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$\mathbb{A}^1$-homotopy invariance of algebraic K-theory with coefficients and du Val singularities

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C. Weibel, and Thomason and Trobaugh, proved (under some assumptions) that algebraic K-theory with coefficients is $\mathbb{A}^1$-homotopy invariant. We generalize this result from schemes to the broad setting of dg categories. Along the way, we extend the Bass–Quillen fundamental theorem as well as Stienstra's foundational work on module structures over the big Witt ring to the setting of dg categories. Among other cases, the above $\mathbb{A}^1$-homotopy invariance result can now be applied to sheaves of (not necessarily commutative) dg algebras over stacks. As an application, we compute the algebraic K-theory with coefficients of dg cluster categories using solely the kernel and cokernel of the Coxeter matrix. This leads to a complete computation of the algebraic K-theory with coefficients of the du Val singularities parametrized by the simply laced Dynkin diagrams. As a byproduct, we obtain vanishing and divisibility properties of algebraic K-theory (without coefficients).

1. Introduction and statement of results

Let k be a base commutative ring, X a quasicompact, quasiseparated k-scheme, and l^ν a prime power. As proved by Weibel [1982, page 391; 1981, Theorem 5.2] and by Thomason and Trobaugh [1990, Theorems 9.5–9.6], we have the following result:

Theorem 1.1. (i) *When $1/l \in k$, the projection morphism $X[t] \to X$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(X; \mathbb{Z}/l^\nu) \to \mathbb{K}(X[t]; \mathbb{Z}/l^\nu)$.*

(ii) *When l is nilpotent in k, the projection morphism $X[t] \to X$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(X) \otimes \mathbb{Z}[1/l] \to \mathbb{K}(X[t]) \otimes \mathbb{Z}[1/l]$.*

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The proof of Theorem 1.1 is quite involved! The affine case, established by Weibel, makes use of a convergent right half-plane spectral sequence, of a universal coefficient sequence, of the Bass–Quillen fundamental theorem (see [Grayson 1976, page 236]), and more importantly of Stienstra's foundational work [1982, §8] on module structures over the big Witt ring. The extension to quasicompact, quasiseparated schemes, later established by Thomason and Trobaugh [1990, §9.1], is based on a powerful method known as "reduction to the affine case".

The first goal of this article is to generalize Theorem 1.1 from schemes to the broad setting of dg categories. Consult Sections 2–3 for applications and computations.

Statement of results. A *differential graded* (*dg*) *category* $\mathcal{A}$, over the base commutative ring k, is a category enriched over complexes of k-modules; see Section 4. Every (dg) k-algebra A gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes, since the category of perfect complexes $\mathrm{perf}(X)$ of every quasicompact, quasiseparated k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; see [Keller 2006, §4.4]. Given a dg category $\mathcal{A}$, let us write $\mathcal{A}[t]$ for the tensor product $\mathcal{A} \otimes k[t]$. Our first main result is the following:

Theorem 1.2. (i) *When $1/l \in k$, the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^{\nu}) \to \mathbb{K}(\mathcal{A}[t]; \mathbb{Z}/l^{\nu})$.*

(ii) *When l is nilpotent in k, the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(\mathcal{A}) \otimes \mathbb{Z}[1/l] \to \mathbb{K}(\mathcal{A}[t]) \otimes \mathbb{Z}[1/l]$.*

For the proof of Theorem 1.2, we adapt the Bass–Quillen fundamental theorem, as well as Stienstra's foundational work on module structures over the big Witt ring, to the broad setting of dg categories; see Theorems 8.4 and 9.1, respectively. These results are of independent interest. Except in Theorem 9.1, we work more generally with a localizing invariant; see Definition 5.1.

2. Applications and computations

The second goal of this article is to explain how Theorem 1.2 leads naturally to several applications and computations.

Sheaves of dg algebras. Let X be a quasicompact, quasiseparated k-scheme and $\mathcal{S}$ a sheaf of (not necessarily commutative) dg $\mathcal{O}_X$-algebras. In addition to $\mathrm{perf}_{\mathrm{dg}}(X)$, we can consider the dg category $\mathrm{perf}_{\mathrm{dg}}(\mathcal{S})$ of perfect complexes of $\mathcal{S}$-modules; see [Tabuada and Van den Bergh 2015, §6]. By applying Theorem 1.2 to the dg category $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(\mathcal{S})$, we obtain the following generalization of Theorem 1.1:

Theorem 2.1. (i) *When $1/l \in k$, the projection morphism $\mathcal{S}[t] \to \mathcal{S}$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(\mathcal{S}; \mathbb{Z}/l^{\nu}) \to \mathbb{K}(\mathcal{S}[t]; \mathbb{Z}/l^{\nu})$.*

(ii) *When l is nilpotent in k, the projection morphism $\mathcal{S}[t] \to \mathcal{S}$ gives rise to an homotopy equivalence of spectra $\mathbb{K}(\mathcal{S}) \otimes \mathbb{Z}[1/l] \to \mathbb{K}(\mathcal{S}[t]) \otimes \mathbb{Z}[1/l]$.*

Remark 2.2 (orbifolds and stacks). Given an orbifold, or more generally a stack $\mathcal{X}$, we can also consider the associated dg category $\mathrm{perf}_{\mathrm{dg}}(\mathcal{X})$ of perfect complexes. Therefore, Theorem 2.1 holds more generally for every sheaf $\mathcal{S}$ of dg $\mathcal{O}_{\mathcal{X}}$-algebras.

DG orbit categories. Given a dg category $\mathcal{A}$ and a dg functor $F : \mathcal{A} \to \mathcal{A}$ which induces an equivalence of categories $\mathrm{H}^0(F) : \mathrm{H}^0(\mathcal{A}) \xrightarrow{\sim} \mathrm{H}^0(\mathcal{A})$, recall from [Keller 2005, §5.1] the construction of the associated dg orbit category $\mathcal{A}/F^{\mathbb{Z}}$. Thanks to Theorem 1.2, all the results established in [Tabuada 2015a] can now be applied to algebraic K-theory with coefficients. For example, Theorem 1.5 of [Tabuada 2015a] gives rise to the result:

Theorem 2.3. *When $1/l \in k$, we have a distinguished triangle of spectra*:

$$\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu) \xrightarrow{\mathbb{K}(F;\mathbb{Z}/l^\nu)-\mathrm{Id}} \mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu) \longrightarrow \mathbb{K}(\mathcal{A}/F^{\mathbb{Z}}; \mathbb{Z}/l^\nu) \longrightarrow \Sigma\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu).$$

When l is nilpotent in k, the same holds with $\mathbb{K}(-; \mathbb{Z}/l^\nu)$ replaced by $\mathbb{K}(-) \otimes \mathbb{Z}[1/l]$.

Remark 2.4 (fundamental isomorphism). When F is the identity dg functor, the dg orbit category $\mathcal{A}/F^{\mathbb{Z}}$ reduces to $\mathcal{A}[t, 1/t]$ and the above distinguished triangle splits. Thus, we obtain a fundamental isomorphism between $\mathbb{K}(\mathcal{A}[t, 1/t]; \mathbb{Z}/l^\nu)$ and the direct sum $\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu) \oplus \Sigma\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu)$. When l is nilpotent in k, the same holds with $\mathbb{K}(-; \mathbb{Z}/l^\nu)$ replaced by $\mathbb{K}(-) \otimes \mathbb{Z}[1/l]$.

DG cluster categories. Let k be an algebraically closed field, Q a finite quiver without oriented cycles, kQ the path k-algebra of Q, $\mathcal{D}^b(kQ)$ the bounded derived category of finitely generated right kQ-modules, and $\mathcal{D}^b_{\mathrm{dg}}(kQ)$ the canonical dg enhancement of $\mathcal{D}^b(kQ)$. Consider the dg functors

$$\tau^{-1}\Sigma^m : \mathcal{D}^b_{\mathrm{dg}}(kQ) \longrightarrow \mathcal{D}^b_{\mathrm{dg}}(kQ), \quad m \geq 0,$$

where τ is the Auslander–Reiten translation. Following Keller [2005, §7.2], the *dg (m)-cluster category $\mathcal{C}_Q^{(m)}$ of Q* is defined as the dg orbit category

$$\mathcal{D}^b_{\mathrm{dg}}(kQ)/(\tau^{-1}\Sigma^m)^{\mathbb{Z}}.$$

In the same vein, the *(m)-cluster category of Q* is defined as $\mathrm{H}^0(\mathcal{C}_Q^{(m)})$. These (dg) categories play, nowadays, a key role in the representation theory of finite-dimensional algebras; see Reiten's ICM address [2010]. As proved by Keller and Reiten [2008, §2], the (m)-cluster categories (with $m \geq 1$) can be conceptually characterized as those $(m+1)$-Calabi–Yau triangulated categories containing a cluster-tilting object whose endomorphism algebra has a quiver without oriented cycles.

As explained in [Tabuada 2015a, Corollary 2.11], in the particular case of dg cluster categories, Theorem 2.3 reduces to the following one:

Theorem 2.5. *When $l \neq \mathrm{char}(k)$, we have a distinguished triangle of spectra*

$$\bigoplus_{r=1}^{v} \mathbb{K}(k;\mathbb{Z}/l^\nu) \xrightarrow{(-1)^m \Phi_Q - \mathrm{Id}} \bigoplus_{r=1}^{v} \mathbb{K}(k;\mathbb{Z}/l^\nu) \to \mathbb{K}(\mathcal{C}_Q^{(m)};\mathbb{Z}/l^\nu) \to \bigoplus_{r=1}^{v} \Sigma\mathbb{K}(k;\mathbb{Z}/l^\nu),$$

where v stands for the number of vertices of Q and Φ_Q for the Coxeter matrix of Q. When $l = \mathrm{char}(k)$, the same holds with $\mathbb{K}(-;\mathbb{Z}/l^\nu)$ replaced by $\mathbb{K}(-)\otimes\mathbb{Z}[1/l]$.

