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Let Γ be a discrete group. Assuming rational injectivity of the Baum–Connes assembly map, we provide new lower bounds on the rank of the positive scalar curvature bordism group and the relative group in Stolz’ positive scalar curvature sequence for $B\Gamma$. The lower bounds are formulated in terms of the part of degree up to 2 in the group homology of Γ with coefficients in the $\mathbb{C}\Gamma$ -module generated by finite order elements. Our results use and extend work of Botvinnik and Gilkey which treated the case of finite groups. Further crucial ingredients are a real counterpart to the delocalized equivariant Chern character and Matthey’s work on explicitly inverting this Chern character in low homological degrees.

1. Introduction

There exists a natural comparison mapping between the positive scalar curvature (psc) sequence of Stolz (top row) to the analytic surgery sequence of Higson and Roe (bottom row):

$$\begin{array}{ccccccccc}
 \Omega_n^{\text{spin}}(B\Gamma) & \longrightarrow & R_n^{\text{spin}}(B\Gamma) & \xrightarrow{\partial} & P_{n-1}^{\text{spin}}(B\Gamma) & \longrightarrow & \Omega_{n-1}^{\text{spin}}(B\Gamma) & \longrightarrow & R_{n-1}^{\text{spin}}(B\Gamma) \\
 \downarrow \beta & & \downarrow \alpha & & \downarrow \rho & & \downarrow \beta & & \downarrow \alpha \\
 KO_n(B\Gamma) & \xrightarrow{\nu} & KO_n(C_r^*\Gamma) & \xrightarrow{\partial} & S_{n-1}^{\mathbb{R}}(\Gamma) & \longrightarrow & KO_{n-1}(B\Gamma) & \xrightarrow{\nu} & KO_{n-1}(C_r^*\Gamma)
 \end{array} \tag{1.1}$$

This diagram was first established by Piazza and Schick [2014, Theorem 1.28] for complex K-theory and n even. It was extended by Xie and Yu [2014, Theorem B] and by the second-named author [2016b, Theorem 3.1.13] to cover all dimensions and the real case. See also Zenobi [2017, Remark 6.2].

We briefly explain the constituents in the diagram above. Start with Stolz’ psc sequence. The group $\Omega_n^{\text{spin}}(B\Gamma)$ is the singular *spin bordism group* of the classifying space $B\Gamma$. That is, it consists of bordism classes of pairs (M, ϕ) , where M is a closed spin manifold of dimension n and $\phi : M \rightarrow B\Gamma$ a continuous map. The *psc spin bordism group* $P_n^{\text{spin}}(B\Gamma)$ consists of bordism classes of (M, ϕ, g) ,

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where (M, ϕ) is as before and $g \in \mathcal{R}^+(M)$ is a metric of psc. Here we require bordisms to have metrics of positive scalar curvature with product structure near the boundary. *Stolz' relative group* $R_{n+1}^{\text{spin}}(\text{B}\Gamma)$ consists of bordism classes of triples (W, ϕ, g) , where W is a compact spin manifold of dimension $(n+1)$, $\phi : W \rightarrow \text{B}\Gamma$ a continuous map, and $g \in \mathcal{R}^+(\partial W)$ a metric of psc on the boundary. The horizontal arrows in Stolz' sequence are the evident forgetful maps.

The (real version of the) *analytic surgery sequence* of Higson and Roe consists of the *real K-homology* of $\text{B}\Gamma$, the *topological K-theory of the reduced group C*-algebra* of Γ , and the *analytic structure group* of Γ . We denote the latter by $S_*^{\mathbb{R}}(\Gamma)$. It is defined in such a way that it fits into a long exact sequence together with the real *Novikov assembly map* $\nu : \text{KO}_*(\text{B}\Gamma) \rightarrow \text{KO}_*(C_r^*\Gamma)$. We also use their complex counterparts, which we denote by $K_*(\text{B}\Gamma)$, $K_*(C_r^*\Gamma)$, and $S_*^{\mathbb{C}}(\Gamma)$.

The groups $P_{n-1}^{\text{spin}}(\text{B}\Gamma)$ and $R_n^{\text{spin}}(\text{B}\Gamma)$ classify psc metrics up to bordism and concordance, respectively, on spin manifolds with fundamental group Γ . For the latter see [Rosenberg and Stolz 2001, Theorem 5.1]. Alas, at present there are no tools known that allow a computation of these groups (not even in simple special cases). However, the comparison (1.1) allows us to obtain lower bounds on these groups using the index-theoretic information contained in the sequence of Higson and Roe. To that end, it is important to know something about the size of the image of the *relative index map* $\alpha : R_n^{\text{spin}}(\text{B}\Gamma) \rightarrow \text{KO}_n(C_r^*\Gamma)$ and the *higher ρ -invariant* $\rho : P_{n-1}^{\text{spin}}(\text{B}\Gamma) \rightarrow S_{n-1}^{\mathbb{R}}(\Gamma)$.

The first case where something can be said is the class of finite groups. Indeed, let H be a finite group. Let $R(H)$ denote its complex representation ring. Let $R_0^q(H)$ be the subgroup generated by those representations ρ of virtual dimension 0 such that its character χ_ρ satisfies $\chi_\rho(\gamma^{-1}) = (-1)^q \chi_\rho(\gamma)$ for all $\gamma \in H$. Botvinnik and Gilkey [1995, Theorem 2.1] showed that the rank of the positive scalar curvature bordism group $P_{2q+4k-1}^{\text{spin}}(\text{B}H)$ is bounded below by the rank of $R_0^q(H)$, where $k \geq 1, q \in \{0, 1\}$ with $4k + 2q \geq 6$. They used relative η -invariants. These are numerical invariants that are related to the higher ρ -invariant via certain trace maps; see for instance [Higson and Roe 2010]. In fact, Botvinnik and Gilkey's result [1995] implies that both

$$\rho : P_{n-1}^{\text{spin}}(\text{B}H) \rightarrow S_{n-1}^{\mathbb{R}}(H) \quad \text{and} \quad \alpha : R_n^{\text{spin}}(\text{B}H) \rightarrow \text{KO}_n(C_r^*H)$$

are *rationally surjective* for $n \geq 6$ (we explain this in Section 4). Moreover, recently Weinberger and Yu [2015] and Xie and Yu [2017] gave lower bounds for a large class of infinite groups based on the number of torsion elements with pairwise different orders. We also refer to [Piazza and Schick 2007] for lower bounds on the positive scalar bordism group based on the L^2 - ρ -invariant.

The results mentioned above only yield information for n even. Using product formulas one can obtain further ad hoc examples of nontrivial relative indices and

ρ -invariants by taking certain products; see [Zeidler 2016a, Corollary 6.10; Zenobi 2017, Corollary 5.24]. For instance, taking products with the circle allows one to also produce examples for n odd.

