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# Right-angled Artin subgroups of right-angled Coxeter and Artin groups

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We determine when certain natural classes of subgroups of right-angled Coxeter groups (RACGs) and right-angled Artin groups (RAAGs) are themselves RAAGs. We characterize finite-index *visual RAAG* subgroups of 2-dimensional RACGs. As an application, we show that any 2-dimensional, one-ended RACG with planar defining graph is quasi-isometric to a RAAG if and only if it is commensurable to a RAAG. Additionally, we give new examples of RACGs with nonplanar defining graphs which are commensurable to RAAGs.

Finally, we give a new proof of a result of Dyer: every subgroup generated by conjugates of RAAG generators is itself a RAAG.

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## 1 Introduction

Let  $\Gamma$  be a finite simplicial graph with vertex set  $V(\Gamma)$  and edge set  $E(\Gamma)$ . The *right-angled Artin group* (RAAG for short) associated to  $\Gamma$  is the group  $A_\Gamma$  given by the presentation

$$A_\Gamma = \langle V(\Gamma) \mid st = ts \text{ for all } (s, t) \in E(\Gamma) \rangle.$$

This article is concerned with the following question. Given a finite set  $S$  of elements in a group, when is the group generated by  $S$  isomorphic to a RAAG in the “obvious” way (ie with  $S$  as the “standard” RAAG generating set)? To make this precise, we define the notion of *RAAG system*.

**Definition 1.1** (RAAG system) Let  $G$  be any group with generating set  $S$ . Let  $\Delta$  be the graph whose vertex set is in bijection with  $S$  and which has an edge between distinct  $s, t \in S \equiv V(\Delta)$  if and only if  $s$  and  $t$  commute. We call  $\Delta$  the *commuting graph* associated to  $S$ . There is a canonical homomorphism  $\phi: A_\Delta \rightarrow G$  extending the bijection  $V(\Delta) \rightarrow S$ . We say that  $(G, S)$  is a *RAAG system* if  $\phi$  is an isomorphism. In particular,  $(A_\Gamma, V(\Gamma))$  is a RAAG system for any RAAG  $A_\Gamma$ .

The *right-angled Coxeter group* (RACG for short) associated to the finite simplicial graph  $\Gamma$  is the group  $W_\Gamma$  given by the presentation

$$W_\Gamma = \langle V(\Gamma) \mid s^2 = 1 \text{ for all } s \in V(\Gamma), st = ts \text{ for all } (s, t) \in E(\Gamma) \rangle.$$

In this article we study subgroups  $G$  generated by particular natural subsets  $S$  of right-angled Coxeter and Artin groups, and we give characterizations for when  $(G, S)$  is a RAAG system or a finite-index RAAG system.

A theorem of Davis and Januszkiewicz [2000] states that every RAAG is commensurable to some RACG. This leads to the following question addressing the converse:

**Question 1.2** Which RACGs are commensurable to RAAGs?

A RACG that is commensurable to a RAAG is, in particular, quasi-isometric to a RAAG. By considering different quasi-isometry invariants, one sees that the converse to the Davis–Januszkiewicz theorem above is far from being true. For instance, there are many RACGs that are one-ended hyperbolic (such as virtual hyperbolic surface groups), while no RAAG is both one-ended and hyperbolic. Furthermore, RAAGs have linear, quadratic or infinite divergence [Behrstock and Charney 2012], whereas the divergence of a RACG can be a polynomial of any degree [Dani and Thomas 2015]. Restricting to RACGs of at most quadratic divergence is still not enough to guarantee they are quasi-isometric to RAAGs. For instance, the Morse boundary of a RAAG with quadratic divergence is always totally disconnected [Charney and Sultan 2015; Cordes and Hume 2017], while the Morse boundary of a RACG of quadratic divergence can have nontrivial connected components [Behrstock 2019]. The above examples show that there are numerous families of RACGs which are not quasi-isometric and, hence, not commensurable to any RAAG. Within the subclass of one-ended RACGs with planar, triangle-free defining graphs, Nguyen and Tran [2019] characterize those quasi-isometric to RAAGs. [Theorem B](#) below answers [Question 1.2](#) in this setting.

We note that every RACG (indeed, every Coxeter group) is virtually special, and therefore has a finite-index subgroup which is a subgroup of a RAAG [Haglund and Wise 2010]. However, this subgroup is not of finite index in the RAAG, which would be required for establishing commensurability.

One approach to proving that a RACG is commensurable to a RAAG is to look for finite-index subgroups that are isomorphic to RAAGs. We focus on a class of subgroups of RACGs, introduced by LaForge [2017] in his PhD thesis, that are logical candidates for being RAAGs. Given a RACG defined by a graph  $\Gamma$  and two nonadjacent vertices  $s, t \in V(\Gamma)$ , it follows that  $st$  is an infinite-order element of  $W_\Gamma$ . There is then a correspondence between edges of the complement graph  $\Gamma^c$  with such infinite-order elements of  $\Gamma$ . Given a subgraph  $\Lambda$  of  $\Gamma^c$ , let  $G$  be the subgroup generated by  $E(\Lambda)$  (thought of as infinite-order elements of  $W_\Gamma$ ). As  $G$  is generated by the edges of  $\Lambda$ , we may as well assume that  $\Lambda$  has no isolated vertices. A natural question is:

**Question 1.3** When is  $(G, E(\Lambda))$  a finite-index RAAG system?

If  $(G, E(\Lambda))$  is indeed a RAAG system, then  $G$  is called a *visual RAAG subgroup* of  $W_\Gamma$ . LaForge obtained some necessary conditions for such subgroups to be visual RAAGs.

We say that  $W_\Gamma$  is 2-dimensional if  $\Gamma$  is triangle-free. Our first main theorem gives an exact characterization of the finite-index visual RAAG subgroups of 2-dimensional RACGs in terms of graph-theoretic conditions:

**Theorem A** *Let  $W_\Gamma$  be a 2-dimensional RACG. Let  $\Lambda$  be a subgraph of  $\Gamma^c$  with no isolated vertices, and let  $G$  be the subgroup generated by  $E(\Lambda)$ . Then the following are equivalent.*

- (1)  $(G, E(\Lambda))$  is a RAAG system and  $G$  is finite index in  $W_\Gamma$ .
- (2)  $(G, E(\Lambda))$  is a RAAG system and  $G$  has index either two or four in  $W_\Gamma$  (and exactly four if  $W_\Gamma$  is not virtually free).
- (3)  $\Lambda$  has at most two components and satisfies conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$ .

The conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$  in the above theorem are algorithmically checkable graph-theoretic conditions on  $\Gamma$  and  $\Lambda$ . See [Section 3](#) for precise definitions of these conditions.

In [Section 5](#) we provide several applications to concrete families of RACGs. In particular we prove:

**Theorem B** *Let  $W_\Gamma$  be a 2-dimensional, one-ended RACG with planar defining graph. Then  $W_\Gamma$  is quasi-isometric to a RAAG if and only if it contains an index 4 subgroup isomorphic to a RAAG.*

A complete description of which RACGs considered in [Theorem B](#) are quasi-isometric to RAAGs is given by Nguyen and Tran [[2019](#), [Theorem 1.2](#)]. [Theorem B](#) shows these are actually commensurable to RAAGs.

We also give two families of RACGs defined by nonplanar graphs which contain finite-index RAAG subgroups (see [Corollaries 5.1](#) and [5.2](#)). These cannot be obtained by applying the Davis–Januszkiewicz constructions to the defining graphs of the RAAGs they are commensurable to. For the family in [Corollary 5.1](#), we use work of Bestvina, Kleiner and Sageev on RAAGs [[Bestvina et al. 2008](#)], to conclude the RACGs are quasi-isometrically distinct. We believe that the methods from this article may be used to further study commensurability of RACGs.

The proof of [Theorem A](#) consists of two main parts. One part involves obtaining an understanding of when  $G$  is of finite index, leading to conditions  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . To obtain these, we use *completions of subgroups*, introduced in [[Dani and Levcovitz 2021](#)]. The other aspect consists of obtaining criteria to recognize when  $(G, E(\Lambda))$  is a RAAG system. To do so, we prove the following theorem by careful analysis of disk diagrams:

**Theorem C** *Let  $W_\Gamma$  be a RACG. Let  $\Lambda$  be a subgraph of  $\Gamma^c$  with no isolated vertices and at most two components. Then the subgroup  $(G, E(\Lambda)) < W_\Gamma$  is a RAAG system if and only if  $\mathcal{R}_1$ – $\mathcal{R}_4$  are satisfied.*

Conditions  $\mathcal{R}_1$ ,  $\mathcal{R}_2$ , and a condition more or less equivalent to  $\mathcal{R}_3$  were known to be necessary for  $(G, E(\Lambda))$  to be a RAAG system by work of LaForge [[2017](#)]. We show in [Example 3.13](#) that they are not sufficient. We introduce a fourth graph-theoretic condition  $\mathcal{R}_4$  to obtain a complete characterization of all visual RAAG subgroups defined by subgraphs of  $\Gamma^c$  with at most two components. The bulk of the proof of [Theorem C](#) consists of showing that the conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$  are sufficient.

Note that, unlike in [Theorem A](#), there is no assumption on the dimension of the RACGs in [Theorem C](#). On the other hand, there is an additional assumption in [Theorem C](#), namely that the subgraph  $\Lambda$  of  $\Gamma$  can have at most two components.

When  $\Lambda$  contains more than two components, the situation becomes much more complex. We show that additional graph-theoretic conditions are necessary to generalize the [Theorem C](#) to this setting (see [Lemmas 3.32](#) and [3.34](#)). Remarkably, a consequence of these conditions is that if  $\Gamma$  is triangle-free and  $(G, E(\Lambda))$  is a finite-index RAAG system, then  $\Lambda$  can have at most two components. This fact is crucial to the proof of [Theorem A](#), which does not have any assumption on the number of components of  $\Lambda$ . Additionally, we are aware that even more conditions are necessary than those in this article, but we do not have a complete conjectural list of conditions that would be sufficient to characterize visual RAAGs.

We next turn our attention to RAAG subgroups of RAAGs. A classical theorem on Coxeter groups, proven independently by Deodhar [\[1989\]](#) and Dyer [\[1990\]](#), states that reflection subgroups of Coxeter groups (ie those generated by conjugates of generators) are themselves Coxeter groups. In fact, Dyer proves an analogous result for the class of groups defined by *reflection systems* (see [\[Dyer 1990\]](#) for the definition), which includes Coxeter groups as well as RAAGs. Specifically, he shows that subgroups generated by conjugates of standard generators are themselves in this class. As RAAGs are the only torsion-free groups in this class, one obtains the following result. Here, we define a *generalized RAAG reflection* to be an element of a RAAG  $A_\Delta$  that is conjugate to a generator in  $V(\Delta)$ .

**Theorem D** [\[Dyer 1990\]](#) *Let  $\mathcal{T}$  be a finite set of generalized RAAG reflections in the RAAG  $A_\Gamma$ . Then the subgroup  $G < A_\Gamma$  generated by  $\mathcal{T}$  is a RAAG.*

We thank Luis Paris for informing us that this result is contained in [\[Dyer 1990\]](#), and the explanation in the preceding paragraph. We include our proof of [Theorem D](#), as our geometric approach is very different from that of Dyer, which is algebraic and uses cocycles. Our proof uses a characterization of RAAG systems in terms of the deletion condition, given by Basarab [\[2002\]](#). We use disk diagrams to show that subgroups generated by generalized RAAG reflections satisfy the criteria in Basarab's characterization.

We note that, although  $G$  (from [Theorem D](#)) is a RAAG,  $(G, \mathcal{T})$  is not necessarily a RAAG system and in general  $G$  is not isomorphic to the RAAG  $A_\Delta$  where  $\Delta$  is the commuting graph corresponding to  $\mathcal{T}$ . Kim and Koberda [\[2013\]](#) show that there exists a subgroup of  $G$  (generated by sufficiently high powers of the elements of  $\mathcal{T}$ ) which is isomorphic to  $A_\Delta$ .

Genevois, as well as an anonymous referee, pointed out to us that a proof of [Theorem D](#) may be possible using [\[Genevois 2017, Theorem 10.54\]](#) (see also [\[Genevois 2019, Theorem 3.24\]](#)).

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## 2 Background

### 2.1 Basic terminology and notation

Let  $G$  be a group with generating set  $S$ . We say that  $w = s_1 \cdots s_n$ , with  $s_i \in (S \cup S^{-1})$  for  $1 \leq i \leq n$ , is a *word over  $S$*  or a *word in  $G$* . If the words  $w$  and  $w'$  represent the same element of  $G$ , then we say that  $w'$  is an *expression* for  $w$  and write  $w' \simeq w$ . We say the word  $w = s_1 \cdots s_n$  is *reduced* (or *reduced over  $S$*  for emphasis) if given  $w' = t_1 \cdots t_m \simeq w$ , it follows that  $n \leq m$ .

### 2.2 Right-angled Coxeter and Artin groups

Coxeter groups can be characterized as those groups which are generated by involutions and which satisfy the deletion condition; see [Definition 2.1](#) below (for a proof of this fact, see [\[Davis 2015, Theorem 3.3.4\]](#)). By work of Basarab [\[2002\]](#), RAAGs can be characterized in a similar manner (see [Theorem 2.2](#) below). This characterization will be utilized in [Section 6](#).

**Definition 2.1** (deletion condition) Let  $G$  be a group generated by  $S$ . We say that  $(G, S)$  satisfies the *deletion condition* if, given any word  $w$  over  $S$ , either  $w$  is reduced or  $w = s_1 \cdots s_k$  and there exist  $1 \leq i < j \leq k$  such that  $s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$  is an expression for  $w$ .

The result below directly follows from a result of Basarab.

**Theorem 2.2** [\[Basarab 2002\]](#) Let  $G$  be a group generated by  $S$  such that  $S \cap S^{-1} = \emptyset$  and  $1 \notin S$ . Then  $(G, S)$  is a RAAG system if and only if

- (1) every  $s$  in  $S$  has infinite order, and
- (2)  $(G, S)$  satisfies the deletion condition.

**Proof** If  $(G, S)$  is a RAAG system, then  $G$  is torsion-free [\[Charney 2007\]](#), so (1) holds. Furthermore,  $(G, S)$  satisfies (2) by [\[Basarab 2002, Corollary 1.4.2\]](#) (see also [\[Bahls 2005, page 31, Exercise 17\]](#) for a simpler proof in this setting). The converse also follows from a direct application of [\[Basarab 2002, Corollary 1.4.2\]](#). □

We now define certain moves which can be performed on a word that produce another expression for it. These moves provide a solution to the word problem for RAAGs and RACGs (see [Theorem 2.4](#) below).

**Definition 2.3** (Tits moves) Let  $G$  be a group generated by  $S$ . Let  $w = s_1 \cdots s_n$  be a word over  $S$ . If  $s_i$  and  $s_{i+1}$  commute for some  $1 \leq i < n$ , then the word  $s_1 \cdots s_{i-1} s_{i+1} s_i s_{i+2} \cdots s_n$  is an expression for  $w$  obtained by a *swap operation* performed on  $w$ , which *swaps*  $s_i$  and  $s_{i+1}$ . If  $s_i = s_{i+1}^{-1}$  for some  $1 \leq i < n$ , then  $s_1 \cdots s_{i-1} s_{i+2} \cdots s_n$  is an expression for  $w$  obtained by a *deletion operation* performed on  $w$ . A *Tits move* is either a swap operation or a deletion operation. We say a word is *Tits reduced* if no sequence of Tits moves can be performed on the word to obtain an expression with fewer generators.

**Theorem 2.4** below shows that RAAGs and RACGs admit a nice solution to the word problem. This solution to the word problem for RACGs is a well-known result of Tits [1969], a version of which holds more generally for all Coxeter groups. The result below in the setting of RAAGs follows from a theorem of Basarab [2002, Theorem 1.4.1] which generalizes Tits' result (see also [Green 1990, Theorem 3.9]).

**Theorem 2.4** [Tits 1969; Basarab 2002] *Let  $A_\Gamma$  be either a RAAG or a RACG. Then:*

- (1) *If  $w_1$  and  $w_2$  are reduced words over  $V(\Gamma)$  representing the same element of  $G$ , then  $w_2$  can be obtained from  $w_1$  by Tits swap moves.*
- (2) *Given any word  $w$  over  $V(\Gamma)$ , a reduced expression for  $w$  can be obtained by applying Tits moves to  $w$ .*

We will often not refer directly to the above theorem, and we will instead simply say that a given RAAG or RACG *admits a Tits solution to the word problem*.

The next two lemmas are well known and will often be implicitly assumed.

**Lemma 2.5** *Let  $A_\Gamma$  either be a RAAG or RACG. Then  $s, t \in V(\Gamma)$  commute as elements of  $A_\Gamma$  if and only if  $(s, t)$  is an edge of  $\Gamma$ .*

**Proof** One direction of the claim follows from the definitions of a RAAG and a RACG. If  $A_\Gamma$  is a RACG, then the other direction follows from [Björner and Brenti 2005, Proposition 4.1.2].

Now suppose that  $A_\Gamma$  is a RAAG, and let  $s, t \in V(\Gamma)$  be nonadjacent vertices. Suppose, for a contradiction, that  $w = sts^{-1}t^{-1} \simeq 1$ . Let  $D$  be a disk diagram with boundary  $w$  (see Section 2.3 for a reference for disk diagrams). This disk diagram contains exactly two intersecting hyperplanes: one labeled by  $s$  and one labeled by  $t$ . However, this is a contradiction as a pair of hyperplanes whose labels are nonadjacent vertices of  $\Gamma$  cannot intersect.  $\square$

**Lemma 2.6** *Let  $W_\Gamma$  be a RACG, and let  $s, t, q, r \in V(\Gamma)$  be such that  $s$  and  $t$  do not commute, and  $r$  and  $q$  do not commute. Then  $(st)(qr) \simeq (qr)(st)$  if and only if*

- (1) *there is a square in  $\Gamma$  formed by  $s, q, t$ , and  $r$ ;*
- (2)  *$t = q$  and  $s = r$ ; or*
- (3)  *$t = r$  and  $s = q$ .*



**Proof** Clearly each of (1), (2) and (3) implies that  $(st)(qr) \simeq (qr)(st)$ .

To prove the converse, suppose that  $(st)(qr) \simeq (qr)(st)$ . Suppose first that  $t = q$ , and consequently  $stqr \simeq sr$ . As  $s$  and  $t$  do not commute and  $q$  and  $r$  do not commute, this is only possible if  $r = t$ . Thus, (2) holds.

If  $s = q$ , as  $qrts \simeq tsqr$ , we apply the same argument to conclude that  $t = r$ , showing (3) holds. By similar arguments, if  $s = r$  then  $t = q$ , and if  $t = r$  then  $s = q$ . Thus, we may assume that  $s, t, q$  and  $r$  are all distinct vertices of  $\Gamma$ . In this case we again conclude by Tits' solution to the word problem, that if  $stqr \simeq qrst$  then  $s, q, t$  and  $r$  form a square in  $\Gamma$ . □

### 2.3 Disk diagrams

We give a brief background on disk diagrams as they are used in our setting, and we refer the reader to [Sageev 1995; Wise 2021] for the general theory of disk diagrams over cube complexes. We then give some preliminary lemmas that are needed in later sections.

Let  $A_\Delta$  be a RAAG, and let  $w = s_1 \cdots s_n$ , with  $s_i \in V(\Delta)$ , be a word equal in  $A_\Delta$  to the identity, ie  $w \simeq 1$ . There exists a Van Kampen diagram  $D$  with boundary label  $w$ , and we call this planar 2-complex a *disk diagram in  $A_\Delta$  with boundary label  $w$* . We now describe some additional properties of  $D$  in our setting. The edges of  $D$  are oriented and labeled by generators in  $V(\Delta)$ . A *path in  $D$*  is a path  $\gamma$  in the 1-skeleton of  $D$ , traversing edges  $e_1, \dots, e_m$ , and the label of  $\gamma$  is the word  $a_1 \cdots a_m$  where, for each  $1 \leq i \leq m$ ,  $a_i$  is the label of  $e_i$  if  $e_i$  is traversed along its orientation, and  $a_i^{-1}$  is the label of  $e_i$  if  $e_i$  is traversed opposite to its orientation. Every cell in  $D$  is a square that has a boundary path with label  $aba^{-1}b^{-1}$  for some commuting generators  $a$  and  $b$  in  $V(\Delta) \cup V(\Delta)^{-1}$ .

There is a base vertex  $p \in \partial D$  and an orientation on  $D$ , such that the smallest closed path  $\delta$  which traverses the boundary of  $D$  in the clockwise orientation starting at  $p$  and traversing every edge outside the interior of  $D$  has label  $w$ . We call  $\delta$  the *boundary path* of  $D$ . Note that if  $D$  contains an edge  $e$  not contained in a square, then necessarily  $\delta$  traverses  $e$  exactly twice.

If  $W_\Gamma$  is a RACG and  $w$  is a word over  $V(\Gamma)$  equal in  $W_\Gamma$  to the identity, then we define a disk diagram  $D$  in  $W_\Gamma$  with boundary  $w$  similarly. However, as each generator in  $V(\Gamma)$  is an involution, we do not need to orient the edges of  $D$ .

Let  $D$  be a disk diagram and  $q = [0, 1] \times [0, 1]$  be a square in  $D$ . The subset  $\{\frac{1}{2}\} \times [0, 1] \subset q$  (similarly,  $[0, 1] \times \{\frac{1}{2}\} \subset q$ ) is a *midcube*. The midpoint of an edge in  $D$  is also defined to be a *midcube*. A *hyperplane* in  $D$  is a minimal nonempty collection  $H$  of midcubes in  $D$  with the property that given any midcube  $m \in H$  and a midcube  $m'$  in  $D$  such that  $m \cap m'$  is contained in an edge of  $D$ , it follows that  $m' \in H$ . We say that  $H$  is dual to an edge  $e$  if the midpoint of  $e$  is in  $H$ .

Since opposite edges in every square in  $D$  have the same label, it follows that every edge intersecting a fixed hyperplane  $H$  has the same label. We call this the *label of the hyperplane*. Since adjacent sides



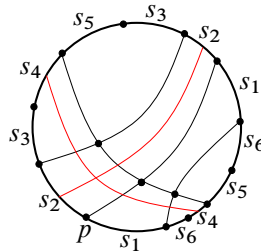


Figure 1: A disk diagram in a RACG with boundary the word  $s_2s_3s_4s_5s_3s_2s_1s_6s_5s_4s_6s_1$  and base vertex  $p$ . Two hyperplanes are shown in red. As these hyperplanes intersect, it must be that  $s_4$  commutes with  $s_2$ .

of a square have distinct labels which commute, it follows that no hyperplane self-intersects, and if two hyperplanes intersect, then their labels correspond to distinct, commuting generators. (See Figure 1 for an example of a disk diagram and some of its hyperplanes.)

**Definition 2.7** (maps preserving boundary combinatorics) Let  $D$  and  $D'$  be disk diagrams, and let  $\delta$  and  $\delta'$  respectively be their boundary paths. Let  $E = \{e_1, \dots, e_m\}$  (resp.  $E' = \{e'_1, \dots, e'_n\}$ ) be the edges traversed by  $\delta$  (resp.  $\delta'$ ). More precisely,  $e_i$  (resp.  $e'_i$ ) is the  $i^{\text{th}}$  edge traversed by  $\delta$  (resp.  $\delta'$ ) for each  $i$ . Observe that every hyperplane of  $D$  is dual to two edges  $e_j, e_k \subset E$  for some  $j \neq k$ . (It could be that  $e_j = e_k$ , thought of as edges of  $D$ .) A similar statement holds for  $D'$ .

Let  $F \subset E$  and  $F' \subset E'$ , and let  $\psi: F \rightarrow F'$  be a bijection. We say that  $\psi$  preserves boundary combinatorics if for every pair of edges  $e, f \in F$  which are dual to the same hyperplane of  $D$ , their images  $\psi(e)$  and  $\psi(f)$  are dual to the same hyperplane of  $D'$ .

Note that if  $\Psi$  preserves boundary combinatorics, then  $\Psi^{-1}$  does as well.

A pair of hyperplanes  $H$  and  $H'$  in a disk diagram  $D$  form a *bigon* if they intersect in at least two distinct points. The following lemma, first proven in [Sageev 1995, Theorem 4.3], guarantees that we can always choose a disk diagram without bigons. The boundary combinatorics statement below is guaranteed by the proof of this fact in [Wise 2021, Lemma 2.3, Corollary 2.4].

