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Constructions stemming from nonseparating planar graphs and their Colin de Verdière invariant

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A planar graph G is said to be nonseparating if there exists an embedding of G in $\mathbb{R}^2$ such that, for any cycle $\mathscr{C} \subset G$, all vertices of $G \setminus \mathscr{C}$ are within the same connected component of $\mathbb{R}^2 \setminus \mathscr{C}$. Dehkordi and Farr classified the nonseparating planar graphs as either outerplanar graphs, subgraphs of wheel graphs, or subgraphs of elongated triangular prisms. We use maximal nonseparating planar graphs to construct examples of maximal linkless graphs and maximal knotless graphs. We show that, for a maximal nonseparating planar graph G with $n \geq 7$ vertices, the complement cG is $(n-7)$–apex. This implies that the Colin de Verdière invariant of the complement cG satisfies $\mu(cG) \leq n-4$. We show this to be an equality. As a consequence, the conjecture of Kotlov, Lovász and Vempala that, for a simple graph G, $\mu(G) + \mu(cG) \geq n-2$ is true for 2–apex graphs G for which $G - \{u, v\}$ is planar nonseparating. It also follows that complements of nonseparating planar graphs of order at least nine are intrinsically linked. We prove that the complements of nonseparating planar graphs G of order at least ten are intrinsically knotted.

57M15; 05C10

1 Introduction

All graphs in this paper are finite and simple. A graph is *intrinsically linked* (IL) if every embedding of it in $\mathbb{R}^3$ (or S^3) contains a nontrivial 2–component link. A graph is *linklessly embeddable* if it is not intrinsically linked (nIL). A graph is *intrinsically knotted* (IK) if every embedding of it in $\mathbb{R}^3$ (or S^3) contains a nontrivial knot. The combined work of Conway and Gordon [1983], Sachs [1984] and Robertson, Seymour and Thomas [Robertson et al. 1993] fully characterize IL graphs: a graph is IL if and only if it contains a graph in the Petersen family as a minor. The Petersen family consists of seven graphs obtained from the complete graph K_6 by ∇Y moves and $Y\nabla$ moves, as described in Figure 1.

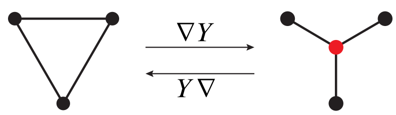


Figure 1: ∇Y and $Y\nabla$ moves.

The ∇Y move and the $Y\nabla$ move preserve the IL property. While K_7 and $K_{3,3,1,1}$ together with many other minor minimal IK graphs have been found [Goldberg et al. 2014; Conway and Gordon 1983; Foisy 2002], a characterization of IK graphs is not fully known. While the ∇Y move preserves the IK property [Motwani et al. 1988], the $Y\nabla$ move doesn't preserve it [Flapan and Naimi 2008]. A graph is said to be k–apex if it can be made planar by removing k vertices. If G and H denote two simple graphs with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$, respectively, then the *sum* $G+H$ denotes the simple graph with vertex set $V(G)\sqcup V(H)$ and edge set $E(G)\sqcup E(H)\sqcup L$, where L denotes the set of all edges with one endpoint in $V(G)$ and the other in $V(H)$.

A planar graph G is *nonseparating* if there exists an embedding of G in $\mathbb{R}^2$ such that, for any cycle $\mathscr{C}\subset G$, all vertices of $G\setminus\mathscr{C}$ are within the same connected component of $\mathbb{R}^2\setminus\mathscr{C}$. By [Dehkordi and Farr 2021], a nonseparating planar graph is one of three types:

(1) an outerplanar graph,

(2) a subgraph of a wheel,

(3) a subgraph of an elongated triangular prism.

In Section 2, we consider sums between maximal nonseparating planar graphs and small empty graphs, complete graphs or paths to construct maximal linklessly embeddable graphs and maximal knotlessly embeddable graphs. A simple graph G is called *maximal linklessly embeddable* (maxnIL) if it is not a proper subgraph of a nIL graph of the same order. A simple graph G is called *maximal knotlessly embeddable* (maxnIK) if it is not a proper subgraph of a nIK graph of the same order. Constructions and properties of maxnIL graphs can also be found in [Aires 2021; Naimi et al. 2023], and for maxnIK graphs in [Eakins et al. 2023].

Colin de Verdière [1990] introduced the graph invariant μ, which is based on spectral properties of matrices associated with the graph G. He showed that μ is monotone under taking minors and that planarity is characterized by the inequality $\mu\leq 3$. By [Lovász and Schrijver 1998; Robertson et al. 1993], it is known that linkless embeddability is characterized by the inequality $\mu\leq 4$. By reformulating the definition of μ in terms of vector labelings, Kotlov, Lovász and Vempala [Kotlov et al. 1997] related the topological properties of a graph to the μ invariant of its complement: for G a simple graph on n vertices,

(1) if G is planar, then $\mu(cG)\geq n-5$;

(2) if G is outerplanar, then $\mu(cG)\geq n-4$;

(3) if G is a disjoint union of paths, then $\mu(cG)\geq n-3$.

For G a graph with n vertices $v_1, v_2, \dots v_n$, cG denotes the *complement of* G in the complete graph K_n. The graph cG has the same set of vertices as G and $E(cG)=\{v_iv_j \mid v_iv_j\notin E(G)\}$.

By [Battle et al. 1962], the complement of a planar graph with nine vertices is not planar. This is also implied by the inequality $\mu(cG)\geq n-5$. Here we show a stronger inequality for maximal nonseparating planar graphs. In Section 3, we prove two theorems.

Theorem 1 *If G is a maximal nonseparating planar graph with $n \geq 7$ vertices, then cG is $(n-7)$–apex.*

Theorem 1 establishes the upper bound $\mu(cG) \leq n-4$ for G a maximal nonseparating planar graph, since $\mu \leq 3$ for planar graphs and adding one vertex increases the value of μ by at most one [van der Holst et al. 1999]. We prove this is an equality.

