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Knot Floer homology, link Floer homology and link detection

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We give new link detection results for knot and link Floer homology, inspired by recent work on Khovanov homology. We show that knot Floer homology detects $T(2,4)$, $T(2,6)$, $T(3,3)$, $L7n1$ and the link $T(2,2n)$ with the orientation of one component reversed. We show link Floer homology detects $T(2,2n)$ and $T(n,n)$, for all n. Additionally, we identify infinitely many pairs of links such that both links in the pair are each detected by link Floer homology but have the same Khovanov homology and knot Floer homology. Finally, we use some of our knot Floer detection results to give topological applications of annular Khovanov homology.

57K10, 57K18

1 Introduction

Knot and link Floer homology are invariants of links in S^3; see Ozsváth and Szabó [31; 32] and Rasmussen [34]. There are a number of formal similarities between these Floer theoretic invariants and the combinatorial Khovanov homology. Recently, Khovanov homology has been shown to detect a number of simple links; see Baldwin, Dowlin, Levine, Lidman and Sazdanovic [2], Li, Xie and Zhang [23], Martin [26] and Xie and Zhang [38; 40]. Some of these detection results have used knot and link Floer homology without going so far as to determine whether knot or link Floer homology detects the relevant link. Inspired by this work, we give such detection results for knot and link Floer homology. We remind the reader that the knot Floer homology of a link L is computed using an associated knot, called the knotification of L, in a connected sum of copies of $S^1 \times S^2$, while the link Floer homology of L is computed directly from the link L in S^3.

Previously it was known that knot Floer homology detects the unknot (see Ozsváth and Szabó [30]), the trefoil (see Ghiggini [7]), the figure eight knot [7], the Hopf link (see Ni [28] and [30]) and the unlink (see Hedden and Watson [15] and Ni [29]). Link Floer homology was known to detect the trivial n–braid together with its braid axis (see Baldwin and Grigsby [3]) and determine if a link is split; see Wang [37]. It was also known that a stronger version of link Floer homology, CFL^∞, detects the Borromean rings and the Whitehead link; see Gorsky, Lidman, B Liu and Moore [11].

We prove the following knot Floer homology detection results:

Theorem 4.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$, then L is isotopic to $T(2,4)$.*

Throughout, we take the links $T(2,2n)$ to be oriented as the closure of the 2–braids σ_1^{2n}.

Theorem 5.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,6))$, *then* L *is isotopic to* $T(2,6)$.

Let J_n be the link obtained from $T(2,2n)$ by reversing the orientation on one of the components. Then:

Theorem 3.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(J_n)$ *for some* n, *then* L *is isotopic to* J_n.

Theorem 7.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(3,3))$, *then* L *is isotopic to* $T(3,3)$.

Theorem 8.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(L7n1)$, *then* L *is isotopic to* $L7n1$.

We also prove the following link Floer homology detection results:

Theorem 3.2 *If* $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2,2n))$ *for some* n, *then* L *is isotopic to* $T(2,2n)$.

Theorem 6.1 *If* $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(n,n))$, *then* L *is isotopic to* $T(n,n)$.

Proposition 9.2 *Suppose link Floer homology detects a link* L, *and that if permuting some collection of Alexander gradings of* $\widehat{\mathrm{HFL}}(L)$ *induces an isomorphism on* $\widehat{\mathrm{HFL}}(L)$ *then there is a symmetry of* L *that exchanges the corresponding components. Then link Floer homology detects* $L \# H$ *for each choice of component of* L *to connect sum with.*

Throughout, we view the Hopf link as $T(2,2)$ and endow it with the associated orientation. A consequence of these detection results is that every link currently known to be detected by Khovanov homology is also detected by either knot or link Floer homology. This leads to the following natural question:

Question 1.1 Is there a link which Khovanov homology detects but which neither knot nor link Floer homology detects?

On the other hand, we show that there are infinitely many links detected by link Floer homology but which are detected by neither Khovanov homology nor knot Floer homology.

Theorem 9.4 *There exist infinitely many pairs of links* (L, L') *such that link Floer homology detects* L *and* L' *but* $\mathrm{Kh}(L) \cong \mathrm{Kh}(L')$ *and* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(L')$.

Finally, we use some of our torus link detection results to derive applications to annular Khovanov homology. Annular Khovanov homology is an invariant of links in the thickened annulus $A \times I$, sometimes thought of as $S^3 \setminus U$ where U is an unknot or the annular axis. To do this, we utilize a generalization of the Ozsváth–Szabó spectral sequence, which relates annular Khovanov homology and knot Floer homology of the lift of the annular axis $\widetilde{U}$ in $\Sigma(L)$, the double branched cover of L; see Grigsby and Wehrli [14] and Roberts [35].

Theorem 10.4 *Let* $L \subseteq A \times I \subseteq S^3$ *be an annular link. If* $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2}, \mathbb{Z}/2\mathbb{Z})$, *then* L *is isotopic to* $\widehat{\sigma_1\sigma_2}$ *in* $A \times I$.

Theorem 10.6 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If* $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2^{-1}}, \mathbb{Z}/2\mathbb{Z})$*, then L is isotopic to $\widehat{\sigma_1\sigma_2^{-1}}$ in $A \times I$.*

Theorem 10.7 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If* $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2\sigma_3}, \mathbb{Z}/2\mathbb{Z})$*, then L is isotopic to $\widehat{\sigma_1\sigma_2\sigma_3}$ in $A \times I$.*

Theorem 10.8 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If* $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}, \mathbb{Z}/2\mathbb{Z})$*, then L is isotopic to $\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}$ in $A \times I$.*

Notice that $\widehat{\sigma_1\sigma_2}$, $\widehat{\sigma_1\sigma_2^{-1}}$, $\widehat{\sigma_1\sigma_2\sigma_3}$ and $\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}$ all represent the unknot when considered in S^3, but these braid closures are all nontrivial knots in $A \times I$.

This paper is organized as follows. In Section 2 we briefly review knot and link Floer homology. In Section 3 we prove that knot Floer homology detects J_n, and link Floer homology detects $T(2, 2n)$. In Section 4 we prove that knot Floer homology detects $T(2, 4)$. In Section 5 we prove that knot Floer homology detects $T(2, 6)$. In Section 6 we prove that link Floer homology detects $T(n, n)$. In Section 7 we prove that knot Floer homology detects $T(3, 3)$. In Section 8 we prove that knot Floer homology detects $L7n1$. In Section 9 we prove that there are infinite families of links detected by link Floer homology that also have the same Khovanov homology and knot Floer homology. Finally, in Section 10 we prove the annular Khovanov homology results using some of our knot Floer detection results.

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2 Knot Floer homology and link Floer homology

Knot Floer homology and link Floer homology are invariants of links in S^3, defined using a version of Lagrangian Floer homology [31; 32; 34]. They are categorifications of the single variable and multivariable Alexander polynomials, respectively. Here we briefly highlight the key features of knot Floer homology and link Floer homology that we use to obtain our detection results. We work with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Let L be an oriented link in S^3, with components $L_1, L_2, \dots, L_n$. The link Floer homology of L is a multigraded vector space

$$\widehat{\mathrm{HFL}}(L) = \bigoplus_{d, A_1, \dots A_n} \widehat{\mathrm{HFL}}_d(L; A_1, \dots A_n).$$

The grading denoted by "d" above is called the Maslov or algebraic grading, while the A_i gradings are called the Alexander gradings. Each A_i satisfies $2A_i + \ell k(L_i, L - L_i) \in 2\mathbb{Z}$.

The knot Floer homology of L is a vector space bigraded by a Maslov grading and a single Alexander grading. The knot Floer homology of L can be obtained by projecting the link Floer homology onto the diagonal of the multi-Alexander gradings, which becomes the Alexander grading, and adding $\frac{1}{2}(n-1)$ to the Maslov grading.

We use a number of formal properties of knot and link Floer homology in proving our link detection results. The first of these is that link Floer homology has a symmetry relating the component of the homology supported in grading $(m, A_1, \dots, A_n)$ with the component of the homology supported in grading $\bigl(m - 2\sum_{i=1}^{n} A_i, -A_1, \dots, -A_n\bigr)$. Knot Floer homology enjoys the same symmetry property, since it can be defined by projecting the multi-Alexander gradings onto the diagonal. There is also a Künneth formula for computing the link Floer homology of a connected sum in terms of a tensor product of link Floer homologies.