In the main results of this paper, we give new systematic lower bounds for all $n \geq 7$ on the image of the relative index and the ρ -invariant based on the part of degree up to 2 of a certain group homology. The result of Botvinnik and Gilkey [1995] yields the 0-dimensional part. Then the idea is that degrees 1 and 2 can be obtained from this by taking products with circles and oriented surfaces. We use the *Baum–Connes assembly map* $\mu : K_*^\Gamma(\underline{E}\Gamma) \rightarrow K_*(C_r^*\Gamma)$, the *delocalized Chern character* of Baum and Connes [1988], and — most centrally — its explicit partial inverse in degrees up to 2 due to Matthey [2004].

To state our results, we start with some preparations. Let Γ be a discrete group and denote by Γ_{fin} the set of elements of finite order of Γ . Let $F\Gamma$ be the set of all finitely supported functions $\Gamma_{\text{fin}} \rightarrow \mathbb{C}$. Letting Γ act by conjugation on Γ_{fin} turns $F\Gamma$ into a $\mathbb{C}\Gamma$ -module. The delocalized equivariant Chern character yields an isomorphism

$$\text{ch}_\Gamma : K_p^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{k \in \mathbb{Z}} H_{p+2k}(\Gamma; F\Gamma). \tag{1.2}$$

It was first introduced by Baum and Connes [1988] but we will instead work with the “handcrafted Chern character” of Matthey [2004]. Matthey [2004] also constructed maps

$$\beta_p^{(t)} : H_p(\Gamma; F\Gamma) \rightarrow K_p^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}$$

for $p \in \{0, 1, 2\}$ which are right-inverse to the delocalized Chern character. Moreover, he defined explicit maps

$$\beta_p^{(a)} : H_p(\Gamma; F\Gamma) \rightarrow K_p(C_r^*\Gamma)$$

which satisfy $\beta_p^{(a)} = (\mu \otimes \mathbb{C}) \circ \beta_p^{(t)}$ for $p \in \{0, 1, 2\}$. They thereby describe the Baum–Connes assembly map explicitly in low homological degrees.

To use these maps for our purposes, we need to adapt the above to real K-homology. To that end, for $q \in \{0, 1\}$, let

$$F^q\Gamma = \{f \in F\Gamma \mid f(\gamma) = (-1)^q f(\gamma^{-1}) \ \forall \gamma \in \Gamma_{\text{fin}}\}.$$

Then $F\Gamma = F^0\Gamma \oplus F^1\Gamma$ as $\mathbb{C}\Gamma$ -modules. We can now state our main result and its corollaries.

Theorem 1.3. *For each $p \in \{0, 1, 2\}$, $q \in \{0, 1\}$ and $k \geq 1$ with $4k + 2q \geq 6$, there exists a linear map*

$$\beta_{p,q,k}^{(\text{psc})} : H_p(\Gamma; F^q\Gamma) \rightarrow R_{p+2q+4k}^{\text{spin}}(\text{B}\Gamma) \otimes \mathbb{C}$$

such that the following diagram commutes:

$$\begin{array}{ccc}
 H_p(\Gamma; F^q \Gamma) & \xrightarrow{\quad} & H_p(\Gamma; F\Gamma) \\
 \downarrow \beta_{p,q,k}^{(\text{psc})} & & \downarrow \beta_p^{(a)} \\
 R_{p+2q+4k}^{\text{spin}}(\mathbf{B}\Gamma) \otimes \mathbb{C} & & \\
 \downarrow \alpha \otimes \mathbb{C} & & \\
 KO_{p+2q}(C_r^* \Gamma) \otimes \mathbb{C} & \xrightarrow{c \otimes \mathbb{C}} & K_p(C_r^* \Gamma) \otimes \mathbb{C}
 \end{array}$$

Here $c : KO_*(C_r^* \Gamma) \rightarrow K_*(C_r^* \Gamma)$ is the complexification map from real to complex K-theory. We implicitly use that complex K-theory is 2-periodic and real K-theory is rationally 4-periodic.

Remark 1.4. We do not claim that our maps $\beta_{p,q,k}^{(\text{psc})}$ are canonical (unlike the original maps of Matthey). Indeed, their construction depends on choosing preimages under the surjective map $\alpha \otimes \mathbb{Q} : R_*^{\text{spin}}(\mathbf{B}H) \otimes \mathbb{Q} \rightarrow KO_*(C_r^* H) \otimes \mathbb{Q}$ for each finite cyclic group H . However, after fixing these choices it is in principle possible to trace through the construction to obtain explicit formulas for $\beta_{p,q,k}^{(\text{psc})}$ similarly as in Matthey’s work.

In any case, the existence of $\beta_{p,q,k}^{(\text{psc})}$ implies lower bounds and surjectivity results:

Corollary 1.5. *Let $n \geq 7$. If the rational Baum–Connes assembly map $\mu \otimes \mathbb{Q}$ is injective, then the rank of $R_n^{\text{spin}}(\mathbf{B}\Gamma)$ is at least the dimension of*

$$\begin{cases}
 H_0(\Gamma; F^0 \Gamma) \oplus H_2(\Gamma; F^1 \Gamma), & n \equiv 0 \pmod{4}, \\
 H_1(\Gamma; F^0 \Gamma), & n \equiv 1 \pmod{4}, \\
 H_0(\Gamma; F^1 \Gamma) \oplus H_2(\Gamma; F^0 \Gamma), & n \equiv 2 \pmod{4}, \\
 H_1(\Gamma; F^1 \Gamma), & n \equiv 3 \pmod{4}.
 \end{cases}$$

Corollary 1.6. *Let $n \geq 7$. If the rational Baum–Connes assembly map $\mu \otimes \mathbb{Q}$ is injective, then the rank of $P_{n-1}^{\text{spin}}(\mathbf{B}\Gamma)$ is at least the dimension of*

$$\begin{cases}
 H_0(\Gamma; F_0^0 \Gamma) \oplus H_2(\Gamma; F^1 \Gamma), & n \equiv 0 \pmod{4}, \\
 H_1(\Gamma; F_0^0 \Gamma), & n \equiv 1 \pmod{4}, \\
 H_0(\Gamma; F^1 \Gamma) \oplus H_2(\Gamma; F_0^0 \Gamma), & n \equiv 2 \pmod{4}, \\
 H_1(\Gamma; F^1 \Gamma), & n \equiv 3 \pmod{4},
 \end{cases}$$

where $F_0^0 = \{f \in F^0 \mid f(1) = 0\}$.

In comparison, Botvinnik and Gilkey [1995, Theorem 0.1] imply that for a finite group H and $n \geq 6$ even, the rank of $P_{n-1}^{\text{spin}}(\mathbf{B}H)$ is bounded below by the dimension of $H_0(H; F_0^0 H)$ if $n \equiv 0 \pmod{4}$ or $H_0(H; F^1 H)$ if $n \equiv 2 \pmod{4}$.

