\mathscr{O}}(\bar{c}_i)=\beta_{\mathscr{O}}(\bar{c}_i^{\perp})$ for $i=1,2$. By the Wu formula, $\mathrm{Sq}^2_{\mathscr{O}}(\bar{c}_2)=\bar{c}_1\bar{c}_2$ and therefore $\beta_{\mathscr{O}}(\bar{c}_1\bar{c}_2)=0$. Since Bocksteins of squares are trivial by the derivation property, this means that the only nontrivial untwisted Bockstein classes are $\beta_{\mathscr{O}}(\bar{c}_1)=p_1$ and $\beta_{\mathscr{O}}(\bar{c}_2)$.

With twisted coefficients, we have the class $\beta_{\det\mathscr{S}_2^\vee}(1)$. The other twisted Bockstein classes $\beta_{\det\mathscr{S}_2^\vee}(\bar{c}_i)$ are trivial: we can check on reduction mod 2, the case $i=1$ follows directly from the Wu formula and the case $i=2$ follows from $\bar{c}_3=0$. The other potential 2–torsion classes are products of the form $\tau_{\varnothing}\beta_{\mathscr{O}}$. We check on mod 2 reduction that $\rho(\tau_{\varnothing}\beta_{\mathscr{O}}(\bar{c}_1))=\bar{c}_1^3=0$ but $\rho(\tau_{\varnothing}\beta_{\mathscr{O}}(\bar{c}_2))=\bar{c}_1^2\bar{c}_2=\bar{c}_2^2\neq 0$. So the torsion classes with twisted coefficients are $\beta_{\det\mathscr{S}_2^\vee}(1)$ in degree 1 and $\beta_{\det\mathscr{S}_2^\vee}(1)\beta_{\mathscr{O}}(\bar{c}_2)$ in degree 4.

As a consequence of the above, we can now also determine the kernel of the boundary maps on the Chow ring. The Chern classes $c_1^{(\perp)}$ and $c_2^{(\perp)}$ have nontrivial Bocksteins and hence do not lift to the Chow–Witt ring. As noted above, the classes c_1^2, c_1c_2 and c_2^2 have trivial Steenrod squares and therefore

$$\ker\partial_{\mathscr{O}}=\mathbb{Z}[c_1c_2,c_1^2,c_2^2,(2)].$$

On the other hand, $\ker\partial_{\det\mathscr{S}_2^\vee}$ is the submodule generated by (2), c_1 and c_2.

Finally, we can turn to the description of $\boldsymbol{I}^\bullet$–cohomology. We already determined the nontrivial Bockstein classes above. In addition to these, the remaining characteristic classes for $\boldsymbol{I}^\bullet$–cohomology are the Pontryagin classes $p_0=1\in H^0(\mathrm{Gr}(2,4),\boldsymbol{I}^0)$,

$$p_1^{(\perp)}=\beta_{\mathscr{O}}(\bar{c}_1^{(\perp)})\in H^2(\mathrm{Gr}(2,4),\boldsymbol{I}^2)\quad\text{and}\quad p_2^{(\perp)}\in H^4(\mathrm{Gr}(2,4),\boldsymbol{I}^4)$$

and the Euler classes $e_2,e_2^{\perp}\in H^2(\mathrm{Gr}(2,4),\boldsymbol{I}^2(\det\mathscr{E}_2^\vee))$.

The relations encoded in the Whitney sum formula $p \cdot p^{\perp} = 1$ are

$$\beta(\bar{c}_1) = \beta(\bar{c}_1^{\perp}), \quad p_2 + \beta(\bar{c}_1)^2 + p_2^{\perp} = 0, \quad p_2^2 = 0.$$

There is also a relation $2p_2\beta(\bar{c}_1) = 0$ which is trivially satisfied. From $\bar{c}_1^3 = 0$ above we find that $\mathrm{Sq}^2(\bar{c}_1)^2 = \bar{c}_1^4 = 0$ and therefore $\beta(\bar{c}_1)^2 = 0$. In particular, $p_2 = -p_2^{\perp}$. Consequently, the only nontorsion classes are $p_0 = 1$, p_2, e_2 and $e_2^{\perp}$, with $e_2^2 = (e_2^{\perp})^2 = p_2$.

A posteriori, we can now note that $\mathrm{Gr}(2,4)$ is an orientable variety, and a Poincaré duality pattern as in singular cohomology is satisfied for the $\boldsymbol{I}^{\bullet}$–cohomology.

Example 5.13 As another example, we work out the Steenrod squares for $\mathrm{Gr}(2,5)$. The relevant relations arising from the Whitney sum formula are

$$\bar{c}_1 = \bar{c}_1^{\perp}, \quad \bar{c}_2^{\perp} = \bar{c}_2 + \bar{c}_1^2, \quad \bar{c}_3^{\perp} = \bar{c}_1^3, \quad \bar{c}_2^2 + \bar{c}_1^2\bar{c}_2 + \bar{c}_1^4 = \bar{c}_2\bar{c}_1^3 = 0.$$

Now we go through the individual degrees in $\mathrm{Ch}^{\bullet}(\mathrm{Gr}(2,5))$:

(1) Degree 1 has $\bar{c}_1$ and $\mathrm{Sq}^2(\bar{c}_1) = \bar{c}_1^2$; this class doesn't lift to $H^1(\mathrm{Gr}(2,5), \boldsymbol{I}^1)$.

