Algebraic & Geometric Topology

Volume 24 (2024)

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Nick Georgakopoulos

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Algebraic & Geometric Topology **24**:1 (2024) 277–323
DOI: 10.2140/agt.2024.24.277
Published: 18 March 2024

The RO(C_4) cohomology of the infinite real projective space

NICK GEORGAKOPOULOS

Following the Hu–Kriz method of computing the C_2 genuine dual Steenrod algebra $\pi_\star(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{C_2}$, we calculate the C_4–equivariant Bredon cohomology of the classifying space $\mathbb{R}P^{\infty\rho} = B_{C_4}\Sigma_2$ as an RO(C_4) graded Green-functor. We prove that as a module over the homology of a point (which we also compute), this cohomology is not flat. As a result, it can't be used as a test module for obtaining generators in $\pi_\star(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{C_4}$ as Hu and Kriz use it in the C_2 case. Their argument for the Borel equivariant dual Steenrod algebra does generalize, however, and we give a complete description of $\pi_\star(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{hC_{2^n}}$ for any $n \geq 2$.

55N91, 55P91

1 Introduction

Historically, computations in stable equivariant homotopy theory have been much more difficult than their nonequivariant counterparts, even when the groups involved are as simple as possible (ie cyclic). In recent years, there has been a resurgence in such calculations for power-2 cyclic groups C_{2^n}, owing to the crucial involvement of C_8–equivariant homology in the solution of the Kervaire invariant problem; see Hill, Hopkins and Ravenel [8].

The case of $G = C_2$ is the simplest and most studied one, partially due to its connections to motivic homotopy theory over $\mathbb{R}$ by means of realization functors; see Heller and Ormsby [6]. It all starts with the $\mathrm{RO}(C_2)$ homology of a point, which was initially described in Lewis [12]. The types of modules over it that can arise as the equivariant homology of spaces were described in May [14], and this description was subsequently used in the computation of the $\mathrm{RO}(C_2)$ homology of C_2–surfaces in Hazel [5]. The C_2–equivariant dual Steenrod algebra (in characteristic 2) was computed in Hu and Kriz [10] and gives rise to a C_2–equivariant Adams spectral sequence that has been more recently leveraged in Isaksen, Wang and Xu [11]. Another application of the Hu–Kriz computation is the definition of equivariant Dyer–Lashof operations by Wilson [17] in the $\mathbb{F}_2$–homology of C_2–spectra with symmetric multiplication. Many of these results rely on the homology of certain spaces being free as modules over the homology of a point, and there is a robust theory of such free spectra, described in Hill [7].

The case of $G = C_4$ has been much less explored and is indeed considerably more complicated. This can already be seen in the homology of a point in integer coefficients (see Zeng [18] and Georgakopoulos [2]) and the case of $\mathbb{F}_2$ coefficients is not much better (compare Sections 3.1 and 3.2 for the C_2 and C_4 cases, respectively). The greater complexity in the ground ring (or to be more precise, ground Green functor), means that modules over it can also be more complicated and indeed, certain freeness results that are easy to obtain in the C_2 case no longer hold when generalized to C_4 (compare Section 4.1 with Sections 6–8).

The computation of the dual Steenrod algebra relies on the construction of Milnor generators. Nonequivariantly, the Milnor generators ξ_i of the mod 2 dual Steenrod algebra can be defined through the completed coaction of the dual Steenrod algebra on the cohomology of $B\Sigma_2 = \mathbb{R}P^\infty$: one has that $H^*(BC_{2+};\mathbb{F}_2) = \mathbb{F}_2[x]$ and the completed coaction $\mathbb{F}_2[x] \to (H\mathbb{F}_2)_*(H\mathbb{F}_2)[\![x]\!]$ is

$$x \mapsto \sum_i x^{2^i} \otimes \xi_i.$$

In the C_2–equivariant case, the space replacing $B\Sigma_2$ is the equivariant classifying space $B_{C_2}\Sigma_2$. This is still $\mathbb{R}P^\infty$ but now equipped with a nontrivial C_2 action (described in Section 4.1). Over the homology of a point, we no longer have a polynomial algebra on a single generator x, but rather a polynomial algebra on two generators c and b, modulo the relation

$$c^2 = a_\sigma c + u_\sigma b,$$

where a_σ and u_σ are the C_2–Euler and orientation classes respectively (defined in Section 2). As a module, this is still free over the homology of a point, and the completed coaction is

$$c \mapsto c \otimes 1 + \sum_i b^{2^i} \otimes \tau_i, \quad b \mapsto \sum_i b^{2^i} \otimes \xi_i.$$

The τ_i and ξ_i are the C_2–equivariant analogues of the Milnor generators, and Hu and Kriz show that they span the genuine dual Steenrod algebra.

For C_4, the cohomology of $B_{C_4}\Sigma_2$ is significantly more complicated (see Section 6) and most importantly is *not* a free module over the homology of a point. In fact, it's not even flat (Proposition 5.3), bringing into question whether we even have a coaction by the dual Steenrod algebra in this case.

There is another related reason to consider the space $B_{C_4}\Sigma_2$. In [17], Wilson describes a framework for equivariant total power operations over an $H\mathbb{F}_2$–module A equipped with a symmetric multiplication. The total power operation is induced from a map of spectra

$$A \to A^{t\,\Sigma_{[2]}},$$

where $(-)^{t\,\Sigma_{[2]}}$ is a variant Tate construction defined in [17].

In the nonequivariant case, $A \to A^{t\,\Sigma_{[2]}}$ induces a map $A_* \to A_*((x))$ and the Dyer–Lashof operations Q^i can be obtained as the components of this map:

$$Q(x) = \sum_i Q^i(x)x^i.$$

In the C_2–equivariant case, we have a map $A_\bigstar \to A_\bigstar[c, b^{\pm}]/(c^2 = a_\sigma c + u_\sigma b)$ and we get power operations

$$Q(x) = \sum_i Q^{i\rho}(x)b^i + \sum_i Q^{i\rho+\sigma}(x)cb^i.$$

When $A = H\mathbb{F}_2$, $A_\bigstar[c, b^{\pm}]/(c^2 = a_\sigma c + u_\sigma b)$ is the cohomology of $B_{C_2}\Sigma_2$ localized away from the class b.

For C_4 we would have to use the cohomology of $B_{C_4}\Sigma_2$ (localized at a certain class) but that is no longer free, meaning that the resulting power operations would have extra relations between them, further complicating the other arguments in [17].

The computation of $H^\bigstar(B_{C_4}\Sigma_{2+}; \mathbb{F}_2)$ also serves as a test case of RO(G) homology computations for equivariant classifying spaces where G is not of prime order. We refer the reader to Shulman [16], Chonoles [1], Wilson [17], and Sankar and Wilson [15] for such computations in the $G = C_p$ case.

As for the organization of this paper, Section 2 describes the conventions and notation that we shall be using throughout this document, as well as the Tate diagram for a group G and a G–equivariant spectrum.

Sections 3.1 and 3.3 describe the Tate diagram for C_2 and C_4, respectively, using coefficients in the constant Mackey functor $\mathbb{F}_2$.

In Section 4 we define equivariant classifying spaces $B_G H$ and briefly explain the elementary computation of the cohomology of $B_{C_2}\Sigma_2$ (this argument also appears in [17]).

In Section 5 we present the result of the computation of $H^\bigstar(B_{C_4}\Sigma_{2+}; \mathbb{F}_2)$ and prove that it's not flat as a Mackey functor module over $(H\mathbb{F}_2)_\bigstar$. Sections 6–8 contain the proofs of the computation of the cohomology of $B_{C_4}\Sigma_2$.

We have included three appendices in the end. Appendix A contains pictures of the spectral sequence converging to $H^{\bigstar}(B_{C_4}\Sigma_{2+};\mathbb{F}_2)$ while Appendix B contains a detailed description of $H^{\bigstar}(S^0;\mathbb{F}_2)$, which is the ground Green functor over which all our Mackey functors are modules.

Appendix C contains the description of the G–equivariant Borel dual Steenrod algebra $(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^{hG}_{\bigstar}$ where $G = C_{2^n}$ and $n \geq 1$. This is independent of the rest of the paper and is related to our work in the following way: In [10], the Borel dual Steenrod algebra is a key ingredient in the computation of the genuine dual Steenrod algebra over the group $G = C_2$; the other key ingredient is the computation of $H^{\bigstar}(B_{C_2}\Sigma_{2+};\mathbb{F}_2)$. For $G = C_{2^n}$, $n \geq 2$, the Borel equivariant description admits a straightforward generalization as we show in Appendix C.

To aid in the creation of the first two appendices, we extensively used the computer program of [2] available here. In fact, we have introduced new functionality in the software that computes the RO(G)–graded homology of spaces such as $B_{C_4}\Sigma_2$ given an explicit equivariant CW decomposition (such as we discuss in Section 6.1). This assisted in the discovery of a nontrivial d^2 differential in the spectral sequence of $B_{C_4}\Sigma_2$ (see Remark 7.8), although the provided proof is independent of the computer computation.

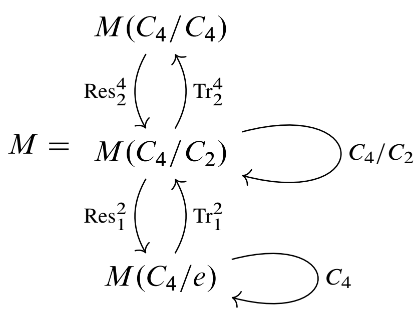
Acknowledgements We would like to thank Dylan Wilson for answering our questions regarding his paper [17] as well as [10]. We would also like to thank Peter May for his numerous editing suggestions, which vastly improved the readability of this paper. Finally, we are indebted to the referee for carefully reviewing an earlier version of this paper and suggesting a multitude of useful corrections and improvements.

2 Conventions and notations

We will use the letter k to denote the field $\mathbb{F}_2$, the constant Mackey functor $k = \underline{\mathbb{F}}_2$ and the corresponding Eilenberg–Mac Lane spectrum Hk. The meaning should always be clear from the context.

All our homology and cohomology will be in k coefficients.

The data of a C_4 Mackey functor M can be represented by a diagram displaying the values of M on orbits, its restriction and transfer maps and the actions of the Weyl groups. We shall refer to $M(C_4/C_4)$, $M(C_4/C_2)$ and $M(C_4/e)$ as the top, middle and bottom levels of the Mackey functor M, respectively. The Mackey functor diagram takes the form



If X is a G–spectrum then $X_\bigstar$ denotes the RO(G)–graded G–Mackey functor, defined on orbits as

$$X_\bigstar(G/H) = X_\bigstar^H = \pi^H(S^{-\bigstar} \wedge X) \quad = [S^\bigstar, X]^H.$$

The index $\bigstar$ will always be an element of the real representation ring RO(G).

RO(C_4) is spanned by the irreducible representations 1, σ and λ, where σ is the 1–dimensional sign representation and λ is the 2–dimensional representation given by rotation by $\pi/2$.

For $V=\sigma$ or $V=\lambda$, denote by $a_V \in k_{-V}^{C_4}$ the Euler class induced by the inclusion of north and south poles $S^0 \hookrightarrow S^V$; also denote by $u_V \in k_{|V|-V}^{C_4}$ the orientation class generating the Mackey functor $k_{|V|-V} = k$. We will use the notation $\overline{a}_V, \overline{u}_V$ to denote the restrictions of a_V, u_V to middle level, and $\overline{\overline{u}}_V$ to denote the restriction of u_V to bottom level. This notation is consistent with its use in [9].

We also write $a_{\sigma_2} \in k_{-\sigma_2}^{C_2}$ and $u_{\sigma_2} \in k_{1-\sigma_2}^{C_2}$ for the C_2 Euler and orientation classes, where σ_2 is the sign representation of C_2.

The Gold relation [8] in k coefficients takes the form

$$a_\sigma^2 u_\lambda = 0.$$

Let EG be a contractible free G–space and $\widetilde{E}G$ be the cofiber of the collapse map $EG_+ \to S^0$. We use the notation

$$X_h = EG_+ \wedge X, \quad \widetilde{X} = \widetilde{E}G \wedge X, \quad X^h = F(EG_+, X) \quad \text{and} \quad X^t = \widetilde{X}^h.$$

The Tate diagram [4] then takes the form

$$\begin{array}{ccccc} X_h & \longrightarrow & X & \longrightarrow & \widetilde{X} \\ \downarrow\simeq & & \downarrow & & \downarrow \\ X_h & \longrightarrow & X^h & \longrightarrow & X^t \end{array}$$

The square on the right is a homotopy pullback diagram and is called the Tate square.

Applying $\pi_\bigstar^G$ on the Tate diagram gives

$$\begin{array}{ccccc} X_{hG\bigstar} & \longrightarrow & X_\bigstar^G & \longrightarrow & \widetilde{X}_\bigstar^G \\ \downarrow\simeq & & \downarrow & & \downarrow \\ X_{hG\bigstar} & \longrightarrow & X_\bigstar^{hG} & \longrightarrow & X_\bigstar^{tG} \end{array}$$

3 The Tate diagram for C_2 and C_4

3.1 The Tate diagram for C_2

For $X=k$ and $G=C_2$ the corners of the Tate square are

$$k_\bigstar^{C_2} = k[a_{\sigma_2}, u_{\sigma_2}] \oplus k\left\{\frac{\theta_{\sigma_2}}{a_{\sigma_2}^i u_{\sigma_2}^j}\right\}_{i,j\geq 0}, \quad \widetilde{k}_\bigstar^{C_2} = k[a_{\sigma_2}^\pm, u_{\sigma_2}], \quad k_\bigstar^{hC_2} = k[a_{\sigma_2}, u_{\sigma_2}^\pm], \quad k_\bigstar^{tC_2} = k[a_{\sigma_2}^\pm, u_{\sigma_2}^\pm],$$

where $\theta_{\sigma_2} = \mathrm{Tr}_1^2(\overline{u}_{\sigma_2}^{-2})$. The map $k_h \to k$ in the Tate diagram induces

$$k_{hC_2\bigstar} = \Sigma^{-1} k_\bigstar^{tC_2}/k_\bigstar^{hC_2} \to k_\bigstar^{C_2}, \quad a_{\sigma_2}^{-i} u_{\sigma_2}^{-j} \mapsto \frac{\theta_{\sigma_2}}{a_{\sigma_2}^i u_{\sigma_2}^{j-1}}.$$

3.2 The RO(C_4) homology of a point

The RO(C_4) homology of a point (in k coefficients) is significantly more complicated than the RO(C_2) one; see [2] for the integer coefficient case. Appendix B contains a very detailed description of it, and the goal in this subsection is to provide a more compact version. We have also included a summary table at the end of the subsection.

The top level is

$$(1)\quad k_\bigstar^{C_4} \doteq k\left[a_\sigma, u_\sigma, a_\lambda, u_\lambda, \frac{u_\lambda}{u_\sigma^{1+i}}, \frac{a_\sigma^2}{a_\lambda^{1+i}}\right] \oplus k[a_\lambda^{\pm}]\left\{\frac{\theta}{a_\sigma^i u_\sigma^j}\right\} \oplus k\left\{\frac{(\theta/a_\lambda)a_\sigma^{1+\epsilon}}{u_\sigma^i a_\lambda^j}\right\} \oplus k[u_\sigma^{\pm}]\left\{\frac{(\theta/a_\lambda)a_\sigma^{1+\epsilon}}{a_\lambda^j u_\lambda^{1+m}}\right\},$$

where the indices i, j, m range in $0, 1, 2, \dots$ and ϵ ranges in $0, 1$.

The use of $\doteq$ as opposed to $=$ is meant to signify some subtlety present in (1) that needs to be clarified before the equality can be used. This subtlety has to do with how quotients are defined (cf [2]) and how elements multiply (the multiplicative relations). For example, the first summand in (1) is not actually a polynomial algebra, but rather a quotient of one, owing to the families of relations

$$\frac{u_\lambda}{u_\sigma^i} \cdot \frac{a_\sigma^2}{a_\lambda^j} = 0, \quad u_\sigma \cdot \frac{u_\lambda}{u_\sigma^{1+i}} = \frac{u_\lambda}{u_\sigma^i}, \quad a_\lambda \cdot \frac{a_\sigma^2}{a_\lambda^{1+i}} = \frac{a_\sigma^2}{a_\lambda^i},$$

where $i, j \ge 0$ (and $u_\lambda/u_\sigma^0 = u_\lambda$ and $a_\sigma^2/a_\lambda^0 = a_\sigma^2$).

We begin this process of carefully interpreting (1) by first noting that the middle level $\overline{u}_\sigma$ and bottom level $\overline{\overline{u}}_\sigma, \overline{\overline{u}}_\lambda$ are invertible. The element θ is then defined as

$$\theta = \mathrm{Tr}_2^4(\overline{u}_\sigma^{-2}).$$

We further introduce the elements

$$x_{n,m} = \mathrm{Tr}_1^4(\overline{\overline{u}}_\sigma^{-n} \overline{\overline{u}}_\lambda^{-m}) \quad \text{for } n \ge 0, m \ge 1.$$

Observe that

$$x_{n,m} = \frac{x_{0,1}}{u_\sigma^n u_\lambda^{m-1}}.$$

The relation between $x_{0,1} = \mathrm{Tr}_1^4(\overline{\overline{u}}_\lambda^{-1})$ and θ is

$$x_{0,1} = a_\sigma^2 \frac{\theta}{a_\lambda} = \theta \frac{a_\sigma^2}{a_\lambda}.$$

With this notation, the second curly bracket in (1) contains elements of the form

$$\frac{x_{n,1}}{a_\lambda^i} \quad \text{and} \quad \frac{x_{n,1}}{a_\sigma a_\lambda^i},$$

and the third contains

$$\frac{x_{n,m}}{a_\lambda^i} \quad \text{and} \quad \frac{x_{n,m}}{a_\sigma a_\lambda^i} \quad \text{for } m > 1.$$

The behavior of the $x_{n,m}$ depends crucially on whether $m = 1$ or not: $x_{n,1}u_\sigma = 0$ but $x_{n,m}u_\sigma \neq 0$ for $m > 1$; the $x_{n,1}$ are infinitely a_σ divisible, since

$$\frac{x_{n,1}}{a_\sigma^2} = \frac{\theta}{u_\sigma^n a_\lambda},$$

while the $x_{n,m}$ for $m > 1$ can only be divided by a_σ once. That's why we separate them into two distinct summands in (1).

The third curly bracket in (1) for $\epsilon = 0$ consists of quotients of

$$s := \frac{(\theta/a_\lambda)a_\sigma}{u_\lambda} u_\sigma = \frac{x_{0,2}u_\sigma}{a_\sigma},$$

which is the mod 2 reduction of the element s from [2]. Note that $su_\lambda = sa_\lambda = 0$.

The quotients in the RHS of (1) are all chosen coherently (cf [2]), that is, we always have the cancellation property

$$z \cdot \frac{y}{xz} = \frac{y}{x}.$$

We also have that

$$\frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}$$

as long as $xz \neq 0$ — this condition is necessary: $(\theta/a_\lambda)a_\sigma \neq 0$ is not $(\theta a_\sigma)/a_\lambda$, as $\theta a_\sigma = 0$.

To compute any product of two elements in the RHS of (1) we follow the following procedure:

- If both elements involve θ then the product is automatically 0.
- If neither element involves θ, perform all possible cancellations and use the relation

  $$\frac{u_\lambda}{u_\sigma^i} \cdot \frac{a_\sigma^2}{a_\lambda^j} = 0,$$

  where i, j range in $0, 1, 2, \ldots$.
- If only one element involves θ, perform all possible cancellations and use

  $$\frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}$$

  as long as xz appears in (1). If the resulting element appears in (1) then that's the product; if not, then the product is 0.

These are all the remarks needed to properly interpret the formula in (1) for the top level $k_\bigstar^{C_4}$.

The middle level is

$$k_\bigstar^{C_2} \doteq k[\bar{a}_\lambda, \bar{u}_\lambda, \sqrt{\bar{a}_\lambda \bar{u}_\lambda}, \bar{u}_\sigma^{\pm}] \oplus k[\bar{u}_\sigma^{\pm}]\left\{\frac{v}{\bar{a}_\lambda^i \bar{u}_\lambda^j \sqrt{\bar{a}_\lambda \bar{u}_\lambda}^{\,\epsilon}}\right\}. \tag{2}$$

element	also known as	degree in $k_\bigstar$	restriction	is transfer of
θ	-	$2\sigma-2$	0	$\overline{u}_\sigma^{-2}$
s	$(x_{0,2}u_\sigma)/a_\sigma$	$2\lambda-3$	$v/\sqrt{\overline{a}_\lambda\overline{u}_\lambda}$	-
a_σ^2/a_λ	-	$\lambda-2\sigma$	$v\overline{u}_\sigma^2$	-
$x_{0,1}$	$(a_\sigma^2/a_\lambda)\theta$	$\lambda-2$	0	v
$x_{0,2}$	$x_{0,1}/u_\lambda$	$2\lambda-4$	0	$v/\overline{u}_\lambda$
$(a_\sigma u_\lambda)/u_\sigma$	-	$1-\lambda$	0	$\sqrt{\overline{a}_\lambda\overline{u}_\lambda}$
v	θ_{σ_2}	$\lambda-2$	0	$\overline{\overline{u}}_\lambda^{-1}$

Table 1

Here, $\sqrt{\overline{a}_\lambda\overline{u}_\lambda}$ is the (unique) element whose square is $\overline{a}_\lambda\overline{u}_\lambda$, and v is defined by $v=\mathrm{Tr}_1^2(\overline{\overline{u}}_\lambda^{-1})$. Further,

$$\mathrm{Tr}_2^4(\sqrt{\overline{a}_\lambda\overline{u}_\lambda})=\frac{a_\sigma u_\lambda}{u_\sigma},\quad \mathrm{Tr}_2^4(v)=x_{0,1},\quad \overline{s}:=\mathrm{Res}_2^4(s)=\frac{v}{\sqrt{\overline{a}_\lambda\overline{u}_\lambda}},\quad \mathrm{Res}_2^4\left(\frac{a_\sigma^2}{a_\lambda}\right)=v\overline{u}_\sigma^2.$$

The interpretation of (2) is complete. We note that the restriction $\mathrm{Res}_2^4\colon k_\bigstar^{C_4}\to k_\bigstar^{C_2}$ makes $k_\bigstar^{C_2}$ into a $k_\bigstar^{C_4}$–module,

$$k_\bigstar^{C_2}=\frac{k_\bigstar^{C_4}[u_\sigma^{-1}]}{a_\sigma}\{1,\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\}.$$

In terms of the notation of the C_2 generators,

$$\overline{a}_\lambda=a_{\sigma_2}^2,\quad \overline{u}_\lambda=u_{\sigma_2}^2,\quad \sqrt{\overline{a}_\lambda\overline{u}_\lambda}=a_{\sigma_2}u_{\sigma_2},\quad v=\theta_{\sigma_2}.$$

Finally, the bottom level is very simple:

$$k_\bigstar^e=k[\overline{\overline{u}}_\lambda^\pm,\overline{\overline{u}}_\sigma^\pm].$$

We conclude with Table 1 giving the interesting/important elements of $k_\bigstar^{C_4}$ (outside of $a_\sigma,u_\sigma,a_\lambda,u_\lambda$). For more details, consult Appendix B.