As proved by Suslin [1984, Corollary 3.13], we have $\mathbb{K}_i(k;\mathbb{Z}/l^\nu) \simeq \mathbb{Z}/l^\nu$ when $i \geq 0$ is even and $\mathbb{K}_i(k;\mathbb{Z}/l^\nu) = 0$ otherwise. Consequently, making use of the long exact sequence of algebraic K-theory groups with coefficients associated to the above distinguished triangle of spectra, we obtain the following result:

Corollary 2.6. *Consider the (matrix) homomorphism*

$$(-1)^m \Phi_Q - \mathrm{Id} : \bigoplus_{r=1}^{v} \mathbb{Z}/l^\nu \longrightarrow \bigoplus_{r=1}^{v} \mathbb{Z}/l^\nu. \tag{2.7}$$

When $l \neq \mathrm{char}(k)$, we have the following computation:

$$\mathbb{K}_i(\mathcal{C}_Q^{(m)};\mathbb{Z}/l^\nu) \simeq \begin{cases} \text{cokernel (2.7)} & \text{if } i \geq 0 \text{ even},\\ \text{kernel (2.7)} & \text{if } i \geq 0 \text{ odd},\\ 0 & \text{if } i < 0.\end{cases}$$

Corollary 2.6 provides a complete computation of the algebraic K-theory with coefficients of all dg orbit categories! Roughly speaking, all the information is encoded in the Coxeter matrix of the quiver. Note also that the kernel and cokernel of (2.7) have the same finite order. In particular, one is trivial if and only if the other one is trivial. Thanks to Corollary 2.6, this implies that the groups $\mathbb{K}_i(\mathcal{C}_Q^{(m)};\mathbb{Z}/l^\nu)$, $i \geq 0$, are either all trivial or all nontrivial.

3. Du Val singularities

The third goal of this article is to explain how Corollary 2.6 provides us a complete computation of the algebraic K-theory with coefficients of the du Val singularities.

Let k be an algebraically closed field of characteristic zero. Recall that the *du Val* singularities[1] [1934a; 1934b; 1934c] are the isolated singularities of the singular affine hypersurfaces $R := k[x, y, z]/(f)$ parametrized by the simply laced Dynkin diagrams:

type	A_n, $n \geq 1$	D_n, $n \geq 4$	E_6	E_7	E_8
f	$x^{n+1} + yz$	$x^{n-1} + xy^2 + z^2$	$x^4 + y^3 + z^2$	$x^3y + y^3 + z^2$	$x^5 + y^3 + z^2$

[1] Also known as *rational double points* or *ADE singularities*.

Let $\underline{\mathrm{MCM}}(R)$ denote the stable category of maximal Cohen–Macaulay R-modules. Thanks to the work of Buchweitz [1986] and Orlov [2004; 2009], this category is also known as the category of singularities $\mathcal{D}^{\mathrm{sing}}(R)$ or equivalently as the category of matrix factorizations $\mathrm{MF}(k[x, y, z], f)$. Roughly speaking, $\underline{\mathrm{MCM}}(R)$ encodes the crucial information concerning the isolated singularity of the singular affine hypersurface R.

Let Q be a *Dynkin quiver*, i.e., a quiver whose underlying graph is a Dynkin diagram of type A, D, or E. As explained by Keller [2005, §7.3], $\underline{\mathrm{MCM}}(R)$ is equivalent to the category of finitely generated projective modules over the preprojective algebra $\Lambda(Q)$ and to the (0)-cluster category of Q. We conclude that the algebraic K-theory of the du Val singularities is given by the algebraic K-theory of the dg (0)-cluster categories $\mathcal{C}^{(0)}_{A_n}$, $\mathcal{C}^{(0)}_{D_n}$, $\mathcal{C}^{(0)}_{E_6}$, $\mathcal{C}^{(0)}_{E_7}$ and $\mathcal{C}^{(0)}_{E_8}$. In these cases, the homomorphisms (2.7) correspond to the following matrices (see [Auslander et al. 1995, pages 289–290]):

$$A_n\colon\ 1 \to 2 \to \cdots \to n-1 \to n \qquad \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ -1 & -1 & \ddots & \ddots & \vdots \\ -1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & 0 & \cdots & 0 & -1 \end{bmatrix}_{n\times n}$$

$$D_n\colon\ \begin{array}{lll} 1 & & \\ & \searrow & \\ & & 3 \to 4 \to \cdots \to n \\ & \nearrow & \\ 2 & & \end{array} \qquad \begin{bmatrix} -2 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -2 & 1 & 0 & \ddots & \ddots & \vdots \\ -1 & -1 & 0 & 1 & \ddots & \ddots & \vdots \\ -1 & -1 & 1 & -1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & -1 & 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n\times n}$$

$$E_6\colon\ \begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & 2 & \to & 4 & \to & 5 & \to & 6 \end{array} \qquad \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ -2 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 1 & -1 & 1 \\ -1 & 0 & -1 & 1 & 0 & -1 \end{bmatrix}$$

$$E_7\colon\ \begin{array}{ccccccccccc} & & & & 3 & & & & & & \\ & & & & \downarrow & & & & & & \\ 1 & \to & 2 & \to & 4 & \to & 5 & \to & 6 & \to & 7 \end{array} \qquad \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$E_8: \quad \begin{array}{c} \qquad\quad 3 \\ \qquad\quad \downarrow \\ 1 \to 2 \to 4 \to 5 \to 6 \to 7 \to 8 \end{array} \qquad \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Thanks to Corollary 2.6, the computation of the algebraic K-theory with coefficients of the du Val singularities reduces then to the computation of the (co)kernels of the above explicit matrix homomorphisms! We now compute the type A_n and leave the remaining cases to the reader.

Theorem 3.1. *Let k be an algebraically closed field of characteristic zero and $n \geq 1$ a positive integer. Under these assumptions and notations, we have*

$$\mathbb{K}_i(\mathcal{C}_{A_n}^{(0)}; \mathbb{Z}/l^\nu) \simeq \begin{cases} \mathbb{Z}/\gcd(n+1, l^\nu) & \text{if } i \geq 0, \\ 0 & \text{if } i < 0. \end{cases}$$

Consequently, the group $\mathbb{K}_i(\mathcal{C}_{A_n}^{(0)}; \mathbb{Z}/l^\nu)$ is nontrivial if and only if $l \mid (n+1)$ and $i \geq 0$.

Intuitively, Theorem 3.1 shows that the algebraic K-theory with $\mathbb{Z}/l^\nu$-coefficients of the isolated singularity of the affine hypersurface $k[x, y, z]/(x^{n+1} + yz)$ measures the l-divisibility of the integer $n + 1$. To the best of the author's knowledge, these computations are new in the literature. They lead to the following vanishing and divisibility properties of algebraic K-theory (without coefficients):

Corollary 3.2. (i) *For every $i \geq 0$, at least one of the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}_{A_n}^{(0)})$ and $\mathbb{K}_{i-1}(\mathcal{C}_{A_n}^{(0)})$ is nontrivial.*

(ii) *For every $l \nmid (n+1)$ the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}_{A_n}^{(0)})$, $i \in \mathbb{Z}$, are uniquely l-divisible, i.e., they are $\mathbb{Z}[1/l]$-modules.*

Roughly speaking, Corollary 3.2 shows that at least half of the groups $\mathbb{K}_i(\mathcal{C}_{A_n}^{(0)})$ are nontrivial and moreover that they are "large" from the divisibility viewpoint.

Proof. Consider the following universal coefficient sequences (see Section 5):

$$0 \to \mathbb{K}_i(\mathcal{C}_{A_n}^{(0)}) \otimes_{\mathbb{Z}} \mathbb{Z}/l \to \mathbb{K}_i(\mathcal{C}_{A_n}^{(0)}; \mathbb{Z}/l) \to \{l\text{-torsion in } \mathbb{K}_{i-1}(\mathcal{C}_{A_n}^{(0)})\} \to 0.$$

Let l be a prime factor of $n + 1$. Thanks to Theorem 3.1, the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}_{A_n}^{(0)}; \mathbb{Z}/l)$, $i \geq 0$, are nontrivial. Therefore, (i) follows from the above short exact sequences. Let l be a prime number which does not divide $n + 1$. Thanks to Theorem 3.1, the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}_{A_n}^{(0)}; \mathbb{Z}/l)$, $i \in \mathbb{Z}$, are trivial. Therefore, (ii) follows also from the above short exact sequences. □

A cyclic quotient singularity. Let the cyclic group $\mathbb{Z}/3$ act on the power series ring $k[\![x, y, z]\!]$ by multiplication with a primitive third root of unit. As proved by Keller and Reiten [2008, §2], the stable category of maximal Cohen–Macaulay modules $\underline{\mathrm{MCM}}(R)$ over the fixed point ring $R := k[\![x, y, z]\!]^{\mathbb{Z}/3}$ is equivalent to the (1)-cluster category of the generalized Kronecker quiver $Q : 1 \substack{\to\\\to\\\to} 2$. In this case the above homomorphism (2.7) is given by the matrix $\left[\begin{smallmatrix} -9 & 3 \\ -3 & 0 \end{smallmatrix}\right]$.