(2) Degree 2 has $\bar{c}_2$ with $\mathrm{Sq}^2(\bar{c}_2) = \bar{c}_1\bar{c}_2$ and $\bar{c}_1^2$ with trivial Sq^2. So the latter class lifts to a torsion class $\beta(\bar{c}_1) \in H^2(\mathrm{Gr}(2,5), \boldsymbol{I}^2)$.

(3) Degree 3 has $\bar{c}_1^3$ with $\mathrm{Sq}^2(\bar{c}_1^3) = \bar{c}_1^4$ and $\bar{c}_1\bar{c}_2$ with trivial Sq^2. So the latter class lifts to a torsion class $\beta(\bar{c}_2) \in H^3(\mathrm{Gr}(2,5), \boldsymbol{I}^3)$.

(4) Degree 4 has $\bar{c}_1^4$ and $\bar{c}_1^2\bar{c}_2$, both with trivial Sq^2. The class $\bar{c}_2^2 = \bar{c}_1^4 + \bar{c}_1^2\bar{c}_2$ lifts to the Pontryagin class, and $\bar{c}_1^4$ lifts to $\beta(\bar{c}_1^3) \in H^4(\mathrm{Gr}(2,5), \boldsymbol{I}^4)$.

(5) Degree 5 has $\bar{c}_1^5$ with $\mathrm{Sq}^2(\bar{c}_1^5) = \bar{c}_1^6$ and consequently this class doesn't lift to $\boldsymbol{I}^5$–cohomology.

(6) Degree 6 has $\bar{c}_1^6$ with trivial Sq^2, and this class lifts to the integral class $\beta(\bar{c}_1^5) \in H^6(\mathrm{Gr}(2,5), \boldsymbol{I}^6)$.

This recovers exactly the pattern for integral cohomology as discussed in eg [9]. In addition to that, we can use the formulas from Theorem 1.1 to determine the cup product of the torsion classes. Computations as above could be done to determine the cohomology with twisted coefficients as well.

6 Chow–Witt rings of finite Grassmannians: proofs

In this section, we will now prove the claims about the structure of $\boldsymbol{I}^{\bullet}$–cohomology of the Grassmannians discussed in Section 4. The overall argument is again to use the decomposition of $\boldsymbol{I}$–cohomology into the image of β and the $\boldsymbol{W}$–cohomology. We compute the $\boldsymbol{W}$–cohomology using a version of the inductive procedure used by Sadykov to compute rational cohomology of the real Grassmannians; see [23]. The base case $k = 1$ is the case of projective space which basically follows from [14]. The inductive step compares the cohomology of the Grassmannians $\mathrm{Gr}(k-1,n)$ and $\mathrm{Gr}(k,n)$ via a space which appears as sphere bundle of the tautological quotient and subbundle over $\mathrm{Gr}(k-1,n)$ and $\mathrm{Gr}(k,n)$, respectively. The image of β is detected on the mod 2 Chow ring, which determines the multiplication with torsion classes.

6.1 Localization sequence for inductive proof

As a preparation for the inductive computation of $\boldsymbol{W}$–cohomology, we set up the relevant localization sequences which allow to compare cohomology of different Grassmannians.

Proposition 6.1 (1) *There are isomorphisms*

$$\mathrm{Gr}(k,n) \cong \mathrm{Gr}(n-k,n).$$

(2) *Denote by* $q\colon \mathscr{S}_k \to \mathrm{Gr}(k,n)$ *and* $p\colon \mathscr{Q}_{n-k+1} \to \mathrm{Gr}(k-1,n)$ *the respective tautological bundles, and by* $z_q\colon \mathrm{Gr}(k,n) \to \mathscr{S}_k$ *and* $z_p\colon \mathrm{Gr}(k-1,n) \to \mathscr{Q}_{n-k+1}$ *the respective zero-sections. Then there is an* $\mathbb{A}^1$*–weak equivalence of associated sphere bundles*

$$\mathscr{S}_k \setminus z_q(\mathrm{Gr}(k,n)) \simeq \mathscr{Q}_{n-k+1} \setminus z_p(\mathrm{Gr}(k-1,n)).$$

Moreover, under this weak equivalence, we have a correspondence of pullbacks of tautological vector bundles $q^*\mathscr{S}_k \cong p^*\mathscr{S}_{k-1} \oplus \mathbb{O}$.

Proof (1) For a vector space V of dimension n, the natural bijection between k–dimensional subspaces of V and $(n-k)$–dimensional subspaces of its dual $V^\vee$ induces a natural isomorphism

$$\mathrm{Gr}(k,V) \cong \mathrm{Gr}(n-k,V^\vee).$$

Choosing an isomorphism $V \cong V^\vee$ induces an isomorphism $\mathrm{Gr}(n-k,V^\vee) \cong \mathrm{Gr}(n-k,V)$. This provides the claimed isomorphisms. Note that these are not natural.