Example 1.7. We describe an explicit example illustrating the nontrivial content of Corollaries 1.5 and 1.6. Let Σ_g denote the oriented surface of genus $g \geq 1$. Let n be a positive integer. Consider the group $\Gamma = \pi_1(\Sigma_g) \times \mathbb{Z}/n\mathbb{Z}$. Then the Baum–Connes assembly map for Γ is an isomorphism.¹ So our results are applicable. Next we explicitly compute the homology groups that appear in the corollaries for this example. Start with the group homology of Γ with trivial coefficients \mathbb{C} . The group homology of $\mathbb{Z}/n\mathbb{Z}$ is torsion in all positive degrees. Hence the Künneth theorem implies that $H_*(\Gamma; \mathbb{C}) \cong H_*(\Sigma_g; \mathbb{C})$. Thus the homology of Γ is $H_p(\Gamma; \mathbb{C}) \cong \mathbb{C}$ for $p \in \{0, 2\}$, $H_1(\Gamma; \mathbb{C}) \cong \mathbb{C}^{2g}$, and zero in degrees greater than 2. To proceed, observe that any finite order element of Γ is of the form $(1, t^k)$, where 1 denotes the neutral element of $\pi_1(\Sigma_g)$ and t the generator of $\mathbb{Z}/n\mathbb{Z}$. The action by conjugation is trivial on these elements. We deduce that $F\Gamma$ is isomorphic to the trivial $\mathbb{C}\Gamma$ -module \mathbb{C}^n . By counting dimensions, we see that $F^0\Gamma \cong \mathbb{C}^{\lfloor n/2 \rfloor + 1}$, $F_0^0\Gamma \cong \mathbb{C}^{\lfloor n/2 \rfloor}$, $F^1\Gamma \cong \mathbb{C}^{\lceil n/2 \rceil - 1}$ as trivial $\mathbb{C}\Gamma$ -modules. Together with the computation of $H_*(\Gamma; \mathbb{C})$, we deduce

$$\begin{aligned} H_0(\Gamma; F^0\Gamma) &\cong \mathbb{C}^{\lfloor n/2 \rfloor + 1}, & H_0(\Gamma; F_0^0\Gamma) &\cong \mathbb{C}^{\lfloor n/2 \rfloor}, & H_0(\Gamma; F^1\Gamma) &\cong \mathbb{C}^{\lceil n/2 \rceil - 1}, \\ H_1(\Gamma; F^0\Gamma) &\cong \mathbb{C}^{2g(\lfloor n/2 \rfloor + 1)}, & H_1(\Gamma; F_0^0\Gamma) &\cong \mathbb{C}^{2g\lfloor n/2 \rfloor}, & H_1(\Gamma; F^1\Gamma) &\cong \mathbb{C}^{2g(\lceil n/2 \rceil - 1)}, \\ H_2(\Gamma; F^0\Gamma) &\cong \mathbb{C}^{\lfloor n/2 \rfloor + 1}, & H_2(\Gamma; F_0^0\Gamma) &\cong \mathbb{C}^{\lfloor n/2 \rfloor}, & H_2(\Gamma; F^1\Gamma) &\cong \mathbb{C}^{\lceil n/2 \rceil - 1}. \end{aligned}$$

This shows that for $n \geq 3$ all homology groups which appear in the conclusion of Corollaries 1.5 and 1.6 are nontrivial.

Corollary 1.8. *Let $n \geq 7$. Let the rational homological dimension of Γ be at most 2. Then, if the rational Baum–Connes assembly map $\mu \otimes \mathbb{Q}$ is surjective, the rational relative index map*

$$\alpha \otimes \mathbb{Q} : R_n^{\text{spin}}(B\Gamma) \otimes \mathbb{Q} \rightarrow KO_n(C_r^*\Gamma) \otimes \mathbb{Q}$$

is surjective.

If $\mu \otimes \mathbb{Q}$ is an isomorphism, then the rational higher ρ -invariant

$$\rho \otimes \mathbb{Q} : P_{n-1}^{\text{spin}}(B\Gamma) \otimes \mathbb{Q} \rightarrow S_{n-1}^{\mathbb{R}}(\Gamma) \otimes \mathbb{Q}$$

is also surjective.

2. The delocalized equivariant Pontryagin character

In this section, we exhibit the *delocalized equivariant Pontryagin character*, which is the real counterpart to the delocalized equivariant Chern character. It is obtained from the delocalized Chern character by precomposing it with complexification.

¹This follows readily from [Higson and Kasparov 2001] because $\pi_1(\Sigma_g) \times \mathbb{Z}/n\mathbb{Z}$ is a-T-menable. However, we should note that the case of surface groups goes back to the original article of Baum and Connes [2000].

Start with some preparations. The rationalized equivariant real K-homology $KO_* \otimes \mathbb{Q}$ is 4-periodic. Indeed, it is a module over

$$KO_*(pt) \otimes \mathbb{Q} \cong \mathbb{Q}[\alpha, \beta, \beta^{-1}] / \langle \alpha^2 - 4\beta \rangle$$

with $\alpha \in KO_4(pt)$, $\beta \in KO_8(pt)$, and module multiplication with $\alpha/2$ implements the 4-periodicity. We will implicitly use this 4-periodicity whenever convenient.

The complexification $c : KO_*(pt) \rightarrow K_*(pt) \cong \mathbb{Z}[\xi, \xi^{-1}]$ satisfies $c(\alpha) = 2\xi^2$ and $c(\beta) = \xi^4$, where $\xi \in K_2(pt)$.

Complex K-homology rationally decomposes into two copies of real K-homology:

Proposition 2.1. *Complexification yields an isomorphism of proper equivariant homology theories:*

$$c := c + \xi^{-1} c : (KO_* \oplus KO_{*+2}) \otimes \mathbb{Q} \xrightarrow{\cong} K_* \otimes \mathbb{Q}. \tag{2.2}$$

The decomposition (2.2) can be used to decompose the equivariant delocalized Chern character and thereby obtain the *delocalized Pontryagin character*:

Proposition 2.3. *The equivariant delocalized Chern character composed with complexification yields an isomorphism*

$$\underline{ph}_\Gamma := \underline{ch}_\Gamma \circ c : KO_p^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{k \in \mathbb{Z}} H_{p+4k}(\Gamma; F^0\Gamma) \oplus H_{p+2+4k}(\Gamma; F^1\Gamma).$$

Because Matthey’s maps are right-inverse to the delocalized Chern character, Proposition 2.3 immediately implies that they decompose as follows:

Corollary 2.4. *Using the identification (2.2), Matthey’s inverse maps [2004]*

$$\beta_p^{(t)} : H_p(\Gamma; F\Gamma) \rightarrow K_p^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}, \quad p \in \{0, 1, 2\},$$

restrict to maps

$$\beta_{p,q}^{(t)} : H_p(\Gamma; F^q\Gamma) \rightarrow KO_{p+2q}^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}, \quad p \in \{0, 1, 2\}, q \in \{0, 1\}.$$

Analogously to the complex case we write $\beta_{p,q}^{(a)} := \mu \otimes \mathbb{C} \circ \beta_{p,q}^{(t)}$. By abuse of notation $\mu : KO_{p+2q}^\Gamma(\underline{E}\Gamma) \rightarrow KO_{p+2q}^\Gamma(C_1^*\Gamma)$ denotes the real version of the Baum–Connes assembly map here.