Proposition 3.3. *We have the following computation*:

$$\mathbb{K}_i(\mathcal{C}_Q^{(1)}; \mathbb{Z}/l^\nu) \simeq \begin{cases} \mathbb{Z}/3 \times \mathbb{Z}/3 & \text{if } i \geq 0 \text{ and } l = 3, \\ 0 & \text{otherwise}. \end{cases}$$

To the best of the author's knowledge, the above computation is new in the literature. Similarly to Corollary 3.2, for every $i \geq 0$ at least one of the algebraic K-theory groups $\mathbb{K}_i(\mathcal{C}_Q^{(1)})$ and $\mathbb{K}_{i-1}(\mathcal{C}_Q^{(1)})$ is nontrivial, and, for every prime number $l \neq 3$, the groups $\mathbb{K}_i(\mathcal{C}_Q^{(1)})$, $i \in \mathbb{Z}$, are uniquely l-divisible.

Remark 3.4. After the circulation of this manuscript, Christian Haesemeyer kindly informed the author that some related computations concerning the G-theory of a local ring of finite Cohen–Macaulay type have been performed by Viraj Navkal [2013].

4. Preliminaries

Throughout the article, k will be a base commutative ring. Unless stated differently, all tensor products will be taken over k.

Dg categories. Let $\mathcal{C}(k)$ be the category of cochain complexes of k-modules. A *differential graded* (*dg*) *category* $\mathcal{A}$ is a $\mathcal{C}(k)$-enriched category and a *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a $\mathcal{C}(k)$-enriched functor; consult Keller's ICM survey [2006]. In what follows, $\mathrm{dgcat}(k)$ stands for the category of (small) dg categories and dg functors.

Let $\mathcal{A}$ be a dg category. The category $\mathrm{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and $\mathrm{H}^0(\mathcal{A})(x, y) := H^0\mathcal{A}(x, y)$. The dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects as $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. A *right $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathrm{dg}}(k)$ of cochain complexes of k-modules. Let us write $\mathcal{C}(\mathcal{A})$ for the category of right $\mathcal{A}$-modules. As explained in [Keller 2006, §§3.1–3.2], the category $\mathcal{C}(\mathcal{A})$ carries a projective Quillen model structure in which the weak equivalences and fibrations are defined objectwise. The *derived category* $\mathcal{D}(\mathcal{A})$ *of* $\mathcal{A}$ is the associated homotopy category or, equivalently, the localization of $\mathcal{C}(\mathcal{A})$ with respect to the (objectwise) quasi-isomorphisms. The full triangulated subcategory of compact objects will be denoted by $\mathcal{D}_c(\mathcal{A})$.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence of (triangulated) categories $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$; see [Keller 2006, §4.6]. As proved in [Tabuada 2005, Theorem 5.3], $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let $\mathrm{Hmo}(k)$ be the associated homotopy category.

The tensor product $\mathcal{A}\otimes\mathcal{B}$ of dg categories is defined as follows: the set of objects is the cartesian product and $(\mathcal{A}\otimes\mathcal{B})((x,w),(y,z)) := \mathcal{A}(x,y)\otimes\mathcal{B}(w,z)$. As explained in [Keller 2006, §2.3 and §4.3], this construction gives rise to symmetric monoidal categories $(\mathrm{dgcat}(k), -\otimes-, k)$ and $(\mathrm{Hmo}(k), -\otimes^{\mathbb{L}}-, k)$.

An *$\mathcal{A}$-$\mathcal{B}$-bimodule* is a dg functor $\mathrm{B}:\mathcal{A}\otimes^{\mathbb{L}}\mathcal{B}^{\mathrm{op}}\to\mathcal{C}_{\mathrm{dg}}(k)$ or, equivalently, a right $(\mathcal{A}^{\mathrm{op}}\otimes^{\mathbb{L}}\mathcal{B})$-module. A standard example is the $\mathcal{A}$-$\mathcal{B}$-bimodule

$$ {}_F\mathrm{B}:\mathcal{A}\otimes^{\mathbb{L}}\mathcal{B}^{\mathrm{op}}\to\mathcal{C}_{\mathrm{dg}}(k), \quad (x,w)\mapsto\mathcal{B}(w,F(x)), \tag{4.1} $$

associated to a dg functor $F:\mathcal{A}\to\mathcal{B}$. Finally, let us denote by $\mathrm{rep}(\mathcal{A},\mathcal{B})$ the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes^{\mathbb{L}}\mathcal{B})$ consisting of those $\mathcal{A}$-$\mathcal{B}$-bimodules B such that $\mathrm{B}(x,-)\in\mathcal{D}_c(\mathcal{B})$ for every object $x\in\mathcal{A}$.

Exact categories. Let $\mathcal{E}$ be an exact category in the sense of [Quillen 1973, §2]. The following examples will be used in the sequel:

Example 4.2. Let A be a k-algebra. Recall from [Quillen 1973, §2] that the category $\mathrm{P}(A)$ of finitely generated projective right A-modules carries a canonical exact structure.

(i) Let $\mathrm{End}(A)$ be the category of endomorphisms in $\mathrm{P}(A)$. The objects are the pairs (M,f), with $M\in\mathrm{P}(A)$ and f an endomorphism of M. The morphisms $(M,f)\to(M',f')$ are the A-linear maps $h:M\to M'$ such that $hf=f'h$. Note that $\mathrm{End}(A)$ inherits naturally from $\mathrm{P}(A)$ an exact structure making the forgetful functor $\mathrm{End}(A)\to\mathrm{P}(A)$, $(M,f)\mapsto M$, exact.

(ii) Let $\mathrm{Nil}(A)$ be the category of nilpotent endomorphisms in $\mathrm{P}(A)$. By construction, $\mathrm{Nil}(A)$ is a full exact subcategory of $\mathrm{End}(A)$.

Following [Keller 2006, §4.4], the *bounded derived dg category $\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})$ of $\mathcal{E}$* is defined as Drinfeld's dg quotient $\mathcal{C}^b_{\mathrm{dg}}(\mathcal{E})/Ac^b_{\mathrm{dg}}(\mathcal{E})$ of the dg category of bounded cochain complexes over $\mathcal{E}$ by the full dg subcategory of acyclic complexes.

Notation 4.3. Let $\mathcal{E}$ be an exact category. In order to simplify the exposition, let us write $\mathcal{E}_{\mathbf{dg}}$ instead of $\mathcal{D}^b_{\mathrm{dg}}(\mathcal{E})$. By construction, we have $\mathrm{H}^0(\mathcal{E}_{\mathbf{dg}})\simeq\mathcal{D}^b(\mathcal{E})$. Note that when $\mathcal{E}=\mathrm{P}(A)$ we have a Morita equivalence between $\mathrm{P}(A)_{\mathbf{dg}}$ and A.

Every exact functor $\mathcal{E}\to\mathcal{E}'$ gives rise to a dg functor $\mathcal{E}_{\mathbf{dg}}\to\mathcal{E}'_{\mathbf{dg}}$. In the same vein, every multiexact functor $\mathcal{E}\times\cdots\times\mathcal{E}'\to\mathcal{E}''$ gives rise to a dg functor $\mathcal{E}_{\mathbf{dg}}\otimes^{\mathbb{L}}\cdots\otimes^{\mathbb{L}}\mathcal{E}'_{\mathbf{dg}}\to\mathcal{E}''_{\mathbf{dg}}$.

Algebraic K-theory with coefficients. Let Spt be the homotopy category of spectra and $\mathbb{S}$ the sphere spectrum. Given a small dg category $\mathcal{A}$, its *nonconnective algebraic K-theory spectrum* $\mathbb{K}(\mathcal{A})$ is defined by applying Schlichting's construction [2006, §12.1] to the Frobenius pair associated to the category of those cofibrant right $\mathcal{A}$-modules which become compact in the derived category $\mathcal{D}(\mathcal{A})$. Let us denote by $\mathbb{K} : \mathrm{dgcat}(k) \to \mathrm{Spt}$ the associated functor. Given a prime power l^ν, the algebraic K-theory with $\mathbb{Z}/l^\nu$-coefficients is defined as[2]

$$\mathbb{K}(-; \mathbb{Z}/l^\nu) : \mathrm{dgcat}(k) \to \mathrm{Spt}, \quad \mathcal{A} \mapsto \mathbb{K}(\mathcal{A}) \wedge^{\mathbb{L}} \mathbb{S}/l^\nu, \tag{4.4}$$

where $\mathbb{S}/l^\nu$ stands for the mod l^ν Moore spectrum of $\mathbb{S}$. In the same vein, we have the functor $\mathbb{K}(-) \otimes \mathbb{Z}[1/l] : \mathrm{dgcat}(k) \to \mathrm{Spt}$ defined by the homotopy colimit

$$\mathbb{K}(\mathcal{A}) \otimes \mathbb{Z}[1/l] := \mathrm{hocolim}\big(\mathbb{K}(\mathcal{A}) \xrightarrow{\cdot l} \mathbb{K}(\mathcal{A}) \xrightarrow{\cdot l} \cdots\big).$$

When $\mathcal{A} = \mathrm{perf}_{\mathrm{dg}}(X)$, with X a quasicompact, quasiseparated k-scheme, $\mathbb{K}(\mathcal{A})$ agrees with $\mathbb{K}(X)$; see [Keller 2006, §5.2; Schlichting 2006, §8]. Consequently, $\mathbb{K}(\mathcal{A}; \mathbb{Z}/l^\nu)$ agrees with $\mathbb{K}(X; \mathbb{Z}/l^\nu)$ and $\mathbb{K}(\mathcal{A}) \otimes \mathbb{Z}[1/l]$ agrees with $\mathbb{K}(X) \otimes \mathbb{Z}[1/l]$.