**Lemma 2.8** [Sageev 1995; Wise 2021] Given a disk diagram  $D$  with boundary label  $w$ , there exists a disk diagram  $D'$  also with boundary label  $w$  such that  $D'$  does not contain any bigons. Moreover, the natural bijection between the edges traversed by the boundary paths of  $D$  and  $D'$  induced by the label  $w$  preserves boundary combinatorics.

**Remark 2.9** In light of Lemma 2.8, for the rest of this paper we will always assume that any disk diagrams we consider do not have bigons.

**Remark 2.10** Let  $\alpha$  be a path with label  $s_1 \cdots s_n$  in some disk diagram. The “edge of  $\alpha$  with label  $s_i$ ” is understood to be the  $i^{\text{th}}$  edge  $\alpha$  traverses (even though there may be several edges of  $\alpha$  with the same label as this edge). A similar statement holds when we refer to subpaths of  $\alpha$ .

Given a disk diagram with boundary label  $w$ , we will often want to produce a new disk diagram with boundary label  $w'$ , where  $w'$  is obtained from  $w$  by a Tits move, and such that boundary combinatorics are preserved on appropriate subsets of the boundary paths. The following lemma exactly describes how we can perform these operations.

**Lemma 2.11** *Let  $D$  be a disk diagram over the group  $W$ , where  $W$  is either a RACG or a RAAG. Suppose the boundary path of  $D$  traverses the edges  $e_1, \dots, e_n$  and has label  $w = s_1 \cdots s_n$ .*

- (1) *If  $s_r$  and  $s_{r+1}$  (taken modulo  $n$ ) are distinct and commute for some  $1 \leq r \leq n$ , then there is a disk diagram  $D'$  whose boundary path traverses the edges  $e'_1, \dots, e'_n$  and has label  $s_1 \cdots s_{i+1} s_i \cdots s_n$ . Furthermore, the map  $\psi$  preserves boundary combinatorics, where  $\psi$  is defined by  $\psi(e_r) = e'_{r+1}$ ,  $\psi(e_{r+1}) = e'_r$ , and  $\psi(e_j) = e'_j$  for  $j \neq r, r + 1$ .*
- (2) *If  $s_r = s_{r+1}^{-1}$  (taken modulo  $n$ ) for some  $1 \leq r \leq n$ , then there is a disk diagram  $D'$  with boundary label  $s_1 \cdots s_{r-1} s_{r+2} \cdots s_n$ . Moreover, the natural map from edges traversed by the boundary path of  $D'$  to edges traversed by the boundary path of  $D$  preserves boundary combinatorics.*
- (3) *Given any generator (or inverse of a generator)  $s$  and any  $r$ , with  $1 \leq r \leq n$ , it follows that there exists a disk diagram  $D'$  with boundary label  $s_1 \cdots s_r (s s^{-1}) s_{r+1} \cdots s_n$ . Moreover, the natural map from edges traversed by the boundary path of  $D$  to the edges traversed by the boundary path of  $D'$  preserves boundary combinatorics.*

**Proof** We first prove (1). Let  $q$  be a square whose edges are labeled consecutively by  $s_r, s_{r+1}, s_r^{-1}$  and  $s_{r+1}^{-1}$ . We form the disk diagram  $D'$  by identifying consecutive edges of  $q$  labeled by  $s_r$  and  $s_{r+1}$  to the edges of  $\partial D$  labeled by  $s_r$  and  $s_{r+1}$  (these edges must be distinct as  $s_r \neq s_{r+1}$ ). The claim is readily checked.

We next prove (2). Let  $e$  and  $f$  be the edges of  $\partial D$  labeled respectively by  $s_r$  and  $s_{r+1}$ . Suppose first that  $e$  and  $f$  are distinct. In this case, form the disk diagram  $D'$  by identifying  $e$  and  $f$ , ie “fold” these edges together. On the other hand, if  $e = f$ , then as  $D$  has boundary label  $w$ , it must follow that  $e$  is a spur, ie an edge attached to  $D$  that is not contained in any square and which contains a vertex of valence 1. In this case we can remove the edge  $e$  from  $D$  to obtain  $D'$ . In either case, the claim is readily checked.

To show (3), form  $D'$  by inserting a spur edge with label  $s$  to the vertex traversed by the boundary path of  $D$  between  $s_r$  and  $s_{r+1}$ . □

### 3 Visual RAAG subgroups of right-angled Coxeter groups

In this and the next section we study visual RAAG subgroups of RACGs, as described in the introduction. We begin by describing some notation that will be used throughout these sections.

Let  $\Gamma$  be a graph, and let  $W_\Gamma$  be the corresponding RACG. Let  $\Gamma^c$  denote the complement of  $\Gamma$ , that is, the graph with the same vertex set as  $\Gamma$ , which has an edge between two (distinct) vertices if and only if

the corresponding vertices are not adjacent in  $\Gamma$ . Let  $\Lambda$  be a subgraph of  $\Gamma^c$  with no isolated vertices, ie one in which every vertex of  $\Lambda$  is contained in some edge.

We form a new graph  $\Theta = \Theta(\Gamma, \Lambda)$  which we think of as a graph containing the edges of both  $\Gamma$  and  $\Lambda$ . More formally,  $V(\Theta) = V(\Gamma)$  and  $E(\Theta) = E(\Gamma) \cup E(\Lambda)$ . Note that as  $E(\Lambda) \subset \Gamma^c$ , it follows that  $\Theta$  is simplicial. We refer to edges of  $\Theta$  that correspond to edges of  $\Gamma$  (resp.  $\Lambda$ ) as  $\Gamma$ -edges (resp.  $\Lambda$ -edges).

A  $\Lambda$ -edge between vertices  $a$  and  $b$  corresponds to an inverse pair of infinite-order elements of  $W_\Gamma$ , namely  $ab$  and  $ba$ . By a slight abuse of terminology, we will use the term  $\Lambda$ -edge to refer to one of these elements and vice versa. We identify  $E(\Lambda)$  with a subset of  $W_\Gamma$  by arbitrarily choosing one of the two infinite-order elements corresponding to each  $\Lambda$ -edge, and we define  $G^\Theta$  to be the subgroup of  $W_\Gamma$  generated by  $E(\Lambda)$ . As we are dealing with subgroups generated by  $E(\Lambda)$ , there is no loss in generality in assuming that  $\Lambda$  has no isolated vertices. The goal of this section is to study when  $(G^\Theta, E(\Lambda))$  is a RAAG system.

Let  $\Delta$  be the commuting graph corresponding to  $E(\Lambda)$  (as defined in the introduction), and let  $A_\Delta$  be the corresponding RAAG. Recall that, by definition,  $(G^\Theta, E(\Lambda))$  is a RAAG system if and only if the natural homomorphism  $\phi: A_\Delta \rightarrow G^\Theta$  extending the bijection between  $V(\Delta)$  and  $E(\Lambda)$  is an isomorphism. As  $\phi$  is always surjective, we would like to understand when  $\phi$  is injective.

For the remainder of this section, we fix  $\Gamma, \Lambda, \Theta, A_\Delta$ , and  $\phi$  as above. Furthermore, we will use the following terminology. The path  $\gamma$  in  $\Theta$  *visiting vertices*  $x_1, x_2, \dots, x_n$  is defined to be the path which starts at  $x_1$ , passes through the remaining vertices in the order listed, and ends at  $x_n$ . We say that  $\gamma$  is simple if  $x_i \neq x_j$  for  $i \neq j$ , and that  $\gamma$  is a loop if  $x_1 = x_n$ . Finally,  $\gamma$  is a cycle if it is a loop with  $n \geq 3$ , such that  $x_i \neq x_j$  unless  $\{i, j\} = \{1, n\}$ . We call a path (resp. cycle) in  $\Theta$  consisting only of  $\Gamma$ -edges a  $\Gamma$ -path (resp.  $\Gamma$ -cycle). We define  $\Lambda$ -paths and  $\Lambda$ -cycles similarly.

We begin by describing some graph-theoretic conditions on  $\Theta$  which are consequences of either  $G^\Theta$  being a RAAG or of  $(G^\Theta, E(\Lambda))$  being a RAAG system.

Conditions  $\mathcal{R}_1$  and  $\mathcal{R}_2$ , defined below, when combined, are equivalent to LaForge's star-cycle condition. LaForge [2017, Lemma 8.2.1] proves that  $\mathcal{R}_1$  and  $\mathcal{R}_2$  are necessary conditions for  $(G^\Theta, E(\Lambda))$  to be a RAAG system. We include proofs here for completeness.

**Definition 3.1** (condition  $\mathcal{R}_1$ ) We say that  $\Theta$  satisfies *condition  $\mathcal{R}_1$*  if it does not contain a  $\Lambda$ -cycle.

**Lemma 3.2** [LaForge 2017] *If  $(G^\Theta, E(\Lambda))$  is a RAAG system, then  $\Theta$  satisfies  $\mathcal{R}_1$ .*

**Proof** Suppose  $\Theta$  does not satisfy  $\mathcal{R}_1$ . Then it contains a  $\Lambda$ -cycle, say with vertices  $a_1, \dots, a_k$ , where  $k \geq 3$ , such that for each  $i \pmod k$ ,  $a_i$  is connected to  $a_{i+1}$  by a  $\Lambda$ -edge. Let  $g_i$  be the generator of  $A_\Delta$  (or its inverse) corresponding to the (oriented)  $\Lambda$ -edge  $a_i a_{i+1}$ . As the  $a_i$ 's are along a cycle, no  $\Lambda$ -edge is repeated, and we have that  $g_i \neq g_j^{-1}$  for all  $i \neq j$ . This, together with the fact that RAAGs

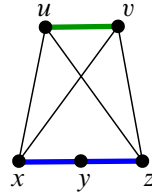


Figure 2

satisfy the deletion condition (see [Theorem 2.2](#)), implies that  $g = g_1g_2 \cdots g_k$  is a nontrivial element of  $A_\Delta$ . Moreover,  $\phi(g) = (a_1a_2)(a_2a_3) \cdots (a_k a_1) = 1$ , so  $g$  is in the kernel of  $\phi$ , and therefore  $\phi$  is not injective. □

**Definition 3.3** (condition  $\mathcal{R}_2$ ) We say that  $\Theta$  satisfies *condition  $\mathcal{R}_2$*  if each component of  $\Lambda \subset \Theta$  (with the natural inclusion) is an induced subgraph of  $\Theta$ .

**Lemma 3.4** [[LaForge 2017](#)] If  $G^\Theta$  is a RAAG, then  $\Theta$  satisfies  $\mathcal{R}_2$ .

**Proof** Suppose  $\Theta$  does not satisfy  $\mathcal{R}_2$ , and let  $u$  and  $v$  be a pair of vertices in a component of  $\Lambda$ , such that  $u$  and  $v$  are adjacent in  $\Theta$ . It follows that  $u$  and  $v$  are connected by a  $\Gamma$ -edge, and therefore they commute. Since  $u$  and  $v$  are in the same component of  $\Lambda$ , there is a simple  $\Lambda$ -path from  $u$  to  $v$  whose vertices (in order) are  $u = a_1, \dots, a_k = v$ . Note that  $k \geq 3$ , since  $\Theta$  is a simplicial graph. For  $1 \leq i \leq k - 1$ , let  $g_i$  be the generator of  $A_\Delta$  (or its inverse) corresponding to the  $\Lambda$ -edge  $a_i a_{i+1}$ , and let  $g = g_1g_2 \cdots g_{k-1}$ . The element  $g$  is a nontrivial element of  $A_\Delta$ , as RAAGs satisfy the deletion condition by [Theorem 2.2](#).

We now have that  $\phi(g)^2 = ((a_1a_2)(a_2a_3) \cdots (a_{k-1}a_k))^2 = (a_1a_k)^2 = (uv)^2 = 1$ , since  $u$  and  $v$  commute. This implies that  $G^\Theta$  has torsion. Thus,  $G^\Theta$  cannot be a RAAG as RAAGs are torsion-free [[Charney 2007](#)]. □

Our next condition,  $\mathcal{R}_3$ , is motivated by the following example.

**Example 3.5** Let  $\Theta$  be the graph in [Figure 2](#), where the  $\Gamma$  edges are black and the  $\Lambda$  edges are colored. Since  $u$  and  $v$  each commute with  $x$  and  $z$ , the commutator  $[uv, xz]$  represents the trivial element in  $W_\Gamma$ . Now observe that  $[uv, xz] \simeq (uv)(xy)(yz)(vu)(zy)(yz)$ , which is a product of  $\Lambda$ -edges, and therefore represents an element  $g$  of  $G^\Theta$ . Now we can see that  $(G^\Theta, E(\Lambda))$  is not a RAAG system: if it were, then it would be possible to show that  $g$  is trivial in  $G^\Theta$  using only swap and deletion moves involving RAAG generators. However, since  $y$  does not commute with  $u$  and  $v$ , no such moves are possible (see [Lemma 2.6](#)). On the other hand, if there had been  $\Gamma$  edges, from  $y$  to both  $u$  and  $v$ , then there would be no contradiction.

A  $\Lambda$ -edge word similar to the one in the above example can be constructed whenever  $\Gamma$  has a square whose vertices alternate between two components of  $\Lambda$ . The example suggests that for such a  $\Lambda$  to define

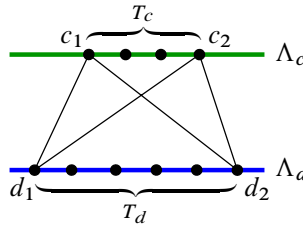


Figure 3: In the figure, the colored parts consist of  $\Lambda$ -edges, and the black parts consist of  $\Gamma$ -edges. The condition  $\mathcal{R}_3$  says that if  $\Theta$  contains a black square as shown, then every vertex of  $T_c$  is joined by a  $\Gamma$ -edge to every vertex of  $T_d$ .

a RAAG, the “intermediate” vertices in  $\Lambda$  between the endpoints of the square must all mutually commute. This is made precise in the definition of  $\mathcal{R}_3$  (Definition 3.8) and Lemma 3.11 below. Before stating these, we introduce some terminology, which will be used throughout this section.

**Definition 3.6** (2-component paths and cycles) We say the  $\Gamma$ -path  $\gamma$  in  $\Theta$  is a 2-component path if  $\gamma$  visits vertices (in order)  $c_1, d_1, c_2, d_2, \dots, c_n, d_n$  for some  $n \geq 1$  (where  $d_n$  could be omitted if  $n > 1$ ) such that the  $c_i$ ’s all lie in a single component  $\Lambda_c$  of  $\Lambda$ , and the  $d_i$ ’s all lie in a single component  $\Lambda_d \neq \Lambda_c$  of  $\Lambda$ . If it is important to emphasize the components visited by  $\gamma$ , we will call it a  $\Lambda_c \Lambda_d$ -path.

A 2-component loop is a 2-component path visiting  $c_1, d_1, \dots, c_n, d_n, c_{n+1}$  such that  $c_1 = c_{n+1}$ . A 2-component cycle is a 2-component loop which is a  $\Gamma$ -cycle. A 2-component cycle of length four will be called a 2-component square.

**Definition 3.7** ( $\Lambda$ -convex hull) We define the  $\Lambda$ -convex hull of a set  $X \subset V(\Theta)$  to be the convex hull of  $X$  in  $\Lambda$ .

**Definition 3.8** (condition  $\mathcal{R}_3$ ) We say that  $\Theta$  satisfies condition  $\mathcal{R}_3$  if the following holds for every 2-component square in  $\Theta$ . Consider a 2-component square in  $\Theta$  visiting vertices  $c_1, d_1, c_2$  and  $d_2$ , where  $c_1, c_2 \in \Lambda_c, d_1, d_2 \in \Lambda_d$ , and  $\Lambda_c$  and  $\Lambda_d$  are distinct components of  $\Lambda$ . Then the graph  $\Gamma$  contains the join of  $V(T_c)$  and  $V(T_d)$ , where  $T_c$  and  $T_d$  are the  $\Lambda$ -convex hulls of  $\{c_1, c_2\}$  and  $\{d_1, d_2\}$  respectively. (See Figure 3.)

We will often need to utilize an expression for a word in  $W_\Gamma$  which is the product of  $\Lambda$ -edges. This construction is the content of the following definition.

**Definition 3.9** ( $\Lambda$ -edge words) Suppose  $\Theta$  satisfies condition  $\mathcal{R}_1$ , and let  $w$  be a word in  $W_\Gamma$  such that  $w = (a_1 a'_1)(a_2 a'_2) \cdots (a_n a'_n)$ , where  $a_i$  and  $a'_i$  are in the same  $\Lambda$ -component of  $\Theta$  for each  $1 \leq i \leq n$ . As  $\Theta$  satisfies  $\mathcal{R}_1$ , there is a unique simple  $\Lambda$ -path from  $a_i$  to  $a'_i$ . Let  $a_i = a^i_1, \dots, a^i_{m_i} = a'_i$  be the vertices visited by this path. Form the word

$$w' = ((a^1_1 a^1_2)(a^1_2 a^1_3) \cdots (a^1_{(m_1-1)} a^1_{m_1})) \cdots ((a^n_1 a^n_2)(a^n_2 a^n_3) \cdots (a^n_{(m_n-1)} a^n_{m_n})).$$

We call  $w'$  the  $\Lambda$ -edge word associated to  $w$ . Note that  $w'$  is well-defined, as long as  $\Theta$  satisfies  $\mathcal{R}_1$ . In particular if  $(G^\Theta, E(\Lambda))$  is a RAAG system, then  $w'$  is well-defined by Lemma 3.2. Also note that  $w \simeq w'$  and  $w'$  is a product of  $\Lambda$ -edges.

**Remark 3.10** Suppose that  $\Theta$  satisfies  $\mathcal{R}_1$  and that  $a, a' \in \Theta$  are two vertices in the same  $\Lambda$ -component. Let  $w' = (a_1a_2)(a_2a_3)\cdots(a_{n-1}a_n)$  be the  $\Lambda$ -edge word associated to  $w = aa'$  (in particular  $a = a_1, a_2, \dots, a_n = a'$  is the unique simple  $\Lambda$ -path from  $a$  to  $a'$ ). We remark that given a  $\Lambda$ -edge  $xy$  of  $\Theta$ , there is at most one occurrence of one of  $xy$  or  $yx$  in  $w'$ . This fact will be relevant in the proofs of the next two lemmas.

Before diving into the next lemma, we briefly discuss some of the ideas used in its proof, and the proof of Lemma 3.16. In each case, we will have a word  $w$  over the RACG  $W_\Gamma$  representing the identity element. We then find a  $\Lambda$ -edge word  $w'$  associated to  $w$  as in Definition 3.9. The word  $w'$  has a natural decomposition into  $\Lambda$ -edges,  $w' = (s_1s'_1)\cdots(s_ns'_n)$ . Moreover, there is a RAAG generator  $g_i \in \Delta$  associated to each  $s_i s'_i = \phi(g_i)$ . By a slight abuse of notation, we also think of  $w' = g_1 \cdots g_n$  as a word over the RAAG  $A_\Delta$ . Doing so, we consider a disk diagram  $D$  in the RAAG  $A_\Delta$  with boundary  $g_1 \cdots g_n$ . The edges of  $D$  are labeled by the  $g_i$ 's. To simplify things, by another abuse of notation we also think of these edges as labeled by the  $\Lambda$ -edges  $s_i s'_i$ . We use the intersection patterns of hyperplanes in  $D$  to deduce commuting relations between the generators of the RAAG. Consequently, this gives us commuting relations between the  $\Lambda$ -edges and for generators in the RACG  $W_\Gamma$ .

**Lemma 3.11** *If  $(G^\Theta, E(\Lambda))$  is a RAAG system, then  $\Theta$  satisfies  $\mathcal{R}_3$ .*

**Remark 3.12** (comparison of Lemma 3.11 with Laforge's chain-chord condition) LaForge [2017, Lemma 8.2.3] introduced a necessary condition, called the chain-chord condition, which, if interpreted in the language of joins and 2-component cycles, is close to our condition  $\mathcal{R}_3$ . We note that there are errors in the statement and proof of [LaForge 2017, Lemma 8.2.3].

**Proof of Lemma 3.11** Suppose there is a 2-component square  $\gamma$  in  $\Theta$  visiting vertices  $c_1, d_1, c_2$  and  $d_2$  as in condition  $\mathcal{R}_3$ . Let  $\Lambda_c$  and  $\Lambda_d$  be the components of  $\Lambda$  respectively containing  $\{c_1, c_2\}$  and  $\{d_1, d_2\}$ . Let  $T_c$  and  $T_d$  be the  $\Lambda$  convex hulls respectively of  $\{c_1, c_2\}$  and  $\{d_1, d_2\}$ . By Lemma 3.2, there is a unique simple  $\Lambda$ -path from  $c_1$  to  $c_2$  (resp.  $d_1$  to  $d_2$ ) and this path is equal to  $T_c$  (resp.  $T_d$ ).

Let  $w$  denote the commutator  $[c_1c_2, d_1d_2]$ . The existence of  $\gamma$  tells us that  $c_1$  and  $c_2$  both commute with  $d_1$  and  $d_2$ , so  $w$  represents the identity in  $W_\Gamma$ .

Let  $w_1, w_2$  and  $w'$  be the  $\Lambda$ -edge words associated to respectively  $c_1c_2, d_1d_2$  and  $w$ . As  $\phi$  is injective,  $w'$  represents the trivial element of  $A_\Delta$ , and there is a disk diagram  $D$  over  $A_\Delta$  with boundary label  $w'$ . We warn that the edges of  $D$  are labeled by  $\Lambda$ -edges, ie generators of  $A_\Delta$ . We will analyze hyperplanes of this diagram.

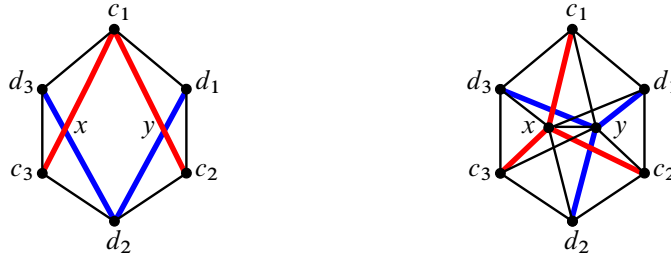


Figure 4: The graph on the left concerns Example 3.13 and the graph of the right concerns Example 3.14.

Let  $p_{w_1}, p_{w_2}, p_{w_1^{-1}}$  and  $p_{w_2^{-1}}$  be the paths in  $\partial D$  with labels  $w_1, w_2, w_1^{-1}$  and  $w_2^{-1}$  respectively. For  $i \in \{1, 2\}$ , the word  $w_i$  (thought of as a word over  $V(\Delta) = E(\Lambda)$ ) does not contain any repeated letters (or their inverses) in  $V(\Delta)$  by Remark 3.10. Consequently, a hyperplane is dual to at most one edge of  $p_{w_1}$  (resp.  $p_{w_2}, p_{w_1^{-1}}$  and  $p_{w_2^{-1}}$ ). Furthermore,  $w_1$  and  $w_2$  are words over  $E(T_c)$  and  $E(T_d)$  respectively. As  $\Lambda_c$  and  $\Lambda_d$  are distinct components of  $\Lambda$ , a hyperplane dual to an edge of  $p_{w_1}$  must be dual to an edge of  $p_{w_1^{-1}}$  and vice versa. A similar statement holds for hyperplanes dual to  $p_{w_2}$  and  $p_{w_2^{-1}}$ .

It follows that every hyperplane dual to  $p_{w_1}$  intersects every hyperplane dual to  $p_{w_2}$ . Consequently, every  $\Lambda$ -edge in the word  $w_1$  commutes with every  $\Lambda$ -edge in the word  $w_2$ . Since  $\phi$  is a homomorphism, the Coxeter group elements corresponding to these  $\Lambda$ -edges must commute as well. By Lemma 2.6 each vertex of  $T_c$  commutes with each vertex of  $T_d$ . □

The next example shows that the conditions obtained so far are not sufficient for  $(G^\Theta, E(\Lambda))$  to be a RAAG system.

**Example 3.13** Let  $\Gamma$  be a hexagon, and let  $\Lambda$  be the graph with two components shown on the left side in Figure 4. It is clear that  $\mathcal{R}_1, \mathcal{R}_2$ , and  $\mathcal{R}_3$  are satisfied. However, by considering the word

$$w = (c_1c_2)(d_1d_2)(c_2c_3)(d_2d_3)(c_3c_1)(d_3d_1)$$

we can see that  $(G^\Theta, E(\Lambda))$  is not a RAAG system. Specifically, the commutation relations specified by  $\Gamma$ -edges show that  $w \simeq 1$  in  $W_\Gamma$ . Moreover  $w$  can be expressed as a product of  $\Lambda$ -edges using the  $\Lambda$ -edge words corresponding to each parenthetical element. However, it is not possible to reduce this word to the empty word using just swap and deletion moves involving the  $\Lambda$ -edges, and as a result,  $(G^\Theta, E(\Lambda))$  cannot be a RAAG system. A rigorous proof of this fact follows from Lemma 3.16 below.