The proofs of Propositions 2.1 and 2.3 can essentially be reduced to the case of finite groups. Thus we need some facts about equivariant K-homology of a finite group and to fix some notation. For more details on the following we refer for example to [Bruner and Greenlees 2010, Chapter 2].

Let $R(H)$ denote the complex representation ring and $RO(H)$ its real counterpart. Let $c : RO(H) \rightarrow R(H)$ be complexification and $r : R(H) \rightarrow RO(H)$ be realification. Let $\tau : R(H) \rightarrow R(H)$ be the map induced by complex conjugation.

Then $c \circ r = 1 + \tau$ and $r \circ c = 2$. Let $R(H)/(1 + \tau)$ be a shorthand for the quotient group $R(H)/(1 + \tau)(R(H))$. Then the equivariant K-homology of a point satisfies

$$KO_i^H(\text{pt}) \otimes \mathbb{Q} \cong \begin{cases} RO(H) \otimes \mathbb{Q}, & i \equiv 0 \pmod{4}, \\ R(H)/(1 + \tau) \otimes \mathbb{Q}, & i \equiv 2 \pmod{4}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.5}$$

$$K_i^H(\text{pt}) \otimes \mathbb{Q} \cong \begin{cases} R(H) \otimes \mathbb{Q}, & i \equiv 0 \pmod{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Complexification $c : KO_i^H(\text{pt}) \rightarrow K_i^H(\text{pt})$ is given by complexification of representations for $i \equiv 0 \pmod{4}$, and by the map $1 - \tau : R(H)/(1 + \tau) \rightarrow R(H)$ for $i \equiv 2 \pmod{4}$.

We are now ready to prove the propositions of this section.

Proof of Proposition 2.1. To show that \mathbf{c} is an isomorphism of proper equivariant homology theories, it suffices to show that

$$\mathbf{c} = c + \xi^{-1} c : (KO_i^H(\text{pt}) \oplus KO_{i+2}^H(\text{pt})) \otimes \mathbb{Q} \rightarrow K_i^H(\text{pt}) \otimes \mathbb{Q} \tag{2.7}$$

is an isomorphism for every finite group H and every $i \in \mathbb{Z}$. The map

$$(KO_i^\bullet \oplus KO_{i+2}^\bullet) \otimes \mathbb{Q} \rightarrow (KO_{i+2}^\bullet \oplus KO_{i+4}^\bullet) \otimes \mathbb{Q}, \quad x \oplus y \mapsto y \oplus (\alpha/2) \cdot x$$

defines a 2-periodicity on the left-hand side which corresponds to Bott periodicity after applying $\mathbf{c} = c + \xi^{-1} c$. Hence it suffices to check (2.7) for $i \in \{0, 1\}$. For $i = 1$, both sides of (2.7) are zero by (2.5) and (2.6). It remains to check $i = 0$. In this case \mathbf{c} corresponds to the map

$$(RO(H) \oplus R(H)/(1 + \tau)) \otimes \mathbb{Q} \rightarrow R(H) \otimes \mathbb{Q}, \quad x \oplus [y] \mapsto c(x) + y - \tau(y).$$

This is an isomorphism because the map

$$R(H) \otimes \mathbb{Q} \rightarrow (RO(H) \oplus R(H)/(1 + \tau)) \otimes \mathbb{Q}, \quad z \mapsto \frac{1}{2}(r(z) \oplus [z])$$

is an explicit inverse. □

Proof of Proposition 2.3. Start with the (nonequivariant) Pontryagin character isomorphism

$$\text{ph} = \text{ch} \circ c : KO_p(X) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{k \in \mathbb{Z}} H_{p+4k}(X; \mathbb{Q}). \tag{2.8}$$

It is, by definition, the Chern character applied after complexification. In particular, the proposition holds if Γ is torsion-free.

Next we deal with the case where $\Gamma = H = \mathbb{Z}/n\mathbb{Z}$ is a finite cyclic group. Then the Chern character yields an isomorphism $\underline{\text{ch}}_H : R(H) = K_0^H(\text{pt}) \rightarrow H_0(H; FH)$ as all the other homology groups vanish. We have $H_0(H; FH) = FH = \mathbb{C}^H$ and $\underline{\text{ch}}_H$ associates to a representation $[\rho] \in R(H)$ its character χ_ρ . As the character of

any real representation is symmetric and the character of an element in the image of $(1 - \tau) : R(H) \rightarrow R(H)$ is antisymmetric, we conclude that

$$\underline{\text{ch}}_\Gamma(c(\text{KO}_{2q}^H(\text{pt}))) = H_0(H; F^q H). \tag{2.9}$$

Next, let G be some group and consider $\Gamma := G \times \mathbb{Z}/n\mathbb{Z}$ and $y \times z \in \text{KO}_{p+2q}^\Gamma(\underline{E}\Gamma)$ with $y \in \text{KO}_p(BG) \cong \text{KO}_p^G(EG)$ and $z \in \text{KO}_{2q}^{\mathbb{Z}/n\mathbb{Z}}(\text{pt})$. There is a natural isomorphism

$$\text{K}_p^{G \times \mathbb{Z}/n\mathbb{Z}}(\underline{E}G) \cong \text{K}_p^G(\underline{E}G) \otimes R(\mathbb{Z}/n\mathbb{Z}) \cong \text{K}_p^G(\underline{E}G) \otimes \text{K}_0^{\mathbb{Z}/n\mathbb{Z}}(\text{pt})$$

and a commutative diagram

$$\begin{array}{ccc} \text{K}_p(BG) \otimes \text{K}_0^{\mathbb{Z}/n\mathbb{Z}}(\text{pt}) \otimes \mathbb{C} & \xrightarrow{\times} & \text{K}_p^{G \times \mathbb{Z}/n\mathbb{Z}}(\underline{E}G) \otimes \mathbb{C} \\ \downarrow \text{ch} \otimes \text{ch}_{\mathbb{Z}/n\mathbb{Z}} & & \downarrow \text{ch}_\Gamma \\ \bigoplus_{k \in \mathbb{Z}} H_{p+2k}(G; \mathbb{C}) \otimes H_0(\mathbb{Z}/n\mathbb{Z}; F\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\times} & \bigoplus_{k \in \mathbb{Z}} H_{p+2k}(\Gamma; F\Gamma) \end{array}$$

In view of (2.2), (2.8), and (2.9) this diagram restricts to

$$\begin{array}{ccc} \text{KO}_p(BG) \otimes \text{KO}_{2q}^{\mathbb{Z}/n\mathbb{Z}}(\text{pt}) \otimes \mathbb{C} & \xrightarrow{\times} & \text{KO}_{p+2q}^{G \times \mathbb{Z}/n\mathbb{Z}}(\underline{E}G) \otimes \mathbb{C} \\ \downarrow \text{ph} \otimes \text{ch}_{\mathbb{Z}/n\mathbb{Z}} \circ c & & \downarrow \text{ch}_\Gamma \circ c \\ \bigoplus_{k \in \mathbb{Z}} H_{p+4k}(G; \mathbb{C}) \otimes H_0(\mathbb{Z}/n\mathbb{Z}; F^q \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{\times} & \bigoplus_{k \in \mathbb{Z}} H_{p+4k}(\Gamma; F^q \Gamma) \end{array}$$