Theorem 2 *For G a maximal nonseparating planar graph with $n \geq 7$ vertices, $\mu(cG) = n-4$.*

Kotlov et al. [1997] conjectured that, for a simple graph G, $\mu(G) + \mu(cG) \geq n-2$. We revisit results about μ to show the conjecture is true for planar graphs and 1–apex graphs. As a consequence of Theorem 2, the conjecture holds for 2–apex graphs G for which $G - \{u, v\}$ is planar nonseparating. Theorem 2 also implies that, for G a maximal nonseparating planar graph with nine vertices, $\mu(cG) = 5 > 4$, and thus cG is intrinsically linked. While the relationship between the μ invariant and intrinsic linkness is well understood, the same is not true for intrinsic knottedness. The inequality $\mu(cG) \geq n-5$ for planar graphs G implies that complements of planar graphs with ten vertices are intrinsically linked. Theorem 2 establishes that, for G a maximal nonseparating planar graph with ten vertices, $\mu(cG) = 6$, but this does not imply that cG is intrinsically knotted. There are known IK graphs with $\mu = 5$ [Foisy 2003; Mattman et al. 2021], as well as nIK graphs with $\mu = 6$ [Flapan and Naimi 2008]. In Section 4, we do a case-by-case analysis to prove the following theorem:

Theorem 3 *If G is a nonseparating planar graph on ten vertices, then cG is intrinsically knotted.*

Since the complement of a nonseparating planar graph contains the complement of a maximal nonseparating planar graph of the same order as a subgraph, it suffices to prove Theorem 3 for maximal nonseparating planar graphs, namely

(1) maximal outerplanar graphs,

(2) the wheel graph,

(3) elongated triangular prisms.

A similar approach to that presented in Section 4 works to prove that:

(a) If G is a nonseparating planar graph on seven vertices, then cG is not outerplanar.

(b) If G is a nonseparating planar graph on eight vertices, then cG is nonplanar.

(c) If G is a nonseparating planar graph on nine vertices, then cG is intrinsically linked.

For outerplanar graphs G with at most nine vertices, these results can also be obtained using the graph invariant μ, since, for such graphs G, $\mu(cG) \geq n-4$ [Kotlov et al. 1997].

2 MaxnIL and maxnIK graphs

In this section, we use maximal nonseparating planar graphs to build examples of maxnIL and maxnIK graphs. Jørgensen [1989] and Dehkordi and Farr [2021] considered the class of graphs of the type $H + E_2$, where E_2 denotes the graph with two vertices and no edges and H is an elongated prism. Jørgensen proved that these graphs are maximal with no K_6 minors. Dehkordi and Farr proved that these graphs are maxnIL. Here we add to this type of example by taking the sum of maximal nonseparating planar graphs with small empty graphs, complete graphs and paths. Sachs [1984] proved that 1–apex graphs are nIL and 2–apex graphs are nIK. A theorem of Mader [1968] shows that a graph G with n vertices and $4n-9$ edges, with $n \geq 6$, contains a K_6 minor, and a graph G with n vertices and $5n-14$ edges, with $n \geq 7$, contains a K_7 minor. We combine these results into the following useful lemma:

Lemma 4 *A maximal 1–apex graph is maxnIL. A maximal 2–apex graph is maxnIK.*

A vertex of a graph H which is incident to all the other vertices of H is a *cone*. We also say that v cones over the subgraph induced by all the vertices of H minus v. Let W_n denote the wheel graph of order $n \geq 4$. Let P_2 be the graph with vertex set $V(P_2) = \{u, v, w\}$ and edge set $E(P_2) = \{\{u, w\}, \{v, w\}\}$. Let K_3 denote the complete graph on vertices $\{u, v, w\}$. Using Lemma 4, we derive the following result:

Theorem 5 (1) *The graph $G \simeq W_n + E_2$ is maxnIL.*

(2) *If H is a maximal outerplanar graph of order $n \geq 4$, then $G \simeq H + K_2$ is a maxnIL graph.*

(3) *The graph $G \simeq W_n + P_2$ is maxnIK.*

(4) *If H is a maximal outerplanar graph of order $n \geq 4$, then $G \simeq H + K_3$ is a maxnIK graph.*

Proof For the first two cases, the graph G is maximal 1–apex, and thus maxnIL. For the last two cases, the graph G is maximal 2–apex, and thus maxnIK. □

For the elongated prism case, we distinguish two cases, according to the number of nontriangular edges of the triangular prism which are subdivided.

Theorem 6 *Let H denote an elongated prism of order $n \geq 6$ obtained by repeated subdivisions of at most two of three nontriangular edges of the prism graph. Then $G \simeq H + P_2$ is a maxnIK graph.*

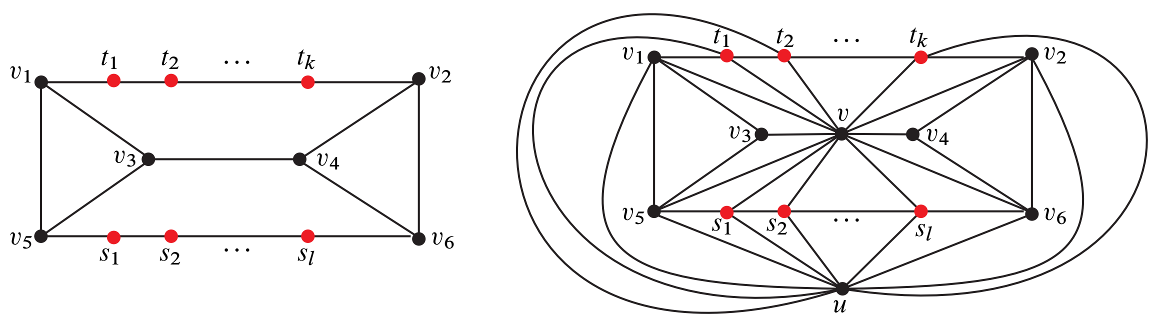


Figure 2: An elongated prism with only two edges subdivided (left) and a planar graph obtained by deleting the vertices t and w of $H + P_2$ (right).

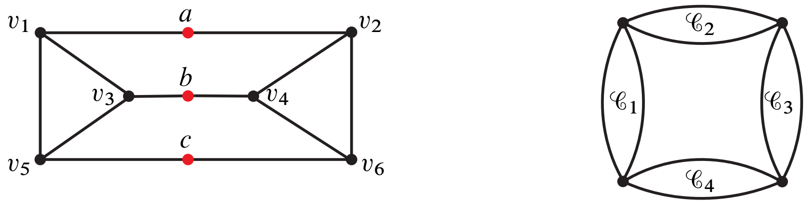


Figure 3: The graph P' obtained by subdividing once each nontriangular edge of the prism graph (left) and the graph D_4 (right).