The main formal property we will use, however, is that the link Floer homology of L admits spectral sequences to the link Floer homologies of its sublinks [3, Lemmas 2.4 and 2.5; 32]. In particular, when L_i is a sublink of L, there is a spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(L - L_i) \otimes V^{|L_i|}$ shifting each Alexander grading by $\frac{1}{2}\,\ell\mathrm{k}(L - L_i, L_i)$. It follows that there is also a spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HF}}(S^3) \otimes V^{n-1}$, or equivalently that there is a spectral sequence from $\widehat{\mathrm{HFK}}(L)$ to $\widehat{\mathrm{HF}}\bigl(\#^{n-1}(S^1 \times S^2)\bigr)$. Here V is the multigraded vector space $\mathbb{F} \oplus \mathbb{F}$ with nonzero Maslov gradings 0 and -1 and multi-Alexander grading $(0, \dots, 0)$.

In addition to enjoying the above algebraic properties, $\widehat{\mathrm{HFK}}(L)$ and $\widehat{\mathrm{HFL}}(L)$ are known to reflect a number of topological properties of L. For starters, there are a number of things we can say about the number of components of L. Since $\widehat{\mathrm{HFL}}(L)$ admits a spectral sequence to V^{n-1}, a link is a knot if and only if $\operatorname{rank}(\widehat{\mathrm{HFK}}(L))$ is odd. Since the Maslov grading for $\widehat{\mathrm{HFL}}(L)$ is integer valued, while the Maslov gradings of $\widehat{\mathrm{HFK}}(L)$ are $\mathbb{Z} + \frac{1}{2}(n-1)$ valued, it follows that if the Maslov gradings of $\widehat{\mathrm{HFK}}(L)$ are contained in $\mathbb{Z} + \frac{1}{2}$ then L has an odd number of components, while if the Maslov gradings of $\widehat{\mathrm{HFK}}(L)$ are contained in grading $\mathbb{Z}$ then L has an even number of components. Finally, since $\widehat{\mathrm{HFK}}(L)$ admits a spectral sequence to $\widehat{\mathrm{HF}}\bigl(\#^{n-1}(S^1 \times S^2)\bigr)$ — which has a generator of Maslov grading $\frac{1}{2}(n-1)$ — we have that $n \le 1 + 2\max\{m : \operatorname{rank}(\widehat{\mathrm{HFK}}_m(L)) \ne 0\}$.

Moreover, since knot Floer homology categorifies the Alexander–Conway polynomial, and the Alexander–Conway polynomial detects the linking number of 2–component links by a result of Hoste [16], it follows that knot Floer homology detects the linking number of 2–component links.

We will also make use of the fact that the link Floer homology of L yields information about the topology of $S^3 - L$; in particular, that link Floer homology detects the Thurston norm of $S^3 - L$ [33]. Finally, if L is not a split link and has a component L_i which is fibered, then the top Alexander grading associated to L_i determines if $L - L_i$ is a braid in the complement of L_i. Specifically, this happens exactly when the rank in maximal nonzero Alexander grading is 2^{n-1} [26, Proposition 1].

3 Knot Floer homology detects J_n

Given that knot Floer homology detects the maximal Euler characteristic of oriented links, it is natural to try and detect links with large Euler characteristic. The unique 1–component link of maximal Euler characteristic is the unknot, which knot Floer homology is known to detect. Links with two components that bound annuli, ie 2–cable links, have high Euler characteristic. The simplest of these are 2–cables of unknots, ie the links $T(2,2n)$ with the orientation of one component reversed. We call these links J_n. In this section we show that knot Floer homology detects each J_n.

Theorem 3.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(J_n)$ for some n, then L is isotopic to J_n.*

For reference, we note that, when n is positive, $\widehat{\mathrm{HFK}}(J_n) \cong \mathbb{F}^n_{3/2}[1] \oplus \mathbb{F}^{2n}_{1/2}[0] \oplus \mathbb{F}^n_{-1/2}[-1]$, where the subscript denotes the Maslov grading of the generator, and $[i]$ denotes the Alexander grading of a summand. When n is negative, $\widehat{\mathrm{HFK}}(J_n)$ can be computed from the above formula using an understanding of how knot Floer homology is affected by mirroring.

A consequence of this detection result is that link Floer homology detects the links $T(2,2n)$. This follows from Theorem 3.1 by considering how link Floer homology changes under reversing the orientation of a single component.

Theorem 3.2 *If $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2,2n))$ for some n, then L is isotopic to $T(2,2n)$.*

To prove Theorem 3.1, we will first prove that L is a 2–component link and that both of the components of L are unknots. Then we will use the fact that knot Floer homology detects genus to show that J_n is detected among 2–component links with unknotted components. The topological argument used here was communicated to the authors by Eugene Gorsky (2020), and also appears in Liu's classification of the links $T(2,2n)$ in terms of surgery to a Heegaard Floer L–space [25].

Lemma 3.3 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(J_n)$ for some n, then L is a 2–component link, and both of the components are unknots.*

Proof First we show that L is a 2–component link. Notice that the parity of the rank of $\widehat{\mathrm{HFK}}(L)$ rules out the case that L is a knot. If L is an n–component link, then there is a spectral sequence from $\widehat{\mathrm{HFK}}(L)$ to $\widehat{\mathrm{HF}}(\#^{n-1}(S^1 \times S^2))$. Because $\widehat{\mathrm{HFK}}(L)$ is only nonzero in Maslov gradings $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$, this spectral sequence can only exist for $n=2$.

To see that both components of L are unknotted, we consider the spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFK}}(K) \otimes V$ where K is a component of L. From this spectral sequence we see that $\widehat{\mathrm{HFK}}(K)$ is 0 in all Maslov gradings, except possibly 0 and 1. Considering how the Maslov grading changes under the symmetry of the Alexander grading for knot Floer homology, we can see that $\widehat{\mathrm{HFK}}(K)$ can only be supported in Alexander grading 0, so K is an unknot. □

With Lemma 3.3 we can now prove Theorem 3.1. The key step is to deduce that J_n is a cable of the unknot.

Proof of Theorem 3.1 Suppose $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,2n))$. By Lemma 3.3, L is a 2–component link. If $n=0$ then the maximal Alexander grading of $\widehat{\mathrm{HFK}}(L)$ is 0, whence L bounds two disjoint disks and L is the 2–component unlink, as desired.

We now consider the $n \neq 0$ case. Here the maximal Alexander grading of $\widehat{\mathrm{HFK}}(L)$ is 1, and we see that the two components of L bound a surface of Euler characteristic 0. Note that the linking number of L, which knot Floer homology detects, is nonzero, so L cannot bound the disjoint union of a disk and a punctured torus. Thus L bounds an annulus, ie it is the twisted 2–cable of some knot. Each component of L is isotopic to the knot that was cabled, and so L is a twisted 2–cable of the unknot by Lemma 3.3. This means that L is J_m for some m. Finally, a simple computation of the respective ranks in each Maslov grading shows that $\widehat{\mathrm{HFK}}(J_m) \cong \widehat{\mathrm{HFK}}(J_n)$ if and only if $m=n$. Thus L is isotopic to J_n. □

4 Knot Floer homology detects $T(2,4)$

Here we will utilize the results of the previous section to obtain a detection result for the torus link $T(2,4)$. The link Floer homology of $T(2,4)$ is shown in Table 1, for reference.

Theorem 4.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$, then L is isotopic to $T(2,4)$.*

To prove this, we show the following lemma:

Lemma 4.2 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$ then L consists of two components, L_1 and L_2, such that each $\widehat{\mathrm{HFK}}(L_i)$ has a unique Maslov index 0 generator. Moreover, that generator is supported in Alexander grading 0 in $\widehat{\mathrm{HFK}}(L_i)$.*

We then show that L has the same link Floer homology as $T(2,4)$, using structural properties of link Floer homology, and apply Theorem 3.2 to complete the proof.

The following lemma will be useful in proving Lemma 4.2:

Lemma 4.3 *Suppose K is a component of a link L such that $\widehat{\mathrm{HFL}}(L)$ is supported in Maslov gradings at most 0 with a unique Maslov grading 0 generator. Then there is a unique Maslov index grading 0 generator in $\widehat{\mathrm{HFK}}(K)$, and it is of nonnegative Alexander grading.*

	-1	0	1
1		$\mathbb{F}_{-1}$	$\mathbb{F}_0$
0	$\mathbb{F}_{-3}$	$\mathbb{F}_{-2}^2$	$\mathbb{F}_{-1}$
-1	$\mathbb{F}_{-4}$	$\mathbb{F}_{-3}$	

Table 1: The link Floer homology of $T(2,4)$. The coordinates give the multi-Alexander grading; the subscript gives the Maslov grading.