Example 3.13 shows that at least one additional condition is needed in order to obtain a characterization of visual RAAGs, and suggests that this condition may be a generalization of  $\mathcal{R}_3$  involving longer 2-component cycles instead of squares. It is tempting to conjecture that, given any  $\Lambda_c\Lambda_d$ -cycle with corresponding  $\Lambda$ -convex hulls  $T_c$  and  $T_d$ , the graph  $\Gamma$  contains the join of  $V(T_c)$  and  $V(T_d)$  (as is the case when the cycle has length four, by Lemma 3.11 above). However, the following example shows this is not necessarily true for longer cycles.



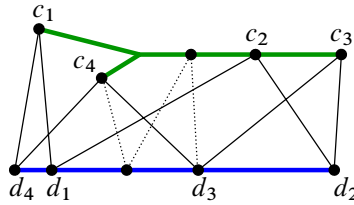


Figure 5: This figure illustrates condition  $\mathcal{R}_4$ . The green subgraph is  $T_c$  and the blue subgraph is  $T_d$ . The condition says that any edge in the 2-component cycle (shown in solid black edges) is part of a square of  $\Gamma$  with two vertices in  $T_c$  and two in  $T_d$ . This is illustrated for the edge from  $d_3$  to  $c_4$ . The dotted lines are  $\Gamma$ -edges which are not necessarily in the 2-component cycle.

**Example 3.14** In Figure 4, let  $\Theta$  be the graph on the right where  $\Gamma$ -edges are black and  $\Lambda$  edges are colored. Observe that  $\Lambda$  has two components, colored red and blue. Consider the 2-component cycle visiting vertices  $c_1, d_1, c_2, d_2, c_3, d_3$  and  $c_1$ . Then  $T_c$  is the entire red tree and  $T_d$  is the entire blue tree. However,  $\Gamma$  does not contain the join of  $V(T_c)$  and  $V(T_d)$ . (For example, there is no edge in  $\Gamma$  connecting  $c_1$  and  $d_2$ .) On the other hand,  $(G^\Theta, E(\Lambda))$  is a RAAG system in this case. (See Corollary 5.1 for a proof.)

Despite the fact that  $\mathcal{R}_3$  does not generalize to a necessary condition on longer cycles in the obvious way, the following weaker statement does turn out to be necessary to guarantee that  $(G^\Theta, E(\Lambda))$  is a RAAG system and is missing from [LaForge 2017].

**Definition 3.15** (condition  $\mathcal{R}_4$ ) We say that  $\Theta$  satisfies condition  $\mathcal{R}_4$  if the following holds. Let  $\gamma$  be any  $\Lambda_c \Lambda_d$ -cycle in  $\Theta$  visiting vertices  $c_1, d_1, c_2, d_2, \dots, c_n, d_n, c_1$  for some  $n \geq 2$ . Let  $T_c$  and  $T_d$  be the  $\Lambda$ -convex hulls of  $\{c_1, \dots, c_n\}$  and  $\{d_1, \dots, d_n\}$  respectively. Then every edge of  $\gamma$  is contained in a 2-component square of  $\Theta$  with two vertices in  $T_c$  and two vertices in  $T_d$ . (See Figure 5.)

The next lemma shows  $\mathcal{R}_4$  is necessary for  $(G^\Theta, E(\Lambda))$  to be a RAAG system.

**Lemma 3.16** If  $(G^\Theta, E(\Lambda))$  is a RAAG system, then  $\Theta$  satisfies  $\mathcal{R}_4$ .

**Proof** Let  $\gamma$  be a  $\Lambda_c \Lambda_d$ -cycle visiting vertices  $c_1, d_1, \dots, c_n, d_n, c_1$ , and let  $T_c$  and  $T_d$  be as in Definition 3.15. Let  $w$  be the word

$$(1) \quad w = (c_1 c_2)(d_1 d_2)(c_2 c_3)(d_2 d_3) \cdots (d_{n-1} d_n)(c_n c_1)(d_n d_1).$$

Then  $w \simeq 1$  in  $W_\Gamma$ . To see this, note that for each  $i$ , we know that  $c_i$  commutes with  $d_{i-1}$  and  $d_i$  (where  $i$  is taken mod  $n$ ). Using this we can cancel the  $c_i$  for  $i > 1$  in pairs to get

$$w \simeq c_1 d_1 d_2 d_2 d_3 \cdots d_{n-1} d_n c_1 d_n d_1 \simeq c_1 d_1 c_1 d_1 \simeq 1.$$

Let  $w'$  be the  $\Lambda$ -edge word associated to  $w$ . Let  $D$  be a disk diagram over  $A_\Delta$  with boundary label  $w'$ . As in the proof of Lemma 3.11, edges of  $D$  are labeled by  $\Lambda$ -edges, which are thought of as generators of  $A_\Delta$ .

Color the part of the boundary of  $D$  and the hyperplanes coming out of it green if they correspond to  $\Lambda$ -edges from  $\Lambda_c$  and blue if they correspond to  $\Lambda$ -edges from  $\Lambda_d$ . Now we see from the structure of  $w'$  that  $\partial D$  alternates between green and blue stretches, and a stretch of a given color corresponds to a simple path in the corresponding component of  $\Lambda$ . It follows from [Remark 3.10](#) that a hyperplane of a given color must start and end in different stretches of that color.

Let  $L = |E(T_c)|$  denote the number of  $\Lambda$ -edges in  $T_c$ . We will prove that condition  $\mathcal{R}_4$  holds for  $\gamma$  by induction on  $(n, L)$ . The conclusion of the lemma is obvious for  $\gamma$  corresponding to  $(2, L)$  for any  $L$ , since the cycle itself is a square. This includes the base case, when  $n = 2$  (ie  $\gamma$  is a square) and  $T_c$  is an edge. Now let  $n > 2$ , and assume the claim is true for all  $(n', L')$  such that either  $n' < n$  or  $n' = n$  and  $L' < L$ .

By [Lemma 3.2](#),  $T_c$  and  $T_d$  are trees. Now suppose  $c_j$  is a leaf of  $T_c$ , and let  $xc_j$  be the  $\Lambda$ -edge incident to  $c_j$  in  $T_c$ . Since  $c_i \neq c_j$  for all  $i \neq j$  (by the definition of a 2-component cycle), we know that  $xc_j$  occurs exactly once in  $w'$  (as part of the subword of  $w'$  representing  $c_{j-1}c_j$ ) and  $c_jx$  occurs exactly once in  $w'$  (as part of the subword representing  $c_jc_{j+1}$ ). It follows there is a unique hyperplane  $H$  labeled  $xc_j$  which is dual to both the path whose label is an expression for  $c_jc_{j+1}$  and the path whose label is an expression for  $c_{j-1}c_j$ . Moreover, the subword  $w''$  of  $w'$  between these two subwords is the product of  $\Lambda$ -edges which is an expression for  $d_{j-1}d_j$ . It follows that every hyperplane dual to the path in  $\partial D$  labeled  $w''$  must intersect the hyperplane  $H$ . By [Lemma 2.6](#), both  $x$  and  $c_j$  commute (in  $W_\Gamma$ ) with each letter of  $V(\Gamma)$  used in the word  $w''$ . In particular,  $d_{j-1}$  and  $d_j$  each commute with  $x$ .

Now there are two possibilities. Suppose first that  $x = c_t$  for some  $t \neq j$ . Since  $t \neq j$  and  $n > 2$  (which implies that  $\gamma$  has more than four edges), it follows that either  $c_t d_{j-1}$  or  $c_t d_j$  is a diagonal of  $\gamma$ . We can use this diagonal to cut  $\gamma$  into two 2-component cycles  $\gamma_1$  and  $\gamma_2$  as follows. Assume  $c_t d_j$  is a diagonal  $\delta$  of  $\gamma$  (the other case is analogous), and let  $\beta_1$  and  $\beta_2$  be the two components of  $\gamma$  obtained by removing the vertices labeled  $c_t$  and  $d_j$ . Set  $\gamma_1 = \beta_1 \cup \delta$  and  $\gamma_2 = \beta_2 \cup \delta$ . Note  $\gamma_1$  and  $\gamma_2$  each have strictly fewer vertices than  $\gamma$ . For  $i = 1, 2$  let  $T_c^i$  and  $T_d^i$  be the components of the  $\Lambda$ -convex hull of  $\gamma_i$  contained respectively in  $\Lambda_c$  and  $\Lambda_d$ . By the induction hypothesis, we see that every edge in  $\gamma_i$  is part of a square in  $\Gamma$  with two vertices in  $T_c^i \subset T_c$  and two in  $T_d^i \subset T_d$ . Since each edge of  $\gamma$  is either an edge of  $\gamma_1$  or of  $\gamma_2$ , the claim follows for this case.

On the other hand, suppose that  $x \neq c_i$  for any  $1 \leq i \leq n$ . Consider the new 2-component cycle  $\gamma'$  obtained from  $\gamma$  by replacing the edges  $d_{j-1}c_j$  and  $c_j d_j$  with  $d_{j-1}x$  and  $x d_j$ . As  $x \neq c_i$  for any  $1 \leq i \leq n$ , this does not violate the requirement that 2-component cycles do not repeat vertices. Let  $T_c'$  and  $T_d'$  be the components of the  $\Lambda$ -convex hull of  $\gamma'$  contained respectively in  $\Lambda_c$  and  $\Lambda_d$ . Since  $c_j$  is a leaf of  $T_c$ , it follows that  $|E(T_c')| < |E(T_c)|$ , and we also have that  $|V(\gamma')| = |V(\gamma)| = n$ . We now apply the induction hypothesis to conclude that each edge of  $\gamma'$  is part of a square of  $\Gamma$  with two vertices in  $T_d' = T_d$  and two vertices in  $T_c' \subset T_c$ . This means that this property holds automatically for all edges of  $\gamma$ , except possibly  $d_{j-1}c_j$  and  $c_j d_j$ . However, these edges are part of the square in  $T_c$  with vertices  $x$ ,  $c_j$ ,  $d_{j-1}$  and  $d_j$ . Thus, the claim follows for this case as well.  $\square$

The following proposition summarizes Lemmas 3.2, 3.4, 3.11 and 3.16.

**Proposition 3.17** *If  $(G^\Theta, E(\Lambda))$  is a RAAG system, then  $\Theta$  satisfies  $\mathcal{R}_1$ – $\mathcal{R}_4$ .*

If  $\Lambda$  has at most two components, then it turns out that there are no additional obstructions to  $(G^\Theta, E(\Lambda))$  being a RAAG system. More precisely:

**Theorem 3.18** *Suppose  $\Lambda$  has at most two components. Then  $(G^\Theta, E(\Lambda))$  is a RAAG system if and only if  $\mathcal{R}_1$ – $\mathcal{R}_4$  are satisfied.*

**Proof outline** Proposition 3.17 constitutes one direction of the theorem. The following strategy will be used to prove that  $\mathcal{R}_1$ – $\mathcal{R}_4$  imply that  $(G^\Theta, E(\Lambda))$  is a RAAG system. We wish to show that the image of every nontrivial element of  $A_\Delta$  under  $\phi$  is nontrivial in  $W_\Gamma$ .

Towards a contradiction, we assume that there exists some nontrivial  $g \in A_\Delta$  such that  $\phi(g) = 1$ . Then there is a disk diagram  $D$  whose boundary label is a word in  $\Lambda$ –edges which represents  $\phi(g)$ . We will put this word in a certain normal form which will be defined in terms of the configuration of hyperplanes in  $D$ .

To define the normal form, we first show that the set of all hyperplanes can be partitioned into subsets that we call “closed chains of hyperplanes” (see Definition 3.20 and Figure 6). Properties of hyperplanes and closed chains can be translated into information about the graph  $\Theta$  and vice versa (see Observations 3.19, 3.23 and 3.24). Next, we prove in Lemma 3.25 that we can fix a particular closed chain  $\mathcal{H}$  which is “maximally nested” in a certain sense. Specifically,  $\mathcal{H}$  has a distinguished hyperplane  $H_0$  such that every other closed chain either intersects  $H_0$  or is separated from the rest of  $\mathcal{H}$  by  $H_0$  (see Figure 8).

Our normal form is defined in terms of the fixed closed chain  $\mathcal{H}$ . We first choose a basepoint  $p$  on  $\partial D$  which is the endpoint of an edge of  $\partial D$  dual to  $H_0$ . (This has the effect of possibly replacing our original element  $g \in A_\Delta$  with a conjugate.) Let  $w$  be the label of  $\partial D$  read clockwise starting at  $p$ . We show in Claim 3.26 that  $w$ ,  $D$  and  $\mathcal{H}$  may be replaced by an equivalent word  $\tilde{w}$  and corresponding disk diagram  $\tilde{D}$  and maximally nested closed chain  $\tilde{\mathcal{H}}$ , with the property that the  $\Lambda$ –edges in  $\tilde{w}$  coming from  $\tilde{\mathcal{H}}$  are “as far right as possible”, ie it is not possible to swap one of these  $\Lambda$ –edges with a  $\Lambda$ –edge to its right by a commutation relation. We consider  $\tilde{w}$  to be a word in normal form representing  $\phi(g)$ .

Finally, to complete the proof of Theorem 3.18, we will show (by analyzing interactions between closed chains in  $\tilde{D}$ ) that if  $\mathcal{R}_1$ – $\mathcal{R}_4$  are satisfied, then the normal form is violated.

Before we embark on the proof, we need to develop some preliminaries on disk diagrams, and on transferring information from the disk diagram  $D$  to the graph  $\Theta$ . In what follows, we assume that  $D$  is a disk diagram whose boundary is a word  $w$  in the RACG  $W_\Gamma$ . Unlike in the proofs of Lemmas 3.11 and 3.16, we are now working in  $W_\Gamma$  rather than  $A_\Delta$ , so the edges and hyperplanes of  $D$  are labeled by generators of  $W_\Gamma$  rather than elements of  $A_\Delta$  corresponding to  $\Lambda$ –edges. As the words  $w$  we consider are the images of elements of  $A_\Delta$  under  $\phi$ , they have a natural decomposition into  $\Lambda$ –edges.

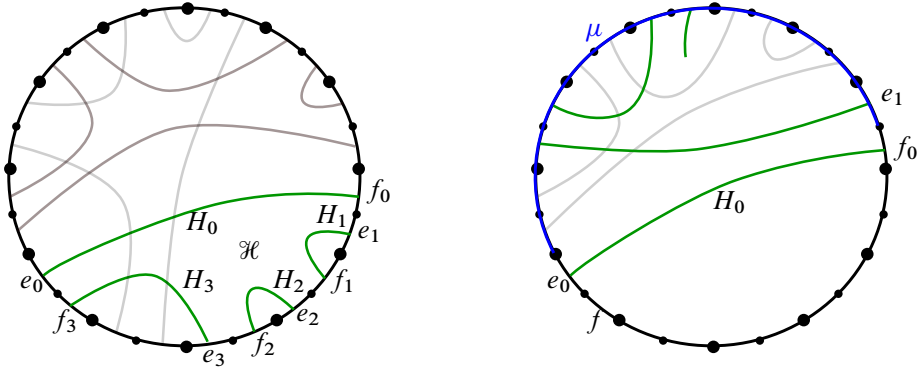


Figure 6: The figure on the left shows a disk diagram  $D$  such that the label of  $\partial D$  has a natural decomposition into  $\Lambda$ -edges, delineated by large black dots. The green hyperplanes form a closed chain of hyperplanes  $\mathcal{H}$ , as defined in Definition 3.20 (here we can take  $\eta = \partial D$ ). Two other closed chains of hyperplanes are shown in gray. The figure on the right shows an impossible configuration pertaining to the proof of Lemma 3.21. The path  $\mu$  from the lemma is colored blue.

We associate a color (red and green) to each component of  $\Lambda$ . Each hyperplane of  $D$  then inherits the color corresponding to the component of  $\Lambda$  in which its label lies. Thus, two edges of  $\partial D$  contained in the same  $\Lambda$ -edge are dual to hyperplanes of the same color.

**Observation 3.19** *If  $\Theta$  satisfies  $\mathcal{R}_2$ , then no two hyperplanes of the same color intersect. This is because if two hyperplanes intersect, then their labels are distinct and commute, and so are connected by a  $\Gamma$ -edge. Thus they cannot be in the same component of  $\Lambda$ , since each component of  $\Lambda$  is an induced subgraph of  $\Theta$ , by  $\mathcal{R}_2$ .*

The hyperplanes of  $D$  can be partitioned into “closed chains of hyperplanes”, as described in Definition 3.20 below. Although the proof of Theorem 3.18 only uses disk diagrams whose boundary labels are words in  $\Lambda$ -edges, the definition below applies to slightly more general disk diagrams, as this will be needed in Section 6.

**Definition 3.20** (chains of hyperplanes) Let  $D$  be a disk diagram whose boundary  $\partial D$  contains a connected subpath  $\eta$  (possibly all of  $\partial D$ ), such that the label of  $\eta$  is a word in  $\Lambda$ -edges. Let  $H_0, \dots, H_n$  be a sequence of distinct hyperplanes in  $D$ . Let  $e_i$  and  $f_i$  be the edges on  $\partial D$  that are dual to  $H_i$ . (See Figure 6 for an illustration when  $n = 3$ .) We say that  $\{H_0, \dots, H_n\}$  is a chain in  $D$ , if for all  $0 \leq i < n$ , the edges  $f_i$  and  $e_{i+1}$  are contained in  $\eta$  and are dual to the same  $\Lambda$ -edge of  $\eta$ . Note that  $e_0$  and  $f_n$  can be dual to edges not contained in  $\eta$ .

Additionally, if  $e_0$  and  $f_n$  are contained in the same  $\Lambda$ -edge of  $\eta$ , we say that  $\{H_0, \dots, H_n\}$  is a closed chain. (Figure 6 shows three closed chains.)

Since the two hyperplanes dual to a  $\Lambda$ -edge have the same color, each chain also inherits a well-defined color.

**Lemma 3.21** *If the label of  $\partial D$  is a word in  $\Lambda$ -edges, then every hyperplane of  $D$  is contained in a unique closed chain. Thus, there is a partition of the hyperplanes of a given color into closed chains.*

**Proof** Let  $H_0$  be a hyperplane of  $D$  dual to edges  $e_0$  and  $f_0$  of  $\partial D$ . Assume  $H_0$  is green. Let  $f$  and  $e_1$  be edges of  $\partial D$  which pair with  $e_0$  and  $f_0$  respectively to form  $\Lambda$ -edges. We claim that  $f$  and  $e_1$  are in the same component of  $D \setminus H_0$ . If not, there would be an odd number of edges in a part  $\mu$  of  $\partial D$  between  $e_0$  and  $f_0$  (see the right side of Figure 6). Since the hyperplanes dual to the two edges of a  $\Lambda$ -edge have the same color, an odd number of these edges would be dual to green hyperplanes. This is a contradiction, since no green hyperplanes can cross  $H_0$  by Observation 3.19, so there must be an even number of edges in  $\mu$  dual to green hyperplanes.

The hyperplane  $H_1$  dual to  $e_1$  is green, and cannot cross  $H_0$ . Let  $f_1$  be the other edge dual to  $H_1$ . If  $f_1 = f$  we have a closed chain. Otherwise, there is an edge  $e_2$  which pairs with  $f_1$  to form a  $\Lambda$ -edge. By the same argument as before,  $e_2$  is in the same component of  $D \setminus H_1$  as  $f_0$  and there is a green hyperplane  $H_2$  dual  $e_2$  and another edge  $f_2$ , such that  $H_2$  does not cross  $H_0$  or  $H_1$  (see the left side of Figure 6). Continuing this process we obtain a sequence of hyperplanes as in Definition 3.20. Since the number of possibilities for  $f_i$  reduces each time, eventually the process stops, with  $f_n = f$  for some  $n$ , and  $H_0, \dots, H_n$  form a closed chain. □

We say that a chain  $\mathcal{K}$  intersects a hyperplane  $H$  if some  $K \in \mathcal{K}$  intersects  $H$ . We say that chains  $\mathcal{K}$  and  $\mathcal{L}$  intersect if  $\mathcal{K}$  intersects some  $H \in \mathcal{L}$ . We will need the following observation:

**Observation 3.22** *If a hyperplane  $H$  intersects a closed chain  $\mathcal{K}$ , then it intersects  $\mathcal{K}$  in exactly two distinct hyperplanes. To see this, note that given a hyperplane  $K \in \mathcal{K}$ , the hyperplanes in  $\mathcal{K} \setminus \{K\}$  all lie in a single component of  $D \setminus K$ . It follows that if  $H$  intersects  $\mathcal{K}$  more than twice, it must intersect some hyperplane of  $\mathcal{K}$  twice. This contradicts the fact that  $D$  has no bigons (see Remark 2.9).*

The following two observations enable us to transfer information from the disk diagram  $D$  to the graph  $\Theta$ .

**Observation 3.23** (chains in  $D$  give  $\Lambda$ -paths in  $\Theta$ ) *Let  $\mathcal{K} = \{K_0, \dots, K_l\}$  be a chain in  $D$ , and for  $0 \leq i \leq l$ , let  $k_i$  be the label of  $K_i$ . Then by the definition of a chain,  $K_i$  and  $K_{i+1}$  are dual to the same  $\Lambda$ -edge in  $\partial D$  for each  $i$ , so there is an edge in  $\Lambda$  between  $k_i$  and  $k_{i+1}$ . It follows that  $\mathcal{K}$  naturally defines a  $\Lambda$ -path in  $\Theta$  visiting vertices  $k_0, k_1, \dots, k_l$ . Moreover, if  $\mathcal{K}$  is a closed chain, then the corresponding  $\Lambda$ -path is a loop. See Figure 7.*

**Observation 3.24** (pairs of intersecting closed chains give 2-component loops in  $\Theta$ ) *Consider two closed chains which intersect, say a red chain  $\mathcal{R}$  and a green chain  $\mathcal{G}$ . Let  $H_1 \in \mathcal{R}$  and  $K_1 \in \mathcal{G}$  be intersecting hyperplanes. By Observation 3.22, the hyperplane  $K_1$  intersects  $\mathcal{R}$  in a second hyperplane  $H_2 \neq H_1$ . Similarly,  $H_2$  intersects  $\mathcal{G}$  in a second hyperplane  $K_2$ . Proceeding in this way, we obtain a polygon with at least four sides, with sides alternating between red and green hyperplanes. See the left side of Figure 7.*

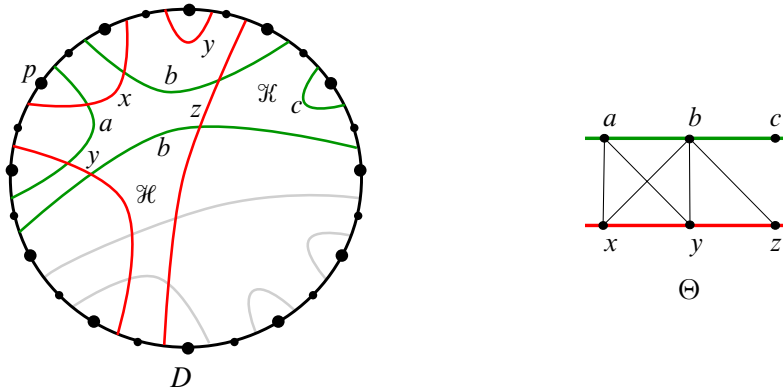


Figure 7: The figure illustrates Observations 3.23 and 3.24. On the left is a disk diagram  $D$  with red and green closed chains called  $\mathcal{H}$  and  $\mathcal{H}$  respectively. The graph on the right is a part of  $\Theta$ . The labels of the hyperplanes in  $D$  correspond to vertices of  $\Theta$  (in particular of  $\Lambda$ ). Starting at the basepoint  $p$  and going around  $\partial D$  clockwise, the closed chain  $\mathcal{H}$  defines a  $\Lambda$ -loop in  $\Theta$  visiting vertices  $x, y, z, y$  and  $x$ , and the closed chain  $\mathcal{H}$  defines a  $\Lambda$ -loop visiting vertices  $a, b, c, b$  and  $a$ . The polygon coming from the intersection of  $\mathcal{H}$  and  $\mathcal{H}$  defines a 2-component loop in  $\Theta$  visiting vertices  $a, x, b, z, b, y$  and  $a$ . Observe that this 2-component loop is not a cycle.