Proof Assume that H is isomorphic to the graph depicted in Figure 2, left, in which the edge $\{v_3, v_4\}$ is not subdivided. Perform a ∇Y move on the triangle induced by the vertices $\{v_3, v_4, u\}$ by deleting the edges $\{v_3, v_4\}$, $\{v_3, u\}$ and $\{v_4, u\}$ and adding a new vertex t incident to all of $\{v_3, v_4, u\}$ to obtain a new graph G'. This graph is 2–apex, since deleting the vertices t and w gives the planar graph of Figure 2, right. Thus, G' is nIK, and so must be G, as the ∇Y move preserves the IK property [Motwani et al. 1988].

To show that G is maximal nIK, one notices that G is isomorphic to a cone w over $H + E_2$. Since $H + E_2$ is maxnIL by [Dehkordi and Farr 2021], adding any edge to G produces a structure of a cone over an IL graph. This structure will contain a minor isomorphic to a graph in either the K_7 family or the $K_{3,3,1,1}$ family, and will therefore be IK. □

Theorem 7 *Let H denote an elongated prism of order $n \geq 9$ obtained by repeated subdivisions of all three nontriangular edges of the prism graph. Then $G \simeq H + P_2$ is an IK graph.*

Proof By repeated edge contractions applied to G, one obtains the minor $S \simeq P' + P_2$, where P' is the graph depicted in Figure 3, left.

Foisy [2002] proved that, if a graph contains a doubly linked D_4 minor in every embedding, the graph must be IK. This result was also proved independently by Taniyama and Yasuhara [2001]. The graph D_4 is depicted in Figure 3, right. An embedding of the graph D_4 is doubly linked if the linking numbers $\mathrm{lk}(C_1, C_3)$ and $\mathrm{lk}(C_2, C_4)$ are both nonzero mod 2. We used a Mathematica program written by Naimi to show that S has a doubly linked D_4 minor in every embedding. □

3 The μ invariant

In this section we determine the value of the μ invariant for complements of maximal nonseparating planar graphs. By [van der Holst et al. 1999], if G is planar with n vertices, then $\mu(cG) \geq n-5$. We first show the inequality $\mu(cG) \leq n-4$ for graphs G which are maximal nonseparating planar. In Theorem 2, we show this is in fact an equality.

Kotlov et al. [1997] conjectured that, for a simple graph G, $\mu(G) + \mu(cG) \geq n-2$. We review that the conjecture holds for planar graphs and 1–apex graphs. We show that, as a consequence of Theorem 2, the conjecture holds for 2–apex graphs G for which $G - \{u, v\}$ is planar nonseparating.

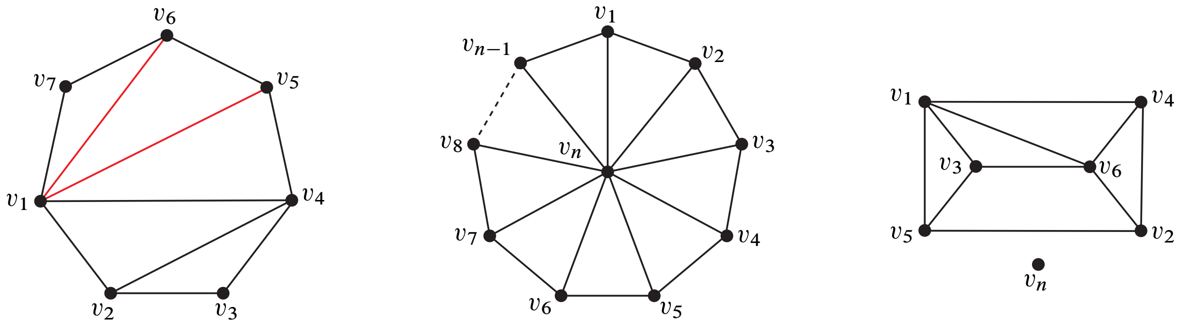


Figure 4: A maximal outerplanar graph with seven vertices (left), the graph G, a wheel with n vertices (center), and $cG \setminus \{v_7, v_8, \dots, v_{n-1}\}$ (right).

Theorem 1 *If G is a maximal nonseparating planar graph with $n \geq 7$ vertices, then cG is $(n-7)$–apex.*

Proof We treat the three types in turn:

Outerplanar case Any maximal outerplanar graph H of order $n \geq 3$ can be represented by a triangulated n–cycle in the plane (with the unbounded face containing all vertices). The n–cycle contains at least one 2–chord, an edge which forms a triangle with two adjacent edges along the cycle. We say that the 2–chord isolates the vertex which is part of the triangle but is not incident to the 2–chord. For example, in Figure 4, left, the 2–chord v_1v_6 isolates the vertex v_7 and the 2–chord v_1v_5 of $H - \{v_7\}$ isolates v_6. The complement of the unique maximal outerplanar graph with five vertices is P_3, a path with three edges, together with an isolated vertex. It follows that the complement of any maximal outerplanar graph with seven vertices is planar, since the deletion of two vertices gives a path with three edges and an isolated vertex. For example, after the deleting the vertices v_7 and v_6, the complement of the graph in Figure 4, left, is the path $v_1v_3v_5v_2$ together with the isolated vertex v_4. Starting with a maximal outerplanar graph with $n \geq 7$ vertices, one can recursively delete $n-7$ isolated vertices and obtain a maximal outerplanar graph of order 7. The same sequence of $n-7$ vertex deletions gives a planar subgraph of cG. Thus, cG is $(n-7)$–apex.

Wheel case Let G be the wheel on n vertices. Then $cG \simeq (K_{n-1} \setminus C_{n-1}) \cup K_1$. Let $\{v_1, v_2, \dots, v_{n-1}\}$ be the vertices of C_{n-1} in consecutive order, as in Figure 4, center. Then $cG \setminus \{v_7, v_8, \dots, v_{n-1}\}$ is a planar graph (the triangular prism added one edge, together with an isolated vertex) and thus cG is $(n-7)$–apex. See Figure 4, right.

