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# Stable $\mathbb{A}^1$ -connectivity over Dedekind schemes

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We show that  $\mathbb{A}^1$ -localization decreases the stable connectivity by at most one over a Dedekind scheme with infinite residue fields. For the proof, we establish a version of Gabber's geometric presentation lemma over a henselian discrete valuation ring with infinite residue field.

## Introduction

**Background.** Morel [2005] formulated the following property on a scheme  $S$ , called the *stable  $\mathbb{A}^1$ -connectivity property*:

*The  $\mathbb{A}^1$ -localization of a connected spectrum on the smooth Nisnevich site over  $S$  is still connected.*

Here, the notion of connectivity refers to the associated Nisnevich homotopy sheaves or equivalently to the connectivity of the Nisnevich stalks. Further, he proved this property for  $S = \text{Spec}(k)$ , where  $k$  is a field. This celebrated result is known as the *stable  $\mathbb{A}^1$ -connectivity theorem* [Morel 2005, Theorem 6.1.8] and has diverse implications. Most of the content from Morel's monograph [2012] is based on this result, such as the unstable  $\mathbb{A}^1$ -connectivity theorem and its implication, the Hurewicz theorem in  $\mathbb{A}^1$ -homotopy theory [Morel 2012, Theorem 6.37]. This leads to a computation of the 0-line of the stable homotopy groups of motivic spheres as the Milnor–Witt  $K$ -theory  $K_*^{\text{MW}}(S)$  of the base  $S$  [Morel 2012, Corollary 6.43]. More immediately, the  $\mathbb{A}^1$ -connectivity theorem implies the vanishing of the negative lines which is analogous to the vanishing of the negative stable homotopy groups of the sphere in topology.

Morel [2005, Conjecture 2] conjectured that the stable  $\mathbb{A}^1$ -connectivity property holds over every regular base. However, Ayoub [2006] constructed a counterexample to this conjecture (see Remark 4.4 below).

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**Aim and results.** In this paper, we want to replace Morel’s stable  $\mathbb{A}^1$ -connectivity property by the following weaker property on a base scheme  $S$  of Krull-dimension  $d$  which is consistent with Ayoub’s counterexample:

*The  $\mathbb{A}^1$ -localization of a  $d$ -connected spectrum on the smooth Nisnevich site over  $S$  is still connected.*

We refer to this property as the *shifted stable  $\mathbb{A}^1$ -connectivity property*. In other words,  $S$  has this property if  $\mathbb{A}^1$ -localization lowers the connectivity by at most the dimension of  $S$ . [Question 4.12](#) below asks whether every regular base scheme has this shifted stable  $\mathbb{A}^1$ -connectivity property. Morel’s stable connectivity theorem is a positive answer in the case  $d = 0$ . In the main theorem of this paper, we give a positive answer in the one-dimensional case, assuming infinite residue fields (see also [Theorem 4.16](#)):

**Theorem A.** *A Dedekind scheme with only infinite residue fields has the shifted stable  $\mathbb{A}^1$ -connectivity property: if  $E$  is an  $i$ -connected spectrum, then its  $\mathbb{A}^1$ -localization  $L^{\mathbb{A}^1}E$  is  $(i - 1)$ -connected.*

Examples for such base schemes are algebraic curves over infinite fields in geometric settings, or  $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{nr}})$  for  $\mathbb{Z}_p^{\mathrm{nr}}/\mathbb{Z}_p$  the maximal unramified extension in more arithmetic settings.

Morel’s proof of the  $\mathbb{A}^1$ -connectivity theorem needs a strong geometric input referred to as *Gabber’s geometric presentation lemma* and written up in [[Colliot-Thélène et al. 1997](#), Theorem 3.1.1]. These authors show how Gabber’s presentation result leads to universal exactness of certain Cousin complexes. In particular, they derive the Bloch–Ogus theorem and the Gersten conjecture for algebraic  $K$ -theory for smooth varieties over a field, as first proved by Quillen [[1973](#), Theorem 5.11]. In [Section 2](#), we prove a version of this presentation result over a henselian discrete valuation ring with infinite residue fields (compare [Theorem 2.1](#)):

**Theorem B.** *Let  $\mathfrak{o}$  be a henselian discrete valuation ring with infinite residue field and let  $\sigma$  denote the closed point of  $S = \mathrm{Spec}(\mathfrak{o})$ . Let  $X$  be a smooth  $S$ -scheme of finite type and let  $Z \hookrightarrow X$  be a proper closed subscheme. Let  $z$  be a point in  $Z$ . If  $z$  lies in the special fibre  $Z_\sigma$ , suppose that  $Z_\sigma \neq X_\sigma$ . Then, Nisnevich-locally around  $z$ , there exists a smooth  $\mathfrak{o}$ -scheme  $V$  of finite type and a cartesian square*

$$\begin{array}{ccc} X \setminus Z & \longrightarrow & X \\ \downarrow & & \downarrow p \\ \mathbb{A}_V^1 \setminus p(Z) & \longrightarrow & \mathbb{A}_V^1 \end{array}$$

*such that  $p$  is étale, the restriction  $p|_Z : Z \hookrightarrow \mathbb{A}_V^1$  is a closed subscheme and  $Z$  is finite over  $V$ . In particular, this square is a Nisnevich-distinguished square.*

The proof is based on [Colliot-Thélène et al. 1997, Theorem 3.1.1] combined with a Noether normalization over a Dedekind base; cf. [Kai 2015, Theorem 4.6].

Apart from this geometric input to the proof of Theorem A, we need a second key ingredient of a more homotopical kind: In Section 3, we examine a vanishing result for the nonsheafified homotopy classes of the  $\mathbb{A}^1$ -localization of a connected spectrum. This is a slight generalization of the argument in [Morel 2005, Lemma 4.3.1] to arbitrary noetherian base schemes of finite Krull-dimension. As a byproduct, we obtain that the  $S^1$ - and the  $\mathbb{P}^1$ -homotopy t-structure over any base scheme is left complete, i.e., a presheaf of spectra is recovered as the homotopy limit over its Postnikov truncations (see Corollaries 3.6 and 3.8).

### 1. Preliminaries

In this paper, our base scheme  $S$  is always a noetherian scheme of finite Krull-dimension. Let  $\text{Sm}_S$  be the category of smooth schemes of finite type over  $S$ . The category  $\text{Sm}_S$  is essentially small and sometimes we choose a small skeleton implicitly without mentioning. Let  $\text{sPre}_+(S)$  be the category of pointed simplicial presheaves on  $\text{Sm}_S$ . We mostly ignore  $S$  in the notation. For an object  $U \in \text{Sm}_S$ , let  $U_+$  denote the presheaf  $\text{hom}_{\text{Sm}_S}(-, U)$  considered as a discrete simplicial set with an additional disjoint basepoint. Whenever we speak of a category having all limits and colimits we actually mean that it has all *small* limits and all *small* colimits.

**Model structures.** In contrast to the foundational address [Morel and Voevodsky 1999] of  $\mathbb{A}^1$ -homotopy theory, we use projective analogues of the unstable model structures and obtain the (pointed) *objectwise*, *Nisnevich-local* and  $\mathbb{A}^1$ -*Nisnevich-local* model structure (see [Dundas et al. 2003, Section 2]). Throughout the whole text, let  $L^{\text{ob}}$ ,  $L^s$  and  $L^{\mathbb{A}^1}$  denote fixed (pointed) objectwise, Nisnevich-local and  $\mathbb{A}^1$ -Nisnevich-local fibrant replacement functors, respectively. Given a symbol  $\tau \in \{\text{ob}, s, \mathbb{A}^1\}$ , a nonnegative integer  $n$  and  $F \in \text{sPre}_+$ , define the *n-th  $\tau$ -homotopy sheaf*  $\pi_n^\tau(F)$  of  $F$  as the Nisnevich sheafification of the *n-th  $\tau$ -homotopy presheaf*

$$[(-)_+ \wedge S^n, L^\tau F].$$

Here, the brackets denote (pointed) objectwise homotopy classes. Notice that  $\pi_n^{\text{ob}}(F) \cong \pi_n^s(F)$ . Whenever the objectwise model structure is considered, we omit the symbol  $^{\text{ob}}$  from the notation.

Let  $\text{Spt}_{S^1}(S)$  be the category of (nonsymmetric)  $S^1$ -spectra on the category  $\text{sPre}_+(S)$  [Hovey 2001, Definition 1.1]. The functor  $(-)_0$  sending an  $S^1$ -spectrum to its zeroth level and the  $S^1$ -suspension spectrum functor  $\Sigma_{S^1}^\infty$  fit into an adjunction  $\Sigma_{S^1}^\infty : \text{sPre}_+ \rightleftarrows \text{Spt}_{S^1} : (-)_0$ . For an integer  $n \geq 0$ , there is also an adjunction  $[-n] : \text{Spt}_{S^1} \rightleftarrows \text{Spt}_{S^1} : [n]$  of shift functors defined for an integer  $n$

by  $E[n]_m := E_{n+m}$  whenever  $n + m \geq 0$  and  $E[n]_m := *$  otherwise. Following the general procedure of [Hovey 2001], we equip the category  $\text{Spt}_{S^1}(S)$  with stable model structures (see Hovey's Definition 3.3) having homotopy categories  $\mathcal{SH}_{S^1}^{\text{ob}}(S)$ ,  $\mathcal{SH}_{S^1}^s(S)$  and  $\mathcal{SH}_{S^1}^{\mathbb{A}^1}(S)$ . The two above-mentioned adjunctions turn into Quillen adjunctions. Each of these stable homotopy categories is a triangulated category with distinguished triangles given by the homotopy cofibre sequences [Hovey 1999, Proposition 7.1.6]. In fact, by choosing symmetric spectra as a more elaborate model, these homotopy categories carry the structure of a closed symmetric monoidal category with a compatible triangulation in the sense of [May 2001, Definition 4.1]. For details of this construction we refer to [Hovey 2001; Jardine 2000; Ayoub 2007]. There are functors

$$\begin{aligned} - \wedge \Sigma_{S^1}^\infty(-) &: \text{Spt}_{S^1} \times \text{sPre}_+ \rightarrow \text{Spt}_{S^1}, \\ \underline{\text{hom}}(\Sigma_{S^1}^\infty(-), -) &: \text{sPre}_+^{\text{op}} \times \text{Spt}_{S^1} \rightarrow \text{Spt}_{S^1}, \end{aligned}$$

defined in the obvious way. For a cofibrant  $F \in \text{sPre}_+$ , they fit into a Quillen adjunction

$$- \wedge \Sigma_{S^1}^\infty F : \text{Spt}_{S^1} \rightleftarrows \text{Spt}_{S^1} : \underline{\text{hom}}(\Sigma_{S^1}^\infty F, -)$$

whose derived adjunction models the monoidal structure from before.

Let  $\Omega_{S^1}$  denote the functor  $\underline{\text{hom}}(\Sigma_{S^1}^\infty S^1, -)$ . We mention that a concrete fibrant replacement functor for the stable  $\tau$ -model structure on  $\text{Spt}_{S^1}$  is given by

$$\Theta_{S^1}^\tau E = \text{colim}((L^\tau E) \rightarrow \Omega_{S^1}(L^\tau E)[1] \rightarrow (\Omega_{S^1})^2(L^\tau E)[2] \rightarrow \cdots), \quad (1.1)$$

where  $\tau \in \{\text{ob}, s, \mathbb{A}^1\}$  and where the application of  $L^\tau$  to a spectrum is levelwise [Hovey 2001, Theorem 4.12]. We write  $\Omega_{S^1}^\infty : \text{Spt}_{S^1} \rightarrow \text{sPre}_+$  for the composition of this fibrant replacement functor with  $(-)_0$ .

As for the unstable structures, we define the  $n$ -th stable  $\tau$ -homotopy sheaf  $\pi_n^\tau(E)$  of a spectrum  $E \in \text{Spt}_{S^1}$  as the Nisnevich sheafification of the  $n$ -th stable  $\tau$ -homotopy presheaf

$$[\Sigma_{S^1}^\infty(-)_+[n], L^\tau E].$$

Here, the brackets denote the morphism sets of  $\mathcal{SH}_{S^1}^{\text{ob}}$ . Since it will be evident from the context if the unstable or the stable homotopy sheaf is considered, we do not introduce an extra decoration.

We use the following explicit model for  $L^{\mathbb{A}^1}$  in the stable context introduced in [Morel 2004, Lemma 4.2.4].

**Lemma 1.2** (Morel). *Let  $S$  be an arbitrary base scheme. For each integer  $k \geq 0$ , we set  $L^k(E) := \underline{\text{hom}}(F^{\wedge k}, L^s(E))$  with  $F := \Sigma_{S^1}^\infty C[-1]$ , where  $C$  is a cofibrant replacement of the cofibre of the morphism*

$$S^0 \xrightarrow{0,1} \mathbb{A}^1$$

in  $\text{sPre}_+$ . Then the functor  $L^\infty : \text{Spt}_{S^1} \rightarrow \text{Spt}_{S^1}$  defined by

$$L^\infty(E) := \text{hocolim}_{k \rightarrow \infty} L^k(E)$$

is a fibrant replacement functor for the stable  $\mathbb{A}^1$ -Nisnevich-local model.

**Remark 1.3.** Likewise, the spectrum  $F$  from the above [Lemma 1.2](#) may be defined by the distinguished triangle

$$F \longrightarrow \Sigma_{S^1}^\infty S^0 \xrightarrow{0,1} \Sigma_{S^1}^\infty \mathbb{A}^1.$$

Let  $k \geq 1$  be an integer. After rotation and smashing with the spectrum  $F^{\wedge(k-1)}$ , the above triangle becomes  $\Sigma_{S^1}^\infty \mathbb{A}^1 \wedge F^{\wedge(k-1)}[-1] \rightarrow F^{\wedge k} \rightarrow F^{\wedge(k-1)}$ . Applying  $\underline{\text{hom}}(-, L^s(E))$  yields the distinguished triangle

$$L^{k-1}(E) \rightarrow L^k(E) \rightarrow \underline{\text{hom}}(\Sigma_{S^1}^\infty \mathbb{A}^1, L^{k-1}(E)[1]).$$

Here  $L^{k-1}(E)[1] \simeq L^{k-1}(E[1])$  holds by definition and homotopy-exactness of  $L^s$ .

**Base change.** We briefly recall the construction of base change functors in  $\mathbb{A}^1$ -homotopy theory. For details, see the monograph [\[Ayoub 2007\]](#) and [\[Hu 2001\]](#).

Let  $f : R \rightarrow S$  be a morphism between noetherian schemes of finite Krull-dimension. There is an adjunction

$$f^* : \text{sPre}_+(S) \rightleftarrows \text{sPre}_+(R) : f_*$$

where the *direct image* functor  $f_*$  is defined by  $(f_*G)(-) := G(- \times_S R)$  and where the left adjoint *inverse image* is determined by  $f^*(U_+) := (U \times_S R)_+$  for  $U \in \text{Sm}_S$ . The functor  $f^*$  is strong symmetric monoidal with respect to the smash product and there is a natural isomorphism  $f_* \underline{\text{hom}}_+(f^*F, G) \cong \underline{\text{hom}}_+(F, f_*G)$  [\[Fausk et al. 2003, \(3.4\)\]](#). If the morphism  $f : R \rightarrow S$  is smooth and of finite type, the inverse image functor has a left adjoint

$$f_{\sharp} : \text{sPre}_+(R) \rightleftarrows \text{sPre}_+(S) : f^*$$

determined by  $f_{\sharp}(V_+) = f_{\sharp}(V_+ \rightarrow R) := (V \sqcup S \rightarrow R \sqcup S \rightarrow S) = f_{\sharp}^{\text{unpointed}}(V)_+$ . We emphasize that throughout the whole text, the functor  $f_{\sharp}$  is considered in this pointed sense: it does not only postcompose with  $f$  but also quotients out the base-point along  $f$ . In the case of a smooth  $f$  of finite type, the inverse image is given by  $f^*F = F \wedge R_+$ , and one has a *projection formula*  $f_{\sharp}(G \wedge f^*F) \cong f_{\sharp}G \wedge F$  (see, e.g., [\[Hoyois 2017, Section 5.1\]](#)) and a natural isomorphism

$$f^* \underline{\text{hom}}_+(A, B) \cong \underline{\text{hom}}_+(f^*A, f^*B) \tag{1.4}$$

by [\[Fausk et al. 2003, Proposition 4.1\]](#). Note that, since  $S$  is noetherian, any open immersion  $R \hookrightarrow S$  is smooth and of finite type.

The adjunction  $(f^*, f_*)$  is a Quillen adjunction for the objectwise, the Nisnevich-local and the  $\mathbb{A}^1$ -Nisnevich-local model structures. If  $f : R \rightarrow S$  is smooth and of finite type, then the adjunction  $(f_{\sharp}, f^*)$  is a Quillen adjunction for the objectwise, the Nisnevich-local and the  $\mathbb{A}^1$ -Nisnevich-local model structures as well and  $f^*$  preserves all weak equivalences; see [Ayoub 2007, Theorem 4.5.10].

**Remark 1.5.** For the projective versions of the model structures, it is easy to see that  $f^*$  and  $f_{\sharp}$  preserve the generating cofibrations and hence all cofibrations. By the same reason, the objectwise acyclic cofibrations are preserved, so  $(f_{\sharp}, f^*)$  and  $(f^*, f_*)$  are Quillen adjunctions for the objectwise structures. In order to see that the right adjoints  $f_*$  and  $f^*$  preserve fibrations for the Nisnevich-local and the  $\mathbb{A}^1$ -Nisnevich-local model structure, it suffices to show that they preserve fibrations between fibrant objects [Dugger 2001, Corollary A.2]. As the right adjoints preserve objectwise fibrations, it suffices to show that they preserve fibrant objects. The fibrant objects of a Bousfield localization may be detected by a particular set  $J'$  of acyclic cofibrations [Hirschhorn 2003, Lemma 3.3.11]. It remains to be shown that the left adjoints preserve these acyclic cofibrations in  $J'$ , which is straightforward; see [Dundas et al. 2003, Definition 2.14].

In particular, Remark 1.5 implies the following lemma.

**Lemma 1.6.** *Suppose  $f : R \rightarrow S$  is a smooth morphism of finite type. For each  $F \in \text{sPre}_+(S)$ , there are canonical (objectwise) weak equivalences*

$$L^S(f^*F) \sim f^*(L^S F) \quad \text{and} \quad L^{\mathbb{A}^1}(f^*F) \sim f^*(L^{\mathbb{A}^1} F)$$

in  $\text{sPre}_+(R)$ .

The spectrum  $S_s^h := \text{Spec}(\mathcal{O}_{S,s}^h)$  of a henselian local ring of a point  $s \in S$  is usually not of finite type over  $S$ . Hence, Lemma 1.6 does not apply directly to the canonical morphism  $\mathfrak{s} : S_s^h \rightarrow S$ . Instead, we treat  $S_s^h$  as a cofiltered limit of the diagram  $\underline{D}$  given by the affine Nisnevich neighbourhoods of  $s$  in  $S$  and invoke the following lemma.

**Lemma 1.7.** *Let  $d : D \rightarrow S$  be a noetherian  $S$ -scheme of finite Krull-dimension. Suppose  $d$  is the limit of a cofiltered diagram  $\underline{D} : \mathcal{I} \rightarrow \text{Sm}_S$  with affine transition morphisms, where  $d_i : D_i \rightarrow S$  denotes the structure morphism of each  $D_i := \underline{D}(i)$ . Assume that each  $D_i$  is quasiseparated. Let  $V \rightarrow D$  be an element of  $\text{Sm}_D$ . Then the following statements hold:*

- (1) *There is a cofinal functor  $\mathcal{I}_V \rightarrow \mathcal{I}$ , a cofiltered diagram  $\underline{V} : \mathcal{I}_V \rightarrow \text{Sm}_S$  with affine transition morphisms and a natural transformation  $\underline{V} \rightarrow \underline{D}|_{\mathcal{I}_V}$  inducing  $V \rightarrow D$  on the limit over  $\mathcal{I}_V$  in  $\text{Sch}_S$ .*
- (2) *For  $V \rightarrow \underline{V}$  as in (1) and for each  $F \in \text{sPre}_+(S)$ , the morphism of diagrams*

$\Gamma(\underline{V}, d^*F) \rightarrow \Gamma(V, d^*F)$  induces a canonical natural isomorphism

$$\Gamma(V, d^*F) \cong \operatorname{colim}_{i \in \mathcal{I}_V} \Gamma(V_i, d_i^*F).$$

(3) For  $V \rightarrow \underline{V}$  as in (1) and for each  $F \in \operatorname{sPre}_+(S)$ , there is a canonical natural isomorphism of pointed (objectwise) homotopy classes

$$[V_+, d^*F] \cong \operatorname{colim}_{i \in \mathcal{I}_V} [V_{i+}, d_i^*F].$$

(4) In (1), open embeddings, étale morphisms, smooth morphisms and Nisnevich-distinguished squares in  $\operatorname{Sm}_D$  can be approximated by their sectionwise counterparts in  $\operatorname{Sm}_S$ .