3.3 The Tate diagram for C_4

Using the notation of the previous subsection, the corners of the Tate square are

$$k_\bigstar^{C_4}\doteq k\left[a_\sigma,u_\sigma,a_\lambda,u_\lambda,\frac{u_\lambda}{u_\sigma^{1+i}},\frac{a_\sigma^2}{a_\lambda^{1+i}}\right]\oplus k[a_\lambda^\pm]\left\{\frac{\theta}{a_\sigma^i u_\sigma^j}\right\}\oplus k\left\{\frac{(\theta/a_\lambda)a_\sigma^{1+\epsilon}}{u_\sigma^i a_\lambda^j}\right\}\oplus k[u_\sigma^\pm]\left\{\frac{(\theta/a_\lambda)a_\sigma^{1+\epsilon}}{a_\lambda^j u_\lambda^{1+m}}\right\},$$

$$\widetilde{k}_\bigstar^{C_4}=a_\lambda^{-1}k_\bigstar^{C_4}\doteq k\left[a_\sigma,u_\sigma,a_\lambda^\pm,u_\lambda,\frac{u_\lambda}{u_\sigma^{1+i}}\right]\oplus k[a_\lambda^\pm]\left\{\frac{\theta}{a_\sigma^i u_\sigma^j}\right\},$$

$$k_\bigstar^{hC_4}=k[a_\sigma,u_\sigma^\pm,a_\lambda,u_\lambda^\pm]/a_\sigma^2,\qquad k_\bigstar^{tC_4}=k[a_\sigma,u_\sigma^\pm,a_\lambda^\pm,u_\lambda^\pm]/a_\sigma^2.$$

The map $k_h\to k$ in the Tate diagram induces

$$k_{hC_4\bigstar}=\Sigma^{-1}k_\bigstar^{tC_4}/k_\bigstar^{hC_4}\to k_\bigstar^{C_4},\quad u_\sigma^{-i}a_\lambda^{-j}u_\lambda^{-m}\mapsto\frac{(\theta/a_\lambda)a_\sigma}{u_\sigma^{i-1}a_\lambda^{j-1}u_\lambda^m}.$$

An important distinction between the Tate diagram of C_4 and of C_2 is that, for the group C_2, the operation $\widetilde{E}C_2 \wedge -$ amounts to taking geometric fixed points: $\Phi^{C_2}k = \widetilde{E}C_2 \wedge k = \widetilde{k}$. This is not the case for C_4, and indeed

$$(\Phi^{C_4}k)_\bigstar = a_\sigma^{-1}\widetilde{k}_\bigstar^{C_4} = k[a_\sigma^\pm, u_\sigma, a_\lambda^\pm].$$

4 Equivariant classifying spaces

For groups G and K, denote by $E_G K$ any $G \times K$ space that is K–free and for which $(E_G K)^\Gamma$ is contractible for any subgroup $\Gamma \subseteq G \times K$ with $\Gamma \cap (\{1\} \times K) = \{1\}$ (a graph subgroup). The spaces $E_G \Sigma_n$ are those appearing in a G–E_∞–operad.

We define the equivariant classifying space

$$B_G K = E_G K / K.$$

4.1 The case of C_2

For $G = C_2$, the spaces $B_{C_2}\Sigma_2$ are used in the computation of the C_2 dual Steenrod algebra by Hu and Kriz [10] and for the construction of the total C_2–Dyer–Lashof operations in [17]. Both use the computation

$$k_{C_2}^\bigstar(B_{C_2}\Sigma_{2+}) = k_{C_2}^\bigstar[c,b]/(c^2 = a_{\sigma_2}c + u_{\sigma_2}b), \tag{3}$$

where c and b are classes in cohomological degrees σ_2 and $1+\sigma_2$, respectively. Let us note here that $B_{C_2}\Sigma_2$ is $\mathbb{R}P^\infty$ with a nontrivial C_2 action; the restrictions of c, b are the generators of degree $1, 2$ of $k^*(\mathbb{R}P^\infty)$.

We shall now summarize this computation, since part of it will be needed for the analogous computation when $G = C_4$, which takes place in Sections 5–8.

Let σ, τ be the sign representations of C_2, Σ_2 respectively, and let $\rho = 1+\sigma$. Then $E_{C_2}\Sigma_2 = S(\infty(\rho \otimes \tau))$; here $S(V)$ denotes the unit G–sphere inside a G–representation V. The graph subgroups of $C_2 \times \Sigma_2$ are C_2 and Δ, and their orbits correspond to the cells

$$\frac{C_2 \times \Sigma_2}{C_2} = S(1 \otimes \tau) \quad \text{and} \quad \frac{C_2 \times \Sigma_2}{\Delta} = S(\sigma \otimes \tau).$$

Wilson [17] defines a filtration on $E_{C_2}\Sigma_2$ given by

$$S(1 \otimes \tau) \subseteq S(\rho \otimes \tau) \subseteq S((\rho+1) \otimes \tau) \subseteq S(2\rho \otimes \tau) \subseteq \cdots,$$

whose quotients (after adjoining disjoint basepoints) are

$$\mathrm{gr}_{2j+1}\, E_{C_2}\Sigma_{2+} = \frac{C_2 \times \Sigma_2}{\Delta}_+ \wedge S^{(j+1)\rho_{C_2}-1}, \quad \mathrm{gr}_{2j}\, E_{C_2}\Sigma_{2+} = \Sigma_{2+} \wedge S^{j\rho_{C_2}}.$$

Taking the quotient by Σ_2 gives a filtration for $B_{C_2}\Sigma_{2+}$ with

$$\mathrm{gr}_{2j+1}\, B_{C_2}\Sigma_{2+} = S^{(j+1)\rho_{C_2}-1}, \quad \mathrm{gr}_{2j}\, B_{C_2}\Sigma_{2+} = S^{j\rho_{C_2}}.$$

Applying $k^\bigstar$ yields a spectral sequence

$$E^1 = k^\bigstar\{e^{j\rho}, e^{j\rho+\sigma}\} \Rightarrow k^\bigstar(B_{C_2}\Sigma_{2+})$$

of modules over the Green functor $k^\bigstar$. The fact that the differentials are module maps gives $E_1 = E_2$ for degree reasons. Furthermore, the vanishing of the $\mathrm{RO}(C_2)$ homology of a point in a certain range gives $E_2 = E_\infty$. The E_∞ page is free as a module over the Green functor $k^\bigstar$, hence there can't be any extension problems, and we get the module structure

$$k^\bigstar(B_{C_2}\Sigma_{2+}) = k^\bigstar\{e^{j\rho}, e^{j\rho+\sigma}\}.$$

It's easier to prove (using the homotopy fixed point spectral sequence) that

$$k^{h\bigstar}(B_{C_2}\Sigma_{2+}) = k^{h\bigstar}[w],$$

where w has cohomological degree 1. The map $k \to k^h$ from Section 2 induces

$$k^\bigstar(B_{C_2}\Sigma_{2+}) \to k^{h\bigstar}(B_{C_2}\Sigma_{2+}),$$

which is the localization which inverts u_{σ_2}. Thus we can see that $c = e^\sigma$ maps to $u_{\sigma_2}w$ (or $a_{\sigma_2} + u_{\sigma_2}w$), $b = e^\rho$ maps to $a_{\sigma_2}w + u_{\sigma_2}w^2$ and conclude that

$$k^\bigstar_{C_2}(B_{C_2}\Sigma_{2+}) = k^\bigstar_{C_2}[c, b]/(c^2 = a_{\sigma_2}c + u_{\sigma_2}b).$$

$B_{C_2}\Sigma_2$ is a C_2–H–space so $k^\bigstar_{C_2}(B_{C_2}\Sigma_{2+})$ is a Hopf algebra (since it is flat over $k^\bigstar_{C_2}$). For degree reasons, we can see that

$$\Delta(c) = c\otimes 1 + 1\otimes c, \quad \Delta(b) = b\otimes 1 + 1\otimes b, \quad \epsilon(c) = \epsilon(b) = 0.$$

(We can add a_{σ_2} to c to force $\epsilon(c) = 0$.) The primitive elements are spanned by c, b^{2^i}.

5 The cohomology of $B_{C_4}\Sigma_2$

In the next section we shall construct a cellular decomposition of $B_{C_4}\Sigma_2$ giving rise to a spectral sequence computing $k^\bigstar(B_{C_4}\Sigma_{2+})$. Here's the result of the computation, describing $k^\bigstar(B_{C_4}\Sigma_{2+})$ as a Green functor algebra over $k^\bigstar$.

Proposition 5.1 *There exist elements*

$$e^a \in k^{\sigma+\lambda}_{C_4}(B_{C_4}\Sigma_{2+}), \quad e^u \in k^{\sigma+\lambda-2}_{C_4}(B_{C_4}\Sigma_{2+}), \quad e^\lambda \in k^{\lambda}_{C_4}(B_{C_4}\Sigma_{2+}), \quad e^\rho \in k^{\rho}_{C_4}(B_{C_4}\Sigma_{2+})$$

such that

$$k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+}) = \frac{k^\bigstar_{C_4}\left[e^a, \dfrac{e^u}{u^i_\sigma}, \dfrac{e^\lambda}{u^i_\sigma}, e^\rho\right]_{i\geq 0}}{S}.$$

The relation set S consists of two types of relations (*we use indices i, $j \geq 0$*):

- ***Module relations*** $$\begin{cases} \dfrac{a_\sigma^2}{a_\lambda^j}\dfrac{e^u}{u_\sigma^i}=0, \\ \dfrac{(\theta/a_\lambda)a_\sigma}{u_\sigma^{i-2}a_\lambda^{j-1}}e^a+\dfrac{s}{u_\sigma^{i-1}a_\lambda^{j-2}}e^u=\dfrac{a_\sigma^2}{a_\lambda^j}\dfrac{e^\lambda}{u_\sigma^i}. \end{cases}$$

- ***Multiplicative relations*** $$\begin{cases} \dfrac{e^u}{u_\sigma^i}\dfrac{e^u}{u_\sigma^j}=\dfrac{u_\lambda}{u_\sigma^{i+j-2}}e^\lambda, \\ \dfrac{e^\lambda}{u_\sigma^i}\dfrac{e^u}{u_\sigma^j}=\dfrac{u_\lambda}{u_\sigma^{i+j}}e^a+a_\lambda\dfrac{e^u}{u_\sigma^{i+j}}, \\ e^a\dfrac{e^u}{u_\sigma^i}=\dfrac{u_\lambda}{u_\sigma^{i-1}}e^\rho+a_\sigma\dfrac{u_\lambda}{u_\sigma^i}e^a, \\ \dfrac{e^\lambda}{u_\sigma^i}\dfrac{e^\lambda}{u_\sigma^j}=\dfrac{u_\lambda}{u_\sigma^{i+j+1}}e^\rho+a_\sigma\dfrac{u_\lambda}{u_\sigma^{i+j+2}}e^a+a_\lambda\dfrac{e^\lambda}{u_\sigma^{i+j}}, \\ e^a\dfrac{e^\lambda}{u_\sigma^i}=\dfrac{e^u}{u_\sigma^{i+1}}e^\rho+a_\sigma\dfrac{u_\lambda}{u_\sigma^{i+1}}e^\rho, \\ (e^a)^2=u_\sigma e^\lambda e^\rho+a_\sigma\dfrac{e^u}{u_\sigma}e^\rho+u_\sigma a_\lambda e^\rho+a_\sigma a_\lambda e^a. \end{cases}$$

The middle level of $k^\bigstar(B_{C_4}\Sigma_{2+})$ *is generated by the restrictions of* $e^a, e^u, e^\lambda, e^\rho$, *which we denote by* $\overline{e}^a, \overline{e}^u, \overline{e}^\lambda, \overline{e}^\rho$, *respectively, and two fractional elements*:

$$k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})=\frac{k^\bigstar_{C_2}\left[\overline{e}^a,\overline{e}^u,\overline{e}^\lambda,\overline{e}^\rho,\dfrac{\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^u}{\overline{u}_\lambda},\dfrac{\overline{a}_\lambda\overline{u}_\sigma^{-1}\overline{e}^u+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^\lambda}{\overline{u}_\lambda}\right]}{\mathrm{Res}_2^4(S)}.$$

Here, $\mathrm{Res}_2^4(S)$ *denotes the relation set obtained by applying the ring homomorphism* Res_2^4 *on each relation of* S. *That is*, *we have the module relations*

$$\frac{v}{\overline{a}_\lambda^i}\overline{e}^u=\frac{v}{\overline{a}_\lambda^i}\overline{e}^\lambda=0 \quad \text{for any } i\ge 0,$$

and the multiplicative relations

$$\begin{aligned} &(\overline{e}^u)^2=\overline{u}_\sigma^2\overline{u}_\lambda\overline{e}^\lambda, && \overline{e}^\lambda\overline{e}^u=\overline{u}_\lambda\overline{e}^a+\overline{a}_\lambda\overline{e}^u, && \overline{e}^a\overline{e}^u=\overline{u}_\lambda\overline{u}_\sigma\overline{e}^\rho, \\ &(\overline{e}^\lambda)^2=\overline{u}_\lambda\overline{u}_\sigma^{-1}\overline{e}^\rho+\overline{a}_\lambda\overline{e}^\lambda, && \overline{e}^a\overline{e}^\lambda=\overline{u}_\sigma^{-1}\overline{e}^u\overline{e}^\rho, && (\overline{e}^a)^2=\overline{u}_\sigma\overline{e}^\lambda\overline{e}^\rho+\overline{u}_\sigma\overline{a}_\lambda\overline{e}^\rho. \end{aligned}$$

As for the Mackey functor structure, *the Weyl group* C_4/C_2 *action on the generators is trivial and we have*

- ***Mackey functor relations*** $$\begin{cases} \mathrm{Tr}_2^4\left(\overline{u}_\sigma^{-i}\dfrac{\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^u}{\overline{u}_\lambda}\right)=a_\sigma\dfrac{e^u}{u_\sigma^{i+1}}, \\ \mathrm{Tr}_2^4\left(\overline{u}_\sigma^{-i}\dfrac{\overline{a}_\lambda\overline{u}_\sigma^{-1}\overline{e}^u+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^\lambda}{\overline{u}_\lambda}\right)=a_\sigma\dfrac{e^\lambda}{u_\sigma^{i+1}}. \end{cases}$$

Finally, the bottom level is

$$k_e^{\bigstar}(B_{C_4}\Sigma_{2+}) = k_e^{\bigstar}[\mathrm{Res}_1^4(e^u)]$$

with trivial Weyl group C_4 action and Mackey functor relations obtained by applying Res_1^4 to the multiplicative relations of S:

$$\mathrm{Res}_1^4 e^\lambda = \bar{\bar{u}}_\sigma^{-2}\bar{\bar{u}}_\lambda^{-1}\,\mathrm{Res}_1^4(e^u)^2, \quad \mathrm{Res}_1^4 e^a = \bar{\bar{u}}_\sigma^{-2}\bar{\bar{u}}_\lambda^{-2}\,\mathrm{Res}_1^4(e^u)^3, \quad \mathrm{Res}_1^4 e^\rho = \bar{\bar{u}}_\lambda^{-3}\bar{\bar{u}}_\sigma^{-3}\,\mathrm{Res}_1^4(e^u)^4.$$

Note: for every quotient y/x there is a defining relation $x\cdot(y/x) = y$. We have omitted these implicit module relations from the description above.

The best description of the middle level is in terms of the generators c, b of

$$k_{C_2}^{\star}(B_{C_2}\Sigma_{2+}) = \frac{k_{C_2}^{\star}[c,b]}{c^2 = a_{\sigma_2}c + u_{\sigma_2}b}.$$

Here, $\star$ ranges in $\mathrm{RO}(C_2)$, and to get $k_{C_2}^{\bigstar}(B_{C_4}\Sigma_{2+})$ for $\bigstar$ in $\mathrm{RO}(C_4)$, we have to restrict to $\mathrm{RO}(C_2)$ representations of the form $n + 2m\sigma_2$. In this way,

$$k_{C_2}^{\bigstar}(B_{C_4}\Sigma_{2+}) = k_{C_2}^{\star}(B_{C_2}\Sigma_{2+})[\bar{u}_\sigma^{\pm}],$$

where $\star$ needs to be restricted to oriented C_2 representations: $\bigstar = n + m\sigma + k\lambda$ in $\mathrm{RO}(C_4)$ corresponds to $\star = n + m + 2k\sigma_2$ in $\mathrm{RO}(C_2)$. The correspondence of generators is

$$\bar{e}^a = \bar{u}_\sigma(a_{\sigma_2}b + bc), \quad \bar{e}^u = \bar{u}_\sigma u_{\sigma_2}c, \quad \bar{e}^\lambda = c^2, \quad \bar{e}^\rho = \bar{u}_\sigma b^2.$$

We can also express the map to homotopy fixed points in terms of our generators:

Proposition 5.2 *There is a choice of the degree-1 element w in*

$$k^{hC_4\bigstar}(B_{C_4}\Sigma_{2+}) = k^{hC_4\bigstar}[w]$$

such that the localization map $k_{C_4}^{\bigstar}(B_{C_4}\Sigma_{2+}) \to k^{hC_4\bigstar}(B_{C_4}\Sigma_{2+})$ induced by $k \to k^h$ and inverting u_σ and u_λ is

$$e^u \mapsto u_\sigma u_\lambda w, \qquad e^\lambda \mapsto u_\lambda w^2,$$
$$e^a \mapsto u_\sigma u_\lambda w^3 + u_\sigma a_\lambda w, \quad e^\rho \mapsto u_\sigma u_\lambda w^4 + a_\sigma u_\lambda w^3 + u_\sigma a_\lambda w^2 + a_\sigma a_\lambda w.$$

Proposition 5.3 *The module $k^{\bigstar}(B_{C_4}\Sigma_{2+})$ is not flat over $k^{\bigstar}$.*

Proof Let $R = k^{\bigstar}$ and $M = k^{\bigstar}(B_{C_4}\Sigma_{2+})$. Consider the map of $\mathrm{RO}(C_4)$–graded Mackey functors $f\colon R \to \Sigma^{2\sigma-\lambda}R$ given on top level by multiplication with a_σ^2/a_λ, and determined on the lower levels by restricting (so it's multiplication with $v\bar{u}_\sigma^2$ on the middle level and 0 on the bottom level). If M is a flat R–module then we have an exact sequence

$$0 \to M \boxtimes_R \mathrm{Ker}(f) \to M \xrightarrow{f} \Sigma^{2\sigma-\lambda}M.$$

Here, $\boxtimes$ is the box product of Mackey functors and $\boxtimes_R$ is the corresponding box product over R–modules.

The restriction functor Res^4_2 from R–modules to $\mathrm{Res}^4_2\, R$–modules is exact and symmetric monoidal, so we replace M, R and $\mathrm{Ker}(f)$ by $\mathrm{Res}^4_2\, M$, $\mathrm{Res}^4_2\, R$ and $\mathrm{Res}^4_2\,\mathrm{Ker}(f)$, respectively, and have an exact sequence of C_2 Mackey functors. Using the notation involving the C_2 generators c and b, and writing $a = a_{\sigma_2}$ and $u = u_{\sigma_2}$, we have

$$M = \bigoplus_{i\geq 0} R\{b^{2i}, cb^{2i+1}\} \oplus \bigoplus_{i\geq 0} R\{ab^{2i+1}, ub^{2i+1}, acb^{2i}, ucb^{2i}\}/\sim.$$

The map f maps each summand to itself, so we may replace M by $R\{c, ab, ub, acb, ucb\}/\sim$ and continue to have the same exact sequence as above. The top level then is

$$0 \to (M \boxtimes_R \mathrm{Ker}(f))(C_2/C_2) \to M(C_2/C_2) \xrightarrow{v\bar{u}_\sigma^2} M(C_2/C_2)$$

and v acts trivially on ab, ub, ac, uc, ie on $M(C_2/C_2)$, so we get

$$(M \boxtimes_R \mathrm{Ker}(f))(C_2/C_2) = M(C_2/C_2).$$

We compute from definition that $(M \boxtimes_R \mathrm{Ker}(f))(C_2/C_2)$ is isomorphic to $M(C_2/C_2) \otimes_{R(C_2/C_2)} I$, where $I := \mathrm{Ker}(R \xrightarrow{v} R)$. But $M(C_2/C_2) \otimes_{R(C_2/C_2)} I \to M(C_2/C_2)$ has image $IM(C_2/C_2)$ hence

$$IM(C_2/C_2) = M(C_2/C_2).$$

This contradicts the fact that $ab = e'^{\lambda+1}$ is not divisible by any element of the ideal I (since $e'^{\lambda+1}$ is only divisible by $\bar{u}_\sigma^{\pm i} \in R$, which are not in I). □

This proof does not depend on the explicit computation of $k^{\bigstar}_{C_4}(B_{C_4}\Sigma_{2+})$ but rather on the fact that, while $k^{\star}_{C_2}(B_{C_2}\Sigma_{2+})$ is free over $k^{\star}_{C_2}$, where $\star \in \mathrm{RO}(C_2)$, this is no longer the case when restricting $\star$ to range over the image of $\mathrm{RO}(C_4) \to \mathrm{RO}(C_2)$.