(2) Since we only want to establish an $\mathbb{A}^1$–weak equivalence, we can replace the Grassmannians $\mathrm{Gr}(k,n)$ and $\mathrm{Gr}(k-1,n)$ by the quotients $\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$ and $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k+1})$, respectively. The pullback of the vector bundle $\mathscr{S}_k$ over $\mathrm{GL}_n/(\mathrm{GL}_k \times \mathrm{GL}_{n-k})$ is the associated bundle for the Stiefel variety $\mathrm{GL}_n/\mathrm{GL}_{n-k}$ viewed as GL_k–torsor and the natural GL_k–representation on $\mathbb{A}^k$. As in the setup of the localization sequence before Proposition 3.1, the complement of the zero section is then, up to a torsor under a unipotent group, $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k})$ because GL_{k-1} is the stabilizer of a line in $\mathbb{A}^k$.

A similar argument works for $\mathrm{Gr}(k-1,n)$. The vector bundle $\mathscr{Q}_{n-k+1}$ over $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k+1})$ is the associated bundle for the Stiefel variety $\mathrm{GL}_n/\mathrm{GL}_{k-1}$ and the natural representation of GL_{n-k+1} on $\mathbb{A}^{n-k+1}$. The complement of the zero section can then, up to a torsor under a unipotent group, be identified with $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k})$. This yields the required $\mathbb{A}^1$–weak equivalence. By an argument similar to the one in the setup for the localization sequence before Proposition 3.1, the pullback of the universal bundle $\mathscr{S}_k$ to $\mathscr{S}_k \setminus \mathrm{Gr}(k,n)$ will split off a direct summand, and the remainder is the tautological rank $k-1$ bundle on $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k})$. On the other hand, the pullback of the tautological rank $k-1$ bundle on $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k+1})$ to $\mathrm{GL}_n/(\mathrm{GL}_{k-1} \times \mathrm{GL}_{n-k})$ will still be the tautological rank $k-1$ bundle. □

Remark 6.2 In abuse of notation, we will denote the total spaces of both sphere bundles

$$\mathcal{Q}_{n-k+1} \setminus z_p(\mathrm{Gr}(k-1,n)) \quad \text{and} \quad \mathcal{S}_k \setminus z_q(\mathrm{Gr}(k,n))$$

by $\mathcal{S}(k,n)$. This is justified by Proposition 6.1 because they are $\mathbb{A}^1$–equivalent and hence they will have isomorphic $\boldsymbol{W}$–cohomology.

We obtain two localization sequences, relating the Grassmannians $\mathrm{Gr}(k,n)$ and $\mathrm{Gr}(k-1,n)$ to their associated sphere bundles. This, combined with the equivalence between the sphere bundles, is the relevant input for the induction step for the computation of the $\boldsymbol{W}$–cohomology of the Grassmannians.

Proposition 6.3 (1) *For any line bundle* $\mathcal{L}$ *on* $\mathrm{Gr}(k-1,n)$, *there is a long exact localization sequence*

$$\cdots \xrightarrow{e^{\perp}_{n-k+1}} H^{\bullet}(\mathrm{Gr}(k-1,n),\boldsymbol{W}(\mathcal{L})) \xrightarrow{p^*} H^{\bullet}(\mathcal{S}(k,n),\boldsymbol{W}(\mathcal{L}))$$
$$\xrightarrow{\partial} H^{\bullet-n+k}(\mathrm{Gr}(k-1,n),\boldsymbol{W}(\mathcal{L}\otimes\det\mathcal{Q}_{n-k+1})) \xrightarrow{e^{\perp}_{n-k+1}} H^{\bullet+1}(\mathrm{Gr}(k-1,n),\boldsymbol{W}(\mathcal{L})) \to \cdots .$$

(2) *For any line bundle* $\mathcal{L}$ *on* $\mathrm{Gr}(k,n)$, *there is a long exact localization sequence*

$$\cdots \xrightarrow{e_k} H^{\bullet}(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathcal{L})) \xrightarrow{q^*} H^{\bullet}(\mathcal{S}(k,n),\boldsymbol{W}(\mathcal{L}))$$
$$\xrightarrow{\partial} H^{\bullet-k+1}(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathcal{L}\otimes\det\mathcal{S}_k)) \xrightarrow{e_k} H^{\bullet+1}(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathcal{L})) \to \cdots .$$

Similar localization sequences are true for the other cohomology theories considered in this paper, but we will not need those.

6.2 Inductive computation of $\boldsymbol{W}$–cohomology

We now determine the structure of the total $\boldsymbol{W}$–cohomology ring of $\mathrm{Gr}(k,n)$. The argument completely follows the computation of rational cohomology of $\mathrm{Gr}_k(\mathbb{R}^n)$ in [23]. Some formulas for oriented Grassmannians related to the ones below can already be found in Ananyevskiy's computation for η–inverted theories; see [1].