We conclude

$$\underline{\text{ch}}_\Gamma(c(y \times z)) \in \bigoplus_{k \in \mathbb{Z}} H_{p+4k}(\Gamma; F^q \Gamma). \tag{2.10}$$

Now let Γ be general. The equivariant K-homology $\text{K}_p^\Gamma(\underline{E}\Gamma)$ is generated by elements of the form $\varphi_*(y \times z)$ with $G \subseteq \Gamma$ some subgroup, $y \in \text{K}_p(BG)$, $z \in \text{K}_0^{\mathbb{Z}/n\mathbb{Z}}(\text{pt})$ and $\varphi : G \times \mathbb{Z}/n\mathbb{Z} \rightarrow \Gamma$ some group homomorphism. This follows from [Matthey 2004, Theorem 1.3 and Section 7]. Using (2.2), this implies that $\text{KO}_i^\Gamma(\underline{E}\Gamma)$ is generated by elements of the form $\varphi_*(y \times z)$ with $y \in \text{KO}_p(BG)$, $z \in \text{KO}_{2q}^{\mathbb{Z}/n\mathbb{Z}}(\text{pt})$ such that $i = p + 2q$. Thus (2.10) implies

$$(\underline{\text{ch}}_\Gamma \circ c)(\text{KO}_p^\Gamma(\underline{E}\Gamma)) \subseteq \bigoplus_{k \in \mathbb{Z}} H_{p+4k}(\Gamma; F^0 \Gamma) \oplus H_{p+2+4k}(\Gamma; F^1 \Gamma). \tag{2.11}$$

Now the proposition follows by combining (2.2) and (2.11) and using the fact that $\underline{\text{ch}}_\Gamma$ is an isomorphism. \square

3. Matthey’s maps

In this section, we exhibit the real versions of Matthey’s maps from Corollary 2.4 more explicitly. We start with a brief summary of the material in [Matthey 2004]

that leads to his definition of

$$\beta_p^{(t)} : H_p(\Gamma; F(\Gamma)) \rightarrow K_p^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}, \quad p \in \{0, 1, 2\}.$$

Let $G^{(0)} := 1$, $G^{(1)} := \mathbb{Z}$ and $G^{(2),g} := \Gamma_g := \pi_1(\Sigma_g)$, where Σ_g is the oriented surface of genus $g \geq 1$. To simplify the notation, we let $G^{(2)}$ stand for $G^{(2),g}$ for some g . Moreover, let $G_n^{(p)} := G^{(p)} \times \mathbb{Z}/n\mathbb{Z}$.

There is an isomorphism $H_p(\Gamma; F\Gamma) \cong \bigoplus_C H_p(BZ_C; \mathbb{C})$. Here the direct sum runs over all conjugacy classes of finite order elements and Z_C denotes the centralizer of some element in the conjugacy class C . An element $x \in H_p(\Gamma; F\Gamma)$ is called *homogeneous* if it is contained in one of the direct summands. For each $p \in \{0, 1, 2\}$ and $n \geq 0$, there is a certain fundamental class $[G_n^{(p)}] \in H_p(G_n^{(p)}; FG_n^{(p)})$. These have the property that for $p \in \{0, 1, 2\}$ any homogeneous element $x \in H_p(\Gamma; F\Gamma)$ can be written as $x = \phi_*[G_n^{(p)}]$ for some $G_n^{(p)}$ and some group homomorphism $\phi : G_n^{(p)} \rightarrow \Gamma$.

Moreover, there is a certain K-homological fundamental class

$$[G_n^{(p)}]_{\mathbb{K}} \in K_p^{G_n^{(p)}}(\underline{E}G_n^{(p)}) \otimes \mathbb{C}.$$

Setting $\phi_*[G_n^{(p)}] \mapsto \phi_*[G_n^{(p)}]_{\mathbb{K}}$ defines a map $H_p(\Gamma; F(\Gamma)) \rightarrow K_p^\Gamma(\underline{E}\Gamma) \otimes \mathbb{C}$ that is right-inverse to the equivariant Chern character. This is Matthey’s definition of $\beta_p^{(t)}$.

To describe how this map decomposes in the real case, we need to recall the definition of the K-homological fundamental class $[G_n^{(p)}]_{\mathbb{K}}$. We start with the fundamental classes $[G^{(p)}]_{\mathbb{K}} \in K_p(\underline{E}G^{(p)})$, which are defined as

$$\begin{aligned} [G^{(0)}]_{\mathbb{K}} &:= 1 \in K_0(\underline{E}G^{(0)}) = K_0(\text{pt}), \\ [G^{(1)}]_{\mathbb{K}} &:= [S^1]_{\mathbb{K}} \in K_1(S^1) \cong K_1^{\mathbb{Z}}(\underline{E}\mathbb{Z}), \\ [G^{(2)}]_{\mathbb{K}} &:= [\Sigma_g]_{\mathbb{K}} \in K_2(\Sigma_g) \cong K_2^{\Gamma_g}(\underline{E}\Gamma_g). \end{aligned}$$

That is, $[G^{(p)}]_{\mathbb{K}}$ is the K-homological fundamental class of the point, the circle or an oriented surface of positive genus. Observe that we may take $\underline{E}G_n^{(p)} = \underline{E}G^{(p)}$ by letting $\mathbb{Z}/n\mathbb{Z}$ act trivially. Then we set

$$[G_n^{(p)}]_{\mathbb{K}} := \sum_{l=0}^{n-1} ([G^{(p)}]_{\mathbb{K}} \times [\omega_n^l]) \otimes \omega_n^{-l} \in K_p^{G_n^{(p)}}(\underline{E}G_n^{(p)}) \otimes \mathbb{C},$$

where $\omega_n := e^{2\pi i/n}$ is the primitive n -th root of unity and $[\omega_n^l] \in R(\mathbb{Z}/n\mathbb{Z}) = K_0^{\mathbb{Z}/n\mathbb{Z}}(\text{pt})$ the corresponding representation.