(5) For each  $F \in \operatorname{sPre}_+(S)$ , there are canonical (objectwise) weak equivalences

$$L^S(d^*F) \sim d^*(L^S F) \quad \text{and} \quad L^{\mathbb{A}^1}(d^*F) \sim d^*(L^{\mathbb{A}^1} F)$$

in  $\operatorname{sPre}_+(D)$ .

*Proof.* (1) This follows from [EGA IV<sub>3</sub> 1966, Theorem 8.8.2, Proposition 17.7.8]. In fact, we may (and always will) even assume that  $\mathcal{I}_V = \mathcal{I} \downarrow_{i_0}$  for a suitable object  $i_0 \in \mathcal{I}$  and  $\underline{V} = V_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I} \downarrow_{i_0}}$  for a suitable smooth morphism  $V_{i_0} \rightarrow D_{i_0}$ .

(2) We may assume  $F$  to be simplicially discrete, i.e., a presheaf. As we may write  $F$  as the colimit over representable presheaves, and pullback- as well as section-functors preserve colimits in the category of presheaves, we may assume that  $F$  is representable by a suitable object  $U \rightarrow S$  in  $\operatorname{Sm}_S$ . Then  $d^*F = U \times_S D$  and  $d_i^*F = U \times_S D_i$ , and (2) follows from (1) and [EGA IV<sub>3</sub> 1966, Theorem 8.8.2].

(3) Let us first observe that  $d^*$  preserves objectwise fibrant objects. Indeed, this holds for the  $d_i^*$  by Remark 1.5. Taking sections and applying (2), it suffices to observe that a filtered colimit of fibrant simplicial sets is again fibrant. The assertion of (3) now follows by taking homotopies with respect to the functorial standard cylinder  $(-) \times \Delta^1$ .

(4) Let  $f : V' \rightarrow V$  be an open embedding (resp. an étale or smooth morphism) in  $\operatorname{Sm}_D$ . We apply (1) first to the structural map  $V \rightarrow D$  of the target and then to  $V' \rightarrow V$  itself. We get approximations  $\underline{V}, \underline{V}' : \mathcal{I}_f \rightarrow \operatorname{Sm}_S$  and a natural transformation  $\underline{f} : \underline{V}' \rightarrow \underline{V}$  inducing  $f$  after taking limits. By [EGA IV<sub>3</sub> 1966, Proposition 8.6.3] (resp. [EGA IV<sub>3</sub> 1966, Proposition 17.7.8]) we may assume that  $\underline{f}$  is sectionwise an open embedding (resp. an étale or smooth morphism). As in (1), we may assume  $\underline{f} = f_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I} \downarrow_{i_0}}$ .

Let  $f$  be étale and  $j : U \hookrightarrow V$  an open immersion inducing a Nisnevich-distinguished square. Choose an approximation  $\underline{f} = f_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I} \downarrow_{i_0}}$  of  $f$  as above. By possibly enlarging  $i_0$ , we can find an approximation  $\underline{j} = j_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I} \downarrow_{i_0}}$  of  $j$



by open immersions. We get a levelwise pullback square

$$\begin{array}{ccc}
 \underline{U} \times_{\underline{V}} \underline{V}' & \longrightarrow & \underline{V}' \\
 \downarrow & & \downarrow \underline{f} \\
 \underline{U} & \xrightarrow{j} & \underline{V}
 \end{array}$$

In particular, the sectionwise definition of  $\underline{f}^*(\underline{Z}) \rightarrow \underline{Z} := \underline{V} \setminus \underline{U}$  (sectionwise with the reduced structure) gives a well defined approximation of  $f^*(Z) \rightarrow Z := V \setminus U$ . By [EGA IV<sub>3</sub> 1966, Corollary 8.8.2.4] we may even assume that this approximation is sectionwise an isomorphism, i.e., the above square of approximations is sectionwise a Nisnevich-distinguished square.

(5) Note that the first assertion is equivalent to  $d^*$  preserving Nisnevich-local fibrant objects and that the second assertion is equivalent to  $d^*$  preserving  $\mathbb{A}^1$ -Nisnevich-local fibrant objects. Let  $F \in \text{sPre}_+$  be Nisnevich-local fibrant. We have to show that  $d^*F$  sends Nisnevich-distinguished squares to homotopy pullback squares of simplicial sets. Let  $Q$  be a Nisnevich-distinguished square in  $\text{Sm}_D$ . By (4),  $Q$  may be approximated by a diagram  $\underline{Q}$  of Nisnevich-distinguished squares. By (2), we have  $(d^*F)(Q) \cong \text{colim}(d_i^*F)(Q_i)$ . Again, as the  $d_i^*$  admit Quillen left adjoints for the Nisnevich-local model, it suffices to show that a filtered colimit of homotopy pullback squares of simplicial sets is again a homotopy pullback square. This, in turn, follows from the fact that those colimits preserve categorical pullback squares, fibrations and weak equivalences of simplicial sets. This shows  $L^S(d^*F) \sim d^*(L^S F)$ .

For the second assertion it suffices to show that  $d^*$  preserves  $\mathbb{A}^1$ -invariant simplicial presheaves. This is the case for the  $d_i^*$  as they admit left adjoints  $d_{i,\sharp}$ . The assertion follows directly from (2). □

We need the following glueing property. Let  $S$  be a base scheme of finite Krull dimension and  $i : Z \hookrightarrow S$  a closed subscheme with complementary open immersion  $j : U \hookrightarrow S$ . For a pointed simplicial presheaf  $F \in \text{sPre}_+(S)$ , there is a homotopy cofibre sequence

$$j_{\sharp} j^* F \rightarrow F \rightarrow i_* L_{\mathbb{A}^1} i^* F \tag{1.8}$$

for the *pointed*  $\mathbb{A}^1$ -Nisnevich-local model structure. This fact follows (e.g., by [Hoyois 2017, Section 5.1]) from the unpointed analogue due to Morel and Voevodsky [1999, Theorem 3.2.21] (see also [Ayoub 2006, Theorem 4.5.36]).

For a morphism  $f : R \rightarrow S$  of noetherian schemes of finite Krull dimension, there is also an adjunction

$$f^* : \text{Spt}_{S^1}(S) \rightleftarrows \text{Spt}_{S^1}(R) : f_*$$

on the level of spectra where one defines  $f^*(E)_n := f^*(E_n)$  and  $f_*(D)_n := f_*(D_n)$  with obvious structure maps. If the morphism  $f : R \rightarrow S$  is smooth and of finite type, there is an adjunction

$$f_{\sharp} : \text{Spt}_{S^1}(R) \rightleftarrows \text{Spt}_{S^1}(S) : f^*$$

with  $f_{\sharp}(D)_n := f_{\sharp}(D_n)$  and structure maps given by the projection formula. The adjunction  $(f^*, f_*)$  is a Quillen adjunction for the stable objectwise, the stable Nisnevich-local and the stable  $\mathbb{A}^1$ -Nisnevich-local model structures, respectively. If  $f : R \rightarrow S$  is smooth and of finite type, then the adjunction  $(f_{\sharp}, f^*)$  is a Quillen adjunction for these stable model structures as well and  $f^*$  preserves all stable weak equivalences; see [Ayoub 2007, Theorem 4.5.23].

We have the following analogue of Lemma 1.6 and Lemma 1.7 in the stable setting.

**Lemma 1.9.** *Let  $\underline{D}$  and  $V \rightarrow D$  be as in Lemma 1.7. Let  $f : R \rightarrow S$  be either smooth of finite type or the canonical map  $\lim \underline{D} \rightarrow S$ . Let  $E \in \text{Spt}_{S^1}(S)$  be a spectrum. Then the following statements hold:*

- (1)  $L^{\text{ob}}(f^*E) \sim f^*(L^{\text{ob}}E)$ ,  $L^s(f^*E) \sim f^*(L^sE)$  and  $L^{\mathbb{A}^1}(f^*E) \sim f^*(L^{\mathbb{A}^1}E)$ .
- (2) *With the notation of Lemma 1.7(1), there is a canonical natural isomorphism of pointed (stable objectwise) homotopy classes*

$$[\Sigma_{S^1}^{\infty}(V_+), d^*E] \cong \text{colim}_{i \in \mathcal{I}_V} [\Sigma_{S^1}^{\infty}(V_{i+}), d_i^*E].$$

*Proof.* The assertions follow from (1.4) and the explicit form of a fibrant replacement (1.1) using Lemma 1.6 and Lemma 1.7. □

**Corollary 1.10.** *Let  $E \in \text{Spt}_{S^1}(S)$  be a spectrum. Then the following statements are equivalent:*

- (1) *The homotopy sheaf  $\pi_0(E)$  is trivial.*
- (2) *For all schemes  $V \in \text{Sm}_S$  with structure morphism  $p : V \rightarrow S$  and all points  $v \in V$  with canonical morphism  $\mathfrak{v} : V_v^h := \text{Spec}(\mathcal{O}_{V,v}^h) \rightarrow V$ , the homotopy sheaf  $\pi_0(\mathfrak{v}^*p^*E)$  is trivial.*
- (3) *For all points  $s \in S$  with canonical morphism  $\mathfrak{s} : S_s^h \rightarrow S$ , the homotopy sheaf  $\pi_0(\mathfrak{s}^*E)$  is trivial.*

*Proof.* First suppose (2) holds. We want to show (1), i.e., we have to show that the Nisnevich stalk at  $(V, v)$  of the sheaf  $\pi_0(E)$  is trivial for all such  $(V, v)$ . By (2) of the previous lemma, we get

$$\pi_0(E)_{(V,v)} = \text{colim}_{f:(W,w) \rightarrow (V,v)} [\Sigma_{S^1}^{\infty}(W_+), f^*p^*E] \cong [\Sigma_{S^1}^{\infty}(V_{v,+}^h), \mathfrak{v}^*p^*E],$$

where the colimit runs over the Nisnevich neighbourhoods of  $(V, v)$ . The identity  $\text{id}_{(V_v^h, v)}$  is cofinal in the Nisnevich neighbourhoods of  $(V_v^h, v)$ , so we obtain  $\Sigma_{S^1}^\infty(V_{v,+}^h, \mathfrak{v}^* p^* E) = \pi_0(\mathfrak{v}^* p^* E)_{(V_v^h, v)}$ , which is trivial by assumption.

For the implication (1)  $\Rightarrow$  (2), suppose that  $\pi_0(E) = 0$ . Let  $V_i \rightarrow V$  be the diagram given by the affine Nisnevich neighbourhoods of  $(V, v)$ . In particular, we have  $\lim_i V_i \cong V_v^h$ . By Lemma 1.7(4), every object in  $\text{Sm}_{V_v^h}$  has the form  $W_v^h := W \times_V V_v^h$  for a suitable  $W \in \text{Sm}_V$ . Take a point  $w \in W_v^h$  and let  $w_0$  be its image in  $W$ . Again by Lemma 1.7(4), we find a diagram of étale maps  $W_j \rightarrow W$  such that  $(W_v^h)_w^h \cong \lim_j (W_j)_v^h \cong \lim_j \lim_i W_j \times_V V_i$ . Using Lemma 1.9(2), we compute

$$\begin{aligned} \pi_0(\mathfrak{v}^* p^* E)_{(W_v^h, w)} &\cong \text{colim}_j [\Sigma_{S^1}^\infty((W_j)_v^h, \mathfrak{v}^* p^* E)] \\ &\cong \text{colim}_j \text{colim}_i [\Sigma_{S^1}^\infty((W_j \times_V V_i)_+, p^* E)]. \end{aligned}$$

The pro-object  $\{W_j \times_V V_i\}_{i,j}$  in  $\text{Sm}_S$  induces a Nisnevich point  $\alpha$  of  $\text{Sh}(\text{Sm}_S)$ . Note that this point may not correspond to the henselian scheme  $X_x^h$  for some  $X \in \text{Sm}_S$  and  $x \in X$  but to a subextension of the strict henselization  $W_{\bar{w}_0}^{sh} / W_{w_0}^h$ . The assumption  $\pi_0(E) = 0$  now implies

$$\text{colim}_j \text{colim}_i [\Sigma_{S^1}^\infty((W_j \times_V V_i)_+, p^* E) \cong \alpha(\pi_0(E)) = 0.$$

As a special case we get the implication (3)  $\Rightarrow$  (2). Finally, the reverse implication (2)  $\Rightarrow$  (3) is trivial. □

**$\mathbb{P}^1$ -spectra.** In this subsection, we briefly recall a model for the  $\mathbb{P}^1$ -stable *motivic homotopy category*. As an underlying category of this model structure, we use  $(\mathbb{G}_m, S^1)$ -bispectra, i.e., the category  $\text{Spt}_{\mathbb{G}_m}(\text{Spt}_{S^1}(S))$  of  $\mathbb{G}_m$ -spectra with entries in  $\text{Spt}_{S^1}$ ; see [Hovey 2001, Definition 1.1]. Here, by abuse of notation,  $\mathbb{G}_m$  denotes the  $S^1$ -suspension spectrum of a cofibrant replacement of the pointed object  $(\mathbb{G}_m, 1)$  of  $\text{sPre}_+(S)$ ; see [Morel 2004, Remark 5.1.10]. Again, by an abuse of notation, we abbreviate this category by  $\text{Spt}_{\mathbb{P}^1}(S)$  and call its objects  $\mathbb{P}^1$ -spectra. Similarly to the passage from  $\text{sPre}_+$  to  $S^1$ -spectra, the zeroth entry of a  $\mathbb{P}^1$ -spectrum and the  $\mathbb{G}_m$ -suspension spectrum functor fit into an adjunction

$$\Sigma_{\mathbb{G}_m}^\infty : \text{Spt}_{S^1} \rightleftarrows \text{Spt}_{\mathbb{P}^1} : (-)_0. \tag{1.11}$$

For  $q \geq 0$ , there is also an adjunction  $\langle -q \rangle : \text{Spt}_{\mathbb{P}^1} \rightleftarrows \text{Spt}_{\mathbb{P}^1} : \langle q \rangle$  of shift functors defined for an integer  $q$  by  $E\langle q \rangle_m := E_{q+m}$  whenever  $q + m \geq 0$  and  $E\langle q \rangle_m := *$  otherwise. Again by the general procedure of [Hovey 2001], we equip  $\text{Spt}_{\mathbb{P}^1}$  with the stable model structure induced via (1.11) by the stable  $\mathbb{A}^1$ -Nisnevich-local structure on  $\text{Spt}_{S^1}$  [Hovey 2001, Definition 3.3]. Its homotopy category  $\mathcal{SH}(S)$  is the ( $\mathbb{P}^1$ -stable) *motivic homotopy category*. The two above-mentioned adjunctions turn into Quillen adjunctions for these structures, respectively.

The motivic homotopy category  $\mathcal{SH}$  is a triangulated category with distinguished triangles again given by the homotopy cofibre sequences. Note that here the triangulated shift is again induced by the simplicial shift [1] and not by the  $\mathbb{G}_m$ -shift  $\langle 1 \rangle$ .

Finally, let us mention a concrete fibrant replacement functor for the above model structure on  $\text{Spt}_{\mathbb{P}^1}$ . This is completely analogous to the  $S^1$ -stabilization process from  $\text{sPre}_+$  to  $\text{Spt}_{S^1}$ . Let  $E \in \text{Spt}_{\mathbb{P}^1}$ . By [Hovey 2001, Theorem 4.12], we may use the functor

$$\Theta_{\mathbb{G}_m} E = \text{colim}(E \rightarrow \Omega_{\mathbb{G}_m} E \langle 1 \rangle \rightarrow (\Omega_{\mathbb{G}_m})^2 E \langle 2 \rangle \rightarrow \dots)$$

if each level of  $E$  is already a fibrant spectrum in  $\text{Spt}_{S^1}$ . Otherwise we can first apply the stable  $\mathbb{A}^1$ -Nisnevich-local fibrant replacement functor  $\Theta_{S^1}^{\mathbb{A}^1}$  levelwise. We write  $\Omega_{\mathbb{G}_m}^\infty : \text{Spt}_{\mathbb{P}^1} \rightarrow \text{Spt}_{S^1}$  for the composition of this fibrant replacement functor with  $(-)_0$  from (1.11).

**Preliminaries on t-structures.** We briefly recall the definition of a homological t-structure and basic properties. Details can be found in [Gelfand and Manin 2003].

**Definition 1.12.** A (homological) t-structure on a triangulated category  $\mathcal{D}$  is a pair of full subcategories  $\mathcal{D}_{\leq 0}$  and  $\mathcal{D}_{\geq 0}$  which are closed under isomorphisms in  $\mathcal{D}$  such that the following axioms hold, where for an integer  $n$ , one sets  $\mathcal{D}_{\geq n} := \mathcal{D}_{\geq 0}[n]$  and  $\mathcal{D}_{\leq n} := \mathcal{D}_{\leq 0}[n]$ .

- (1) For all  $X \in \mathcal{D}_{\geq 0}$  and all  $Y \in \mathcal{D}_{\leq -1}$  we have  $\text{hom}_{\mathcal{D}}(X, Y) = 0$ .
- (2)  $\mathcal{D}_{\geq 0}$  is closed under [1] (i.e.,  $\mathcal{D}_{\geq 1} \subseteq \mathcal{D}_{\geq 0}$ ) and dually  $\mathcal{D}_{\leq -1} \subseteq \mathcal{D}_{\leq 0}$ .
- (3) For all  $Y \in \mathcal{D}$  there exists a distinguished  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  with  $X \in \mathcal{D}_{\geq 0}$  and  $Z \in \mathcal{D}_{\leq -1}$ .

Set  $\mathcal{D}_{=n} := \mathcal{D}_{\geq n} \cap \mathcal{D}_{\leq n}$  and call  $\mathcal{D}_{=0}$  the *heart* of the t-structure. A t-structure is called *nondegenerate* if  $\bigcap_{n \geq 0} \mathcal{D}_{\geq n} = \{0\}$  and  $\bigcap_{n \leq 0} \mathcal{D}_{\leq n} = \{0\}$ . A t-structure is called *left complete* if for all  $X \in \mathcal{D}$  the canonical morphism

$$X \rightarrow \text{holim}_{n \rightarrow \infty} X_{\leq n}$$

is an isomorphism. Dually, a t-structure is called *right complete* if for all  $X \in \mathcal{D}$  the canonical morphism  $\text{hocolim}_{n \rightarrow -\infty} X_{\geq n} \rightarrow X$  is an isomorphism.