Elongated prism case Let G be an elongated prism with $n \geq 7$ vertices. Without loss of generality, let $v_1v_3v_5$ be one of two induced triangles of G. Let a, b and c denote their respective neighbors in $V(G) \setminus \{v_1, v_3, v_5\}$, as in Figure 5, left. Deleting all vertices but $\{v_1, v_3, v_5, a, b, c\}$ in cG gives the subgraph of the outerplanar graph with six vertices in Figure 5, right. Deleting any $n-7$ vertices of cG none of which is in the set $\{v_1, v_3, v_5, a, b, c\}$ yields a planar graph, and thus cG is $(n-7)$–apex. □

Corollary 8 *For G a maximal nonseparating planar graph with $n \geq 7$ vertices, $\mu(cG) \leq n-4$.*

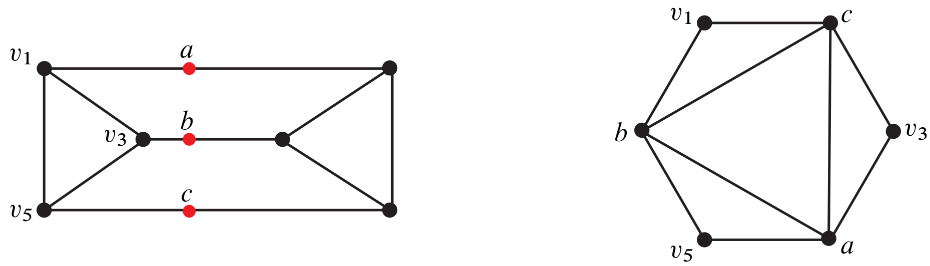


Figure 5: An elongated prism (left) and the subgraph induced by $\{v_1, v_3, v_5, a, b, c\}$ in cG (right).

Proof By Theorem 1, cG is $(n-7)$–apex. Let H be the planar subgraph of cG obtained by deleting $n-7$ vertices. Then $\mu(H) \leq 3$ and $\mu(cG) \leq 3 + (n-7) = n-4$, since adding one vertex to a graph increases the value of μ by at most one (see [van der Holst et al. 1999, Theorem 2.7]). □

Corollary 8 establishes an upper bound of $n-4$ for the values of μ of complements of maximal nonseparating planar graphs on n vertices. We show that $n-4$ is the actual value of μ. We use [van der Holst et al. 1999, Theorem 5.5], whice says that, for H a graph on n vertices and $\nu(H) := n - \mu(cH) - 1$, the inequality $\nu(H) \leq 2$ holds if and only if H does not contain as a subgraph any of the five graphs in Figure 6. We also use that, for a graph G with at least one edge, $\mu(G + K_1) = \mu(G) + 1$ by [van der Holst et al. 1999, Theorem 2.7].

Theorem 2 *For G a maximal nonseparating planar graph with $n \geq 7$ vertices, $\mu(cG) = n-4$.*

Proof Corollary 8 established the inequality $\mu(cG) \leq n-4$. Here we show that $\mu(cG) \geq n-4$. If G is outerplanar, then $\mu(cG) \geq n-4$ [Kotlov et al. 1997]. If G is the wheel graph on n vertices, $cG = cC_{n-1} \cup K_1$. By [van der Holst et al. 1999, Theorem 5.5], $\nu(C_{n-1}) \leq 2$ and we have

$$\mu(cG) = \mu(cC_{n-1}) = n - 1 - \nu(C_{n-1}) - 1 \geq n-4.$$

For elongated prisms, we distinguish two cases, according to the number of nontriangular edges of the prism which are being subdivided:

Case 1 Consider G the elongated prism in Figure 7, left, with exactly one nontriangular edge of the prism graph subdivided, v_1v_2, If at least two vertices are added along v_1v_2, as in Figure 7, left, consider

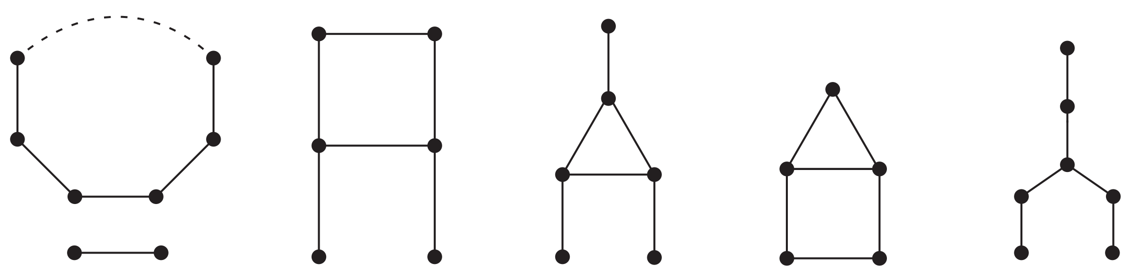


Figure 6: Five graphs.

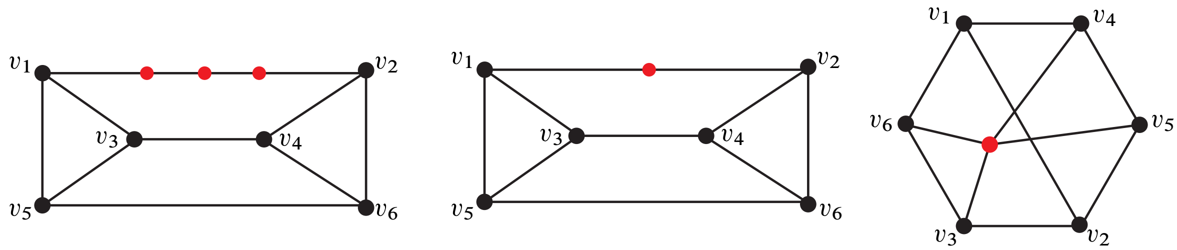


Figure 7: An elongated prism with one nontriangular edge subdivided by more than one vertex (left), an elongated prism with one nontriangular edge subdivided by exactly one vertex (center) and the complement of that graph (right).

the graph $H = G - \{v_1, v_2\}$. Then $\mu(cH) = (n-2) - \nu(H) - 1 \geq n-5$, by [van der Holst et al. 1999, Theorem 5.5]. Since in cG the set of adjacent vertices $\{v_1, v_2\}$ cones over cH, $\mu(cG) \geq n-4$ by [van der Holst et al. 1999, Theorem 2.7]. If only one vertex is added along the one edge, as in Figure 7, center, the set of adjacent vertices $\{v_1, v_2\}$ no longer cones over cH. However, in this case, cG contains a K_4 minor, and thus $\mu(cG) \geq 3$. See Figure 7, right.