To obtain the real counterparts to this, we first observe how $[\omega_n^l] \in K_0^{\mathbb{Z}/n\mathbb{Z}}(\text{pt})$ decomposes under the isomorphism from Proposition 2.1, (2.5) and (2.6). Indeed,

$$\begin{aligned} \mathbf{K}_0^{\mathbb{Z}/n\mathbb{Z}}(\text{pt}) \otimes \mathbb{Q} &\cong (\mathbf{KO}_0^{\mathbb{Z}/n\mathbb{Z}}(\text{pt}) \oplus \mathbf{KO}_2^{\mathbb{Z}/n\mathbb{Z}}(\text{pt})) \otimes \mathbb{Q} \\ &\cong (\mathbf{RO}(\mathbb{Z}/n\mathbb{Z}) \oplus \mathbf{R}(\mathbb{Z}/n\mathbb{Z})/(1 + \tau)) \otimes \mathbb{Q}. \end{aligned}$$

Given an element $x \in \mathbf{R}(\mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{Q}$, we write $\mathfrak{R}x := r(x)/2 \in \mathbf{RO}(\mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{Q}$ and $\mathfrak{S}x \in \mathbf{R}(\mathbb{Z}/n\mathbb{Z})/(1 + \tau) = \mathbf{KO}_2^{\mathbb{Z}/n\mathbb{Z}}(\text{pt})$ for the class represented by $x/2$. Then we have $x = c(\mathfrak{R}x + \mathfrak{S}x)$. We define

$$\begin{aligned} [G_n^{(p)}]_{\mathbf{KO}}^0 &:= \sum_{l=0}^{n-1} ([G^{(p)}]_{\mathbf{KO}} \times \mathfrak{R}[\omega_n^l]) \otimes \omega_n^{-l} \in \mathbf{KO}_p^{G_n^{(p)}}(\underline{\mathbf{E}}G_n^{(p)}) \otimes \mathbb{C}, \\ [G_n^{(p)}]_{\mathbf{KO}}^1 &:= \sum_{l=0}^{n-1} ([G^{(p)}]_{\mathbf{KO}} \times \mathfrak{S}[\omega_n^l]) \otimes \omega_n^{-l} \in \mathbf{KO}_{p+2}^{G_n^{(p)}}(\underline{\mathbf{E}}G_n^{(p)}) \otimes \mathbb{C}. \end{aligned} \tag{3.1}$$

Here $[G^{(p)}]_{\mathbf{KO}}$ denotes the KO-fundamental class of the point, the circle or a surface, respectively. We find that $[G_n^{(p)}]_{\mathbf{K}} = [G_n^{(p)}]_{\mathbf{KO}}^0 \oplus [G_n^{(p)}]_{\mathbf{KO}}^1$ under the isomorphism from Proposition 2.1. The homological fundamental class also decomposes as $[G_n^{(p)}] = [G_n^{(p)}]^0 \oplus [G_n^{(p)}]^1$ according to

$$\mathbf{H}_p(G_n^{(p)}; \mathbf{F}G_n^{(p)}) = \mathbf{H}_p(G_n^{(p)}; \mathbf{F}^0 G_n^{(p)}) \oplus \mathbf{H}_p(G_n^{(p)}; \mathbf{F}^1 G_n^{(p)}).$$

From this discussion we deduce:

Proposition 3.2. *The real versions of Matthey’s maps from Corollary 2.4 are given by*

$$\beta_{p,q}^{(t)} : \mathbf{H}_p(\Gamma; \mathbf{F}^q(\Gamma)) \rightarrow \mathbf{KO}_{p+2q}^\Gamma(\underline{\mathbf{E}}\Gamma) \otimes \mathbb{C}, \quad \phi_*[G_n^{(p)}]^q \mapsto \phi_*[G_n^{(p)}]_{\mathbf{KO}}^q.$$

Remark 3.3. The element $[G_n^{(p)}]_{\mathbf{KO}}^0$ can be rewritten as

$$[G_n^{(p)}]_{\mathbf{KO}}^0 = \sum_{l=0}^{\lfloor n/2 \rfloor} ([G^{(p)}]_{\mathbf{KO}} \times \mathfrak{R}[\omega_n^l]) \otimes 2 \cos(2\pi l/n).$$

A similar equation involving the sine function holds for $[G_n^{(p)}]_{\mathbf{KO}}^1$. Thus it would be possible to restrict to real coefficients. However, we shall not use this any further, and continue to keep complex coefficients everywhere.

4. Secondary index classes of psc metrics for finite groups

The proposition below is essentially due to Botvinnik and Gilkey [1995], albeit formulated in a slightly different way. In the proof, we briefly explain for the convenience of the reader how its statement can be deduced from the result in the literature:

Proposition 4.1. *Suppose that H is a finite group and $n \geq 6$. Then the ρ -invariant $\rho : \mathbf{P}_{n-1}^{\text{spin}}(\mathbf{B}H) \rightarrow \mathbf{S}_{n-1}^{\mathbb{R}}(H)$ is rationally surjective.*

Proof. We only need to consider n to be even because the analytic structure group of a finite group rationally vanishes in odd degrees. Let $n = 4k + 2q$ with $k \geq 1$ and $q \in \{0, 1\}$. Each (virtual) unitary representation π of H induces a trace functional $\text{tr}_\pi : K_0(\mathbb{C}H) \rightarrow \mathbb{Z}$. If π is of virtual dimension 0, then tr_π extends to a functional $\eta_\pi : S_1^{\mathbb{C}}(H) \rightarrow \mathbb{R}$ on the complex version of the analytic structure group; see [Higson and Roe 2010]. By the construction of η_π , the composition $\eta_\pi \circ \rho : P_{4k+2q-1}^{\text{spin}}(\mathbb{B}H) \rightarrow \mathbb{R}$ recovers the relative η -invariant used in [Botvinnik and Gilkey 1995].

Since finite groups satisfy the Baum–Connes conjecture, the Higson–Roe sequence rationally becomes a short exact sequence:

$$0 \rightarrow \mathbb{Q} \rightarrow K_0(\mathbb{C}H) \otimes \mathbb{Q} \rightarrow S_1^{\mathbb{C}}(H) \otimes \mathbb{Q} \rightarrow 0. \tag{4.2}$$

Let $R_0(H)$ denote the space of virtual unitary representations of dimension 0. The pairing $R(H) \otimes \mathbb{Q} \times K_0(\mathbb{C}H) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$, $(\pi, x) \mapsto \text{tr}_\pi(x)$ is nondegenerate. We conclude from this and (4.2) that the pairing

$$R_0(H) \otimes \mathbb{R} \times S_1^{\mathbb{C}}(H) \otimes \mathbb{R} \rightarrow \mathbb{R}, \quad (\pi, x) \mapsto \eta_\pi(x) \tag{4.3}$$

is also nondegenerate. The complex analytic structure group admits a decomposition $S_1^{\mathbb{C}}(H) \otimes \mathbb{Q} \cong (S_1^{\mathbb{R}}(H) \oplus S_3^{\mathbb{R}}(H)) \otimes \mathbb{Q}$ analogous to Proposition 2.1. Applying this to (4.3) yields a nondegenerate pairing

$$R_0^q(H) \otimes \mathbb{R} \times S_{2q-1}^{\mathbb{R}}(H) \otimes \mathbb{R} \rightarrow \mathbb{R}, \quad (\pi, x) \mapsto \eta_\pi(x) \tag{4.4}$$

for each $q \in \{0, 1\}$.