**Remark 1.13.** The adjunctions

$$\begin{aligned} \text{inclusion} : \mathcal{D}_{\geq n} &\rightleftarrows \mathcal{D} & : \tau_{\geq n} \\ \tau_{\leq n} : \mathcal{D} &\rightleftarrows \mathcal{D}_{\leq n} & : \text{inclusion} \end{aligned}$$

turn  $\mathcal{D}_{\geq n}$  into a coreflective and  $\mathcal{D}_{\leq n}$  into a reflective subcategory of  $\mathcal{D}$ . The counit of the first adjunction is denoted by  $(-)_{\geq n} : \mathcal{D} \rightarrow \mathcal{D}$  and called the *n-skeleton*. The

unit of the second adjunction is denoted by  $(-)\leq_n$  and called the  $n$ -coskeleton. The skeleton and the coskeleton induce a distinguished triangle

$$X_{\geq n} \rightarrow X \rightarrow X_{\leq n-1} \rightarrow (X_{\geq n})[1].$$

**Remark 1.14.** Let  $\mathcal{D}$  be a triangulated category obtained from the homotopy category of a stable model category together with a t-structure. If the t-structure is left complete, then  $\bigcap_{n \geq 0} \mathcal{D}_{\geq n} = \{0\}$ , which can be seen as follows. Take  $X \in \bigcap_{n \geq 0} \mathcal{D}_{\geq n}$  and suppose that  $X \rightarrow \text{holim } X_{\leq n}$  is an isomorphism. The homotopy limit of the diagram

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ X_{\geq n+1} & \xrightarrow{\cong} & X & \longrightarrow & X_{\leq n} \\ \downarrow & & \parallel & & \downarrow \\ X_{\geq n} & \xrightarrow{\cong} & X & \longrightarrow & X_{\leq n-1} \end{array}$$

of triangles is the triangle  $\text{holim } X_{\geq n} \rightarrow X \rightarrow \text{holim } X_{\leq n}$ . Since the homotopy limit of weak equivalences is a weak equivalence, the first morphism  $\text{holim } X_{\geq n} \rightarrow X$  of this triangle is an isomorphism. This implies  $\text{holim } X_{\leq n} \cong 0$  and hence  $X \cong 0$ . In the same way, right completeness implies  $\bigcap_{n \leq 0} \mathcal{D}_{\leq n} = \{0\}$ .

For the converse, consider [Lurie 2017, Proposition 1.2.1.19]: Suppose that  $\mathcal{D}_{\geq 0}$  is stable under countable homotopy products. Then  $\bigcap_{n \geq 0} \mathcal{D}_{\geq n} = \{0\}$  implies left completeness. Dually, if  $\mathcal{D}_{\leq 0}$  is stable under countable homotopy coproducts, the relation  $\bigcap_{n \leq 0} \mathcal{D}_{\leq n} = \{0\}$  implies right completeness.

**Proposition 1.15** [Ayoub 2007, Proposition 2.1.70]. *Let  $\mathcal{D}$  be a triangulated category with coproducts and let  $\mathcal{S}$  be a set of compact objects of  $\mathcal{D}$ . Define*

- $\mathcal{D}_{\leq -1}$  as the full subcategory of those  $Y$  of  $\mathcal{D}$  with  $\text{hom}_{\mathcal{D}}(\mathcal{S}[n], Y) = 0$  for all  $n \geq 0$  and all  $S \in \mathcal{S}$ ,
- $\mathcal{D}_{\geq 0}$  as the full subcategory of those  $X$  of  $\mathcal{D}$  with  $\text{hom}_{\mathcal{D}}(X, Y) = 0$  for all  $Y \in \mathcal{D}_{\leq -1}$ .

The pair  $\mathcal{D}_{\leq 0} = \mathcal{D}_{\leq -1}[1]$  and  $\mathcal{D}_{\geq 0}$  forms a t-structure. The category  $\mathcal{D}_{\geq 0}$  is the full subcategory of  $\mathcal{D}$  generated under extensions, (small) sums and cones from  $\mathcal{S}$  and in particular  $\mathcal{S} \subseteq \mathcal{D}_{\geq 0}$ . Moreover, the truncation functor  $\tau_{\leq -1}$  is given by  $\tau_{\leq -1}(X) := \text{hocolim}_{k \rightarrow \infty} \Phi^k(X)$ , where  $\Phi(X)$  is defined as the cone

$$\coprod_{\substack{\text{Hom}(\mathcal{S}[n], X) \\ S \in \mathcal{S}, n \geq 0}} \mathcal{S}[n] \rightarrow X \rightarrow \Phi(X).$$

**Remark 1.16.** Let  $\mathcal{D}$  be a triangulated category obtained from the homotopy category of a stable model category and let  $\mathcal{S}$  be a set of compact objects of  $\mathcal{D}$ . The

t-structure obtained from the previous [Proposition 1.15](#) satisfies the property that  $\mathcal{D}_{\leq 0}$  is stable under countable homotopy coproducts. If  $\mathcal{D}$  has an underlying cofibrantly generated model category and  $\mathcal{S}$  equals (up to shifts) the set of cofibres of the generating cofibrations, then  $\bigcap_{n \leq 0} \mathcal{D}_{\leq n} = \{0\}$  by [[Hovey 1999](#), Theorem 7.3.1] and  $\mathcal{D}$  is right complete by [Remark 1.14](#). It is however usually a nontrivial issue to show left completeness of a t-structure obtained from [Proposition 1.15](#).

**Canonical t-structures on  $S^1$ -spectra.** In this subsection we recall some basic properties of canonical t-structures on  $S^1$ -spectra arising in  $\mathbb{A}^1$ -homotopy theory.

**Definition 1.17.** Consider the set  $\mathcal{S} := \{\Sigma_{S^1}^\infty U_+ \mid U \in \text{Sm}_{\mathcal{S}}\}$ . The *objectwise t-structure* (resp. *Nisnevich-local t-structure*,  *$\mathbb{A}^1$ -Nisnevich-local t-structure*) on  $\mathcal{S}\mathcal{H}_{S^1}^{\text{ob}}$  (resp.  $\mathcal{S}\mathcal{H}_{S^1}^s$ ,  $\mathcal{S}\mathcal{H}_{S^1}^{\mathbb{A}^1}$ ) is obtained by applying [Proposition 1.15](#) to the triangulated category  $\mathcal{S}\mathcal{H}_{S^1}^{\text{ob}}$  (resp.  $\mathcal{S}\mathcal{H}_{S^1}^s$ ,  $\mathcal{S}\mathcal{H}_{S^1}^{\mathbb{A}^1}$ ) and to  $\mathcal{S}$ .

**Remark 1.18.** In [[Morel 2005](#)] the Nisnevich-local t-structure on  $\mathcal{S}\mathcal{H}_{S^1}^s$  is called the *standard t-structure*. In [[Morel 2004](#), Section 4.3] the  $\mathbb{A}^1$ -Nisnevich-local t-structure on  $\mathcal{S}\mathcal{H}_{S^1}^{\mathbb{A}^1}$  is called the *homotopy t-structure* (on  $S^1$ -spectra).

**Remark 1.19.** By definition we have

$$\mathcal{S}\mathcal{H}_{S^1 \leq -1}^{\text{ob}} = \{Y \in \mathcal{S}\mathcal{H}_{S^1}^{\text{ob}} \mid Y \text{ has trivial homotopy presheaves } [\Sigma_{S^1}^\infty(-)_+[i], Y] \text{ for all } i \geq 0\}.$$

Applying the classical [[Margolis 1983](#), Proposition 3.6] objectwise, we get

$$\mathcal{S}\mathcal{H}_{S^1 \geq 0}^{\text{ob}} = \{X \in \mathcal{S}\mathcal{H}_{S^1}^{\text{ob}} \mid X \text{ has trivial homotopy presheaves } [\Sigma_{S^1}^\infty(-)_+[i], X] \text{ for all } i \leq -1\}.$$

The objectwise t-structure is clearly nondegenerate as there are no nonzero spectra without nontrivial homotopy presheaves. The objectwise t-structure is right complete by [Remark 1.16](#) and left complete by [Remark 1.14](#) as  $\mathcal{S}\mathcal{H}_{S^1 \geq 0}^{\text{ob}}$  is stable under countable homotopy products.

**Remark 1.20.** Again, by definition we have

$$\mathcal{S}\mathcal{H}_{S^1 \leq -1}^s = \{Y \in \mathcal{S}\mathcal{H}_{S^1}^s \mid Y \text{ has trivial homotopy presheaves } [\Sigma_{S^1}^\infty(-)_+[i], L^s Y] \text{ for all } i \geq 0\}$$

and using on Nisnevich stalks the classical result [[Margolis 1983](#), Proposition 3.6], we get

$$\begin{aligned} \mathcal{S}\mathcal{H}_{S^1 \geq 0}^s &= \{X \in \mathcal{S}\mathcal{H}_{S^1}^s \mid X \text{ has trivial homotopy sheaves } \pi_i^s X \text{ for all } i \leq -1\}, \\ \mathcal{S}\mathcal{H}_{S^1 \leq -1}^s &= \{Y \in \mathcal{S}\mathcal{H}_{S^1}^s \mid Y \text{ has trivial homotopy sheaves } \pi_i^s Y \text{ for all } i \geq 0\}. \end{aligned}$$

The Nisnevich-local t-structure is clearly nondegenerate as all nonzero spectra have at least one nontrivial homotopy sheaf. The Nisnevich-local t-structure is right complete by [Remark 1.16](#) and left complete by, e.g., [[Spitzweck 2014](#), Lemma 4.4].

**Remark 1.21.** A Nisnevich-local fibrant replacement functor  $L^s$  respects only the truncation from above, i.e., if  $E$  is in  $\mathcal{SH}_{S^1 \leq -1}^{\text{ob}}$ , then the spectrum  $L^s E$  is in  $\mathcal{SH}_{S^1 \leq -1}^s$ . The analogous statement is not true for the positive part  $\mathcal{SH}_{S^1 \geq 0}^{\text{ob}}$ , which can be seen as follows: By Hilbert’s Theorem 90, there is an isomorphism  $\text{Pic}(X) \cong H_{\text{Nis}}^1(X, \mathbb{G}_m)$ . The Eilenberg–Mac Lane spectrum  $H\mathbb{G}_m$  is in the heart of the objectwise t-structure but

$$H_{\text{Nis}}^1(X, \mathbb{G}_m) = [\Sigma_{S^1}^\infty X_+, L^s H\mathbb{G}_m[1]] = [\Sigma_{S^1}^\infty X_+[-1], L^s H\mathbb{G}_m]$$

and certainly there are schemes  $X$  with nontrivial Picard group.

**Remark 1.22.** By definition and [Remark 1.20](#), one has

$$\begin{aligned} \mathcal{SH}_{S^1 \leq -1}^{\mathbb{A}^1} &= \{Y \in \mathcal{SH}_{S^1}^{\mathbb{A}^1} \mid Y \text{ has trivial homotopy presheaves} \\ &\quad [\Sigma_{S^1}^\infty(-)_+[i], L^{\mathbb{A}^1} Y] \text{ for all } i \geq 0\} \\ &= \{Y \in \mathcal{SH}_{S^1}^{\mathbb{A}^1} \mid Y \text{ has trivial homotopy sheaves } \pi_i^{\mathbb{A}^1} Y \text{ for all } i \geq 0\}. \end{aligned}$$

The  $\mathbb{A}^1$ -Nisnevich-local t-structure is right complete by [Remark 1.16](#) and we have  $\bigcap_{n \leq 0} \mathcal{SH}_{S^1 \leq n}^{\mathbb{A}^1} = \{0\}$ . It will be shown in [Corollary 3.6](#) that the  $\mathbb{A}^1$ -Nisnevich-local t-structure is left complete and hence nondegenerate.

**Definition 1.23.** We define

$$\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1, \pi} := \{X \in \mathcal{SH}_{S^1}^{\mathbb{A}^1} \mid X \text{ has trivial homotopy sheaves } \pi_i^{\mathbb{A}^1} X \text{ for all } i \leq -1\}.$$

**Remark 1.24.** The full subcategory  $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1, \pi}$  in  $\mathcal{SH}_{S^1}^{\mathbb{A}^1}$  is closed under homotopy colimits and extensions. There is an inclusion  $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1, \pi} \subseteq \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1}$  due to [[Spitzweck 2014](#), Lemmas 4.1 and 4.3]. Conversely, the other implication  $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1} \subseteq \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1, \pi}$  holds if and only if  $L^{\mathbb{A}^1} \Sigma_{S^1}^\infty U_+ \in \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1, \pi}$  for all  $U \in \text{Sm}_S$ . Unfortunately, there are schemes  $S$  such that these two equivalent conditions do not hold (see [Remark 4.4](#)). However, they hold true over the spectrum of a field  $S$  [[Morel 2005](#), Theorem 6.1.8] and we have  $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1} = \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1, \pi}$  in that case.

**Homotopy t-structures on  $\mathbb{P}^1$ -spectra.** In this subsection we recall the homotopy t-structure on the motivic homotopy category  $\mathcal{SH}$ . We remind the reader that  $\langle q \rangle$  denotes the  $\mathbb{G}_m$ -shift operation.

**Definition 1.25.** The *homotopy t-structure* on  $\mathcal{SH}$  is the t-structure obtained by applying [Proposition 1.15](#) to the triangulated category  $\mathcal{SH}$  and the set

$$\mathcal{S} = \{\Sigma_{\mathbb{P}^1}^\infty(U_+) \langle q \rangle \mid U \in \text{Sm}_S \text{ and } q \in \mathbb{Z}\}.$$

**Remark 1.26.** We use the name “homotopy t-structure” in order to agree with the terminology of [Morel 2004; 2005].

**Remark 1.27.** Unravelling the definitions, one identifies

$$\begin{aligned} \mathcal{SH}_{\leq -1} &= \{Y \in \mathcal{SH} \mid \Omega_{\mathbb{G}_m}^\infty(Y\langle q \rangle) \in \mathcal{SH}_{S^1 \leq -1}^{\mathbb{A}^1} \text{ for all } q \in \mathbb{Z}\} \\ &= \{Y \in \mathcal{SH} \mid (\operatorname{colim}_k \Omega_{\mathbb{G}_m}^k Y_{k+q}) \in \mathcal{SH}_{S^1 \leq -1}^{\mathbb{A}^1} \text{ for all } q \in \mathbb{Z}\}. \end{aligned}$$

In particular,

$$\Omega_{\mathbb{G}_m}^\infty(\mathcal{SH}_{\leq -1}) \subseteq \mathcal{SH}_{S^1 \leq -1}^{\mathbb{A}^1} \quad \text{and} \quad \Sigma_{\mathbb{G}_m}^\infty(\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1}) \subseteq \mathcal{SH}_{\geq 0},$$

using [Ayoub 2006, Lemma 2.1.16] for the latter.

**Remark 1.28.** Over a field, using [Morel 2004, Lemma 4.3.11] and the equality  $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1} = \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1, \pi}$  from Remark 1.24, we can also identify

$$\mathcal{SH}_{\geq 0} = \{X \in \mathcal{SH} \mid \Omega_{\mathbb{G}_m}^\infty(X\langle q \rangle) \in \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1} \text{ for all } q \in \mathbb{Z}\};$$

cf. [Morel 2004, Section 5.2]. In particular, we have  $\Omega_{\mathbb{G}_m}^\infty(\mathcal{SH}_{\geq 0}) \subseteq \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1}$  in this case.

**Remark 1.29.** The homotopy t-structure on the motivic homotopy category is right complete by Remark 1.16 and we have  $\bigcap_{n \leq 0} \mathcal{SH}_{\leq n} = \{0\}$ . It will be shown in Corollary 3.8 that the homotopy t-structure on the motivic homotopy category is also left complete and hence nondegenerate.

## 2. Gabber presentations over henselian discrete valuation rings

Throughout this section, fix a henselian discrete valuation ring  $\mathfrak{o}$  with maximal ideal  $\mathfrak{m} \trianglelefteq \mathfrak{o}$ , local uniformizer  $\pi \in \mathfrak{m}$ , residue field  $\mathbb{F} = \mathfrak{o}/\mathfrak{m}$  and field of fractions  $k$ . Assume that  $\mathbb{F}$  is an infinite field. Let  $S$  be the spectrum of  $\mathfrak{o}$  and denote by  $\sigma$  the closed point of  $S$  and by  $\eta$  the generic point of  $S$ . We want to prove the following version of Gabber’s geometric presentation lemma over  $\mathfrak{o}$ .

**Theorem 2.1.** *Let  $\mathfrak{o}$  be a henselian discrete valuation ring with infinite residue field. Let  $X/\mathfrak{o}$  be a smooth  $\mathfrak{o}$ -scheme of finite type and let  $Z \hookrightarrow X$  be a proper closed subscheme. Let  $z$  be a point in  $Z$ . If  $z$  lies in the special fibre, suppose that  $Z_\sigma \neq X_\sigma$ . Then, Nisnevich-locally around  $z$ , there exists a smooth  $\mathfrak{o}$ -scheme  $V$  of finite type and a cartesian square*

$$\begin{array}{ccc} X \setminus Z & \longrightarrow & X \\ \downarrow & & \downarrow p \\ \mathbb{A}_V^1 \setminus p(Z) & \longrightarrow & \mathbb{A}_V^1 \end{array}$$



such that  $p$  is étale, the restriction  $p|_Z : Z \hookrightarrow \mathbb{A}_V^1$  is a closed subscheme and  $Z$  is finite over  $V$ . In particular, this square is a Nisnevich-distinguished square and therefore, the induced canonical morphism  $X/(X \setminus Z) \rightarrow \mathbb{A}_V^1/(\mathbb{A}_V^1 \setminus p(Z))$  is an isomorphism of Nisnevich sheaves.

**Remark 2.2.** The essential case of [Theorem 2.1](#) is that of an effective Cartier-divisor  $Z \hookrightarrow X$  (see the proof of [Theorem 2.4](#) below). Earlier results for relative effective Cartier-divisors  $Z \hookrightarrow X$  over discrete valuation rings are [[Gillet and Levine 1987](#), Lemma 1] and [[Dutta 1995](#), Theorem 3.4]. These results are even Zariski local and do not assume infinite residue fields. However, they do not include an analogue for the crucial finiteness claim of  $Z/V$ .

The map  $p$  in [Theorem 2.1](#) will be provided by a careful choice of suitable linear projections. Before we give a short outline of the proof, let us first recall some basic facts about linear projections.

**Linear projections.** Denote by  $\mathbb{A}_{x_1/x_0, \dots, x_N/x_0, S} = \mathbb{A}_{\underline{x}/x_0, S}$  the affine  $N$ -space  $\mathbb{A}_S^N$  with coordinates  $x_1/x_0, \dots, x_N/x_0$  and by  $\mathbb{P}_{x_0: \dots : x_N, S} = \mathbb{P}_{\underline{x}, S}$  the projective  $N$ -space of  $\mathbb{P}_S^N$  with homogeneous coordinates  $x_0 : \dots : x_N$ . We get the standard open embedding  $\mathbb{A}_{\underline{x}/x_0, S} \hookrightarrow \mathbb{P}_{\underline{x}, S}$ . By abuse of notation, we identify  $x_1/x_0, \dots, x_N/x_0$  with  $x_1, \dots, x_N$  and write just  $\mathbb{A}_{\underline{x}, S}$  for  $\mathbb{A}_{\underline{x}/x_0, S}$  (and similarly for other coordinates).