Bass's construction. Let $H : \mathrm{dgcat}(k) \to \mathrm{Ab}$ be a functor with values in the category of abelian groups. Following [Bass 1968, §XII], consider the sequence of functors $N^p H : \mathrm{dgcat}(k) \to \mathrm{Ab}$, $p \geq 0$, defined by $N^0 H(\mathcal{A}) := H(\mathcal{A})$ and

$$N^p H(\mathcal{A}) := \mathrm{kernel}\big(N^{p-1} H(\mathcal{A}[t]) \xrightarrow{\mathrm{id} \otimes (t=0)} N^{p-1} H(\mathcal{A})\big), \quad p \geq 1. \tag{4.5}$$

Note that the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to a splitting $N^{p-1} H(\mathcal{A}[t]) \simeq N^p H(\mathcal{A}) \oplus N^{p-1} H(\mathcal{A})$. Let $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ be the category of nonnegatively graded chain complexes of abelian groups. Following Bass, we also have the functor

$$N^\bullet H : \mathrm{dgcat}(k) \to \mathrm{Ch}_{\geq 0}(\mathbb{Z}), \quad \mathcal{A} \mapsto N^\bullet H(\mathcal{A}),$$

where the chain complex $N^\bullet H(\mathcal{A})$ is defined by $N^0 H(\mathcal{A}) := H(\mathcal{A})$ and, for $p \geq 1$,

$$N^p H(\mathcal{A}) := \bigcap_{i=1}^{p} \mathrm{kernel}\big(H(\mathcal{A}[t_1, \ldots, t_p]) \xrightarrow{\mathrm{id} \otimes (t_i = 0)} H(\mathcal{A}[t_1, \ldots, \hat{t}_i, \ldots, t_p])\big),$$

$$N^p H(\mathcal{A}) \longrightarrow N^{p-1} H(\mathcal{A}), \quad t_i \mapsto \begin{cases} 1 - \sum_{i=2}^{p} t_i & \text{if } i = 1, \\ t_{i-1} & \text{if } i \neq 1. \end{cases}$$

Note that the above two definitions of $N^p H(\mathcal{A})$ are isomorphic. In what follows we will simply write $NH(\mathcal{A})$ instead of $N^1 H(\mathcal{A})$.

[2]Given any two prime numbers p and q, we have $\mathbb{S}/pq \simeq \mathbb{S}/p \oplus \mathbb{S}/q$ in Spt. Therefore, without loss of generality, we can (and will) work solely with one prime power l^ν.

5. Proof of Theorem 1.2

We will work often with the following general notion:

Definition 5.1. A functor $E : \mathrm{dgcat}(k) \to \mathrm{Spt}$ is called a *localizing invariant* if it inverts Morita equivalences and sends short exact sequences of dg categories (see [Keller 2006, §4.6]) to distinguished triangles of spectra

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \;\mapsto\; E(\mathcal{A}) \to E(\mathcal{B}) \to E(\mathcal{C}) \xrightarrow{\partial} \Sigma E(\mathcal{A}).$$

Thanks to the work of Blumberg and Mandell [2012], Keller [1998; 1999], Schlichting [2006], Thomason and Trobaugh [1990], and others, examples include not only nonconnective algebraic K-theory (with coefficients) but also Hochschild homology, cyclic homology, negative cyclic homology, periodic cyclic homology, topological Hochschild homology, topological cyclic homology, etc. Given an integer $q \in \mathbb{Z}$, the abelian group $\mathrm{Hom}_{\mathrm{Spt}}(\Sigma^q \mathbb{S}, E(\mathcal{A}))$ will be denoted by $E_q(\mathcal{A})$.

The proof of Theorem 1.2 is divided into four steps:

(I) Spectral sequence.

(II) Universal coefficient sequence.

(III) Fundamental theorem.

(IV) Module structure over the big Witt ring.

In order to simplify the exposition, we develop each one of these steps in a different section. Making use of Steps I–IV, we then conclude the proof of Theorem 1.2 in Section 10.

6. Step I: spectral sequence

Let E be a localizing invariant and $\Delta_n := k[t_0, \dots, t_n]/\big(\sum_{i=0}^n t_i - 1\big)$, $n \geq 0$, the simplicial k-algebra with faces and degeneracies given by the formulas

$$d_r(t_i) := \begin{cases} t_i & \text{if } i < r, \\ 0 & \text{if } i = r, \\ t_{i-1} & \text{if } i > r, \end{cases} \qquad \text{and} \qquad s_r(t_i) := \begin{cases} t_i & \text{if } i < r, \\ t_i + t_{i+1} & \text{if } i = r, \\ t_{i+1} & \text{if } i > r. \end{cases}$$

Out of this data, we can construct the $\mathbb{A}^1$-*homotopization* of E:

$$E^h : \mathrm{dgcat}(k) \to \mathrm{Spt}, \quad \mathcal{A} \mapsto \mathrm{hocolim}_n\, E(\mathcal{A} \otimes \Delta_n).$$

Note that E^h comes equipped with a natural 2-morphism $E \Rightarrow E^h$. As explained in [Tabuada 2015b, Proposition 5.2], E^h remains a localizing invariant and the canonical dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to an homotopy equivalence of spectra $E^h(\mathcal{A}) \to E^h(\mathcal{A}[t])$.

Given an integer $q \in \mathbb{Z}$, consider the functor $E_q : \mathrm{dgcat}(k) \to \mathrm{Ab}$ and the associated nonnegatively graded chain complex of abelian groups

$$0 \longleftarrow E_q(\mathcal{A}) \xleftarrow{d_0 - d_1} E_q(\mathcal{A}[t]) \longleftarrow \cdots \xleftarrow{(-1)^r \sum_r d_r} E_q(\mathcal{A} \otimes \Delta_n) \longleftarrow \cdots . \tag{6.1}$$

Under the isomorphisms

$$\Delta_n \xrightarrow{\sim} k[t_1, \ldots, t_n], \qquad t_i \mapsto \begin{cases} 1 - \sum_{i=1}^n t_i & \text{if } i = 0, \\ t_i & \text{if } i \neq 0, \end{cases}$$

the (Moore) normalization of (6.1) identifies with $N^\bullet E_q(\mathcal{A})$. Consequently, following [Quillen 1966], we obtain a standard convergent right half-plane spectral sequence $E^1_{pq} = N^p E_q(\mathcal{A}) \Rightarrow E^h_{p+q}(\mathcal{A})$. In the particular case of algebraic K-theory with coefficients, we have the convergent spectral sequence

$$E^1_{pq} = N^p \mathbb{K}_q(\mathcal{A}; \mathbb{Z}/l^\nu) \Rightarrow \mathbb{K}^h_{p+q}(\mathcal{A}; \mathbb{Z}/l^\nu). \tag{6.2}$$

Similarly, since $\pi_q(\mathbb{K}(\mathcal{A}) \otimes \mathbb{Z}[1/l]) \simeq \mathbb{K}_q(\mathcal{A})_{\mathbb{Z}[1/l]}$, we have the spectral sequence

$$E^1_{pq} = N^p \mathbb{K}_q(\mathcal{A})_{\mathbb{Z}[1/l]} \Rightarrow \mathbb{K}^h_{p+q}(\mathcal{A})_{\mathbb{Z}[1/l]}. \tag{6.3}$$

Remark 6.4. The preceding constructions and spectral sequences have their roots in the work of Anderson [1973], in the definition of homotopy K-theory (see [Weibel 1989]), and in the work of Suslin and Voevodsky [1996].

7. Step II: universal coefficient sequence

Let E be a localizing invariant. Similarly to (4.4), consider the functor

$$E(-; \mathbb{Z}/l^\nu) : \mathrm{dgcat}(k) \to \mathrm{Spt}, \quad \mathcal{A} \mapsto E(\mathcal{A}) \wedge^{\mathbb{L}} \mathbb{S}/l^\nu.$$

For every dg category $\mathcal{A}$ we have a distinguished triangle of spectra

$$E(\mathcal{A}) \xrightarrow{\cdot l^\nu} E(\mathcal{A}) \longrightarrow E(\mathcal{A}; \mathbb{Z}/l^\nu) \xrightarrow{\partial} \Sigma E(\mathcal{A}). \tag{7.1}$$

Consequently, the associated long exact sequence (obtained by applying the functor $\mathrm{Hom}_{\mathrm{Spt}}(\mathbb{S}, -)$ to (7.1)) breaks up into short exact sequences

$$0 \to E_q(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \to E_q(\mathcal{A}; \mathbb{Z}/l^\nu) \to \{l^\nu\text{-torsion in } E_{q-1}(\mathcal{A})\} \to 0.$$

Note that since the above distinguished triangle of spectra (7.1) is functorial on $\mathcal{A}$, we have moreover the short exact sequences

$$0 \to N^p E_q(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Z}/l^\nu \to N^p E_q(\mathcal{A}; \mathbb{Z}/l^\nu) \to \{l^\nu\text{-torsion in } N^p E_{q-1}(\mathcal{A})\} \to 0.$$

Remark 7.2. The preceding universal coefficient sequences are well known. In the case where $E = \mathbb{K}$, they were established by Thomason [1985, Appendix A].

8. Step III: fundamental theorem

Recall that we have the exact functors:

$$\mathrm{Nil}(k) \subset \mathrm{End}(k) \to \mathrm{P}(k), \qquad (M, f) \mapsto M, \tag{8.1}$$

$$\mathrm{P}(k) \to \mathrm{Nil}(k) \subset \mathrm{End}(k), \qquad M \mapsto (M, 0). \tag{8.2}$$

Let E be a localizing invariant and $\mathrm{Nil}(k)_{\mathbf{dg}}$, $\mathrm{P}(k)_{\mathbf{dg}}$ the dg categories introduced at Notation 4.3. Given a dg category $\mathcal{A}$ and an integer q, consider the abelian group

$$\mathrm{Nil}\, E_q(\mathcal{A}) := \mathrm{kernel}\big(E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \xrightarrow{\mathrm{id} \otimes (8.1)} E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}}) \simeq E_q(\mathcal{A})\big).$$

Note that since $(8.1) \circ (8.2) = \mathrm{id}$, the morphism

$$E(\mathcal{A}) \simeq E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}}) \xrightarrow{\mathrm{id} \otimes (8.2)} E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \tag{8.3}$$

gives rise to a splitting $E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \simeq \mathrm{Nil}\, E_q(\mathcal{A}) \oplus E_q(\mathcal{A})$.