Finally, let $4k + 2q \geq 6$. Then [Botvinnik and Gilkey 1995, Theorem 2.1] implies that the composition

$$P_{4k+2q-1}^{\text{spin}}(\mathbb{B}H) \otimes \mathbb{R} \xrightarrow{\rho \otimes \mathbb{R}} S_{2q-1}^{\mathbb{R}}(H) \otimes \mathbb{R} \xrightarrow{\bigoplus_i \eta_{\pi_i}} \mathbb{R}^{\dim R_0^q(H)}$$

is surjective, where $(\pi_i)_i$ is a basis of $R_0^q(H) \otimes \mathbb{R}$. Since the pairing (4.4) is nondegenerate, the latter map in this composition is an isomorphism. Thus $\rho \otimes \mathbb{R}$ must be surjective as well. □

Corollary 4.5. *Suppose that H is a finite group and $n \geq 6$. Then the relative index map $\alpha : R_n^{\text{spin}}(\mathbb{B}H) \rightarrow KO_n(C_r^*H)$ is rationally surjective.*

Proof. Again we only need to consider n to be even and let $n = 4k + 2q \geq 6$. For a finite group H , the groups $\Omega_l^{\text{spin}}(\mathbb{B}H)$ and $KO_l(\mathbb{B}H)$ are torsion for $l \not\equiv 0 \pmod{4}$. Moreover, $\beta \otimes \mathbb{Q} : \Omega_{4k}^{\text{spin}}(\mathbb{B}H) \otimes \mathbb{Q} \rightarrow KO_0(\mathbb{B}H) \otimes \mathbb{Q}$ is surjective because $KO_0(\mathbb{B}H) \otimes \mathbb{Q} \cong KO_0(\text{pt}) \otimes \mathbb{Q}$ is generated by the class represented by any product of Kummer surfaces. By Proposition 4.1, the ρ -invariant

$$\rho \otimes \mathbb{Q} : P_{4k+2q-1}^{\text{spin}}(\mathbb{B}H) \otimes \mathbb{Q} \rightarrow S_{2q-1}^{\mathbb{R}}(H) \otimes \mathbb{Q}$$

is also surjective. Thus we have a diagram of exact sequences

$$\begin{array}{ccccccc}
 \Omega_{4k+2q}^{\text{spin}}(\mathbf{B}H) \otimes \mathbb{Q} & \longrightarrow & \mathbf{R}_{4k+2q}^{\text{spin}}(\mathbf{B}H) \otimes \mathbb{Q} & \longrightarrow & \mathbf{P}_{4k+2q-1}^{\text{spin}}(\mathbf{B}H) \otimes \mathbb{Q} & \longrightarrow & 0 \\
 \downarrow \beta \otimes \mathbb{Q} & & \downarrow \alpha \otimes \mathbb{Q} & & \downarrow \rho \otimes \mathbb{Q} & & \downarrow \\
 \mathbf{K}O_{2q}(\mathbf{B}H) \otimes \mathbb{Q} & \longrightarrow & \mathbf{K}O_{2q}(\mathbf{R}H) \otimes \mathbb{Q} & \longrightarrow & \mathbf{S}_{2q-1}^{\mathbb{R}}(H) \otimes \mathbb{Q} & \longrightarrow & 0
 \end{array}$$

The four lemma implies that $\alpha \otimes \mathbb{Q}$ must be surjective as well. □

5. Proof of main results

Our main result, Theorem 1.3, follows immediately from Proposition 3.2 and the following lemma.

Lemma 5.1. *For each $n \in \mathbb{N}$, $p \in \{0, 1, 2\}$, $q \in \{0, 1\}$ and $k \geq 1$ with $4k + 2q \geq 6$, there exists $[G_n^{(p)}]_{\text{psc}}^q \in \mathbf{R}_{p+2q+4k}^{\text{spin}}(\mathbf{B}G_n^{(p)}) \otimes \mathbb{C}$ with*

$$\alpha([G_n^{(p)}]_{\text{psc}}^q) = \mu([G_n^{(p)}]_{\mathbf{K}O}^q) \in \mathbf{K}O_{2q}(\mathbf{C}_r^*G_n^{(p)}) \otimes \mathbb{C}.$$

Proof. Let

$$x_{n,l}^q := \begin{cases} \Re[\omega_n^l], & q = 0 \\ \Im[\omega_n^l], & q = 1 \end{cases} \in \mathbf{K}O_{2q}^{\mathbb{Z}/n\mathbb{Z}}(\text{pt}).$$

By Corollary 4.5, we can choose an element $y_{n,l}^{q,k} \in \mathbf{R}_{2q+4k}^{\text{spin}}(\mathbf{B}\mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{Q}$ such that $\alpha(y_{n,l}^{q,k}) = \mu(x_{n,l}^q) \in \mathbf{K}O_{2q}(\mathbf{C}_r^*(\mathbb{Z}/n\mathbb{Z})) \otimes \mathbb{Q}$. Now let $[G^{(0)}]_{\Omega} := [\text{pt}] \in \Omega_0^{\text{spin}}(\text{pt})$, $[G^{(1)}]_{\Omega} := [S^1] \in \Omega_1^{\text{spin}}(\mathbf{B}\mathbb{Z})$ and $[G^{(2),g}]_{\Omega} := [\Sigma_g] \in \Omega_2^{\text{spin}}(\mathbf{B}\Gamma_g) \otimes \mathbb{Q}$. Note that for the latter we need to choose one from the 2^{2g} different spin structures on the oriented surface. However, rationally the element $[\Sigma_g]$ is independent of this choice. Taking direct products yields a map $\Omega_l^{\text{spin}}(X) \otimes \mathbf{R}_m^{\text{spin}}(Y) \xrightarrow{\times} \mathbf{R}_{l+m}^{\text{spin}}(X \times Y)$. Using this, we let

$$[G_n^{(p)}]_{\text{psc}}^q := \sum_{l=0}^{n-1} ([G^{(p)}]_{\Omega} \times y_{n,l}^{q,k}) \otimes \omega_n^{-l} \in \mathbf{R}_{p+2q+4k}^{\text{spin}}(\mathbf{B}G_n^{(p)}) \otimes \mathbb{C}. \quad \square$$