Let  $\mathbb{A}_{x_1, \dots, x_N, S}^\vee = \mathbb{A}_{x_1^*, \dots, x_N^*, S}$  be the dual affine space, i.e., for any  $\sigma$ -algebra  $A$ ,  $\mathbb{A}_{x_1, \dots, x_N, S}^\vee(A)$  is the free  $A$ -module generated by the coordinate functions  $x_1, \dots, x_N$  of  $\mathbb{A}_{\underline{x}, S}$ . Dually, we can view  $\mathbb{A}_{\underline{x}, S}(A)$  as the free  $A$ -module generated by the dual coordinate functions  $x_1^*, \dots, x_N^*$ . To be more precise, take  $r$  copies of  $\mathbb{A}_{\underline{x}, S}^\vee$  and denote the  $j$ -th copy by  $\mathbb{A}_{\underline{x}, S}^{\vee, (j)}$  with coordinate functions  $x_{i,j}^* := x_i^*$  for  $1 \leq i \leq N$ . Mapping  $t_j \mapsto \sum_i x_i \otimes x_{i,j}^*$  defines the dual pairing  $\langle -, - \rangle : \mathbb{A}_{\underline{x}, S} \times_S \mathbb{A}_{\underline{x}, S}^{\vee, (j)} \rightarrow \mathbb{A}_{t_j, S}$ . Via this pairing, each  $A$ -point  $u$  of  $\mathbb{A}_{\underline{x}, S}^{\vee, (j)}$  induces a linear  $A$ -morphism (in abuse of notation also denoted by)  $u$  defined as the composition

$$\mathbb{A}_{\underline{x}, A} \xrightarrow{\text{id} \times u} \mathbb{A}_{\underline{x}, A} \times_A \mathbb{A}_{\underline{x}, A}^{\vee, (j)} \xrightarrow{\langle -, - \rangle} \mathbb{A}_{t_j, A} \quad \text{via} \quad t_j \mapsto \sum_i \langle x_i^*, u \rangle x_i.$$

This map is precisely the linear form given by the  $A$ -point  $u$  seen as the corresponding linear combination of the  $x_i$  in  $\mathbb{A}_{\underline{x}, S}^{\vee, (j)}(A) = \bigoplus_i A x_i$ . We define

$$\mathfrak{E}_r := \mathbb{A}_{\underline{x}, S}^{\vee, (1)} \times_S \dots \times_S \mathbb{A}_{\underline{x}, S}^{\vee, (r)}$$

and look at it as the space of linear projections  $\mathbb{A}_{\underline{x}, S} \rightarrow \mathbb{A}_{t_1, \dots, t_r, S}$ . Indeed, each  $A$ -point  $\underline{u}$  of  $\mathfrak{E}_r$  induces a linear  $A$ -morphism

$$\underline{u} : \mathbb{A}_{\underline{x}, A} \rightarrow \mathbb{A}_{t_1, \dots, t_r, A} \quad \text{via} \quad t_j \mapsto \sum_i \langle x_i^*, u_j \rangle x_i.$$

Mapping  $t_0 \mapsto x_0$ , this extends to a rational map

$$\underline{u} : \mathbb{P}_{\underline{x}, A} \dashrightarrow \mathbb{P}_{t_0: \dots : t_r, A}$$

with locus of indeterminacy  $L_{\underline{u}} := V_+(x_0, u_1, \dots, u_r) \subseteq H_\infty$ , where  $u_j$  is the linear form in the coordinates  $x_1, \dots, x_N$  corresponding to the  $j$ -th component of  $\underline{u}$  and  $H_\infty \subset \mathbb{P}_{x,A}$  is the hyperplane at infinity  $V_+(x_0)$ .

Assume  $Y \hookrightarrow \mathbb{A}_{x,A}$  is a (reduced) closed subscheme with (reduced) projective closure  $\bar{Y} \hookrightarrow \mathbb{P}_{x,A}$  such that  $\bar{Y} \cap L_{\underline{u}} = \emptyset$ . Then  $\underline{u}$  induces regular maps

$$p_{\underline{u}} : Y \rightarrow \mathbb{A}_{t_1, \dots, t_r, A} \quad \text{and} \quad \bar{p}_{\underline{u}} : \bar{Y} \rightarrow \mathbb{P}_{t_0, \dots, t_r, A}$$

satisfying  $p_{\underline{u}} = \bar{p}_{\underline{u}} \times_{\mathbb{P}_{t,A}} \mathbb{A}_{t,A}$ . Observe that [Shafarevich 1994, Theorem I.5.3.7] remains true in our setting:

**Lemma 2.3.** *For any  $\underline{u} \in \mathfrak{E}_r(A)$  and any closed  $Y \hookrightarrow \mathbb{A}_{x,S}$  with  $\bar{Y} \cap L_{\underline{u}} = \emptyset$ , the linear projections  $p_{\underline{u}}$  and  $\bar{p}_{\underline{u}}$  are finite maps.*

*Proof.* It suffices to show that  $\bar{p}_{\underline{u}}$  is finite. As a map between projective schemes over  $S$ ,  $\bar{p}_{\underline{u}}$  itself is projective. It remains to show that  $\bar{p}_{\underline{u}}$  is quasifinite. Let  $\bar{\sigma}$  be a geometric point of  $S$  over  $\sigma$ , and likewise  $\bar{\eta}$  over  $\eta$ . By [Shafarevich 1994, Theorem I.5.3.7],  $\bar{\sigma}^* \bar{p}_{\underline{u}}$  and  $\bar{\eta}^* \bar{p}_{\underline{u}}$  are finite. It follows that  $\bar{p}_{\underline{u}}$  is finite on the special fibre  $\sigma^* \bar{p}_{\underline{u}}$  and the geometric fibre  $\eta^* \bar{p}_{\underline{u}}$ , hence quasifinite.  $\square$

**Outline of the proof of Theorem 2.1.** The proof of Theorem 2.1 principally follows the proof of Gabber's geometric presentation lemma over fields in [Colliot-Thélène et al. 1997]. The crucial part of Theorem 2.1 turns out to be the finiteness claim for  $Z/V$ . We make the Ansatz  $p = p_{\underline{u}}$  for a closed embedding  $i : X \hookrightarrow \mathbb{A}_{x_1, \dots, x_N, S}$  and a linear projection  $\underline{u} \in \mathfrak{E}_n(\mathfrak{o})$ , for  $n$  the relative dimension of  $X/S$ . Using Lemma 2.3, one can show that the property

*the induced map  $p_{(u_1, \dots, u_{n-1})}|_Z : Z \rightarrow \mathbb{A}_{t_1, \dots, t_{n-1}, S}$  is finite*

is open in our space of linear projections  $\mathfrak{E}_n$  (see the proof of Lemma 2.11, below). If  $W \hookrightarrow \mathfrak{E}_n$  is the corresponding open locus, we first need to make sure that the special fibre  $W_\sigma$  is nonempty — because the residue field  $\mathbb{F}$  is infinite and  $\mathfrak{E}_n$  isomorphic to an affine space,  $W(\mathfrak{o})$  is automatically nonempty in this case. We will see in the proof of Lemma 2.11 that  $W_\sigma$  is nonempty, as soon as  $X$  over  $S$  is fibrewise dense inside its closure in  $\mathbb{P}_{x_0, \dots, x_N, S}$ . In fact, we just need this closure to be dense on the special fibre, but this obviously is equivalent to fibrewise density. Therefore, special care needs to be taken about the choice of our initial closed embedding  $i : X \hookrightarrow \mathbb{A}_{x,S}$ . In Proposition 2.6 we will provide a closed embedding of this type, but the price to pay is that we need to replace  $(X, z)$  by a suitable Nisnevich neighbourhood.

Let  $F = F_{\underline{u}}$  denote the set of preimages under  $p_{(u_1, \dots, u_{n-1})}$  of  $p_{(u_1, \dots, u_{n-1})}(z)$  in  $Z$ . The next goal is to find  $\underline{u}$  in  $W(\mathfrak{o})$  such that  $p_{\underline{u}}$  is étale around each point of  $F$  and the restriction of  $p_{\underline{u}}$  to  $F$  is universally injective. This again corresponds to nonempty open conditions in (the special fibre of)  $\mathfrak{E}_n$  (see the proof

of Lemma 2.12). Shrinking  $W$  accordingly, we may assume that this is the case for all  $\underline{u} \in W(\mathfrak{o})$  (see Proposition 2.9).

Making use of the finiteness of  $p_{(u_1, \dots, u_{n-1})}|_Z$ , we get an open neighbourhood  $V = V_{\underline{u}}$  of  $p_{(u_1, \dots, u_{n-1})}(z)$  in  $\mathbb{A}_{t_1, \dots, t_{n-1}, S}$  such that  $p_{\underline{u}}$  is étale at all points in the intersection  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  and the restriction of  $p_{\underline{u}}$  induces a closed embedding  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V) \hookrightarrow \mathbb{A}_V^1$  (see Lemma 2.13). Finally, replacing  $(X, z)$  by a suitable Zariski-neighbourhood, we establish in Lemma 2.14 the remaining properties claimed in Theorem 2.1.

Summing up, we will in fact show slightly more, proving the following version of [Colliot-Thélène et al. 1997, Theorem 3.2.2] over  $\mathfrak{o}$ .

**Theorem 2.4.** *Let  $X = \text{Spec}(A)/S$  be a smooth affine  $S$ -scheme of finite type, fibrewise of pure dimension  $n$  and let  $Z = \text{Spec}(B) \hookrightarrow X$  be a proper closed subscheme. Let  $z$  be a point in  $Z$ . If  $z$  lies in the special fibre, suppose that  $Z_\sigma \neq X_\sigma$ . Then, Nisnevich-locally around  $z$ , there exists a closed embedding  $X \hookrightarrow \mathbb{A}_S^n$  and a Zariski-open subset  $W \subseteq \mathfrak{E}_n$  with  $W(\mathfrak{o}) \neq \emptyset$  such that the following holds:*

*For all  $\underline{u} \in W(\mathfrak{o})$  with linear projections*

$$p_{\underline{u}} = p_{(u_1, \dots, u_{n-1})} \times_S p_{u_n} : X \rightarrow \mathbb{A}_S^n = \mathbb{A}_S^{n-1} \times_S \mathbb{A}_S^1,$$

*there are Zariski-open neighbourhoods  $V \subseteq \mathbb{A}_S^{n-1}$  containing  $p_{(u_1, \dots, u_{n-1})}(z)$  and  $U \subseteq p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  containing  $z$  satisfying.*

- (1)  $p_{(u_1, \dots, u_{n-1})}|_Z : Z \rightarrow \mathbb{A}_S^{n-1}$  is finite,
- (2)  $Z \cap U = Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)$ ,
- (3)  $p_{\underline{u}}|_U : U \rightarrow \mathbb{A}_S^n$  is étale and restricts to a closed embedding  $Z \cap U \hookrightarrow \mathbb{A}_V^1$  and
- (4)  $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z \cap U)) \cap U = Z \cap U$ .

The proof of Theorem 2.4 follows the proof in [Colliot-Thélène et al. 1997] and the outline sketched above.

*Proof.* Clearly, we may assume that both  $X$  and  $Z$  are connected. Next, observe that the case of  $z$  lying in the generic fibre  $X_\eta$  of  $X/S$  is already covered by [Colliot-Thélène et al. 1997, Theorem 3.2.2]. Thus, we may assume that  $z$  lies in the special fibre  $X_\sigma$  of  $X/S$ . Finally, observe that we may enlarge  $Z$ . In particular, picking any element  $f$  in the kernel of  $A \rightarrow B$  with  $f \neq 0$  in the special fibre  $A \otimes_{\mathfrak{o}} \mathbb{F}$ , we may assume  $B = A/f$ , i.e.,  $Z = V(f)$ .

We follow the outline of the proof sketched above: Up to a refinement by a suitable Nisnevich neighbourhood  $(X', z') \rightarrow (X, z)$ , Proposition 2.6 provides a closed embedding  $i_0 : X' \hookrightarrow \mathbb{A}_{x_0, \dots, x_N, S}$  such that  $Z' = Z \times_X X'$  is fibrewise dense in its Zariski-closure  $\bar{Z}'$  in  $\mathbb{P}_{x_0, \dots, x_N, S}$ . Replacing our base-point  $z$  by a specialization to a closed point  $z_0$  in the image of  $X' \rightarrow X$ , we can assume that  $z$  is closed itself (see Reduction 2.8). Further, we replace  $(X, z)$  by  $(X', z')$ , i.e., we assume

$X' = X$ . Composing the closed embedding  $i_0 : X \hookrightarrow \mathbb{A}_{\underline{x}, S}$  with a linear projection  $\mathbb{A}_{\underline{x}, S} \rightarrow \mathbb{A}_{t_1, \dots, t_n, S}$ , corresponding to an  $\mathfrak{o}$ -point  $\underline{u}$  of the space of linear projections  $\mathfrak{E}_n$ , we get maps

$$\begin{array}{ccc} X & \xrightarrow{p_{\underline{u}}} & \mathbb{A}_{t_1, \dots, t_n, S} \\ & \searrow p_{(u_1, \dots, u_{n-1})} & \downarrow \text{pr} \\ & & \mathbb{A}_{t_1, \dots, t_{n-1}, S} \end{array}$$

Here  $n$  is the dimension of  $X$ . [Proposition 2.9](#) provides an open  $W$  in our space of linear projections  $\mathfrak{E}_n$  with  $W(\mathfrak{o})$  nonempty, and for each linear projection  $\underline{u}$  in  $W(\mathfrak{o})$  the restriction  $p_{(u_1, \dots, u_{n-1})}|_Z$  is finite (i.e., part (1) in [Theorem 2.4](#)),  $p_{\underline{u}}$  is étale around

$$F = F_{\underline{u}} = p_{(u_1, \dots, u_{n-1})}^{-1}(p_{(u_1, \dots, u_{n-1})}(z)) \cap Z$$

and  $p_{\underline{u}}|_F : F \rightarrow p_{\underline{u}}(F)$  is universally injective.

Fix any such  $\underline{u}$  in  $W(\mathfrak{o})$ . In [Lemma 2.13](#), we will replace  $\mathbb{A}_{t_1, \dots, t_{n-1}, S}$  by a Zariski-neighbourhood  $V = V_{\underline{u}}$  of  $p_{(u_1, \dots, u_{n-1})}(z)$  such that  $p_{\underline{u}}$  is étale around every point of  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  and such that the induced restriction

$$p_{\underline{u}}|_{Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)} : Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V) \hookrightarrow \mathbb{A}_{t_n, V}$$

is a closed embedding. In [Lemma 2.14](#), we will shrink  $p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  to a Zariski-neighbourhood  $U_1$  of  $z$  satisfying the analogue of (4) in [Theorem 2.4](#), i.e.,

$$p_{\underline{u}}^{-1}(p_{\underline{u}}(Z \cap U_1)) \cap U_1 = Z \cap U_1,$$

without changing  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V) = Z \cap U_1$ . In particular,  $p_{\underline{u}}$  restricts to a closed embedding  $Z \cap U_1 \rightarrow \mathbb{A}_{t_n, V}$ . Since  $p_{\underline{u}}$  is étale already around every point of  $Z \cap U_1$ , we may shrink  $U_1$  a bit more (by intersecting it with the open étale locus of  $p_{\underline{u}}$ ) to get the desired Zariski-neighbourhood  $U = U_{\underline{u}}$  of  $(X, z)$  satisfying (2), (3) and (4) in [Theorem 2.4](#).  $\square$

**Towards the finiteness part.** The key part in the proof of [Theorem 2.4](#) is the finiteness assertion (1). By [Lemma 2.3](#), we need to find a closed embedding  $i_0 : X \hookrightarrow \mathbb{A}_{\underline{x}, S}$  and an  $\mathfrak{o}$ -point  $\underline{u} \in \mathfrak{E}_n(\mathfrak{o})$  such that the closure  $\bar{Z}$  of  $Z$  in  $\mathbb{P}_{\underline{x}, S}$  intersects  $L_{(u_1, \dots, u_{n-1})}$  trivially. Unfortunately, it is not enough to require that the fibrewise closure of  $Z$  in  $\mathbb{P}_{\underline{x}, S}$  misses  $L_{(u_1, \dots, u_{n-1})}$ . Indeed,  $\bar{Z}$  might not be the fibrewise projective closure of  $Z$  over  $S$ —the special fibre  $\bar{Z}_{\sigma}$  might be strictly larger than the projective closure of  $Z_{\sigma}$ , as seen in the next example.

**Example 2.5.** Let  $A = \mathfrak{o}[x_1]$  and  $B = \mathfrak{o}[x_1]/(\pi x_1^2 + x_1 + 1)$ . Then  $\text{Spec}(B) \subset \mathbb{P}_{x_0: x_1, S}$  is fibrewise closed but at least one solution of  $\pi x_1^2 + x_1 + 1$  in  $k^{\text{alg}}$  specializes to  $\infty$  in  $\mathbb{P}_{x_0: x_1, \mathbb{F}}$ , i.e.,  $\text{Spec}(B) \subset \mathbb{P}_{x_0: x_1, S}$  is not closed.

To avoid these difficulties, we need to make a careful choice for the embedding  $i_0 : X \hookrightarrow \mathbb{A}_{x,S}$ .

**Proposition 2.6.** *Nisnevich-locally around  $z$ , there exists a closed embedding*

$$i_0 : X \hookrightarrow \mathbb{A}_{x,S}$$

with  $Z$  fibrewise dense over  $S$  inside its closure  $\bar{Z}$  in  $\mathbb{P}_{x,S}$ .

*Proof.* We need to adapt [Kai 2015, Theorem 4.6] to our situation. Since  $z$  lies in the special fibre and  $\mathfrak{o}$  is henselian, [loc. cit.] gives us an affine Nisnevich neighbourhood  $(Z', z') \rightarrow (Z, z)$  and a closed embedding  $\bar{i}_0 : Z' \hookrightarrow \mathbb{A}_{x_1, \dots, x_m, S}$ , such that  $Z'$  is fibrewise dense over  $S$  inside its closure  $\bar{Z}'$  in  $\mathbb{P}_{x_0, \dots, x_m, S}$ . Since both  $Z$  and  $Z'$  are affine, the underlying étale morphism  $Z' \rightarrow Z$  is standard smooth, i.e.,  $Z' = \text{Spec}(B')$  with  $B' = B[t_1, \dots, t_s]/(\bar{g}_1, \dots, \bar{g}_s)$  and invertible Jacobi-determinant  $\det(\{\partial \bar{g}_i / \partial t_j\}_{i,j}) \in B'^{\times}$ .

We want to extend  $(Z', z')$  to a Nisnevich neighbourhood  $(X', z')$  of  $(X, z)$ . Since the Jacobi-determinant  $\det(\{\partial \bar{g}_i / \partial t_j\}_{i,j})$  is invertible in  $B'$ , it is nontrivial in  $B' \otimes k(z')$ . Choose a lift  $g_i \in A[\underline{t}]$  for each  $\bar{g}_i$  and set  $A' := A[\underline{t}]/(g_1, \dots, g_s)$  and  $X' = \text{Spec}(A')$ . By construction,  $B' = A' \otimes_A B$ , so  $z'$  induces a point (also denoted by)  $z'$  in  $X'$ . Since  $\det(\{\partial g_i / \partial t_j\}_{i,j}) \equiv \det(\{\partial \bar{g}_i / \partial t_j\}_{i,j}) \neq 0$  in  $A' \otimes k(z')$ , the Jacobi-determinant  $\det(\{\partial g_i / \partial t_j\}_{i,j})$  is invertible around  $z'$  in  $X'$ . By shrinking  $X'$  without changing  $Z'$  (since the Jacobi-determinant is invertible on the latter), we may assume that  $(Z', z') \rightarrow (Z, z)$  extends to an affine Nisnevich neighbourhood  $(X', z') \rightarrow (X, z)$ . Further, lifting the images of  $x_i$  in  $B'$  to  $A'$ ,  $\bar{i}_0$  extends to a map  $i'_0 : X' \rightarrow \mathbb{A}_{x_1, \dots, x_n, S}$ .