Theorem 6.4 *Let* F *be a perfect field of characteristic* $\neq 2$ *and let* $1 \le k < n$. *The total* $\boldsymbol{W}$*–cohomology ring* $\bigoplus_{i,\mathcal{L}} H^i(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathcal{L}))$ *has the following presentation, as a commutative* $\mathbb{Z}\oplus\mathrm{Pic}(\mathrm{Gr}(k,n))/2$*–graded* $W(F)$*–algebra:*

(1) *For* k *and* n *even, the total* $\boldsymbol{W}$*–cohomology ring* $\bigoplus_{j,\mathcal{L}} H^j(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathcal{L}))$ *is isomorphic to*

$$\frac{W(F)[p_2,\ldots,p_k,e_k,p^{\perp}_2,\ldots,p^{\perp}_{n-k},e^{\perp}_{n-k}]}{(p\cdot p^{\perp}=1,\, e_k\cdot e^{\perp}_{n-k}=0,\, e_k^2=p_k,\, (e^{\perp}_{n-k})^2=(p^{\perp}_{n-k})^2)}.$$

(2) *If* n *is odd,*

$$\bigoplus_{j,\mathcal{L}} H^j(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathcal{L})) \cong \begin{cases} \frac{W(F)[p_2,\ldots,p_k,e_k,p^{\perp}_2,\ldots,p^{\perp}_{n-k-1}]}{(p\cdot p^{\perp}=1,\, e_k^2=p_k)} & \text{if } k \text{ is even,} \\ \frac{W(F)[p_2,\ldots,p_{k-1},p^{\perp}_2,\ldots,p^{\perp}_{n-k},e^{\perp}_{n-k}]}{(p\cdot p^{\perp}=1,\, (e^{\perp}_{n-k})^2=(p^{\perp}_{n-k})^2)} & \text{if } k \text{ is odd.} \end{cases}$$

(3) *For k and $n-k$ odd,*

$$\bigoplus_{j,\mathscr{L}} H^j(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L})) \cong \frac{W(F)[p_2,\dots,p_{k-1},p_2^\perp,\dots,p_{n-k-1}^\perp]}{(p\cdot p^\perp=1)} \otimes \bigwedge[R].$$

Here the notation is the one of Theorem 5.7, ie the bidegrees of the even Pontryagin classes p_{2i} are $(4i,0)$, the bidegrees of the Euler classes e_k and $e_{n-k}^\perp$ are $(k,1)$ and $(n-k,1)$, respectively, and the class R in the last case has bidegree $(n-1,0)$.

Remark 6.5 The description of the $\boldsymbol{I}^\bullet$–cohomology ring in Theorem 5.7 is compatible with the above claims via the natural projection $H^j(\mathrm{Gr}(k,n),\boldsymbol{I}^j(\mathscr{L})) \to H^j(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L}))$. Moreover, Theorem 5.7 implies Theorem 6.4.

The following is an analogue of Proposition 5.5 for the above cohomology; it will be used in the inductive proof of Theorem 6.4.

Proposition 6.6 *Let $1\le k<n$. Consider the morphism*

$$e_{n-k}^\perp\colon H^{\bullet-n+k}(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L})) \to H^\bullet(\mathrm{Gr}(k,n),\boldsymbol{W}(\mathscr{L}\otimes\det\mathscr{S}_{n-k}^\vee))$$

given by multiplication with the Euler class.

(1) *The cokernel is the quotient of the cohomology algebra modulo the ideal $\langle e_{n-k}^\perp\rangle$.*

(2) *If $k\equiv n-k\equiv 0 \bmod 2$, then the kernel of $e_{n-k}^\perp$ is the ideal $\langle e_k\rangle$. The cokernel is generated by the classes $p_2,\dots,p_k,e_k,p_2^\perp,\dots,p_{n-k-2}^\perp$ modulo the relations $p\cdot p^\perp=1$ and $e_k^2=p_k$. The classes in the kernel are products of e_k with a class in the cokernel.*

(3) *If $k+1\equiv n-k\equiv 0 \bmod 2$, then the cokernel is generated by the Pontryagin classes $p_2,\dots,p_{k-1}$, $p_2^\perp,\dots,p_{n-k-2}^\perp$ modulo the relation $p\cdot p^\perp=1$. The kernel is the ideal $\langle p_{k-1}e_{n-k}^\perp\rangle$.*

(4) *If $n-k\equiv 1 \bmod 2$, the multiplication map is 0. The kernel and cokernel are the whole cohomology algebra.*

Proof This follows directly from the explicit presentation of Theorem 6.4. □

Proof of Theorem 6.4 Fix a natural number n. The claim for $\mathrm{Gr}(k,n)$ is proved by induction on k.