Since an intersecting pair of hyperplanes corresponds to an edge of  $\Gamma$ , a 2-colored polygon of the type we just constructed defines a 2-component loop in  $\Theta$  (where each edge of the 2-component loop comes from a corner of the 2-colored polygon). See Figure 7. We warn that the 2-component loop obtained from a 2-colored polygon in  $D$  may not be a 2-component cycle. (Note that a 2-component cycle is a 2-component loop in which all of the vertices are distinct, and there are at least two vertices in each component.)

In order to define a normal form for the word  $u$  from the proof outline, we will need to choose a closed chain in  $D$  with some special properties:

**Lemma 3.25** *Let  $u$  and  $D$  be as in the proof outline. There exists a closed chain  $\mathcal{H}$  of  $D$ , containing a distinguished hyperplane  $H_0$ , such that given any closed chain  $\mathcal{H} \neq \mathcal{H}$ , either*

- (1)  $\mathcal{H}$  and  $\mathcal{H} \setminus \{H_0\}$  lie in different components of  $D \setminus H_0$ , or
- (2)  $\mathcal{H}$  intersects  $H_0$ .

**Proof** We iteratively construct a sequence of closed chains  $\mathcal{H}^1, \mathcal{H}^2, \dots$  with distinguished hyperplanes  $H_0^1, H_0^2, \dots$  such that for all  $i > 1$ ,

- (i)  $\mathcal{H}^i$  and  $\mathcal{H}^{i-1} \setminus \{H_0^{i-1}\}$  lie in the same component of  $D \setminus H_0^{i-1}$ , and
- (ii)  $H_0^{i-1}$  and  $\mathcal{H}^i \setminus \{H_0^i\}$  lie in different components of  $D \setminus H_0^i$ .

Let  $\mathcal{H}^1$  and  $H_0^1$  be arbitrary. Now for any  $j$ , if  $\mathcal{H}^j$  and  $H_0^j$  do not satisfy the conclusion of the lemma, then there must exist another closed chain  $\mathcal{H}^{j+1}$  which lies entirely in  $C_j$ , where  $C_j$  is the component

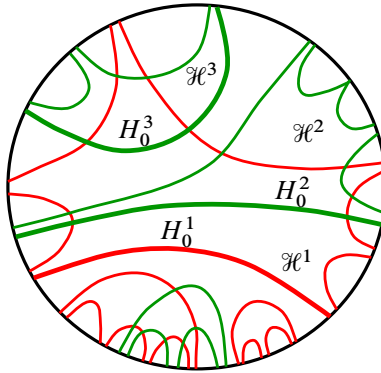


Figure 8: The figure illustrates the procedure for finding  $\mathcal{H}$  and  $H_0$  in Lemma 3.25. Each closed chain in the sequence is labeled in its interior. The hyperplanes  $H_0^i$  are shown in bold. In this example  $\mathcal{H} = \mathcal{H}^3$  has the desired property.

of  $D \setminus H_0^j$  containing  $\mathcal{H} \setminus \{H_0^j\}$ . (Figure 8 illustrates this for  $j = 1, 2$ .) There is a unique hyperplane in  $\mathcal{H}^{j+1}$  satisfying condition (ii) above with  $i = j + 1$ , and we set this equal to  $H_0^{j+1}$ . Thus, we can produce a longer sequence of closed chains with properties (i) and (ii).

By construction, there is a nesting of components  $C_1 \supset C_2 \supset C_3 \supset \dots$ , and it follows that  $H_0^1, H_0^2, \dots$  are distinct hyperplanes in  $D$ . As  $D$  has finitely many hyperplanes, this process can only be repeated finitely many times. Thus,  $\mathcal{H}^j$  satisfies the claim for some  $j$ . □

We are now ready to prove the theorem.

**Proof of Theorem 3.18** As discussed, we need to show that if  $\mathcal{R}_1\text{--}\mathcal{R}_4$  are satisfied, then the map  $\phi: A_\Delta \rightarrow G^\Theta$  is injective. Let  $g \in A_\Delta$  be a nontrivial element. Let  $v = v_1 v_2 \dots v_n$  be a reduced word over the set of the generators of  $A_\Delta$ , which represents  $g$ . By the definition of  $A_\Delta$ , we have that  $\phi(v_i)$  is a  $\Lambda$ -edge of  $\Theta$ , for  $1 \leq i \leq n$ . Then  $u = \phi(v_1)\phi(v_2)\dots\phi(v_n)$  is a concatenation of  $\Lambda$ -edges which represents  $\phi(g)$ . Towards a contradiction, we assume that  $u$  represents the identity element of  $W_\Gamma$ . Then there is a disk diagram  $D$  whose boundary label (read clockwise starting from some basepoint) is  $u$ . By Lemma 2.8 we may assume that  $D$  has no bigons.

An element has trivial image under  $\phi$  if and if every element of its conjugacy class does. Thus, we may assume that  $g$  is of minimal length in its conjugacy class, where the length of an element is defined to be the minimal length of a word representing it.

We partition the hyperplanes of  $D$  into closed chains. (See Lemma 3.21.) By Lemma 3.25, we can choose a chain  $\mathcal{H}$ , with distinguished hyperplane  $H_0$ , such that given any other chain  $\mathcal{K}$ , either  $H_0$  separates  $\mathcal{K}$  from  $\mathcal{H} \setminus \{H_0\}$ , or  $\mathcal{K}$  intersects  $H_0$ . Let  $a_0, a_1, \dots, a_s$  be the labels of the hyperplanes of  $\mathcal{H}$ , starting from  $H_0$ , and proceeding in order in the clockwise direction around  $\partial D$ . Then the  $\Lambda$ -edges  $a_0 a_1, a_1 a_2, \dots, a_{s-1} a_s, a_s a_0$  appear in  $\partial D$  in that order, possibly interspersed with some other  $\Lambda$ -edges.



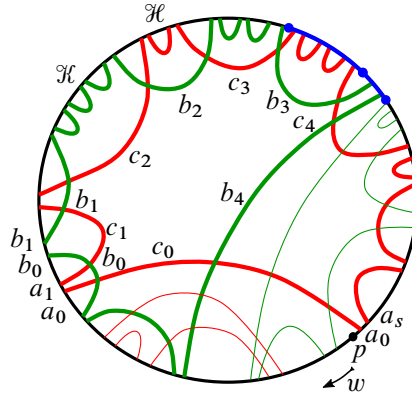


Figure 9: This example illustrates the proof of [Theorem 3.18](#). The chain  $\mathcal{H}$  satisfying [Claim 3.26](#) is shown in thick red lines. In particular, no  $\Lambda$ -edge from  $\mathcal{H}$  (except possibly the last one) commutes with the  $\Lambda$ -edge appearing after it in  $\partial D$ . The chain  $\mathcal{K}$ , which contributes the first  $\Lambda$ -edge not in  $\mathcal{H}$  (after  $a_0a_1$ ), is shown in thick green lines. The polygon formed by the intersection of  $\mathcal{H}$  and  $\mathcal{K}$  induces a 2-component loop which visits (in this example)  $c_0, b_0, c_1, b_1, c_2, b_2, c_3, b_3, c_4, b_4, c_0$ . The blue subpaths of  $\partial D$  are the subpaths defined in [Claim 3.28](#), with  $i = 4$ .

Let  $p$  denote the vertex on  $\partial D$  which is the endpoint of the  $\Lambda$ -edge from  $\mathcal{H}$  labeled  $a_s a_0$ , read clockwise. (See [Figure 9](#).) Let  $w$  be the word labeling  $\partial D$  clockwise, starting from  $p$ . Then  $w$  is a cyclic conjugate of  $u$ . Let  $x$  be the corresponding cyclic conjugate of  $v$ . Since  $v$  was chosen to be reduced, and since  $g$  (the element of  $A_\Delta$  represented by  $v$ ) is of minimal length in its conjugacy class by assumption, it follows that  $x$  is reduced.

We now show that we can modify  $D$  in such a way that the resultant boundary label is a word representing  $w = \phi(x)$  which is in a certain normal form:

**Claim 3.26** *There exists a disk diagram  $\tilde{D}$  such that the following hold.*

- (1) *There is a closed chain  $\tilde{\mathcal{H}}$  in  $\tilde{D}$  which has a distinguished hyperplane  $\tilde{H}_0$  satisfying the criterion in [Lemma 3.25](#). The labels of the hyperplanes of  $\tilde{\mathcal{H}}$  starting from  $\tilde{H}_0$  and proceeding clockwise, are  $a_0, \dots, a_s$  (ie they are the same labels as the labels of the hyperplanes in  $\mathcal{H}$ ).*
- (2) *Let  $\tilde{p}$  be the endpoint of the  $\Lambda$ -edge  $a_s a_0$  from  $\tilde{\mathcal{H}}$ , and let  $\tilde{w}$  be the word labeling  $\partial \tilde{D}$  in the clockwise direction starting from  $\tilde{p}$ . Then  $\tilde{w} = \phi(\tilde{x})$ , where  $\tilde{x}$  is a reduced word in  $A_\Delta$  obtained from  $x$  by Tits swap moves.*
- (3) *The  $\Lambda$ -edges from  $\tilde{\mathcal{H}}$  appear as far right as possible in  $\tilde{w}$ . More formally, the word  $\tilde{w}$  has no subword of the form  $a_i a_{i+1} b b'$  such that  $a_i a_{i+1}$  is one of the  $\Lambda$ -edges coming from  $\tilde{\mathcal{H}}$  with  $0 \leq i \leq s$  (with indices mod  $s$ ),  $a_i a_{i+1} \neq b b'$ , and  $a_i a_{i+1}$  commutes with  $b b'$ .*

**Proof** We construct  $\tilde{D}$  iteratively, starting with  $D$ . If  $\mathcal{H}, p, w$  and  $x$  are as defined above, then the first two conditions in the claim are satisfied. If (3) is not satisfied, then  $w$  has a subword  $a_i a_{i+1} b b'$  as in (3). Since  $a_s a_0$  is the last  $\Lambda$ -edge of  $w$ , we conclude that  $a_i a_{i+1} \neq a_s a_0$ .

Note that  $a_i a_{i+1} \neq bb'$  (by condition (3)) and  $a_i a_{i+1} \neq (bb')^{-1}$  (since  $x$  is reduced and  $w = \phi(x)$ ). Then it follows from Lemma 2.6, that each of  $a_i$  and  $a_{i+1}$  commutes with each of  $b$  and  $b'$ . By applying Lemma 2.11(1) four times, we obtain a new disk diagram  $D'$  such that the label of  $\partial D'$  is obtained from the label of  $\partial D$  by swapping the  $\Lambda$ -edges  $a_i a_{i+1}$  and  $bb'$ . Moreover, the natural map  $\psi$  from the edges of  $\partial D$  to the edges of  $\partial D'$  (defined in Lemma 2.11(1)) preserves boundary combinatorics. By applying Lemma 2.8 if necessary, we may assume that  $D'$  has no bigons, so hyperplanes in  $D'$  intersect at most once.

Since boundary combinatorics are preserved,  $\psi$  induces a bijection between the hyperplanes dual to  $\partial D$  and those dual to  $\partial D'$ . Since the transition from  $D$  to  $D'$  involves swapping a pair of  $\Lambda$ -edges, the label of  $\partial D'$  is still a product of  $\Lambda$ -edges, and so the hyperplanes of  $D'$  can be partitioned into closed chains of hyperplanes. Moreover,  $\psi$  induces a bijection between the closed chains of hyperplanes in  $D$  and  $D'$ .

If  $\mathcal{H}'$  and  $H'_0$  denote the images of  $\mathcal{H}$  and  $H_0$  respectively under  $\psi$ , it is clear that the labels of the hyperplanes of  $\mathcal{H}'$ , starting from  $H'_0$  and proceeding clockwise, are  $a_0, \dots, a_s$ . We now prove that  $\mathcal{H}'$  together with  $H'_0$  still satisfies the criterion in Lemma 3.25 required in (1).

Let  $\mathcal{K}'$  be a closed chain in  $D'$ , and let  $\mathcal{K}$  be its preimage in  $D$ . Our choice of  $\mathcal{K}$  implies that either  $H_0$  separates  $\mathcal{K}$  from  $\mathcal{K} \setminus \{H_0\}$ , or  $\mathcal{K}$  intersects  $H_0$ . In the former case,  $H'_0$  still separates  $\mathcal{K}'$  from  $\mathcal{K}' \setminus \{H'_0\}$ . This is because the swap performed does not involve any hyperplanes from chains which do not intersect  $H_0$ , since (as noted above)  $a_i a_{i+1} \neq a_s a_0$ .

On the other hand, suppose that  $\mathcal{K}$  intersects  $H_0$ . By Observation 3.22, there are exactly two hyperplanes  $K_1$  and  $K_2$  in  $\mathcal{K}$  which intersect  $H_0$ . If  $K_j$ , for  $j = 1, 2$ , is not dual to the  $\Lambda$ -edge labeled by  $bb'$ , then the image of  $K_j$  intersects  $H'_0$ . Moreover, if  $i \neq 0$ , then it follows that the images of  $K_1$  and  $K_2$  in  $D'$  intersect the hyperplane  $H'_0$ . Thus, we only need to consider the case where the  $\Lambda$ -edge  $a_0 a_1$  is swapped, and (up to relabeling)  $K_1$  is dual to  $b$  and  $K_2$  is dual to  $b'$ . In this case,  $K_1$  and  $K_2$  are dual to the same  $\Lambda$ -edge. It follows that no hyperplane in  $\mathcal{K} \setminus \{K_1, K_2\}$  is contained in the same component of  $D \setminus H_0$  as  $\mathcal{K} \setminus \{H_0\}$ . Thus, in  $D'$ , no hyperplane of  $\mathcal{K}'$  is contained in the same component of  $D' \setminus H'_0$  as  $\mathcal{K}' \setminus \{H'_0\}$ . We have shown that  $\mathcal{K}'$ , with distinguished hyperplane  $H'_0$ , satisfies the conclusion of Lemma 3.25.

Let  $p'$  be the vertex on  $\partial D'$  which is the endpoint of the  $\Lambda$ -edge from  $\mathcal{H}'$  labeled  $a_s a_0$ . Since the swap performed did not involve  $a_s a_0$ , the label  $w'$  of  $\partial D'$ , read clockwise from  $p'$ , is obtained from  $w$  by swapping a single pair of  $\Lambda$ -edges, and its preimage in  $x'$  in  $A_\Delta$  is obtained from  $x$  by swapping one pair of generators. This shows (2).

We have established that  $D'$ , together with  $\mathcal{H}'$ , satisfies (1) and (2) of Claim 3.26. If (3) still fails, we may repeat the process above. Since each individual iteration involves moving one  $\Lambda$ -edge from the image of  $\mathcal{H}$  to the right, this process eventually stops. After finitely many iterations, we arrive at a disk diagram  $\tilde{D}$  such that all three conditions hold. □

For the rest of the proof we assume, without loss of generality, that  $D$ ,  $\mathcal{H}$ ,  $p$ ,  $w$  and  $x$  satisfy the conclusion of Claim 3.26.

We now analyze closed chains which intersect  $\mathcal{H}$ . First consider the case that there are no such chains. This includes the case when  $\Lambda$  has a single component. Since  $\mathcal{H}$  is a closed chain, it defines a loop in  $\Lambda$ . (See Observation 3.23.) On the other hand, since no chains intersect  $\mathcal{H}$ , the union of the edges of  $\partial D$  dual to the hyperplanes of  $\mathcal{H}$  is a continuous subpath (with label  $(a_0a_1)(a_1a_2)\cdots(a_s a_0)$ ). Applying the following claim to this subpath, we conclude that the  $\Lambda$ -loop defined by  $\mathcal{H}$  is a cycle. This contradicts  $\mathcal{R}_1$ . (The claim will be used again later in this proof.)

**Claim 3.27** *Let  $v$  be a subpath of  $\partial D$  labeled by a product of  $\Lambda$ -edges. Suppose there exists a closed chain  $\mathcal{X}$ , such that each edge of  $v$  is dual to a hyperplane in  $\mathcal{X}$ . It follows that the label of  $v$  is  $(x_1x_2)\cdots(x_{n-1}x_n)$ , where  $x_1, x_2, \dots, x_n$  are the labels of the hyperplanes of  $\mathcal{X}$  dual to  $v$ , in order. Furthermore, the  $\Lambda$ -path through vertices  $x_1, \dots, x_n$  is simple.*

**Proof** The claim about the label of  $v$  is immediate. If the path through vertices  $x_1, \dots, x_n$  is not simple, then there is a  $\Lambda$ -loop through vertices  $x_i, x_{i+1}, \dots, x_{i+j} = x_i$  for some  $i$  and  $j$ . By  $\mathcal{R}_1$ , the image of this loop in  $\Lambda$  is a tree. Let  $x_r$  be a leaf of this tree, with  $i < r < j$ . It follows that  $x_{r-1} = x_{r+1}$ . Consequently, the label of  $v$  (and therefore of the word  $w$ ) has a subword  $(x_{r-1}x_r)(x_r x_{r-1})$ . This is a contradiction, as it implies that the preimage  $x$  of  $w$  in  $A_\Delta$  is not reduced. □

Thus, we may assume that there is at least one chain intersecting  $\mathcal{H}$ . In particular,  $\Lambda$  has two components: say a red component  $\Lambda_a$  which contains the labels of  $\mathcal{H}$ , and a green component  $\Lambda_b$ . By Claim 3.26, each chain intersecting  $\mathcal{H}$  intersects  $H_0$ . Let  $\mathcal{K}$  be the “first” such chain, in the sense that the first  $\Lambda$ -edge from a chain other than  $\mathcal{H}$  appearing in  $w$  to the right of  $a_0a_1$  is from  $\mathcal{K}$ . (See Figure 9.) By  $\mathcal{R}_2$  and Observation 3.19, we conclude that  $\mathcal{K}$  is green. Let  $b_0, \dots, b_{s'}$  be the labels of the hyperplanes of  $\mathcal{K}$ , where  $b_0b_1$  is the label of the first  $\Lambda$ -edge from  $\mathcal{K}$  appearing in  $w$  to the right of  $a_0a_1$ .

Now consider the 2-colored polygon in  $D$  whose sides alternate between hyperplanes in  $\mathcal{H}$  and  $\mathcal{K}$ , as described in Observation 3.24. Let  $c_0, d_0, \dots, c_k, d_k$  be the labels of these sides, where  $c_0 = a_0, d_0 = b_0$ , and  $c_0, \dots, c_k$  (resp.  $d_0, \dots, d_k$ ) is a subsequence of  $a_0, \dots, a_s$  (resp. of  $b_0, \dots, b_{s'}$ ). (See Figure 9.)

The following technical claim about the hyperplanes dual to certain subpaths of  $\partial D$  associated to this 2-colored polygon will be needed in what follows:

**Claim 3.28** *For  $0 \leq i \leq k$ , let  $e_i$  and  $f_i$  (resp.  $e'_i$  and  $f'_i$ ) be the edges dual to the hyperplane of  $\mathcal{H}$  labeled  $c_i$  (resp. the hyperplane of  $\mathcal{K}$  labeled  $d_i$ ), where  $e_i$  (resp.  $e'_i$ ) appears before  $f_i$  (resp.  $f'_i$ ) reading clockwise from  $p$ .*

*For  $i > 0$ , let  $\eta_i$  be the subpath of  $\partial D$  from (and including)  $f_{i-1}$  to (and including)  $e_i$ , and let  $\mu_i$  be the subpath of  $\partial D$  from the endpoint of  $\eta_i$  to (and including)  $e'_i$ . (See Figure 10.) Then every edge of  $\eta_i$  (resp.  $\mu_i$ ) is dual to a hyperplane in  $\mathcal{H}$  (resp.  $\mathcal{K}$ ).*

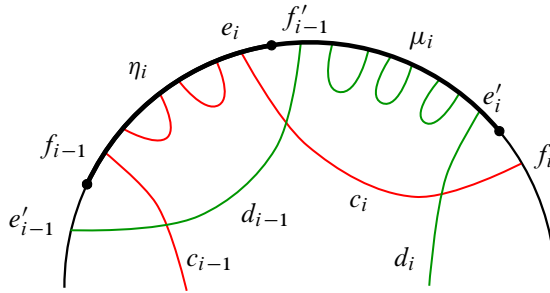


Figure 10: The paths  $\eta_i$  and  $\mu_i$  from Claim 3.28 are shown in bold, delineated by dots. We remark that if  $i = k$ , then there could be additional hyperplanes not in  $\mathcal{H}$  or  $\mathcal{H}$  between the endpoint of  $\mu_k$  and the start of the edge  $f_i$ .

**Proof** Suppose there is some hyperplane  $L$  dual to an edge  $e$  of  $\eta_i$  such that the closed chain  $\mathcal{L}$  containing  $L$  is not equal to  $\mathcal{H}$ . From the definition of  $\eta_i$ , we conclude that  $e$  is on the same side of  $H_0$  as  $\mathcal{H} \setminus \{H_0\}$  in  $D$ , and by our choice of  $\mathcal{H}$  (and Lemma 3.25), it follows that  $\mathcal{L}$  intersects  $H_0$ . Therefore,  $L$  is green by Observation 3.19.

Let  $K$  denote the hyperplane of  $\mathcal{H}$  labeled  $d_{i-1}$ . Then  $K$  separates  $e$  from  $\mathcal{H} \setminus \{K\}$ , so  $L \notin \mathcal{H}$ , ie  $\mathcal{L} \neq \mathcal{H}$ . If  $d_{i-1} = b_0$ , ie if  $K$  does intersect  $H_0$ , then our choice of  $\mathcal{H}$  implies that  $K$  is the first hyperplane not in  $\mathcal{H}$  dual to  $\partial D$  after the  $\Lambda$ -edge  $a_0 a_1$ , so such an  $L$  cannot exist. On the other hand, if  $K$  does not intersect  $H_0$ , then  $K$  separates  $e$  from  $H_0$ . So, in order to intersect  $H_0$ , the chain  $\mathcal{L}$  must also intersect  $\mathcal{H}$ , which is a contradiction, since  $\mathcal{L}$  and  $\mathcal{H}$  are both green.

Now suppose  $L \in \mathcal{L} \neq \mathcal{H}$  is dual to an edge  $e$  of  $\mu_i$ . Since  $\mu_i$  is only defined for  $i > 0$ , it is on the same side of  $H_0$  as  $\mathcal{H} \setminus \{H_0\}$ , and consequently, the same holds for  $e$ . Therefore, we conclude as before that  $L$  is green.

Additionally, we conclude as before that the hyperplane  $K \in \mathcal{H}$  labeled  $d_{i-1}$  does not intersect  $H_0$ . Now consider the subchain of  $\mathcal{H}'$  of  $\mathcal{H}$  consisting of the hyperplanes dual to all but the last edge  $e'_i$  of  $\mu_i$ . Since  $K$  does not intersect  $H_0$ , it follows that  $e$  is separated from  $H_0$  by some hyperplane in  $\mathcal{H}'$ . Thus, in order to intersect  $H_0$ ,  $\mathcal{L}$  must intersect  $\mathcal{H}$ , which is again a contradiction.  $\square$

The 2-colored polygon obtained above gives a 2-component loop in  $\Theta$ , as described in Observation 3.24. A priori this loop may not be a 2-component cycle, ie it is possible that  $c_i = c_j$  or  $d_i = d_j$  for some  $i$  and  $j$ . However, we now show that it contains a cycle. We will then be able to apply  $\mathcal{R}_4$  to this cycle to make progress towards obtaining a contradiction to the normal form in Claim 3.26.

**Claim 3.29** Consider the 2-component loop in  $\Theta$  visiting  $c_0, d_0, \dots, c_k, d_k, c_{k+1} = c_0$  defined above. There exist  $0 \leq l \leq k - 1$  and  $m \geq 2$ , such that one of the two following subsequences of vertices (with indices taken mod  $k + 1$ ) defines a 2-component cycle in  $\Theta$ :

- (1)  $c_l, d_l, c_{l+1}, \dots, d_{l+m-1}, c_{l+m} = c_l$ ;
- (2)  $d_l, c_{l+1}, d_{l+1}, \dots, c_{l+m}, d_{l+m} = d_l$ .