Unfortunately, there is no reason for  $i'_0$  to be a closed embedding. To repair this, choose a closed embedding  $X' \hookrightarrow \mathbb{A}_{y_1, \dots, y_r, S}$  over  $S$ , i.e., generators  $a_j$  of  $A'$  as an  $\mathfrak{o}$ -algebra. Recall that we assumed  $B = A/f$  for  $f$  in  $A$  nonzero in the special fibre  $A \otimes_{\mathfrak{o}} \mathbb{F}$  (see the proof of Theorem 2.4). Writing  $a^{\underline{j}} = a_1^{j_1} \cdot \dots \cdot a_r^{j_r}$ , any element of the ideal  $f \cdot A'$  is of the form  $\sum_j \lambda_j \cdot f a^{\underline{j}}$ , where  $\lambda_j \in \mathfrak{o}$  and  $\underline{j}$  runs over a finite subset of  $\mathbb{N}^r$ . Mapping  $y^{(\underline{j})} \mapsto f a^{\underline{j}}$ , we get a map  $X' \rightarrow \mathbb{A}_{\{y^{(\underline{j})} | \underline{j} \in \mathbb{N}^r\}, S}$  into a copy of the infinite affine space over  $S$ . Together with the map  $i'_0 : X' \rightarrow \mathbb{A}_{x_1, \dots, x_n, S}$ , we get a closed embedding

$$i_{\infty} : X' \hookrightarrow \mathbb{A}_{x_1, \dots, x_m, S} \times_S \mathbb{A}_{\{y^{(\underline{j})} | \underline{j} \in \mathbb{N}^r\}, S} \cong \mathbb{A}_S^{\infty}$$

into the fibre product. Indeed,  $i'_0 : \mathfrak{o}[x_1, \dots, x_m] \rightarrow A'$  is surjective modulo  $f$  and  $\mathfrak{o}[y^{(\underline{j})} | \underline{j} \in \mathbb{N}^r] \rightarrow A'$  has image  $\mathfrak{o}[f \cdot A']$  by construction. By Lemma 2.7,  $i_{\infty}$  induces  $i_0 : X' \hookrightarrow \mathbb{A}_{x_1, \dots, x_m, S} \times_S \mathbb{A}_{y^{(\underline{j}_1)}, \dots, y^{(\underline{j}_l)}, S}$ , still a closed embedding for suitable  $\underline{j}_1, \dots, \underline{j}_l \in \mathbb{N}^r$ . Setting  $x_{m+l} := y^{(\underline{j}_s)}$  and  $N := m+l$ , we have constructed a closed embedding  $i_0 : X' \hookrightarrow \mathbb{A}_{x_1, \dots, x_N, S}$  such that  $i_0|_{Z'}$  factors over  $\bar{i}_0 : Z' \hookrightarrow \mathbb{A}_{x_1, \dots, x_m, S} = V(x_{m+1}, \dots, x_N) \subseteq \mathbb{A}_{x_1, \dots, x_N, S}$ . In particular, the closure of  $Z'$  in  $\mathbb{P}_{x_0, \dots, x_N, S}$  is just

$\bar{Z}'$  inside the linear subspace  $V_+(x_{m+1}, \dots, x_N) \subseteq \mathbb{P}_{x_0: \dots: x_N, S}$ , so  $Z'$  is fibrewise dense over  $S$  inside this closure.  $\square$

**Lemma 2.7.** *Let  $C$  be an  $\sigma$ -algebra of finite type. Let  $\iota: \text{Spec}(C) \hookrightarrow \mathbb{A}_{t_1, t_2, \dots, S} = \mathbb{A}_S^\infty$  be a closed embedding and let  $\text{pr}_{\leq N}: \mathbb{A}_{t_1, t_2, \dots, S} \rightarrow \mathbb{A}_{t_1, \dots, t_N, S} = \mathbb{A}_S^N$  be the canonical projection. Then  $\text{pr}_{\leq N} \circ \iota$  is a closed embedding for  $N \gg 0$ .*

*Proof.* Suppose that  $C$  is generated as an  $\sigma$ -algebra by  $c_1, \dots, c_r \in C$ . Since the corresponding map on algebras  $\sigma[t_1, t_2, \dots] \rightarrow C$  is surjective, we can find polynomials  $f_i \in \sigma[t_1, t_2, \dots]$  mapping to  $c_i$ . Pick  $N \gg 0$  such that all the  $f_i$  lie inside  $\sigma[t_1, \dots, t_N]$ . Then  $\iota$  restricted to  $\sigma[t_1, \dots, t_N]$  is still surjective, hence the claim.  $\square$

**Choosing linear projections.** In the next step, we want to find the Zariski-open subset  $W \subseteq \mathfrak{E}_n$  parametrizing the linear projections  $p_{\underline{u}}$  in [Theorem 2.4](#). To do so, let us first make one further reduction.

**Reduction 2.8.** By [Proposition 2.6](#), there is a Nisnevich neighbourhood  $(X', z') \rightarrow (X, z)$  and a closed embedding  $X' \hookrightarrow \mathbb{A}_{\underline{x}, S}$  such that  $Z' = Z \times_X X'$  is fibrewise dense in its Zariski-closure  $\bar{Z}'$  in  $\mathbb{P}_{\underline{x}, S}$ . Let  $z_0$  be a specialization of  $z$  in the image of  $X' \rightarrow X$ . We can find a point  $z'_0$  in  $Z'$  such that  $k(z'_0) = k(z_0)$ , i.e.,  $(X', z'_0) \rightarrow (X, z_0)$  is a Nisnevich neighbourhood, too. The Nisnevich localization  $(X', z') \rightarrow (X, z)$  will be the only non-Zariski-localization in the proof of [Theorem 2.4](#). Thus we may assume that  $z$  is a closed point in the following. Further, from now on we may identify  $X' = X$ .

The Zariski-open subset  $W \subseteq \mathfrak{E}_n$  in [Theorem 2.4](#) will be provided in the following proposition.

**Proposition 2.9.** *Let  $X = \text{Spec}(A)/S$  be a connected smooth affine  $S$ -scheme of finite type, fibrewise of pure dimension  $n$ ,  $f$  an element in  $A$  which is nonzero in  $A \otimes_{\sigma} \mathbb{F}$  and  $Z = \text{Spec}(B = A/f) \hookrightarrow X$  the closed embedding. Let  $z$  be a closed point in the special fibre of  $Z$ . Suppose there is a closed embedding  $i_0: X \hookrightarrow \mathbb{A}_{\underline{x}, S}$  such that  $Z$  is fibrewise dense over  $S$  inside its closure  $\bar{Z}$  in  $\mathbb{P}_{\underline{x}, S}$ .*

*Then there is a Zariski-open subset  $W \subseteq \mathfrak{E}_n$  with  $W(\sigma) \neq \emptyset$ , such that for all  $\underline{u} \in W(\sigma)$  the following hold:*

- (1)  $p_{(u_1, \dots, u_{n-1})}|_Z: Z \rightarrow \mathbb{A}_{t_1, \dots, t_{n-1}, S}$  is finite,
- (2)  $p_{\underline{u}}$  is étale at all points of  $F = p_{(u_1, \dots, u_{n-1})}^{-1}(p_{(u_1, \dots, u_{n-1})}(z)) \cap Z$  and
- (3)  $p_{\underline{u}}|_F: F \rightarrow p_{\underline{u}}(F)$  is radicial.

Let us first fix the following notation.

**Remark 2.10.** For  $Y/S$  a smooth scheme, denote by  $\text{red}: Y(\sigma) \rightarrow Y(\mathbb{F}) = Y_\sigma(\mathbb{F})$  the reduction map we get by precomposing with the closed point  $\sigma$ . Because  $\sigma$  is henselian and  $Y/S$  smooth, this reduction map is always surjective.

*Proof of Proposition 2.9.* We divide Proposition 2.9 into two parts: Lemma 2.11 will provide an open  $W_1$  of  $\mathfrak{E}_n$  such that  $W_1(\mathfrak{o})$  is nonempty and every  $\underline{u}$  in  $W_1(\mathfrak{o})$  satisfies claim (1), while Lemma 2.12 will provide an open  $W_2$  such that  $W_2(\mathfrak{o})$  is nonempty and every  $\underline{u}$  in  $W_2(\mathfrak{o})$  satisfies claims (2) and (3) in Proposition 2.9. The intersection  $W = W_1 \cap W_2$  has all the properties claimed by Proposition 2.9. For the nonemptiness of  $W(\mathfrak{o})$ , recall that the reduction map  $W(\mathfrak{o}) \twoheadrightarrow W(\mathbb{F})$  is surjective and  $W_1(\mathbb{F}) \cap W_2(\mathbb{F})$  is nonempty as the special fibre of  $W_1 \cap W_2$  is a nonempty open subscheme of an affine space over the infinite field  $\mathbb{F}$ .  $\square$

**Lemma 2.11.** *Under the assumptions of Proposition 2.9, there is a Zariski-open subset  $W_1 \subseteq \mathfrak{E}_n$  with  $W_1(\mathfrak{o}) \neq \emptyset$ , such that for all  $\underline{u} \in W_1(\mathfrak{o})$  the restriction  $p_{(u_1, \dots, u_{n-1})}|_Z : Z \rightarrow \mathbb{A}_{t_1, \dots, t_{n-1}, S}$  is finite.*

**Lemma 2.12.** *Under the assumptions of Proposition 2.9, there is a Zariski-open subset  $W_2 \subseteq \mathfrak{E}_n$  with  $W_2(\mathfrak{o}) \neq \emptyset$ , such that  $p_{\underline{u}}$  is étale at all points of  $F$  and  $p_{\underline{u}}|_F : F \rightarrow p_{\underline{u}}(F)$  is radical for all  $\underline{u} \in W_2(\mathfrak{o})$ .*

*Proof of Lemma 2.11.* This is just a version of the arguments leading to [Grayson 1978, Proposition 1.1]. Recall that the  $j$ -th factor  $\mathbb{A}_{\underline{x}, S}^{\vee, (j)}$  of  $\mathfrak{E}_n$  is  $\mathbb{A}_{x_{1,j}^*, \dots, x_{N,j}^*, S}$ , i.e.,  $\mathfrak{E}_n = \mathbb{A}_{\{x_{i,j}^* \mid 1 \leq i \leq N, 1 \leq j \leq n\}, S}$ . Define

$$\mathbb{L} := V_+ \left( x_0, \sum_j x_{i,j}^* \otimes x_j \mid 1 \leq i < n \right) \subseteq \mathfrak{E}_n \times_S H_\infty \quad \text{and} \quad \bar{Z}_\infty := \bar{Z} \cap H_\infty.$$

Here,  $H_\infty = V_+(x_0) \subset \mathbb{P}_{\underline{x}, S}$  is the hyperplane at infinity. By construction,  $\mathbb{L} \rightarrow \mathfrak{E}_n$  has fibre  $\mathbb{L}_{\underline{u}} = L_{(u_1, \dots, u_{n-1})}$  over  $\underline{u} \in \mathfrak{E}_n(\mathfrak{o})$ . Since the projection  $\text{pr} : \mathfrak{E}_n \times_S H_\infty \rightarrow \mathfrak{E}_n$  is projective, hence closed,

$$W_1 := \mathfrak{E}_n \setminus \text{pr}(\mathbb{L} \cap (\mathfrak{E}_n \times_S \bar{Z}_\infty))$$

is open. Again by construction, for any  $\underline{u} \in W_1(\mathfrak{o})$ ,  $L_{(u_1, \dots, u_{n-1})} \cap \bar{Z} = \emptyset$ , so  $p_{(u_1, \dots, u_{n-1})}|_Z : Z \rightarrow \mathbb{A}_{t_1, \dots, t_{n-1}, S}$  is finite by Lemma 2.3.

It remains to show that  $W_1(\mathfrak{o}) \neq \emptyset$ . The reduction map  $\text{red} : W_1(\mathfrak{o}) \twoheadrightarrow W_1(\mathbb{F})$  is surjective, so we have to show  $W_1(\mathbb{F}) = W_{1,\sigma}(\mathbb{F}) \neq \emptyset$ . The special fibre  $W_{1,\sigma}$  equals  $\mathfrak{E}_{n,\sigma} \setminus \text{pr}(\mathbb{L}_\sigma \cap (\mathfrak{E}_{n,\sigma} \times_{\mathbb{F}} \bar{Z}_{\infty,\sigma}))$ . Further,  $Z_\sigma \subset \bar{Z}_\sigma$  is dense by assumption so  $\bar{Z}_\sigma$  is the closure  $\bar{Z}_\sigma$  of  $Z_\sigma$  inside  $\mathbb{P}_{\underline{x}, \mathbb{F}}$ . It follows that  $\bar{Z}_{\infty,\sigma} = \bar{Z}_\sigma \cap H_{\infty,\sigma}$ , i.e., we are in the situation of [Grayson 1978, Proposition 1.1] and  $W_{1,\sigma}(\mathbb{F}) \neq \emptyset$ .  $\square$

Lemma 2.12 can easily be derived from [Colliot-Thélène et al. 1997, Lemmas 3.4.1 and 3.4.2] applied over the special fibre.

*Proof of Lemma 2.12.* As a closed embedding of smooth  $S$ -schemes,  $i_0 : X \hookrightarrow \mathbb{A}_{\underline{x}, S}$  is regular. Let  $I = (f_1, \dots, f_{N-n}) \trianglelefteq \mathfrak{o}[\underline{x}]$  be the ideal of  $i_0$  for  $f_1, \dots, f_{N-n}$  a regular sequence. Write  $A = \mathcal{O}(X)$  over  $\mathfrak{o}[\underline{t}]$  (via  $p_{\underline{u}}$ ) as  $\mathfrak{o}[\underline{t}][\underline{x}]/(f_i, u_j - t_j \mid i, j)$ .

Then  $p_{\underline{u}}$  is étale at a point  $x \in X$  if it is standard smooth around  $x$ , i.e., if the Jacobi-determinant

$$\det\left(\left\{\frac{\partial f_i}{\partial x_s}\right\}_{i,s} \mid \left\{\frac{\partial(u_j - t_j)}{\partial x_s}\right\}_{j,s}\right) = \det\left(\left\{\frac{\partial f_i}{\partial x_s}\right\}_{i,s} \mid \left\{\frac{\partial(u_j)}{\partial x_s}\right\}_{j,s}\right)$$

is invertible in  $\mathcal{O}_{X,x}$ . We may write the latter determinant as  $df_1 \wedge \cdots \wedge df_{N-n} \wedge du_1 \wedge \cdots \wedge du_n$  in  $\Omega_{\mathfrak{o}[\underline{x}]/\mathfrak{o}}^N \otimes_{\mathfrak{o}[\underline{x}]} \mathcal{O}_{X,x}$ . Since  $X \hookrightarrow \mathbb{A}_{\underline{x},S}$  is a smooth pair, the conormal sequence

$$0 \rightarrow I/I^2 \otimes_A \mathcal{O}_{X,x} \rightarrow \Omega_{\mathfrak{o}[\underline{x}]/\mathfrak{o}}^1 \otimes_{\mathfrak{o}[\underline{x}]} \mathcal{O}_{X,x} \rightarrow \Omega_{A/\mathfrak{o}}^1 \otimes_A \mathcal{O}_{X,x} \rightarrow 0$$

is split exact and

$$\Omega_{\mathfrak{o}[\underline{x}]/\mathfrak{o}}^N \otimes_{\mathfrak{o}[\underline{x}]} \mathcal{O}_{X,x} = \bigwedge^{N-n} (I/I^2 \otimes_A \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} (\Omega_{A/\mathfrak{o}}^n \otimes_A \mathcal{O}_{X,x}).$$

Note that  $I/I^2$  is free over  $A$  with basis given by the regular sequence  $f_1, \dots, f_{N-n}$ . In particular,  $f_1 \wedge \cdots \wedge f_n$  is invertible in  $\bigwedge^{N-n} (I/I^2 \otimes_A \mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$  and

$$df_1 \wedge \cdots \wedge df_{N-n} \wedge du_1 \wedge \cdots \wedge du_n = (f_1 \wedge \cdots \wedge f_n) \otimes (d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_n)$$

is invertible if and only if  $d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_n$  is invertible in  $\Omega_{A/\mathfrak{o}}^n \otimes_A \mathcal{O}_{X,x} = \mathcal{O}_{X,x}$  for  $\bar{u}_j$  the image of  $t_j$  under  $\mathfrak{o}[\underline{t}] \rightarrow A$ . By Nakayama's lemma, this is equivalent to  $d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_n \neq 0$  in  $\Omega_{X/S}^n \otimes_{\mathcal{O}_X} k(x)$ . Suppose  $x$  is contained in  $F$ . Since  $z$  lies in the special fibre, so does  $x$  and  $\Omega_{X/S}^n \otimes_{\mathcal{O}_X} k(x) = \Omega_{X_\sigma/\mathbb{F}}^n \otimes_{\mathcal{O}_{X_\sigma}} k(x)$ . Summing up,  $p_{\underline{u}}$  is étale at  $x \in F$  if  $d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_n \neq 0$  in  $\Omega_{X_\sigma/\mathbb{F}}^n \otimes_{\mathcal{O}_{X_\sigma}} k(x)$ . Thus, we are in fact in the situation of [Colliot-Thélène et al. 1997, Lemma 3.4.1], i.e., we get a nonempty open subset  $\bar{W}'_2 \subseteq \mathfrak{E}_{n,\sigma} \otimes_{\mathbb{F}} \mathbb{F}^{\text{alg}}$  with  $p_{\underline{u}}$  étale around  $F$  for all  $\underline{u}$  with  $\text{red}(\underline{u}) \in \bar{W}'_2(\mathbb{F}^{\text{alg}})$ . Here,  $\mathbb{F}^{\text{alg}}/\mathbb{F}$  is an algebraic closure.

For the universal injectivity, observe that  $p_{\underline{u}}|_F = p_{\text{red}(\underline{u})}|_F$ . Thus, we are in the situation of [Colliot-Thélène et al. 1997, Lemma 3.4.2], i.e., we get a nonempty open subset  $\bar{W}''_2 \subseteq \mathfrak{E}_{n,\sigma} \otimes_{\mathbb{F}} \mathbb{F}^{\text{alg}}$  with  $p_{\underline{u}}|_{F \otimes_{\mathbb{F}} \mathbb{F}^{\text{alg}}}$  (universally) injective for all  $\underline{u}$  with  $\text{red}(\underline{u}) \in \bar{W}''_2(\mathbb{F}^{\text{alg}})$ .

Finally, by a standard descent argument (see the proof of [Colliot-Thélène et al. 1997, Lemma 3.4.3]) for the intersection  $\bar{W}'_2 \cap \bar{W}''_2$ , we get an open subset  $\bar{W}_2 \subseteq \mathfrak{E}_{n,\sigma}$  with  $\bar{W}_2(\mathbb{F}) \neq \emptyset$ , such that  $p_{\underline{u}}$  is étale at all points of  $F$  and  $p_{\underline{u}}|_F$  is universally injective for all  $\underline{u}$  with  $\text{red}(\underline{u}) \in \bar{W}_2(\mathbb{F})$ . Let  $W_2 \subseteq \mathfrak{E}_n$  be any open subset with special fibre  $W_{2,\sigma} = \bar{W}_2$ . Then  $\underline{u} \in W_2(\mathfrak{o})$  if and only if  $\text{red}(\underline{u}) \in \bar{W}_2(\mathbb{F})$  and  $W_2(\mathfrak{o}) \neq \emptyset$ , since the reduction map is surjective.  $\square$

**Choosing neighbourhoods.** Fix a linear projection  $p_{\underline{u}}$  for an  $\mathfrak{o}$ -point  $\underline{u}$  in the open subset  $W \subseteq \mathfrak{E}_n$  provided by Proposition 2.9. In the following, we construct the open neighbourhoods  $V$  and  $U$  in Theorem 2.4.

As in [Colliot-Thélène et al. 1997], we first secure  $V \subseteq \mathbb{A}_{t_1, \dots, t_{n-1}, S}$  in Lemma 2.13 and an open neighbourhood  $z \in U_1 \subseteq p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  covering Theorem 2.4 parts



(2) and (4) in [Lemma 2.14](#). If we define  $U$  as the intersection of  $U_1$  with the étale locus of  $p_{\underline{u}}$ , the pair  $V$  and  $U$  will finally satisfy claims (2), (3) and (4) of [Theorem 2.4](#). The proofs can almost literally be transferred from [[Colliot-Thélène et al. 1997](#)].

**Lemma 2.13** (cf. [[Colliot-Thélène et al. 1997](#), Lemma 3.5.1]). *Under the assumptions of [Proposition 2.9](#) and any choice of linear projection  $\underline{u}$  in  $W(\mathfrak{o})$ , there is a Zariski-open neighbourhood  $V \subseteq \mathbb{A}_{t_1, \dots, t_{n-1}, S}$  of  $p_{(u_1, \dots, u_{n-1})}(z)$  such that  $p_{\underline{u}}$  is étale in a Zariski-open neighbourhood of  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  and restricts to a closed embedding  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V) \hookrightarrow \mathbb{A}_V^1$ .*

*Proof.* We will get  $V$  as  $V_1 \cap V_2$ , where  $V_1 \subseteq \mathbb{A}_{t_1, \dots, t_{n-1}, S}$  is an open neighbourhood of  $p_{(u_1, \dots, u_{n-1})}(z)$  such that  $p_{\underline{u}}$  is étale at all points of  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V_1)$  and  $V_2 \subseteq \mathbb{A}_{t_1, \dots, t_{n-1}, S}$  is an open neighbourhood such that  $p_{\underline{u}}$  restricts to a closed embedding  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V_2) \rightarrow \mathbb{A}_{V_2}^1$ .