Theorem 8.4 (fundamental theorem). *Given a localizing invariant E, we have $NE_{q+1}(\mathcal{A}) \simeq \mathrm{Nil}\, E_q(\mathcal{A})$.*

The remainder of this section is devoted to the proof of Theorem 8.4. Let $\mathbb{P}^1$ be the projective line over the base commutative ring k, with $i : \mathrm{Spec}(k[t]) \hookrightarrow \mathbb{P}^1$ and $j : \mathrm{Spec}(k[1/t]) \hookrightarrow \mathbb{P}^1$ the classical Zariski open cover.

Proposition 8.5. *We have a short exact sequence of dg categories*

$$0 \longrightarrow \mathrm{Nil}(k)_{\mathbf{dg}} \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1) \xrightarrow{\mathbb{L}j^*} \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[1/t])) \longrightarrow 0. \tag{8.6}$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1)_Z & \longrightarrow & \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1) & \xrightarrow{\mathbb{L}j^*} & \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[1/t])) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathbb{L}i^*} & & \downarrow & & \\ 0 & \to & \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))_{Z'} & \to & \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t])) & \to & \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t, 1/t])) & \to & 0 \end{array}$$

where Z (resp. Z') stands for the complement of $\mathrm{Spec}(k[1/t])$ in $\mathbb{P}^1$ (resp. of $\mathrm{Spec}(k[t, 1/t])$ in $\mathrm{Spec}(k[t])$) and $\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1)_Z$ (resp. $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))_{Z'}$) stands for the dg category of those perfect complexes of $\mathcal{O}_{\mathbb{P}^1}$-modules (resp. $\mathcal{O}_{\mathrm{Spec}(k[t])}$-modules) which are supported on Z (resp. on Z'). As proved by Thomason and Trobaugh [1990, Theorems 2.6.3 and 7.4], both rows are short exact sequences of dg categories and the left-hand side vertical dg functor is a Morita equivalence. It remains then only to show that $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))_{Z'}$ is Morita equivalent to $\mathrm{Nil}(k)_{\mathbf{dg}}$.

Let us write $\mathbb{H}_{1,t}(k[t])$ for the exact category of finitely presented $k[t]$-modules of projective dimension ≤ 1 that are annihilated by some power t^n of t. Following

[Schlichting 2011, §§3.1.8–3.1.11], we have a short exact sequence of dg categories

$$0 \longrightarrow \mathbb{H}_{1,t}(k[t])_{\mathbf{dg}} \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t])) \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t, 1/t])) \longrightarrow 0.$$

Making use of Keller's characterization [2006, Theorem 4.11(i)] of short exact sequences of dg categories, we conclude that $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[t]))_{Z'}$ is Morita equivalent to $\mathbb{H}_{1,t}(k[t])_{\mathbf{dg}}$. As proved by Grayson and Quillen [Grayson 1976, page 236], we have an equivalence of exact categories $\mathrm{Nil}(k) \to \mathbb{H}_{1,t}(k[t])$, $(M, f) \mapsto M_f$, where M_f stands for the $k[t]$-module M on which t acts as f. Consequently, we obtain an induced equivalence of dg categories $\mathbb{H}_{1,t}(k[t])_{\mathbf{dg}} \simeq \mathrm{Nil}(k)_{\mathbf{dg}}$. This concludes the proof of Proposition 8.5. □

As proved by Drinfeld [2004, Proposition 1.6.3], the functor

$$\mathcal{A} \otimes^{\mathbb{L}} - : \mathrm{Hmo}(k) \to \mathrm{Hmo}(k)$$

is well defined and preserves short exact sequences of dg categories. Consequently, (8.6) gives rise to the short exact sequence of dg categories

$$0 \longrightarrow \mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}} \longrightarrow \mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1 \xrightarrow{\mathrm{id} \otimes \mathbb{L} j^*} \mathcal{A}[1/t] \longrightarrow 0, \tag{8.7}$$

where $\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1$ stands for $\mathcal{A} \otimes^{\mathbb{L}} \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1)$. By applying the functor E to (8.7), we obtain a distinguished triangle of spectra

$$E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \to E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1) \to E(\mathcal{A}[1/t]) \xrightarrow{\partial} \Sigma E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}). \tag{8.8}$$

Now, recall from [Orlov 1992, §2] that we have two fully faithful dg functors

$$\iota_{-1} : \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \to \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1), \quad M \mapsto \mathbb{L}p^*(M) \otimes^{\mathbb{L}} \mathcal{O}_{\mathbb{P}^1}(-1),$$
$$\iota_0 : \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \to \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1), \quad M \mapsto \mathbb{L}p^*(M),$$

where $p \colon \mathbb{P}^1 \to \mathrm{Spec}(k)$ stands for the projection morphism. The dg functor ι_{-1} induces a Morita equivalence between $\mathrm{perf}_{\mathrm{dg}}(\mathrm{pt})$ and Drinfeld's dg quotient $\mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1)/\iota_0 \, \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt})$. Following [Tabuada 2008, §13], we obtain a *split* short exact sequence of dg categories (see also [Orlov 1992, Theorem 2.6])

$$0 \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \underset{\iota_0}{\overset{r}{\leftrightarrows}} \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1) \underset{s}{\overset{\iota_{-1}}{\leftrightarrows}} \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \longrightarrow 0, \tag{8.9}$$

where r is the right adjoint of ι_0, $r \circ \iota_0 = \mathrm{id}$, ι_{-1} is right adjoint of s, and $\iota_{-1} \circ s = \mathrm{id}$. By first applying the functor $\mathcal{A} \otimes^{\mathbb{L}} -$ to (8.9), and then the functor E to the resulting split short exact sequence of dg categories, we obtain the isomorphism

$$[E(\mathrm{id} \otimes \iota_0), E(\mathrm{id} \otimes \iota_{-1})] : E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1). \tag{8.10}$$

The proof of the following general lemma is clear:

Lemma 8.11. *If* $(f, g) : A \oplus A \xrightarrow{\sim} B$ *is an isomorphism in an additive category, then* $(f, f-g) : A \oplus A \xrightarrow{\sim} B$ *is also an isomorphism.*

By applying Lemma 8.11 to (8.10), we obtain the isomorphism

$$[E(\mathrm{id} \otimes \iota_0), E(\mathrm{id} \otimes \iota_0) - E(\mathrm{id} \otimes \iota_{-1})] : E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1). \quad (8.12)$$

Proposition 8.13. *The composition*

$$E(\mathcal{A}) \xrightarrow{(8.3)} E(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \longrightarrow E(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1)$$

agrees with $E(\mathrm{id} \otimes \iota_0) - E(\mathrm{id} \otimes \iota_{-1})$.

Proof. As proved in [Tabuada 2005, Corollary 5.10], there is a bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B})$ and the set of isomorphism classes of the category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. Under this bijection, the composition law of $\mathrm{Hmo}(k)$ corresponds to the bifunctor

$$\mathrm{rep}(\mathcal{A}, \mathcal{B}) \times \mathrm{rep}(\mathcal{B}, \mathcal{C}) \to \mathrm{rep}(\mathcal{A}, \mathcal{C}), \quad (\mathrm{B}, \mathrm{B}') \mapsto \mathrm{B} \otimes^{\mathbb{L}}_{\mathcal{B}} \mathrm{B}'. \quad (8.14)$$

Since the $\mathcal{A}$-$\mathcal{B}$-bimodules (4.1) belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$, we have the $\otimes$-functor

$$\mathrm{dgcat}(k) \to \mathrm{Hmo}(k), \quad \mathcal{A} \mapsto \mathcal{A}, \ F \mapsto {}_F\mathrm{B}. \quad (8.15)$$

The *additivization* $\mathrm{Hmo}_0(k)$ *of* $\mathrm{Hmo}(k)$ is the additive category with the same objects and abelian groups of morphisms given by $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A}, \mathcal{B}) := K_0 \,\mathrm{rep}(\mathcal{A}, \mathcal{B})$, where $K_0 \,\mathrm{rep}(\mathcal{A}, \mathcal{B})$ stands for the Grothendieck group of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. The composition law is induced by the above bitriangulated functor (8.14) and the symmetric monoidal structure by bilinearity from $\mathrm{Hmo}(k)$. Note that we also have the symmetric monoidal functor

$$\mathrm{Hmo}(k) \to \mathrm{Hmo}_0(k), \quad \mathcal{A} \mapsto \mathcal{A}, \ \mathrm{B} \mapsto [\mathrm{B}]. \quad (8.16)$$

Let us denote by $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ the composition of (8.15) with (8.16).

Now, consider the composition of dg functors

$$\iota : \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \simeq \mathrm{P}(k)_{\mathbf{dg}} \xrightarrow{(8.2)} \mathrm{Nil}(k)_{\mathbf{dg}} \longrightarrow \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1).$$

Thanks to Proposition 8.17, below, and to the fact that U is a $\otimes$-functor, it suffices now to show that $U(\iota)$ agrees with $U(\iota_0) - U(\iota_{-1})$. As explained in [Grayson 1976, page 237], we have a short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \iota(\mathrm{pt}) \to 0$. Consequently, we obtain a short exact of dg functors

$$0 \to \iota_{-1} \to \iota_0 \to \iota \to 0, \quad \iota_{-1}, \iota_0, \iota : \mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \to \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1).$$

By the construction of the additive category $\mathrm{Hmo}_0(k)$, we conclude that $[{}_\iota\mathrm{B}] = [{}_{\iota_0}\mathrm{B}] - [{}_{\iota_1}\mathrm{B}]$, i.e., that $U(\iota) = U(\iota_0) - U(\iota_{-1})$. This achieves the proof. □

Proposition 8.17. *Given a localizing invariant* $E : \mathrm{dgcat}(k) \to \mathrm{Spt}$, *there is an (unique) additive functor* $\bar{E} : \mathrm{Hmo}_0(k) \to \mathrm{Spt}$ *such that* $\bar{E} \circ U \simeq E$.