The base case is the case $\mathbb{P}^{n-1}=\mathrm{Gr}(1,n)$, in which case the claim follows directly from the computations in [14] — realizing for instance $H^i(\mathbb{P}^n,\boldsymbol{W}(\mathscr{L}))\cong H^i(\mathbb{P}^n,\boldsymbol{I}^{i-1}(\mathscr{L}))$. In both cases there are only two nontrivial groups, one of them is $H^0(\mathbb{P}^{n-1},\boldsymbol{W})\cong W(F)$. If n is even, then $\mathbb{P}^{n-1}$ is orientable, and the other nontrivial cohomology groups is $H^{n-1}(\mathbb{P}^{n-1},\boldsymbol{W})\cong W(F)$ (nontwisted coefficients), generated by the orientation class R. If n is odd, the other nontrivial cohomology group is $H^{n-1}(\mathbb{P}^{n-1},\boldsymbol{W}(\det\mathscr{S}_1))\cong W(F)$, generated by $e_{n-1}^\perp$.[8]

[8]Note the similarity with the rational cohomology of the real projective space.

For the inductive step, assume that Theorem 6.4 holds for $\mathrm{Gr}(k-1,n)$. We have to make a case distinction depending on parities of k and n.

If $n-k+1$ and $k-1$ are even Then the Euler classes e_{k-1} and $e^{\perp}_{n-k+1}$ are nonzero. The kernel and cokernel of $e^{\perp}_{n-k+1}$ are described in parts (1) and (2) of Proposition 6.6. As an algebra over the image of the cokernel of $e^{\perp}_{n-k+1}$, the cohomology of $\mathscr{S}(k,n)=\mathscr{Q}_{n-k+1}\setminus z_p(\mathrm{Gr}(k-1,n))$ is an exterior algebra, generated by 1 and the class R in degree $(n-1,0)$ which is a lift of e_{k-1} along ∂. This follows from the localization sequence for the bundle $\mathscr{Q}_{n-k+1}$; see point (1) of Proposition 6.3.

For the second localization, for the bundle $\mathscr{S}_k$, we first note that the Euler classes e_k and $e^{\perp}_{n-k}$ are zero. We check what we can say about the map q^*: We have the Pontryagin classes $p_2,\dots,p_{k-1}$, and these are mapped to their counterparts in the cohomology of $\mathscr{S}_k\setminus z_q(\mathrm{Gr}(k,n))$, by Proposition 6.1. By exactness, all the classes in the image of the restriction morphism q^* will have trivial image under the boundary map ∂. Also, the class R from degree $(n-1,0)$ has image under ∂ in degree $(n-k,1)$; in the case at hand, $n-k$ is odd, so there are no nontrivial elements in this degree and therefore $\partial R=0$.

Now we need to determine which classes have nontrivial image under ∂. The class e_{k-1} from the cokernel of $e^{\perp}_{n-k+1}$ necessarily maps to 1 under ∂. By the derivation property, more generally a product $p\cdot e_{k-1}$ of the Euler class with a polynomial p in the Pontryagin classes $p_2,\dots,p_{k-1}$ will map under ∂ to p, viewed as an element of the cohomology of $\mathrm{Gr}(k,n)$.

At this point, we see that the $\boldsymbol{W}$–cohomology of $\mathrm{Gr}(k,n)$ is indeed generated by the characteristic classes listed in the theorem statement: the Pontryagin classes p_i and $p_i^{\perp}$ are mapped to their counterparts in the cohomology of $\mathscr{S}(k,n)$; and the same is true for the class R. The only missing generator of the cohomology of $\mathscr{S}(k,n)$ is the Euler class e_{k-1}, but we saw above that this class has nontrivial boundary. It follows similarly, that the only relation is given by the Whitney sum formula.

If both $n-k+1$ and $k-1$ are odd Then the Euler classes e_{k-1} and $e^{\perp}_{n-k+1}$ are zero. In particular, via the first localization sequence for the bundle $\mathscr{Q}_{n-k+1}\to\mathrm{Gr}(k-1,n)$, the cohomology of $\mathscr{S}(k,n)$ consists of two copies of the cohomology of $\mathrm{Gr}(k-1,n)$; one of the copies is obviously generated by 1 in degree $(0,0)$, the other generated by a class in bidegree $(n-k,1)$ which is a lift of $1\in H^0$ along the boundary map.

Now for the second bundle $\mathscr{S}_k\to\mathrm{Gr}(k,n)$, both Euler classes e_k and $e^{\perp}_{n-k+1}$ are nontrivial. We check what we can say about the restriction map q^* in the corresponding localization sequence. In the cohomology of $\mathrm{Gr}(k,n)$, we have the Pontryagin classes and these are mapped under q^* to their counterparts in the cohomology of $\mathscr{S}(k,n)$. The class in bidegree $(n-k,1)$ (which arose as lift of 1 in the first localization sequence) lifts to the Euler class $e^{\perp}_{n-k}$.

The class R from the cohomology of $\mathscr{S}(k,n)$ has nontrivial boundary; its degree is $(n-1,0)$ and its image under the boundary map has degree $(n-k,1)$, so the class R is mapped exactly to the Euler class $e^{\perp}_{n-k}$.

Consequently, this establishes the claimed presentation of the $\boldsymbol{W}$–cohomology ring.