6 A cellular decomposition of $B_{C_4}\Sigma_2$

We denote the generators of C_4 and Σ_2 by g and h, respectively; let also τ be the sign representation of Σ_2, and $\rho = 1 + \sigma + \lambda$ the regular representation of C_4.

The graph subgroups of $C_4 \times \Sigma_2$ are $C_4 = \langle g\rangle$, $C_2 = \langle g^2\rangle$, $\Delta = \langle gh\rangle$, $\Delta' = \langle g^2 h\rangle$ and e.

Since $\rho \otimes \tau$ contains a trivial representation when restricted to any of these graph subgroups, we have a model for the universal space

$$E_{C_4}\Sigma_2 = S(\infty(\rho \otimes \tau)),$$

and $B_{C_4}\Sigma_2$ is $\mathbb{R}P^\infty$ with nontrivial C_4 action

$$g(x_1, x_2, x_3, x_4, \dots) = (x_1, -x_2, -x_4, x_3, \dots).$$

$S(\infty(\rho\otimes\tau))$ is the space

$$S(\infty)=\Big\{(x_n):\text{finitely supported and }\sum_i x_i^2=1\Big\}$$

with $C_4\times\Sigma_2$ action

$$g(x_1,x_2,x_3,x_4,x_5,\dots)=(x_1,-x_2,-x_4,x_3,x_5,\dots),\quad h(x_1,x_2,\dots)=(-x_1,-x_2,\dots).$$

We shall use the notation $(x_1,\dots,x_n)$ for the point $(x_1,\dots,x_n,0,0,\dots)\in S(\infty)$. Moreover, the subspace of $S(\infty)_+$ where only $x_1,\dots,x_n$ are allowed to be nonzero shall be denoted by $\{(x_1,\dots,x_n)\}$.

We now describe a cellular decomposition of $E_{C_4}\Sigma_{2+}$ where the orbits are $C_4\times\Sigma_{2+}/H\wedge S^V$, where V is a C_4 representation.

- Start with $\{(x_1)\}$ the union of two points $(1),(-1)$ and the basepoint. This is $C_4\times\Sigma_2/C_{4+}$.
- $\{(x_1)\}$ includes in $\{(x_1,x_2)\}=S(1+\sigma)_+$. The cofiber is the wedge of two circles, corresponding to x_2 being positive or negative, and the action is
$$g(x_1,+)=(x_1,-),\quad h(x_1,+)=(-x_1,-).$$
After applying the self-equivalence given by $f(x_1,+)=(x_1,+)$ and $f(x_1,-)=(-x_1,-)$, the action becomes
$$g(x_1,+)=(-x_1,-),\quad h(x_1,+)=(x_1,-).$$
This is exactly $C_4\times\Sigma_2/\Delta_+\wedge S^\sigma$.
- $\{(x_1,x_2)\}$ includes in $\{(x_1,x_2,x_3,0),(x_1,x_2,0,x_4)\}$. The cofiber is the wedge of four spheres corresponding to the sign of the nonzero coordinate among the last two coordinates. If we number the spheres from 1 to 4 and use $(x,y)^i$ coordinates to denote them for $i=1,2,3,4$ then
$$g(x,y)^i=(x,-y)^{i+1},\quad h(x,y)^i=(-x,-y)^{i+2}.$$
Applying the self-equivalence
$$f(x,y)^1=(x,y)^1,\quad f(x,y)^2=(-y,x)^2,\quad f(x,y)^3=(-x,-y)^3,\quad f(x,y)^4=(y,-x)^4,$$
the action becomes $g(x,y)^i=(-y,x)^{i+1}$ and $h(x,y)^i=(x,y)^{i+2}$, ie we have $C_4\times\Sigma_2/\Delta'_+\wedge S^\lambda$.
- $\{(x_1,x_2,x_3,0),(x_1,x_2,0,x_4)\}$ includes in $\{(x_1,x_2,x_3,x_4)\}=S(\rho\otimes\tau)$ and the cofiber is the wedge of four S^3's corresponding to the signs of x_3,x_4. Analogously to the item above, we get the space $C_4\times\Sigma_2/\Delta'_+\wedge S^{1+\lambda}$.
- The process now repeats: $\{(x_1,x_2,x_3,x_4)\}$ includes in $\{(x_1,x_2,x_3,x_4,x_5)\}$ and the cofiber is the wedge of two S^4's corresponding to the sign of x_5 and we get $C_4\times\Sigma_2/C_{4+}\wedge S^{1+\sigma+\lambda}$. And so on.

We get the decomposition of $B_{C_4}\Sigma_{2+}$ where the associated graded is

$$\mathrm{gr}_{4j}=S^{j\rho},\quad \mathrm{gr}_{4j+1}=S^{j\rho+\sigma},\quad \mathrm{gr}_{4j+2}=\Sigma^{j\rho+\lambda}C_4/C_{2+},\quad \mathrm{gr}_{4j+3}=\Sigma^{j\rho+1+\lambda}C_4/C_{2+}.$$

This filtration gives a spectral sequence of $k^\star$–modules converging to $k^\star(B_{C_4}\Sigma_{2+})$, which we shall analyze in the next section.

6.1 A decomposition using trivial spheres

The cellular decomposition of $B_{C_4}\Sigma_2$ we just established consists of one cell in every dimension, where by "cell" we mean a space of the form $(C_4/H)_+ \wedge S^V$ for H a subgroup of C_4 and V a real nonvirtual C_4–representation; let us call this a "type I" decomposition. It is also possible to obtain a decomposition using only "trivial spheres", namely with cells of the form $(C_4/H)_+ \wedge S^n$; we shall refer to this as a "type II" decomposition. A type I decomposition can be used to produce a type II decomposition by replacing each type I cell $(C_4/H)_+ \wedge S^V$ with its type II decomposition. This is useful for computer-based calculations, since type II decompositions lead to chain complexes as opposed to spectral sequences — $k_*((C_4/H)_+ \wedge S^V)$ is concentrated in a single degree if and only if V is trivial. Equipped with a type II decomposition, the computer program of [2] can calculate the additive structure of $k^\bigstar(B_{C_4}\Sigma_{2+})$ in a finite range (this can be helpful with our spectral sequence calculations: see Remark 7.8).

We note however that a minimal type I decomposition may expand to a nonminimal type II decomposition; this is the case for $B_{C_4}\Sigma_2$, where the minimal type II decomposition uses $2d+3$ cells in each dimension $d \geq 1$, while the one obtained by expanding the type I decomposition uses $3d+3$ cells in each dimension $d \geq 1$. It is the minimal decomposition that we have used as input for the computer program of [2].

7 The spectral sequence for $B_{C_4}\Sigma_2$

Applying $k^\bigstar$ on the filtration of $B_{C_4}\Sigma_{2+}$ gives a spectral sequence

$$E_1^{V,s} = k^V \operatorname{gr}_s \Rightarrow k^V B_{C_4}\Sigma_{2+}.$$

The differential d^r has (V,s) bidegree $(1,r)$ so it goes 1 unit to the right and r units up in (V,s) coordinates.

Before we can write down the E_1 page, we will need some notation. For a G–Mackey functor M and subgroup $H \subseteq G$, let $M_{G/H}$ denote the G–Mackey functor defined on orbits as $M_{G/H}(G/K) = M(G/H \times G/K)$; the restriction, transfer and Weyl group action in $M_{G/H}$ are induced from those in M. Equivalently, $M_{G/H}$ can be thought of as restricting M to an H Mackey functor and then inducing up to a G–Mackey functor.

For $G = C_4$ and $H = C_2$, the bottom level of M_{C_4/C_2} is

$$M_{C_4/C_2}(C_4/e) = M(C_4/e \times C_4/C_2) = M(C_4/e) \oplus M(C_4/e) = M(C_4/e)\{x, y\},$$

where x and y are used to distinguish the two copies of $M(C_4/e)$, ie so that any element of $M_{C_4/C_2}(C_4/e)$ can be uniquely written as $mx + m'y$ for $m, m' \in M(C_4/e)$. The Weyl group $W_{C_4}e = C_4$ acts as

$$g(mx + m'y) = (gm)(gx) + (gm')(gy) = (gm)y + (gm')x,$$

ie $y = gx$ for a fixed generator $g \in C_4$.

We can then describe M_{C_4/C_2} in terms of M and the computation of the restriction and transfer on x, which are shown in the diagram

$$M_{C_4/C_2} = \begin{array}{c} M(C_4/C_2)\{x+gx\} \\ {\scriptstyle m(x+gx)\mapsto m(x+gx)}\big\downarrow \;\; \big\uparrow{\scriptstyle mx\mapsto m(x+gx)} \\ M(C_4/C_2)\{x,gx\} \circlearrowleft C_4/C_2 \\ {\scriptstyle mx\mapsto \mathrm{Res}(m)x}\big\downarrow \;\; \big\uparrow{\scriptstyle mx\mapsto \mathrm{Tr}(m)x} \\ M(C_4/e)\{x,gx\} \circlearrowleft C_4 \end{array}$$

where in each map, m is any element of the appropriate level $M(C_4/H)$, with $H \subseteq C_4$.

If $M = R$ is a Green functor, then R_{C_4/C_2} is an R–module. Its top level, namely $R(C_4/C_2)\{x+gx\}$, is an $R(C_4/C_4)$–module via extension of scalars along the restriction map $\mathrm{Res}^4_2\colon R(C_4/C_4) \to R(C_4/C_2)$.

7.1 The E_1 page

The rows in the E_1 page are

$$E_1^{\bigstar,4j} = k^{\bigstar-j\rho}, \qquad E_1^{\bigstar,4j+1} = k^{\bigstar-j\rho-\sigma},$$
$$E_1^{\bigstar,4j+2} = (k^{\bigstar-j\rho-\lambda})_{C_4/C_2}, \quad E_1^{\bigstar,4j+3} = (k^{\bigstar-j\rho-\lambda-1})_{C_4/C_2}.$$

We will write $e^{j\rho}$, $e^{j\rho+\sigma}$, $e^{j\rho+\lambda}$ and $e^{j\rho+\lambda+1}$ for the unit elements corresponding to the E_1 terms above, living in degrees $V = j\rho, j\rho+\sigma, j\rho+\lambda, j\rho+\lambda+1$ and filtrations $s = 4j, 4j+1, 4j+2, 4j+3$, respectively. We also write $\overline{e}^V, \overline{\overline{e}}^V$ for their restrictions to the middle and bottom levels respectively. In this way,

$$E_1^{\bigstar,*} = k^{\bigstar}\{e^{j\rho}, e^{j\rho+\sigma}\} \oplus (k^{\bigstar})_{C_4/C_2}\{e^{j\rho+\lambda}, e^{j\rho+\lambda+1}\},$$

and the three levels of the Mackey functor $E_1^{\bigstar,*}$, from top to bottom, are

$$k^{\bigstar}_{C_4}\{e^{j\rho}, e^{j\rho+\sigma}\} \oplus k^{\bigstar}_{C_2}\{e^{j\rho+\lambda}(x+gx), e^{j\rho+\lambda+1}(x+gx)\},$$
$$k^{\bigstar}_{C_2}\{\overline{e}^{j\rho}, \overline{e}^{j\rho+\sigma}\} \oplus k^{\bigstar}_{C_2}\{\overline{e}^{j\rho+\lambda}x, \overline{e}^{j\rho+\lambda}gx, \overline{e}^{j\rho+\lambda+1}x, \overline{e}^{j\rho+\lambda+1}gx\},$$
$$k^{\bigstar}_{e}\{\overline{\overline{e}}^{j\rho}, \overline{\overline{e}}^{j\rho+\sigma}\} \oplus k^{\bigstar}_{e}\{\overline{\overline{e}}^{j\rho+\lambda}x, \overline{\overline{e}}^{j\rho+\lambda}gx, \overline{\overline{e}}^{j\rho+\lambda+1}x, \overline{\overline{e}}^{j\rho+\lambda+1}gx\}.$$

For the top level, $k^{\bigstar}_{C_2}$ is a $k^{\bigstar}_{C_4}$–module through the restriction $\mathrm{Res}^4_2\colon k^{\bigstar}_{C_4} \to k^{\bigstar}_{C_2}$:

$$k^{\bigstar}_{C_2} = \frac{k^{\bigstar}_{C_4}[u_\sigma^{-1}]}{a_\sigma}\{1, \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\}.$$

It's important to note that this is *not* a cyclic $k_{\bigstar}^{C_4}$–module.

At this point, the reader may want to look over pictures of the E_1 page that we have included in Appendix A. We will reference them in the following subsections.

7.2 The d^1 differentials

In this subsection, we explain how the d^1 differentials on each level are computed. We shall need this crucial remark.

Remark 7.1 The restriction of the C_4 action on $B_{C_4}\Sigma_2$ to $C_2 \subseteq C_4$ results in a C_2 space equivalent to $B_{C_2}\Sigma_2$. The equivariant cohomology of this space is known from Section 4.1 and we shall use this result to compute the middle level spectral sequence for $B_{C_4}\Sigma_2$. Further restricting to the trivial group $e \subseteq C_4$, we get the nonequivariant space $\mathbb{R}P^\infty$ and this will be used to compute the bottom level spectral sequence.

Proposition 7.2 *The nontrivial d^1 differentials are generated by*

$$d^1(\overline{e}^{j\rho+\sigma}) = v\overline{u}_\sigma \overline{e}^{j\rho+\lambda}(x+gx),$$
$$d^1(e^{j\rho+\sigma}) = v\overline{u}_\sigma e^{j\rho+\lambda}(x+gx),$$
$$d^1(\overline{e}^{j\rho+\lambda}x) = \overline{e}^{j\rho+\lambda+1}(x+gx).$$

Proof First of all, the bottom level spectral sequence is concentrated on the diagonal and the nontrivial d^1 differentials are $k\{x, gx\} \to k\{x, gx\}$, $x \mapsto x + gx$, since $k^*(\mathbb{R}P^\infty)$ is k in every nonnegative degree. See Figures 1, 2, 3, 4 and 5.

The d^1 differentials on middle and top level are computed from the fact that they are $k^\bigstar$–module maps, hence determined on

$$e^{j\rho}, \quad e^{j\rho+\sigma}, \quad \overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}^{\,\epsilon} e^{j\rho+\lambda+\epsilon'}(x+gx)$$

for the top level ($\epsilon, \epsilon' = 0, 1$), and on

$$\overline{e}^{j\rho}, \quad \overline{e}^{j\rho+\sigma}, \quad \overline{e}^{j\rho+\lambda}x, \quad \overline{e}^{j\rho+\lambda+1}x$$

for the middle level. We remark that because $k^\bigstar_{C_2}$ is not a cyclic $k^\bigstar_{C_4}$–module, it does not suffice to compute the top level d^1 on $e^{j\rho}, e^{j\rho+\sigma}, e^{j\rho+\lambda}, e^{j\rho+\lambda+1}$.

The d^1 differentials from row $4j$ to row $4j+1$ are all determined by $d^1\colon ke^{j\rho} \to k^{1-\sigma}e^{j\rho+\sigma}$. Note that $k^{1-\sigma}$ is generated by $0 \mid \overline{u}_\sigma^{-1} \mid \overline{\overline{u}}_\sigma^{-1}$ — this notation was defined in [2] and expresses the generators of all three levels from top to bottom, separated by vertical columns. The d^1 is trivial on bottom level, and using the fact that it commutes with restriction we can see that it's trivial in all levels. See Figure 1 and degrees $V = 0, 1$.

Similarly, the d^1 differentials from row $4j+1$ to row $4j+2$ are all determined by $d^1\colon ke^{j\rho+\sigma} \to (k^{\sigma-\lambda+1})_{C_4/C_2}e^{j\rho+\lambda}$. Note that $(k^{\sigma-\lambda+1})_{C_4/C_2}$ is generated by

$$v\overline{u}_\sigma(x+gx) \mid v\overline{u}_\sigma(x, gx) \mid \overline{\overline{u}}_\sigma\overline{\overline{u}}_\lambda^{-1}(x, gx).$$

The differential is trivial on bottom level, but on middle level the C_2 computation gives $k^{\sigma}_{C_2}(B_{C_4}\Sigma_{2+})=0$ forcing the differential to be nontrivial (the only other way to kill $E_1^{\sigma-\lambda+1,4j+2}(C_4/C_2)=k^2$ is for the d^1 differential from row $4j+2$ to $4j+3$ to be the identity $k^2\to k^2$ on middle level, which can't happen as we show in the next paragraph). Thus

$$d^1(\overline{e}^{j\rho+\sigma})=v\overline{u}_\sigma\overline{e}^{j\rho+\lambda}(x+gx)\quad\text{and}\quad d^1(e^{j\rho+\sigma})=v\overline{u}_\sigma e^{j\rho+\lambda}(x+gx).$$

See Figure 2 and degrees $V=\sigma,\sigma+1$.

The d^1 differentials from row $4j+2$ to row $4j+3$ are determined by

$$d^1: k_{C_4/C_2}\overline{u}_\sigma^{-i}e^{j\rho+\lambda}\to k_{C_4/C_2}\overline{u}_\sigma^{-i}e^{j\rho+\lambda+1},$$
$$d^1: k_{C_4/C_2}\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}\to k_{C_4/C_2}\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda+1}.$$

On bottom level, these d^1's all are $x\mapsto x+gx$ and the commutation with restriction and transfer gives

$$d^1(\overline{e}^{j\rho+\lambda}x)=\overline{e}^{j\rho+\lambda+1}(x+gx),\quad d^1(\overline{u}_\sigma^{-i}e^{j\rho+\lambda}(x+gx))=0,\quad d^1(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}(x+gx))=0.$$

See Figure 3 and degrees $V=\lambda,\lambda+1$.

Finally, the d^1 differentials from row $4j+3$ to row $4j+4$ are determined by

$$d^1: k_{C_4/C_2}\overline{u}_\sigma^{-i}e^{j\rho+\lambda+1}\to k^{1-\sigma}e^{j\rho+\rho},\quad d^1: k_{C_4/C_2}\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda+1}\to k^{2-\sigma+\lambda}e^{j\rho+\rho}.$$

These are trivial on the bottom level and by the commutation with restriction and transfer we can see that they are trivial on all levels. See Figure 4 and degrees $V=\rho-1,\rho$. □

This settles the E_1 page computation.

7.3 Bottom level computation

We can immediately conclude that the bottom level spectral sequence collapses in E_2, giving a single k in every RO(C_4) degree. Thus there are no extension problems and the C_4 (Weyl group) action is trivial.

7.4 Middle level computation

By Remark 7.1 and comparing the description of the middle level E_2 with that of $k^{\bigstar}_{C_2}(B_{C_2}\Sigma_{2+})$ of Section 4.1 we can see that the middle level spectral sequence collapses on $E_2=E_\infty$.

To go from E_∞ to $k^{\bigstar}_{C_4}(B_{C_2}\Sigma_{2+})$ we need to be able to choose unique lifts for the permanent cycles when they are multiple candidates. To be consistent in our choices, we use the following rules.

7.4.1 Choosing unique lifts If we have a middle level element $\alpha\in E_\infty^{s,V}$ and the group $E_\infty^{t,V}$ vanishes for $t>s$, then α lifts uniquely to $k^{\bigstar}_{C_2}(B_{C_4}\Sigma_{2+})$. If on the other hand $E_\infty^{t,V}\neq 0$ for some $t>s$, then there are multiple lifts of α. In that case, we pick the lift for which there are no exotic restrictions (if possible). For example, if $\mathrm{Res}^2_1(\alpha)=0$ in E_∞ and there is a unique lift β of α such that $\mathrm{Res}^2_1(\beta)=0$, then we use β as our lift of α.

For the purposes of the following proposition let us temporarily write $a \rightsquigarrow b$ where b is the notation for unique lift for a.

Proposition 7.3 *There are unique lifts*

$$\overline{e}^{j\rho} \rightsquigarrow \overline{e}^{j\rho}, \qquad \overline{u}_\lambda \overline{e}^{j\rho+\sigma} \rightsquigarrow \overline{e}^{j,u}, \qquad \overline{a}_\lambda \overline{e}^{j\rho+\sigma} \rightsquigarrow \widetilde{e}^{j,a},$$

$$\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j\rho+\sigma} \rightsquigarrow e^{j,au}, \quad \overline{e}^{j\rho+\lambda}(x+gx) \rightsquigarrow \overline{e}^{j\rho+\lambda}, \quad \overline{e}^{j\rho+\lambda+1}x = \overline{e}^{j\rho+\lambda+1}gx \rightsquigarrow e'^{j\rho+\lambda+1}.$$

These lifts generate $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$ as a $k^\bigstar_{C_2}$–module, and we have the relation

$$v\overline{e}^{j\rho+\lambda} = 0.$$

Proof We shall only explain a few of these, as most are immediate from Section 7.4.1 and the description of the E_2 page. The elements $\overline{u}^i_\sigma \overline{e}^{j\rho+\sigma}$ don't survive, but every other multiple of $\overline{e}^{j\rho+\sigma}$ does (since v annihilates them). These multiples are generated (as a $k^\bigstar_{C_2}$–module) by

$$\overline{a}_\lambda \overline{e}^{j\rho+\sigma}, \quad \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j\rho+\sigma}, \quad \overline{u}_\lambda \overline{e}^{j\rho+\sigma}.$$

Note that we don't need to include elements involving $v\overline{e}^{j\rho+\sigma}$, since

$$\frac{v}{\overline{a}^i_\lambda \sqrt{\overline{a}_\lambda \overline{u}_\lambda}^{\,\epsilon} u^k_\lambda}\overline{e}^{j\rho+\sigma} = \frac{v}{\overline{a}^i_\lambda \sqrt{\overline{a}_\lambda \overline{u}_\lambda}^{\,\epsilon} u^{k+1}_\lambda}\overline{u}_\lambda \overline{e}^{j\rho+\sigma},$$

where $i, k \geq 0$ and $\epsilon = 0, 1$.

For each $j \geq 0$, the element $\overline{a}_\lambda \overline{e}^{j\rho+\sigma}$ has two distinct lifts. On E_∞ we have that $\mathrm{Res}^2_1(\overline{a}_\lambda \overline{e}^{j\rho+\sigma}) = 0$, and on $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$ only one of the two lifts has trivial restriction.