Let  $U' \subseteq X$  be the étale locus of  $p_{\underline{u}}$ . Since  $\underline{u} \in W(\mathfrak{o})$ ,  $U'$  is an open neighbourhood of  $F = p_{(u_1, \dots, u_{n-1})}^{-1}(p_{(u_1, \dots, u_{n-1})}(z)) \cap Z$  in  $X$  ([Proposition 2.9](#)). As  $p_{(u_1, \dots, u_{n-1})}|_Z$  is finite,  $p_{(u_1, \dots, u_{n-1})}(Z \setminus U')$  is closed and we set

$$V_1 := \mathbb{A}_{t_1, \dots, t_{n-1}, S} \setminus p_{(u_1, \dots, u_{n-1})}(Z \setminus U').$$

By construction,  $V_1$  is a Zariski-open neighbourhood of the image  $p_{(u_1, \dots, u_{n-1})}(z)$  of  $z$  and moreover  $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V_1) \subseteq Z \cap U'$  is contained in the étale locus  $U'$  of  $p_{\underline{u}}$ .

To get the neighbourhood  $V_2$ , consider  $p_{\underline{u}}|_Z : Z \rightarrow \mathbb{A}_{t, S} = \mathbb{A}_{t_n, \mathbb{A}_{t_1, \dots, t_{n-1}, S}}$  as a family of maps over  $\mathbb{A}_{t_1, \dots, t_{n-1}, S}$ . Since  $Z/\mathbb{A}_{t_1, \dots, t_{n-1}, S}$  is finite, the property “ $p_{\underline{u}}|_Z$  is a closed embedding” is Zariski-open in the base  $\mathbb{A}_{t_1, \dots, t_{n-1}, S}$  by Nakayama’s lemma. Thus we have to show that the fibre of this family

$$p_{\underline{u}}|_F : Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(p_{(u_1, \dots, u_{n-1})}(z)) = F \rightarrow \mathbb{A}_{t_n, p_{(u_1, \dots, u_{n-1})}(z)}$$

over  $p_{(u_1, \dots, u_{n-1})}(z)$  is a closed embedding. But  $p_{\underline{u}}(F) \subset \mathbb{A}_{t_n, p_{(u_1, \dots, u_{n-1})}(z)}$  is closed as a finite set of closed points and  $p_{\underline{u}}|_F : F \rightarrow p_{\underline{u}}(F)$  is a closed embedding as by [Proposition 2.9](#), it is radicial and  $p_{\underline{u}}$  is étale and hence unramified at each point of  $F$ .  $\square$

**Lemma 2.14** (cf. [[Colliot-Thélène et al. 1997](#), Lemma 3.6.1]). *Under the assumptions of [Proposition 2.9](#) and any choice of linear projection  $\underline{u}$  in  $W(\mathfrak{o})$ , let*

$$Z' := Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V) \quad \text{and} \quad U_1 := p_{(u_1, \dots, u_{n-1})}^{-1}(V) \setminus (p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \setminus Z').$$

*Then  $U_1 \subseteq p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  is a Zariski-open neighbourhood of the point  $z$  and we have  $Z \cap U_1 = Z'$  and  $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \cap U_1 = Z'$ .*

*Proof.* By definition of  $U_1$ ,  $z$  lies inside  $U_1$ ,  $Z \cap U_1 = Z'$  and  $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \cap U_1 = Z'$ . It remains to show that  $U_1 \subseteq p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  is open. By [Lemma 2.13](#),  $p_{\underline{u}}$  restricts

to a closed embedding  $Z' \rightarrow \mathbb{A}_{t_n, V}$ , so  $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \subset p_{(u_1, \dots, u_{n-1})}^{-1}(V)$  is closed. We need to show that  $Z' \subseteq p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))$  is open.

To this end, consider the étale locus  $U''$  of

$$p_{\underline{u}}|_{p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))} : p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \rightarrow p_{\underline{u}}(Z').$$

By [Lemma 2.13](#),  $p_{\underline{u}}$  is étale at all points of  $Z'$ . Thus, the base change  $p_{\underline{u}}|_{p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))}$  is still étale at all points of  $Z'$ , that is,  $Z'$  is contained inside the open subset  $U'' \subseteq p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))$ . But  $Z' \rightarrow p_{\underline{u}}(Z')$  is an isomorphism by [Lemma 2.13](#), so both  $Z'/p_{\underline{u}}(Z')$  and  $U''/p_{\underline{u}}(Z')$  are étale and hence  $Z' \subseteq U''$  is open.  $\square$

### 3. Objectwise stable $\mathbb{A}^1$ -connectivity

In this section, we derive connectivity results for homotopy presheaves (i.e., “objectwise” connectivity results). These are used in the proof of our main theorem in the following section. Moreover, we show the left completeness of the  $\mathbb{A}^1$ -Nisnevich-local t-structure on  $S^1$ - and  $\mathbb{P}^1$ -spectra. Throughout this section, let  $S$  be an arbitrary noetherian scheme of finite dimension.

**Results for  $S^1$ -spectra.** We start with objectwise connectivity results for  $S^1$ -spectra.

**Proposition 3.1.** *Let  $U \in \text{Sm}_S$  be a scheme of dimension  $e$ . Then given  $E$  in  $\mathcal{SH}_{S^1}^{>e}(S)$ , one has*

$$[\Sigma_{S^1}^{\infty}(U_+)[i], L^{\mathbb{A}^1}E] = 0,$$

where  $L^{\mathbb{A}^1}$  is a fibrant replacement functor for the stable  $\mathbb{A}^1$ -Nisnevich-local model structure.

**Remark 3.2.** [Proposition 3.1](#) gives a connectivity result for a  $U$ -section of the homotopy presheaf  $[\Sigma_{S^1}^{\infty}(-_+)[i], L^{\mathbb{A}^1}E]$  with respect to the dimension of  $U$ . However, we are interested in a connectivity result depending only on the dimension of the base scheme  $S$ . The price we have to pay for this is to sheafify the homotopy presheaf, i.e., eventually we are interested in connectivity results for the Nisnevich stalks of the presheaf  $[\Sigma_{S^1}^{\infty}(-_+)[i], L^{\mathbb{A}^1}E]$ . Unfortunately we cannot apply [Proposition 3.1](#) directly to the stalks as their dimension is unbounded.

*Proof of [Proposition 3.1](#).* We work with the explicit model  $L^{\infty}$  of [Lemma 1.2](#) as an  $\mathbb{A}^1$ -Nisnevich-local fibrant replacement functor  $L^{\mathbb{A}^1}$ . By homotopy-exactness of  $L^{\infty}$ , we have to show that

$$[\Sigma_{S^1}^{\infty}(U_+), \text{hocolim}_{k \rightarrow \infty} L^k(E)] = 0$$

for  $U \in \text{Sm}_S$  of dimension  $e$  and  $E \in \mathcal{SH}_{S^1}^{>e}$ . Since  $\Sigma_{S^1}^{\infty}(U_+)$  is compact, every homotopy class in question is represented by some  $\Sigma_{S^1}^{\infty}(U_+) \rightarrow L^k(E)$ . Hence, it

suffices to show, that for every  $k \geq 0$ ,

$$[\Sigma_{S^1}^\infty(U_+), L^k E] = 0.$$

We argue by induction on  $k \geq 0$  for all  $U \in \text{Sm}_S$  of dimension  $e$  and all spectra  $E \in \mathcal{SH}_{S^1 > e}^s$  at once.

For  $k = 0$  the statement follows directly from [Lemma 3.3](#) below. Let  $k \geq 1$ . The distinguished triangle in [Remark 1.3](#) induces the long exact sequence

$$\begin{aligned} \cdots \rightarrow [\Sigma_{S^1}^\infty(U_+), L^{(k-1)} E] &\rightarrow [\Sigma_{S^1}^\infty(U_+), L^k E] \\ &\rightarrow [\Sigma_{S^1}^\infty(U_+ \wedge \mathbb{A}^1), L^{(k-1)} E[1]] \rightarrow \cdots \end{aligned}$$

The abelian group on the left-hand side vanishes by the induction hypothesis on  $k$ . In order to see the vanishing of the right-hand side, we observe that

$$U \sqcup \mathbb{A}^1 \cong U_+ \vee \mathbb{A}^1 \rightarrow (U \times \mathbb{A}^1) \sqcup \mathbb{A}^1 \cong U_+ \times \mathbb{A}^1 \rightarrow U_+ \wedge \mathbb{A}^1$$

and therefore  $U_+ \rightarrow (U \times \mathbb{A}^1)_+ \rightarrow U_+ \wedge \mathbb{A}^1$  is a homotopy cofibre sequence in  $\text{sPre}_+(S)$ . This yields a distinguished triangle after applying the left Quillen functor  $\Sigma_{S^1}^\infty$ . Consider the long exact sequence obtained by an application of  $[-, L^{(k-1)} E[1]]$  to this triangle. It suffices to show the vanishing of both of the abelian groups  $[\Sigma_{S^1}^\infty(U_+)[1], L^{(k-1)} E[1]]$  and  $[\Sigma_{S^1}^\infty(U \times \mathbb{A}^1)_+, L^{(k-1)} E[1]]$ . For the first group, this follows from the inductive hypothesis on  $k$  and likewise for the second, since the dimension of  $U \times \mathbb{A}^1$  is  $e + 1$  and  $E[1] \in \mathcal{SH}_{S^1 > e+1}^s$ .  $\square$

**Lemma 3.3.** *Let  $U \in \text{Sm}_S$  be a scheme of dimension  $e$ . Then for  $D \in \mathcal{SH}_{S^1 > i+e}^s(S)$ , one has*

$$[\Sigma_{S^1}^\infty(U_+)[i], L^s D] = 0.$$

*Proof.* It suffices to show that  $[\Sigma_{S^1}^\infty(U_+), L^s D] = 0$  for  $D \in \mathcal{SH}_{S^1 > e}^s$ . Indeed,  $[\Sigma_{S^1}^\infty(U_+)[i], L^s D] \cong [\Sigma_{S^1}^\infty(U_+), L^s(D[-i])]$  as  $L^s$  is homotopy-exact. Recall that the Nisnevich-cohomological dimension is bounded by the Krull-dimension, i.e., for any sheaf  $G$  of abelian groups on  $\text{Sm}_S$  and  $n > \dim(U)$ , we have

$$[\Sigma_{S^1}^\infty(U_+), L^s H G[n]] = H_{\text{Nis}}^n(U, G) = 0;$$

see, e.g., [\[Thomason and Trobaugh 1990, Lemma E.6\(c\)\]](#).

By the left completeness of the Nisnevich-local structure, there is a filtration

$$\begin{array}{ccccccc} 0 \simeq \text{holim}_{n \rightarrow \infty} L^s D_{\geq n} & \longrightarrow & \cdots & \longrightarrow & L^s D_{\geq e+2} & \longrightarrow & L^s D_{\geq e+1} = L^s D \\ & & & & \downarrow & & \downarrow \\ & & \cdots & & L^s H \pi_{e+2}(D)[e+2] & & L^s H \pi_{e+1}(D)[e+1] \end{array}$$

and a surjection  $0 = [\Sigma_{S^1}^\infty(U_+), \text{holim}_n L^s D_{\geq n}] \twoheadrightarrow \lim_n [\Sigma_{S^1}^\infty(U_+), L^s D_{\geq n}]$  by the Milnor- $\lim^1$ -sequence. Hence,  $\lim_n [\Sigma_{S^1}^\infty(U_+), L^s D_{\geq n}] = 0$ . For  $i \geq 1$ , there is a

long exact sequence

$$\begin{aligned} \cdots \rightarrow [\Sigma_{S^1}^\infty(U_+), L^s D_{\geq e+i+1}] &\rightarrow [\Sigma_{S^1}^\infty(U_+), L^s D_{\geq e+i}] \\ &\rightarrow [\Sigma_{S^1}^\infty(U_+), L^s H\pi_{e+i}(D)[e+i]] \rightarrow \cdots, \end{aligned}$$

where the abelian group on the right-hand side is zero by the above-mentioned result on the Nisnevich-cohomological dimension. For this reason, the projection  $\lim[\Sigma_{S^1}^\infty(U_+), L^s D_{\geq n}] \rightarrow [\Sigma_{S^1}^\infty(U_+), L^s D_{\geq e+1}]$  is surjective and therefore we get  $[\Sigma_{S^1}^\infty(U_+), L^s D_{\geq e+1}] = 0$  as desired.  $\square$

**Corollary 3.4.** *Let  $U \in \text{Sm}_S$  be an  $S$ -pointed scheme of dimension  $e$ . Then for  $D \in \mathcal{SH}_{S^1}^s_{>i+e}(S)$ , one has*

$$[\Sigma_{S^1}^\infty(U)[i], L^s D] = 0.$$

*Proof.* The basepoint  $s : S \rightarrow U$  is a splitting of the structure morphism  $p : U \rightarrow S$ . In particular,  $\dim(U) \geq \dim(S)$ . Consider the distinguished triangle

$$\Sigma_{S^1}^\infty(S_+) \rightarrow \Sigma_{S^1}^\infty(U_+) \rightarrow \Sigma_{S^1}^\infty(U) \rightarrow \Sigma_{S^1}^\infty(S_+)[1].$$

If  $\dim(U) > \dim(S)$  then the assertion follows from the previous [Lemma 3.3](#) applied to the entries  $\Sigma_{S^1}^\infty(U_+)$  and  $\Sigma_{S^1}^\infty(S_+)[1]$  of the triangle. Now we consider the case  $\dim(U) = \dim(S)$ . Because of the splitting  $s : S \rightarrow U$ ,  $p$  is surjective. Since  $p$  is smooth of relative dimension zero, it follows that  $p$  is étale. Thus, the section  $s$  itself is étale. As it is also a closed immersion, the image of  $s$  is a component of  $U$ , i.e.,  $U \cong (U')_+$  for some  $U' \in \text{Sm}_S$  with  $\dim(U') \leq \dim(U)$ . The result then follows from the previous [Lemma 3.3](#) applied to  $\Sigma_{S^1}^\infty(U'_+)$ .  $\square$

We get the following analogue as a corollary to [Proposition 3.1](#).

**Corollary 3.5.** *Let  $U \in \text{Sm}_S$  be an  $S$ -pointed scheme of dimension  $e$ . Then for  $E \in \mathcal{SH}_{S^1}^s_{>i+e}(S)$ , one has*

$$[\Sigma_{S^1}^\infty(U)[i], L^{\mathbb{A}^1} E] = 0.$$

*Proof.* The proof is literally the same as that of [Corollary 3.4](#) using [Proposition 3.1](#) instead of [Lemma 3.3](#).  $\square$

**Corollary 3.6.** *The  $\mathbb{A}^1$ -Nisnevich-local  $t$ -structure on  $\mathcal{SH}_{S^1}^{\mathbb{A}^1}(S)$  is left complete and hence nondegenerate. In particular,*

$$L^{\mathbb{A}^1} \text{holim}_{n \rightarrow \infty}(E_{\leq n}) \sim \text{holim}_{n \rightarrow \infty} L^{\mathbb{A}^1}(E_{\leq n}).$$

*Proof.* First note that the truncation functors of the  $\mathbb{A}^1$ -Nisnevich-local  $t$ -structure are (after inclusion to  $\mathcal{SH}_{S^1}^s$ ) given by  $L^{\mathbb{A}^1}((-)_{\leq n})$  (see [Proposition 1.15](#)). Consider a spectrum  $E \in \text{Spt}_{S^1}(S)$ . To see that  $L^{\mathbb{A}^1} E \rightarrow \text{holim}_n L^{\mathbb{A}^1}(E_{\leq n})$  is an isomorphism in  $\mathcal{SH}_{S^1}^s$ , we may equivalently show  $\text{holim}_n L^{\mathbb{A}^1}(E_{\geq n}) \simeq 0$ , which is

implied by the triviality of the group  $\pi_i(\text{holim}_n L^{\mathbb{A}^1}(E_{\geq n}))(U)$  for every integer  $i$  and every  $U \in \text{Sm}_S$ . Equivalently, we show that  $\pi_i \text{holim}_n(L^{\mathbb{A}^1}(E_{\geq n})(U))$  is trivial. Proposition 3.1 yields  $[\Sigma_{S^1}^\infty U_+[i], L^{\mathbb{A}^1}(E_{\geq n})] = 0$  for all integers  $n > i + \dim(U)$ . Hence, we obtain  $\lim_n \pi_i(L^{\mathbb{A}^1}(E_{\geq n})(U)) = 0$ . Using Milnor’s  $\lim^1$ -sequence, it follows that the group  $\pi_i \text{holim}_n(L^{\mathbb{A}^1}(E_{\geq n})(U))$  is trivial. Indeed, the  $\lim^1$ -term is trivial as the occurring groups are eventually zero.  $\square$

**Results for  $\mathbb{P}^1$ -spectra.** In this subsection, we show some analogous statements to those of the preceding section for  $\mathbb{P}^1$ -spectra. The results of this subsection are not needed for the rest of the paper but are of independent interest.

**Proposition 3.7.** *Let  $U \in \text{Sm}_S$  be a scheme of dimension  $e$ . For  $E \in \mathcal{SH}_{>i+e}(S)$ , one has*

$$[\Sigma_{\mathbb{P}^1}^\infty(U_+[i]\langle q \rangle), E]_{\mathcal{SH}} = 0$$

for all  $q \in \mathbb{Z}$ .