If $n-k+1$ is even and $k-1$ is odd Then the Euler class $e^{\perp}_{n-k+1}$ is nontrivial. The kernel and cokernel of $e^{\perp}_{n-k+1}$ are described in Proposition 6.6. The cokernel is generated by the Pontryagin classes, and the kernel is the ideal $\langle p_{k-2}e^{\perp}_{n-k+1}\rangle$. The class $p_{k-2}e^{\perp}_{n-k+1}$ has degree $(n-k+3,1)$ and consequently lifts along the boundary map ∂ to a class in degree $(2n-3,0)$.

Now for the second bundle $\mathcal{S}_k \to \mathrm{Gr}(k,n)$, the Euler class e_k is also nontrivial. We check what happens in the associated localization sequence. For the moment, call the right-hand side of the isomorphism in (2) of the statement the "candidate presentation". The cokernel of the multiplication by e_k on the candidate presentation is generated by the Pontryagin classes which map to their counterparts in the cohomology of $\mathcal{S}(k,n)$. The kernel of the Euler class on the candidate presentation is the ideal generated by $e_k p^{\perp}_{n-k-1}$ in degree $(2n-k-2,1)$.

The Pontryagin classes in the cokernel all map to their counterparts under the restriction map q^*. The class in degree $(2n-3,0)$ (which arose as a lift of $p_{k-2}e^{\perp}_{n-k+1}$) maps to $e_k p^{\perp}_{n-k-1}$ in degree $(2n-k-2,1)$.

Consequently, we see that the description of the cohomology of $\mathrm{Gr}(k,n)$, given in Theorem 6.4, is true if and only if it is true for $\mathrm{Gr}(k-1,n)$. Therefore, this argument also settles the case where $n-k+1$ is odd and $k-1$ is even. □

6.3 Putting the pieces together

We are now in the position to prove the theorems about the structure of $\boldsymbol{I}$–cohomology of the Grassmannians $\mathrm{Gr}(k,n)$.

Proof of Proposition 5.8 For all characteristic classes except R the claims on their reductions follow from Theorem 3.24(2). The injectivity of ρ on the image of $\beta_{\mathcal{L}}$ follows from Theorem 6.4, in combination with Lemma 2.4, via the splitting of $\boldsymbol{I}$–cohomology as direct sum of $\boldsymbol{W}$–cohomology and the image of $\beta_{\mathcal{L}}$.

It remains to identify the reduction of R. This follows by tracing through the inductive proof of Theorem 5.7, noting that R arises via boundary maps from Euler classes. The key point to note is that the reduction of R in the mod 2 Chow ring must be divisible by both $\bar{c}_k$ and $\bar{c}^{\perp}_{n-k}$, which implies the claim. Alternatively, the identification can be deduced from [23, Remarks 4 and 5] using the real cycle class map isomorphisms of [17]. □

Proof of Theorem 5.7 Again, we use the splitting of $\boldsymbol{I}$–cohomology as direct sum of $\boldsymbol{W}$–cohomology and the image of $\beta_{\mathcal{L}}$ which follows from Theorem 6.4, in combination with Lemma 2.4.

Relations (1)–(4) claimed in the theorem are satisfied because they are already satisfied on the level of $B\mathrm{GL}_n$ by Theorem 3.24 and the Whitney sum formulas in Propositions 3.3 and 3.28. Relation (5) involving R has two components: the claim on multiplication with torsion classes follows from the injectivity of ρ on the image of β given by Proposition 5.8, and the claim $R^2=0$ in $\boldsymbol{W}$–cohomology follows from Theorem 6.4. The image of R^2 under the projection to $\mathrm{Im}\,\beta$ can be computed in mod 2 Chow theory, where we have $\rho(R^2)=\bar{c}_{k-1}\bar{c}^{\perp}_{n-k}\bar{c}_k\bar{c}^{\perp}_{n-k-1}=0$. In particular, we get a well-defined map from the candidate presentation (with generators listed in (G1)–(G5) and relations (R1)–(R5) of Theorem 5.7) to the total $\boldsymbol{I}$–cohomology ring of $\mathrm{Gr}(k,n)$.

To show that the generators listed in Theorem 5.7 generate the $\boldsymbol{I}$–cohomology ring we again first show that all the torsion classes in the image of β are accounted for. Knowing the mod 2 Chow ring of the Grassmannians — see Proposition 5.4 — this follows as in the proof of Theorem 3.24 by considering the image of $\mathrm{Sq}^2_{\mathscr{L}}$. Then the surjectivity for $\boldsymbol{W}$–cohomology follows from Theorem 6.4.

To show injectivity, ie that all relations in the cohomology ring are accounted for, we note that the $W(F)$–torsionfree part generated by the Pontryagin classes, as well as Euler classes or R (whenever applicable), has exactly the relations (2), (3) and (5), by Theorem 6.4. So it suffices to investigate relations among classes in the image of β. Since ρ is injective on the image of $\beta_{\mathscr{L}}$, it suffices to show that all relations appearing in $\mathrm{Ch}^\bullet(\mathrm{Gr}(k,n))$ arise from those for $B\mathrm{GL}_n$ and the Whitney sum formulas. This follows from the presentation of the mod 2 Chow rings in Proposition 5.4 and Theorem 3.24. □

6.4 An example

We discuss the argument for nonorientable Grassmannians in the special case comparing $\mathbb{P}^4$ and $\mathrm{Gr}(2,5)$. The following computation also indicates how one may go about establishing the formulas for $\boldsymbol{I}$–cohomology directly without the β–$\boldsymbol{W}$–decomposition. A complete version of this argument can be found in the first version of the present paper on the arXiv [27].