Similarly, the elements $\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j\rho+\sigma}$ have trivial restriction on E_∞ and unique lifts with trivial restriction on $k^\bigstar_{C_2}(B_{C_4}\Sigma_{2+})$. □

Remark 7.4 We should explain the notation used for the generators above. First, the elements $\overline{e}^{j,u}$ and $\overline{e}^{j\rho+\lambda}$ will turn out to be the restrictions of top level elements $e^{j,u}$ and $e^{j\rho+\lambda}$ respectively, both in E_∞ and in $k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+})$, hence their notation. Second, the elements $e^{j,au}$ and $e'^{j\rho+\lambda+1}$ are never restrictions, neither in E_∞ nor in $k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+})$, so their notation is rather ad hoc: the au in $e^{j,au}$ serves as a reminder of the $\sqrt{\overline{a}_\lambda \overline{u}_\lambda}$ in $e^{j,au} = \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j\rho+\sigma}$, while the prime $'$ in $e'^{j\rho+\lambda+1}$ is used to distinguish them from the top level generators $e^{j\rho+\lambda+1}$ that the $e'^{j\rho+\lambda+1}$ transfer to. Finally, the elements $\widetilde{e}^{j,a}$ are restrictions of top level elements $e^{j,a}$ in E_∞, but not in $k^\bigstar_{C_4}(B_{C_4}\Sigma_{2+})$ due to nontrivial Mackey functor extensions (exotic restrictions). That's why we denote them by $\widetilde{e}^{j,a}$ as opposed to $\overline{e}^{j,a}$; the $\overline{e}^{j,a}$ are reserved for $\mathrm{Res}^4_2(e^{j,a}) = \widetilde{e}^{j,a} + \overline{u}_\sigma e'^{j\rho+\lambda+1}$; see Lemma 8.4.

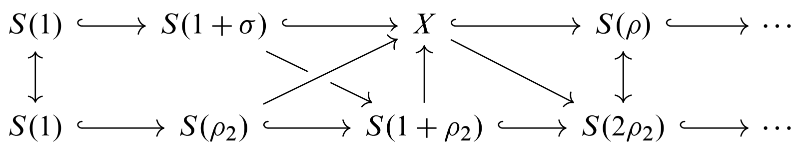
For convenience, when $j = 0$ we write $\widetilde{e}^{a}$, e^{au} and $\overline{e}^{u}$ in place of $\widetilde{e}^{0,a}$, $e^{0,au}$ and $\overline{e}^{0,u}$, respectively.

Now recall that $k^\bigstar_{C_2}(B_{C_2}\Sigma_{2+})$ is freely generated over $k^\bigstar_{C_2}$ under the elements $e^{j\rho_2}$ and $e^{j\rho_2+\sigma_2}$; see Section 4.1. We shall write our middle level C_4 generators in terms of the C_2 generators.

Proposition 7.5 *We have*

$$\begin{aligned}
\overline{e}^{j\rho} &= \overline{u}_\sigma^j e^{2j\rho_2}, & \overline{e}^{j,u} &= \overline{u}_\sigma^{j+1} u_{\sigma_2} e^{2j\rho_2+\sigma_2},\\
e^{j,au} &= \overline{u}_\sigma^{j+1} a_{\sigma_2} e^{2j\rho_2+\sigma_2}, & \widetilde{e}^{j,a} &= \overline{u}_\sigma^{j+1} a_{\sigma_2} e^{(2j+1)\rho_2},\\
\overline{e}^{j\rho+\lambda} &= \overline{u}_\sigma^j a_{\sigma_2} e^{2j\rho_2+\sigma_2} + \overline{u}_\sigma^j u_{\sigma_2} e^{(2j+1)\rho_2}, & e'^{j\rho+\lambda+1} &= \overline{u}_\sigma^j e^{(2j+1)\rho_2+\sigma_2}.
\end{aligned}$$

Proof The map $f\colon E_{C_4}\Sigma_2 \to E_{C_2}\Sigma_2$, $f(x_1,x_2,x_3,x_4,\dots) = (x_1,x_3,x_2,x_4,\dots)$ is a $C_2\times\Sigma_2$–equivariant homeomorphism and induces a map on filtrations



(The downwards arrows are f while the arrows in the opposite direction are f^{-1}.) To keep the notation tidy, we verify the correspondence of generators for $j=0$.

In the C_4 spectral sequence, we have $\widetilde{e}^a\overline{u}_\sigma^{-1}$ and $e'^{\lambda+1}$ in degree $\lambda+1$ and filtrations 1 and 3, respectively. In the C_2 spectral sequence, we have $a_{\sigma_2}e^{\sigma_2+1}$ and $e^{2\sigma_2+1}$ in the same degree and filtrations 2 and 3, respectively. The correspondence of filtrations gives

$$e'^{\lambda+1} = e^{2\sigma_2+1} \quad\text{and}\quad \widetilde{e}^a\overline{u}_\sigma^{-1} = a_{\sigma_2}e^{\sigma_2+1} + \epsilon e^{2\sigma_2+1},$$

where $\epsilon = 0,1$. Applying restriction on the second equation reveals that $\epsilon=0$ and thus $\widetilde{e}^a\overline{u}_\sigma^{-1} = a_{\sigma_2}e^{\sigma_2+1}$. The correspondence of filtrations in degrees $\lambda-1$ and λ gives

$$\overline{e}^u\overline{u}_\sigma^{-1} = \epsilon_1 a_{\sigma_2}u_{\sigma_2} + u_{\sigma_2}e^{\sigma_2}, \quad e^{au}\overline{u}_\sigma^{-1} = \epsilon_2 a_{\sigma_2}e^{\sigma_2} + \epsilon_3 u_{\sigma_2}e^{\sigma_2+1}, \quad \overline{e}^\lambda = \epsilon_4 a_{\sigma_2}e^{\sigma_2} + \epsilon_5 u_{\sigma_2}e^{\sigma_2+1},$$

where $\epsilon_i = 0,1$. Applying restriction shows that

$$\epsilon_3 = 0 \quad\text{and}\quad \epsilon_5 = 1,$$

which further forces $\epsilon_2 = 1$. Looking at degree $2\lambda+\sigma-2$ in the C_4 spectral sequence, we see that we have a relation

$$\overline{a}_\lambda\overline{e}^u = \overline{u}_\lambda\widetilde{e}^a + \epsilon_6\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\,\overline{u}_\sigma\overline{e}^\lambda + \epsilon_7\overline{u}_\sigma\overline{u}_\lambda e'^{\lambda+1},$$

where again $\epsilon_i = 0,1$. Combining the equations above we conclude that

$$\overline{a}_\lambda\overline{e}^u = \overline{u}_\lambda\widetilde{e}^a + \sqrt{\overline{a}_\lambda\overline{u}_\lambda}\,\overline{u}_\sigma\overline{e}^\lambda,$$

and

$$\overline{e}^u\overline{u}_\sigma^{-1} = u_{\sigma_2}e^{\sigma_2}, \quad e^{au}\overline{u}_\sigma^{-1} = a_{\sigma_2}e^{\sigma_2}, \quad \overline{e}^\lambda = a_{\sigma_2}e^{\sigma_2} + u_{\sigma_2}e^{\sigma_2+1}.$$

To compute the ϵ_i we used the freeness of $k_{C_2}^\star(B_{C_2}\Sigma_{2+})$ over $k_{C_2}^\star$. ◻

As a corollary we obtain the relations

$$\begin{aligned}
\overline{u}_\lambda \widetilde{e}^{j,a} &= \overline{a}_\lambda \overline{e}^{j,u} + \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{u}_\sigma \overline{e}^{j\rho+\lambda}, \\
\overline{u}_\lambda e^{j,au} &= \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^{j,u}, \\
\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,e^{j,au} &= \overline{a}_\lambda \overline{e}^{j,u}, \\
\overline{a}_\lambda e^{j,au} &= \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\widetilde{e}^{j,a} + \overline{a}_\lambda \overline{u}_\sigma \overline{e}^{j\rho+\lambda}, \\
\frac{v}{\overline{a}_\lambda^i} \overline{e}^{j\rho+\lambda} &= 0.
\end{aligned}$$

Thus, $k_{C_2}^{\bigstar}(B_{C_4}\Sigma_{2+})$ is spanned as a $k_{C_2}^{\bigstar}$–module by $\overline{e}^{j\rho}$, $\widetilde{e}^{j,a}$, $e^{j,au}$, $\overline{e}^{j,u}$, $\overline{e}^{j\rho+\lambda}$ and $e'^{j\rho+\lambda+1}$ under the relations above. The bottom level $k_e^{\bigstar}(B_{C_4}\Sigma_{2+})$ is free on the restrictions of $\overline{e}^{j\rho}$, $\overline{e}^{j,u}$, $\overline{e}^{j\rho+\lambda}$ and $e'^{j\rho+\lambda+1}$.

The C_4/C_2 (Weyl group) action is trivial: the only extensions that may arise are $g\widetilde{e}^{j,a} = \widetilde{e}^{j,a} + \epsilon e'^{j\rho+\lambda+1}$ and $ge^{j,au} = e^{j,au} + \epsilon' \overline{e}^{j\rho+\lambda}$ where $\epsilon, \epsilon' = 0, 1$; applying restriction shows that $\epsilon = \epsilon' = 0$.

The cup product structure can be understood in terms of the C_2 generators c and b of Section 4.1. As an algebra, $k_{C_2}^{\bigstar}(B_{C_4}\Sigma_{2+})$ is generated by $\widetilde{e}^a$, e^{au}, $\overline{e}^u$, $\overline{e}^\lambda$, $e'^{\lambda+1}$ and $\overline{e}^\rho$ under multiplicative relations that are implied by the correspondence of generators:

$$\begin{aligned}
&e^\rho = \overline{u}_\sigma b^2, &&\widetilde{e}^a = \overline{u}_\sigma a_{\sigma_2} b, &&e^{au} = \overline{u}_\sigma a_{\sigma_2} c, \\
&\overline{e}^u = \overline{u}_\sigma u_{\sigma_2} c, &&\overline{e}^\lambda = c^2 = a_{\sigma_2} c + u_{\sigma_2} b, &&e'^{\lambda+1} = cb.
\end{aligned}$$

Remark 7.6 The reader may notice that this description of the middle level $k_{C_2}^{\bigstar}(B_{C_4}\Sigma_{2+})$ is rather different from the one given in Proposition 5.1. Let us now explain this discrepancy. First, the relation

$$\overline{u}_\lambda e^{au} = \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^u$$

allows us to replace e^{au} by the quotient

$$\frac{\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^u}{\overline{u}_\lambda},$$

which is why e^{au} does not appear in Proposition 5.1 but $(\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^u)/\overline{u}_\lambda$ does. Second, in Lemma 8.4, we shall see that $\widetilde{e}^a + \overline{u}_\sigma e'^{\lambda+1}$ is the restriction of a top level generator e^a, which we denote by $\overline{e}^a$. We can replace the generator $\widetilde{e}^a$ by the element $\overline{e}^a$ and get the relation

$$\overline{u}_\sigma \overline{u}_\lambda e'^{\lambda+1} = \overline{u}_\lambda \overline{e}^a + \overline{a}_\lambda \overline{e}^u + \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{u}_\sigma \overline{e}^\lambda.$$

Thus we can replace the generator $e'^{\lambda+1}$ by the quotient

$$\frac{\overline{u}_\sigma^{-1} \overline{a}_\lambda \overline{e}^u + \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{e}^\lambda}{\overline{u}_\lambda},$$

which is what we do in the description of the middle level $k_{C_2}^{\bigstar}(B_{C_4}\Sigma_{2+})$ found in Proposition 5.1. For our convenience, we shall continue to use the generators $\widetilde{e}^{j,a}$, $e^{j,au}$ and $e'^{j\rho+\lambda+1}$ in the following subsections, instead of their replacements.

7.5 Top level differentials

In this subsection, we compute the top level of the E_∞ page.

From Section 7.2, we know that (the top level of) the E_2 page is generated by

$$e^{j\rho}, \quad \alpha e^{j\rho+\sigma}, \quad \overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}^{\epsilon} e^{j\rho+\lambda+\epsilon'},$$

where $i, j \geq 0$, $\epsilon, \epsilon' = 0, 1$ and $\alpha \in \mathrm{Ker}(k_\bigstar^{C_4} \xrightarrow{\mathrm{Res}_2^4} k_\bigstar^{C_2} \xrightarrow{v} k_\bigstar^{C_2})$. We also have the relation

$$ve^{j\rho+\lambda} = 0.$$

For degree reasons, the elements $e^{j\rho}$ survive the spectral sequence.

The elements $\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1}$ and $\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda} e^{j\rho+\lambda+1}$ are transfers, hence also survive (by the middle level computation of Section 7.4).

If $\alpha \in k_\bigstar^{C_4}$ is a transfer then so are the elements $\alpha e^{j\rho+\sigma}$ and thus they survive. The elements $\alpha \in k_{C_4}^\bigstar \setminus \{u_\sigma^m, m \geq 0\}$ that are not transfers can be broken into three categories:

- multiples of a_λ,
- multiples of u_λ/u_σ^i,
- $a_\sigma u_\sigma^i$.

Proposition 7.7 *The elements $a_\lambda e^{j\rho+\sigma}$ survive the spectral sequence, while the elements $a_\sigma u_\sigma^i e^{j\rho+\sigma}$ support nontrivial differentials*

$$d^2(a_\sigma u_\sigma^i e^{j\rho+\sigma}) = v\overline{u}_\sigma^{i+2} e^{j\rho+\lambda+1} \quad \text{for } i, j \geq 0.$$

Proof The elements $a_\lambda e^{j\rho+\sigma}$ can only support $d^3(a_\lambda e^{j\rho+\sigma}) = e^{(j+1)\rho}$ and applying restriction shows that this cannot happen.

Fix $j \geq 0$. For degree reasons, the only differential $a_\sigma e^{j\rho+\sigma}$ can support is $d^2(a_\sigma e^{j\rho+\sigma}) = v\overline{u}_\sigma^2 e^{j\rho+\lambda+1}$. If $a_\sigma e^{j\rho+\sigma}$ survives then it lifts to a unique element α of $k_{C_4}^{j\rho+2\sigma}(B_{C_4}\Sigma_{2+})$, while $a_\lambda e^{j\rho+\sigma}$ has two possible lifts to $k_{C_4}^{j\rho+2\sigma}(B_{C_4}\Sigma_{2+})$ that differ by $\mathrm{Tr}_2^4(u_\sigma e'^{j\rho+\lambda+1})$. Both lifts have the same restriction, which by Lemma 8.4 is computed to be $\widetilde{e}^{j,a} + \overline{u}_\sigma e'^{j\rho+\lambda+1}$ (the proof of the lemma works regardless of the survival of $a_\sigma e^{j\rho+\sigma}$). Now one of those lifts, that we shall call β, satisfies

$$\frac{a_\sigma^2}{a_\lambda}\beta = a_\sigma\alpha$$

in $k_{C_4}^{j\rho+3\sigma}(B_{C_4}\Sigma_{2+})$. Applying Res_2^4 gives that

$$\mathrm{Res}_2^4\left(\frac{a_\sigma^2}{a_\lambda}\right)\mathrm{Res}_2^4(\beta) = 0 \Longrightarrow v\overline{u}_\sigma^2(\widetilde{e}^{j,a} + \overline{u}_\sigma e'^{j\rho+\lambda+1}) = 0 \Longrightarrow v\overline{u}_\sigma^3 e'^{j\rho+\lambda+1} = 0,$$

which contradicts the computation of the module structure of the middle level. □

This differential is depicted by a dashed arrow in Figure 5, top. It does not appear in Figure 5 center and bottom.

Remark 7.8 The nonsurvival of $a_\sigma e^\sigma$ is consistent with the computation that $k_{C_4}^{2\sigma}$ has dimension 1 (spanned by $a_\sigma^2 e^0$) by the computer program of [2].

All the other elements of E_2 survive the spectral sequence:

Proposition 7.9 *The elements* $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$, $\overline{u}_\sigma^{-i}e^{j\rho+\lambda}$ *and* $\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}$ *survive the spectral sequence for* $i, j \geq 0$.

Proof We work page by page. On E_2 we have

$$d^2(\overline{u}_\sigma^{-i}e^{j\rho+\lambda}) = \epsilon_1\frac{\theta}{a_\sigma u_\sigma^{i-2}}e^{(j+1)\rho}, \quad d^2(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}) = \epsilon_2\frac{\theta}{a_\sigma^2 u_\sigma^{i-3}}a_\lambda e^{(j+1)\rho} + \epsilon_3\frac{u_\lambda}{u_\sigma^{i+1}}e^{(j+1)\rho},$$

where $\epsilon_i = 0, 1$. Multiplying by a_σ and using that $a_\sigma\overline{u}_\sigma^{-i}e^{j\rho+\lambda} = 0$ and that $a_\sigma\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda} = 0$ shows that $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$.

On E_3 we have

$$d^3(\overline{u}_\sigma^{-i}e^{j\rho+\lambda}) = \epsilon_1\frac{\theta}{a_\sigma^2 u_\sigma^{i-2}}e^{(j+1)\rho+\sigma},$$
$$d^3(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}) = \epsilon_2\frac{\theta}{a_\sigma^3 u_\sigma^{i-3}}a_\lambda e^{(j+1)\rho+\sigma},$$
$$d^3\left(\frac{u_\lambda}{u_\sigma^i}e^{j\rho+\sigma}\right) = \epsilon_3\frac{\theta}{a_\sigma^2 u_\sigma^{i-4}}e^{(j+1)\rho},$$

where again $\epsilon_i = 0, 1$. We see that $\epsilon_1 = \epsilon_2 = 0$ by multiplication with a_σ, while $\epsilon_3 = 0$ can be seen by multiplying with a_σ^2.

The pattern of higher differentials is the same as in E_2 and E_3, and the same arguments show that there are no higher differentials. □

In conclusion:

Corollary 7.10 *The* E_∞ *page is generated as a* $k_{C_4}^\bigstar$*–module by*

$$e^{j\rho}, \quad a_\lambda e^{j\rho+\sigma}, \quad (u_\lambda/u_\sigma^i)e^{j\rho+\sigma}, \quad \overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}^{\epsilon}e^{j\rho+\lambda+\epsilon'},$$

where $i, j \geq 0$ *and* $\epsilon, \epsilon' = 0, 1$. *We have relations*

$$v\overline{u}_\sigma e^{j\rho+\lambda} = v\overline{u}_\sigma^2 e^{j\rho+\lambda+1} = 0.$$

8 Lifts and extension problems

8.1 Coherent lifts

If we have a top level element $\alpha \in E_\infty^{s,V}$ and $E_\infty^{t,V}$ vanishes for $t > s$, then α lifts uniquely to $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$. If on the other hand $E_\infty^{t,V}$ does not vanish for some $t > s$, then there are multiple choices of lifts of α.

When it comes to fractions y/x, we should make sure our choices of lifts are "coherent". Let us explain what that means with an example. The element $u_\lambda e^\sigma$ has a unique lift x_0, while $(u_\lambda/u_\sigma^i)e^\sigma$ has multiple distinct lifts if $i\ge 5$. If we choose x_i to lift $(u_\lambda/u_\sigma^i)e^\sigma$ then it will always be true that $u_\sigma^i x_i = x_0$; however, we shouldn't write $x_i = x_0/u_\sigma^i$ unless we can also guarantee that

$$u_\sigma x_i = x_{i-1}.$$

This expresses the coherence of fractions (also discussed in Section 3.2 and Appendix B) which is the cancellation property,

$$u_\sigma \frac{u_\lambda}{u_\sigma^i} e^\sigma = \frac{u_\lambda}{u_\sigma^{i-1}} e^\sigma.$$

This holds on E_∞, and we also want it to hold on $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$.

One more property enjoyed by the $(u_\lambda/u_\sigma^i)e^\sigma$ is that $a_\sigma^2(u_\lambda/u_\sigma^i)e^\sigma = 0$; it turns out that there are unique lifts x_i of $(u_\lambda/u_\sigma^i)e^\sigma$ such that $a_\sigma^2 x_i = 0$, and those lifts also satisfy the coherence property $u_\sigma x_i = x_{i-1}$:

Proposition 8.1 *For $i,j\ge 0$, there are unique lifts $e^{j,u}/u_\sigma^i$ and $e^{j\rho+\lambda}/u_\sigma^i$ of the elements $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$ and $\overline{u}_\sigma^{-i}e^{j\rho+\lambda}$, respectively, that satisfy*

$$a_\sigma^2 \frac{e^{j,u}}{u_\sigma^i} = 0 \quad \text{and} \quad a_\sigma^2 \frac{e^{j\rho+\lambda}}{u_\sigma^i} = 0.$$

These lifts are also coherent.

Proof Fix $i,j\ge 0$. We first deal with lifts of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$.

- **Existence** Fix $\bigstar$ to be the RO(C_4) degree of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$ and write F^s for the decreasing filtration on $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ defining the spectral sequence, namely

$$E_\infty^{s,\bigstar} = F^s/F^{s+1}.$$

We start with any random lift $\alpha_0\in F^{4j+1}$ of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$; if $a_\sigma^2\alpha_0=0$ then we are done. Otherwise take s_0 maximal with $a_\sigma^2\alpha_0\in F^{s_0}$; since $a_\sigma^2(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}=0$ we have $s_0>4j+1$. In fact $s_0>4j+2$ since $E^{4j+2,\bigstar}=0$.

We now prove that $s_0>4j+3$: $E^{4j+3,\bigstar}$ is spanned by $\overline{u}_\sigma^{3-i}e^{j\rho+\lambda+1}$ so we need to investigate the possibility $a_\sigma^2\alpha = \overline{u}_\sigma^{3-i}e^{j\rho+\lambda+1}$ on $E^{4j+3,\bigstar}$. Multiplying by u_σ^i reduces us to the case $i=0$, where $u_\sigma^i\alpha$ is the unique lift of $u_\lambda e^{j\rho+\sigma}$. But we can see directly that $(a_\sigma^2/a_\lambda)u_\sigma^i\alpha=0$ for degree reasons, hence $a_\sigma^2u_\sigma^i\alpha=0$ as well.