Case 2 Assume G is obtained from the triangular prism by subdividing edges v_1v_2 and v_5v_6 along the way, as in Figure 8, left. The graph $H = G - \{v_1, v_6\}$ is a path with $n-2$ vertices, so $\mu(cH) \geq n-5$ [Kotlov et al. 1997]. In cG, the set of adjacent vertices $\{v_1, v_6\}$ cones over cH, yielding $\mu(cG) \geq \mu(cH) + 1 \geq n-4$ by [van der Holst et al. 1999, Theorem 2.7]. □

We briefly discuss the state of a conjecture of [Kotlov et al. 1997], that, for a simple graph G on n vertices, $\mu(G) + \mu(cG) \geq n-2$. By [Kotlov et al. 1997; Colin de Verdière 1990; van der Holst et al. 1999], the conjecture holds if either one of G or cG is planar. We note that the conjecture holds if $\mu(G) \geq n-6$ or $\mu(cG) \geq n-6$. Assume $\mu(G) \geq n-6$. If $\mu(cG) \geq 4$, then $\mu(G) + \mu(cG) \geq n-2$; if $\mu(cG) < 4$, $\mu(G)$ is planar, and the conjecture holds.

Proposition 9 *The conjecture holds for* 1*–apex graphs.*

Proof Let G be a 1–apex graphs with n vertices and $H = G - \{v\}$ planar. Then $\mu(cH) \geq (n-1) - 5 = n-6$ [Kotlov et al. 1997]. We have that cH, the complement of H in K_{n-1}, is a subgraph of cG, the complement of G in K_n, since cG may have additional edges incident to v, and so $n-6 \leq \mu(cH) \leq \mu(cG)$. Thus, the conjecture holds for G. □

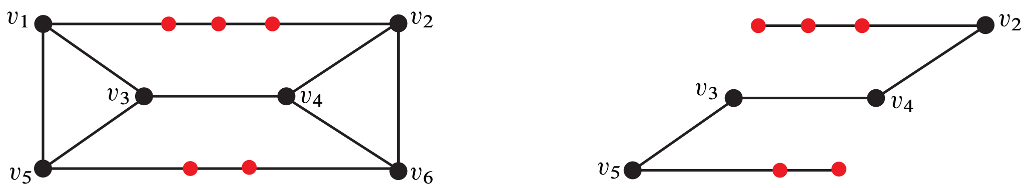


Figure 8: An elongated prism G with two subdivided edges (left) and $H = G - \{v_1, v_6\}$ (right).

Corollary 10 *Let G be a 2–apex graph with n vertices with $H = G - \{u, v\}$ planar nonseparating. Then $\mu(G) + \mu(cG) \geq n - 2$.*

Proof Since H is planar nonseparating, by Theorem 2, $\mu(cH) \geq (n-2) - 4 = n - 6$, with equality if H is maximal. We have that cH, the complement of H in K_{n-2}, is a subgraph of cG, the complement of G in K_n, since cG may have additional edges incident to u and v, and so $\mu(cG) \geq \mu(cH) \geq n - 6$. Thus, the conjecture holds for G. □

4 Graphs of order ten

The relationship between the μ invariant and the property of being intrinsic knotted is not well understood. While Theorem 2 establishes that, for G a maximal nonseparating planar graph with ten vertices, $\mu(cG) = 6$, this information has no bearing on whether cG is intrinsically knotted. Flapan and Naimi [2008] prove that the IK property is not preserved by the $Y\nabla$ move by showing a graph in the K_7 family which is not intrinsically knotted. Since $\mu(K_7) = 6$ and both the ∇Y move and the $Y\nabla$ move preserve μ for $\mu \geq 4$ [van der Holst et al. 1999], this nIK graph has $\mu = 6$. On the other hand, Foisy [2003] and Mattman et al. [2021] provide examples of IK graphs with $\mu = 5$. In this section, we do a case-by-case analysis to prove that, for G a maximal nonseparating planar graph with ten vertices, cG is intrinsically knotted. We recall that the ∇Y move preserves the IK property. In some cases, graphs are shown to be IK because they are obtained through one or more ∇Y moves from IK graphs such as K_7 or $K_{3,3,1,1}$. In other cases, graphs G are shown to be IK because the graphs obtained from G by one or more $Y\nabla$ moves contain K_7 or $K_{3,3,1,1}$ minors.

Lemma 11 *If G is a maximal outerplanar graph with ten vertices, then cG is intrinsically knotted.*

Proof We label the vertices of G by $v_1, v_2, \ldots, v_9, v_{10}$ in clockwise order around the cycle $\mathscr{C}$ bordering the outer face of a planar embedding. See Figure 9. We organize the proof according to the longest chord of $\mathscr{C}$. The length of a chord is defined as the length of the shortest path in $\mathscr{C}$ between the endpoints of the chord. In each case we show the complement cG contains an intrinsically knotted graph as a minor. We remark that, within any triangulation of the disk bounded by $\mathscr{C}$, out of a total of seven chords, at most six have length 2 or 3. Thus there exist chords of length 4 or 5.

Case (a) If the cycle $\mathscr{C}$ has a chord of length 5, we may assume without loss of generality that $v_1v_6 \in E(G)$. Consider the cycles $\mathscr{C}_1 := v_1v_6v_7v_8v_9v_{10}$ and $\mathscr{C}_2 := v_1v_2v_3v_4v_5v_6$. We note that $\mathscr{C}$ necessarily contains a 3–chord or a 4–chord with one endpoint at v_1 or v_6 and the other endpoint among the vertices of $\mathscr{C}_i$ for $i = 1, 2$. We distinguish six cases, according to whether there are any 4–chords at all and whether these chords share one of their ends:

(a1) Assume there exists a 4–chord incident to v_1 or v_6, say $v_1v_5 \in E(G)$.