Proof The Maslov grading 0 generator must persist under the spectral sequence from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(K) \otimes V^{|L|-1}$, as else it cannot persist to $\widehat{\mathrm{HF}}(S^3) \otimes V^{|L|-1}$. If this generator sat in a negative Alexander grading then the symmetry properties of knot Floer homology would imply that there is a positive Maslov index generator in $\widehat{\mathrm{HFL}}(K) \otimes V^{|L|-1}$. However there are no positive Maslov index generators in $\widehat{\mathrm{HFL}}(L)$, and so there are none in $\widehat{\mathrm{HFL}}(K) \otimes V^{|L|-1}$. □

Proof of Lemma 4.2 Suppose L is an n–component link such that $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$. Then $n \leq 2$ since $\widehat{\mathrm{HFK}}(L)$ admits a spectral sequence to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$. Indeed, since $\mathrm{rank}(\widehat{\mathrm{HFK}}(L))$ is odd for knots, $n = 2$. Since knot Floer homology detects the linking number of 2–component links, it follows that $\ell\mathrm{k}(L_1, L_2) = 2$.

There is only one generator in Maslov grading 0 and it must survive in the spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(L_i) \otimes V$. We call this generator θ_0. The bi-Alexander grading for θ_0 is then $\big(A_1 + \frac{1}{2}l, A_2 + \frac{1}{2}l\big)$, where A_i is the Alexander grading of the generator in Maslov grading 0 in $\widehat{\mathrm{HFK}}(L_i)$ and l is the linking number between the components. Since $A_1 + \frac{1}{2}l + A_2 + \frac{1}{2}l = 2$ and $l = 2$, it follows that $A_1 + A_2 = 0$. By Lemma 4.3, $A_1 = A_2 = 0$, as desired. □

To complete our proof of Theorem 4.1 we show that if L has the same knot Floer homology as $T(2,4)$ then L also has the same link Floer homology as $T(2,4)$. This result, combined with Theorem 3.2, proves Theorem 4.1.

Proof of Theorem 4.1 Suppose L is a link such that $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,4))$. We seek to understand $\widehat{\mathrm{HFL}}(L)$. From the argument in the proof of Lemma 4.2, the only Maslov grading 0 generator of $\widehat{\mathrm{HFL}}(L)$ sits in bi-Alexander grading $(1,1)$.

Since there are spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFK}}(K_i) \otimes V$ for each i, there are also generators of $\widehat{\mathrm{HFL}}(L)$ in (A_1, A_2) gradings $(1,0)$ and $(0,1)$. The symmetry of $\widehat{\mathrm{HFL}}(L)$ gives generators at $(-1,-1)$, $(-1,0)$ and $(0,-1)$ as well. With these 6 generators determined, there are now 2 more generators to add so that the link Floer homology has rank 8. To maintain an even rank in each A_i grading, they both must be added at the same bigrading. The only way to do this and maintain symmetry is to add them at $(0,0)$, so that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2,4))$, and Theorem 3.2 shows L is isotopic to $T(2,4)$. □

5 Knot Floer homology detects $T(2,6)$

In the previous section we showed that knot Floer homology detects the torus link $T(2,4)$. The torus link $T(2,6)$ is then a natural candidate for detection results. In this section we show that knot Floer homology indeed detects $T(2,6)$.

Theorem 5.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(2,6))$, then L is isotopic to $T(2,6)$.*

Note that the maximal Maslov grading of $\widehat{\mathrm{HFK}}(L)$ is $\frac{1}{2}$, while $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$ has a Maslov grading $\frac{1}{2}(n-1)$ generator. Since $\widehat{\mathrm{HFK}}(L)$ admits a spectral sequence to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$, where n is the number of components of L, L has at most 2 components. Indeed, as $\operatorname{rank}(\widehat{\mathrm{HFK}}(L))$ is even, L has exactly two components. Since knot Floer homology detects the linking number of 2–component links, the linking number is 3.

From here, the proof of Theorem 5.1 amounts to an algebraic argument showing that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFK}}(L)$, and applying Theorem 3.2.

For reference, after renormalizing the Maslov gradings to agree with the link Floer homology, the knot Floer homology of $T(2,6)$ is: rank one in (M, A) gradings $(0,3)$ and $(-6,-3)$; rank two in (M, A) gradings $(-1,2)$, $(-2,1)$, $(-3,0)$, $(-4,-1)$ and $(-5,-2)$; and rank 0 in all other bigradings.

Proof of Theorem 5.1 Suppose that L has the same knot Floer homology as $T(2,6)$. As in the proof that knot Floer homology detects $T(2,4)$, we have that $A_1 + A_2 + \ell k(L_1, L_2) = 3$, where each A_i is the Alexander grading of the Maslov index 0 generator in $\widehat{\mathrm{HFK}}(L_i)$. Thus $A_1 + A_2 = 0$, and Lemma 4.3 implies that $A_1 = A_2 = 0$.

We now show that L has the same link Floer homology as $T(2,6)$, so it follows from Theorem 3.2 that L is isotopic to $T(2,6)$.

Since the linking number is 3 and the Maslov index 0 generator sits in Alexander grading 0 in the knot Floer homology of each component, it follows that, in $\widehat{\mathrm{HFL}}(L)$, the Maslov index 0 generator sits in Alexander bigrading $\left(\frac{3}{2}, \frac{3}{2}\right)$. The Maslov index -1 generators in $\widehat{\mathrm{HFK}}(L)$ must be in bi-Alexander gradings $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, \frac{1}{2}\right)$. There is also a Maslov index -2 generator in Alexander grading $\left(\frac{1}{2}, \frac{1}{2}\right)$. Consider the remaining Maslov index -2 generator. Suppose it sits in Alexander grading $(y, 1-y)$. Observe that there must be Maslov index -3 generators sitting in Alexander gradings $(y-1, 1-y)$ and $(y, -y)$. The symmetry property of $\widehat{\mathrm{HFK}}(L)$ then implies that $y = \frac{1}{2}$. The symmetry properties of $\widehat{\mathrm{HFL}}$ then show that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(2,6))$, and so by Theorem 3.2 L is isotopic to $T(2,6)$, as desired. □

6 Link Floer homology detects $T(n, n)$

In the previous section we showed that link Floer homology detects the $T(2,2n)$ torus links, motivated by detection results for $T(2,2)$, $T(2,4)$ and $T(2,6)$. The torus link $T(2,2)$ can also be viewed as one of the simplest links in the family of $T(n,n)$ torus links. In this section we show that link Floer homology detects the links $T(n,n)$. We use a characterization of $T(n+1,n+1)$ as an n–braid for $T(n,n)$ union the braid axis.

J Licata gave a computation of $\widehat{\mathrm{HFL}}(T(n,n))$ without the Maslov gradings of certain generators [24]. The computation of the Maslov gradings of these generators was subsequently completed by Gorsky and Hom [10]. We prove that link Floer homology detects $T(n,n)$ using only certain structural properties of the link Floer homology. It follows from this that there are many graded vector spaces that do not arise as the link Floer homology of any link. In particular, we will be interested in multigraded vector spaces B_n exhibiting the following four properties:

(1) There is a unique Maslov grading 0 generator.

(2) The multi-Alexander grading of the Maslov grading 0 generator is $\bigl(\frac{1}{2}(n-1), \frac{1}{2}(n-1), \dots, \frac{1}{2}(n-1)\bigr)$.

(3) B_n has support contained only in multi-Alexander gradings $(A_1, A_2, \dots, A_n)$ satisfying $A_i \leq \frac{1}{2}(n-1)$ for all i.

(4) B_n has rank 2^{n-1} in A_i grading $\frac{1}{2}(n-1)$.

Note that $\widehat{\mathrm{HFL}}(T(n,n))$ satisfies these properties. Observe that if L is any link whose link Floer homology satisfies all of the above conditions, then L is not a split link, so each component L_i of L is a braid axis for $L - L_i$.

Theorem 6.1 *If $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(n,n))$, then L is isotopic to $T(n,n)$.*

The main ingredient of this proof is a result stating that, under certain circumstances, if the link Floer homology of a link has certain algebraic properties then the linking numbers of certain components with the rest of the link are positive.