As $s_0>4j+3$, we can see directly that $F^{s_0}/F^{s_0+1}=E_\infty^{s_0,\bigstar}$ is generated by an element βe^V where $\beta\in k_{C_4}^{\bigstar-V}$ is divisible by a_σ^2. If $\alpha'\in F^{s_0}$ is a lift of $(\beta/a_\sigma^2)e^V$ then $\alpha_1=\alpha_0+\alpha'$ is a lift of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$. If $a_\sigma^2\alpha_1=0$ then we are done, otherwise $a_\sigma^2\alpha_1\in F^{s_1}$ for $s_1>s_0$ so we get α_2 by the same argument as above. Since $F^s=0$ for large enough s, this inductive process will eventually end with the desired lift.

- **Uniqueness** If α and α' are two lifts of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$, then their difference is a finite sum p of elements $\beta' e^V$, where each $\beta' \in k_{C_4}^\bigstar$ is a fraction with a_σ^2 in its denominator. If $a_\sigma^2\alpha = a_\sigma^2\alpha' = 0$ then $a_\sigma^2 p = 0 \Longrightarrow a_\sigma^2\beta' = 0 \Longrightarrow \beta' = 0 \Longrightarrow p = 0$.

- **Coherence** Unfix i and let x_i be the lift of $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$ with $a_\sigma^2 x_i = 0$. Then $u_\sigma x_i$ is a lift of $(u_\lambda/u_\sigma^{i-1})e^{j\rho+\sigma}$ and $a_\sigma^2(u_\sigma x_i) = 0$, hence by uniqueness,

$$u_\sigma x_i = x_{i-1}.$$

The case of $\overline{u}_\sigma^{-i} e^{j\rho+\lambda}$ is near identical to what we did above for $(u_\lambda/u_\sigma^i)e^{j\rho+\sigma}$. The changes are as follows. First, $s_0 > 4j+2$ (instead of $s_0 > 4j+1$). Next, we can see that $s_0 > 4j+4$ if $i > 1$, and multiplying by u_σ also proves the $i = 0, 1$ cases (this replaces the argument that showed $s_0 > 4j+3$). The rest of the arguments are identical. □

8.2 Top-level generators

The elements $e^{j\rho}$ have unique lifts to $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$, which we continue to denote by $e^{j\rho}$.

On the other hand, for each $j \geq 0$ there are two possible lifts of $a_\lambda e^{j\rho+\sigma}$. There is no good way to make a unique choice at this point, so we shall write $e^{j,a}$ for either.

In this subsection we shall prove:

Proposition 8.2 *The $k_{C_4}^\bigstar$–module $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ is generated by*

$$e^{j\rho}, \quad e^{j,a}, \quad \frac{e^{j,u}}{u_\sigma^i}, \quad \frac{e^{j\rho+\lambda}}{u_\sigma^i}, \quad \text{where } i, j \geq 0.$$

By Corollary 7.10 it suffices to prove that the $k_{C_4}^\bigstar$–module generated by the elements $e^{j\rho}$, $e^{j,a}$, $e^{j,u}/u_\sigma^i$ and $e^{j\rho+\lambda}/u_\sigma^i$ contains lifts of the elements

$$\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1}, \quad \sqrt{\overline{a}_\lambda\overline{u}_\lambda}\,\overline{u}_\sigma^{-i} e^{j\rho+\lambda}, \quad \sqrt{\overline{a}_\lambda\overline{u}_\lambda}\,\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1} \quad \in E_\infty.$$

Lemma 8.3 *The elements*

$$\frac{e^{j\rho+\lambda+1}}{u_\sigma^i} := \mathrm{Tr}_2^4(\overline{u}_\sigma^{-i} e'^{j\rho+\lambda+1})$$

are coherent lifts of $\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1} \in E_\infty$. Furthermore,

$$a_\sigma \frac{e^{j\rho+\lambda}}{u_\sigma^i} = \frac{e^{j\rho+\lambda+1}}{u_\sigma^{i-1}}.$$

Proof We see directly that $\mathrm{Tr}_2^4(\overline{u}_\sigma^{-i} e'^{j\rho+\lambda+1})$ lift $\overline{u}_\sigma^{-i} e^{j\rho+\lambda+1}$ and coherence follows from the Frobenius relations.

Next, we see directly from the E_∞ page that $e^{j\rho+\lambda}$ is not in the image of the transfer Tr_2^4. Since $\mathrm{Ker}(a_\sigma) = \mathrm{Im}(\mathrm{Tr}_2^4)$ in $k_{C_4}^{\bigstar}(B_{C_4}\Sigma_{2+})$, we must have a module extension of the form

$$a_\sigma e^{j\rho+\lambda} = u_\sigma e^{j\rho+\lambda+1}.$$

By Proposition 8.1, $a_\sigma^2 e^{j\rho+\lambda}/u_\sigma^i = 0$, hence $a_\sigma e^{j\rho+\lambda}/u_\sigma^i$ is a transfer. The equation above shows that $a_\sigma e^{j\rho+\lambda}/u_\sigma^i \neq 0$, and the only way $a_\sigma e^{j\rho+\lambda}/u_\sigma^i$ can be a nonzero transfer is for $a_\sigma e^{j\rho+\lambda}/u_\sigma^i = \mathrm{Tr}_2^4(\overline{u}_\sigma^{-i+1} e'^{j\rho+\lambda+1})$. □

Before we can lift the rest of the E_∞ generators, we will need the following exotic restriction:

Lemma 8.4 *Both choices of $e^{j,a}$ have the same (exotic) restriction*

$$\mathrm{Res}_2^4(e^{j,a}) = \widetilde{e}^{j,a} + \overline{u}_\sigma e'^{j\rho+\lambda+1}.$$

Proof The two choices of $e^{j,a}$ differ by $u_\sigma e^{j\rho+\lambda+1} = \mathrm{Tr}_2^4(\overline{u}_\sigma e'^{j\rho+\lambda+1})$ hence have the same restriction. From the E_∞ page,

$$\mathrm{Res}_2^4(e^{j,a}) = \widetilde{e}^{j,a} + \epsilon\overline{u}_\sigma e'^{j\rho+\lambda+1},$$

where $\epsilon = 0, 1$. Transferring this gives

$$\mathrm{Tr}_2^4(\widetilde{e}^{j,a}) = \epsilon u_\sigma e^{j\rho+\lambda+1}.$$

Now transferring the middle level relation

$$\overline{u}_\lambda \widetilde{e}^{j,a} = \overline{a}_\lambda \overline{e}^{j,u} + \sqrt{\overline{a}_\lambda \overline{u}_\lambda}\,\overline{u}_\sigma \overline{e}^{j\rho+\lambda}$$

shows that

$$u_\lambda \,\mathrm{Tr}_2^4(\widetilde{e}^{j,a}) = a_\sigma u_\lambda e^{j\rho+\lambda}$$

and thus $\mathrm{Tr}_2^4(\widetilde{e}^{j,a}) \neq 0$, which proves $\epsilon = 1$. □

Lemma 8.5 *The elements*

$$\frac{e_{\surd}^{j\rho+\lambda}}{u_\sigma^i} := \frac{u_\lambda}{u_\sigma^{i+1}} e^{j,a} + a_\lambda \frac{e^{j,u}}{u_\sigma^{i+1}}$$

are coherent lifts of $\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda \overline{u}_\lambda}\, e^{j\rho+\lambda} \in E_\infty$.

Proof Fix $i, j \geq 0$ and fix $\bigstar$ to be the degree of the element

$$\frac{u_\lambda}{u_\sigma^i} e^{j,a} + a_\lambda \frac{e^{j,u}}{u_\sigma^i}.$$

This element is by definition in filtration $4j+1$, however its projection to $E_\infty^{4j+1,\bigstar}$ is

$$\frac{u_\lambda}{u_\sigma^i} a_\lambda e^{j\rho+\sigma} + a_\lambda \frac{u_\lambda}{u_\sigma^i} e^{j\rho+\sigma} = 0,$$

so it is actually in filtration $4j+2$. But observe that $E_\infty^{4j+2,\star}$ is generated by $\overline{u}_\sigma^{-i+1}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda}$, so it suffices to check that

$$\frac{u_\lambda}{u_\sigma^i}e^{j,a}+a_\lambda\frac{e^{j,u}}{u_\sigma^i}$$

is not 0 when projected to $E_\infty^{4j+2,\star}$. Multiplying by u_σ^i reduces us to the case $i=0$, and then

$$\begin{aligned}\operatorname{Res}_2^4(u_\lambda e^{j,a}+a_\lambda e^{j,u}) &= \overline{u}_\lambda\tilde{e}^{j,a}+\overline{u}_\sigma\overline{u}_\lambda e'^{j\rho+\lambda+1}+\overline{a}_\lambda\overline{e}^{j,u}\\ &= \overline{u}_\sigma\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^{j\rho+\lambda}+\overline{u}_\sigma\overline{u}_\lambda e'^{j\rho+\lambda+1}\end{aligned}$$

using Lemma 8.4 and the middle level computation of Section 7.4. Projecting this restriction to $E_\infty^{4j+2,\star}$ returns

$$\overline{u}_\sigma\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{e}^{j\rho+\lambda}\neq 0,$$

as desired.

Coherence of $e_{\surd}^{j\rho+\lambda}/u_\sigma^i$ follows from the coherence of u_λ/u_σ^i and $e^{j,u}/u_\sigma^i$. □

Lemma 8.6 *The elements*

$$\frac{e_{\surd}^{j\rho+\lambda+1}}{u_\sigma^i}:=\operatorname{Tr}_2^4(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e'^{j\rho+\lambda+1})$$

are coherent lifts of $\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}e^{j\rho+\lambda+1}\in E_\infty$. *Furthermore,*

$$a_\sigma\frac{e_{\surd}^{j\rho+\lambda}}{u_\sigma^i}=\frac{e_{\surd}^{j\rho+\lambda+1}}{u_\sigma^{i-1}}.$$

Proof The fact that these transfers are lifts follows from the E_∞ page; coherence follows from the Frobenius relations. We check the equality directly:

$$\begin{aligned}a_\sigma\frac{e_{\surd}^{j\rho+\lambda}}{u_\sigma^i} &= \frac{a_\sigma u_\lambda}{u_\sigma^{i+1}}e^{j,a}+a_\lambda\frac{a_\sigma e^{j,u}}{u_\sigma^{i+1}}\\ &= \operatorname{Tr}_2^4(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda})e^{j,a}+a_\lambda\operatorname{Tr}_2^4(e^{j,au}\overline{u}_\sigma^{-i})\\ &= \operatorname{Tr}_2^4(\overline{u}_\sigma^{-i}\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\tilde{e}^{j,a}+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{u}_\sigma^{-i+1}e'^{j\rho+\lambda+1}+\overline{a}_\lambda e^{j,au}\overline{u}_\sigma^{-i})\\ &= \operatorname{Tr}_2^4(\overline{a}_\lambda\overline{u}_\sigma^{-i+1}\overline{e}^{j\rho+\lambda}+\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{u}_\sigma^{-i+1}e'^{j\rho+\lambda+1})\\ &= \operatorname{Tr}_2^4(\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\overline{u}_\sigma^{-i+1}e'^{j\rho+\lambda+1})\\ &= \frac{e_{\surd}^{j\rho+\lambda+1}}{u_\sigma^{i-1}}.\end{aligned}$$

We used the middle level relation $\overline{a}_\lambda e^{j,au}=\sqrt{\overline{a}_\lambda\overline{u}_\lambda}\tilde{e}^{j,a}+\overline{a}_\lambda\overline{u}_\sigma\overline{e}^{j\rho+\lambda}$ and the fact that $\overline{u}_\sigma^{-i}\overline{e}^{j\rho+\lambda}$ is the restriction of $e^{j\rho+\lambda}/u_\sigma^i$, which follows from the same fact on E_∞. □

Lemmas 8.3, 8.5 and 8.6 combined with Corollary 7.10 prove Proposition 8.2.

8.3 Mackey functor structure

Proposition 8.7 *The Mackey functor structure of $k^\star(B_{C_4}\Sigma_{2+})$ is determined by*

$$\mathrm{Res}^4_2(e^{j\rho})=\overline{e}^{j\rho},\qquad \mathrm{Res}^4_2\left(\frac{e^{j,u}}{u_\sigma^i}\right)=\overline{e}^{j,u}\overline{u}_\sigma^{-i},\qquad \mathrm{Res}^4_2\left(\frac{e^{j\rho+\lambda}}{u_\sigma^i}\right)=\overline{e}^{j\rho+\lambda}\overline{u}_\sigma^{-i},$$

$$\mathrm{Tr}^4_2(e^{j,au}\overline{u}_\sigma^{-i})=a_\sigma\frac{e^{j,u}}{u_\sigma^{i+1}},\quad \mathrm{Tr}^4_2(e'^{j\rho+\lambda+1}\overline{u}_\sigma^{-i})=a_\sigma\frac{e^{j\rho+\lambda}}{u_\sigma^{i+1}},\qquad \mathrm{Res}^4_2(e^{j,a})=\widetilde{e}^{j,a}+\overline{u}_\sigma e'^{j\rho+\lambda+1},$$

where $i,j\geq 0$.

Proof We can see directly that there are no Mackey functor extensions for $e^{j\rho}$, $e^{j,u}/u_\sigma^i$ and $e^{j\rho+\lambda}/u_\sigma^i$. The rest were established in the previous two subsections, apart from

$$\mathrm{Tr}^4_2(e^{j,au}\overline{u}_\sigma^{-i})=a_\sigma\frac{e^{j,u}}{u_\sigma^{i+1}}.$$

To see this, recall that $a_\sigma^2(e^{j,u}/u_\sigma^i)=0$, hence $a_\sigma(e^{j,u}/u_\sigma^i)$ is a transfer. Moreover, $a_\sigma(e^{j,u}/u_\sigma^i)\neq 0$, which is seen on the E_∞ page, and the only way that $a_\sigma(e^{j,u}/u_\sigma^i)$ can be a nonzero transfer is for $a_\sigma(e^{j,u}/u_\sigma^i)=\mathrm{Tr}^4_2(e^{j,au}\overline{u}_\sigma^{-i})$. □

We did not list $\mathrm{Tr}^4_2(\widetilde{e}^{j,a})=a_\sigma e^{j\rho+\lambda}$ as this immediately follows by applying Tr^4_2 on

$$\mathrm{Res}^4_2(e^{j,a})=\widetilde{e}^{j,a}+\overline{u}_\sigma e'^{j\rho+\lambda+1}.$$

8.4 Top level module relations

With the exception of relations expressing coherence (ie $u_\sigma(e^{j,u}/u_\sigma^i)=e^{j,u}/u_\sigma^{i-1}$ and $u_\sigma(e^{j\rho+\lambda}/u_\sigma^i)=e^{j\rho+\lambda}/u_\sigma^{i-1}$), the rest of the module relations are given as follows.

Proposition 8.8 *The $k^\star_{C_4}$–module $k^\star_{C_4}(B_{C_4}\Sigma_{2+})$ is generated by*

$$e^{j\rho},\quad e^{j,a},\quad \frac{e^{j,u}}{u_\sigma^i},\quad \frac{e^{j\rho+\lambda}}{u_\sigma^i},$$

under the relations

$$\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j,u}}{u_\sigma^i}=0\quad\text{and}\quad \frac{(\theta/a_\lambda)a_\sigma}{u_\sigma^{i-2}a_\lambda^{m-1}}e^{j,a}+\frac{s}{u_\sigma^{i-1}a_\lambda^{m-2}}e^{j,u}=\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j\rho+\lambda}}{u_\sigma^i}$$

for $i,j,m\geq 0$.

Proof For $m>0$, we have the possible extensions

$$\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j,u}}{u_\sigma^i}=\sum_*\epsilon_*\frac{\theta}{a_\sigma^*u_\sigma^*a_\lambda^*}e^{*\rho+*},$$

where each $*$ denotes a nonnegative index (with different instances of $*$ being possibly different indices) and each $\epsilon_* = 0, 1$. Thus, multiplication by a_λ is an isomorphism for both sides — recall that a_λ acts invertibly on elements of the form $\theta/(a_\sigma^* u_\sigma^*)$ and a_σ^2 — which reduces us to $m = 1$. For $m = 1$ and $i > 0$ there are no extensions as there are no elements of the degree of $(a_\sigma^2/a_\lambda)(e^{j,u}/u_\sigma^i)$ in the right-hand side; in other words, $\epsilon_* = 0$ for all $*$. This establishes

$$\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j,u}}{u_\sigma^i} = 0.$$

Similarly, if $m > 0$, we have the possible extensions

$$\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j\rho+\lambda}}{u_\sigma^i} = \frac{s}{u_\sigma^{i-2}a_\lambda^{m-1}} e_{\surd}^{j\rho+\lambda} + \sum_* \epsilon_* \frac{\theta}{a_\sigma^* u_\sigma^* a_\lambda^*} e^{*\rho+*},$$

and multiplying with a_λ^m reduces us to

$$a_\sigma^2 \frac{e^{j\rho+\lambda}}{u_\sigma^i} = \sum_* \epsilon_* \frac{\theta}{a_\sigma^* u_\sigma^* a_\lambda^{*-m}} e^{*\rho+*}.$$

But

$$a_\sigma^2 \frac{e^{j\rho+\lambda}}{u_\sigma^i} = a_\sigma \frac{e^{j\rho+\lambda+1}}{u_\sigma^{i-1}} = a_\sigma \operatorname{Tr}_2^4(e'^{j\rho+\lambda+1} \overline{u_\sigma}^{-i+1}) = 0,$$

hence $\epsilon_* = 0$ for all $*$. Thus

$$\frac{a_\sigma^2}{a_\lambda^m}\frac{e^{j\rho+\lambda}}{u_\sigma^i} = \frac{s}{u_\sigma^{i-2}a_\lambda^{m-1}} e_{\surd}^{j\rho+\lambda},$$

and substituting

$$e_{\surd}^{j\rho+\lambda} = \frac{u_\lambda}{u_\sigma} e^{j,a} + a_\lambda \frac{e^{j,u}}{u_\sigma}$$

gives the desired relation. For $i = m = 1$ we get $(a_\sigma^2/a_\lambda)(e^{j\rho+\lambda+1}/u_\sigma) = 0$. □

As special cases, for $i, j, m \geq 0$ we get the relations

$$a_\sigma^2 \frac{e^{j\rho+\lambda}}{u_\sigma^i} = 0, \quad \frac{a_\sigma^2}{a_\lambda}\frac{e^{j\rho+\lambda}}{u_\sigma} = 0, \quad \frac{\frac{\theta}{a_\lambda} a_\sigma}{u_\sigma^i} e^{j,a} = \frac{a_\sigma^2}{a_\lambda}\frac{e^{j\rho+\lambda}}{u_\sigma^{i+2}}, \quad \frac{s}{a_\lambda^m} e^{j,u} = \frac{a_\sigma^2}{a_\lambda^{m+2}}\frac{e^{j\rho+\lambda}}{u_\sigma}.$$

8.5 Top level cup products

Proposition 8.9 *As a $k_{C_4}^\bigstar$ algebra, $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ is generated by e^a, e^u/u_σ^i, e^λ/u_σ^i and e^ρ.*

Proof First of all, $e^{j\rho} = (e^\rho)^j$ since there are no extensions in degree $j\rho$ (to see that $(e^\rho)^j \neq 0$, apply restriction). Let A be the algebra span of e^a, e^u/u_σ^i, e^λ/u_σ^i and e^ρ. To see that $e^{j,a} \in A$ observe

$$e^{j\rho}e^a = \epsilon a_\sigma a_\lambda e^{j\rho} + e^{j,a} + \epsilon' u_\sigma e^{j\rho+\lambda+1},$$

and since

$$e^{j\rho+\lambda+1} = \operatorname{Tr}_2^4(e'^{j\rho+\lambda+1}) = \operatorname{Tr}_2^4(e^{j\rho}e'^{\lambda+1}) = e^{j\rho}e^{\lambda+1},$$

we get that $e^{j,a} \in A$ regardless of the status of ϵ and ϵ'.

Now suppose by induction that all elements in filtration $\leq 4j$ are in A. We have that

$$e^{j\rho}\frac{e^u}{u_\sigma^i}=\cdots+\frac{e^{j,u}}{u_\sigma^i}+\sum\epsilon_*\frac{\theta}{a_\sigma^*u_\sigma^*a_\lambda^*}e^{*\rho}+\sum\epsilon_*'\frac{\theta}{a_\sigma^*u_\sigma^*a_\lambda^*}e^{*,a},$$

where $\cdots$ are in filtration $<4j+1$, hence in A. Since $e^{*\rho},e^{*,a}\in A$ for any $*\geq 0$, we get $e^{j,u}/u_\sigma^i\in A$. This establishes that everything in filtration $\leq 4j+1$ is in A.

Finally,

$$e^{j\rho}\frac{e^\lambda}{u_\sigma^i}=\cdots+\frac{e^{j\rho+\lambda}}{u_\sigma^i}+\sum\epsilon_*\frac{\theta}{a_\sigma^*u_\sigma^*a_\lambda^*}e^{*\rho}+\sum\epsilon_*'\frac{\theta}{a_\sigma^*u_\sigma^*a_\lambda^*}e^{*,a},$$

where $\cdots$ are in filtration $<4j+2$, so by the same argument $e^{j\rho+\lambda}/u_\sigma^i\in A$ as well. This completes the induction step. □

Inverting u_σ and u_λ gives

$$k^{hC_4\bigstar}[e^\rho,e^a,e^u,e^\lambda]$$

modulo relations, which is isomorphic to

$$k^{hC_4\bigstar}(B_{C_4}\Sigma_{2+})=k[a_\sigma,a_\lambda,u_\sigma^\pm,u_\lambda^\pm,w]/a_\sigma^2,\quad\text{where }|w|=1.$$

There are two possible choices for w, differing by $a_\sigma u_\sigma^{-1}$, but both work equally well for the following arguments.