First, we consider the localization sequence associated to the tautological rank 4 bundle on $\mathbb{P}^4$ which has the form

$$\cdots \to H^j(\mathbb{P}^4, \boldsymbol{I}^j(\mathscr{L})) \to H^j(T, \boldsymbol{I}^j(\mathscr{L})) \to H^{j-3}\big(\mathbb{P}^4, \boldsymbol{I}^{j-4}(\mathscr{L}(1))\big) \xrightarrow{e_4^\perp} \cdots,$$

where T is the complement of the zero section of the rank 4 bundle on $\mathbb{P}^4$. From the shape of the localization sequence, we see that there are isomorphisms

$$\begin{aligned} H^j(\mathbb{P}^4, \boldsymbol{I}^j(\mathscr{L})) &\cong H^j(T, \boldsymbol{I}^j(\mathscr{L})) && \text{for } j \le 2, \\ H^{j+3}(T, \boldsymbol{I}^{j+3}(\mathscr{L})) &\cong H^j\big(\mathbb{P}^4, \boldsymbol{I}^{j-1}(\mathscr{L}(1))\big) && \text{for } j \ge 2. \end{aligned}$$

The complicated bit is given by two exact sequences. First,

$$0 \to H^3(\mathbb{P}^4, \boldsymbol{I}^3(\mathscr{L})) \to H^3(T, \boldsymbol{I}^3(\mathscr{L})) \to H^0\big(\mathbb{P}^4, \boldsymbol{I}^{-1}(\mathscr{L}(1))\big) \to H^4(\mathbb{P}^4, \boldsymbol{I}^3(\mathscr{L})).$$

In the case where $\mathscr{L} = \mathbb{O}$, then the first and third terms in the exact sequence are trivial and so is $H^3(T, \boldsymbol{I}^3)$. In the case where $\mathscr{L} = \mathbb{O}(1)$, the third term is $W(F) \cdot 1$ and the last term is $W(F) \cdot e_4^\perp$, so multiplication with the Euler class $e_4^\perp$ is an isomorphism. Consequently, we have an isomorphism $H^3(T, \boldsymbol{I}^3(1)) \cong H^3(\mathbb{P}^4, \boldsymbol{I}^3(1)) \cong \mathbb{Z}/2\mathbb{Z}$, generated by e_1^3.

The second exact sequence is

$$H^0\big(\mathbb{P}^4, \boldsymbol{I}^0(\mathscr{L}(1))\big) \to H^4(\mathbb{P}^4, \boldsymbol{I}^4(\mathscr{L})) \to H^4(T, \boldsymbol{I}^4(\mathscr{L})) \to H^1\big(\mathbb{P}^4, \boldsymbol{I}^0(\mathscr{L}(1))\big) \to 0.$$

In the case where $\mathscr{L}=\mathbb{O}$, the first and last terms in the exact sequence are trivial and we get an isomorphism $H^4(T,\boldsymbol{I}^4)\cong H^4(\mathbb{P}^4,\boldsymbol{I}^4)\cong\mathbb{Z}/2\mathbb{Z}$, generated by e_1^4. In the case where $\mathscr{L}=\mathbb{O}(1)$, the first morphism is multiplication by the Euler class which is an isomorphism. In particular, we get an isomorphism $H^4(T,\boldsymbol{I}^4(1))\to H^1(\mathbb{P}^4,\boldsymbol{I}^0(1))\cong 0$.

Now we can consider the localization sequence for the tautological rank 2 bundle on $\mathrm{Gr}(2,5)$ which has the form

$$\cdots\xrightarrow{e_2} H^j(\mathrm{Gr}(2,5),\boldsymbol{I}^j(\mathscr{L}))\to H^j(T,\boldsymbol{I}^j(\mathscr{L}))\to H^{j-1}\big(\mathrm{Gr}(2,5),\boldsymbol{I}^{j-2}(\mathscr{L}(1))\big)\xrightarrow{e_2}\cdots.$$

Because of cohomology vanishing in negative degrees, $H^0(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}))\cong H^0(T,\boldsymbol{I}^0(\mathscr{L}))$, and we note that this is isomorphic to the respective cohomology of $\mathbb{P}^4$. Next, there is an exact sequence

$$0\to H^1(\mathrm{Gr}(2,5),\boldsymbol{I}^1(\mathscr{L}))\to H^1(T,\boldsymbol{I}^1(\mathscr{L})).$$

For $\mathscr{L}=\mathbb{O}$, the last group is trivial, implying triviality of $H^1(\mathrm{Gr}(2,5),\boldsymbol{I}^1(\mathscr{L}))$. For $\mathscr{L}=\mathbb{O}(1)$, the last group is $\mathbb{Z}/2\mathbb{Z}$. The explicit generator $\beta_{\mathbb{O}(1)}(1)$ maps to a generator of the last group and this implies that $H^1(\mathrm{Gr}(2,5),\boldsymbol{I}^1(\mathscr{L}))\cong\mathbb{Z}/2\mathbb{Z}$.