Lemma 6.2 *Let L be a link with components L_i for $1 \leq i \leq n$. Suppose that $\widehat{\mathrm{HFL}}(L)$ has a unique generator of Maslov index 0 with A_i grading $x \geq 0$. Suppose $\widehat{\mathrm{HFL}}(L)$ is supported in A_i gradings at most x. Then $\ell\mathrm{k}(L_i, L_j) \geq 0$ for all j.*

Proof Let θ_0 denote the unique Maslov index 0 generator. The vector space $\widehat{\mathrm{HF}}(S^3) \otimes V^{n-1}$ is nonzero in Maslov grading 0, so all other intermediate vector spaces with spectral sequences fitting between $\widehat{\mathrm{HFL}}(L)$ and $\widehat{\mathrm{HF}}(S^3) \otimes V^{n-1}$ must also be nonzero in this Maslov grading. Because θ_0 is the only generator in this Maslov grading, it must survive in every such spectral sequence.

Consider the spectral sequence to $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ obtained by forgetting the component L_j. The A_i grading on $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ will be shifted by $\frac{1}{2}\,\ell\mathrm{k}(L_i, L_j)$ for $i \neq j$. We will show that this shift must be nonnegative.

Because θ_0 survives this spectral sequence, $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ will have top A_i grading $\frac{1}{2}(n-1)$. Considering the A_i grading on $\widehat{\mathrm{HFL}}(L)$, we see that $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ will have bottom A_i grading no smaller than $\frac{1}{2}(-n+1)$. Since the A_i grading on $\widehat{\mathrm{HFL}}(L - L_j) \otimes V$ must be symmetric about a nonnegative number, the shift applied to the Alexander grading must be nonnegative, so $\ell\mathrm{k}(L_i, L_j) \geq 0$. □

With this result on the nonnegativity of linking numbers, we can proceed with the proof of Theorem 6.1. We will proceed by induction, using the characterization of $T(n+1, n+1)$ as the link consisting of the unique n–braid for $T(n,n)$ together with the braid axis.

Proof of Theorem 6.1 Suppose that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(n,n))$. Lemma 6.2 tells us that $\ell\mathrm{k}(L_i, L_j) \geq 0$ for every distinct i and j. Moreover, because L is not split and each component L_i of L is a braid axis for $L - L_i$, we have $\ell\mathrm{k}(L_i, L_j) \neq 0$.

The top nonzero A_i grading is $\frac{1}{2}(n-1)$. The relationship between the top nonzero A_i grading and the Seifert genus of L_i implies that

$$\tfrac{1}{2}(n-1) \geq g(L_i) + \sum_{j \neq i} \tfrac{1}{2}\,\ell\mathrm{k}(L_i, L_j)2.$$

However, because $\ell\mathrm{k}(L_i, L_j) > 0$,

$$g(L_i) + \sum_{j \neq i} \tfrac{1}{2}\,\ell\mathrm{k}(L_i, L_j)2 \geq g(L_i) + \tfrac{1}{2}n - 1,$$

with equality when $\ell\mathrm{k}(L_i, L_j) = 1$ for all j. Combining these inequalities gives that $g(L_i) = 0$, and $\ell\mathrm{k}(L_i, L_j) = 1$ for all i, j.

We now know that L is an n–component link where each component is an unknot, each component is a braid axis for the rest of the link, and the linking number between any two components is 1. The torus link $T(n,n)$ is the only n–component link satisfying all of these conditions. This can be verified by induction on n. Specifically, check explicitly that $T(2,2)$ is the only such 2–component link, then view L as a braid axis of some n–braid representing an n–component link satisfying the same properties. □

7 Knot Floer homology detects $T(3,3)$

In previous sections we showed that, for some of the first members of the family of $T(2,2n)$ torus links, the link Floer homology detection results can be strengthened to knot Floer homology detection results. In this section we do the same for $T(3,3)$, the third member of the $T(n,n)$ family. The knot Floer homology of $T(3,3)$ is given as follows:

$$\widehat{\mathrm{HFK}}(T(3,3), i) = \begin{cases} \mathbb{F}_1 & \text{for } i = 3, \\ \mathbb{F}_0^3 & \text{for } i = 2, \\ \mathbb{F}_{-1}^3 & \text{for } i = 1, \\ \mathbb{F}_{-1} \oplus \mathbb{F}_{-2}^3 & \text{for } i = 0, \\ \mathbb{F}_{-3}^3 & \text{for } i = -1, \\ \mathbb{F}_{-4}^3 & \text{for } i = -2, \\ \mathbb{F}_{-5} & \text{for } i = -3, \\ 0 & \text{otherwise.} \end{cases}$$

See [24] for a discussion of this result.

Theorem 7.1 *If* $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(3,3))$, *then L is isotopic to* $T(3,3)$.

To prove this, we will use various spectral sequence arguments to show that L has the same link Floer homology as $T(3,3)$. The above theorem then follows immediately from Theorem 6.1.

Proof Suppose L is an n–component link such that $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(T(3,3))$.

We first argue that $n = 3$. Note that the maximal Maslov grading of a generator of $\widehat{\mathrm{HFK}}(L)$ is 1. Thus $n \leq 3$, as else $\widehat{\mathrm{HFK}}(L)$ would not admit a spectral sequence to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$. Also $n \neq 2$, since the Maslov gradings of $\widehat{\mathrm{HFK}}(L)$ are supported in integer gradings. Moreover L cannot be a knot, since $\mathrm{rank}(\widehat{\mathrm{HFK}})(L)$ is even. Thus $n = 3$.

Let L_1, L_2 and L_3 be the components of L. We now seek to determine the structure of $\widehat{\mathrm{HFL}}(L)$.

The symmetry of $\widehat{\mathrm{HFL}}(L)$ implies that the unique generator in Maslov grading -2 and Alexander grading 0 sits in multi-Alexander grading $(0,0,0)$. Similarly the symmetry implies that at least one of the Maslov grading -3 generators also sits at multigrading $(0,0,0)$.

Since the Maslov grading 0 generator in knot Floer homology is of Alexander grading 3, the Maslov grading 0 generator in link Floer homology sits in Alexander multigrading $(x, y, 3-x-y)$ for some pair of integers (x, y). In order that the link Floer homology admits the requisite spectral sequences, there are Maslov grading -1 generators in multi-Alexander gradings $(x, y, 2-x-y)$, $(x-1, y, 3-x-y)$ and $(x, y-1, 3-x-y)$.

Now, observe that each Maslov grading -1 generator has at least one distinct Alexander grading from the unique Maslov grading 0 generator. In order to admit the requisite spectral sequences, there must be Maslov grading -2 generators with $(A_1, A_2) = (x, y-1), (x-1, y)$, $(A_1, A_3) = (x-1, 3-x-y), (x, 2-x-y)$ and $(A_2, A_3) = (y-1, 3-x-y), (y, 2-x-y)$. A direct computation shows that at most one of these corresponds to the generator in multigrading $(0,0,0)$. Thus there are Maslov index -2 generators in Alexander gradings $(x-1, y-1, 3-x-y)$, $(x, y-1, 2-x-y)$ and $(x-1, y, 2-x-y)$.

By a similar argument, we can see that there is a Maslov index -3 generator in multi-Alexander grading $(x-1, y-1, 2-x-y)$. If $(x, y) \neq (1,1)$ this determines the entire link Floer homology of L. If $x = y = 1$ then the remaining Maslov index -3 generators must be of multi-Alexander grading $(0,0,0)$ to ensure that each (A_i, A_j) grading is of even rank, so again the link Floer homology of L is determined.

Consider the Maslov index -3 generator in multi-Alexander grading $(0,0,0)$, θ_{-3}. Since θ_{-3} does not persist under the spectral sequence to $\widehat{\mathrm{HFL}}(L_i) \otimes V^{\otimes 2}$ for any i, there must be a Maslov index -2 generator in nonzero Alexander grading with each Alexander grading at least 0, or a Maslov index -4 generator with each Alexander grading at most 0. Observe that these two conditions are equivalent by the symmetry of the link Floer homology of L. Thus $x-1, y-1, 3-x-y \geq 0$, $x, y-1, 2-x-y \geq 0$, or $x-1, y-1, 2-x-y \geq 0$. By permuting the components, we may take $x-1, y-1, 3-x-y \geq 0$. There are only three solutions: $(x, y) = (1,1)$, $(x, y) = (2,1)$ or $(x, y) = (1,2)$. If $(x, y) = (1,1)$, then $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(T(3,3))$.

We complete the proof by excluding the cases $(x, y) = (2,1)$ and $(x, y) = (1,2)$. After permuting components, we may take $x = 1$ and $y = 2$ without loss of generality.