Proposition 8.10 *After potentially replacing the generators e^a, e^u/u_σ^i and e^λ/u_σ^i with algebra generators in the same degrees of $k_{C_4}^{\bigstar}(B_{C_4}\Sigma_{2+})$ and satisfying the same already established relations, the localization map*

$$k_{C_4}^{\bigstar}(B_{C_4}\Sigma_{2+})\to k^{hC_4\bigstar}(B_{C_4}\Sigma_{2+})$$

is given by

$$\begin{aligned}&e^u\mapsto u_\sigma u_\lambda w, && e^\lambda\mapsto u_\lambda w^2,\\ &e^a\mapsto u_\sigma u_\lambda w^3+u_\sigma a_\lambda w, && e^\rho\mapsto u_\sigma u_\lambda w^4+a_\sigma u_\lambda w^3+u_\sigma a_\lambda w^2+a_\sigma a_\lambda w.\end{aligned}$$

Proof Using the C_2 result (see Section 4.1), we have the correspondence on the middle level generators:

- $\overline{e}^u\mapsto\overline{u}_\sigma\overline{u}_\lambda w$,
- $e^\lambda\mapsto\overline{u}_\lambda w^2$,
- $\mathrm{Res}_2^4(e^a)\mapsto\overline{u}_\sigma(\overline{u}_\lambda w^3+\overline{a}_\lambda w)$,
- $\overline{e}^\rho\mapsto\overline{u}_\sigma(\overline{a}_\lambda w^2+\overline{u}_\lambda w^4)$,

from which we can deduce that the correspondence on top level is

- $e^u\mapsto u_\sigma u_\lambda w+\epsilon_1 a_\sigma u_\lambda$,
- $e^\lambda\mapsto u_\lambda w^2+\epsilon_2 a_\sigma u_\sigma^{-1}u_\lambda w$,
- $e^a\mapsto u_\sigma u_\lambda w^3+\epsilon_3 a_\sigma u_\lambda w^2+u_\sigma a_\lambda w$,
- $e^\rho\mapsto u_\sigma u_\lambda w^4+\epsilon_4 a_\sigma u_\lambda w^3+u_\sigma a_\lambda w^2+\epsilon_5 a_\sigma a_\lambda w$,

where the ϵ_i range in $0,1$.

We may add $\epsilon_1 a_\sigma u_\lambda / u_\sigma^i$ to e^u/u_σ^i to force $\epsilon_1 = 0$; we may add $\epsilon_2 a_\sigma e^u / u_\sigma^{i+2}$ to e^λ / u_σ^i to force $\epsilon_2 = 0$, and we may add $\epsilon_3 a_\sigma e^\lambda$ to e^a to force $\epsilon_3 = 0$.

It remains to prove that $\epsilon_4 = \epsilon_5 = 1$. This is a computation based on the Bockstein homomorphism $\beta\colon k_{C_4}^\bigstar(X) \to k_{C_4}^{\bigstar+1}(X)$. For $X = S^0$ we have

$$\beta(a_\sigma) = \beta(a_\lambda) = \beta(u_\lambda) = 0 \quad \text{and} \quad \beta(u_\sigma) = a_\sigma.$$

For $X = B_{C_4}\Sigma_{2+}$, we see that $\beta(e^\rho) = 0$ for degree reasons ($k_{C_4}^{\rho+1}(B_{C_4}\Sigma_{2+}) = 0$) and in the homotopy fixed points, $\beta(w) = w^2$ and $\beta(w^3) = w^4$. Thus, applying β on $e^\rho \mapsto u_\sigma u_\lambda w^4 + \epsilon_4 a_\sigma u_\lambda w^3 + u_\sigma a_\lambda w^2 + \epsilon_5 a_\sigma a_\lambda w$ shows that $\epsilon_4 = \epsilon_5 = 1$. □

Proposition 8.11 *In $k_{C_4}^\bigstar(B_{C_4}\Sigma_{2+})$ we have the multiplicative relations*

$$\frac{e^u}{u_\sigma^i}\frac{e^u}{u_\sigma^j} = \frac{u_\lambda}{u_\sigma^{i+j-2}} e^\lambda, \qquad \frac{e^\lambda}{u_\sigma^i}\frac{e^u}{u_\sigma^j} = \frac{u_\lambda}{u_\sigma^{i+j}} e^a + a_\lambda \frac{e^u}{u_\sigma^{i+j}},$$

$$e^a \frac{e^u}{u_\sigma^i} = \frac{u_\lambda}{u_\sigma^{i-1}} e^\rho + a_\sigma \frac{u_\lambda}{u_\sigma^i} e^a, \qquad \frac{e^\lambda}{u_\sigma^i}\frac{e^\lambda}{u_\sigma^j} = \frac{u_\lambda}{u_\sigma^{i+j+1}} e^\rho + a_\sigma \frac{u_\lambda}{u_\sigma^{i+j+2}} e^a + a_\lambda \frac{e^\lambda}{u_\sigma^{i+j}},$$

$$e^a \frac{e^\lambda}{u_\sigma^i} = \frac{e^u}{u_\sigma^{i+1}} e^\rho + a_\sigma \frac{u_\lambda}{u_\sigma^{i+1}} e^\rho, \qquad (e^a)^2 = u_\sigma e^\lambda e^\rho + a_\sigma \frac{e^u}{u_\sigma} e^\rho + u_\sigma a_\lambda e^\rho + a_\sigma a_\lambda e^a.$$

Proof First,

$$\frac{e^u}{u_\sigma^i}\frac{e^u}{u_\sigma^j} = \epsilon_0 a_\lambda \frac{u_\lambda}{u_\sigma^{i+j-2}} + \epsilon_1 a_\sigma \frac{u_\lambda}{u_\sigma^{i+j}} e^u + \epsilon_2 \frac{u_\lambda}{u_\sigma^{i+j-2}} e^\lambda + \cdots,$$

where $\epsilon_i = 0, 1$ and $\cdots$ is the sum of elements mapping to 0 in homotopy fixed points, but all having denominator a_σ^2. Mapping to homotopy fixed points shows $\epsilon_0 = \epsilon_1 = 0$ and $\epsilon_2 = 1$, while multiplying by a_σ^2 trivializes the LHS (by $a_\sigma^2(e^u/u_\sigma^i) = 0$) and thus shows that $\cdots = 0$.

The same argument applied to

$$\frac{e^\lambda}{u_\sigma^i}\frac{e^u}{u_\sigma^j} = \epsilon_0 \frac{\theta a_\lambda^2}{a_\sigma u_\sigma^{i+j-2}} + \epsilon_1 a_\sigma a_\lambda \frac{u_\lambda}{u_\sigma^{i+j}} + \epsilon_2 a_\lambda \frac{e^u}{u_\sigma^{i+j}} + \epsilon_3 \frac{u_\lambda}{u_\sigma^{i+j}} e^a + \epsilon_4 a_\sigma \frac{u_\lambda}{u_\sigma^{i+j}} e^\lambda + \cdots$$

shows that

$$\frac{e^\lambda}{u_\sigma^i}\frac{e^u}{u_\sigma^j} = \epsilon_0 \frac{\theta a_\lambda^2}{a_\sigma u_\sigma^{i+j-2}} + a_\lambda \frac{e^u}{u_\sigma^{i+j}} + \frac{u_\lambda}{u_\sigma^{i+j}} e^a.$$

There are two ways to show that $\epsilon_0 = 0$: the first is to multiply with $a_\sigma u_\sigma^{i+j-2}$ and compute $a_\sigma e^\lambda (e^u/u_\sigma^2)$ using $a_\sigma e^\lambda = \mathrm{Tr}_2^4(e'^{\lambda+1})$ together with the Frobenius relation and our knowledge of the multiplicative structure of the middle level from Section 7.4. The alternative is to observe that in the spectral sequence, if a and b live in filtrations $\geq n$ then so does ab. Before the modifications to the generators done in the proof of Proposition 8.10, $e^u/u_\sigma^i, e^a$ were in filtration ≥ 1 and e^λ/u_σ^i were in filtration ≥ 2. Thus, with the original generators, the extension for $e^\lambda e^u/u_\sigma^2$ does not involve the filtration 0 term $a_\lambda^2\theta/a_\sigma$. This is true even after performing the modifications prescribed in the proof of Proposition 8.10, since said modifications never involve terms with θ. Thus $\epsilon_0 = 0$.

Similarly we have

$$e^a\frac{e^u}{u_\sigma^i}=\epsilon_0\frac{\theta a_\lambda^2}{u_\sigma^{i-4}}+\epsilon_1\frac{\theta a_\lambda}{a_\sigma u_\sigma^{i-3}}e^a+\epsilon_2 a_\sigma a_\lambda\frac{e^u}{u_\sigma^i}+\epsilon_3 a_\sigma\frac{u_\lambda}{u_\sigma^i}e^a+\epsilon_4 a_\lambda\frac{e^\lambda}{u_\sigma^{i-2}}+\epsilon_5\frac{u_\lambda}{u_\sigma^{i-1}}e^\rho+\cdots$$

for $i\geq 3$, and mapping to homotopy fixed points and multiplying by a_σ^2 shows that

$$e^a\frac{e^u}{u_\sigma^i}=\epsilon_0\frac{\theta a_\lambda^2}{u_\sigma^{i-4}}+\epsilon_1\frac{\theta a_\lambda}{a_\sigma u_\sigma^{i-3}}e^a+a_\sigma\frac{u_\lambda}{u_\sigma^i}e^a+\frac{u_\lambda}{u_\sigma^{i-1}}e^\rho.$$

Multiplying by a_σ and using that $a_\sigma(e^u/u_\sigma^i)=\mathrm{Tr}_2^4(e^{au}\overline{u}_\sigma^{-i})$ shows that $\epsilon_1=0$. To show $\epsilon_0=0$, we use the filtration argument above.

These arguments also work with

$$\frac{e^\lambda}{u_\sigma^i}\frac{e^\lambda}{u_\sigma^j}=\epsilon_0\frac{\theta a_\lambda^2}{u_\sigma^{i+j-2}}+\epsilon_1\frac{\theta a_\lambda}{a_\sigma u_\sigma^{i+j-2}}e^a+\epsilon_2 a_\sigma a_\lambda\frac{e^u}{u_\sigma^{i+j-2}}+\epsilon_3 a_\sigma\frac{u_\lambda}{u_\sigma^{i+j+2}}e^a+\epsilon_4 a_\lambda\frac{e^\lambda}{u_\sigma^{i+j}}+\epsilon_5\frac{u_\lambda}{u_\sigma^{i+j+1}}e^\rho+\cdots,$$

$$e^a\frac{e^\lambda}{u_\sigma^i}=\epsilon_6\frac{\theta a_\lambda}{a_\sigma u_\sigma^{i-2}}e^a+\epsilon_7 a_\sigma a_\lambda\frac{e^\lambda}{u_\sigma^{i-1}}+\epsilon_8\frac{a_\sigma u_\lambda}{u_\sigma^{i+1}}e^\rho+\epsilon_9\frac{\theta a}{a_\sigma a_\lambda u_\sigma^{i-3}}e^\rho+\epsilon_{10}\frac{u_\lambda}{u_\sigma^{i+1}}e^\rho e^u+\cdots,$$

$$(e^a)^2=\epsilon_{11}a_\sigma^2 a_\lambda^2+\epsilon_{12}a_\sigma a_\lambda e^a+\epsilon_{13}u_\sigma a_\lambda e^\rho+\epsilon_{14}a_\sigma\frac{e^u}{u_\sigma}e^\rho+\epsilon_{16}u_\sigma e^\lambda e^\rho,$$

to complete the proof. □

We also have the nontrivial Bockstein

$$\beta(e^u/u_\sigma)=e^\lambda.$$

Appendix A Pictures of the spectral sequence

In this appendix, we have included 15 pictures of the E_1 page of the spectral sequence from Section 7. On each page, the three levels of the spectral sequence are drawn in three separate figures from top to bottom, using (V,s) coordinates. For notational simplicity and due to limited space, we suppress the e^V's and x, gx's from the generators. The e^V's can be recovered by looking at the filtration s (e.g. in filtration $s=4j$ we get $e^{j\rho}$) and to denote the presence of 2–dimensional vector spaces $k\{x,gx\}$ we write k^2 next to each generator. This k^2 is actually a $k[C_4/C_2]$ when considering the C_4 action; we only write k^2 in the diagrams as it is shorter, which helps with alignment.

For example, in Figure 1, top, there is an element $x_{0,1}/u_\sigma^2$ in coordinates $(5,5)$. This represents the fact that the top level of $E_1^{5,5}$ is generated by $(x_{0,1}/u_\sigma^2)e^{\rho+\sigma}$. In Figure 1, center, we have $v\overline{u}_\sigma^{-2}$ in the same

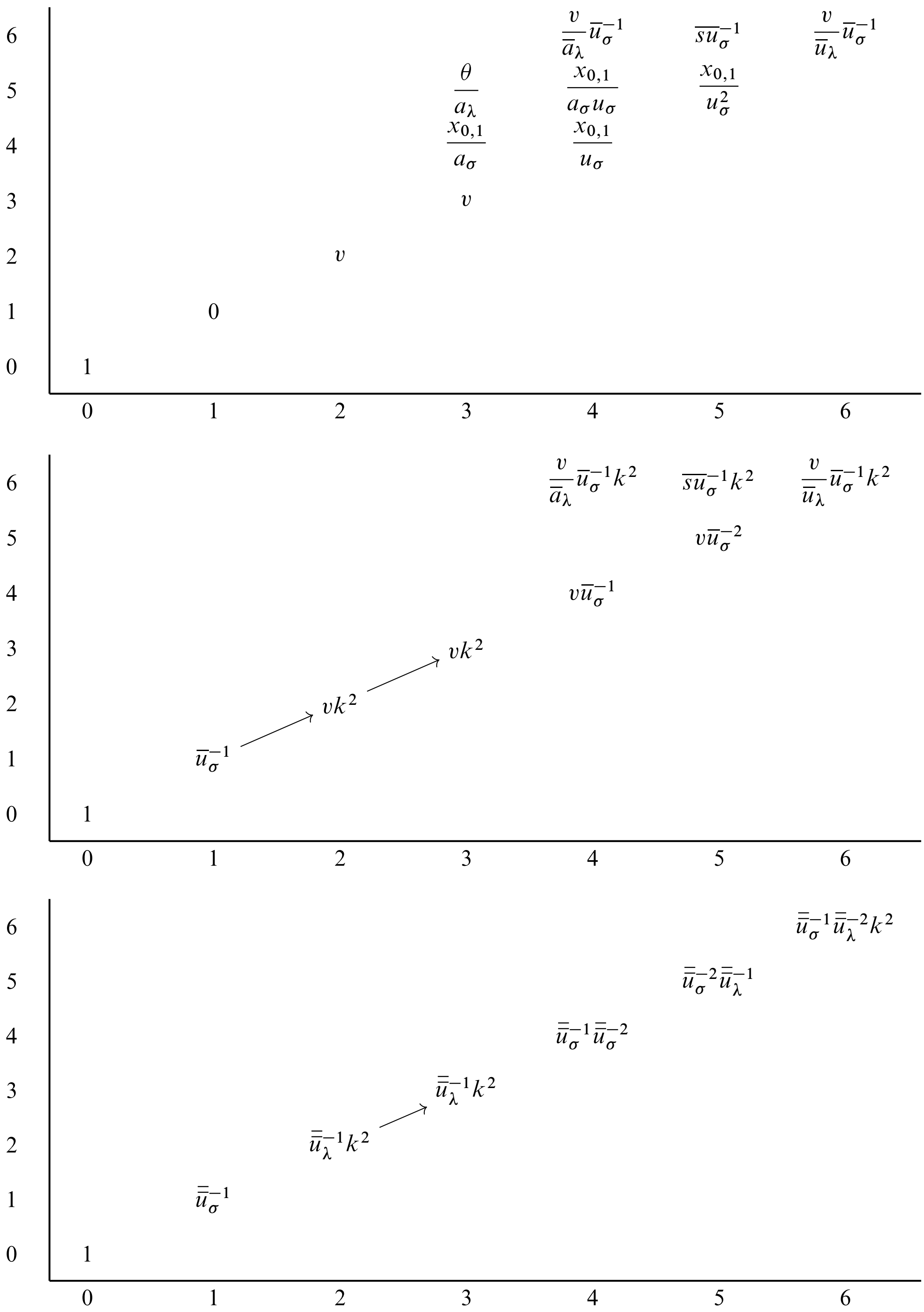


Figure 1: Top: $\underline{E}_1^{*,s}(C_4/C_4)$, $* \in \mathbb{Z}$. Center: $\underline{E}_1^{*,s}(C_4/C_2)$, $* \in \mathbb{Z}$. Bottom: $\underline{E}_1^{*,s}(C_4/e)$, $* \in \mathbb{Z}$.

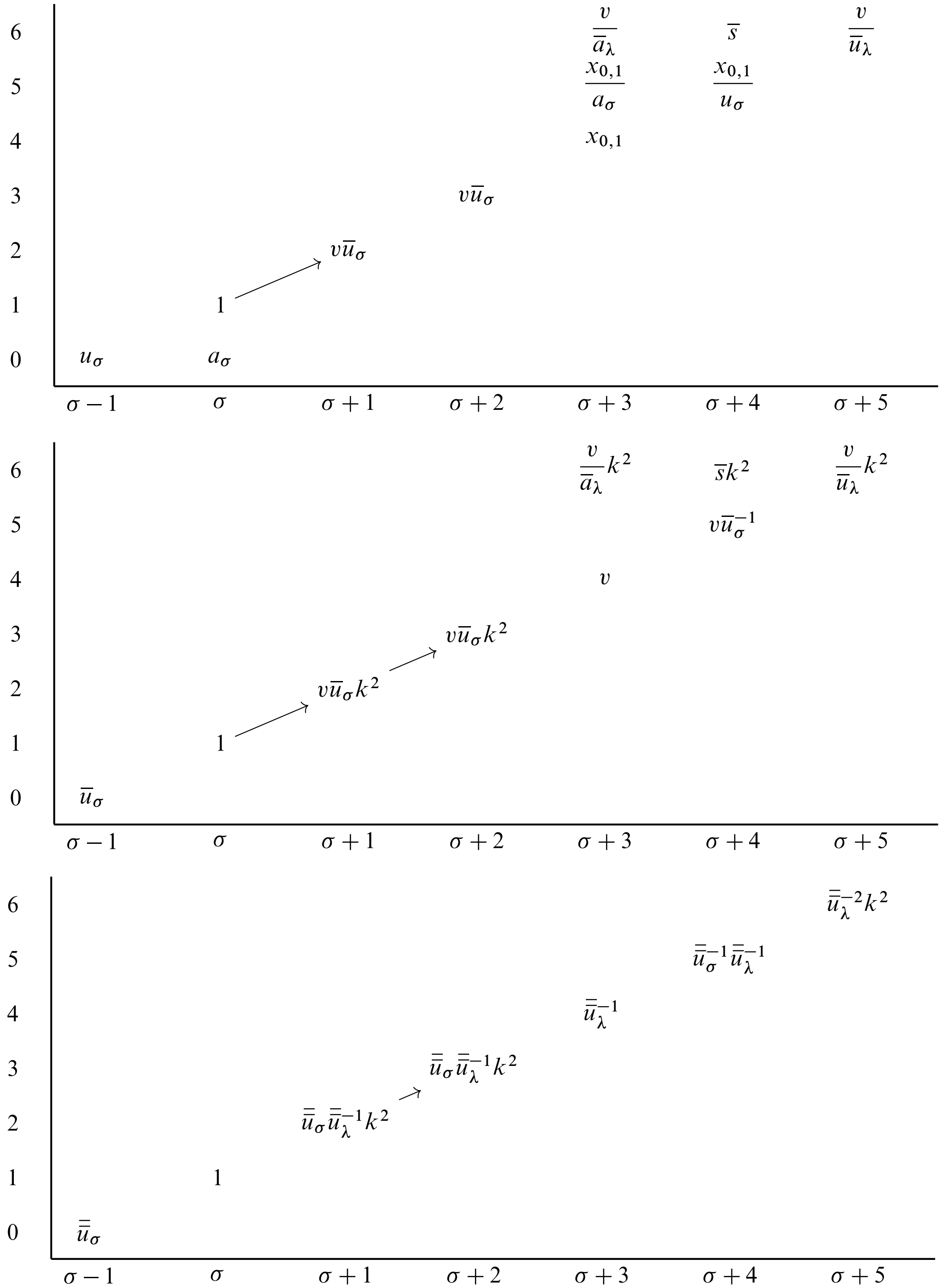


Figure 2: Top: $\underline{E}_1^{\sigma+*,s}(C_4/C_4)$, $*\in\mathbb{Z}$. Center: $\underline{E}_1^{\sigma+*,s}(C_4/C_2)$, $*\in\mathbb{Z}$. Bottom: $\underline{E}_1^{\sigma+*,s}(C_4/e)$, $*\in\mathbb{Z}$.