(i) If $v_1v_7 \in E(G)$ (see Figure 9, far left), then the complement cG contains as a subgraph the graph obtained through two ∇Y moves from K_7 with vertex set $\{v_2, v_3, v_4, v_8, v_9, v_{10}, v_6\}$: one ∇Y

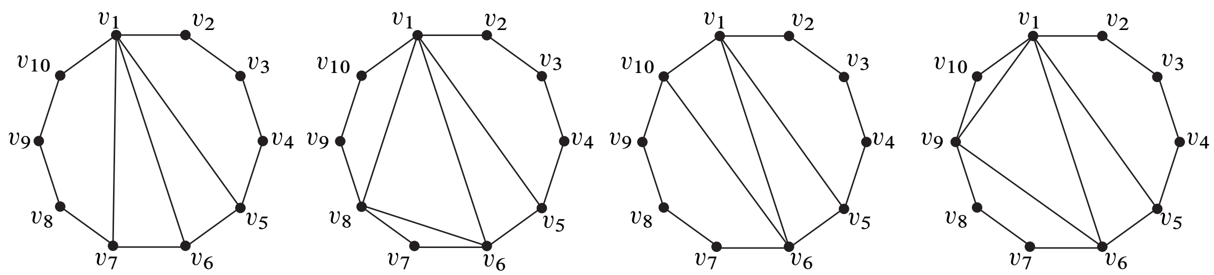


Figure 9: Outerplanar graphs with ten vertices.

move over the triangle $v_2v_3v_4$ with new vertex v_7 and one ∇Y move over the triangle $v_8v_9v_{10}$ with new vertex v_5.

(ii) If $v_1v_7 \notin E(G)$ and $v_1v_8 \in E(G)$ (see Figure 9, center left), then, in cG, delete any edges incident to v_5 except v_5v_8, v_5v_9 and v_5v_{10}, then perform a $Y\nabla$ move at v_5 to create a graph containing the triangle $v_8v_9v_{10}$. This graph contains a $K_{3,3,1,1}$ minor with partition $\{v_2, v_3, v_4\}$, $\{v_6, v_7, v_8\}$, $\{v_9\}$, $\{v_{10}\}$.

(iii) If $v_6v_{10} \in E(G)$ (see Figure 9, center right), then, in cG, delete any edges incident to v_1 except v_1v_7, v_1v_8 and v_1v_9, then perform a $Y\nabla$ move at v_1 to create a graph containing the triangle $v_7v_8v_9$. Further, delete any edges incident to v_6 except v_2v_6, v_3v_6 and v_4v_6, then perform a $Y\nabla$ move at v_6 to create a graph containing the triangle $v_2v_3v_4$. Within this new graph, contract v_5v_{10} to a new vertex t to obtain a K_7 minor with vertices $\{v_2, v_3, v_4, v_7, v_8, v_9, t\}$.

(iv) If $v_6v_{10} \notin E(G)$ and $v_6v_9 \in E(G)$ (see Figure 9, far right), then, in cG, delete any edges incident to v_6 except v_6v_2, v_6v_3 and v_6v_4, then perform a $Y\nabla$ move at v_6 to create a graph containing the triangle $v_2v_3v_4$. Within this new graph, contract the edge v_5v_9 to a vertex t, and contract the edge v_1v_7 to a vertex t_7 to obtain a K_7 minor with vertices $\{v_2, v_3, v_4, t_7, v_8, v_{10}, t\}$.

(a2) Assume there is no 4–chord of $\mathscr{C}$ incident to v_1 or v_6. There are two 3–chords of $\mathscr{C}$ incident to v_1 or v_6 and endpoints in each $\mathscr{C}_1$ and $\mathscr{C}_2$. Assume $v_1v_4 \in E(G)$.

(i) If $v_1v_8 \in E(G)$ (see Figure 10, far left), for any choice of edges which triangulate the quadrilaterals $v_1v_2v_3v_4$ and $v_8v_9v_{10}v_1$, the complement cG contains as a subgraph the graph Cousin 12 of

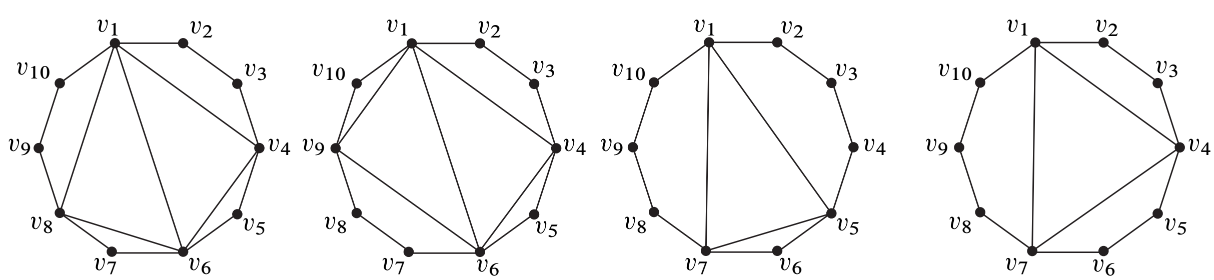
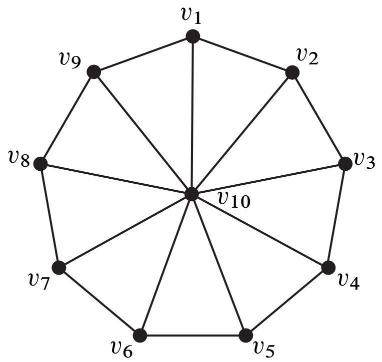


Figure 10: Outerplanar graphs with ten vertices.



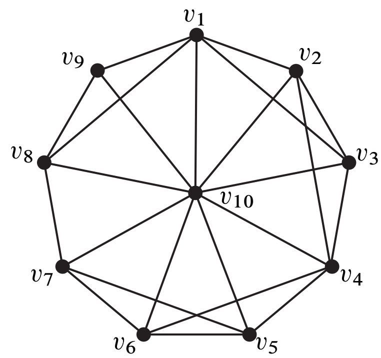


Figure 11: A wheel graph with ten vertices (left) and the complement of $E_9 + e$ in K_{10} (right)

.

$K_{3,3,1,1}$ described in [Goldberg et al. 2014]. This is a minor minimal IK graph with nine vertices obtained from $K_{3,3,1,1}$ by two ∇Y moves followed by a $Y\nabla$ move.