**Proof** Observe that since  $c_{k+1} = c_0$ , the following set is nonempty:

$$\{j \mid c_i = c_{i+j} \text{ or } d_i = d_{i+j} \text{ for some } 0 \leq i < k - 1 \text{ and } 1 \leq j \leq k + 1\}.$$

Let  $m$  denote its minimum value. We first show that  $m \geq 2$ , or equivalently that, for each  $0 \leq i \leq k - 1$ , both  $c_i \neq c_{i+1}$  and  $d_i \neq d_{i+1}$  are true. Suppose  $c_i = c_{i+1}$  for some  $i$ . Consider the path  $\eta_{i+1}$  from **Claim 3.28**. It is labeled by  $\Lambda$ -edges, and every edge in it is dual to a hyperplane from  $\mathcal{H}$ . Then by **Claim 3.27**, it follows that  $\eta_{i+1}$  defines a simple  $\Lambda$ -path from the vertex  $c_i$  to the vertex  $c_{i+1}$ . However, this contradicts the assumption that  $c_i = c_{i+1}$ . This proves that for all  $0 \leq i \leq k - 1$ , we have  $c_i \neq c_{i+1}$ . The proof that  $d_i \neq d_{i+1}$  is similar.

Now if  $l$  is such that  $c_l = c_{l+m}$  (the case when  $d_l = d_{l+m}$  is similar), then it readily follows from the minimality of  $m$  that the vertices  $c_l, d_l, c_{l+1}, \dots, c_{l+m-1}, d_{l+m-1}$  are distinct, and therefore define the desired cycle. □

Continuing the proof of the theorem, we can now assume  $\Theta$  has a 2-component cycle  $\gamma$  as in (1) from **Claim 3.29**. (The case in which  $\Theta$  has a 2-component cycle as in (2) is similar.) Let  $T_c$  and  $T_d$  be the  $\Lambda$ -convex hulls of  $\{c_l, \dots, c_{l+m-1}\}$  and  $\{d_l, \dots, d_{l+m-1}\}$  respectively. Then  $T_c$  and  $T_d$  are trees by  $\mathcal{R}_1$ . Let  $c_j$  be a leaf of  $T_c$  with  $c_j \neq c_0$ . Then  $c_j$  labels a hyperplane  $H_t \in \mathcal{H}$  for some  $t \neq 0$ , so  $a_t = c_j$ . Similarly,  $c_{j-1}$  labels a hyperplane  $H_{t-r}$  of  $\mathcal{H}$ , while  $d_{j-1}$  and  $d_j$  label hyperplanes  $K_{t'}$  and  $K_{t'+r'}$  respectively of  $\mathcal{H}$ , where  $d_{j-1} = b_{t'}$  and  $d_j = b_{t'+r'}$ .

Consider the paths  $\eta_j$  and  $\mu_j$  defined in **Claim 3.28**. The last  $\Lambda$ -edge of  $\eta_j$  is  $a_{t-1}a_t$ . By **Claim 3.28**, the first edge of  $\mu_j$  is dual to a hyperplane in  $\mathcal{H}$ . It follows that this must be  $K_{t'}$ , with label  $b_{t'}$ , for otherwise  $K_{t'}$  would separate this edge from  $\mathcal{H} \setminus K_{t'}$ . It follows that the first  $\Lambda$ -edge of  $\mu_j$  is  $b_{t'}b_{t'+1}$ , and that the word  $w$  has a subword  $a_{t-1}a_t b_{t'} b_{t'+1}$ .

To complete the proof, we will show that the presence of this subword violates the normal form established in **Claim 3.26(3)**. Since the labels of  $\mathcal{H}$  and  $\mathcal{K}$  are from different components of  $\Lambda$ , it is immediate that  $a_{t-1}a_t \neq b_{t'}b_{t'+1}$ . We now show that  $a_{t-1}a_t$  and  $b_{t'}b_{t'+1}$  commute.

The 2-component cycle  $\gamma$  in  $\Theta$  contains an edge with endpoints  $a_t$  and  $b_{t'}$ . Applying  $\mathcal{R}_4$  to this edge, we conclude that there is a 2-component square visiting  $a_t, b_{t'}, a$  and  $b$ , where  $a \in T_c$  and  $b \in T_d$ . Next, applying  $\mathcal{R}_3$  to this 2-component square, we see that  $b_{t'}$  commutes with the vertices of the  $\Lambda$ -convex hull of  $\{a_t, a\}$ . **Claims 3.27** and **3.28** together imply that the path  $\eta_j$  induces a simple  $\Lambda$ -path visiting vertices  $a_{t-r}, a_{t-r+1}, \dots, a_t$ . Consequently, the vertices along this path, and in particular  $a_{t-1}$ , are in  $T_c$ . Moreover,  $a_{t-1}$  is the unique vertex of  $T_c$  adjacent to  $a_t$ , since  $a_t = c_j$  is a leaf of  $T_c$ . It follows that  $a_{t-1}$  is contained in the  $\Lambda$ -convex hull (which is the same as the  $T_c$ -convex hull) of  $\{a_t, a\}$ . Thus,  $a_{t-1}$  and  $b_{t'}$  commute. The same reasoning, applied to the edge of  $\gamma$  with endpoints  $a_t$  and  $b_{t'+r'}$ , implies that  $a_{t-1}$  and  $b_{t'+r'}$  commute.

Using the  $\Gamma$ -edges whose existence is implied by these two additional commutation relations, we obtain a 2-component square visiting  $a_t, b_{t'}, a_{t-1}$  and  $b_{t'+r'}$ . Applying  $\mathcal{R}_3$  to this square, we conclude that  $a_t$

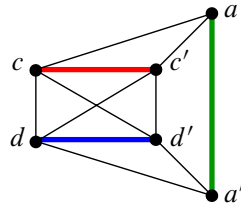


Figure 11: This figure illustrates condition  $\mathcal{R}_5$ . The red, blue and green segments are respectively  $T_c$ ,  $T_d$  and  $T_a$ . Condition  $\mathcal{R}_5$  states that any  $\Lambda$ -edge contained in the green segment must either commute with every  $\Lambda$ -edge in the red segment or must commute with every  $\Lambda$ -edge in the blue segment.

and  $a_{t-1}$  commute with each vertex in the  $\Lambda$ -convex hull of  $\{b_{t'}, b_{t'+r'}\}$ . By Claim 3.27, we see that the path  $\mu_j$  from Claim 3.28 defines a simple  $\Lambda$ -path visiting  $b_{t'}, b_{t'+1}, \dots, b_{t'+r'}$ . It follows that  $b_{t'+1}$  is in the convex hull of  $\{b_{t'}, b_{t'+r'}\}$ , and consequently,  $a_t$  and  $a_{t-1}$  commute with  $b_{t'+1}$ .

Putting together the commutation relations established in the previous paragraphs, we conclude that  $a_{t-1}a_t$  commutes with  $b_{t'}b_{t'+1}$ . This contradicts the fact that we have chosen  $D$  so that it satisfies (3) of Claim 3.26. □

### 3.1 Three or more $\Lambda$ -components

In the case that  $\Lambda$  contains at most two components, Theorem 3.18 shows that  $\mathcal{R}_1$ – $\mathcal{R}_4$  are necessary and sufficient conditions that guarantee  $(G^\Theta, E(\Lambda))$  is a RAAG system. In this subsection, we do not place any restriction on the number of components of  $\Lambda$ . We give an additional necessary condition  $\mathcal{R}_5$  for  $(G^\Theta, E(\Lambda))$  to be a RAAG system, and Example 3.31 shows this condition is independent of conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$ . The authors are aware that *even more* conditions are required in order to generalize Theorem 3.18 to this setting. These extra conditions are not included here, as they are complicated and the authors do not believe to yet possess the complete list of the necessary and sufficient conditions for this generalization.

We further show in this subsection that if  $\Theta$  contains certain subgraphs and  $(G^\Theta, E(\Lambda))$  is a RAAG system, then  $\Gamma$  must necessarily contain a triangle. These results are needed in the next section.

**Definition 3.30** (condition  $\mathcal{R}_5$ ) We say that  $\Theta$  satisfies *condition  $\mathcal{R}_5$*  if the following holds. Let  $\Lambda_a, \Lambda_c$  and  $\Lambda_d$  be distinct components of  $\Lambda$ . Suppose we have vertices  $a, a' \in \Lambda_a, c, c' \in \Lambda_c$  and  $d, d' \in \Lambda_d$ , such that  $\Theta$  contains a 2-component square visiting  $c, d, c'$  and  $d'$ . Furthermore, suppose that  $c$  and  $c'$  are each adjacent to  $a$  in  $\Gamma$  and that  $d$  and  $d'$  are each adjacent to  $a'$  in  $\Gamma$ . (See Figure 11.) Let  $T_a, T_c$  and  $T_d$  be the  $\Lambda$ -convex hulls of  $\{a, a'\}, \{c, c'\}$  and  $\{d, d'\}$  respectively. Then given any  $\Lambda$ -edge  $xx'$  of  $T_a$ , the graph  $\Gamma$  contains either the join of  $\{x, x'\}$  with  $V(T_c)$  or the join of  $\{x, x'\}$  with  $V(T_d)$ .

The following is a concrete example showing that when  $\Lambda$  has more than two components, the conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$  are not sufficient to guarantee that  $(G^\Theta, E(\Lambda))$  is a RAAG system.

**Example 3.31** Let  $\Gamma$  be the graph whose vertex set is  $\{a, a', c, c', d, d'\}$  and whose edge set is the set of black edges in Figure 11. Let  $\Lambda \subset \Gamma^c$  consist of exactly three  $\Lambda$ -edges:  $aa'$ ,  $cc'$  and  $dd'$ . Then  $\Theta = \Theta(\Gamma, \Lambda)$  satisfies conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$  and does not satisfy condition  $\mathcal{R}_5$ . By Lemma 3.32 below,  $(G^\Theta, E(\Lambda))$  is not a RAAG system.

We now show that condition  $\mathcal{R}_5$  is necessary.

**Lemma 3.32** *If  $(G^\Theta, E(\Lambda))$  is a RAAG system, then  $\Theta$  satisfies condition  $\mathcal{R}_5$ .*

**Proof** By Theorem 3.18, we may assume that  $\Theta$  satisfies conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$ . Let  $a, a' \in \Lambda_a, c, c' \in \Lambda_c$  and  $d, d' \in \Lambda_d$  be as in Definition 3.30. Define the words  $z_a = a'a, z_c = cc', z_d = dd'$  and  $z = [z_a z_c z_a^{-1}, z_d]$ . By the commuting relations imposed in Definition 3.30, it follows that  $z \simeq 1$  in  $W_\Gamma$ . Let  $w_a, w_c, w_d$  and  $w$  be the  $\Lambda$ -edge words corresponding respectively to  $z_a, z_c, z_d$  and  $z$ . Let  $D$  be a disk diagram over  $A_\Delta$  with boundary label  $w$ .

Let  $\gamma_c, \zeta_c, \gamma_d$  and  $\zeta_d$  be the paths in  $\partial D$  labeled respectively by  $w_c, w_c^{-1}, w_d$  and  $w_d^{-1}$ . Note that no hyperplane is dual to two distinct edges of  $\gamma_c$  (resp.  $\zeta_c, \gamma_d$  and  $\zeta_d$ ). This follows as  $z_c$  is a word in unique  $\Lambda$ -edges. Thus, every hyperplane dual to  $\gamma_c$  (resp.  $\gamma_d$ ) is also dual to  $\zeta_c$  (resp.  $\gamma_d$ ).

Let  $\alpha$  be a path in  $\partial D$  between  $\gamma_c$  and  $\gamma_d$  (which is labeled by  $w_a$ ). Again, no hyperplane is dual to two distinct edges of  $\alpha$ . Let  $xx'$  be a  $\Lambda$ -edge of  $T_a$ , and let  $H$  be the unique hyperplane dual to  $\alpha$  with label  $xx'$ . Note that either  $H$  intersects every hyperplane dual to  $\gamma_c$  or  $H$  intersects every hyperplane dual to  $\gamma_d$ . Furthermore, every  $\Lambda$ -edge of  $T_c$  (resp.  $T_d$ ) is the label of a hyperplane dual to  $\gamma_c$  (resp.  $\gamma_d$ ). The claim now follows from Lemma 2.6, and the fact that intersecting hyperplanes correspond to commuting generators of  $A_\Delta$ . □

The following corollary shows that if  $\Theta$  contains a configuration like that in the hypothesis of condition  $\mathcal{R}_5$ , then  $\Gamma$  must contain a triangle. This corollary is a warm-up to the more complicated Lemma 3.34.

**Corollary 3.33** *Suppose  $(G^\Theta, E(\Lambda))$  is a RAAG system and  $\Theta$  contains a set of vertices  $\{a, a', b, b', c, c'\}$  satisfying the hypothesis of  $\mathcal{R}_5$ . Then  $\Gamma$  contains a triangle.*

**Proof** Let  $P = \{a, a', c, c', d, d'\}$  be a subset of vertices of  $\Theta$  satisfying the hypothesis of  $\mathcal{R}_5$ . We call such a  $P$  a *configuration* in  $\Theta$ . Keeping the same notation as in Definition 3.30, we call the number of vertices of  $T_a$  the *complexity* of  $P$ , and we prove the claim by induction on complexity. Note that  $a = a'$  is possible in the hypothesis of  $\mathcal{R}_5$ , so the lowest possible complexity is  $N = 1$ . The corollary follows in this case, as  $\Gamma$  then contains a triangle spanned by the vertices  $a = a', c$  and  $d$ .

Now let  $N > 1$  and suppose the claim is true for all configurations  $P$  of smaller complexity. As  $N > 1$ , there is a vertex  $y$  such that  $ay$  is a  $\Lambda$ -edge of  $T_a$ . By Lemma 3.32, either  $y$  is adjacent in  $\Gamma$  to both  $c$  and  $c'$ , or  $y$  is adjacent in  $\Gamma$  to both  $d$  and  $d'$ . In either case, we see that  $\Theta$  contains a configuration of smaller complexity. □



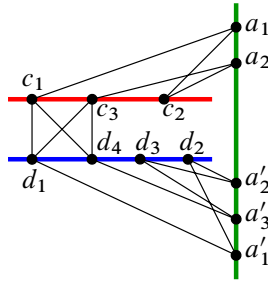


Figure 12: This figure illustrates the configuration described in Lemma 3.34 in the case  $n = 3$  and  $m = 4$ . The black edges are edges of  $\Gamma$ . The red, green and blue parts consist of  $\Lambda$ -edges, and are all contained in  $\Lambda$ . The different colors indicate that they are in three distinct components of  $\Lambda$ .

The next lemma shows that if  $\Theta$  contains certain subgraphs which generalize the configurations in the hypothesis of  $\mathcal{R}_5$ , then  $\Gamma$  must contain a triangle.

**Lemma 3.34** *Let  $\Lambda_a$ ,  $\Lambda_c$  and  $\Lambda_d$  be distinct components of  $\Lambda$ . Suppose  $\Theta$  has a  $\Lambda_a \Lambda_c$ -path visiting  $c_1, a_1, c_2, \dots, a_{n-1}, c_n$ , and a  $\Lambda_a \Lambda_d$ -path visiting  $d_1, a'_1, d_2, \dots, a'_{m-1}, d_m$ , where  $c_i \in \Lambda_c$ ,  $d_i \in \Lambda_d$  and  $a_i, a'_i \in \Lambda_a$  for all appropriate  $i$ . Further suppose that  $\Theta$  contains a 2-component square visiting  $c_1, d_1, c_n$  and  $d_n$ . (See Figure 12.) If  $(G^\Theta, E(\Lambda))$  is a RAAG system, then  $\Gamma$  has a triangle.*

**Proof** By Theorem 3.18 and Lemma 3.32, we may assume that  $\Theta$  satisfies conditions  $\mathcal{R}_1$ – $\mathcal{R}_5$ . Let  $A = \{a_1, \dots, a_{n-1}, a'_1, \dots, a'_{m-1}\}$ ,  $C = \{c_1, \dots, c_n\}$  and  $D = \{d_1, \dots, d_m\}$  be vertices of  $\Theta$  as in the statement of the lemma. We call such a triple  $(A, B, C)$  a configuration of  $\Theta$ . Let  $T_a$ ,  $T_c$  and  $T_d$  be the  $\Lambda$ -convex hulls of  $A$ ,  $C$  and  $D$  respectively. We define the complexity of  $(A, C, D)$  to be the integer  $N = |C| + |D| + |T_a|_E + |T_c|_E + |T_d|_E$ , where  $|X|_E$  denotes the number of edges in a graph  $X$ . The proof will be by induction on complexity of configurations.

By hypothesis, we have that  $n, m \geq 2$  and  $|T_c|_E, |T_d|_E \geq 1$ . If  $n = m = 2$  then  $\Gamma$  contains a triangle by Corollary 3.33. In particular, the base case follows.

We now fix a configuration  $A = \{a_1, \dots, a_{n-1}, a'_1, \dots, a'_{m-1}\}$ ,  $C = \{c_1, \dots, c_n\}$  and  $D = \{d_1, \dots, d_m\}$  as above of complexity  $N$ , and we assume that the result holds for configurations of smaller complexity. By the previous paragraph, we may also assume (up to relabeling) that  $n > 2$ . We prove the lemma by showing that either  $\Gamma$  contains a triangle or  $\Theta$  contains a configuration of smaller complexity.

Define  $\alpha_{ac}$  and  $\alpha_{ad}$  to respectively be the hypothesized  $\Lambda_a \Lambda_c$ -path and  $\Lambda_a \Lambda_d$ -path. We may assume that  $\alpha_{ac}$  and  $\alpha_{ad}$  are simple paths, for if not, we would be able to excise a loop to obtain a configuration of smaller complexity.

We claim that for all  $1 < i < n$ , we may assume that  $c_i$  does not lie on the simple path in  $\Lambda_c$  from  $c_1$  to  $c_n$ . For suppose there exists such a vertex  $c_i$ . By  $\mathcal{R}_3$ , it follows that  $c_i$  commutes with both  $d_1$  and  $d_m$ .

There then exists a  $\Lambda_a \Lambda_c$  path visiting  $c_1, a_1, \dots, a_{i-1}, c_i$ , and it follows that  $\Theta$  contains a configuration of smaller complexity (obtained by replacing  $\alpha_{ac}$  with this new path). Thus we may make this assumption without loss of generality. Furthermore, as  $n \geq 3$ , there exists an integer  $j$  such that  $c_j$  is a leaf vertex of  $T_c$  and such that  $1 < j < m$ . We fix such a vertex  $c_j$ .

Define the word  $z_c$  to be

$$z_c = (a'_1 a_1)(c_1 c_2)(a_1 a_2)(c_2 c_3)(a_2 a_3) \cdots (a_{n-2} a_{n-1})(c_{n-1} c_n)(a_{n-1} a'_1)$$

and define the word  $z_d$ , depending on the value of  $m$ , to be

$$z_d = \begin{cases} d_1 d_2 & \text{if } m = 2, \\ (d_1 d_2)(a'_1 a'_2)(d_2 d_3)(a'_2 a'_3) \cdots (a'_{m-2} a'_{m-1})(d_{m-1} d_m)(a'_{m-1} a'_1) & \text{if } m > 2. \end{cases}$$

In  $W_\Gamma$  we have that  $z_c \simeq a'_1 c_1 c_n a'_1$  and  $z_d \simeq a'_1 d_1 d_m a'_1$ . Let  $z = [z_c, z_d]$ . Note that as  $c_1$  and  $c_n$  commute with  $d_1$  and  $d_m$  in  $W_\Gamma$ ,

$$z \simeq [a'_1 c_1 c_n a'_1, a'_1 d_1 d_m a'_1] \simeq a'_1 [c_1 c_n, d_1 d_m] a'_1 \simeq 1.$$

Let  $w_c, w_d$  and  $w$  be the  $\Lambda$ -edge words associated to  $z_c, z_d$  and  $z$  respectively. Let  $D$  be a disk diagram over  $A_\Delta$  with boundary label  $w$ . Let  $\gamma_c, \zeta_c, \gamma_d$  and  $\zeta_d$  be the subpaths of  $\partial D$  labeled respectively by  $w_c, w_c^{-1}, w_d$  and  $w_d^{-1}$ .

Let  $yc_j$  be the  $\Lambda$ -edge of  $T_c$  incident to  $c_j$ . Since  $\alpha_{ac}$  does not repeat vertices and since  $c_j$  is a leaf of  $T_c$ , it follows that  $w_c$  contains exactly two occurrences of the letter  $y$  contained in the subword labeled by  $(yc_j)(a_{j-1}x_1)(x_1x_2) \cdots (x_l a_j)(c_j y)$ , where the  $x_i$ 's are vertices in  $\Lambda_a$ . In particular, there are exactly four edges of  $\partial D$  (two on  $\gamma_c$  and two on  $\zeta_c$ ) labeled by either  $yc_j$  or  $c_j y$ . Correspondingly, there are exactly two hyperplanes,  $H$  and  $H'$  in  $D$  labeled  $yc_j$ .

We claim that we may assume that  $H$  is dual to both  $\gamma_c$  and  $\zeta_c$ , and the same is true for  $H'$ . For suppose otherwise, and suppose that  $H$  is dual to two edges of  $\gamma_c$ . (The case of  $H'$  is similar.) It follows that any hyperplane dual to the subpath of  $\gamma_c$  labeled by  $(a_{j-1}x_1)(x_1x_2) \cdots (x_l a_j)$  (which lies between the endpoints of  $H$ ) must intersect  $H$ . Thus, in particular,  $(a_{j-1}x_1)$  and  $(x_l a_j)$  commute with  $yc_j$ , and applying Lemma 2.6, we conclude that  $y$  commutes with both  $a_{j-1}$  and  $a_j$ . We now show that we can replace  $\alpha_{ac}$  with a new  $\Lambda_a \Lambda_c$ -path from  $c_1$  to  $c_n$  such that  $|T_a|_E$  is reduced, and thus  $\Theta$  contains a smaller complexity configuration. If  $y$  is not equal to any  $c_k$  for any  $1 \leq k \leq m$ , then we obtain this path by simply replacing  $c_j$  with  $y$  in  $\alpha_{ac}$ . On the other hand, if  $y = c_k$  for some  $k$ , then we replace  $\alpha_{ac}$  with the  $\Lambda_a \Lambda_c$  path visiting  $c_1, a_1, \dots, c_k, a_j, c_{j+1}, a_{j+1}, \dots, a_{n-1}, c_n$  if  $k < j$  and perform a similar replacement if  $k > j$ . In either case, we have produced a configuration of smaller complexity. Thus, we now assume that each of  $H$  and  $H'$  is dual to both  $\gamma_c$  and  $\zeta_c$ .

Let  $Q$  and  $Q'$  be the hyperplanes in  $D$  dual to the edges of  $\gamma_c$  labeled by  $a_{j-1}x_1$  and  $x_l a_j$  respectively. If both  $Q$  and  $Q'$  intersect  $H \cup H'$ , then we can conclude, as above, that  $y$  commutes with both  $a_{j-1}$  and  $a_j$ . We can then find a smaller complexity configuration as in the previous paragraph. Thus, we can

assume that either  $Q$  or  $Q'$  is dual to both  $\gamma_c$  and  $\zeta_c$ . We assume that  $Q$  has this property (the case of  $Q'$  is similar).

We now examine hyperplanes dual to  $\gamma_d$  and  $\zeta_d$ . If  $m = 2$ , then the unique hyperplane whose label contains  $d_1$  is dual to both  $\gamma_d$  and  $\zeta_d$ , and this hyperplane intersects both  $H$  and  $Q$ . Thus,  $d_1$  commutes with both  $c_j$  and  $a_{j-1}$ . Since  $c_j$  commutes with  $a_{j-1}$ , it follows that  $\Gamma$  contains a triangle. On the other hand, if  $m > 2$  by the same reasoning as before, we can assume there is a leaf vertex  $d_{j'}$  of  $T_d$  and a hyperplane with label  $y'd_{j'}$  that intersects both  $\gamma_d$  and  $\zeta_d$ . This then implies that  $d_{j'}$  commutes with both  $c_j$  and  $a_{j-1}$  and consequently,  $\Gamma$  contains a triangle.  $\square$

### 4 Finite-index visual RAAGs

As in the previous section, given a simplicial graph  $\Gamma$  and a subgraph  $\Lambda$  of  $\Gamma^c$  with no isolated vertices, we set  $\Theta = \Theta(\Gamma, \Lambda)$ , and let  $G^\Theta$  be the subgroup generated by  $E(\Lambda)$ . Our goal is to characterize graphs  $\Lambda \subset \Gamma^c$  such that  $(G^\Theta, E(\Lambda))$  is a RAAG system and  $G^\Theta$  has finite index in  $W_\Gamma$ .