Since $\mathrm{rank}(\widehat{\mathrm{HFK}}(L)) = 18$, we have $\mathrm{rank}(\widehat{\mathrm{HFK}}(L_i)) \leq \frac{9}{2}$. It follows that each component is an unknot or a trefoil. Observe that if a component is a trefoil then it must be $T(2,3)$, as there are no positive Maslov index generators in $\widehat{\mathrm{HFL}}(L)$. Indeed there can be no $T(2,3)$ component, as this would require there to be an Alexander grading 2 less than an Alexander grading of the Maslov index 0 generator containing a summand $\mathbb{F}_{-2} \oplus \mathbb{F}_{-3}^2 \oplus \mathbb{F}_{-4}$, which does not occur. Thus each component is an unknot. From here we can compute the linking numbers from the Alexander gradings of the Maslov grading 0 generator.

We find $\ell\mathrm{k}(L_1, L_3) = -1 = -\ell\mathrm{k}(L_2, L_3)$, and $\ell\mathrm{k}(L_1, L_2) = 3$. Since L is not a split link, $L - L_3$ is a 2–braid in the complement of L_3. Each of L_1 and L_2 are unknots and $\ell\mathrm{k}(L_1, L_2) = 3$, so $L - L_3$ is $T(2,6)$ as an unoriented link. However, $\operatorname{rank}(\widehat{\mathrm{HFK}}(T(2,6) \otimes V)) = 24$, so $T(2,6)$ cannot be a sublink of L and $(x, y) \neq (1, 2)$. □

8 Knot Floer homology detects $L7n1$

We have now shown that knot Floer homology detects a number of the low crossing number links that Khovanov homology is known to detect. In this section we continue this task, showing that knot Floer homology detects the link $L7n1$.

Theorem 8.1 *If $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(L7n1)$, then L is isotopic to $L7n1$.*

Our proof relies on the observation that $L7n1$ can be realized as a 2–braid representing $T(2,3)$ together with the braid axis.

Note that $\widehat{\mathrm{HFK}}(L)$ admits a spectral sequence to $\widehat{\mathrm{HF}}\big(\#^{n-1}(S^1 \times S^2)\big)$ — where n is the number of components of L — and that the knot Floer homology of a knot is of odd rank. It follows that L has two components. Since knot Floer homology detects the linking number of 2–component knots, it follows that the linking number of L is two. From here we break up the proof of Theorem 8.1 into the following lemmas:

Lemma 8.2 *Suppose L is a 2–component link such that $\widehat{\mathrm{HFK}}(L7n1) \cong \widehat{\mathrm{HFK}}(L)$. Then $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L7n1)$.*

Lemma 8.3 *Suppose L satisfies $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L7n1)$. Then L is isotopic to $L7n1$.*

The combination of these lemmas immediately gives the proof of Theorem 8.1.

L7n1 has homology as computed in [32] and shown in Table 2.

Lemma 8.2 is proven by combining the symmetry and parity properties of link Floer homology.

Proof of Lemma 8.2 Since L has 2 components, $\widehat{\mathrm{HFL}}(L)$ has exactly 2 Alexander gradings.

2		$\mathbb{F}_{-1}$	$\mathbb{F}_0$
1		$\mathbb{F}_{-2}$	$\mathbb{F}_{-1}$
0		$\mathbb{F}_{-2} \oplus \mathbb{F}_{-3}$	
-1	$\mathbb{F}_{-5}$	$\mathbb{F}_{-4}$	
-2	$\mathbb{F}_{-6}$	$\mathbb{F}_{-5}$	
	-1	0	1

Table 2: The link Floer homology of L7n1.

Let θ_0 be the Maslov grading 0 generator. This generator θ_0 has bi-Alexander grading $\left(\frac{3}{2}+x, \frac{3}{2}-x\right)$ for some x. Indeed, there must be generators sitting in gradings $\left(\frac{1}{2}+x, \frac{3}{2}-x\right)$ and $\left(\frac{3}{2}+x, \frac{1}{2}-x\right)$, each of Maslov index -1. Together with the symmetry properties of link Floer homology, this determines the Alexander bigradings of 6 generators. The same symmetry properties also imply that the 2 generators in Alexander grading 0 must have bi-Alexander grading $(0,0)$. Thus, up to choice of x, we need only specify the location of 1 more generator to determine the link Floer homology of L. Since each Alexander grading needs to be of even rank, the remaining Maslov grading -2 element must be in bi-Alexander grading $\left(\frac{1}{2}+x, \frac{1}{2}-x\right)$. Moreover, since the Maslov grading -3 component, θ_{-3}, cannot persist in the spectral sequence to $\widehat{\mathrm{HF}}(S^3)\otimes V$, it follows that $x \in \left\{\frac{1}{2}, 0, -\frac{1}{2}\right\}$, for the Alexander gradings obstruct the existence of generators y with $\langle \partial y, \theta_{-3}\rangle \neq 0$ and $\langle \partial\theta_{-3}, y\rangle \neq 0$ unless x is in this range. Indeed, $x \neq 0$, since otherwise we would have an element with Alexander grading in $\mathbb{Z}$, and another with Alexander grading in $\mathbb{Z}+\frac{1}{2}$. If $x=\frac{1}{2}$ then we have the link Floer homology of $L7n1$, while if $x=-\frac{1}{2}$ we can switch the components and thereby obtain the link Floer homology of $L7n1$. □

We complete the proof of Theorem 8.1 by showing that link Floer homology detects the link $L7n1$. We use the fact that $L7n1$ is the closure of a braid for $T(2,3)$ together with its braid axis.

Proof of Lemma 8.3 Suppose a link L, with components L_1 and L_2, satisfies $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L7n1)$. The rank in each of the maximal A_i gradings is 2, so L is either split or the L is exchangeably braided. L cannot be split, as if it were, then at least 1 component of L would be an unknot, for reasons of rank. But if this were the case, then $\widehat{\mathrm{HFL}}(L)$ would be supported in A_i grading 0 for at least one i, which is not the case. Thus each component is a braid axis for the other, and in particular $\ell k(L_1, L_2) \neq 0$.

Observe that $\mathrm{rank}(\widehat{\mathrm{HFK}}(L_i)) \leq 5$, with equality if and only if the spectral sequence corresponding to L_i collapses on the E_1 page. If the spectral sequence collapses on the E_1 page, then $\widehat{\mathrm{HFK}}(L_i)$ would have no shift applied to its Alexander grading as it is already symmetric around grading 0. Therefore, if $\mathrm{rank}(\widehat{\mathrm{HFK}}(L_i)) = 5$, then $\ell k(L_1, L_2) = 0$, a contradiction.

Thus $\mathrm{rank}(\widehat{\mathrm{HFK}}(L_i)) < 5$, and the link has components that are either unknots or trefoils. Observe that any trefoil component must be $T(2,3)$, as there are no generators of positive Maslov grading.

Suppose L_1 and L_2 are both unknots. The shifts in Alexander grading coming from the spectral sequences from $\widehat{\mathrm{HFL}}(L)$ to $\widehat{\mathrm{HFL}}(L_i)\otimes V$ would imply that $\ell k(L_1, L_2) = 4$ and $\ell k(L_2, L_1) = 2$, a contradiction.

Observe that L_1 cannot be a trefoil, for there is no Maslov grading -2 generator in an A_1 grading 2 less than the A_1 grading of the unique grading 0 element. Thus L_2 is $T(2,3)$ and $\ell k(L_2, L_1) = 2$. Since L_2 is a 2–braid closure in the complement of L_1 and L_2 is $T(2,3)$, the link L must be L7n1. □

9 Connected sums with a Hopf link

In this section we deduce some properties of link Floer homology under the operation of taking a connected sum with a Hopf link. We then explore some applications of these properties to the question of link detection. Our main application is to find two infinite families of links which are not detected by Khovanov homology or knot Floer homology, but which are detected by link Floer homology. Throughout this section we let H denote the Hopf link.