(ii) If $v_6v_9 \in E(G)$ (see Figure 10, center left), obtain a K_7 minor of cG by contracting the edges v_1v_8, v_2v_6 and v_4v_9.

Case (b) Assume the cycle $\mathscr{C}$ has no chord of length 5. Then it has at least a chord of length 4. Assume $v_1v_7 \in E(G)$. Up to symmetry, we recognize two cases.

(b1) If $v_1v_5 \in E(G)$ (see Figure 10, center right), then the complement cG contains the graph obtained through two ∇Y moves from K_7 with vertex set $\{v_2, v_3, v_4, v_6, v_8, v_9, v_{10}\}$: one ∇Y move over the triangle $v_2v_3v_4$ with new vertex v_7 and one ∇Y move over the triangle $v_8v_9v_{10}$ with new vertex v_5.

(b2) If $v_1v_4, v_4v_7 \in E(G)$ (see Figure 10, far right), then, in cG, delete any edge incident to v_4 except v_4v_8, v_4v_9 and v_4v_{10}, then perform a $Y\nabla$ move at v_4 to create a graph containing the triangle $v_8v_9v_{10}$. Within this graph, contract the edges v_1v_5 to t_5 and v_2v_7 to t_2 obtain a K_7 with vertex set $\{t_2, v_3, t_5, v_6, v_8, v_9, v_{10}\}$. □

Lemma 12 *If G is a wheel with ten vertices, then cG is intrinsically knotted.*

Proof The graph $E_9 + e$ is a minor minimal intrinsically knotted graph with nine vertices described in [Goldberg et al. 2014]. The complement of $E_9 + e$ in K_{10} contains the 10–wheel as a subgraph. See Figure 11. Thus, the complement cG contains $E_9 + e$ as a subgraph and therefore it is intrinsically knotted. □

Lemma 13 *If G is an elongated triangular prism with ten vertices, then cG is intrinsically knotted.*

Proof An elongated prism with ten vertices is obtained by subdividing the three nontriangular edges of the prism with four vertices. These four vertices can be added in four different ways:

(a) on three different edges,

(b) on two edges with a 2-2 partition,

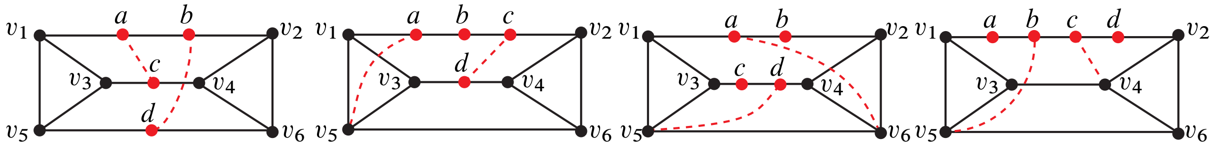


Figure 12: Elongated prisms with ten vertices. Dashed edges are edges of the complement graph.

(c) on two edges with a 3-1 partition,

(d) all on one edge.

See Figure 12. In each case, we show that cG contains a $K_{3,3,1,1}$ minor.

Case (a) The four vertices are added on three different edges of the elongated prism, as in Figure 12, far left. Within cG, contract the edge ac to the vertex t and bd to u to obtain a $K_{3,3,1,1}$ minor of cG given by the partition $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$, $\{t\}$, $\{u\}$.

Case (b) The four vertices are added to two edges of the elongated prism with a 2-2 partition, as in Figure 12, center left. Within cG, contract dv_5 to t_5 and av_6 to t_6 to obtain a $K_{3,3,1,1}$ minor of cG given by the partition $\{v_1, v_3, c\}$, $\{v_2, v_4, b\}$, $\{t_5\}$, $\{t_6\}$.

Case (c) The four vertices are added to two edges of the elongated prism with a 3-1 partition, as in Figure 12, center right. Within cG, contract av_5 to t_5 and cd to t to obtain a $K_{3,3,1,1}$ minor of cG given by the partition $\{v_1, v_3, t_5\}$, $\{v_2, v_4, v_6\}$, $\{b\}$, $\{t\}$.

Case (d) The four vertices are added all on one edge of the elongated prism, as in Figure 12, far right. Within cG, contract bv_5 to t_5 and cv_4 to t_4 to obtain a $K_{3,3,1,1}$ minor of cG given by the partition $\{v_1, v_3, a\}$, $\{v_2, v_6, d\}$, $\{t_4\}$, $\{t_5\}$. □

Since $cG \subseteq cH$ for H a subgraph of G of the same order, Lemmas 11, 12 and 13 give the following theorem:

Theorem 3 *If G is a nonseparating planar graph on ten vertices, then cG is intrinsically knotted.*

Corollary 14 *For $n \geq 10$, the complement of a nonseparating planar graph on n vertices is IK.*

Remark 15 The bound $n \geq 10$ in Corollary 14 is the best possible. If G is the 9–wheel, then $cG \setminus v = K_8 \setminus C_8$. Here v is the isolated point within the complement of the wheel. Since it has 20 edges, $K_8 \setminus C_8$ is 2–apex and it is therefore knotlessly embeddable [Mattman 2011]. As cG is isomorphic to $K_8 \setminus C_8$ with the isolated vertex v added, it is also 2–apex, and thus nIK.

Note that, in the proof of Theorem 3, we've showed that the complements of nonseparating planar graphs of order 10 all have minor minimal intrinsically knotted minors of smaller order. From this it follows that there are no minor minimal intrinsically knotted (MMIK) graphs of order ten or more with nonseparating planar complements. On the other hand, by the combined work of [Blain et al. 2007; Conway and Gordon

1983; Campbell et al. 2008; Foisy 2002; Goldberg et al. 2014; Kohara and Suzuki 1992; Mattman et al. 2017], the eleven MMIK graphs of order at most 9 are known. Considering their complements, there are just four MMIK graphs with nonseparating planar complements.

Corollary 16 *There are exactly four minor minimal intrinsically knotted graphs whose complements are nonseparating planar*: K_7, $K_{3,3,1,1}$, K_7^{∇} (*the graph obtained by performing a single* ∇Y*–move on* K_7) *and* $G_{9,28}$ (*the complement of a* 7*–cycle and an independent edge inside* K_9).