Proof. Recall from [Tabuada 2005] that a functor $E : \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an additive category, is called an *additive invariant* if it inverts Morita equivalences and sends split short exact sequences of dg categories to direct sums. As proved in [Tabuada 2005, Theorems 5.3 and 6.3], the functor $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ is the *universal additive invariant*, i.e., given any additive category D there is an equivalence of categories

$$U^* : \mathrm{Fun}_{\mathrm{additive}}(\mathrm{Hmo}_0(k), \mathrm{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D}),$$

where the left-hand side denotes the category of additive functors and the right-hand side the category of additive invariants. The proof follows now from the fact that every localizing invariant is in particular an additive invariant. □

The distinguished triangle (8.8) gives rise to the long exact sequence

$$\cdots \to E_{q+1}(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1) \to E_{q+1}(\mathcal{A}[1/t]) \to E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \to E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathbb{P}^1) \to \cdots$$

Note that the two compositions

$$\mathrm{perf}_{\mathrm{dg}}(\mathrm{pt}) \underset{\iota_{-1}}{\overset{\iota_0}{\rightrightarrows}} \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^1) \xrightarrow{\mathbb{L} j^*} \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k[1/t])) \tag{8.18}$$

agree with the inverse image dg functor induced by $\mathrm{Spec}(k[1/t]) \to \mathrm{pt}$. Making use of the isomorphism (8.12), we conclude that the above long exact sequence breaks up into shorter exact sequences

$$0 \to E_{q+1}(\mathcal{A}) \to E_{q+1}(\mathcal{A}[1/t]) \to E_q(\mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}}) \to E_q(\mathcal{A}) \to 0. \tag{8.19}$$

Moreover, making use of Proposition 8.13, we observe that the last morphism in (8.19) corresponds to the projection $\mathrm{Nil}\, E_q(\mathcal{A}) \oplus E_q(\mathcal{A}) \to E_q(\mathcal{A})$. Consequently, (8.19) can be further reduced to a short exact sequence

$$0 \longrightarrow E_{q+1}(\mathcal{A}) \longrightarrow E_{q+1}(\mathcal{A}[1/t]) \longrightarrow \mathrm{Nil}\, E_q(\mathcal{A}) \longrightarrow 0.$$

From this short exact sequence we obtain, finally, the sought-for isomorphism

$$N E_{q+1}(\mathcal{A}) \simeq \mathrm{cokernel}\big(E_{q+1}(\mathcal{A}) \to E_{q+1}(\mathcal{A}[1/t])\big) \simeq \mathrm{Nil}\, E_q(\mathcal{A}).$$

This concludes the proof of Theorem 8.4.

9. Step IV: module structure over the big Witt ring

Recall from [Bloch 1977, page 192] the construction of the big Witt ring $W(R)$ of a commutative ring R. As an additive group, $W(R)$ is equal to $(1 + tR[\![t]\!], \times)$. The multiplication $*$ is uniquely determined by naturality, formal factorization of the elements of $W(R)$ as $h(t)=\prod_{n\geq 1}(1-a_nt^n)$, and the equality $(1-at)*h(t)=h(at)$. The zero element is $1+0t+\cdots$ and the unit element is $(1-t)$.

Theorem 9.1. *Given a dg category $\mathcal{A}$, the abelian groups* $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$, $q\in\mathbb{Z}$, *carry a $W(k)$-module structure.*

The remainder of this section is devoted to the proof of Theorem 9.1. Recall from [Grayson 1976] that for every positive integer $n\geq 1$ we have a *Frobenius* functor

$$F_n:\mathrm{End}(k)\to\mathrm{End}(k),\quad (M,f)\mapsto(M,f^n),$$

as well as a *Verschiebung* functor

$$V_n:\mathrm{End}(k)\to\mathrm{End}(k),\quad (M,f)\mapsto\left(M^{\oplus n},\begin{bmatrix}0&\cdots&\cdots&0&f\\1&\ddots&&&0\\0&\ddots&\ddots&&\vdots\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&\cdots&0&1&0\end{bmatrix}_{n\times n}\right).$$

Both these functors are exact and preserve the full subcategory of nilpotent endomorphisms $\mathrm{Nil}(k)$. Moreover, the following diagrams are commutative:

$$\begin{array}{ccc}\mathrm{End}(k)&\xrightarrow{F_n}&\mathrm{End}(k)\\ {\scriptstyle(8.1)}\downarrow&&\downarrow{\scriptstyle(8.1)}\\ \mathrm{P}(k)&=\!=\!=&\mathrm{P}(k)\end{array}\qquad\begin{array}{ccc}\mathrm{End}(k)&\xrightarrow{V_n}&\mathrm{End}(k)\\ {\scriptstyle(8.1)}\downarrow&&\downarrow{\scriptstyle(8.1)}\\ \mathrm{P}(k)&\xrightarrow[M\mapsto M^{\oplus n}]{}&\mathrm{P}(k)\end{array}\tag{9.2}$$

Following [Grayson 1976], let $\mathrm{End}_0(k)$ be the kernel of $K_0\,\mathrm{End}(k)\xrightarrow{(8.1)}K_0\mathrm{P}(k)$. Note that since $(8.1)\circ(8.2)=\mathrm{id}$, the homomorphism $K_0\mathrm{P}(k)\xrightarrow{(8.2)}K_0\,\mathrm{End}(k)$ gives rise to a splitting $K_0\,\mathrm{End}(k)\simeq\mathrm{End}_0(k)\oplus K_0\mathrm{P}(k)$. Note also that the image in $\mathrm{End}_0(k)$ of an endomorphism (M,f) is given by $[(M,f)]-[(M,0)]$. Under these notations, we have induced Frobenius and Verschiebung homomorphisms $F_n,V_n:\mathrm{End}_0(k)\to\mathrm{End}_0(k)$. Consider also the biexact functor

$$\mathrm{End}(k)\times\mathrm{Nil}(k)\to\mathrm{Nil}(k),\quad ((M,f),(M',f'))\mapsto(M\otimes M',f\otimes f'),\tag{9.3}$$

and the associated commutative diagram

$$\begin{array}{ccc}\mathrm{End}(k)\times\mathrm{Nil}(k)&\xrightarrow{(9.3)}&\mathrm{Nil}(k)\\ {\scriptstyle(8.1)\times(8.1)}\downarrow&&\downarrow{\scriptstyle(8.1)}\\ \mathrm{P}(k)\times\mathrm{P}(k)&\xrightarrow[(M,M')\mapsto M\otimes M']{}&\mathrm{P}(k)\end{array}\tag{9.4}$$

Given a dg category $\mathcal{A}$, (9.2) and (9.4) give rise to the commutative diagrams

$$\begin{array}{ccc} \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} & \xrightarrow{\mathrm{id}\otimes F_n} & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \\ \downarrow & & \downarrow \\ \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} & = & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \end{array} \qquad \begin{array}{ccc} \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} & \xrightarrow{\mathrm{id}\otimes V_n} & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \\ \downarrow & & \downarrow \\ \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} & \longrightarrow & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \end{array}$$

$$\begin{array}{ccc} \mathrm{End}(k)_{\mathbf{dg}}\otimes^{\mathbb{L}}\mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} & \longrightarrow & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)_{\mathbf{dg}} \\ \downarrow & & \downarrow \\ \mathrm{P}(k)_{\mathbf{dg}}\otimes^{\mathbb{L}}\mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} & \longrightarrow & \mathcal{A}\otimes^{\mathbb{L}}\mathrm{P}(k)_{\mathbf{dg}} \end{array} \tag{9.5}$$

In what follows, we will still denote by $F_n, V_n : \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$ the induced Frobenius and Verschiebung homomorphisms. Thanks to the work of Waldhausen [1985, page 342], a pairing of dg categories gives rise to a pairing on algebraic K-theory groups; see [Tabuada 2013, §4.2]. Therefore, since $\mathrm{End}_0(k)$ is the kernel of the homomorphism $K_0(\mathrm{End}(k)_{\mathbf{dg}}) \xrightarrow{(8.1)} K_0(\mathrm{P}(k)_{\mathbf{dg}})$, we obtain from (9.5) the bilinear pairings

$$-\cdot - : \mathrm{End}_0(k)\times \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}), \quad q\in\mathbb{Z}. \tag{9.6}$$

Remark 9.7 ($\mathrm{End}_0(k)$-module structure). The tensor product of k-modules gives rise naturally to a symmetric monoidal structure on the exact categories $\mathrm{P}(k)$ and $\mathrm{End}(k)$, making the forgetful functor (8.1) symmetric monoidal. Therefore, the abelian group $\mathrm{End}_0(k)$ comes equipped with an induced ring structure. Moreover, by construction, the bilinear pairings (9.6) endow the abelian groups $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$, $q\in\mathbb{Z}$, with an $\mathrm{End}_0(k)$-module structure.