Proposition 9.1 *A link L can be expressed as $L' \# H$ if and only if there is an Alexander grading in $\widehat{\mathrm{HFL}}(L)$ where the span of its nonzero grading levels is $\left\{\frac{-1}{2}, \frac{1}{2}\right\}$.*

Proof This observation follows directly from the connection between link Floer homology and the Thurston norm. A component has a span of its nonzero grading levels $\left\{\frac{-1}{2}, \frac{1}{2}\right\}$ if and only if that component bounds a disk which intersects the rest of the link in a single point. This is equivalent to expressing L as $L' \# H$. □

This observation has the following immediate consequence:

Proposition 9.2 *Suppose link Floer homology detects a link L, and that if permuting some collection of Alexander gradings of $\widehat{\mathrm{HFL}}(L)$ induces an isomorphism on $\widehat{\mathrm{HFL}}(L)$, then there is a symmetry of L that exchanges the corresponding components. Then link Floer homology detects $L \# H$ for each choice of component of L to connect sum with.*

Proof Suppose L' is a link such that $\widehat{\mathrm{HFL}}(L') \cong \widehat{\mathrm{HFL}}(L \# H)$. Consider the span of the Alexander grading associated to the new unknotted component. Proposition 9.1 implies that $L' = L'' \# H$ for some link L'' and some choice of component of L'' to connect sum onto. It follows from the Künneth formula that $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L'')$, whence L'' is L by assumption. Indeed, we have also assumed that if permuting some collection of Alexander gradings of $\widehat{\mathrm{HFL}}(L)$ induces an isomorphism of $\widehat{\mathrm{HFL}}(L)$, then there is a symmetry of L that exchanges the corresponding components. Thus if different choices of component on which to connect sum H give the same links with the same link Floer homology, the resulting links are isotopic. It follows that link Floer homology detects $L \# H$ irrespective of which component of L is used for the connected sum. □

Remark 9.3 While Proposition 9.1 allows one to determine if a link L can be decomposed as a smaller link L' connect sum with a Hopf link, there is in general an issue with determining where the connected sum is occurring. An instructive example is the case that L' is the disjoint union of two different knots with the same knot Floer homology. In this case the two choices of where to connect sum with a Hopf link produce topologically distinct links which have the same link Floer homology. While in this example link Floer homology does not detect L', we cannot rule out the possibility that something similar could occur for links detected by link Floer homology.

Combining Proposition 9.2 and previous knot Floer detection results immediately gives some new detection results for link Floer homology.

We now provide two infinite families of links which are detected by link Floer homology but are not detected by Khovanov homology or knot Floer homology.

Theorem 9.4 *There exist infinitely many pairs of links (L, L') such that link Floer homology detects L and L' but $\mathrm{Kh}(L) \cong \mathrm{Kh}(L')$ and $\widehat{\mathrm{HFK}}(L) \cong \widehat{\mathrm{HFK}}(L')$.*

To prove Theorem 9.4 we introduce two families of links and show that every link in either of these families is detected by link Floer homology. This is the content of Theorems 9.5 and 9.7. We then highlight explicit examples within these families that neither Khovanov homology nor knot Floer homology can distinguish.

Both families of links are trees of unknots. The first family consists of links L_n given by the tree of unknots corresponding to the graph with $n-1$ vertices, each connected to a fixed vertex. For the second family, let $L_{(a,b)}$ be the tree of unknots corresponding to the graph with $a+b+2$ vertices $\{x_1, x_2, \dots x_a, x, y_1, y_2, \dots y_b, y\}$. Each x_i has a unique edge, connecting it to x. Each y_i has a unique edge, connecting it to y. Finally there is a unique edge connecting x and y.

Theorem 9.5 *For each $n \geq 2$, if $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L_n)$, then L is isotopic to the link L_n.*

Remark 9.6 The links L_n can be viewed as the trivial $(n-1)$–braid together with its braid axis. So Theorem 9.5 was already known, because link Floer homology detects braid closures [26, Proposition 1] and detects the trivial braid amongst braid closures [3, Theorem 3.1]. However, we provide a different proof of Theorem 9.5 because it is a simpler case of the ideas used in the proof of Theorem 9.7.

Proof of Theorem 9.5 Suppose L has the same link Floer homology as L_n. First notice that L cannot be a split link because its Alexander polynomial is nonzero. By the observation that L is not split, we see that each of these $n-1$ components must bound a disk which only intersects the final component of L. Then L must be built from a knot K by connect summing K with $n-1$ Hopf links. It follows from the Künneth formula and the fact that knot Floer homology detects the unknot that L is isotopic to the link L_n. □

Theorem 9.7 *For every pair (a, b) with a and b positive, if $\widehat{\mathrm{HFL}}(L) \cong \widehat{\mathrm{HFL}}(L_{(a,b)})$, then L is isotopic to the link $L_{(a,b)}$.*

Proof First notice that link Floer homology detects the link $L_{(0,b)} = L_{b+1}$. We will now proceed by induction on a.

Suppose that L has the same link Floer homology as $L_{(a,b)}$. First notice that L cannot be a split link because its Alexander polynomial is nonzero. By the observation that L is not split, we see that each of these $a+b$ components must bound a disk which only intersects one of the final two components of L. Call these final components X and Y, based on if their Alexander gradings agree with the Alexander gradings associated to the component in the tree of unknots for the vertex x or y, respectively. Without loss of generality, at least one component bounds a disk that intersects X in a single point. Then L can be written as $L' \# H$, where the connect sum is taken along the component X. A quick computation shows that $\widehat{\mathrm{HFL}}(L') \cong \widehat{\mathrm{HFL}}(L_{(a-1,b)})$. By induction, L' is isotopic to $L_{(a-1,b)}$, whence L is isotopic to $L_{(a,b)}$. □

With these detection results in place, we are now ready to prove Theorem 9.4.

Proof of Theorem 9.4 Consider the links L_n and $L_{(a,b)}$ with $a+b+1=n$. These links are detected by link Floer homology. We now check that $\mathrm{Kh}(L_n) \cong \mathrm{Kh}(L_{(a,b)})$ and $\widehat{\mathrm{HFK}}(L_n) \cong \widehat{\mathrm{HFK}}(L_{(a,b)})$.

Both links can be constructed by starting with an unknot and connect summing a Hopf link n times in total. A simple computation shows that Khovanov homology and knot Floer homology of $L \# H$ do not depend on which component of L the Hopf link is connect summed onto. This shows $\mathrm{Kh}(L_n) \cong \mathrm{Kh}(L_{(a,b)})$ and $\widehat{\mathrm{HFK}}(L_n) \cong \widehat{\mathrm{HFK}}(L_{(a,b)})$. □

10 Applications to annular Khovanov homology

Annular Khovanov homology was defined by Asaeda, Przytycki and Sikora [1] as a categorification of the Kauffman bracket skein module of the thickened annulus. The resulting theory is an invariant of links in the thickened annulus $A \times I$, or alternatively the complement of an unknot in the 3–sphere $S^3 \setminus U$. In particular, annular Khovanov homology is well suited to studying braid closures [3; 13; 17; 18].

In this section we apply some of our earlier knot Floer detection results to show that annular Khovanov homology detects certain braid closures. The proofs will rely on the spectral sequence from annular Khovanov homology of a link L to the knot Floer homology of the lift of the annular axis in $\Sigma(L)$ [13; 35].

Let $\beta_n := \sigma_1\sigma_2\dots\sigma_{n-1}$. We use knot Floer detection results for $T(2,3)$, $T(2,4)$ and $T(2,6)$ to show that annular Khovanov homology detects the closure of the braids β_3, β_4 and β_6. The structure of each proof is similar. First we use properties of annular Khovanov homology to deduce necessary topological properties of the annular knot, like braidedness or unknottedness. Then we use a knot Floer detection result to show that the lift of the annular axis is $T(2,3)$, $T(2,4)$ or $T(2,6)$, respectively. Finally, we translate this into information about the annular link. In this section we use the following result for mapping class groups:

Proposition 10.1 *Suppose γ is an n–braid and β is a periodic n–braid. If $BH(\gamma)$ and $BH(\beta)$ are conjugate then so too are γ and β.*

Remark 10.2 An alternative proof of this proposition was originally communicated to the authors by Marissa Loving and Dan Margalit (2020).

Proof Let β and γ be as in the statement of the proposition. Note that both conjugation and the Birman–Hilden correspondence preserve the Nielsen–Thurston classification, so we know that γ is periodic as well. That is, a power of γ is some power of the full twist Δ^2. Thus there are numbers N and M such that $\beta^N = \gamma^M$.