Suppose the graph  $\Gamma$  contains a vertex  $s$  which is  $\Gamma$ -adjacent to every other vertex of  $\Gamma$ . We say that  $s$  is a *cone vertex*. In this case, it easily follows that  $W_{\Gamma \setminus s}$  has index 2 in  $W_\Gamma$  and that  $s$  cannot be contained in any  $\Lambda$ -edge.

We now recall a construction from [Dani and Levcovitz 2021] which will help us compute the index of  $G^\Theta$ . The construction is general, but for simplicity, and as it is all that we use, we choose to only describe it in the context where  $\Gamma$  is triangle-free. We refer the reader to [Dani and Levcovitz 2021] for full details.

Let  $\Gamma$  be a triangle-free graph. We say a cell complex is  $\Gamma$ -labeled if every edge of the complex is labeled by a vertex of  $\Gamma$ . Let  $X$  be a  $\Gamma$ -labeled complex. Suppose two edges of  $X$  have the same label and a common endpoint. A *fold operation* produces a new complex from  $X$  by naturally identifying these two edges.

Suppose now that  $f_1$  and  $f_2$  are edges of  $X$  which share a common vertex  $u$  and whose labels  $s_1, s_2 \in V(\Gamma)$  have an edge between them in  $\Gamma$ . Let  $c$  be a 2-cube with edges  $c_1, c_2, c_3$  and  $c_4$  such that  $c_i \cap c_{i+1}$  is a vertex of  $c$  for each  $i \pmod 4$ . We label  $c_1$  and  $c_3$  by  $s_1$ , and  $c_2$  and  $c_4$  by  $s_2$ . A *square attachment operation* produces a new complex from  $X$  by attaching  $c$  to  $X$  by identifying  $c_1$  to  $f_1$  and  $c_2$  to  $f_2$ . Note that, unlike in [Dani and Levcovitz 2021], we do not need to define cube attachments for higher-dimensional cubes, as we are in the case that  $\Gamma$  is triangle-free.

Finally, given a collection of 2-cubes in  $X$  with common boundary, we can produce a new complex from  $X$  by naturally identifying every 2-cube in this collection to a single 2-cube. In this case, we say a *cube identification operation* was performed to  $X$ .

We define a  $\Gamma$ -labeled complex  $\Omega_0$  associated to  $G^\Theta$  as follows. First, we enumerate the  $\Lambda$ -edges as  $s_1 t_1, \dots, s_n t_n$ , where  $s_i$  and  $t_i$  are the two endpoints of the  $i^{\text{th}}$   $\Lambda$ -edge. We set  $\Omega_0$  to be a bouquet of  $n$  circles, each of which is subdivided into two edges, such that the  $i^{\text{th}}$  circle has label  $s_i t_i$ .

Next, we describe a series of complexes built iteratively from  $\Omega_0$ . These are

$$\Omega_0 \rightarrow \Omega_1 \rightarrow \Omega_2 \rightarrow \dots$$

For each  $i > 0$ , the complex  $\Omega_i$  is obtained by either a fold, square attachment or square identification operation performed to  $\Omega_{i-1}$ . Furthermore, we assume that the order of operations is as follows: first all possible fold and square identifications are performed, then all possible square attachment operations are applied to the resulting complex, and these processes are alternated (see [Dani and Levcovitz 2021] for details).

Let  $\Omega$  be the direct limit of such a sequence. We call  $\Omega$  a *completion* of  $G^\Theta$ . In [Dani and Levcovitz 2021] we show that properties of  $\Omega$  reflect those of the subgroup  $G^\Theta$ .

The index of  $G^\Theta$  can be determined by properties of  $\Omega$ . We say that a vertex  $u$  of a  $\Gamma$ -labeled complex has *full valence* if for any vertex  $s \in \Gamma$ , there is an edge incident to  $u$  with label  $s$ . Below we present a version of [Dani and Levcovitz 2021, Theorem 6.9] together with [Dani and Levcovitz 2021, Lemma 6.8] under the hypotheses which we will need:

**Theorem 4.1** *Let  $\Gamma$  be a triangle-free graph with no cone vertex. A subgroup  $G < W_\Gamma$  has finite index in  $W_\Gamma$  if and only if  $\Omega$  is finite and every vertex of  $\Omega$  has full valence. Furthermore, if  $G$  is indeed of finite index, then its index is exactly the number of vertices of  $\Omega$ .*

We introduce two new properties below which will help us characterize when  $G^\Theta$  has finite index in  $W_\Gamma$ .

**Definition 4.2** (conditions  $\mathcal{F}_1$  and  $\mathcal{F}_2$ ) We say that  $\Theta = \Theta(\Gamma, \Lambda)$  satisfies *condition  $\mathcal{F}_1$*  if given any  $s \in V(\Theta)$  which is not a cone vertex of  $\Gamma$ , it follows that  $s$  is the endpoint of some  $\Lambda$ -edge. We say that  $\Theta$  satisfies *condition  $\mathcal{F}_2$*  if given any distinct components  $\Lambda_s$  and  $\Lambda_t$  of  $\Lambda$ , and vertices  $s$  of  $\Lambda_s$  and  $t$  of  $\Lambda_t$ , there is a  $\Lambda_s \Lambda_t$ -path in  $\Theta$  from  $s$  to  $t$ .

**Remark 4.3** Suppose  $\Gamma$  is connected,  $\Lambda$  contains exactly two components and that  $\Theta = \Theta(\Gamma, \Lambda)$  satisfies  $\mathcal{R}_2$  and  $\mathcal{F}_1$ . Then  $\Theta$  satisfies  $\mathcal{F}_2$ . For given any two vertices contained in different components of  $\Lambda$ , as  $\Gamma$  is connected, there is a  $\Gamma$ -path between them. Furthermore, this has to be a 2-component path as  $\Theta$  satisfies  $\mathcal{R}_2$ , and the two  $\Lambda$ -components this path visits have to be the ones containing the chosen vertices (as there are only two  $\Lambda$  components). This remark will prove to be useful when verifying whether certain graphs satisfy  $\mathcal{F}_2$ .

**Remark 4.4** Suppose  $\Theta = \Theta(\Gamma, \Lambda)$  satisfies  $\mathcal{F}_2$ , and let  $\Lambda_1$  and  $\Lambda_2$  be distinct  $\Lambda$ -components. Then there exists an  $\Lambda_1 \Lambda_2$ -path between any two distinct vertices of  $\Lambda_1$ . To see this, let  $s$  and  $s'$  be distinct

vertices of  $\Lambda_1$ , and let  $t$  be vertex of  $\Lambda_2$ . By  $\mathcal{F}_2$  there is a  $\Lambda_1\Lambda_2$ -path from  $s$  to  $t$ , and similarly there is a  $\Lambda_1\Lambda_2$ -path from  $t$  to  $s'$ . Combining these two paths gives a  $\Lambda_1\Lambda_2$ -path from  $s$  to  $s'$ .

**Lemma 4.5** *Let  $\Gamma$  be a triangle-free graph with no cone vertex, and let  $\Lambda$  be a subgraph of  $\Gamma^c$  with no isolated vertices, such that  $(G^\Theta, E(\Lambda))$  is a RAAG system. If  $\Lambda$  has at most  $k \leq 2$  components and  $\Theta$  satisfies  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , then  $G^\Theta$  is of index  $2^k$  in  $W_\Gamma$ .*

We remark that this proof readily generalizes to the case of arbitrary  $k$ . However, we only need the case  $k \leq 2$ .

**Proof** Let  $\Omega_0$  be the  $\Gamma$ -labeled complex defined above, and let  $\Omega'$  be the complex obtained from  $\Omega_0$  by all possible fold operations.

Suppose first that  $\Lambda$  has one component. As  $\Lambda$  is connected, it is easily seen that  $\Omega'$  consists of two vertices with an edge labeled by  $s$  between them for  $s \in V(\Lambda)$ . As  $\Lambda$  satisfies  $\mathcal{R}_2$  by Proposition 3.17, no two vertices of  $\Lambda$  have an edge between them in  $\Gamma$ . Thus, no square attachments can be performed to  $\Omega'$ , and it follows that  $\Omega = \Omega'$ . Hence,  $\Omega$  is finite and has exactly two vertices.

Note that by the description of  $\Omega = \Omega'$  above, every vertex of  $\Omega$  is adjacent to every edge of  $\Omega$ . Also note that by condition  $\mathcal{F}_1$ , for every vertex  $s \in \Gamma$  there is some edge in  $\Omega$  labeled by  $s$ . From these two facts we deduce that every vertex of  $\Omega$  has full valence. Thus,  $G^\Theta$  has index 2 in  $W_\Gamma$  by Theorem 4.1.

Now suppose that  $\Lambda$  has two components  $\Lambda_1$  and  $\Lambda_2$ . In this case,  $\Omega'$  is readily seen to be a complex consisting of three vertices,  $u, v_1$  and  $v_2$ , with an edge from  $u$  to  $v_i$  labeled  $s$  corresponding to each vertex  $s$  of  $\Lambda_i$ , for  $i = 1, 2$ . By condition  $\mathcal{F}_1$ , the vertex  $u$  has full valence. Furthermore, by  $\mathcal{R}_2$ , for each  $i \in \{1, 2\}$ , no two edges of  $\Omega'$  that are each adjacent to both  $v_i$  and  $u$  have labels which are adjacent in  $\Gamma$ .

Let  $\Omega''$  be the complex obtained from  $\Omega'$  by performing all possible square attachment operations to  $\Omega'$ , and let  $\Omega'''$  be the complex obtained from  $\Omega''$  by all possible fold and square identification operations. In particular,  $\Omega'' = \Omega_l$  and  $\Omega''' = \Omega_k$  for some  $0 \leq l \leq k$ . Let  $s$  and  $s'$  be distinct vertices of  $\Lambda_1$ , and let  $t$  be any vertex of  $\Lambda_2$ . By condition  $\mathcal{F}_2$ , there is a  $\Lambda_1\Lambda_2$ -path whose vertices are  $s, t_1, s_1, t_2, s_2, \dots, t_m, s_m, t$  where  $s_i \in \Lambda_1$  and  $t_i \in \Lambda_2$  for all  $1 \leq i \leq m$ . Similarly, there is a  $\Lambda_1\Lambda_2$ -path whose vertices are  $s', t'_1, s'_1, t'_2, s'_2, \dots, t'_n, s'_n, t$  where  $s'_i \in \Lambda_1$  and  $t'_i \in \Lambda_2$  for all  $1 \leq i \leq n$ . Thus,  $\Omega''$  must contain length two paths, which do not intersect  $u$ , from  $v_1$  to  $v_2$  with each of the labels

$$t_1s, t_1s_1, t_2s_1, t_2s_2, \dots, t_ms_{m-1}, t_ms_m, t_s m,$$

and similarly length two paths, which do not intersect  $u$ , from  $v_1$  to  $v_2$  with each of the labels

$$t'_1s', t'_1s'_1, t'_2s'_1, t'_2s'_2, \dots, t'_ns'_{m-1}, t'_ns'_m, t'_s m.$$

It follows that the middle vertices of all these paths get folded to a single vertex  $v_3$  in  $\Omega'''$ . This analysis can be done for any  $s, s' \in \Lambda_1$ . Similar paths can also be produced for any  $t, t' \in \Lambda_2$ . It then follows that

$\Omega'''$  consists of exactly 4 vertices:  $u, v_1, v_2$  and  $v_3$ . Furthermore, there is an edge with label  $s$  between  $v_1$  and  $v_3$  for each  $s \in \Lambda_1$ , and there is an edge with label  $t$  between  $v_2$  and  $v_3$  for each  $t \in \Lambda_2$ . Thus, every vertex of  $\Omega'''$  can be seen to have full valence. Additionally, by condition  $\mathcal{R}_2$ , no additional square attachment operations can be performed to  $\Omega'''$ . Hence,  $\Omega = \Omega'''$ . It follows that  $G^\Theta$  has index exactly four in  $W_\Gamma$ . □

The next lemma shows that  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are necessary conditions for  $G^\Theta$  to have finite index.

**Lemma 4.6** *Let  $\Gamma$  be a triangle-free graph with no cone vertex, and let  $\Lambda$  be a subgraph of  $\Gamma^c$  with no isolated vertices, such that  $(G^\Theta, E(\Lambda))$  is a RAAG system. If  $G = G^\Theta$  is of finite index in  $W_\Gamma$ , then  $\Theta$  satisfies  $\mathcal{F}_1$  and  $\mathcal{F}_2$ .*

**Proof** We first check that condition  $\mathcal{F}_1$  holds. Let  $\Omega$  be a completion of  $G := G^\Theta$  as described in the beginning of this section. [Theorem 4.1](#) implies in particular that given any vertex  $s \in \Gamma$  there is an edge of  $\Omega$  with label  $s$ . This implies the vertex  $s$  is contained in some  $\Lambda$ -edge. Thus,  $\mathcal{F}_1$  must hold.

We now check condition  $\mathcal{F}_2$ . Let  $s \in \Lambda_s$  and  $t \in \Lambda_t$  be as in the definition of condition  $\mathcal{F}_2$  ([Definition 4.2](#)). If  $s$  commutes with  $t$ , then there is an edge in  $\Gamma$  between  $s$  and  $t$ , and we are done. So we may assume that  $s$  and  $t$  do not commute.

As  $G$  is of finite index, it follows that there exist  $g_1, \dots, g_n \in W_\Gamma$  such that  $W_\Gamma = Gg_1 \sqcup Gg_2 \sqcup \dots \sqcup Gg_n$ . Let  $w_1, \dots, w_n$  be reduced words representing  $g_1, \dots, g_n$ , and let  $K = \max\{|w_1|, \dots, |w_n|\}$ . Define the word  $h = s_1 t_1 s_2 t_2 \dots s_{K+4} t_{K+4}$  where  $s_i = s$  and  $t_i = t$  for all  $1 \leq i \leq K + 4$ . It readily follows from Tits' solution to the word problem (see [Theorem 2.4](#)) that  $h$  is reduced. Furthermore, we can write  $h \simeq ww'$ , where  $w$  and  $w'$  are words in  $W_\Gamma$  such that  $w' = w_i$  for some  $1 \leq i \leq n$  and  $w$  is a product of  $\Lambda$ -edges representing an element of  $G$ . We can form a disk diagram in  $W_\Gamma$  with boundary label  $hw'^{-1}w^{-1}$ . Let  $\alpha_h, \alpha_w$  and  $\alpha_{w'}$  respectively be the corresponding paths along the boundary of  $D$  with labels respectively  $h, w$  and  $w^{-1}$ .

Note that as  $h$  is reduced, no hyperplane intersects  $\alpha_h$  twice. Also note that any pair of hyperplanes emanating from  $\alpha_h$  cannot intersect as  $s$  and  $t$  do not commute. As  $|h| > |w'| + 4$ , it follows that the hyperplanes  $H_{s_1}, H_{t_1}, H_{s_2}$  and  $H_{t_2}$ , dual respectively to the first four edges of  $\alpha_h$  (namely those labeled by  $s_1, t_1, s_2$  and  $t_2$ ), must each intersect  $\alpha_w$ . It must now be the case that there exists a chain of hyperplanes (see [Definition 3.20](#))  $H_{s_1} = H_0, H_1, \dots, H_m = H_{s_2}$  and another chain of hyperplanes  $H_{t_1} = H'_0, H'_1, \dots, H'_n = H_{t_1}$ . These two chains intersect, and by reasoning similar to that in [Observation 3.24](#), it follows that there is a  $\Lambda_s \Lambda_t$ -path from  $s$  to  $t$ . □

**Lemma 4.7** *Let  $\Gamma$  be a triangle-free graph. Let  $\Lambda$  be a subgraph of  $\Gamma^c$  with no isolated vertices, such that  $(G^\Theta, E(\Lambda))$  is a RAAG system and  $G^\Theta$  has finite index in  $W_\Gamma$ . If  $\Gamma$  contains a cone vertex, then  $\Lambda$  contains exactly one component. If  $W_\Gamma$  is not virtually free, then  $\Lambda$  contains exactly two components. Otherwise,  $\Lambda$  contains at most two components.*

**Proof** Suppose first that  $\Gamma$  contains a cone vertex  $s \in \Gamma$ . We may assume that  $\Gamma$  does not consist of a single edge, as  $\Lambda$  would be empty in that case. As  $\Gamma$  is triangle-free in addition, there can be at most one cone vertex. Since  $\Gamma$  is triangle-free, it follows that  $\Gamma' = \Gamma \setminus s$  is a graph with no edges and is therefore virtually free. Furthermore, every  $\Lambda$ -edge is contained in  $\Gamma'$ , and  $G^\Theta$  is a finite-index subgroup of  $W_{\Gamma'}$ . By Lemma 4.6, we conclude that  $\Theta' = \Theta(\Gamma', \Lambda)$  satisfies condition  $\mathcal{F}_2$ . In particular, there is a  $\Gamma'$ -edge between any two  $\Lambda$  components. As  $\Gamma'$  does not have any edges,  $\Lambda$  has exactly one component and the claim follows in this case.

We now assume that  $\Gamma$  does not contain a cone vertex. Furthermore, by Lemma 4.6 we may assume that  $\Theta = \Theta(\Gamma, \Lambda)$  satisfies  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , and that  $\Theta$  satisfies  $\mathcal{R}_1$ – $\mathcal{R}_4$  by Proposition 3.17.

Suppose now that no two distinct  $\Lambda$ -edges commute. It follows that  $G^\Theta$  is isomorphic to a free group, and since  $G^\Theta$  is of finite index,  $W_\Gamma$  is virtually free. Suppose, for a contradiction, that  $\Lambda$  has three distinct components  $\Lambda_1, \Lambda_2$  and  $\Lambda_3$ . Let  $s$  and  $t$  be distinct vertices of  $\Lambda_1$ . By Remark 4.4 there is a  $\Lambda_1\Lambda_2$ -path  $\alpha_1$  from  $s$  to  $t$  which we can assume does not repeat vertices. Similarly, there is a  $\Lambda_1\Lambda_3$ -path  $\alpha_2$  from  $s$  to  $t$  which does not repeat vertices. Observe that  $s, t \in \alpha_1 \cap \alpha_2$ . Starting at  $s$  and traveling along  $\alpha_1$ , let  $x$  be the first vertex after  $s$  such that  $x \in \alpha_1 \cap \alpha_2$ . Then the subpath  $\alpha'_1$  of  $\alpha_1$  between  $s$  and  $x$  contains exactly two vertices of  $\alpha_1 \cap \alpha_2$ . Let  $\alpha'_2$  be the subpath of  $\alpha_2$  between  $s$  and  $x$ . Note that  $|\alpha'_1|, |\alpha'_2| \geq 2$ , as every other vertex of  $\alpha_1$  is in  $\Lambda_2$  and  $\alpha_2 \cap \Lambda_2 = \emptyset$ . It follows that  $c = \alpha'_1 \cup \alpha'_2$  is a cycle in  $\Gamma$ . Let  $c'$  be a subcycle of  $c$  which is an induced subgraph of  $\Gamma$ . If  $c'$  has three vertices, then this contradicts  $\Gamma$  being triangle-free. On the other hand, if  $c'$  has more than three vertices, then this contradicts  $W_\Gamma$  being virtually free. Thus,  $\Lambda$  can have at most two components and the claim follows in this case.

Suppose now there exist  $\Lambda$ -edges  $a_1a_2$  and  $b_1b_2$  which commute, with  $a_1a_2 \neq (b_1b_2)^{\pm 1}$ . These  $\Lambda$ -edges must be in different components of  $\Lambda$  by condition  $\mathcal{R}_2$  and Lemma 2.6. In this case,  $W_\Gamma$  is not virtually free as it contains a subgroup isomorphic to  $\mathbb{Z}^2$ . Suppose, for a contradiction, that  $\Lambda$  contains at least three distinct  $\Lambda$ -edge components  $\Lambda_1, \Lambda_2$  and  $\Lambda_3$ . Without loss of generality, we may assume that  $a_1b_1 \in \Lambda_1$  and that  $a_2b_2 \in \Lambda_2$ . We will obtain a contradiction by showing that  $\Gamma$  must contain a triangle.

By Lemma 2.6,  $a_1, a_2, b_1$  and  $b_2$  form a square in  $\Gamma$ . By Remark 4.4, there is a  $\Lambda_1\Lambda_3$ -path from  $a_1$  to  $a_2$ . Similarly, there is a  $\Lambda_2\Lambda_3$ -path from  $b_1$  to  $b_2$ . Thus,  $\Gamma$  contains the configuration described in the statement of Lemma 3.34. That lemma then implies that  $\Gamma$  contains a triangle, a contradiction.  $\square$

**Theorem 4.8** *Let  $W_\Gamma$  be a 2-dimensional RACG. Let  $\Lambda$  be a subgraph of  $\Gamma^c$  with no isolated vertices, and let  $G^\Theta$  be the subgroup of  $W_\Gamma$  generated by the  $\Lambda$ -edges. Then the following are equivalent:*

- (1)  $(G^\Theta, E(\Lambda))$  is a RAAG system and  $G^\Theta$  has finite index in  $W_\Gamma$ .
- (2)  $(G^\Theta, E(\Lambda))$  is a RAAG system and  $G^\Theta$  has index either two or four in  $W_\Gamma$  (and exactly four if  $W_\Gamma$  is not virtually free).
- (3)  $\Lambda$  has at most two components and  $\Theta$  satisfies conditions  $\mathcal{R}_1$ – $\mathcal{R}_4, \mathcal{F}_1$  and  $\mathcal{F}_2$ .



**Proof** Clearly (2) implies (1). To see the remaining implications, suppose first that  $\Gamma$  contains a cone vertex  $s$ . Then  $\Gamma' = \Gamma \setminus s$  is a graph with no edges, and  $W_{\Gamma'}$  is an index two subgroup of  $\Gamma$ . Suppose that (1) holds. By Lemma 4.7,  $\Lambda$  has exactly one component. By Theorem 3.18 and Lemma 4.6,  $\Theta' = \Theta(\Gamma', \Lambda)$  satisfies conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . Consequently,  $\Theta$  satisfies these conditions as well. Thus (3) holds. By Lemma 4.5, we know that  $(G^{\Theta'}, E(\Lambda))$  is a RAAG system of index 2 in  $W_{\Gamma'}$ , and thus  $(G^{\Theta}, E(\Lambda))$  is a RAAG system of index four in  $W_{\Gamma}$ . Therefore (2) holds. Finally, if (3) holds then (1) holds by Theorem 3.18 and Lemma 4.5.

Now suppose that  $\Gamma$  does not have a cone vertex. If (1) holds, then by Lemma 4.7,  $\Lambda$  has exactly two components if  $W_{\Gamma}$  is not virtually free and at most two components otherwise. Thus (2) holds by Lemma 4.5. By Theorem 3.18 and Lemma 4.6, (3) holds. Finally if (3) holds, then (1) follows by Theorem 3.18 and Lemma 4.5.  $\square$

**Corollary 4.9** *Let  $W_{\Gamma}$  be a 2–dimensional RACG. Let  $\Lambda$  be a subgraph of  $\Gamma^c$  with no isolated vertices such that the subgroup  $(G, E(\Lambda))$  is a finite-index RAAG system. Then either:*

- (1) *The graph  $\Gamma$  does not contain any edges and  $E(\Lambda)$  is a spanning tree in  $\Gamma^c$ . In particular,  $W_{\Gamma}$  is virtually free.*
- (2) *The group  $W_{\Gamma}$  is not virtually free. Furthermore, the vertices of  $\Gamma$  can be 2–colored by red and blue (ie each edge of  $\Gamma$  connects a red vertex and a blue vertex) and  $G$  is isomorphic to the kernel of the homomorphism  $\Psi: W_{\Gamma} \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle r, b \mid r^2 = b^2 = 1 \rangle$  which maps red and blue generators of  $V(\Gamma)$  to  $r$  and  $b$  respectively.*

**Proof** By Theorem 4.8,  $\Lambda$  has at most two components. Suppose first that  $\Lambda$  contains exactly one component. Again by Theorem 4.8, the graph  $\Theta$  satisfies  $\mathcal{R}_1$ ,  $\mathcal{R}_2$ , and  $\mathcal{F}_1$ . From these conditions, it follows that  $\Gamma$  cannot contain any edges and that  $E(\Lambda)$  is a spanning tree in  $\Gamma^c$ . As  $\Gamma$  does not contain any edges,  $W_{\Gamma}$  is virtually free.