Proof By inspection, the complements of the four graphs are nonseparating. The complements of the remaining seven order 9 graphs are planar, but none of them are nonseparating as:

- They cannot be subgraphs of an elongated prism of order 9 (size 12), since their size (14–15) is too big.
- They all have at least two vertices of degree bigger than 3; thus, they cannot be subgraphs of a wheel graph.
- They all have a K_4 minor; thus, they cannot be outerplanar. □

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References

[Aires 2021] **M Aires**, *On the number of edges in maximally linkless graphs*, J. Graph Theory 98 (2021) 383–388 MR Zbl

[Battle et al. 1962] **J Battle**, **F Harary**, **Y Kodama**, *Every planar graph with nine points has a nonplanar complement*, Bull. Amer. Math. Soc. 68 (1962) 569–571 MR Zbl

[Blain et al. 2007] **P Blain**, **G Bowlin**, **T Fleming**, **J Foisy**, **J Hendricks**, **J Lacombe**, *Some results on intrinsically knotted graphs*, J. Knot Theory Ramifications 16 (2007) 749–760 MR Zbl

[Campbell et al. 2008] **J Campbell**, **T W Mattman**, **R Ottman**, **J Pyzer**, **M Rodrigues**, **S Williams**, *Intrinsic knotting and linking of almost complete graphs*, Kobe J. Math. 25 (2008) 39–58 MR Zbl

[Colin de Verdière 1990] **Y Colin de Verdière**, *Sur un nouvel invariant des graphes et un critère de planarité*, J. Combin. Theory Ser. B 50 (1990) 11–21 MR Zbl

[Conway and Gordon 1983] **J H Conway**, **C M Gordon**, *Knots and links in spatial graphs*, J. Graph Theory 7 (1983) 445–453 MR Zbl

[Dehkordi and Farr 2021] **H R Dehkordi**, **G Farr**, *Non-separating planar graphs*, Electron. J. Combin. 28 (2021) art. id. 1.11 MR Zbl

[Eakins et al. 2023] **L Eakins**, **T Fleming**, **T Mattman**, *Maximal knotless graphs*, Algebr. Geom. Topol. 23 (2023) 1831–1848 MR Zbl

[Flapan and Naimi 2008] **E Flapan**, **R Naimi**, *The Y-triangle move does not preserve intrinsic knottedness*, Osaka J. Math. 45 (2008) 107–111 MR Zbl

[Foisy 2002] **J Foisy**, *Intrinsically knotted graphs*, J. Graph Theory 39 (2002) 178–187 MR Zbl

[Foisy 2003] **J Foisy**, *A newly recognized intrinsically knotted graph*, J. Graph Theory 43 (2003) 199–209 MR Zbl

[Goldberg et al. 2014] **N Goldberg**, **T W Mattman**, **R Naimi**, *Many, many more intrinsically knotted graphs*, Algebr. Geom. Topol. 14 (2014) 1801–1823 MR Zbl

[van der Holst et al. 1999] **H van der Holst**, **L Lovász**, **A Schrijver**, *The Colin de Verdière graph parameter*, from "Graph theory and combinatorial biology" (L Lovász, A Gyárfás, G Katona, A Recski, L Székely, editors), Bolyai Soc. Math. Stud. 7, Bolyai Math. Soc., Budapest (1999) 29–85 MR Zbl

[Jørgensen 1989] **L K Jørgensen**, *Some maximal graphs that are not contractible to* K_6, tech report 89-28, Aalborg Univ. Inst. Elektron. Syst. (1989)

[Kohara and Suzuki 1992] **T Kohara**, **S Suzuki**, *Some remarks on knots and links in spatial graphs*, from "Knots 90" (A Kawauchi, editor), de Gruyter, Berlin (1992) 435–445 MR Zbl

[Kotlov et al. 1997] **A Kotlov**, **L Lovász**, **S Vempala**, *The Colin de Verdière number and sphere representations of a graph*, Combinatorica 17 (1997) 483–521 MR Zbl

[Lovász and Schrijver 1998] **L Lovász**, **A Schrijver**, *A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs*, Proc. Amer. Math. Soc. 126 (1998) 1275–1285 MR Zbl

[Mader 1968] **W Mader**, *Homomorphiesätze für Graphen*, Math. Ann. 178 (1968) 154–168 MR Zbl

[Mattman 2011] **T W Mattman**, *Graphs of* 20 *edges are* 2*–apex, hence unknotted*, Algebr. Geom. Topol. 11 (2011) 691–718 MR Zbl

[Mattman et al. 2017] **T W Mattman**, **C Morris**, **J Ryker**, *Order nine MMIK graphs*, from "Knots, links, spatial graphs, and algebraic invariants" (E Flapan, A Henrich, A Kaestner, S Nelson, editors), Contemp. Math. 689, Amer. Math. Soc., Providence, RI (2017) 103–124 MR Zbl

[Mattman et al. 2021] **T W Mattman**, **R Naimi**, **A Pavelescu**, **E Pavelescu**, *Intrinsically knotted graphs with linklessly embeddable simple minors*, preprint (2021) arXiv 2111.08859

[Motwani et al. 1988] **R Motwani**, **A Raghunathan**, **H Saran**, *Constructive results from graph minors: linkless embeddings*, from "29th annual symposium on foundations of computer science", IEEE, Piscataway, NJ (1988) 398–409

[Naimi et al. 2023] **R Naimi**, **A Pavelescu**, **E Pavelescu**, *New bounds on maximal linkless graphs*, Algebr. Geom. Topol. 23 (2023) 2545–2559 MR

[Robertson et al. 1993] **N Robertson**, **P D Seymour**, **R Thomas**, *Linkless embeddings of graphs in* 3*–space*, Bull. Amer. Math. Soc. 28 (1993) 84–89 MR Zbl

[Sachs 1984] **H Sachs**, *On spatial representations of finite graphs*, from "Finite and infinite sets, II" (A Hajnal, L Lovász, V T Sós, editors), Colloq. Math. Soc. János Bolyai 37, North-Holland, Amsterdam (1984) 649–662 MR Zbl

[Taniyama and Yasuhara 2001] **K Taniyama**, **A Yasuhara**, *Realization of knots and links in a spatial graph*, Topology Appl. 112 (2001) 87–109 MR Zbl

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