Now we consider the fractional Dehn twist coefficients of β and γ. We know that $\mathrm{FDTC}(\beta) = k/m$ for some fixed k and m. The Birman–Hilden correspondence either preserves the fractional Dehn twist coefficient of n–braids or halves it, depending on the parity of n. The fractional Dehn twist coefficient is preserved under conjugation by a combination of [20, Corollary 4.17] and [8, Proposition 5.3], so

$\mathrm{FDTC}(BH(\beta)) = \mathrm{FDTC}(BH(\gamma))$. Thus $\mathrm{FDTC}(\beta) = \mathrm{FDTC}(\gamma) = k/m$. The fractional Dehn twist coefficient is multiplicative under exponentiation, so $\mathrm{FDTC}(\beta^N) = kN/m$ and $\mathrm{FDTC}(\gamma^M) = kM/m$, but $\beta^N = \gamma^M$ so we must have that $M = N$. Finally, N^{th} roots are unique up to conjugation in the braid group [9], so that β and γ are conjugate. □

The spectral sequence from the annular Khovanov homology of an annular link L to the knot Floer homology of the lift of the annular axis in $\Sigma(L)$ is defined with $\mathbb{Z}/2\mathbb{Z}$ coefficients. At times, however, we will work with annular Khovanov homology over $\mathbb{C}$, because with these coefficients annular Khovanov homology has the structure of an $\mathfrak{sl}_2(\mathbb{C})$ representation [12, Proposition 3].

For the readers convenience we recall, from [12, Proposition 14], that

$$\mathrm{AKh}^i(\beta_n, \mathbb{C}) = \begin{cases} V_{(n)}\{n-1\} & \text{for } i = 0, \\ V_{(n-2)}\{n+1\} & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Here $V_{(m)}$ is the $(m+1)$–dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. We now study annular Khovanov homology with $\mathbb{Z}/2$ coefficients. Note that $\mathrm{rank}(\mathrm{AKh}(\beta_n; \mathbb{Z}/2)) \geq \mathrm{rank}(\mathrm{AKh}(\beta_n; \mathbb{C})) = 2n$. Now, $T(2,n)$ can be thought of as a 2–periodic knot with quotient β_n. It follows from [36] that $\mathrm{rank}(\mathrm{Kh}(T(2,n); \mathbb{Z}/2)) \geq \mathrm{rank}(\mathrm{AKh}(\beta_n; \mathbb{Z}/2))$. Indeed, $\mathrm{rank}(\mathrm{Kh}(T(2,n); \mathbb{Z}/2)) = 2n$ by the universal coefficient theorem and [21, Proposition 26], so in fact $\mathrm{rank}(\mathrm{AKh}(\beta_n; \mathbb{Z}/2)) = 2n$, and the above description of annular Khovanov homology is equally valid for $\mathbb{Z}/2$ coefficients.

More is known about the knot Floer homology of genus-1 fibered knots, so we are able to prove a stronger result for the closure of the 3–braid $\beta_3 = \sigma_1\sigma_2$ than β_4 and β_6. We are also able to use the classification of 3–braids representing the unknot to show that annular Khovanov homology also detects the closure of the 3–braid $\sigma_1\sigma_2^{-1}$.

Theorem 10.3 *If L is a 3–braid closure and $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) = 6$, then L is isotopic to $\widehat{\sigma_1\sigma_2}$ or $\widehat{\sigma_1^{-1}\sigma_2^{-1}}$ in $A \times I$.*

Proof The lift $\widetilde{U}_L$ of the braid axis U in $\Sigma(L)$ is a genus-1 fibered knot.

The manifold $\Sigma(L)\backslash\widetilde{U}_L$ is naturally a sutured manifold, where the sutures on $S^3\backslash\widetilde{U}_L$ are two distinct pairs of meridional sutures lifted into the double branched cover from the product sutures on $A \times I$. There is a spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\widehat{\mathrm{SFH}}(-\Sigma(L)\setminus\widetilde{U}_L, \mathbb{Z}/2\mathbb{Z}) \cong \widehat{\mathrm{HFK}}(\widetilde{U}_L, -\Sigma(K), \mathbb{Z}/2\mathbb{Z}) \otimes V$, where V is a 2–dimensional vector space supported in bigradings $(0,0)$ and $(-1,-1)$. Furthermore, the k grading in AKh corresponds to the Alexander grading on $\widehat{\mathrm{SFH}}$ or $\widehat{\mathrm{HFK}}$; see [14, Theorem 2.1; 35, Theorem 1.1 and Proposition 5.3].

From this spectral sequence we can see that $\widehat{\mathrm{HFK}}(\widetilde{U}_K, -\Sigma(K), \mathbb{Z}/2\mathbb{Z})$ has rank no larger than 3. Every genus-1 fibered knot has knot Floer homology at least rank 3 and there are only four genus-1 fibered knots with rank 3 knot Floer homologies. They are the left- and right-handed trefoils in S^3 and two knots in the Poincaré homology sphere [5, Corollary 1.6].

The monodromies of fibered knots are unique up to conjugation. The monodromy of a fibered knot in $\Sigma(L)$ is the image of a braid representing L in $\mathrm{Mod}(S_1^1)$ under the Birman–Hilden correspondence. Finally, because $B_3 \cong \mathrm{Mod}(S_1^1)$, or by Proposition 10.1, conjugate monodromies must come from conjugate braids, so L must be isotopic to the closure of one of the 4–braids on this list that corresponds to one of these four possible fibered knots:

(1) $\sigma_1\sigma_2$,
(2) $\sigma_1^{-1}\sigma_2^{-1}$,
(3) $(\sigma_1\sigma_2)^{-6}\sigma_1\sigma_2$,
(4) $(\sigma_1\sigma_2)^{6}\sigma_1^{-1}\sigma_2^{-1}$.

A computation shows that the ranks of the annular Khovanov homologies of the last two braid closures are larger than 6 [19].

Therefore L is isotopic to $\widehat{\sigma_1\sigma_2}$ or $\widehat{\sigma_1^{-1}\sigma_2^{-1}}$ in $A \times I$. □

The detection result in Theorem 10.4 follows immediately from Theorem 10.3 and previous results about annular Khovanov homology.

Theorem 10.4 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2}$ in $A \times I$.*

Proof If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to a 3–braid closure [13, Corollary 1.2; 39, Corollary 8.4]. The careful reader may worry about coefficients because we are working over $\mathbb{Z}/2\mathbb{Z}$ while [39, Corollary 8.4] is stated for coefficients over $\mathbb{C}$, but the corollary is also true over $\mathbb{Z}/2\mathbb{Z}$. This is because the universal coefficient theorem ensures that, in each annular grading, the rank of annular Khovanov homology over $\mathbb{C}$ is bounded above by the rank over $\mathbb{Z}/2\mathbb{Z}$ and the ranks will have the same parity. In particular, if the annular Khovanov homology in an annular grading is rank 1 over $\mathbb{Z}/2\mathbb{Z}$ then it is also rank 1 over $\mathbb{C}$.

From Theorem 10.3 we have that L is isotopic to $\widehat{\sigma_1\sigma_2}$ or $\widehat{\sigma_1^{-1}\sigma_2^{-1}}$. A simple computation shows that $\mathrm{AKh}(\widehat{\sigma_1^{-1}\sigma_2^{-1}}, \mathbb{Z}/2\mathbb{Z}) \ncong \mathrm{AKh}(\widehat{\sigma_1\sigma_2}, \mathbb{Z}/2\mathbb{Z})$, so L must be isotopic to $\widehat{\sigma_1\sigma_2}$ in $A \times I$. □

One interpretation of Theorem 10.3 is that $\widehat{\sigma_1\sigma_2}$ and $\widehat{\sigma_1^{-1}\sigma_2^{-1}}$ are the simplest 3–braids from the point of view of annular Khovanov homology.

Proposition 10.5 *If L is isotopic to a 3–braid closure in $A \times I$, then $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) \geq 6$.*

Proof From the universal coefficient theorem it follows that $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) \geq \dim(\mathrm{AKh}(L, \mathbb{C}))$, so it suffices to show that $\dim(\mathrm{AKh}(L, \mathbb{C})) \geq 6$. Because L is a 3–braid closure, $\mathrm{AKh}(L, \mathbb{C})$ has dimension one in grading $k = 3$. The $\mathfrak{sl}_2(\mathbb{C})$ action on $\mathrm{AKh}(L, \mathbb{C})$ and the fact that the k grading gives the $\mathfrak{sl}_2(\mathbb{C})$ weights implies that $\mathrm{AKh}(L, \mathbb{C})$ contains an irreducible weight-3 representation of $\mathfrak{sl}_2(\mathbb{C})$, showing that $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) \geq 4$.

The $\mathfrak{sl}_2(\mathbb{C})$ action gives a symmetry in the k gradings. Since $\mathrm{AKh}(L, \mathbb{C})$ is 0 in $k = 0$, the dimension of $\mathrm{AKh}(L, \mathbb{C})$ must be even. It thus only remains to rule out the case that $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) = 4$.