Proposition 9.8. *We have $V_n(\alpha\cdot F_n(\beta)) = V_n(\alpha)\cdot\beta$ for every $\alpha\in\mathrm{End}_0(k)$ and $\beta\in\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$.*

Proof. Let S be the multiplicatively closed subset of $\mathbb{Z}[x,y][s]$ generated by s and $s^n - x^n y$. In what follows, we denote by $\mathrm{End}(\mathbb{Z}[x,y];S)$ the full exact subcategory of $\mathrm{End}(\mathbb{Z}[x,y])$ consisting of those endomorphisms (N,g) for which there exists a polynomial $p(s)\in S$, depending on (N,g), such that $p(g)=0$. The endomorphisms

$$\epsilon_1 := \left(\mathbb{Z}[x,y]^{\oplus n}, \begin{bmatrix} 0 & \cdots & \cdots & 0 & x^n y \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{n\times n} \right), \tag{9.9}$$

$$\epsilon_2 := \left(\mathbb{Z}[x,y]^{\oplus n}, \begin{bmatrix} 0 & \cdots & \cdots & 0 & xy \\ x & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x & 0 \end{bmatrix}_{n\times n} \right), \tag{9.10}$$

belong to $\mathrm{End}(\mathbb{Z}[x,y]; S)$ since they satisfy the equation $s^n - x^n y = 0$. Following [1982, §§5–6], consider the multiexact functor

$$\theta(-,-,-) : \mathrm{End}(\mathbb{Z}[x,y]; S) \times \mathrm{End}(k) \times \mathrm{Nil}(k) \longrightarrow \mathrm{Nil}(k)$$

which sends the triple $((N,g),(M,f),(M',f'))$ to the nilpotent endomorphism $(N \otimes_{\mathbb{Z}[x,y]} M \otimes M', g \otimes \mathrm{id} \otimes \mathrm{id})$, where the left $\mathbb{Z}[x,y]$-module structure on $M \otimes M'$ is given by $x \mapsto f'$ and $y \mapsto f$. Note that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{End}(\mathbb{Z}[x,y]; S) \times \mathrm{End}(k) \times \mathrm{Nil}(k) & \xrightarrow{\theta(-,-,-)} & \mathrm{Nil}(k) \\ \Big\downarrow {\scriptstyle (8.1)\times(8.1)\times(8.1)} & & \Big\downarrow {\scriptstyle (8.1)} \\ P(\mathbb{Z}[x,y]) \times \mathrm{P}(k) \times \mathrm{P}(k) & \xrightarrow[(N,M,M')\mapsto N\otimes_{\mathbb{Z}[x,y]} M \otimes M']{} & \mathrm{P}(k) \end{array} \tag{9.11}$$

Given a dg category $\mathcal{A}$, (9.11) leads to the commutative square

$$\begin{array}{ccc} \mathrm{End}(\mathbb{Z}[x,y]; S)_{\mathbf{dg}} \otimes^{\mathbb{L}} \mathrm{End}(k)_{\mathbf{dg}} \otimes^{\mathbb{L}} \mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}} & \longrightarrow & \mathcal{A} \otimes^{\mathbb{L}} \mathrm{Nil}(k)_{\mathbf{dg}} \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{P}(\mathbb{Z}[x,y])_{\mathbf{dg}} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}} \otimes^{\mathbb{L}} \mathcal{A} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}} & \longrightarrow & \mathcal{A} \otimes^{\mathbb{L}} \mathrm{P}(k)_{\mathbf{dg}} \end{array} \tag{9.12}$$

In the same way that the diagram (9.5) gives rise to the bilinear pairings (9.6), the diagram (9.12) gives rise to the multilinear homomorphisms

$$\mathrm{End}_0(\mathbb{Z}[x,y]; S) \times \mathrm{End}_0(k) \times \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}), \quad q \in \mathbb{Z}. \tag{9.13}$$

Thanks to Lemma 9.16, below, the evaluation of the homomorphism (9.13) at the class $[\epsilon_1] - [(\mathbb{Z}[x,y]^{\oplus n}, 0)] \in \mathrm{End}_0(\mathbb{Z}[x,y]; S)$ reduces to the bilinear pairing

$$\mathrm{End}_0(k) \times \mathrm{Nil}\,\mathbb{K}_n(\mathcal{A}) \longrightarrow \mathrm{Nil}\,\mathbb{K}_n(\mathcal{A}), \quad (\alpha,\beta) \mapsto V_n(\alpha \cdot F_n(\beta)). \tag{9.14}$$

Similarly, the evaluation of (9.13) at $[\epsilon_2] - [(\mathbb{Z}[x,y]^{\oplus n}, 0)]$ reduces to the pairing

$$\mathrm{End}_0(k) \times \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}) \to \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}), \quad (\alpha,\beta) \mapsto V_n(\alpha) \cdot \beta. \tag{9.15}$$

Now, recall from [Almkvist 1974] (see also [Grayson 1978]) that the characteristic polynomial gives rise to an *injective* ring homomorphism

$$\mathrm{End}_0(\mathbb{Z}[x,y]; S) \to W(\mathbb{Z}[x,y]), \quad [(N,g)] - [(N,0)] \mapsto \det(\mathrm{id} - gt).$$

Since the matrices (9.9)–(9.10) have the same characteristic polynomial, namely $1+(x^ny)t^n$, we conclude that $[\epsilon_1]-[(\mathbb{Z}[x,y]^{\oplus n},0)]=[\epsilon_2]-[(\mathbb{Z}[x,y]^{\oplus n},0)]$. This implies that the above pairings (9.14)–(9.15) agree and consequently that $V_n(\alpha\cdot F_n(\beta))=V_n(\alpha)\cdot\beta$ for every $\alpha\in\mathrm{End}_0(k)$ and $\beta\in\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$. $\square$

Lemma 9.16. *We have the commutative diagrams*

$$\begin{array}{ccc} \mathrm{End}(k)\times\mathrm{Nil}(k) & \xrightarrow{\theta(\epsilon_1,-,-)} & \mathrm{Nil}(k) \\ {\scriptstyle \mathrm{id}\times F_n}\downarrow & & \uparrow{\scriptstyle V_n} \\ \mathrm{End}(k)\times\mathrm{Nil}(k) & \xrightarrow[(9.3)]{} & \mathrm{Nil}(k) \end{array} \qquad \begin{array}{ccc} \mathrm{End}(k)\times\mathrm{Nil}(k) & \xrightarrow{\theta(\epsilon_2,-,-)} & \mathrm{Nil}(k) \\ {\scriptstyle V_n\times\mathrm{id}}\downarrow & & \Vert \\ \mathrm{End}(k)\times\mathrm{Nil}(k) & \xrightarrow[(9.3)]{} & \mathrm{Nil}(k) \end{array}$$

Proof. Let $(M,f)\in\mathrm{End}(k)$ and $(M',f')\in\mathrm{Nil}(k)$. By definition of ϵ_1 and ϵ_2, we observe that $\theta(\epsilon_1,(M,f),(M',f'))$ is naturally isomorphic to the endomorphism

$$\left((M\otimes M')^{\oplus n},\ \begin{bmatrix} 0 & \cdots & \cdots & 0 & f\otimes f'^n \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{n\times n}\right)$$

and that $\theta(\epsilon_2,(M,f),(M',f'))$ is naturally isomorphic to the endomorphism

$$\left(M^{\oplus n}\otimes M',\ \begin{bmatrix} 0 & \cdots & \cdots & 0 & f \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}_{n\times n}\otimes f'\right).$$

This achieves the proof. $\square$

Given an integer $m\geq 0$, let $\mathrm{Nil}(k)^m$ be the full exact subcategory of $\mathrm{Nil}(k)$ consisting of those nilpotent endomorphisms (M,f) with $f^m=0$. By construction, we have an exhaustive increasing filtration $\mathrm{Nil}(k)^m\subset\mathrm{Nil}(k)^{m+1}\subset\cdots\subset\mathrm{Nil}(k)$.

Given a dg category $\mathcal{A}$ and an integer $q\in\mathbb{Z}$, let us denote by $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})^m$ the image of the induced homomorphism

$$\mathrm{kernel}\big(\mathbb{K}_q(\mathcal{A}\otimes^{\mathbb{L}}\mathrm{Nil}(k)^m_{\mathbf{dg}})\xrightarrow{\mathrm{id}\otimes(8.1)}\mathbb{K}_q(\mathcal{A}\otimes\mathrm{P}(k)_{\mathbf{dg}})\big)\longrightarrow\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}).$$

Note that $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})=\bigcup_m\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})^m$ and that the Frobenius homomorphism $F_n:\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})\to\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$ vanishes on $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})^m$ whenever $n\geq m$.

Given elements $a\in k$ and $\beta\in\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$, consider the definition

$$(1-at^n)\odot\beta:=V_n\big([(k,a)]-[(k,0)]\big)\cdot\beta, \tag{9.17}$$

where (k,a) stands for the endomorphism of k given by multiplication by a. Thanks to Proposition 9.8, (9.17) agrees with $V_n\big(([(k,a)]-[(k,0)])\cdot F_n(\beta)\big)$. Consequently, whenever $\beta\in \mathrm{Nil}\,\mathbb{K}_n(\mathcal{A})^m$ with $n\geq m$, we have $(1-at^n)\odot\beta=0$. Since $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})=\bigcup_m \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})^m$, we obtain the bilinear pairings — the sum is finite! —

$$\begin{aligned} &W(k)\times \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})\to \mathrm{Nil}\,\mathbb{K}_q(\mathcal{A}),\\ &\Big(\prod_{n\geq 1}(1-a_nt^n),\beta\Big)\mapsto \sum_{n\geq 1}((1-a_nt^n)\odot\beta). \end{aligned} \tag{9.18}$$

Now, recall from [Almkvist 1974] that the injective ring homomorphism

$$\mathrm{End}_0(k)\to W(k),\quad \big([(M,f)]-[(M,0)]\big)\mapsto \det(\mathrm{id}-ft),$$

sends $V_n\big([(k,a)]-[(k,0)]\big)$ to $1-at^n$. Since every element of $W(k)$ can be written uniquely as $\prod_{n\geq 1}(1-a_nt^n)$, we conclude that (9.18) extends (9.6). Moreover, thanks to Remark 9.7, the bilinear pairings (9.18) endow the abelian groups $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$, $q\in\mathbb{Z}$, with a $W(k)$-module structure. This concludes the proof of Theorem 9.1.