*Proof.* Set  $\mathcal{F} := [\Sigma_{\mathbb{P}^1}^\infty(U_+[i]\langle q \rangle), -]_{\mathcal{SH}}$  for abbreviation. By the construction in Proposition 1.15, the class  $\mathcal{SH}_{>i+e}$  is generated under extensions, (small) sums and cones from  $\mathcal{S}[i + e + 1]$ . If  $E$  is obtained from an extension  $E' \rightarrow E \rightarrow E''$  and  $\mathcal{F}$  vanishes on  $E'$  and  $E''$ , then it also vanishes on  $E$ . If  $E$  is a (small) sum of objects  $E'_\alpha$  on which  $\mathcal{F}$  vanishes, we use the homotopy-compactness of  $\Sigma_{\mathbb{P}^1}^\infty(U_+[i]\langle q \rangle)$  to conclude that  $\mathcal{F}(E) = 0$ . Suppose that  $E$  sits in a distinguished triangle  $E' \rightarrow E'' \rightarrow E \rightarrow E'[1]$  and we know the vanishing of  $\mathcal{F}$  on  $E''$  and  $E'[1]$ . Then we know it on  $E$ . Summing up, it suffices to show that  $\mathcal{F}(\mathcal{S}[n]) = 0$  for all  $n \geq i + e + 1$ , i.e.,  $[\Sigma_{\mathbb{P}^1}^\infty(U_+[i]\langle q \rangle), \Sigma_{\mathbb{P}^1}^\infty(V_+[n]\langle q' \rangle)]_{\mathcal{SH}} = 0$  for all  $V \in \text{Sm}_S$  and  $q' \in \mathbb{Z}$ . We compute

$$\begin{aligned} & [\Sigma_{\mathbb{P}^1}^\infty(U_+[i]\langle q \rangle), \Sigma_{\mathbb{P}^1}^\infty(V_+[n]\langle q' \rangle)]_{\mathcal{SH}} \\ & \cong [\Sigma_{\mathbb{P}^1}^\infty(U_+[i-n]), \Sigma_{\mathbb{P}^1}^\infty(V_+\langle q' - q \rangle)]_{\mathcal{SH}} \\ & \cong [\Sigma_{S^1}^\infty(U_+[i-n]), \Omega_{\mathbb{G}_m}^\infty(\Sigma_{\mathbb{G}_m}^\infty \Sigma_{S^1}^\infty(V_+\langle q' - q \rangle))] \\ & \cong [\Sigma_{S^1}^\infty(U_+[i-n]), \text{colim}_k \Omega_{\mathbb{G}_m}^k L^{\mathbb{A}^1}(\Sigma_{S^1}^\infty(V_+) \wedge \mathbb{G}_m^{\wedge(k+q'-q)})] \\ & \cong \text{colim}_k [\Sigma_{S^1}^\infty(U_+[i-n]) \wedge \mathbb{G}_m^{\wedge k}, L^{\mathbb{A}^1}(\Sigma_{S^1}^\infty(V_+) \wedge \mathbb{G}_m^{\wedge(k+q'-q)})], \end{aligned}$$

where the last isomorphism is due to compactness of  $\Sigma_{S^1}^\infty(U_+)$ . Now we use the fact that  $\mathbb{A}^1$ -Nisnevich-locally there is an equivalence  $\mathbb{G}_m \sim \mathbb{P}^1[-1]$ . Hence, it suffices to show that for all but finitely many  $k \geq 0$  (and in particular, we may assume  $k + q' - q \geq 0$ ), one has

$$\begin{aligned} & [\Sigma_{S^1}^\infty(U_+[i-n]) \wedge \mathbb{G}_m^{\wedge k}, L^{\mathbb{A}^1}(\Sigma_{S^1}^\infty(V_+) \wedge \mathbb{G}_m^{\wedge(k+q'-q)})] \\ & \cong [\Sigma_{S^1}^\infty(U_+ \wedge (\mathbb{P}^1)^{\wedge k})[i-n-k], L^{\mathbb{A}^1} \Sigma_{S^1}^\infty(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})] = 0. \end{aligned}$$

By the same arguments as in the proof of [Proposition 3.1](#), this is implied by the vanishing of the group  $[\Sigma_{S^1}^\infty(U \times \mathbb{P}^k)_+ [i - n - k], L^{\mathbb{A}^1} \Sigma_{S^1}^\infty(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})]$ . Since the spectrum  $\Sigma_{S^1}^\infty(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})$  is in  $\mathcal{SH}_{S^1 \geq 0}^s$ , the result follows from [Proposition 3.1](#) as the scheme  $U \times \mathbb{P}^k$  has dimension  $e + k$ .  $\square$

**Corollary 3.8.** *Let  $S$  be a noetherian scheme of finite Krull-dimension. Then the homotopy  $t$ -structure on the motivic homotopy category  $\mathcal{SH}(S)$  is left complete and hence nondegenerate.*

*Proof.* Let  $E \in \mathcal{SH}$ . We have to show that the canonical morphism  $E \rightarrow \text{holim } E_{\leq n}$  is an isomorphism in  $\mathcal{SH}$ . Equivalently, we may show that  $\text{holim } E_{\geq n} \simeq 0$ . By [\[Hovey 1999, Theorem 7.3.1\]](#), this is implied by the vanishing of the homotopy classes

$$[\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle, \text{holim}_{n \rightarrow \infty} E_{\geq n}]$$

in  $\mathcal{SH}$  for all  $U \in \text{Sm}_S$  and all  $i, q \in \mathbb{Z}$ . Using Milnor's  $\text{lim}^1$ -sequence as in [Corollary 3.6](#), this, in turn, is implied by the following statement: for all  $U \in \text{Sm}_S$  and  $i, q \in \mathbb{Z}$  there exists an integer  $n_0$  with  $[\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle, E_{\geq n}] = 0$  for all  $n \geq n_0$ . Setting  $n_0 := i + \dim(U)$ , this is precisely the preceding [Proposition 3.7](#).  $\square$

#### 4. Stalkwise stable $\mathbb{A}^1$ -connectivity

In this section, we derive our main connectivity result for homotopy sheaves (i.e., a “stalkwise” connectivity result). We formulate the *shifted stable  $\mathbb{A}^1$ -connectivity property* on the base scheme and show that this property holds for every Dedekind scheme with infinite residue fields.

**Stable  $\mathbb{A}^1$ -connectivity.** Let us recall the following property on a base scheme  $S$  introduced by Morel [\[2005, Definition 1\]](#).

**Definition 4.1.** A noetherian scheme  $S$  of dimension  $d$  has the *stable  $\mathbb{A}^1$ -connectivity property*, if for every integer  $i$  and every spectrum  $E$  in  $\mathcal{SH}_{S^1 \geq i}^s(S)$ , the  $\mathbb{A}^1$ -localization  $L^{\mathbb{A}^1} E$  is contained in  $\mathcal{SH}_{S^1 \geq i}^s(S)$ .

**Theorem 4.2** [\[Morel 2005, Theorem 6.1.8\]](#). *If  $S$  is the spectrum of a field, then  $S$  has the stable  $\mathbb{A}^1$ -connectivity property.*

**Corollary 4.3.** *If  $S$  is the spectrum of a field, then  $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1}(S) = \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1, \pi}(S)$ .*

**Remark 4.4.** Ayoub [\[2006\]](#) gave examples of base schemes that do not have the stable  $\mathbb{A}^1$ -connectivity property: Let  $S/k$  be a connected normal surface over  $k$  an algebraically closed field, regular away from one closed singular point  $s$ . Let  $S' \rightarrow S$  be a resolution with exceptional divisor  $E$  and let  $E_{\text{red}}$  be the underlying reduced subscheme. Then by [\[op. cit., Corollary 3.3\]](#),  $S$  does not have the stable  $\mathbb{A}^1$ -connectivity property if  $\text{Pic}_{E_{\text{red}}}$  is not  $\mathbb{A}^1$ -invariant. Here,  $\text{Pic}_{E_{\text{red}}}$  is the Nisnevich

sheafification of the presheaf  $U \mapsto \text{Pic}(U \times_s E_{\text{red}})$  on  $\text{Sm}_{k(s)}$ . A family of concrete examples for such a surface  $S$  (due to Barbieri-Viale) is given in the example in [op. cit., Section 3] as hypersurfaces of  $\mathbb{P}_k^3$ . Even worse, it follows from [op. cit., Lemma 1.3] that no  $\mathbb{P}_k^n$  for  $n \geq 3$  has the stable  $\mathbb{A}^1$ -connectivity property.

**Towards stable  $\mathbb{A}^1$ -connectivity.** We saw in Remark 4.4 that connectivity may drop for general base schemes. Thus, it is an interesting question if, for a given base scheme  $S$ , there is at least *some* uniform bound  $r$  for the loss of connectivity, i.e., for  $E$  an  $i$ -connected spectrum, the  $\mathbb{A}^1$ -localization  $L^{\mathbb{A}^1}E$  is at least  $(i - r)$ -connected. In this subsection, we want to discuss a general recipe for finding such a bound, based on Morel’s original work over a field.

**Proposition 4.5.** *Let  $S$  be a noetherian scheme of finite Krull-dimension and let  $r \geq 0$  be an integer. Let  $E \in \mathcal{SH}_{S^1}^s_{\geq i}(S)$  be a spectrum. Suppose for all  $V \in \text{Sm}_S$ , all integers  $k < i - r$  and all  $f \in [\Sigma_{S^1}^\infty V_+, L^{\mathbb{A}^1}E[-k]]$ , Nisnevich-locally in  $V$ , there exists a Zariski-open  $W \hookrightarrow V$  such that*

- (1)  $f|_{\Sigma_{S^1}^\infty W_+} = 0$ , and
- (2)  $\pi_0^{\mathbb{A}^1}(V/W) = 0$ .

Then  $L^{\mathbb{A}^1}E \in \mathcal{SH}_{S^1}^s_{\geq i-r}(S)$ .

*Proof.* We may assume  $k = 0$ . We have to show that the sheaf  $\pi_0^{\mathbb{A}^1}(E)$  is trivial. Take a connected scheme  $V \in \text{Sm}_S$  with structure morphism  $p : V \rightarrow S$  and a point  $v \in V$ . It suffices to show that the Nisnevich stalk of  $\pi_0^{\mathbb{A}^1}(E)$  at  $(V, v)$  is trivial. Let  $f_{(V,v)}$  be a germ in this stalk. Possibly refining  $(V, v)$  Nisnevich-locally, we may assume that  $f_{(V,v)}$  is induced by an element  $f \in [\Sigma_{S^1}^\infty V_+, L^{\mathbb{A}^1}E]$ . After a further Nisnevich refinement of  $(V, v)$ , we find a Zariski-open  $W \hookrightarrow V$  satisfying properties (1) and (2). The homotopy cofibre sequence  $W_+ \rightarrow V_+ \rightarrow V/W$  induces a long exact sequence

$$\dots \rightarrow [\Sigma_{S^1}^\infty V/W, L^{\mathbb{A}^1}E] \rightarrow [\Sigma_{S^1}^\infty V_+, L^{\mathbb{A}^1}E] \rightarrow [\Sigma_{S^1}^\infty W_+, L^{\mathbb{A}^1}E] \rightarrow \dots$$

Since the restriction of  $f$  to  $\Sigma_{S^1}^\infty W_+$  is trivial by (1),  $f$  is the image of an element in the group  $[\Sigma_{S^1}^\infty V/W, L^{\mathbb{A}^1}E]$ , i.e., a morphism  $g : V/W \rightarrow (L^{\mathbb{A}^1}E)_0$  in the (unstable) objectwise (pointed) homotopy category. We want to show the triviality of the germ  $f_{(V,v)}$ , so it is enough to show that  $\pi_0(g)$  is trivial. As the adjunction  $(\Sigma_{S^1}^\infty, (-)_0)$  is a Quillen-adjunction for the  $\mathbb{A}^1$ -Nisnevich-local model,  $(L^{\mathbb{A}^1}E)_0$  is  $\mathbb{A}^1$ -Nisnevich-local. Therefore, the morphism  $g$  factors through  $h : L^{\mathbb{A}^1}(V/W) \rightarrow (L^{\mathbb{A}^1}E)_0$  and it suffices to show that  $\pi_0(h)$  is trivial, which follows from assumption (2).  $\square$

Now let us discuss how to obtain assumptions (1) and (2) from the previous proposition. We start with assumption (2).

We first recall a well-known construction. The singular functor  $\text{Sing} : \text{sPre}(S) \rightarrow \text{sPre}(S)$  is given on  $U$ -sections by the diagonal of the bisimplicial set  $\text{Sing}(F)(U) = F_*(\Delta^\bullet \times U)$ , where  $\Delta^\bullet$  denotes the standard cosimplicial object in  $\text{Sm}_S$ ; see [Morel and Voevodsky 1999, Section 2.3.2] in the analogous situation for simplicial sheaves. An infinite alternating composition of a Nisnevich-local fibrant replacement functor  $L^s$  and  $\text{Sing}$  yields an  $\mathbb{A}^1$ -Nisnevich-local fibrant replacement functor  $L^{\mathbb{A}^1}$  in the unstable setting. We refer to [loc. cit.] for details of this construction.

Note that for Lemma 4.6 and its Corollary 4.7, we work in the unstable setting.

**Lemma 4.6.** *Let  $V \in \text{Sm}_S$  be an irreducible scheme and  $W \hookrightarrow V$  a nonempty open subscheme. Let  $Z = (V \setminus W)_{\text{red}}$  be the reduced complement. Suppose, moreover, that each point  $v$  of  $V$  admits a Nisnevich neighbourhood  $V'$  (with pullback  $W'$  and  $Z'$  to  $V'$ ) and an étale map  $p : V' \rightarrow \mathbb{A}_Y^1$  in  $\text{Sm}_S$  with  $Z' \rightarrow Y$  finite such that*

$$\begin{array}{ccc} W' & \longrightarrow & V' \\ \downarrow & & \downarrow p \\ \mathbb{A}_Y^1 \setminus p(Z') & \longrightarrow & \mathbb{A}_Y^1 \end{array}$$

*is a Nisnevich-distinguished square. Then  $\pi_0(\text{Sing}(a_{\text{Nis}}(V/W)))$  is trivial for  $a_{\text{Nis}}$ , the Nisnevich sheafification.*

*Proof.* We follow the proof of [Morel 2005, Lemma 6.1.4]. Since a simplicial presheaf  $F$  has the same 0-simplices as the simplicial presheaf  $\text{Sing}(F)$ , there is an epimorphism  $[(-)_+, F] \twoheadrightarrow [(-)_+, \text{Sing}(F)]$  of presheaves. As Nisnevich sheafification preserves epimorphisms, we get a natural epimorphism  $\pi_0(F) \twoheadrightarrow \pi_0(\text{Sing}(F))$  of sheaves. Applying this to the discrete simplicial presheaf  $F = a_{\text{Nis}}(V/W)$  and precomposing with the epimorphism  $V = a_{\text{Nis}}V \twoheadrightarrow a_{\text{Nis}}(V/W)$  of sheaves, we get a natural epimorphism of sheaves

$$V \twoheadrightarrow a_{\text{Nis}}(V/W) = \pi_0(a_{\text{Nis}}(V/W)) \twoheadrightarrow \pi_0(\text{Sing}(a_{\text{Nis}}(V/W))).$$

Hence, it suffices to show that for each point  $v \in V$  there exists a Nisnevich neighbourhood  $V'$  such that  $V' \rightarrow \pi_0(\text{Sing}(a_{\text{Nis}}(V'/W)))$  is zero where  $W' := W \times_V V'$ .

Let  $v \in V$  be a point and choose the Nisnevich neighbourhood  $V'$  from the assumption of the proposition. In particular,  $V'/W' \rightarrow \mathbb{A}_Y^1/(\mathbb{A}_Y^1 \setminus Z')$  is an isomorphism of Nisnevich sheaves, so we may assume  $V' = \mathbb{A}_Y^1$  with closed  $Z' \hookrightarrow \mathbb{A}_Y^1$  and  $Z' \rightarrow Y$  finite. By finiteness, the morphism  $Z' \rightarrow \mathbb{A}_Y^1 \hookrightarrow \mathbb{P}_Y^1$  is proper and hence a closed immersion. Therefore, we get a diagram

$$\begin{array}{ccccc} \mathbb{A}_Y^1 \setminus Z' & \longrightarrow & \mathbb{A}_Y^1 & \xrightarrow{q} & a_{\text{Nis}}(\mathbb{A}_Y^1/\mathbb{A}_Y^1 \setminus Z') \\ \downarrow & & \downarrow j & & \downarrow \cong \\ \mathbb{P}_Y^1 \setminus Z' & \longrightarrow & \mathbb{P}_Y^1 & \xrightarrow{q'} & a_{\text{Nis}}(\mathbb{P}_Y^1/\mathbb{P}_Y^1 \setminus Z') \end{array}$$



where the right vertical morphism is an isomorphism of Nisnevich sheaves as the left-hand square is a Zariski- and therefore a Nisnevich-distinguished square.

There exists an elementary  $\mathbb{A}^1$ -homotopy  $Y \times \mathbb{A}^1 \rightarrow \mathbb{P}_Y^1$  from the zero-section  $s_0 : Y \rightarrow \mathbb{A}_Y^1 \hookrightarrow \mathbb{P}_Y^1$  to the section  $s_\infty : Y \rightarrow \mathbb{P}_Y^1$  at infinity, and the latter factorizes over  $\mathbb{P}_Y^1 \setminus Z'$ . As by [Morel and Voevodsky 1999, Lemma 2.3.6] the functor  $\text{Sing}$  turns elementary  $\mathbb{A}^1$ -homotopies into objectwise homotopies, the maps  $\text{Sing}(s_0)$  and  $\text{Sing}(s_\infty)$  are identified in the objectwise homotopy category. The morphism  $\text{Sing}(Y) \xrightarrow{\sim} \text{Sing}(\mathbb{A}_Y^1)$  is an objectwise weak equivalence by [Morel and Voevodsky 1999, Corollary 2.3.5]. Hence, the composition  $\text{Sing}(q' \circ j)$  is the constant map to the point in the objectwise homotopy category. It follows that the same is true for  $\text{Sing}(q)$ , as desired. Note that the cited arguments of [Morel and Voevodsky 1999] are valid for the objectwise structure on simplicial presheaves.  $\square$

Using the epimorphism  $\pi_0(\text{Sing}(a_{\text{Nis}}(V/W))) \twoheadrightarrow \pi_0^{\mathbb{A}^1}(V/W)$  of sheaves [Morel and Voevodsky 1999, Corollary 2.3.22], we get:

**Corollary 4.7.** *In the situation of Lemma 4.6, we have*

$$\pi_0^{\mathbb{A}^1}(V/W) = 0.$$

**Remark 4.8.** As explained in [Morel 2005, Remark 6.1.5],  $\pi_0^{\mathbb{A}^1}(V/W) = 0$  might fail for arbitrary open subschemes  $W \hookrightarrow V$ . For example, let  $S$  be the spectrum of a local ring with closed point  $i : \sigma \hookrightarrow S$  and open complement  $j : W \hookrightarrow S$ . Define  $V := S$  and consider the  $\mathbb{A}^1$ -Nisnevich-local homotopy cofibre sequence  $j_{\#}j^*(V/W) \rightarrow V/W \rightarrow i_*L^{\mathbb{A}^1}i^*(V/W)$  from (1.8). We have  $j_{\#}j^*(S/W) \simeq *$  and therefore  $L^{\mathbb{A}^1}(V/W) \simeq i_*L^{\mathbb{A}^1}(i^*(V/W))$ . On the other hand,  $i^*(S/W) \simeq i^*(S)/i^*(W) \simeq \sigma/\emptyset \simeq S_\sigma^0$ , and hence  $i_*L^{\mathbb{A}^1}(i^*(V/W))$  has nontrivial  $\pi_0$ .

Now we turn to assumption (1) of Proposition 4.5. For the special case that  $S$  is the spectrum of a field, this is an observation of Morel [2004, Lemma 3.3.6]. Please note that the extra claim  $s^*(W) \neq \emptyset$  in the next lemma excludes the obvious obstacle to assumption (2) of Proposition 4.5 in our application (see Remark 4.8).

**Lemma 4.9.** *Let  $S$  be a noetherian scheme of finite Krull-dimension together with a codimension  $c$  point  $s \in S$ . Let  $E \in \mathcal{SH}_{S^1, >c}^s(S)$  be a spectrum. Then for any  $V \in \text{Sm}_S$  with  $s^*(V) \neq \emptyset$  and any  $f \in [\Sigma_{S^1}^\infty V_+, L^{\mathbb{A}^1}E]$  there exists an open subscheme  $W \hookrightarrow V$  with  $f|_{\Sigma_{S^1}^\infty W_+} = 0$  and  $s^*(W) \neq \emptyset$ .*

*Proof.* Let  $\eta_Z \in V$  be a generic point of an irreducible component  $Z$  of  $s^*(V)$ . In particular, the ring  $\mathcal{O}_{V, \eta_Z}$  has dimension  $c$ . We write

$$j : \mathcal{U} := \text{Spec}(\mathcal{O}_{V, \eta_Z}) \cong \lim_{j_i : U_i \hookrightarrow V} U_i \rightarrow V,$$

where the limit on the right-hand side is indexed by the diagram constituted by the open immersions  $j_i : U_i \hookrightarrow V$  with  $U_i$  affine and  $U_i \cap Z \neq \emptyset$ .