If $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) = 4$, then $\mathrm{AKh}(L, \mathbb{C})$ consists only of an irreducible weight-3 representation of $\mathfrak{sl}_2(\mathbb{C})$ which must live in a single homological grading. Because $\mathrm{AKh}(L, \mathbb{C})$ is supported in a single homological grading, the spectral sequence from $\mathrm{AKh}(L, \mathbb{C})$ to $\mathrm{Kh}(L, \mathbb{C})$ collapses. The proof of Theorem 3.1(a) in [3] shows that the only braid closures for which this spectral sequence collapses immediately are closures of trivial braids. A computation shows that $\dim(\mathrm{AKh}(\hat{1}_3, \mathbb{C})) > 6$, so there is no 3–braid closure with $\dim(\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})) = 4$. □

Because the 3–braid closures that are unknotted in S^3 are completely classified, we are able to use the previous results to show that annular Khovanov homology detects the closure of the 3–braid $\sigma_1\sigma_2^{-1}$ in $A \times I$.

Theorem 10.6 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2^{-1}}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2^{-1}}$ in $A \times I$.*

Proof If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2^{-1}}, \mathbb{Z}/2\mathbb{Z})$ then L is isotopic to a 3–braid closure [13, Corollary 1.2; 39, Corollary 8.4]. Computing that the ungraded Euler characteristic of $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ is 2 shows that L is a knot, because the ungraded Euler characteristic is $2^{|L|}$, where $|L|$ is the number of components of L.

Because the rank of $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ is 10 and L is not the trivial braid, $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$ has rank strictly smaller than 10. The only knots in S^3 with the rank of $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$ strictly less than 10 are the unknot and the trefoils [4; 22]. Also, $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ is not nonzero in the correct bigradings for L to be one of the trefoils, so L must be the unknot in S^3.

Up to conjugation, the only 3–braids that represent the unknot are $\sigma_1\sigma_2$, $\sigma_1^{-1}\sigma_2^{-1}$ and $\sigma_1\sigma_2^{-1}$ [27, Theorem 12.1]. The annular Khovanov homology shows that L is not the closure of $\sigma_1\sigma_2$ or $\sigma_1^{-1}\sigma_2^{-1}$, so L is isotopic to $\widehat{\sigma_1\sigma_2^{-1}}$ in $A \times I$. □

We now consider the annular Khovanov homology of the braids $\beta_n = \sigma_1\sigma_2 \ldots \sigma_{n-1}$ in $A \times I$ more generally. We will first show that any braid closure with the same annular Khovanov homology as $\hat{\beta}_n$ must represent an unknot in S^3. We then show that the lift of the braid axis for β_{2n} has the same knot Floer homology as $T(2, 2n)$.

The next two results then follow from the fact that knot Floer homology detects $T(2, 4)$ and $T(2, 6)$.

Theorem 10.7 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2\sigma_3}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2\sigma_3}$ in $A \times I$.*

Theorem 10.8 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}, \mathbb{Z}/2\mathbb{Z})$, then L is isotopic to $\widehat{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}$ in $A \times I$.*

To prove these theorems we first prove two general lemmas.

Lemma 10.9 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If* $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\hat{\beta}_n, \mathbb{Z}/2\mathbb{Z})$, *then L is an unknot in S^3.*

Proof We first compute $\mathrm{AKh}(L, \mathbb{C})$ from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$. Throughout, we will use that the dimension of annular Khovanov homology over $\mathbb{C}$ can be no larger than that over $\mathbb{Z}/2\mathbb{Z}$. Because L is an n–braid closure, $\mathrm{AKh}(L, \mathbb{C})$ must contain a weight-n irreducible $\mathfrak{sl}_2(\mathbb{C})$ representation in grading $i = 0$, of dimension n. Thus $\mathrm{AKh}(L, \mathbb{C})$ must consist only of this representation in grading $i = 0$, because $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ has dimension n in homological grading 0. We therefore have that all of the generators in grading $i = 1$ for $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ must correspond to generators of $\mathrm{AKh}(L, \mathbb{C})$. If not, they would correspond to 2–torsion in $\mathrm{AKh}(L, \mathbb{Z})$, but the torsion contributes dimension in two different homological gradings by the universal coefficient theorem.

A simple computation of annular Khovanov homology verifies that L is not the trivial braid. Thus by [3, Theorem 3.1], we know that the differential ∂_- on $\mathrm{AKh}(L, \mathbb{C})$ inducing the spectral sequence to $\mathrm{Kh}(L)$ must send the highest-weight generator in the grading $i = 0$ to something nonzero. The only generator in the correct quantum grading is the highest-weight generator in the grading $i = 1$, so that must be the image of the highest-weight generator in the grading $i = 0$ under ∂_-. The action ∂_- is part of the action of $\mathfrak{sl}_2(\wedge)$ on $\mathrm{AKh}(L, \mathbb{C})$ and commutes up to sign with the lowering operator f [12, Theorem 1]. This means that the image of ∂_- is spanned by all generators in grading $i = 1$. Thus $\mathrm{Kh}(L)$ is dimension 2, and L is the unknot. $\square$

Lemma 10.10 *Let $L \subseteq A \times I \subseteq S^3$ be an annular link. If* $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{AKh}(\hat{\beta}_{2n}, \mathbb{Z}/2\mathbb{Z})$, *then* $\widehat{\mathrm{HFK}}(\widetilde{U}, \Sigma(L)) \cong \widehat{\mathrm{HFK}}(T(2, 2n))$.

Proof For this computation, we will use the spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\widehat{\mathrm{HFK}}(-\widetilde{U})$ and compute the Maslov gradings of the generators of $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$.

From the construction of the spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\widehat{\mathrm{HFK}}(-\widetilde{U})$ as an iterated mapping cone, it follows that in each i grading on $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ the relative Maslov grading of any two generators agrees with half the difference of the quantum gradings of the generators. It remains to relate the relative Maslov gradings for generators in i grading 0 and 1, and then upgrade this information to an absolute Maslov grading.

The induced differential ∂_- giving the spectral sequence from $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$ to $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$ is part of the total differential on the iterated mapping cone induced by counting pseudoholomorphic polygons. Thus ∂_- lowers the Maslov grading by one. This implies that, for $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$, generators in the same k grading live in the same relative Maslov grading. Since generators in the same k grading or $2A$ grading also have the same Maslov grading, the spectral sequence to $\widehat{\mathrm{HFK}}(-\widetilde{U})$ collapses as all differentials preserve the k grading or $2A$ grading and change the Maslov grading.

To upgrade the above to a statement about the absolute Maslov grading, notice that there are only two generators which survive in the spectral sequence to $\mathrm{Kh}(L, \mathbb{Z}/2\mathbb{Z})$, namely the generators that sit in the

k gradings $-2n$ and $2-2n$. These generators must then be in Maslov gradings 0 and 1, respectively. From here we can pin down the Maslov gradings of the remaining generators of $\mathrm{AKh}(L, \mathbb{Z}/2\mathbb{Z})$.

The claim $\widehat{\mathrm{HFK}}(\widetilde{U}) \cong \widehat{\mathrm{HFK}}(T(2,2n))$ then follows from the fact that $\widehat{\mathrm{HFK}}(-\widetilde{U}) \cong (\widehat{\mathrm{HFK}}(\widetilde{U}))^*$, with the appropriate change in gradings. □

Proof of Theorem 10.7 Suppose L is as in the statement of the theorem. Lemma 10.10 implies that $\widehat{\mathrm{HFK}}(\widetilde{U}, \Sigma(L)) \cong \widehat{\mathrm{HFK}}(T(2,4))$, whence $\widetilde{U}$ is $T(2,4)$ by Theorem 4.1, which is fibered of genus 2. Up to isotopy, fibered link exteriors have unique fibrations by Seifert surfaces; for instance see [6, Chapter 1.4]. Note that the exterior of a link may fiber in different ways if one does not require the fibers to be Seifert surfaces. The monodromies of fibered links are unique up to conjugation. The monodromy of a fibered link in $\Sigma(L)$ is the image of a braid representing L in $\mathrm{Mod}(S_2^2)$ under the Birman–Hilden correspondence. Finally, by Proposition 10.1 conjugate monodromies must come from conjugate braids, so L must be isotopic to β_4. □

The proof of Theorem 10.8 is identical so we omit it.

Remark 10.11 Similar techniques could be applied to show that annular Khovanov homology detects the closure of the braid $\sigma_1 \in B_2$. However, the fact that annular Khovanov homology detects this braid closure already follows from the fact that annular Khovanov homology detects braid closures and the braid index [13, Corollary 1.2; 39, Corollary 8.4], combined with the computations of the annular Khovanov homologies of all 2–braid closures [12, Proposition 15].

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