10. Conclusion of the proof of Theorem 1.2

(i) As explained by Weibel [1981, Proposition 1.2], we have a ring homomorphism $\mathbb{Z}[1/l]\to W(\mathbb{Z}[1/l])$, $\lambda\mapsto(1-t)^\lambda$. Consequently, using the functoriality of $W(-)$ and the assumption $1/l\in k$, we observe that $W(k)$ is a $\mathbb{Z}[1/l]$-module. By combining Theorem 9.1 with Theorem 8.4 (with $E=\mathbb{K}$), we conclude that the groups $N\mathbb{K}_q(\mathcal{A})$, $q\in\mathbb{Z}$, carry a $\mathbb{Z}[1/l]$-module structure. The recursive formula (4.5) (with $H=\mathbb{K}_q$) implies that the groups $N^p\mathbb{K}_q(\mathcal{A})$, $p\geq 1$, are also $\mathbb{Z}[1/l]$-modules. Therefore, making use of the short exact sequences (see Step II)

$$0\to N^p\mathbb{K}_q(\mathcal{A})\otimes_{\mathbb{Z}}\mathbb{Z}/l^\nu\to N^p\mathbb{K}_q(\mathcal{A};\mathbb{Z}/l^\nu)\to\{l^\nu\text{-torsion in } N^p\mathbb{K}_{q-1}(\mathcal{A})\}\to 0,$$

we conclude that the groups $N^p\mathbb{K}_q(\mathcal{A};\mathbb{Z}/l^\nu)$ are trivial. The convergent right half-plane spectral sequence (6.2) then implies that the edge morphisms

$$\mathbb{K}_q(\mathcal{A};\mathbb{Z}/l^\nu)\to \mathbb{K}_q^h(\mathcal{A};\mathbb{Z}/l^\nu)$$

are isomorphisms. The proof follows now from the fact that the canonical dg functor $\mathcal{A}\to\mathcal{A}[t]$ gives rise to a homotopy equivalence of spectra

$$\mathbb{K}^h(\mathcal{A};\mathbb{Z}/l^\nu)\to \mathbb{K}^h(\mathcal{A}[t];\mathbb{Z}/l^\nu);$$

see Step I.

(ii) We start with the following (arithmetic) result:

Lemma 10.1. *When l is nilpotent in k, the abelian groups* $\mathrm{Nil}\,\mathbb{K}_q(\mathcal{A})$ *are l-groups.*

Proof. Recall that the unit of $W(k)$ is $1-t$. Let $m \geq 0$ be a fixed integer. As explained by Weibel [1981, §1.5], whenever l is nilpotent in k there exists an integer $r \gg 0$ (which depends on m) such that $(1-t)^{l^r} \in 1 + t^m k[\![t]\!]$. This implies that the formal factorization of $(1-t)^{l^r}$ only contains factors $(1-a_n t^n)$ with $n \geq m$. As in Step IV, we observe that every element β of $\mathrm{Nil}\, \mathbb{K}_q(\mathcal{A})^m$ is l^r-torsion. Finally, since $\mathrm{Nil}\, \mathbb{K}_q(\mathcal{A}) \simeq \bigcup_m \mathrm{Nil}\, \mathbb{K}_q(\mathcal{A})^m$, we conclude that $\mathrm{Nil}\, \mathbb{K}_q(\mathcal{A})$ is a l-group. □

By combining Lemma 10.1 with Theorem 8.4 (with $E = \mathbb{K}$), we conclude that the abelian groups $N\mathbb{K}_q(\mathcal{A})$, $q \in \mathbb{Z}$, are l-groups. The recursive formula (4.5) (with $H = \mathbb{K}$) implies that the abelian groups $N^p\mathbb{K}_q(\mathcal{A})$, $p \geq 1$, are also l-groups. Therefore, $N^p\mathbb{K}_q(\mathcal{A})_{\mathbb{Z}[1/l]} = 0$. Making use of the convergent right half-plane spectral sequence (6.3), we see that the edge morphisms $\mathbb{K}_q(\mathcal{A})_{\mathbb{Z}[1/l]} \to \mathbb{K}_q^h(\mathcal{A})_{\mathbb{Z}[1/l]}$ are isomorphisms. The proof follows now from the fact that the dg functor $\mathcal{A} \to \mathcal{A}[t]$ gives rise to an homotopy equivalence of spectra

$$\mathbb{K}^h(\mathcal{A}) \otimes \mathbb{Z}[1/l] \to \mathbb{K}^h(\mathcal{A}[t]) \otimes \mathbb{Z}[1/l];$$

see Step I.

11. Proof of Theorem 3.1

Thanks to Corollary 2.6, it suffices to compute the kernel and the cokernel of the (matrix) homomorphism (2.7) in the case where $m = 0$ and $Q = A_n$. The kernel is the solution of the system of linear equations with $\mathbb{Z}/l^\nu$-coefficients

$$\left\{\begin{array}{r} -2x_1 + x_2 = 0 \\ -x_1 - x_2 + x_3 = 0 \\ \vdots \\ -x_1 - x_{j-1} + x_j = 0 \\ \vdots \\ -x_1 - x_{n_1} + x_n = 0 \\ -x_1 - x_n = 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} x_2 = 2x_1 \\ x_3 = 3x_1 \\ \vdots \\ x_j = jx_1 \\ \vdots \\ x_n = nx_1 \\ x_n = -x_1 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{r} (n+1)x_1 = 0 \\ x_2 = 2x_1 \\ \vdots \\ x_j = jx_1 \\ \vdots \\ x_n = nx_1 \end{array}\right\}.$$

From the above resolution of the system, we observe that the kernel is isomorphic to the $(n+1)$-torsion in $\mathbb{Z}/l^\nu$ or equivalently to the cyclic group $\mathbb{Z}/\gcd(n+1, l^\nu)$. Let us now compute the cokernel. Consider the (matrix) homomorphism

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ -1 & -1 & \ddots & \ddots & \vdots \\ -1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & 0 & \cdots & 0 & -1 \end{bmatrix} : \bigoplus_{r=1}^{n} \mathbb{Z} \to \bigoplus_{r=1}^{n} \mathbb{Z}. \tag{11.1}$$

Note that the cokernel of (11.1) is isomorphic to $\mathbb{Z}/(n+1)$. A canonical generator is given by the image of the vector $(0,\ldots,0,-1)\in\bigoplus_{r=1}^{n}\mathbb{Z}$. Using the fact that the functor $-\otimes_{\mathbb{Z}}\mathbb{Z}/l^{\nu}$ is right exact, we conclude that the cokernel of (2.7) is isomorphic to $\mathbb{Z}/(n+1)\otimes_{\mathbb{Z}}\mathbb{Z}/l^{\nu}\simeq\mathbb{Z}/\gcd(n+1,l^{\nu})$. This concludes the proof.

Remark 11.2. Thanks to [Tabuada 2015a, Corollary 2.11], the Grothendieck group of $\mathcal{C}_{A_n}^{(0)}$ identifies with the cokernel of (11.1). We observe that $K_0(\mathcal{C}_{A_n}^{(0)})\simeq\mathbb{Z}/(n+1)$.

12. Proof of Proposition 3.3

Similarly to the proof of Theorem 3.1, it suffices to compute the kernel and cokernel of the (matrix) homomorphism (2.7) in the case where $m=1$ and Q is the generalized Kronecker quiver $1 \rightrightarrows 2$. The kernel is given by the solution of the system of linear equations with $\mathbb{Z}/l^{\nu}$-coefficients

$$\begin{cases}-9x_1+3x_2=0,\\ -3x_1=0.\end{cases}\tag{12.1}$$

Clearly, the solution of (12.1) is (3-torsion in $\mathbb{Z}/l^{\nu}$) $\times$ (3-torsion in $\mathbb{Z}/l^{\nu}$) or, equivalently, the cyclic group $\mathbb{Z}/\gcd(3,l^{\nu})\times\mathbb{Z}/\gcd(3,l^{\nu})$. Note that the latter group is isomorphic to $\mathbb{Z}/3\times\mathbb{Z}/3$ when $l=3$ and is zero otherwise. Let us now compute the cokernel. Consider the (matrix) homomorphism

$$\begin{bmatrix}-9 & 3\\ -3 & 0\end{bmatrix}:\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\oplus\mathbb{Z}.\tag{12.2}$$

The cokernel of (12.2) is isomorphic to $\mathbb{Z}/3\times\mathbb{Z}/3$. Canonical generators are given by the image of the vectors $(1,0)$ and $(-3,1)$. Since the functor $-\otimes_{\mathbb{Z}}\mathbb{Z}/l^{\nu}$ is right exact, we conclude that the cokernel of (2.7) is isomorphic to

$$(\mathbb{Z}/3\times\mathbb{Z}/3)\otimes_{\mathbb{Z}}\mathbb{Z}/l^{\nu}\simeq\mathbb{Z}/3\otimes_{\mathbb{Z}}\mathbb{Z}/l^{\nu}\times\mathbb{Z}/3\otimes_{\mathbb{Z}}\mathbb{Z}/l^{\nu}\simeq\mathbb{Z}/\gcd(3,l^{\nu})\times\mathbb{Z}/\gcd(3,l^{\nu}).$$

Once again, the right-hand side abelian group is isomorphic to $\mathbb{Z}/3\times\mathbb{Z}/3$ when $l=3$ and is zero otherwise. This concludes the proof.

Remark 12.3. As in Remark 11.2, the Grothendieck group of $\mathcal{C}_{Q}^{(1)}$ is identified with the cokernel of (12.2). We observe that $K_0(\mathcal{C}_{Q}^{(1)})\simeq\mathbb{Z}/3\times\mathbb{Z}/3$.

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