For H^2, we have an exact sequence

$$H^0\big(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}(1))\big)\to H^2(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}))\to H^2(T,\boldsymbol{I}^2(\mathscr{L}))\to H^1\big(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}(1))\big).$$

For $\mathscr{L}=\mathbb{O}$, the outer groups are both zero and hence $H^2(\mathrm{Gr}(2,5),\boldsymbol{I}^0(\mathscr{L}))\cong\mathbb{Z}/2\mathbb{Z}$. For $\mathscr{L}=\mathbb{O}(1)$, the first map is an isomorphism mapping 1 to e_2. Note that only using the localization sequence at this point would require knowledge of the restriction morphism $H^1(\mathrm{Gr}(2,5),\boldsymbol{I}^2(\mathscr{L}))\to H^1(T,\boldsymbol{I}^2(\mathscr{L}))$ to show that $H^2\big(\mathrm{Gr}(2,5),\boldsymbol{I}^2(\mathbb{O}(1))\big)$ is isomorphic to $W(F)$ and not a proper quotient.

The remaining cohomology groups can be computed similarly, producing exactly the results from Example 5.13.

6.5 Remarks on oriented Grassmannians

We briefly formulate the analogous results for the Chow–Witt rings of the oriented Grassmannians. Recall that the $\mathbb{A}^1$–fundamental group of the Grassmannians is $\pi_1^{\mathbb{A}^1}(\mathrm{Gr}(k,n))\cong\mathbb{G}_m$, since up to $\mathbb{A}^1$–weak equivalence the Grassmannians are GL_{n-k}–quotients of the Stiefel varieties $\mathrm{GL}_n/\mathrm{GL}_k$ which are highly $\mathbb{A}^1$–connected. The oriented Grassmannians $\widetilde{\mathrm{Gr}}(k,n)$ are the $\mathbb{A}^1$–universal covers of the Grassmannians $\mathrm{Gr}(k,n)$. Explicitly, they are given as the complement of the zero section of the line bundle $\det\mathscr{S}_k$. For the Chow–Witt rings of the oriented Grassmannians $\widetilde{\mathrm{Gr}}(k,n)$, we only have the trivial duality because they are $\mathbb{A}^1$–simply connected.

We can formulate a result analogous to Theorem 5.7 for the oriented Grassmannians. The proof of the result proceeds exactly along the lines of the proofs for $\mathrm{Gr}(k,n)$. Some results concerning the $\boldsymbol{W}$–cohomology of the oriented Grassmannians can be deduced from the work of Ananyevskiy in [1].

Theorem 6.7 *Let F be a perfect field of characteristic $\neq 2$, and let $1 \leq k < n$.*

(1) *There is a cartesian square of $\mathbb{Z}$–graded $\mathrm{GW}(F)$–algebras,*

$$\begin{array}{ccc} \widetilde{\mathrm{CH}}^\bullet(\widetilde{\mathrm{Gr}}(k,n)) & \longrightarrow & \ker \partial \\ \big\downarrow & & \big\downarrow{\scriptstyle \bmod 2} \\ H^\bullet(\widetilde{\mathrm{Gr}}(k,n), \boldsymbol{I}^\bullet) & \xrightarrow[\rho]{} & \mathrm{Ch}^\bullet(\widetilde{\mathrm{Gr}}(k,n)) \end{array}$$

(2) *The cokernel of the Bockstein morphism*

$$\beta\colon \mathrm{CH}^j(\widetilde{\mathrm{Gr}}(k,n)) \to H^{j+1}(\widetilde{\mathrm{Gr}}(k,n), \boldsymbol{I}^{j+1})$$

is described exactly as in Theorem 6.4, except that there is no additional $\mathbb{Z}/2\mathbb{Z}$–grading and the Euler classes are elements of the cohomology with trivial duality.

(3) *The reduction morphism*

$$\rho\colon H^{j+1}(\widetilde{\mathrm{Gr}}(k,n), \boldsymbol{I}^{j+1}) \to \mathrm{CH}^{j+1}(\widetilde{\mathrm{Gr}}(k,n))$$

is injective on the image of the Bockstein morphism β. In particular, the image of Bockstein can be determined from the Wu formula for the Steenrod squares on the mod 2 Chow ring of $\widetilde{\mathrm{Gr}}(k,n)$.

Remark 6.8 A result like the above should be true for all flag varieties (at least in type A). The cokernel of the Bockstein should have the same presentation as the rational cohomology of the real realization (but of course as a $W(F)$–algebra). The Bockstein classes should all be detected on the mod 2 Chow ring so that the structure of the torsion can be determined just from the knowledge of the Steenrod squares.

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