Suppose now that  $\Lambda$  has exactly two components. We color the vertices of one component red and the vertices of the other component blue. By  $\mathcal{R}_2$ , each edge of  $\Gamma$  connects a red vertex and a blue vertex, ie we have a 2–coloring of  $\Gamma$ . Furthermore, by the definition of  $\Psi$ , every  $\Lambda$ –edge (thought of as an element of  $G$ ) is in the kernel of  $\Psi$ . As  $G$  is generated by such elements, it follows that  $G < \ker(\Psi)$ . By Theorem 4.8,  $G$  has index 4 in  $W_{\Gamma}$ . As  $\ker(\Psi)$  has index 4 as well, it follows that  $G$  is isomorphic to  $\ker(\Psi)$ .  $\square$

## 5 Applications

In this section we give concrete families of RACGs containing finite-index RAAG subgroups. These cannot be obtained by applying the Davis–Januszkiewicz constructions to the defining graphs of the RAAGs they are commensurable to.

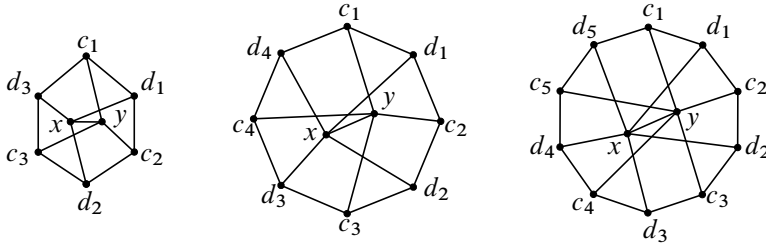


Figure 13: The figure illustrates the graphs  $\Gamma_n$  defined in Corollary 5.1 for  $n = 3, 4, 5$ .

### 5.1 Nonplanar RACGs commensurable to RAAGs

In this subsection, we construct two families of RACGs with nonplanar defining graphs containing finite-index RAAG subgroups. These will serve as a warm-up for Theorem 5.5.

We begin by constructing a family of quasi-isometrically distinct RACGs defined by the sequence of graphs  $\Gamma_n$  (shown in Figure 13) which are commensurable to RAAGs whose defining graphs are cycles.

**Corollary 5.1** (to Theorem 4.8) *For  $n \geq 3$ , let  $\Gamma_n$  be the graph obtained by starting with a  $2n$ -gon whose vertices (in cycle order) are  $c_1, d_1, c_2, d_2, \dots, c_n, d_n$  and adding two vertices  $x$  and  $y$ , such that  $y$  is adjacent to  $c_i$  for each  $i$ ,  $x$  is adjacent to  $d_i$  for each  $i$ , and  $x$  is adjacent to  $y$  (see Figure 13). Then*

- (1) *the RACG  $W_{\Gamma_n}$  has a subgroup of index four that is isomorphic to (and hence is commensurable to) the RAAG  $A_{C_{2n}}$ , where  $C_{2n}$  is a cycle of length  $2n$ ;*
- (2)  *$W_{\Gamma_n}$  is not quasi-isometric to  $W_{\Gamma_m}$  for  $m \neq n$ .*

**Proof** Fix  $n \geq 3$ , and let  $\Gamma$  denote  $\Gamma_n$ . We define a graph  $\Lambda \subset \Gamma^c$  as follows. Let  $\Lambda_x$  be the star graph consisting of the union of the edges of  $\Gamma^c$  from  $x$  to  $c_i$  for each  $i$ . Let  $\Lambda_y$  be the star graph consisting of the edges of  $\Gamma^c$  from  $y$  to  $d_i$  for each  $i$ . Let  $\Lambda = \Lambda_x \cup \Lambda_y$ . (See Figure 4 for an illustration of  $\Lambda$  in the case  $n = 3$ .)

We show below that  $\Theta = \Theta(\Gamma, \Lambda)$  satisfies  $\mathcal{R}_1$ – $\mathcal{R}_4$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . Then it will follow from Theorem 4.8, that  $(G^\Theta, E(\Lambda))$  is a RAAG system, and that  $G^\Theta$  has index four in  $W_\Gamma$ . Moreover, it is easily checked that the commuting graph  $\Delta$  associated to  $\Lambda$  (as defined in Section 2.2) is isomorphic to  $C_{2n}$ . Consequently,  $G^\Theta$  is isomorphic to  $A_{C_{2n}}$ . Thus, this will show (1).

It is easy to verify  $\mathcal{F}_1$ ,  $\mathcal{R}_1$ , and  $\mathcal{R}_2$ . Then by Remark 4.3, it follows that  $\mathcal{F}_2$  holds as well. We now check  $\mathcal{R}_3$ . First note that there are exactly three squares in  $\Gamma$  containing the edge  $c_1d_1$ , and each of these satisfies the property in  $\mathcal{R}_3$ . Now the fact that every square contains an edge of the  $2n$ -gon, together with the symmetry of the diagram, implies that  $\mathcal{R}_3$  holds.

To check  $\mathcal{R}_4$ , let  $\gamma$  be a  $\Lambda_x\Lambda_y$ -cycle and let  $e$  be an edge of  $\gamma$ . By symmetry, we can assume that  $e$  is either  $c_1d_1$ ,  $c_1y$  or  $xy$ . Suppose first that  $e = c_1d_1$ . Then  $\gamma$  contains either  $d_n c_1$  or  $yc_1$ . In both cases,

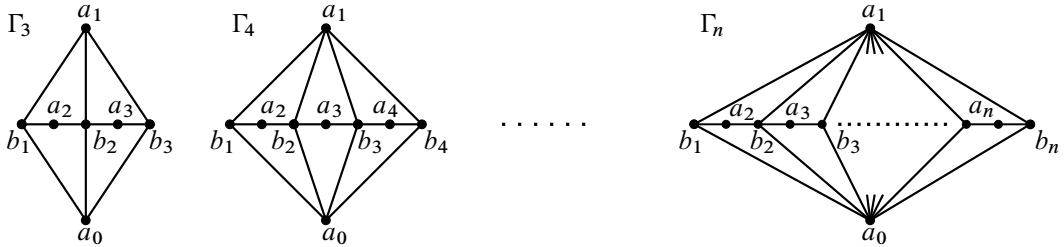


Figure 14: The figure defines the family of graphs  $\Gamma_n$ , for  $n \geq 3$ , used in Corollary 5.2.

the  $\Lambda$ -convex hull of the vertices of  $\gamma$  contains  $y$ . Similarly,  $\gamma$  contains either  $d_1x$  or  $d_1c_2$ , and in both cases the  $\Lambda$ -convex hull of  $\gamma$  contains  $x$ . As  $c_1d_1$  is contained in the square  $c_1d_1xy$ , and  $x$  and  $y$  are in the appropriate convex hulls, it follows that  $\mathcal{R}_4$  holds for the  $\Gamma$ -cycle  $\gamma$  and edge  $e$ .

Suppose now that  $e = c_1y$ . It follows that  $\gamma$  contains either  $yx$  or  $yc_i$  for some  $i > 1$ . In each case,  $x$  is in the  $\Lambda$ -convex hull of  $\gamma$ . Furthermore,  $\gamma$  contains either  $c_1d_1$  or  $c_1d_n$ . In the former case, the square  $c_1d_1xy$  contains  $e$  and has vertices in the  $\Lambda$ -convex hull of the vertices in  $\gamma$ . In the latter case the same argument applies to the square  $c_1d_nxy$ .

Finally, suppose that  $e = xy$ . By symmetry, we may assume that  $\gamma$  contains  $yc_1$ . Furthermore,  $yc_1$  must be followed by either  $c_1d_2$  or  $c_1d_n$  in  $\gamma$ . Then, as in the previous paragraph, either the square  $c_1d_1xy$  or the square  $c_1d_nxy$  contains  $e$  and has vertices in the  $\Lambda$ -convex hull of  $\gamma$ . Thus  $\mathcal{R}_4$  is satisfied in all cases.

We have thus established that (1) holds, by showing that  $\Theta$  satisfies  $\mathcal{R}_1$ - $\mathcal{R}_4$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . Consequently, for each  $n$ , we know that  $W_{\Gamma_n}$  is commensurable, and in particular quasi-isometric, to  $A_{C_{2n}}$ . Claim (2) then follows from [Bestvina et al. 2008]. □

Next, we give a family of RACGs whose defining graphs are not planar and are commensurable to RAAGs which are not atomic (as defined in [Bestvina et al. 2008]).

**Corollary 5.2** *Given  $n \geq 3$  and  $k \geq 1$ , let  $\Delta_{nk}$  be the graph obtained by taking  $k$  copies of  $\Gamma_n$  (defined in Figure 14), and identifying them all along the subgraph induced by  $V(\Gamma_n) \setminus \{a_0\}$ . Thus  $\Delta_{nk}$  has vertices  $a_1, a_2, \dots, a_n, b_1, \dots, b_n$  and also  $a_{01}, \dots, a_{0k}$ . (The left side of Figure 15 shows  $\Delta_{42}$ .) Then  $W_{\Delta_{nk}}$  contains an index four subgroup isomorphic to a RAAG.*

**Proof** Fix  $n \geq 3, k \geq 1$  and let  $\Delta = \Delta_{nk}$ . We define  $\Lambda$ , a subgraph of  $\Delta^c$  consisting of two components. The first component  $\Lambda_a$  is the union of the edges of  $\Delta^c$  of the form  $a_1a_i$ , where  $2 \leq i \leq n$  and  $a_1a_{0j}$  for  $1 \leq j \leq k$ . The second component  $\Lambda_b$  is the path in  $\Delta^c$  visiting  $b_1, b_2, \dots, b_n$ . (See the right side of Figure 15 for an illustration of the case  $n = 4$  and  $k = 2$ ).

Let  $\Theta = \Theta(\Delta, \Lambda)$  be as in the previous sections. We verify the properties  $\mathcal{R}_1$ - $\mathcal{R}_4$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . It will then follow from Theorem 4.8 that the subgroup generated by  $E(\Lambda)$  is an index four visual RAAG subgroup.

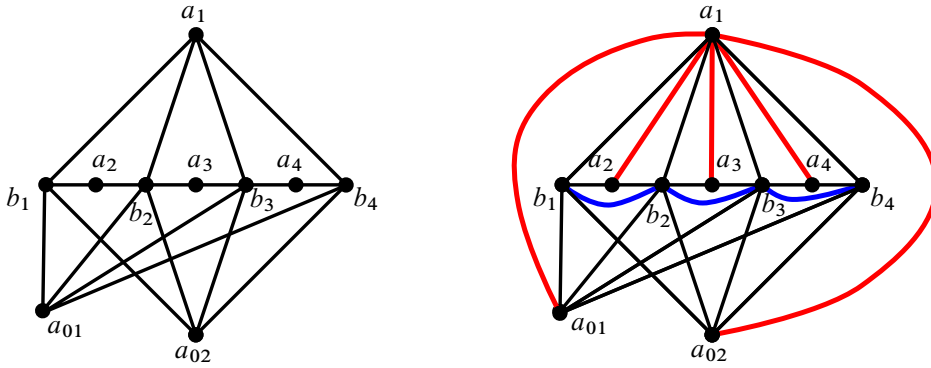


Figure 15: The figure shows  $\Delta_{42}$  on the left, and the two components of  $\Lambda$  for the graph  $\Delta_{42}$  on the right. The component  $\Lambda_a$  is shown in red and the component  $\Lambda_b$  is shown in blue.

The conditions  $\mathcal{R}_1, \mathcal{R}_2$  and  $\mathcal{F}_1$  are immediate. Condition  $\mathcal{F}_2$  holds by Remark 4.3. We now check  $\mathcal{R}_3$ . Each square in  $\Delta$  is of one of the following forms:

- (1)  $b_i a_{i+1} b_{i+1} a_1$  for  $1 \leq i \leq n - 1$ ;
- (2)  $b_i a_{i+1} b_{i+1} a_{0j}$  for  $1 \leq i \leq n - 1$  and  $1 \leq j \leq k$ ;
- (3)  $b_i a_1 b_{i'} a_{0j}$  for  $1 \leq i < i' \leq n$  and  $1 \leq j \leq k$ ;
- (4)  $b_i a_{0j} b_{i'} a_{0j'}$  for  $1 \leq i < i' \leq n$  and  $1 \leq j \leq k$ .

Condition  $\mathcal{R}_3$  follows immediately for the first type of square, as the appropriate  $\Lambda$ -convex hulls do not contain any additional vertices not included in the vertex set of the square. For the second type of square, the convex hull in  $\Lambda_b$  does not contain any additional vertices, but the convex hull in  $\Lambda_a$  contains the additional vertex  $a_1$ , as this vertex lies on the  $\Lambda_a$ -path between  $a_{i+1}$  and  $a_{0j}$ . Since  $a_1$  is adjacent to  $b_i$  and  $b_{i+1}$ , the condition  $\mathcal{R}_3$  is verified for this type of 2-component square. For the third type, the  $\Lambda$ -convex hull of  $\{a_1, a_{0j}\}$  does not contain any additional vertices of  $\Lambda$ , and the  $\Lambda$ -convex hull of  $\{b_i, b_{i'}\}$  contains the additional vertices  $b_{i+1}, \dots, b_{i'-1}$ . Since  $a_1$  and  $a_{0j}$  are adjacent to each of these,  $\mathcal{R}_3$  is verified for this type of 2-component square as well. Finally, for the last type, the  $\Lambda$ -convex hull of  $\{a_{0j}, a_{0j'}\}$  contains the additional vertex  $a_1$ , and the  $\Lambda$ -convex hull of  $\{b_i, b_{i'}\}$  contains the additional vertices  $b_{i+1}, \dots, b_{i'-1}$ . Once again, it is easily verified that  $a_0, a_{1j}, a_{1j'}$  are each adjacent to each of  $b_i, \dots, b_{i'}$ . Thus  $\mathcal{R}_3$  is verified.

Finally, we check  $\mathcal{R}_4$ . Let  $\gamma$  be a  $\Lambda_a \Lambda_b$ -cycle and let  $e$  be an edge of  $\gamma$ . First suppose  $e$  is of the form  $a_i b_i$  for  $2 \leq i \leq n$ . In this case,  $\gamma$  necessarily passes through  $b_{i-1}$  and some  $a$ , where either  $a = a_1$  or  $a = a_{0j}$ , for some  $1 \leq j \leq k$ . Thus the  $\Lambda_a \Lambda_b$ -square  $b_{i-1} a_i b_i a$  satisfies the criterion in  $\mathcal{R}_4$ , since it contains  $e$ , and the two vertices  $a$  and  $b_{i-1}$  are contained in the  $\Lambda$ -convex hull of the vertices of  $\gamma$ . The case where  $e$  is of the form  $a_i b_{i-1}$  for  $2 \leq i \leq n$  is similar.

Now suppose  $e$  is of the form  $a_1 b_i$  for some  $1 \leq i \leq n$ . Then  $\gamma$  necessarily passes through an edge of one of the following forms:  $b_i a_{0j}$  for some  $1 \leq j \leq k$ ,  $b_i a_i$  or  $b_i a_{i+1}$ . In the first of these cases,  $\gamma$

necessarily also passes through a vertex  $b_{i'}$  for some  $i' \neq i$ , and  $a_1 b_i a_0 b_{i'}$  is the desired square. If the edge is of the form  $b_i a_i$  (resp.  $b_i a_{i+1}$ ) then  $\gamma$  must also pass through  $b_{i-1}$  (resp.  $b_{i+1}$ ), and the desired square is  $a_1 b_i a_i b_{i-1}$  (resp.  $a_1 b_i a_{i+1} b_{i+1}$ ). The case where  $e$  is of the form  $a_0 b_i$  for some  $1 \leq i \leq n$  and  $1 \leq j \leq k$  is similar. This completes the verification of  $\mathcal{R}_4$ , and the corollary follows.  $\square$

**Remark 5.3** The RAAGs obtained in the above corollary do not have a tree for defining graph when  $k \geq 2$  and  $n \geq 3$ . This is easy to check by computing the associated commuting graph.

## 5.2 2-Dimensional RACGs with planar defining graph

Nguyen and Tran [2019] characterized exactly which one-ended, 2-dimensional RACGs defined by planar nonjoin, CFS graphs are quasi-isometric to RAAGs. In this subsection, we use their work in conjunction with Theorem 4.8 to prove Theorem B from the introduction. Note that CFS is a graph-theoretic condition introduced in [Dani and Thomas 2015] to characterize RACGs with at most quadratic divergence. We omit the definition, as it is not needed here.

**Remark 5.4** Any one-ended, 2-dimensional RACG that is quasi-isometric to a RAAG must have CFS defining graph. This follows as one-ended RAAGs have either linear or quadratic divergence [Behrstock and Charney 2012], and the defining graph of a 2-dimensional RACG with linear or quadratic divergence is CFS [Dani and Thomas 2015].

Recall that a graph  $\Sigma$  is a *suspension* if  $\Sigma$  decomposes as a join  $\Sigma = \{a_1, a_2\} \star B$  where  $a_1$  and  $a_2$  are nonadjacent vertices. We also say that  $\Sigma$  is the *suspension* of the graph  $B$ . We use the notation  $\Sigma_k(a, b)$  to denote the suspension graph  $\{a_1, a_2\} \star \{b_1, \dots, b_k\}$ , and we say that  $a_1$  and  $a_2$  are the *suspension vertices*.

Let  $\Gamma$  be a graph which is connected, triangle-free, CFS and planar. Suppose that a planar embedding from  $\Gamma$  into the sphere  $\mathbb{S}^2$  is fixed. Nguyen and Tran [2019] constructed a tree  $T$  (this is the *visual decomposition tree* of Section 3 of that paper) associated to  $\Gamma$  with the following properties. The vertices of  $T$  are in bijection with maximal suspension subgraphs of  $\Gamma$ . As  $\Gamma$  is triangle-free, every maximal suspension of  $\Gamma$  is of the form  $\Sigma_k(a, b)$ , where both  $\{a_1, a_2\}$  and  $\{b_1, \dots, b_k\}$  are each sets of disjoint vertices of  $\Gamma$ , and  $k \geq 3$  if  $T$  contains at least two vertices. Moreover, every vertex of  $\Gamma$  is contained in some suspension corresponding to a vertex of  $T$ . Two vertices of  $T$  corresponding to suspensions  $\Sigma = \Sigma_k(a, b)$  and  $\Sigma' = \Sigma_l(c, d)$  are connected by an edge if  $\Sigma \cap \Sigma'$  is a 4-cycle  $C$  which separates  $\mathbb{S}^2$  into two nontrivial components  $B_1$  and  $B_2$ , such that  $\Sigma_1 \setminus C \subset B_1$  and  $\Sigma_2 \setminus C \subset B_2$ . Moreover, it must follow (by the maximality of the suspensions) that  $C = \{a_1, c_1, a_2, c_2\}$ , ie  $C$  contains exactly the suspension vertices of  $\Sigma$  and  $\Sigma'$ .

If  $\Gamma$  (with the above assumptions) is a join, then it readily follows that  $\Gamma$  is quasi-isometric to a RAAG whose defining graph is a tree of diameter at most 2. Nguyen and Tran [2019, Theorem 1.2] showed that

if  $\Gamma$  is not a join, then  $W_\Gamma$  is quasi-isometric to a RAAG if and only if every vertex  $v \in T$  has valence strictly less than  $k$ , where  $\Sigma_k(a, b)$  is the maximal suspension in  $\Gamma$  corresponding to  $v$ . Moreover, they showed that such RAAGs have defining graph a tree of diameter at least 3. Below, we prove such RACGs are in fact commensurable to RAAGs.

**Theorem 5.5** *Let  $W_\Gamma$  be a 2–dimensional, one-ended RACG with planar defining graph  $\Gamma$ . Then  $W_\Gamma$  is quasi-isometric to a RAAG if and only if it contains an index 4 subgroup isomorphic to a RAAG.*

**Proof** One direction of the theorem is obvious. Thus, we prove that if  $W_\Gamma$  satisfies these hypotheses and is quasi-isometric to a RAAG, then  $W_\Gamma$  contains an index 4 subgroup isomorphic to a RAAG. We do this by constructing a subgraph  $\Lambda \subset \Gamma^c$  with two components and satisfying the hypotheses of [Theorem 4.8](#).

Fix a planar embedding of  $\Gamma$  into the sphere  $S^2$ . Note that by [Remark 5.4](#) and the hypotheses of the theorem, it follows that  $\Gamma$  is triangle-free, CFS and planar. Thus, there exists a visual decomposition tree  $T$  associated to  $\Gamma$  as described above. Furthermore, as  $W_\Gamma$  is quasi-isometric to a RAAG, it follows from [[Nguyen and Tran 2019](#), Theorem 1.2] that the valence of a vertex of  $T$  corresponding to the maximal suspension  $\Sigma_k(a, b)$  is less than  $k$ .

Henceforth, to simplify notation, the word suspension will always refer to a maximal suspension, and will consequently correspond to a vertex of  $T$ . Given a suspension  $\Sigma_k(a, b) = \{a_1, a_2\} \star B$  we say that a labeling  $\{b_1, \dots, b_k\}$  of the vertices of  $B$  is *cyclic* if the following holds. If  $C$  is a 4–cycle spanning the vertices  $\{a_1, b_i, a_2, b_{i+1}\}$  for some  $1 \leq i \leq k$  or spanning the vertices  $\{a_1, b_1, a_2, b_k\}$ , then every vertex of  $\Sigma \setminus C$  is contained in a common component of  $S^2 \setminus C$ . Observe that if  $E$  is a cycle corresponding to an edge of  $T$  incident to the vertex of  $T$  given by  $\Sigma_k(a, b)$ , then the planarity of  $\Gamma$  implies that  $E$  is one of the cycles  $C$  mentioned in the previous sentence.

Let  $N$  be the number of vertices of  $T$ . Let  $T_1 \subset \dots \subset T_N = T$  be a nested sequence of subtrees of  $T$  such that  $T_1$  consists of a single vertex of  $T$  and  $T_i$  has exactly  $i$  vertices. Such choices are clearly possible. For each  $1 \leq i \leq n$ , let  $\Gamma_i$  be the subgraph of  $\Gamma$  spanned by every suspension that corresponds to a vertex of  $T_i$ . Note that  $\Gamma_i \subset \Gamma_{i+1}$  for all  $1 \leq i < N$  and that  $\Gamma_N = \Gamma$ . We define a nested sequence of graphs  $\Lambda_1 \subset \dots \subset \Lambda_N$  such that for each  $1 \leq i \leq N$ ,  $\Lambda_i \subset \Gamma_i^c$  and the following hold:

- (1) Let  $C$  be a 4–cycle corresponding to an edge of  $T$  that is incident to  $T_i$ . Then each pair of nonadjacent vertices in  $C$  is contained in a common edge of  $\Lambda_i$ .
- (2) The graph  $\Lambda_i$  contains exactly two components, and  $\Theta_i = \Theta(\Gamma_i, \Lambda_i)$  satisfies conditions  $\mathcal{R}_1$ – $\mathcal{R}_4$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$ .

The theorem clearly follows from this claim by using the graph  $\Lambda = \Lambda_N \subset \Gamma^c$ .

We first define  $\Lambda_1$  corresponding to the vertex  $T_1 = \{v\}$ . Let  $\Sigma = \Sigma_k(a, b) = \{a_1, a_2\} \star \{b_1, \dots, b_k\}$  be the suspension corresponding to  $v$ , and assume that  $\{b_1, \dots, b_k\}$  is cyclic. As the valence of  $v$  in  $T$  is less than  $k$ , by possibly relabeling, we can assume that the 4–cycle  $\{a_1, b_1, a_2, b_k\}$  does not correspond

to an edge of  $T$ . We define one component of  $\Lambda_1$  to be the edge  $(a_1, a_2)$ , and the other component of  $\Lambda_1$  to consist of the edges  $(b_1, b_2), (b_2, b_3), \dots, (b_{k-1}, b_k)$ . By the observation above and our choice of labeling, any 4-cycle  $C$  corresponding to an edge of  $T$  incident to  $v$  is of the form  $\{a_1, b_i, a_2, b_{i+1}\}$  for some  $1 \leq i \leq k-1$ . Thus condition (1) follows. Condition (2) is readily verified.

Suppose now that we have defined the graph  $\Lambda_{n-1}$  corresponding to the tree  $T_{n-1}$  satisfying conditions (1) and (2). We now define  $\Lambda_n$ .

Let  $u$  be the unique vertex in  $T_n \setminus T_{n-1}$ , and let  $u'$  be the unique vertex of  $T_{n-1}$  that is adjacent to  $u$ . Let  $\Sigma = \Sigma_k(a, b) = \{a_1, a_2\} \star \{b_1, \dots, b_k\}$  and  $\Sigma' = \Sigma_l(c, d) = \{c_1, c_2\} \star \{d_1, \dots, d_l\}$  be the suspension graphs corresponding to  $u$  and  $u'$  respectively. Furthermore, suppose these labelings are cyclic. It follows that  $E = \{a_1, c_1, a_2, c_2\}$  is the 4-cycle corresponding to the edge in  $T$  between  $u$  and  $u'$ . By possibly relabeling, we can assume that  $c_1 = b_1, c_2 = b_k, a_1 = d_1$  and  $a_2 = d_l$ . As  $\Lambda_{n-1}$  satisfies (1) above,  $(a_1, a_2)$  and  $(c_1, c_2)$  are edges of  $\Lambda_{n-1}$ .