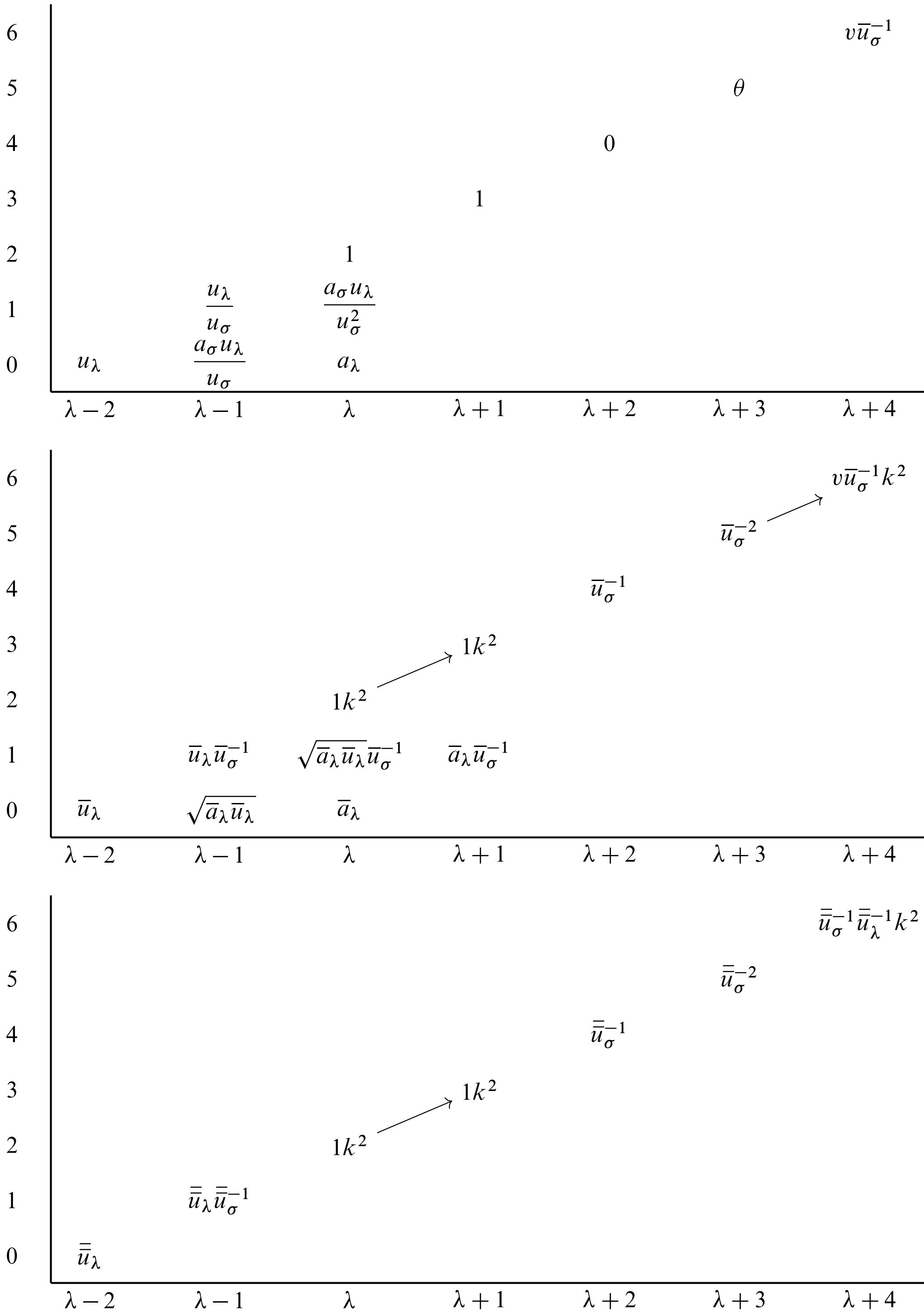


Figure 3: Top: $\underline{E}_1^{\lambda+*,s}(C_4/C_4)$, $*\in\mathbb{Z}$. Center: $\underline{E}_1^{\lambda+*,s}(C_4/C_2)$, $*\in\mathbb{Z}$. Bottom: $\underline{E}_1^{\lambda+*,s}(C_4/e)$, $*\in\mathbb{Z}$.

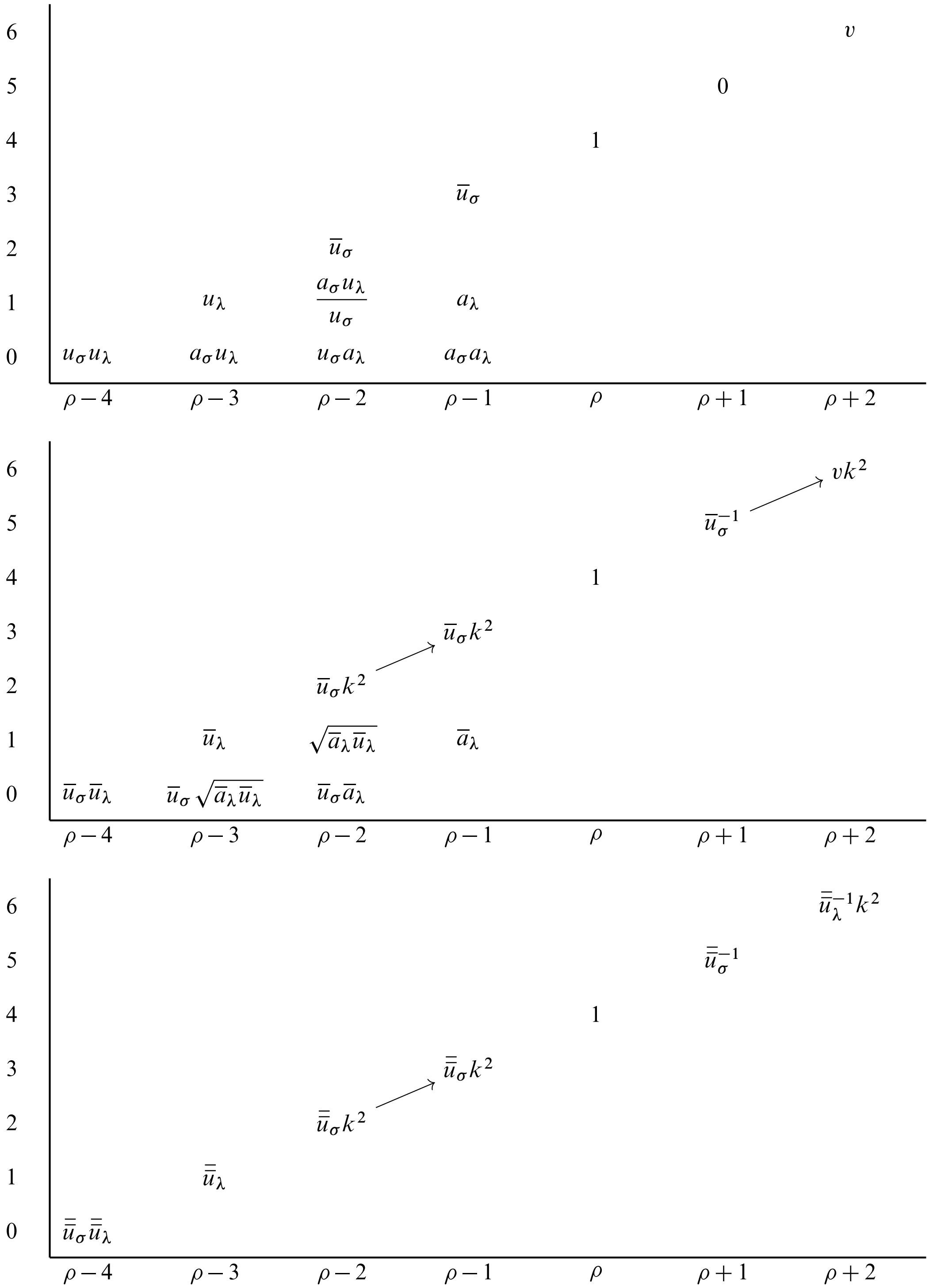


Figure 4: Top: $\underline{E}_1^{\rho+*,s}(C_4/C_4)$, $* \in \mathbb{Z}$. Center: $\underline{E}_1^{\rho+*,s}(C_4/C_2)$, $* \in \mathbb{Z}$. Bottom: $\underline{E}_1^{\rho+*,s}(C_4/e)$, $* \in \mathbb{Z}$.

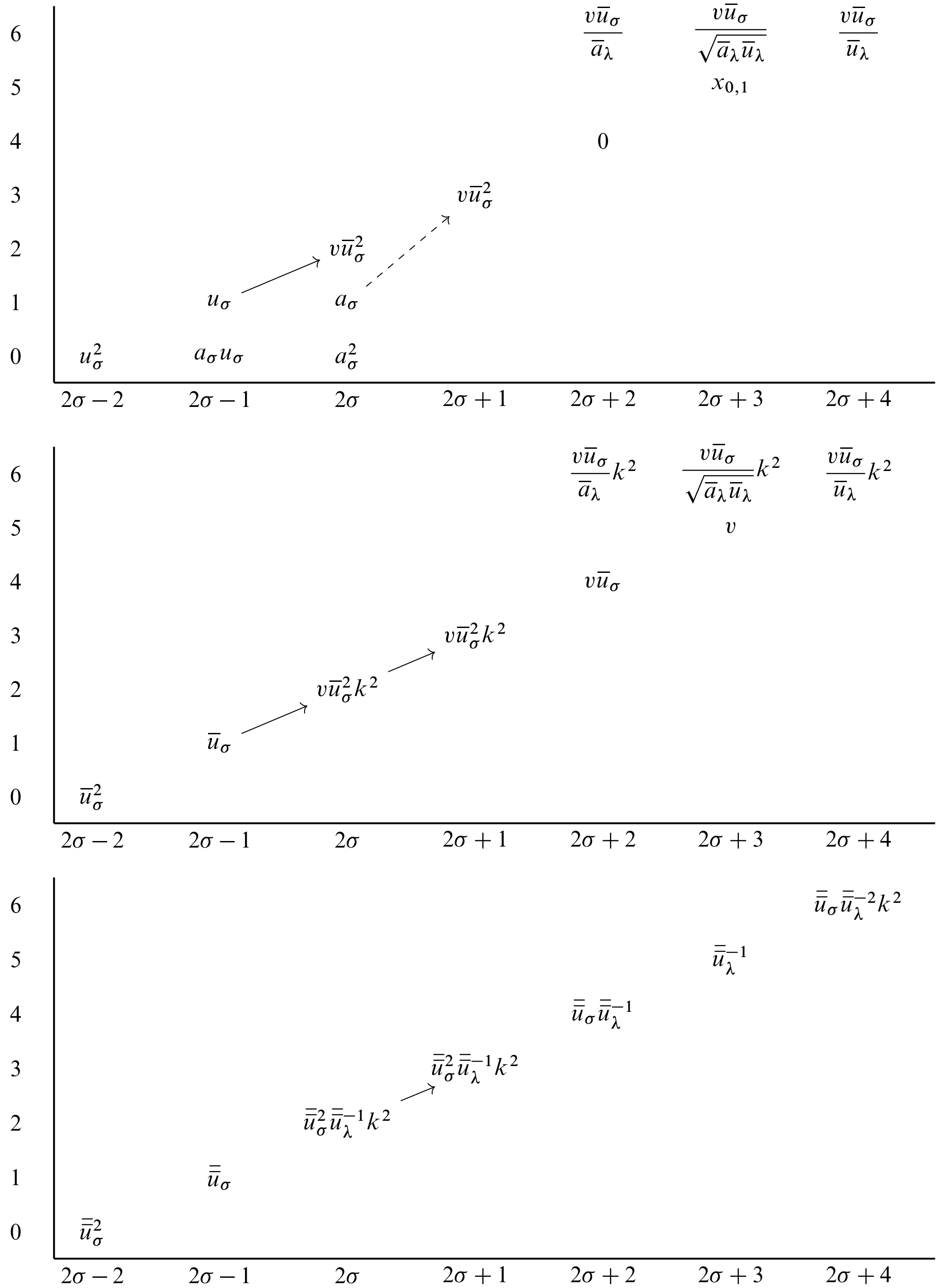


Figure 5: Top: $\underline{E}_1^{2\sigma+*,s}(C_4/C_4)$, $* \in \mathbb{Z}$. Center: $\underline{E}_1^{2\sigma+*,s}(C_4/C_2)$, $* \in \mathbb{Z}$. Bottom: $\underline{E}_1^{2\sigma+*,s}(C_4/e)$, $* \in \mathbb{Z}$.

coordinates, meaning that the middle level of $E_1^{5,5}$ is generated by $(v\overline{u}_\sigma^{-2})\overline{e}^{\rho+\sigma}$. We have

$$\mathrm{Tr}_2^4(v\overline{u}_\sigma^{-2}\overline{e}^{\rho+\sigma}) = \frac{x_{0,1}}{u_\sigma^2}e^{\rho+\sigma}$$

In the top, middle and bottom graphs of Figure 1, if we look at coordinates $(2,2)$ we see v, vk^2 and $\overline{\overline{u}}_\lambda^{-1}k^2$, respectively. This represents the fact that the three levels of $E_1^{2,2}$ are generated by $ve^\lambda(x+gx)$ for the top, $v\overline{e}^\lambda x, v\overline{e}^\lambda gx$ for the middle, and $\overline{\overline{u}}_\lambda^{-1}\overline{\overline{e}}^\lambda x, \overline{\overline{u}}_\lambda^{-1}\overline{\overline{e}}^\lambda gx$ for the bottom level. We have:

$$\mathrm{Tr}_2^4(v\overline{e}^\lambda x) = ve^\lambda(x+gx)$$

These pictures are all obtained automatically by the computer program of [2], available on the author's GitHub page.

Appendix B The $\mathrm{RO}(C_4)$ homology of a point in $\mathbb{F}_2$ coefficients

In this appendix, we write down the detailed computation of $k_\bigstar$ for $\bigstar \in \mathrm{RO}(C_4)$. We use the following notation for Mackey functors (compare with [2]):

$$k = \begin{array}{c} k \\ {}_1\!\downarrow\uparrow_0 \\ k \\ {}_1\!\downarrow\uparrow_0 \\ k \end{array} \qquad k_- = \begin{array}{c} 0 \\ \downarrow\uparrow \\ k \\ {}_1\!\downarrow\uparrow_0 \\ k \end{array} \qquad \langle k\rangle = \begin{array}{c} k \\ \downarrow\uparrow \\ 0 \\ \downarrow\uparrow \\ 0 \end{array} \qquad \overline{\langle k\rangle} = \begin{array}{c} 0 \\ \downarrow\uparrow \\ k \\ \downarrow\uparrow \\ 0 \end{array}$$

$$L = \begin{array}{c} k \\ {}_0\!\downarrow\uparrow_1 \\ k \\ {}_0\!\downarrow\uparrow_1 \\ k \end{array} \qquad p^*L = \begin{array}{c} k \\ {}_0\!\downarrow\uparrow_1 \\ k \\ {}_1\!\downarrow\uparrow_0 \\ k \end{array} \qquad Q = \begin{array}{c} k \\ {}_0\!\downarrow\uparrow_1 \\ k \\ \downarrow\uparrow \\ 0 \end{array} \qquad Q^\sharp = \begin{array}{c} k \\ {}_1\!\downarrow\uparrow_0 \\ k \\ \downarrow\uparrow \\ 0 \end{array}$$

$$L^\sharp = \begin{array}{c} k \\ {}_1\!\downarrow\uparrow_0 \\ k \\ {}_0\!\downarrow\uparrow_1 \\ k \end{array} \qquad k_-^\flat = \begin{array}{c} 0 \\ \downarrow\uparrow \\ 0k \\ {}_0\!\downarrow\uparrow_1 \\ k \end{array}$$

Our notation corresponds to that used in [9] according to Table 2 below. Note that [9] is concerned with the integral homology of a point, while we are working over the characteristic 2. As a result, in this table, the Mackey functors of [9] are levelwise tensored with $k=\mathbb{F}_2$, an operation we denote by $-\otimes k$. We also note that L is dual to k and $k_-^\flat$ is dual to k_-.

Henceforth $n, m \geq 0$. We employ the notation $a \mid b \mid c$ to denote the generators of all three levels of a Mackey functor, from top to bottom, used in [2].

notation in [9]	notation in [2]
$\square \otimes k$	k
$\overline{\square} \otimes k$	k_-
$\bullet \otimes k$	$\langle k \rangle$
$\overline{\bullet} \otimes k$	$\overline{\langle k \rangle}$
$\text{◪} \otimes k = \dot{\text{⬕}} \otimes k$	L
$\text{⬕} \otimes k = \dot{\square} \otimes k$	p^*L
$\blacktriangledown \otimes k$	Q
$\circ \otimes k = \blacktriangle \otimes k$	$Q^\sharp$
$\text{⬒} \otimes k$	$L^\sharp$
$\overline{\text{⬕}} \otimes k$	$k_-^\flat$

Table 2

B.1 $k_* S^{n\sigma + m\lambda}$

We have

$$k_*(S^{n\sigma+m\lambda}) = \begin{cases} k & \text{if } * = n+2m, \\ Q^\sharp & \text{if } n \le * < n+2m \text{ and } *-n \text{ is even}, \\ Q & \text{if } n+1 \le * < n+2m \text{ and } *-n \text{ is odd}, \\ \langle k \rangle & \text{if } 0 \le * < n. \end{cases}$$

Moreover,

- $u_\sigma^n u_\lambda^m \mid \overline{u}_\sigma^n \overline{u}_\lambda^m \mid \overline{\overline{u}}_\sigma^n \overline{\overline{u}}_\lambda^m$ generates $k_{n+2m} = k$,
- $u_\sigma^n a_\lambda^{m-i} u_\lambda^i \mid \overline{u}_\sigma^n \overline{a}_\lambda^{m-i} \overline{u}_\lambda^i \mid 0$ generates $k_{n+2i} = Q^\sharp$ for $0 \le i < m$,
- $a_\sigma u_\sigma^{n-1} a_\lambda^{m-i} u_\lambda^i \mid \overline{u}_\sigma^n \overline{a}_\lambda^{m-i} \overline{u}_\lambda^{i-1} \sqrt{\overline{a}_\lambda \overline{u}_\lambda} \mid 0$ generates $k_{n+2i-1} = Q$ for $1 \le i \le m$, $n > 0$,
- $\dfrac{a_\sigma a_\lambda^{m-i} u_\lambda^i}{u_\sigma} \Big| \overline{a}_\lambda^{m-i} \overline{u}_\lambda^{i-1} \sqrt{\overline{a}_\lambda \overline{u}_\lambda} \mid 0$ generates $k_{2i-1} = Q$ for $1 \le i \le m$, $n = 0$,
- $a_\sigma^{n-i} u_\sigma^i a_\lambda^m \mid 0 \mid 0$ generates $k_i = \langle \mathbb{Z}/2 \rangle$ for $0 \le i < n$.

B.2 $k_* S^{-n\sigma - m\lambda}$

If n and m are not both 0:

$$k_*(S^{-n\sigma-m\lambda}) = \begin{cases} L & \text{if } * = -n-2m \text{ and } m \ne 0, \\ p^*L & \text{if } * = -n-2m \text{ and } n > 1, m = 0, \\ k_- & \text{if } * = -1 \text{ and } n = 1, m = 0, \\ Q^\sharp & \text{if } -n-2m < * < -n-1 \text{ and } *+n \text{ is odd}, \\ Q & \text{if } -n-2m < * < -n-1 \text{ and } *+n \text{ is even}, \\ \langle k \rangle & \text{if } -n-1 \le * < -1 \text{ and } m \ne 0, \\ \langle k \rangle & \text{if } -n+1 \le * < -1 \text{ and } m = 0. \end{cases}$$

Moreover,

- $\mathrm{Tr}_1^4\left(\frac{1}{\overline{\overline{u}}_\sigma^n\overline{\overline{u}}_\lambda^m}\right) \Big| \mathrm{Tr}_1^2\left(\frac{1}{\overline{\overline{u}}_\sigma^n\overline{\overline{u}}_\lambda^m}\right) \Big| \frac{1}{\overline{\overline{u}}_\sigma^n\overline{\overline{u}}_\lambda^m}$ generates $k_{-n-2m} = L$ for $m \neq 0$,
- $\frac{\theta}{u_\sigma^{n-2}} \Big| \overline{u}_\sigma^{-n} \mid \overline{\overline{u}}_\sigma^{-n}$ generates $k_{-n} = p^*L$ for $m = 0, n \geq 2$,
- $0 \mid \overline{u}_\sigma^{-1} \mid \overline{\overline{u}}_\sigma^{-1}$ generates $k_{-1} = k_-$ for $n = 1$, $m = 0$,
- $\frac{s}{u_\sigma^n a_\lambda^{i-2} u_\lambda^{m-i}} \Big| \frac{\overline{s}}{\overline{u}_\sigma^n \overline{a}_\lambda^{i-2} \overline{u}_\lambda^{m-i}} \Big| 0$ generates $k_{-n-2m+2i-3} = Q^\sharp$ for $2 \leq i \leq m$,
- $\frac{x_{0,1}}{u_\sigma^n a_\lambda^{m-i} u_\lambda^{i-1}} \Big| \frac{v}{\overline{u}_\sigma^n \overline{a}_\lambda^{m-i} \overline{u}_\lambda^{i-1}} \Big| 0$ generates $k_{-n-2i} = Q$ for $1 \leq i < m$,
- $\frac{x_{0,1}}{a_\sigma^{n-i} u_\sigma^i a_\lambda^{m-1}} \Big| 0 \mid 0$ generates $k_{-i-2} = \langle k\rangle$ for $0 \leq i \leq n-1$, $m \neq 0$,
- $\frac{\theta}{a_\sigma^{n-i} u_\sigma^{n-2}} \Big| 0 \mid 0$ generates $k_{-i} = \langle k\rangle$ for $2 \leq i < n$, $m = 0$.

B.3 $k_* S^{m\lambda - n\sigma}$

If n, m are both nonzero,

$$k_*(S^{m\lambda-n\sigma}) = \begin{cases} \langle k\rangle & \text{if } 2m-n < * \leq -2, \\ k & \text{if } * = 2m-n \geq -1, \\ \langle k\rangle \oplus k & \text{if } * = 2m-n \leq -2, \\ Q & \text{if } -1 \leq * < 2m-n \text{ and } *+n \text{ is odd}, \\ Q^\sharp & \text{if } -1 \leq * < 2m-n \text{ and } *+n \text{ is even}, \\ \langle k\rangle \oplus Q & \text{if } -n+1 \leq * < 2m-n \text{ and } *+n \text{ is odd, and } * \leq -2, \\ \langle k\rangle \oplus Q^\sharp & \text{if } -n+2 \leq * < 2m-n \text{ and } *+n \text{ is even, and } * \leq -2, \\ Q & \text{if } * = -n \text{ and } n \geq 2, \\ \overline{\langle k\rangle} & \text{if } * = -1 \text{ and } n = 1. \end{cases}$$

Moreover,

- $\frac{u_\lambda^m}{u_\sigma^n} \Big| \frac{\overline{u}_\lambda^m}{\overline{u}_\sigma^n} \Big| \frac{\overline{\overline{u}}_\lambda^m}{\overline{\overline{u}}_\sigma^n}$ generates the k in k_{2m-n},
- $\frac{a_\lambda^i u_\lambda^{m-i}}{u_\sigma^n} \Big| \frac{\overline{a}_\lambda^i \overline{u}_\lambda^{m-i}}{\overline{u}_\sigma^n} \Big| 0$ generates the $Q^\sharp$ in $k_{2m-n-2i}$ for $0 < i < m$,
- $\frac{a_\sigma a_\lambda^i u_\lambda^{m-i}}{u_\sigma^{n+1}} \Big| \frac{\sqrt{\overline{a}_\lambda \overline{u}_\lambda} \overline{a}_\lambda^i \overline{u}_\lambda^{m-i-1}}{\overline{u}_\sigma^n} \Big| 0$ generates the Q in $k_{2m-n-2i-1}$ for $0 \leq i < m$,
- $\frac{\theta a_\lambda^m}{a_\sigma^{n-i} u_\sigma^{i-2}} \Big| 0 \mid 0$ generates the $\langle k\rangle$ in k_{-i} for $2 \leq i < n$,

- $\dfrac{\theta a_\lambda^m}{u_\sigma^{n-2}} \,\Big|\, \overline{a}_\lambda^m \overline{u}_\sigma^{-n} \mid 0$ generates $k_{-n} = Q$ for $n \geq 2$,
- $0 \mid \overline{u}_\sigma^{-1} \overline{a}_\lambda^m \mid 0$ generates $k_{-1} = \overline{\langle k \rangle}$ for $n = 1$.