Let  $p : V \rightarrow S$  denote the structural morphism. We have

$$\begin{aligned}
& \operatorname{colim}_{j_i: U_i \hookrightarrow V} [\Sigma_{S^1}^\infty(U_i \rightarrow S)_+, L^{\mathbb{A}^1} E]_{\operatorname{Spt}_{S^1}(S)} \\
& \cong \operatorname{colim}_{j_i: U_i \hookrightarrow V} [\Sigma_{S^1}^\infty p_\#((U_i \rightarrow V)_+), L^{\mathbb{A}^1} E]_{\operatorname{Spt}_{S^1}(S)} \\
& \cong \operatorname{colim}_{j_i: U_i \hookrightarrow V} [p_\# \Sigma_{S^1}^\infty(U_i \rightarrow V)_+, L^{\mathbb{A}^1} E]_{\operatorname{Spt}_{S^1}(S)} \\
& \cong \operatorname{colim}_{j_i: U_i \hookrightarrow V} [\Sigma_{S^1}^\infty(U_i \rightarrow V)_+, p^*(L^{\mathbb{A}^1} E)]_{\operatorname{Spt}_{S^1}(V)} \\
& \cong \operatorname{colim}_{j_i: U_i \hookrightarrow V} [\Sigma_{S^1}^\infty(U_i \rightarrow V)_+, L^{\mathbb{A}^1}(p^* E)]_{\operatorname{Spt}_{S^1}(V)} \quad (\text{by Lemma 1.9(1)}) \\
& \cong \operatorname{colim}_{j_i: U_i \hookrightarrow V} [j_{i,\#} \Sigma_{S^1}^\infty(U_i \rightarrow U_i)_+, L^{\mathbb{A}^1}(p^* E)]_{\operatorname{Spt}_{S^1}(V)} \quad (j_{i,\#}((U_i \rightarrow U_i)_+) = (U_i \rightarrow V)_+) \\
& \cong \operatorname{colim}_{j_i: U_i \hookrightarrow V} [\Sigma_{S^1}^\infty(U_i \rightarrow U_i)_+, j_i^* L^{\mathbb{A}^1}(p^* E)]_{\operatorname{Spt}_{S^1}(U_i)} \\
& \cong [\Sigma_{S^1}^\infty(\mathcal{U} \rightarrow \mathcal{U})_+, j^* L^{\mathbb{A}^1}(p^* E)]_{\operatorname{Spt}_{S^1}(\mathcal{U})} \quad (\text{by Lemma 1.9(2)}) \\
& \cong [\Sigma_{S^1}^\infty(\mathcal{U} \rightarrow \mathcal{U})_+, L^{\mathbb{A}^1}(j^* p^* E)]_{\operatorname{Spt}_{S^1}(\mathcal{U})} \quad (\text{by Lemma 1.9(1)}).
\end{aligned}$$

Using the Quillen adjoint pair  $(p_\#, p^*)$ , we see that  $p^*$  preserves connectivity. By Lemma 1.9, the same is true for  $j^*$ , so  $j^*(p^* E)$  is contained in  $\mathcal{SH}_{S^1 > c}^s(\mathcal{U})$ . By Proposition 3.1, we get  $[\Sigma_{S^1}^\infty \mathcal{U}_+, j^* p^* E] = 0$  as the scheme  $\mathcal{U}$  has dimension  $c$ .

The restrictions of  $f \in [\Sigma_{S^1}^\infty V_+, L^{\mathbb{A}^1} E]$  induce an element of the set

$$\operatorname{colim}_{j_i} [\Sigma_{S^1}^\infty U_{i,+}, L^{\mathbb{A}^1} E] = 0$$

from the left-hand side of the chain of equations above. This means that there exists an open subscheme  $W := U_i \hookrightarrow V$  with  $W \cap Z \neq \emptyset$  and  $f|_{\Sigma_{S^1}^\infty W_+} = 0$ . Since  $Z \subseteq s^*(V)$ , we have  $s^*(W) \neq \emptyset$ .  $\square$

Finally, let us mention that for connectivity results we may restrict ourselves to local base schemes.

**Lemma 4.10.** *Let  $S$  be a noetherian scheme of finite Krull-dimension and let  $r \geq 0$  be an integer. Let  $E \in \mathcal{SH}_{S^1 \geq i}^s(S)$  be a spectrum. Suppose that for all points  $s \in S$  and  $\mathfrak{s} : S_s^h \rightarrow S$ , we have that  $L^{\mathbb{A}^1} \mathfrak{s}^* E \in \mathcal{SH}_{S^1 \geq i-r}^s(S_s^h)$ . Then  $L^{\mathbb{A}^1} E \in \mathcal{SH}_{S^1 \geq i-r}^s(S)$ .*

*Proof.* After shifting, we can assume that  $i = r$ . We have to show that the sheaf  $\pi_0^{\mathbb{A}^1}(E)$  is trivial. It follows from Corollary 1.10 that  $\pi_0(L^{\mathbb{A}^1} E)$  is trivial if and only if  $\pi_0(\mathfrak{s}^* L^{\mathbb{A}^1} E)$  is trivial for all  $s \in S$ . Hence the claim follows.  $\square$

**Shifted stable  $\mathbb{A}^1$ -connectivity.** In order to obtain a uniform bound for the loss of connectivity of a spectrum  $E$ , we may restrict ourselves to a local base scheme  $S$  by Lemma 4.10. We want to invoke Proposition 4.5. Given a  $V$ -section  $f$  of a homotopy presheaf of  $E$ , we have to search for an open subscheme  $W \hookrightarrow V$

fulfilling assumptions (1) and (2) of [Proposition 4.5](#), i.e.,  $f = 0$  when restricted to  $W$ , and  $V/W$  is  $\mathbb{A}^1$ -Nisnevich-locally connected. For the former condition, we want to apply [Lemma 4.9](#). To avoid the obstacle to the latter condition coming from [Remark 4.8](#), we need to take care that  $W$  has nonempty fibre over the closed point of  $S$ . This point has codimension  $c$  equal to the dimension  $d$  of  $S$ . Thus, again by [Lemma 4.9](#), a natural candidate for a uniform bound on the loss of connectivity of  $E$  is  $c = d$ . This motivates the following definition and question.

**Definition 4.11.** A noetherian scheme  $S$  of dimension  $d$  has the *shifted stable  $\mathbb{A}^1$ -connectivity property*, if for every integer  $i$  and every spectrum  $E$  in  $\mathcal{SH}_{S^1 \geq i}^s(S)$ , the  $\mathbb{A}^1$ -localization  $L^{\mathbb{A}^1}E$  is contained in  $\mathcal{SH}_{S^1 \geq i-d}^s(S)$ .

**Question 4.12.** Let  $S$  be a regular noetherian scheme of dimension  $d$ . Does  $S$  have the shifted stable  $\mathbb{A}^1$ -connectivity property?

**Remark 4.13.** Morel's connectivity theorem ([Theorem 4.2](#) above) provides a positive answer in the case of  $S$  the spectrum of a field. In the case of  $S$  a Dedekind scheme with all residue fields infinite, we get a positive answer by [Theorem 4.16](#) below. Unfortunately, we do not have a positive or negative answer for more general base schemes.

**Remark 4.14.** The example of Ayoub discussed in [Remark 4.4](#) does not provide a negative answer to [Question 4.12](#). In fact, Ayoub gave an example of a base scheme  $S$  (of dimension  $\geq 2$ ) and a spectrum  $E$  whose homotopy sheaves  $\pi_i^{\mathbb{A}^1}(E)$  are not *strictly*  $\mathbb{A}^1$ -invariant. The latter property is a consequence of the *nonshifted* stable  $\mathbb{A}^1$ -connectivity property, i.e., the property that  $\mathbb{A}^1$ -localization does not lower the connectivity at all. However, the proof (see [[Morel 2005](#), Theorem 6.2.7]) that the nonshifted stable  $\mathbb{A}^1$ -connectivity property implies strictly  $\mathbb{A}^1$ -invariant homotopy sheaves does not carry over from the nonshifted to the shifted stable connectivity property of the base.

At least, a positive answer to [Question 4.12](#) would follow from  $\mathbb{A}^1$ -invariance of  $\mathbb{A}^1$ -homotopy sheaves  $\pi_k^{\mathbb{A}^1}(E)$  as the following proposition shows. Note that the property of a Nisnevich sheaf to be strictly  $\mathbb{A}^1$ -invariant is a stronger property than just being  $\mathbb{A}^1$ -invariant (see [Remark 4.14](#)).

**Proposition 4.15.** *Let  $S$  be a noetherian scheme of dimension  $d$ . Let  $i$  be an integer and  $E \in \mathcal{SH}_{S^1 \geq i}^s(S)$  be such that the sheaf  $\pi_k^{\mathbb{A}^1}(E)$  is  $\mathbb{A}^1$ -invariant for all integers  $k < i - d$ . Then  $L^{\mathbb{A}^1}E \in \mathcal{SH}_{S^1 \geq i-d}^s(S)$ .*

*Proof.* First note that for any open immersion  $j : S' \hookrightarrow S$  the functor  $j^*$  preserves  $\mathbb{A}^1$ -invariance of (simplicial) presheaves and  $j^*\pi_0 \cong \pi_0 j^*$ . In particular, our assumptions on the spectrum are stable under restriction to open subschemes of the base. Let  $E$  be a spectrum in  $\mathcal{SH}_{S^1 > d}^s$  with  $\mathbb{A}^1$ -invariant homotopy sheaves  $\pi_k^{\mathbb{A}^1}(E)$  in degrees  $k \leq 0$ . To prove [Proposition 4.15](#), it is again enough to show that  $\pi_0^{\mathbb{A}^1}(E)$

is trivial. We argue by induction on the dimension  $d$  of the base  $S$ . The case  $d = 0$  is [Theorem 4.2](#). Let  $d > 0$ . By [Corollary 1.10](#), we may assume that  $S$  is local with closed point  $i : \sigma \hookrightarrow S$ . Take a connected scheme  $V \in \text{Sm}_S$  with structure morphism  $p : V \rightarrow S$  and a point  $v \in V$ . It suffices to show that the Nisnevich stalk of  $\pi_0^{\mathbb{A}^1}(E)$  at  $(V, v)$  is trivial. By the induction hypothesis, we may assume that  $v$  lies in the fibre over  $\sigma$  as the open complement  $S \setminus \sigma$  has Krull-dimension strictly smaller than  $d$ . Moreover, we may assume that  $i^*(V)$  is connected. Let  $f_{(V,v)}$  be a germ in this stalk. We have to show that  $f_{(V,v)}$  is trivial. After possibly refining  $(V, v)$  Nisnevich-locally, we may assume that  $f_{(V,v)}$  is induced by an element  $f \in [\Sigma_{S^1}^\infty V_+, L^{\mathbb{A}^1} E]$ . By [Lemma 4.9](#), there exists an open subscheme  $W \hookrightarrow V$  with  $f|_{\Sigma_{S^1}^\infty W_+} = 0$  and  $i^*(W) \neq \emptyset$ . Clearly, we may assume that  $v \notin W$ . The cofibre sequence  $W_+ \rightarrow V_+ \rightarrow V/W$  induces an exact sequence

$$0 \rightarrow \tilde{\pi}_0(L^{\mathbb{A}^1} E)(V/W) \rightarrow \tilde{\pi}_0(L^{\mathbb{A}^1} E)(V) \rightarrow \tilde{\pi}_0(L^{\mathbb{A}^1} E)(W)$$

of homotopy sheaves. Here we write  $\tilde{\pi}_0(L^{\mathbb{A}^1} E)(V/W)$  for  $\text{Hom}(V/W, \tilde{\pi}_0(L^{\mathbb{A}^1} E))$ . Since the restriction of  $f$  to  $W$  is trivial, it suffices to show that  $\tilde{\pi}_0(L^{\mathbb{A}^1} E)(V/W)$  is trivial. The  $\mathbb{A}^1$ -Nisnevich-local homotopy cofibre sequence

$$j_{\sharp} j^*(V/W) \rightarrow V/W \rightarrow i_* L^{\mathbb{A}^1} i^*(V/W)$$

from [\(1.8\)](#) induces a long exact sequence

$$\dots \rightarrow [i_* L^{\mathbb{A}^1} i^*(V/W), \pi_0^{\mathbb{A}^1}(E)] \rightarrow [V/W, \pi_0^{\mathbb{A}^1}(E)] \rightarrow [j_{\sharp} j^*(V/W), \pi_0^{\mathbb{A}^1}(E)]$$

by the  $\mathbb{A}^1$ -Nisnevich-local fibrancy of  $\pi_0^{\mathbb{A}^1}(E) = \pi_0(L^{\mathbb{A}^1} E)$ . For the latter, note that a sheaf considered as a discrete simplicial presheaf is Nisnevich-locally fibrant. The right-hand side equals  $[j^*(V/W), j^* \pi_0^{\mathbb{A}^1}(E)]$ , and  $j^* \pi_0^{\mathbb{A}^1}(E) \cong \pi_0^{\mathbb{A}^1}(j^* E)$  is trivial by induction. The triviality of the set on the left-hand side follows from the triviality of  $\pi_0(i_* L^{\mathbb{A}^1} i^*(V/W))$ . By [\[Spitzweck 2014, Proposition 4.2\]](#), the latter is zero if  $\pi_0^{\mathbb{A}^1}(i^*(V/W)) = 0$ . Since  $i^*(V)$  is irreducible and  $i^*(W)$  is nonempty, we conclude by [\[Morel 2005, Lemma 6.1.4\]](#). □

**The one-dimensional case.** Using the Gabber presentation given by [Theorem 2.4](#), we can give a positive answer to [Question 4.12](#) for a Dedekind scheme  $S$  with infinite residue fields.

**Theorem 4.16.** *Let  $S$  be a Dedekind scheme and assume that all of its residue fields are infinite. Then  $S$  has the shifted stable  $\mathbb{A}^1$ -connectivity property:  $E \in \mathcal{SH}_{S^1 \geq i}^s(S)$  implies  $L^{\mathbb{A}^1} E \in \mathcal{SH}_{S^1 \geq i-1}^s(S)$ .*

*Proof.* By [Lemma 4.10](#), we may assume that  $S$  is henselian local of dimension  $\leq 1$  with infinite residue field and closed point  $\sigma$ . The case of dimension zero is covered by [Theorem 4.2](#). Hence we may assume that  $S$  is the spectrum of a

henselian discrete valuation ring. We want to apply [Proposition 4.5](#). Consider an element  $f \in [\Sigma_{S_1}^\infty V_+, L^{\mathbb{A}^1} E]$  for  $V \in \text{Sm}_S$ . We may assume that  $\sigma^*(V) \neq \emptyset$ , since otherwise we argue as Morel in the proof of [Theorem 4.2](#). By [Lemma 4.9](#) applied to the closed point  $\sigma$  of  $S$ , we find an open subscheme  $W \hookrightarrow V$  such that  $f_{\Sigma_{S_1}^\infty W_+} = 0$  and  $\sigma^*(W) \neq \emptyset$ . Let  $i : Z \hookrightarrow V$  be the reduced closed complement of  $W$ . In particular,  $Z_\sigma \neq V_\sigma$ . By [Theorem 2.1](#), the conditions of [Lemma 4.6](#) are fulfilled, and we get the second assumption  $\pi_0^{\mathbb{A}^1}(V/W) = 0$  of [Proposition 4.5](#) as well.  $\square$

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### References

- [Ayoub 2006] J. Ayoub, “Un contre-exemple à la conjecture de  $\mathbb{A}^1$ -connexité de F. Morel”, *C. R. Math. Acad. Sci. Paris* **342**:12 (2006), 943–948. [MR](#) [Zbl](#)
- [Ayoub 2007] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, I*, Astérisque **314**, 2007. [MR](#) [Zbl](#)
- [Colliot-Thélène et al. 1997] J.-L. Colliot-Thélène, R. T. Hoobler, and B. Kahn, “The Bloch–Ogus–Gabber theorem”, pp. 31–94 in *Algebraic K-theory* (Toronto, 1996), edited by V. P. Snaith, Fields Inst. Communications **16**, American Mathematical Society, Providence, RI, 1997. [MR](#) [Zbl](#)
- [Dugger 2001] D. Dugger, “Replacing model categories with simplicial ones”, *Trans. Amer. Math. Soc.* **353**:12 (2001), 5003–5027. [MR](#) [Zbl](#)
- [Dundas et al. 2003] B. I. Dundas, O. Röndigs, and P. A. Østvær, “Motivic functors”, *Doc. Math.* **8** (2003), 489–525. [MR](#) [Zbl](#)
- [Dutta 1995] S. P. Dutta, “On Chow groups and intersection multiplicity of modules, II”, *J. Algebra* **171**:2 (1995), 370–382. [MR](#) [Zbl](#)
- [EGA IV<sub>3</sub> 1966] A. Grothendieck, “Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, III”, *Inst. Hautes Études Sci. Publ. Math.* **28** (1966), 5–255. [MR](#) [Zbl](#)
- [Fausk et al. 2003] H. Fausk, P. Hu, and J. P. May, “Isomorphisms between left and right adjoints”, *Theory Appl. Categ.* **11** (2003), No. 4, 107–131. [MR](#) [Zbl](#)
- [Gelfand and Manin 2003] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, 2nd ed., Springer, 2003. [MR](#) [Zbl](#)
- [Gillet and Levine 1987] H. Gillet and M. Levine, “The relative form of Gersten’s conjecture over a discrete valuation ring: the smooth case”, *J. Pure Appl. Algebra* **46**:1 (1987), 59–71. [MR](#) [Zbl](#)
- [Grayson 1978] D. R. Grayson, “Projections, cycles, and algebraic K-theory”, *Math. Ann.* **234**:1 (1978), 69–72. [MR](#) [Zbl](#)
- [Hirschhorn 2003] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs **99**, American Mathematical Society, Providence, RI, 2003. [MR](#) [Zbl](#)

- [Hovey 1999] M. Hovey, *Model categories*, Mathematical Surveys and Monographs **63**, American Mathematical Society, Providence, RI, 1999. [MR](#) [Zbl](#)
- [Hovey 2001] M. Hovey, “Spectra and symmetric spectra in general model categories”, *J. Pure Appl. Algebra* **165**:1 (2001), 63–127. [MR](#) [Zbl](#)
- [Hoyois 2017] M. Hoyois, “The six operations in equivariant motivic homotopy theory”, *Adv. Math.* **305** (2017), 197–279. [MR](#) [Zbl](#)
- [Hu 2001] P. Hu, “Base change functors in the  $\mathbb{A}^1$ -stable homotopy category”, *Homology Homotopy Appl.* **3**:2 (2001), 417–451. [MR](#) [Zbl](#)
- [Jardine 2000] J. F. Jardine, “Motivic symmetric spectra”, *Doc. Math.* **5** (2000), 445–552. [MR](#) [Zbl](#)
- [Kai 2015] W. Kai, “A moving lemma for algebraic cycles with modulus and contravariance”, preprint, 2015. [arXiv](#)
- [Lurie 2017] J. Lurie, “Higher algebra”, preprint, 2017, available at <http://www.math.harvard.edu/~lurie/papers/HA.pdf>.
- [Margolis 1983] H. R. Margolis, *Spectra and the Steenrod algebra*, North-Holland Mathematical Library **29**, North-Holland Publishing Co., Amsterdam, 1983. [MR](#) [Zbl](#)
- [May 2001] J. P. May, “The additivity of traces in triangulated categories”, *Adv. Math.* **163**:1 (2001), 34–73. [MR](#) [Zbl](#)
- [Morel 2004] F. Morel, “An introduction to  $\mathbb{A}^1$ -homotopy theory”, pp. 357–441 in *Contemporary developments in algebraic K-theory* (Trieste, 2002), edited by M. Karoubi et al., ICTP Lecture Notes **15**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004. [MR](#) [Zbl](#)
- [Morel 2005] F. Morel, “The stable  $\mathbb{A}^1$ -connectivity theorems”, *K-Theory* **35**:1-2 (2005), 1–68. [MR](#) [Zbl](#)
- [Morel 2012] F. Morel,  *$\mathbb{A}^1$ -algebraic topology over a field*, Lecture Notes in Math. **2052**, Springer, 2012. [MR](#) [Zbl](#)
- [Morel and Voevodsky 1999] F. Morel and V. Voevodsky, “ $\mathbb{A}^1$ -homotopy theory of schemes”, *Inst. Hautes Études Sci. Publ. Math.* **90** (1999), 45–143. [MR](#) [Zbl](#)
- [Quillen 1973] D. Quillen, “Higher algebraic K-theory, I”, pp. 85–147 in *Algebraic K-theory, I: Higher K-theories* (Seattle, 1972), edited by H. Bass, Lecture Notes in Math. **341**, Springer, 1973. [MR](#) [Zbl](#)
- [Shafarevich 1994] I. R. Shafarevich, *Basic algebraic geometry 1: Varieties in projective space*, 2nd ed., Springer, 1994. [MR](#) [Zbl](#)
- [Spitzweck 2014] M. Spitzweck, “Algebraic cobordism in mixed characteristic”, preprint, 2014. [arXiv](#)
- [Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, “Higher algebraic K-theory of schemes and of derived categories”, pp. 247–435 in *The Grothendieck Festschrift*, vol. III, edited by P. Cartier et al., Progress in Math. **88**, Birkhäuser, Boston, 1990. [MR](#) [Zbl](#)

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