As the valence of  $u$  is less than  $k$ , there exist some  $1 \leq j < k$  such that the 4-cycle  $\{b_j, a_1, b_{j+1}, a_2\}$  does not correspond to an edge of  $T$ . We define  $\Lambda_n \subset \Gamma_n^c$  to contain every edge of  $\Lambda_{n-1} \subset \Gamma_{n-1}^c \subset \Gamma_n^c$  and additionally the edges

$$(b_1, b_2), (b_2, b_3), \dots, (b_{j-1}, b_j), (b_{j+1}, b_{j+2}), \dots, (b_{k-1}, b_k).$$

This corresponds to adding one or two line segments each to a distinct vertex of  $\Lambda_{n-1}$ . As  $\Lambda_{n-1}$  contains two components (by (2)) and does not contain any cycles (by  $\mathcal{R}_1$ ), it follows that  $\Lambda_n$  contains two components and satisfies  $\mathcal{R}_1$  as well. Furthermore, (1) and condition  $\mathcal{F}_1$  (for  $\Theta_n$ ) follow from directly from our choices. Condition  $\mathcal{F}_2$  then follows from [Remark 4.3](#).

We now check  $\mathcal{R}_2$ . Let  $x, y \in \Lambda_n$  be vertices contained in the same component of  $\Lambda_n$ . If  $x$  and  $y$  are both contained in  $\Lambda_{n-1}$ , then the claim follows as  $\Lambda_{n-1}$  satisfies  $\mathcal{R}_2$  and no new edges are added between vertices of  $\Gamma_{n-1}$  in forming  $\Gamma_n$ . If  $x$  and  $y$  are both contained in  $\Sigma$ , then by construction, they must lie in the same factor of the join  $\Sigma$  and there is no edge between them. The only case left to check is that  $x$  and  $y$  lie in different components of  $\mathbb{S}^2 \setminus E$ . However, in this case there is no edge between  $x$  and  $y$  as  $E$  separates  $x$  from  $y$  in the planar embedding.

We now check that  $\mathcal{R}_3$  holds. Let  $C$  be a 2-component square in  $\Lambda_n$ . As  $E$  separates every vertex of  $\Sigma \setminus E$  from every vertex in  $(\Gamma_{n-1} \setminus E) \subset \Gamma_n$ , it follows that either  $C$  lies in  $\Gamma_{n-1} \subset \Gamma_n$  or  $C$  lies in  $\Sigma$ . In the first case the claim follows as  $\Theta_{n-1}$  satisfies  $\mathcal{R}_3$  (and noting that the convex hull of  $C$  in  $\Lambda_n$  lies in  $\Lambda_{n-1}$ ). In the latter case, the claim is easily verified.

We now check  $\mathcal{R}_4$ . Let  $P$  be a 2-component cycle in  $\Gamma_n$ . If  $P$  lies entirely in  $\Gamma_{n-1}$  then every edge of  $P$  satisfies condition  $\mathcal{R}_4$  as  $\Theta_{n-1}$  satisfies  $\mathcal{R}_4$ . If  $P$  lies entirely in  $\Sigma$ , then  $\mathcal{R}_4$  is easily verified. Thus, we may assume that  $P$  decomposes into two subpaths  $P_1$  and  $P_2$  such that  $P_1 \subset \Gamma_{n-1}$  and  $P_2 \subset \Sigma \setminus E$ . As  $P$  does not repeat vertices, it follows that  $P_2$  consists of just two edges  $(a_1, b_q)$  and  $(a_2, b_q)$  for some  $2 \leq q \leq k$ . As the valence of  $u'$  is less than  $l$ , there exists some  $1 \leq q' < l$  and corresponding 4-cycle



$\{c_1, d_{q'}, c_2, d_{q'+1}\}$  such that every vertex of  $\Gamma_n$  is contained in a common component of  $\mathbb{S}^2 \setminus C$ . From this, we see that  $a_1$  and  $a_2$  are in different components of  $\Gamma_{n-1} \setminus \{c_1, c_2\}$ . Thus,  $P_1$  must either contain  $c_1$  or  $c_2$ . Suppose that  $P_1$  contains  $c_1$  (the other case is similar). The path  $P_1$  does not contain both the edge  $(a_1, c_1)$  and the edge  $(a_2, c_1)$ , for if it did, then  $P$  would either be the equal 4-cycle  $\{a_1, c_1, a_2, b_q\}$  or contain it as a subcycle. In the former case  $P \subset \Sigma$ , a case we have already ruled out, and in the latter case,  $P$  necessarily repeats a vertex (which is not allowed). We now define a cycle  $P'$  depending on which edges  $P_1$  contains. We set  $P' = (P_1 \setminus (a_1, c_1)) \cup (a_2, c_1)$  if  $(a_1, c_1) \subset P_1$ ,  $P' = (P_1 \setminus (a_2, c_1)) \cup (a_1, c_1)$  if  $(a_2, c_1) \subset P_1$ , and  $P' = P_1 \cup (a_1, c_1) \cup (a_2, c_1)$  if  $P_1$  does not contain either of  $(a_1, c_1)$  and  $(a_2, c_1)$ . In each case, it follows that  $P'$  is a cycle in  $\Gamma_{n-1}$  containing every edge of  $P_1$ , except possibly  $(a_1, c_1)$  and  $(a_2, c_1)$ . Additionally, every vertex of  $P'$  is a vertex of  $P$ , so the  $\Lambda$ -convex hull of  $P'$  is contained in the  $\Lambda$ -convex hull of  $P$ . From this and as  $\Gamma_{n-1}$  satisfies  $\mathcal{R}_4$ , it follows that every edge of  $P$  that is contained in  $P_1$  satisfies  $\mathcal{R}_4$  as well. Finally, every edge of  $P \setminus P_1$  can be seen to satisfy  $\mathcal{R}_4$  by using the 4-cycle  $\{a_1, b_q, a_2, c_1\}$ .  $\square$

## 6 Generalized reflection subgroups of RAAGs

Let  $A_\Gamma$  be a RAAG. A *generalized RAAG reflection* is a conjugate of an element of  $V(\Gamma)$ , ie  $ws w^{-1}$  for some  $s \in V(\Gamma) \cup V(\Gamma)^{-1}$  and  $w$  a word in  $A_\Gamma$ . Let  $\mathcal{T}$  be a set of reduced generalized RAAG reflections. We say that  $\mathcal{T}$  is *trimmed* if  $\mathcal{T} \cap \mathcal{T}^{-1} = \emptyset$ , and if given any two distinct generalized RAAG reflections  $ws w^{-1}$  and  $w's'w'^{-1}$  in  $\mathcal{T}$ , no expression for  $w'$  has prefix  $ws^{-1}$  or prefix  $ws$ . The following lemma follows from a straightforward adaptation of the proof of [Dani and Levcovitz 2021, Lemma 10.1] to the setting of RAAGs.

**Lemma 6.1** *Let  $\mathcal{T}$  be a set of generalized RAAG reflections in the RAAG  $A_\Gamma$ , and let  $G$  be the subgroup generated by  $\mathcal{T}$ . Then  $G$  is generated by a trimmed set of generalized RAAG reflections which can be algorithmically obtained from  $\mathcal{T}$ .*

In this section, we give a new proof of a result of Dyer:

**Theorem 6.2** [Dyer 1990] *Let  $\mathcal{T}$  be a finite set of generalized RAAG reflections in  $A_\Gamma$ . Then the subgroup  $G < A_\Gamma$  generated by  $\mathcal{T}$  is a RAAG. Moreover, if  $\mathcal{T}$  is trimmed then  $(G, \mathcal{T})$  is a RAAG system.*

We will use the characterization of RAAGs in Theorem 2.2 to show that  $G$  is a RAAG. We first prove a series of lemmas about disk diagrams of a special type, namely, ones whose boundary labels are words over a trimmed set of generalized RAAG reflections.

The setup for these lemmas is as follows and will be fixed for the rest of this section. We fix a trimmed set  $\mathcal{T}$  of reduced generalized RAAG reflections in  $A_\Gamma$ . Let  $z = r_1 \cdots r_n$  be an expression for the identity element where  $r_i = w_i s_i w_i^{-1} \in \mathcal{T}$  for each  $1 \leq i \leq n$ . Let  $D$  be a disk diagram whose boundary  $\partial D$  is labeled by  $z$ . For  $1 \leq i \leq n$ , let  $p_{r_i}$  be the subpath of  $\partial D$  which is labeled by  $r_i$ . Furthermore let  $p_{w_i}$  and



$p_{w_i^{-1}}$  denote the subpaths of  $\partial D$  labeled  $w_i$  and  $w_i^{-1}$  respectively, and let  $e_i$  denote the edge labeled  $s_i$ . Let  $H_i$  be the hyperplane dual to  $e_i$ , and let  $\mathcal{H} = \{H_i\}_{i=1}^n$  be the collection of all such hyperplanes. Note that as  $r_i$  is a reduced word, no hyperplane is dual to two edges of  $p_{r_i}$  for any  $i$ .

In all of the following lemmas, arithmetic is taken modulo  $n$ .

**Lemma 6.3** *For each  $1 \leq i \leq n$ , the hyperplane  $H_i$  does not intersect a hyperplane dual to  $p_{w_i}$  or a hyperplane dual to  $p_{w_i^{-1}}$*

**Proof** Suppose  $H_i$  intersects a hyperplane  $K$  that is dual to an edge  $f$  of  $p_{w_i}$ . Without loss of generality, we may assume that  $f$  is the edge closest to  $e_i$  out of all possible choices for  $K$ . As no hyperplane is dual to two edges of  $p_{r_i}$ , it follows that every hyperplane dual to an edge of  $p_{w_i}$  which lies between  $e_i$  and  $f$  must intersect  $K$ . Thus,  $w_i$  has suffix the word  $t_1 \cdots t_m$ , where  $t_1$  is the label of  $K$  and  $t_1$  commutes with  $s_i$ , as well as with  $t_j$  for  $2 \leq j \leq m$ . This readily implies that  $r_i$  is not reduced, for in  $r_i = w_i s_i w_i^{-1}$ , an occurrence of the RAAG generator  $t_1$  in  $w_i$  can be canceled with an occurrence of  $t_1^{-1}$  in  $w_i^{-1}$ . However, this is a contradiction as  $r_i$  is reduced by assumption. The argument for hyperplanes dual to  $p_{w_i^{-1}}$  is analogous. □

**Lemma 6.4** *For each  $1 \leq i \leq n$ , the hyperplane  $H_i$  is not dual to  $p_{w_{i+1}}$ ,  $p_{w_{i+1}^{-1}}$ ,  $p_{w_{i-1}^{-1}}$  or  $p_{w_{i-1}}$ .*

**Proof** For a contradiction, suppose  $H_i$  is dual to an edge  $f$  of  $p_{w_{i+1}}$ . By Lemma 6.3, every hyperplane dual to an edge of  $p_{w_i^{-1}}$  must also be dual to  $p_{w_{i+1}}$ . Write  $s_i w_i^{-1} = t_1 \cdots t_m$  and  $w_{i+1} = k_1 \cdots k_l$  where  $t_j \in V(\Gamma)$  for  $1 \leq j \leq m$  and  $k_j \in V(\Gamma)$  for  $1 \leq j \leq l$ . The structure of the hyperplanes in  $D$  implies that  $w_{i+1}$  has an expression which begins with  $t_m^{-1} \cdots t_1^{-1} = w_i s_i^{-1}$ . This is a contradiction as  $\mathcal{T}$  is trimmed. A similar argument shows that  $H_i$  is not dual to  $p_{w_{i-1}^{-1}}$ .

Suppose now that  $H_i$  is dual to  $p_{w_{i-1}}$ . By Lemma 6.3, it follows that  $H_{i-1}$  is dual to  $p_{w_i}$ . However, this is not possible by the same argument as above. Similarly,  $H_i$  cannot be dual to  $p_{w_{i+1}^{-1}}$ . □

The proof of the following lemma is similar to that of the previous one.

**Lemma 6.5** *If  $H_i = H_{i+1}$  for some  $1 \leq i \leq n$  then  $r_i \simeq r_{i+1}^{-1}$ .* □

**Lemma 6.6** *If  $H_i$  intersects  $H_{i+1}$ , then  $r_i$  and  $r_{i+1}$  commute. Furthermore, there is a disk diagram  $D'$  with boundary label  $r_1 \cdots r_{i-1} r_{i+1} r_i r_{i+2} \cdots r_n$ , such that the natural bijection, from  $e_i$ ,  $e_{i+1}$  and the edges traversed by the subpath of the boundary path of  $D$  labeled by  $r_{i+2} \cdots r_n r_1 \cdots r_{i-1}$  to the edges traversed by the corresponding subpaths of the boundary path of  $D'$  with the same labels, preserves boundary combinatorics.*

**Proof** Suppose  $H_i$  intersects  $H_{i+1}$ . By Lemma 6.3, every hyperplane dual to  $p_{w_i^{-1}}$  is either dual to  $p_{w_{i+1}}$  or intersects  $H_{i+1}$ . Similarly, every hyperplane dual to  $p_{w_{i+1}}$  is either dual to  $p_{w_i^{-1}}$  or intersects  $H_i$ . It then readily follows that  $w_i$  has a reduced expression  $ba_1$  and  $w_{i+1}$  has a reduced

expression  $ba_2$ , where  $a_1, a_2$  and  $b$  are words, such that the generators in the word  $a_1s_i$  are all distinct from and commute with the generators in the word  $a_2s_{i+1}$ . Consequently,  $r_i$  commutes with  $r_{i+1}$ .

We now construct the disk diagram  $D'$ . By Tits' solution to the word problem, the expression  $ba_1$  (resp.  $ba_2$ ) can be obtained from  $w_i$  (resp.  $w_{i+1}$ ) by sequentially permuting adjacent letters. Thus, by repeatedly applying Lemma 2.11(1), we obtain a disk diagram with boundary label

$$r_1 \cdots r_{i-1}(ba_1s_ia_1^{-1}b^{-1})(ba_2s_{i+1}a_2^{-1}b^{-1})r_{i+2} \cdots r_n.$$

By repeatedly applying Lemma 2.11(2), we can "cancel"  $b^{-1}b$  and obtain a disk diagram with boundary label

$$r_1 \cdots r_{i-1}(ba_1s_ia_1^{-1})(a_2s_{i+1}a_2^{-1}b^{-1})r_{i+2} \cdots r_n.$$

Then, by repeatedly applying Lemma 2.11(1), we obtain a disk diagram with label

$$r_1 \cdots r_{i-1}(ba_2s_{i+1}a_2^{-1})(a_1s_ia_1^{-1}b^{-1})r_{i+2} \cdots r_n.$$

By Lemma 2.11(3), we obtain a disk diagram with boundary label

$$r_1 \cdots r_{i-1}(ba_2s_{i+1}a_2^{-1}b^{-1})(ba_1s_ia_1^{-1}b^{-1})r_{i+2} \cdots r_n.$$

Finally, by repeatedly applying Lemma 2.11(1), we obtain a disk diagram  $D'$  with boundary label

$$r_1 \cdots r_{i-1}r_{i+1}r_i r_{i+2} \cdots r_n.$$

Note that in each of these steps, the desired boundary combinatorics are preserved. □

**Lemma 6.7** *For every  $1 \leq i \leq n$ , there exists some  $j \neq i$  such that  $H_i = H_j$ .*

**Proof** Suppose we have a disk diagram with boundary label  $z = r_1 \cdots r_n$  such that, for some  $1 \leq i \leq n$ , the hyperplane  $H_i$  is dual to an edge  $f$  of  $\partial D$  where  $f \neq e_j$  for all  $1 \leq j \leq n$ . We call any disk diagram which has such an  $H_i$  a *pathological diagram with pathology caused by  $H_i$* . Given such a diagram, we define  $p$  to be a path along  $\partial D$  between  $e_i$  and  $f$ , which does not include  $e_i$  and  $f$ . We also let  $\mathcal{H}$  denote the set of  $H_j$  such that  $e_j$  is contained in  $p$ .

Given a pathological disk diagram  $D$  we may choose a hyperplane  $H_i$  causing the pathology together with a path  $p$  such that the set  $\mathcal{H}$  is minimal among all possible choices of  $H_i$  and  $p$ . After such a choice, we call  $|\mathcal{H}'|$  the complexity of  $D$ . We will prove that pathological diagrams are not possible by induction on the complexity  $c$  of such a diagram. The base case, when  $c = 0$ , already follows from Lemma 6.4.

Now suppose we are given a pathological disk diagram  $D$  with pathology caused by  $H_i$  such that its complexity is  $c = |\mathcal{H}'| > 0$ , and suppose by induction there do not exist pathological disk diagrams of complexity smaller than  $c$ .

The edge  $f \neq e_i$  of  $\partial D$  that is dual to  $H_i$  lies in a path  $p_{r_{i'}}$  in  $\partial D$  labeled by  $w_{i'}s_{i'}w_{i'}^{-1}$  for some  $1 \leq i' \leq n$  where  $i \neq i'$ . Let  $Q$  denote the hyperplane  $H_{i'}$ . Note that  $Q$  may or may not be in  $\mathcal{H}'$ . We prove our claim by considering two cases:

**Case 1** Every hyperplane in  $\mathcal{H}'$  intersects  $H_i$ .

We first observe that  $\mathcal{H}'$  is nonempty (since the complexity of  $D$  is positive) and does not consist of  $Q$  alone (by Lemma 6.4). Therefore, we may choose  $K \in \mathcal{H}' \setminus Q$  such that no hyperplane in  $\mathcal{H}' \setminus Q$  intersects  $H_i$  between  $K \cap H_i$  and  $H_i \cap e_i$ . Let  $1 \leq l \leq n$  be such that  $K$  is dual to  $e_l \subset p_{r_l} \subset p$ . Then for each  $j$  with  $i < j < l$ , the hyperplane  $H_j$  intersects  $K = H_l$ . Thus, by repeatedly applying Lemma 6.6, we can produce a new disk diagram with boundary label  $r_1 \cdots r_l r_i \cdots r_{l-1} r_{l+1} \cdots r_n$ . Furthermore, this new disk diagram is still pathological and has complexity smaller than  $D$ . However, this is not possible by our induction hypothesis.

**Case 2** Some hyperplane  $K \in \mathcal{H}'$  does not intersect  $H_i$ .

We can choose such a hyperplane  $K$  to be innermost, ie choose  $K \in \mathcal{H}'$  such that  $K$  does not intersect  $H_i$  and such that any hyperplane of  $\mathcal{H}'$  dual to the subpath of  $p$  between the edges dual to  $K$  intersects  $K$ . Since  $H_i$  and  $p$  were chosen to attain the complexity of  $D$ , it follows that  $K$  does not cause a pathology, and is dual to distinct edges  $e_l$  and  $e_{l'}$  in  $p$ , where  $1 \leq l, l' \leq n$ . By relabeling the  $r_j$ 's if necessary, we may assume that  $l < l'$ , and that the subpath of  $\partial D$  from  $e_l$  to  $e_{l'}$  is contained in  $p$ . By repeatedly applying Lemma 6.6, we can produce a new pathological disk diagram  $D'$  with label  $r_1 \cdots r_{l-1} r_{l+1} \cdots r_{l'-1} r_l r_{l'} \cdots r_n$  and where some hyperplane, which we still denote by  $K$ , is dual to both the edge labeled by  $e_l$  and the one labeled by  $e_{l'}$ . By Lemma 6.5,  $r_l \simeq r_{l'}^{-1}$ . Furthermore, by repeatedly applying Lemma 2.11(1) if necessary, we may assume that  $r_l = r_{l'}^{-1}$  is the label of  $\partial D'$ .

We now produce a new disk diagram  $D''$  by identifying the consecutive paths in  $\partial D'$  labeled by  $r_l$  and  $r_{l'}$ , ie we fold these two paths together. If  $K \neq Q$ , then we have produced a new pathological disk diagram with complexity  $c - 2$ , contradicting the induction hypothesis. On the other hand, if  $K = Q$ , note that the image of  $H_i$  in  $D''$  must intersect the path labeled by  $r_i \cdots r_{l-1} r_{l+1} \cdots r_{l'-1}$  in  $\partial D''$ . Moreover we claim that it cannot be dual to an edge labeled by  $e_j$  for  $i < j \leq l' - 1$ . Suppose it is dual to an edge labeled  $e_j$ . It follows that the hyperplane  $H_j$  in  $D$  is dual to an edge  $f'$  in  $p$ , such that  $f' \neq e_k$  for any  $k$ , and such that the images of  $f$  and  $f'$  are identified in  $D''$ . This is a contradiction, as it implies that  $H_j$  causes a pathology of lower complexity than  $H_i$ . Thus, the image of  $H_i$  in  $D''$  causes a pathology of complexity at most  $c - 2$ , which is again a contradiction. □

**Proof of Theorem 6.2** As  $G$  can be generated by a trimmed set of generalized RAAG reflections (by Lemma 6.1), we assume without loss of generality that  $\mathcal{T}$  is trimmed. We will show that  $(G, \mathcal{T})$  is a RAAG system by applying Theorem 2.2. Note that  $\mathcal{T} \cap \mathcal{T}^{-1} = \emptyset$  as  $\mathcal{T}$  is trimmed. We check each condition of that theorem, by proving the corresponding two claims:

- (i) Every  $r \in \mathcal{T}$  has infinite order.

By definition,  $r$  is equal to a reduced word  $ws w^{-1}$  with  $s \in V(\Gamma) \cup V(\Gamma)^{-1}$  and  $w$  a word in  $W_\Gamma$ . It follows that  $ws^n w^{-1}$  is an expression for  $r^n$ . Moreover, as  $r$  is reduced, it readily follows from Theorem 2.4 that  $ws^n w^{-1}$  is reduced as well. Hence,  $r$  has infinite order.

- (ii) Given any word  $w = a_1 \cdots a_m$ , with  $a_i \in \mathcal{T}$ , either  $w$  is reduced over  $\mathcal{T}$  or there is an expression for  $w$  of the form  $a_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_m$ .

Suppose  $w = a_1 \cdots a_m$  is not reduced over  $\mathcal{T}$ . Let  $w' = b_1 \cdots b_k$ , with  $b_i \in \mathcal{T}$  and  $k < m$ , be an expression for  $w$  which is reduced over  $\mathcal{T}$ . Form a disk diagram  $D$  with boundary label  $ww'^{-1}$ .

We relabel the generalized reflections in the word  $ww'^{-1}$  by setting  $r_i = a_i$  for  $1 \leq i \leq m$ , and  $r_{m+i} = b_{k-i+1}^{-1}$  (the  $i^{\text{th}}$  generalized RAAG reflection in  $w'^{-1}$ ) for  $1 \leq i \leq k$ . By Lemma 6.7, every  $H \in \mathcal{H}$  is only dual to edges of  $\partial D$  labeled by  $s_i$  for some  $i$ , where  $r_i = w_i s_i w_i^{-1}$ . As  $m > k$ , there exists some hyperplane  $H \in \mathcal{H}$  that is dual to two edges of the subpath  $p$  of  $\partial D$  labeled by  $w$ . Furthermore, we may choose an innermost such  $H \in \mathcal{H}$ , in the sense that every hyperplane in  $\mathcal{H} \setminus H$  intersects  $p$  at most once.

Let  $e_l$  and  $e_{l'}$  be the edges dual to  $H$  where  $l < l' \leq m$ . By repeatedly applying Lemma 6.6, we produce a disk diagram whose boundary label is

$$r_1 \cdots \hat{r}_l \cdots r_{l'-1} r_l r_{l'} \cdots r_n,$$

such that a hyperplane of  $\mathcal{H}$  is still dual to the images of the edges  $e_l$  and  $e_{l'}$  under the natural map between the boundaries of the disk diagrams. By Lemma 6.5,  $r_l = r_{l'}^{-1}$ . Thus,  $r_1 \cdots \hat{r}_l \cdots \hat{r}_{l'} \cdots r_n$  is an expression for  $ww'^{-1}$ . Consequently,  $r_1 \cdots \hat{r}_l \cdots \hat{r}_{l'} \cdots r_m = a_1 \cdots \hat{a}_l \cdots \hat{a}_{l'} \cdots a_m$  is an expression for  $w$ .  $\square$

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