B.4 $k_* S^{n\sigma - m\lambda}$

If n, m are both nonzero,

$$k_*(S^{n\sigma-m\lambda}) = \begin{cases} Q^\sharp & \text{if } * = n-2 \text{ and } n, m \geq 2, \\ \overline{\langle k \rangle} & \text{if } * = -1 \text{ and } n = 1, m \geq 2, \\ \langle k \rangle \oplus Q & \text{if } n-2m < * < n-2 \text{ and } *+n \text{ is even, and } * \geq 0, \\ \langle k \rangle \oplus Q^\sharp & \text{if } n-2m < * < n-2 \text{ and } *+n \text{ is odd, and } * \geq 0, \\ Q & \text{if } n-2m < * < n-2 \text{ and } *+n \text{ is even, and } * < 0, \\ Q^\sharp & \text{if } n-2m < * < n-2 \text{ and } *+n \text{ is odd, and } * < 0, \\ L \oplus \langle k \rangle & \text{if } * = n-2m \text{ and } n-2m \geq 0 \text{ and } m \geq 2, \\ L & \text{if } * = n-2m \text{ and } n-2m < 0 \text{ and } m \geq 2, \\ L^\sharp & \text{if } * = n-2 \text{ and } n > 1 \text{ and } m = 1, \\ k_-^\flat & \text{if } * = -1 \text{ and } n = 1 \text{ and } m = 1, \\ \langle k \rangle & \text{if } 0 \leq * < n-2m. \end{cases}$$

Moreover,

- $\dfrac{a_\sigma^2 u_\sigma^{n-2}}{a_\lambda^m} \,\Big|\, \dfrac{v\overline{u}_\sigma^n}{\overline{a}_\lambda^{m-1}} \,\Big|\, 0$ generates $k_{n-2} = Q^\sharp$ for $n, m \geq 2$,
- $0 \,\Big|\, \dfrac{v\overline{u}_\sigma}{\overline{a}_\lambda^{m-1}} \,\Big|\, 0$ generates $k_{n-2} = \overline{\langle k \rangle}$ for $n = 1, m \geq 2$,
- $\dfrac{x_{0,2} u_\sigma^n}{a_\lambda^{i-1} u_\lambda^{m-i-1}} \,\Big|\, \dfrac{v\overline{u}_\sigma^n}{\overline{a}_\lambda^{i-1} \overline{u}_\lambda^{m-i}} \,\Big|\, 0$ generates the Q in $k_{n-2m+2i-2}$ for $2 \leq i \leq m-1$,
- $\dfrac{s u_\sigma^n}{a_\lambda^{i-2} u_\lambda^{m-i}} \,\Big|\, \dfrac{\overline{s}\overline{u}_\sigma^n}{\overline{a}_\lambda^{i-2} \overline{u}_\lambda^{m-i}} \,\Big|\, 0$ generates the $Q^\sharp$ in $k_{n-2m+2i-3}$ for $2 \leq i \leq m$,
- $\dfrac{x_{0,2} u_\sigma^n}{u_\lambda^{m-2}} \,\Big|\, \dfrac{v\overline{u}_\sigma^n}{\overline{u}_\lambda^{m-1}} \,\Big|\, \overline{\overline{u}}_\sigma^n \overline{\overline{u}}_\lambda^{-m}$ generates the L in k_{n-2m} for $m \geq 2$,
- $\dfrac{a_\sigma^2 u_\sigma^{n-2}}{a_\lambda} \,\Big|\, v\overline{u}_\sigma^n \mid \overline{\overline{u}}_\sigma^n \overline{\overline{u}}_\lambda^{-1}$ generates $k_{n-2} = L^\sharp$ for $n > 1, m = 1$,
- $0 \mid v\overline{u}_\sigma \mid \overline{\overline{u}}_\sigma \overline{\overline{u}}_\lambda^{-1}$ generates $k_{-1} = k_-^\flat$ for $n = m = 1$,
- $\dfrac{a_\sigma^i u_\sigma^{n-i}}{a_\lambda^m} \,\Big|\, 0 \mid 0$ generates the $\langle k \rangle$ in k_{n-i} for $2 < i \leq n$.

B.5 Subtleties about quotients

In this subsection, we investigate the subtleties regarding quotients y/x, similar to what we did in [2] for the integer coefficient case.

The crux of the matter is as follows: If we have $ax = y$ in $k_{\bigstar}^{C_4}$ then we can immediately conclude that $a = y/x$ as long as a is the *unique* element in its $\mathrm{RO}(C_4)$ degree satisfying $ax = y$. Unfortunately, as we can see from the detailed description of $k_{\bigstar}^{C_4}$, there are degrees $\bigstar$ for which $k_{\bigstar}^{C_4}$ is a two-dimensional vector space, generated by elements a and b both satisfying $ax = bx = y$; in this case a and b are both candidates for y/x and we need to distinguish them somehow. This is done by looking at the products of a and b with other Euler/orientation classes.

For a concrete example, take $k_{-2+4\sigma-\lambda}^{C_4}$, which is k^2 with generators a and b such that

$$u_\sigma a = u_\sigma b = \frac{u_\lambda}{u_\sigma^3},$$

so both a and b are candidates for u_λ/u_σ^4 (for degree reasons, there is a unique choice for u_λ/u_σ^3). To distinguish a and b, we use multiplication by a_σ^2: for one generator, say a, we have $a_\sigma^2 a = 0$, while for the other generator we get $a_\sigma^2 b = \theta a_\lambda$. So now

$$a_\sigma^2(a+b) = a_\sigma^2 b = \theta a_\lambda,$$

and both $a+b$ and b are candidates for $(\theta a_\lambda)/a_\sigma^2$. However, θ/a_σ^2 is defined uniquely and we insist

$$\frac{x}{z}\frac{y}{w} = \frac{xy}{zw}$$

whenever $xy \neq 0$, thus $(\theta a_\lambda)/a_\sigma^2$ is uniquely defined from

$$\frac{\theta a_\lambda}{a_\sigma^2} = \frac{\theta}{a_\sigma^2} a_\lambda.$$

Multiplying with u_σ returns 0 and as $u_\sigma b \neq 0$, we conclude that

$$a + b = \frac{\theta a_\lambda}{a_\sigma^2}.$$

Since $a_\sigma^2 a = 0$ and $a_\sigma^2 b = \theta a_\lambda$, we conclude that

$$a = \frac{u_\lambda}{u_\sigma^4} \quad \text{and} \quad b = \frac{u_\lambda}{u_\sigma^4} + \frac{\theta a_\lambda}{a_\sigma^2}.$$

More generally, we can use u_σ and a_σ multiplication to distinguish

$$\frac{a_\lambda^* u_\lambda^{*>0}}{u_\sigma^*}, \quad \frac{\theta a_\lambda^{*>0}}{a_\sigma^{*\geq 2} u_\sigma^*} \quad \text{and} \quad \frac{a_\lambda^* u_\lambda^{*>0}}{u_\sigma^*} + \frac{\theta a_\lambda^{*>0}}{a_\sigma^{*\geq 2} u_\sigma^*}.$$

Here, $* \geq 0$ is a generic index, ie the 12 total instances of $*$ can all be different; the important thing is that the $*$ indices are chosen so that these three elements are in the same $\mathrm{RO}(C_4)$–degree.

We can also distinguish between

$$\frac{a_\sigma a_\lambda^* u_\lambda^{*>0}}{u_\sigma^*},\quad \frac{\theta a_\lambda^{*>0}}{a_\sigma^{*\geq 2}u_\sigma^*}\quad\text{and}\quad \frac{a_\sigma a_\lambda^* u_\lambda^{*>0}}{u_\sigma^*}+\frac{\theta a_\lambda^{*>0}}{a_\sigma^{*\geq 2}u_\sigma^*}$$

by u_σ and a_σ multiplication, although it's easier to use that only the first of the three elements is a transfer. We distinguish

$$\frac{a_\sigma^{*\geq 2}u_\sigma^*}{a_\lambda^*},\quad \frac{x_{0,2}u_\sigma^*}{a_\lambda^* u_\lambda^*}\quad\text{and}\quad \frac{a_\sigma^{*\geq 2}u_\sigma^*}{a_\lambda^*}+\frac{x_{0,2}u_\sigma^*}{a_\lambda^* u_\lambda^*}$$

by a_λ^i multiplication (which for large enough i annihilates only the second term) and a_σ multiplication (which annihilates only the first term). We similarly distinguish

$$\frac{a_\sigma^{*\geq 2}u_\sigma^*}{a_\lambda^*},\quad \frac{s u_\sigma^*}{a_\lambda^* u_\lambda^*}\quad\text{and}\quad \frac{a_\sigma^{*\geq 2}u_\sigma^*}{a_\lambda^*}+\frac{s u_\sigma^*}{a_\lambda^* u_\lambda^*}$$

by a_λ^i and a_σ^2 multiplication.

Appendix C The C_{2^n} Borel equivariant dual Steenrod algebra

In this appendix we compute the Borel dual Steenrod algebra $(k\wedge k)_\bigstar^{hG}$ for the groups $G=C_{2^n}$ as an RO(G)–graded Hopf algebroid over the Borel homology of a point $k_\bigstar^{hG}$, where $k=H\mathbb{F}_2$. We also compare our result with the description of the Borel Steenrod algebra given in [3], which is dual to our calculation. Henceforth, $G=C_{2^n}$, with $n\geq 1$.

C.1 The Borel homology of a point

The real representation ring RO(G) is spanned by the irreducible representations 1, σ and $\lambda_{s,k}$, where σ is the 1–dimensional sign representation and $\lambda_{s,m}$ is the 2–dimensional representation given by rotation by $2\pi s(m/2^n)$ degrees for $1\leq m$ dividing 2^{n-2} and odd $1\leq s<2^n/m$. Note that 2–locally, $S^{\lambda_{s,m}}\simeq S^{\lambda_{1,m}}$ as C_{2^n}–equivariant spaces, by the s–power map. Therefore, to compute $k_\bigstar(X)$ for $\bigstar\in \mathrm{RO}(C_{2^n})$ it suffices to only consider $\bigstar$ in the span of 1, σ and $\lambda_k:=\lambda_{1,2^k}$ for $0\leq k\leq n-2$ ($\lambda_{n-1}=2\sigma$ and $\lambda_n=2$). For $V=\sigma$ or $V=\lambda_m$, denote by $a_V\in k_{-V}^{C_{2^n}}$ the Euler class induced by the inclusion of north and south poles $S^0\hookrightarrow S^V$; also denote by $u_V\in k_{|V|-V}^{C_{2^n}}$ the orientation class generating the Mackey functor $k_{|V|-V}=k$; see [8]. The orientation classes $u_V\colon k\wedge S^{|V|}\to k\wedge S^V$ are nonequivariant equivalences hence they act invertibly on $k_{hG\bigstar}$, $k_\bigstar^{hG}$ and $k_\bigstar^{tG}$.

Proposition C.1 *For $G=C_{2^n}$ and $n>1$,*

$$k_\bigstar^{hG}=k[a_\sigma,a_{\lambda_0},u_\sigma^\pm,u_{\lambda_0}^\pm,\dots,u_{\lambda_{n-2}}^\pm]/a_\sigma^2,\quad k_\bigstar^{tG}=k[a_\sigma,a_{\lambda_0}^\pm,u_\sigma^\pm,u_{\lambda_0}^\pm,\dots,u_{\lambda_{n-2}}^\pm]/a_\sigma^2,$$

while for $n=1$,

$$k_\bigstar^{hC_2}=k[a_\sigma,u_\sigma^\pm],\quad k_\bigstar^{tC_2}=k[a_\sigma^\pm,u_\sigma^\pm].$$

In all cases, $k_{hG\bigstar}=\Sigma^{-1}k_\bigstar^{tG}/k_\bigstar^{hG}$ (forgetting the ring structure) and the norm map $k_{hG\bigstar}\to k_\bigstar^{hG}$ is trivial.

Proof The homotopy fixed-point spectral sequence becomes

$$H^*(G;k)[u_\sigma^\pm, u_{\lambda_0}^\pm, \dots, u_{\lambda_{n-2}}^\pm] \Rightarrow k_\bigstar^{hG}.$$

We have $H^*(G;k) = k^*BG = k[a]/a^2 \otimes k[b]$, where $|a| = 1$ and $|b| = 2$. The spectral sequence collapses with no extensions and we can identify $a = a_\sigma u_\sigma^{-1}$ and $b = a_{\lambda_0} u_{\lambda_0}^{-1}$. Finally, $\widetilde{E}G = S^{\infty\lambda_0} = \operatorname{colim}(S^{\lambda_0} \xrightarrow{a_{\lambda_0}} S^{\lambda_0} \xrightarrow{a_{\lambda_0}} \cdots)$ so to get $k_\bigstar^{tG}$ we are additionally inverting a_{λ_0}. □

C.2 The Borel dual Steenrod algebra

We now compute the G–Borel dual Steenrod algebra

$$(k \wedge k)_\bigstar^{hG}$$

as a Hopf algebroid over $k_\bigstar^{hG}$, for $G = C_{2^n}$.

We will implicitly be completing it at the ideal generated by a_σ for $G = C_2$, and at the ideal generated by a_{λ_0} for $G = C_{2^n}$ with $n > 1$; see [10, page 373] for more details in the case of $G = C_2$. With this convention, Hu and Kriz computed the C_2–Borel dual Steenrod algebra to be

$$(k \wedge k)_\bigstar^{hC_2} = k_\bigstar^{hC_2}[\xi_i]$$

for $|\xi_i| = 2^i - 1$ (with $\xi_0 = 1$). The generators ξ_i restrict to the Milnor generators in the nonequivariant dual Steenrod algebra and

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k, \quad \epsilon(\xi_i) = 0 \quad \text{for } i \geq 1, \qquad \eta_R(a_\sigma) = a_\sigma, \qquad \eta_R(u_\sigma)^{-1} = \sum_{i=0}^{\infty} a_\sigma^{2^i-1} u_\sigma^{-2^i} \xi_i.$$

Proposition C.2 *For $G = C_{2^n}$, with $n > 1$,*

$$(k \wedge k)_\bigstar^{hG} = k_\bigstar^{hG}[\xi_i]$$

for $|\xi_i| = 2^i - 1$ restricting to the $C_{2^{n-1}}$ generators ξ_i, with

$$\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k, \quad \epsilon(\xi_i) = 0 \quad \text{for } i \geq 1,$$

$$\eta_R(a_\sigma) = a_\sigma, \quad \eta_R(a_{\lambda_0}) = a_{\lambda_0}, \quad \eta_R(u_\sigma) = u_\sigma + a_\sigma \xi_1, \quad \eta_R(u_{\lambda_m}) = u_{\lambda_m} \quad \text{for } m > 0,$$

$$\eta_R(u_{\lambda_0})^{-1} = \sum_i a_{\lambda_0}^{2^i-1} u_{\lambda_0}^{-2^i} \xi_i^2.$$

Proof The computation of $(k \wedge k)_*^{hG} = (k \wedge k)^*(BG)$ follows from the computation of $k_*^{hG} = k^*(BG) = k[a]/a^2 \otimes k[b]$ and the fact that nonequivariantly, $k \wedge k$ is a free k–module. To see that the homotopy fixed point spectral sequence for $k \wedge k$ converges strongly, let $F^i BG$ be the skeletal filtration on the Lens space $BG = S^\infty / C_{2^n}$; we can then compute directly that $\lim_i^1 (k \wedge k)^*(F^i BG) = \lim_i^1 (k[a]/a^2 \otimes k[b]/b^i) = 0$.

Thus we get $(k \wedge k)^{hG}_{\bigstar} = k^{hG}_{\bigstar}[\xi_i]$, and the diagonal Δ and augmentation ϵ are the same as in the nonequivariant case. The Euler classes a_σ and a_{λ_0} are maps of spheres so they are preserved under $\eta_{\boldsymbol{R}}$. The action of $\eta_{\boldsymbol{R}}$ on u_σ and u_{λ_0} can be computed through the right coaction on $k^{hG}_{\bigstar}$: the (completed) coaction of the nonequivariant dual Steenrod algebra on $k^*(BG) = k[a]/a^2 \otimes k[b]$ is

$$a \mapsto a \otimes 1, \quad b \mapsto \sum_i b^{2^i} \otimes \xi_i^2.$$

To verify the formula for the coaction on b we need to check that $\mathrm{Sq}^1(b) = 0$ (the alternative is $\mathrm{Sq}^1(b) = ab$). From the long exact sequence associated to $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$, we can see that the vanishing of the Bockstein on b follows from $H^2(C_{2^n}; \mathbb{Z}/4) = \mathbb{Z}/4$ for $n > 1$.

After identifying $a = a_\sigma u_\sigma^{-1}$ and $b = a_{\lambda_0} u_{\lambda_0}^{-1}$ we get the formula for $\eta_{\boldsymbol{R}}(u_{\lambda_0})$, and also that

$$\eta_{\boldsymbol{R}}(u_\sigma) = u_\sigma + \epsilon a_\sigma \xi_1,$$

where ϵ is either 0 or 1. This is equivalent to

$$\eta_{\boldsymbol{R}}(u_\sigma^{-1}) = u_\sigma^{-1} + \epsilon a_\sigma u_\sigma^{-2} \xi_1,$$

and to see that $\epsilon = 1$ we use the map $k^{hC_2} = k^{h(C_{2^n}/C_{2^{n-1}})} \to k^{hC_{2^n}}$ that sends a_σ and u_σ to a_σ and u_σ, respectively. Finally, to compute $\eta_{\boldsymbol{R}}(u_{\lambda_m})$ for $m > 0$ note that

$$k^{hC_{2^{n-m}}} = k^{hC_{2^n}/C_{2^m}} \to k^{hC_{2^n}}$$

sends a_{λ_0} and u_{λ_0} to $a_{\lambda_m} = 0$ and u_{λ_m}, respectively. □

C.3 Comparison with Greenlees's description

We now compare with the dual description given in [3].

In our notation, the G–spectrum b of [3] is $b = k^h$ and $b^V(X)$ corresponds to $(k^h)^{|V|}_G(X)$; to get $(k^h)^V_G(X)$ we need to multiply with the invertible element $u_V \in k^{hG}_{|V|-V}$. The Borel Steenrod algebra is $b^{\bigstar}_G b = (k^h)^{\bigstar}_G(k^h)$ and the Borel dual Steenrod algebra is $b^G_{\bigstar} b = (k^h)^G_{\bigstar}(k^h) = (k \wedge k)^{hG}_{\bigstar}$.

Greenlees proves that the Borel Steenrod algebra is given by the Massey–Peterson twisted tensor product (see [13]) of the nonequivariant Steenrod algebra k^*k and the Borel cohomology of a point $(k^h)^{\bigstar}_G = k^{hG}_{-\bigstar}$. The twisting has to do with the fact that the action of the Borel Steenrod algebra on $x \in (k^h)^{\bigstar}_G(X)$ is given by

$$(\theta \otimes a)(x) = \theta(ax),$$

where $\theta \in k^*k$ and $a \in k^{hG}_{\bigstar}$. The product of elements $\theta \otimes a$ and $\theta' \otimes a'$ in the Borel Steenrod algebra is not $\theta\theta' \otimes aa'$, since θ does not commute with cup products, but rather satisfies the Cartan formula

$$\theta(ab) = \sum_i \theta_i'(a)\theta_i''(b), \quad \Delta\theta = \sum_i \theta_i' \otimes \theta_i''.$$

Therefore,

$$(\theta \otimes a)(\theta' \otimes a')(x) = \theta(a\theta'(a'x)) = \sum_i \theta_i'(a)(\theta_i''\theta')(a'x),$$

so

$$(\theta \otimes a)(\theta' \otimes a') = \sum_i \theta_i'(a)(\theta_i''\theta' \otimes a') \tag{4}$$

(we have ignored signs as we are working in characteristic 2).

So the Borel Steenrod algebra is $k^*k \otimes k_\bigstar^{hG}$ with twisted algebra structured defined by (4).

Moreover, Greenlees expresses the action of k^*k on $(k^h)_G^\bigstar(X)$ in terms of the action of k^*k on the orientation classes u_V and the usual (nonequivariant) action of k^*k on $(k^h)_G^*(X) = k^*(X \wedge_G EG_+)$. This is done through the Cartan formula: if $x \in (k^h)_G^V(X)$, then $u_V^{-1}x \in (k^h)_G^{|V|}(X)$ and

$$\theta(x) = \theta(u_V u_V^{-1}x) = \sum_i \theta_i'(u_V)\theta_i''(u_V^{-1}x).$$

What remains to compute is $\theta_i'(u_V)$, namely the action of k^*k on orientation classes.

In our case, for $G = C_{2^n}$, we can see that:

Proposition C.3 *The action of k^*k on orientation classes is determined by*

$$\mathrm{Sq}^i(u_\sigma) = \begin{cases} u_\sigma & \text{if } i = 0, \\ a_\sigma & \text{if } i = 1, \\ 0 & \text{otherwise}; \end{cases} \qquad \mathrm{Sq}^i(u_{\lambda_m}) = \begin{cases} u_{\lambda_m} & \text{if } i = 0, \\ a_{\lambda_0} & \text{if } i = 2, m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof Compare with the proof of Proposition C.2. □

The twisting in the case of the Borel dual Steenrod algebra corresponds to the fact that $(k \wedge k)_\bigstar^{hG}$ is a Hopf algebroid and not a Hopf algebra; computationally this amounts to the formula for η_R of Proposition C.2.

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Department of Mathematics, University of Chicago
Chicago, IL, United States

nickgmath@gmail.com

Received: 20 July 2021 Revised: 8 July 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 1 (pages 1–594) 2024