Proof of Theorem 1.3. We have the diagram

$$\begin{array}{ccccc}
 & & \mathbf{H}_p(\Gamma; \mathbf{F}^q\Gamma) & \hookrightarrow & \mathbf{H}_p(\Gamma; \mathbf{F}\Gamma) \\
 & & \downarrow \beta_{p,q}^{(\text{psc})} & & \downarrow \beta_p^{(\text{a})} \\
 & & \mathbf{R}_{p+2q+4k}^{\text{spin}}(\mathbf{B}\Gamma) \otimes \mathbb{C} & & \\
 & \nearrow \beta_{p,q}^{(\text{a})} & \downarrow \alpha \otimes \mathbb{C} & & \\
 \mathbf{K}O_{p+2q}^{\Gamma}(\underline{\mathbf{E}}\Gamma) \otimes \mathbb{C} & \xrightarrow{\mu \otimes \mathbb{C}} & \mathbf{K}O_{p+2q}(\mathbf{C}_r^*\Gamma) \otimes \mathbb{C} & \xrightarrow{c \otimes \mathbb{C}} & \mathbf{K}_p(\mathbf{C}_r^*\Gamma) \otimes \mathbb{C}
 \end{array}$$

where by construction of $\beta_{p,q}^{(t)}$ the outer paths commute. To prove the existence of $\beta_{p,q,k}^{(\text{psc})}$, it suffices to show that the image of $\mu \otimes \mathbb{C} \circ \beta_{p,q}^{(t)}$ is contained in the image of $\alpha \otimes \mathbb{C}$. Proposition 3.2 implies that the image of $\beta_{p,q}^{(t)}$ is generated by elements of the form $\phi_*[G_n^{(p)}]_{\text{KO}}^q$, where $[G_n^{(p)}]_{\text{KO}}^q$ is defined in (3.1) and $\phi : G_n^{(p)} \rightarrow \Gamma$ is a group homomorphism. Thus it suffices to show that the elements $\mu(\phi_*[G_n^{(p)}]_{\text{KO}}^q)$ are contained in the image of $\alpha \otimes \mathbb{C}$. Indeed, Lemma 5.1 states that $\mu([G_n^{(p)}]_{\text{KO}}^q)$ admits a lift to $\mathbf{R}_{p+2q+4k}^{\text{spin}}(\mathbf{B}G_n^{(p)})$. Therefore, by functoriality, we conclude that $\mu(\phi_*[G_n^{(p)}]_{\text{KO}}^q)$ admits a lift to $\mathbf{R}_{p+2q+4k}^{\text{spin}}(\mathbf{B}\Gamma)$. \square

Proof of Corollary 1.5. If $\mu \otimes \mathbb{Q}$ is injective, then $\beta_p^{(a)} = \mu \otimes \mathbb{C} \circ \beta_p^{(t)}$ maps $H_p(\Gamma; \mathbb{F}\Gamma)$ injectively into $\mathbf{K}_p(\mathbf{C}_\Gamma^* \Gamma) \otimes \mathbb{C}$. Thus the diagram in Theorem 1.3 implies that for fixed n the following map must be injective:

$$\sum_{p+2q+4k=n} \beta_{p,q,k}^{(\text{psc})} : \bigoplus_{p+2q \in n+4\mathbb{Z}} H_p(\Gamma; \mathbb{F}^q \Gamma) \rightarrow \mathbf{R}_n^{\text{spin}}(\mathbf{B}\Gamma) \otimes \mathbb{C}. \tag{5.2}$$

Here p, q, k range over $\{0, 1, 2\}, \{0, 1\}, \mathbb{Z}$, respectively. Unpacking this yields the table in the statement of Corollary 1.5. \square

Proof of Corollary 1.6. The image in $\mathbf{K}_p^\Gamma(\underline{\mathbf{E}}\Gamma) \otimes \mathbb{C}$ of the restriction of $\beta_p^{(t)}$ to $H_p(\Gamma; \mathbb{F}_0^0 \Gamma \oplus \mathbb{F}^1 \Gamma)$ intersects trivially with the image of $\mathbf{K}_p(\mathbf{B}\Gamma) \otimes \mathbb{C} \hookrightarrow \mathbf{K}_p^\Gamma(\underline{\mathbf{E}}\Gamma) \otimes \mathbb{C}$. This follows from the decomposition of the handcrafted Chern character based on the Shapiro isomorphism; see [Matthey 2004, Theorem 1.4]. Thus the injectivity of (5.2) together with a diagram chase involving (1.1) implies that the following map must be injective as well:

$$\sum_{p+2q+4k=n} \partial \circ \beta_{p,q,k}^{(\text{psc})} : \bigoplus_{p+2q \in n+4\mathbb{Z}} H_p(\Gamma; \mathbb{F}_0^q \Gamma) \rightarrow \mathbf{P}_{n-1}^{\text{spin}}(\mathbf{B}\Gamma) \otimes \mathbb{C}.$$

Here we use the convention $\mathbb{F}_0^1 \Gamma := \mathbb{F}^1 \Gamma$. \square

Proof of Corollary 1.8. If the rational homological dimension of Γ is at most 2, then the map

$$\sum_{p+2q \in n+4\mathbb{Z}} \beta_{p,q}^{(t)} : \bigoplus_{p+2q \in n+4\mathbb{Z}} H_p(\Gamma; \mathbb{F}\Gamma^q) \rightarrow \mathbf{KO}_n^\Gamma(\underline{\mathbf{E}}\Gamma) \otimes \mathbb{C}$$

is the inverse to the Chern character. In particular, it is surjective. If the rational Baum–Connes assembly map $\mu \otimes \mathbb{Q}$ is also surjective, then this implies that the following is surjective too:

$$\sum_{p+2q \in n+4\mathbb{Z}} \beta_{p,q}^{(a)} = \mu \otimes \mathbb{C} \circ \sum_{p+2q \in n+4\mathbb{Z}} \beta_{p,q}^{(t)} : \bigoplus_{p+2q \in n+4\mathbb{Z}} H_p(\Gamma; \mathbb{F}\Gamma^q) \rightarrow \mathbf{KO}_n(\mathbf{C}_\Gamma^* \Gamma) \otimes \mathbb{C}.$$

Theorem 1.3 implies that for $n \geq 7$, the image of $\beta_{p,q}^{(a)}$ is contained in the image of $\alpha \otimes \mathbb{C}$, which proves surjectivity of $\alpha \otimes \mathbb{C}$ and thus of $\alpha \otimes \mathbb{Q}$.

If $\mu \otimes \mathbb{Q}$ is injective, then $\nu \otimes \mathbb{Q}$ is injective and by exactness the boundary map $\partial \otimes \mathbb{Q} : \mathrm{KO}_n(\mathrm{C}_r^* \Gamma) \otimes \mathbb{Q} \rightarrow \mathrm{S}_{n-1}^{\mathbb{R}}(\Gamma) \otimes \mathbb{Q}$ is surjective. Hence the surjectivity statement for $\rho \otimes \mathbb{Q}$ if $\mu \otimes \mathbb{Q}$ is an isomorphism follows from surjectivity of $\alpha \otimes \mathbb{Q}$ and commutativity of the diagram (1.1). \square

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References

- [Baum and Connes 1988] P. Baum and A. Connes, “Chern character for discrete groups”, pp. 163–232 in *A fête of topology*, edited by Y. Matsumoto et al., Academic Press, Boston, 1988. MR Zbl
- [Baum and Connes 2000] P. Baum and A. Connes, “Geometric K -theory for Lie groups and foliations”, *Enseign. Math. (2)* **46**:1-2 (2000), 3–42. MR Zbl
- [Botvinnik and Gilkey 1995] B. Botvinnik and P. B. Gilkey, “The eta invariant and metrics of positive scalar curvature”, *Math. Ann.* **302**:3 (1995), 507–517. MR Zbl
- [Bruner and Greenlees 2010] R. R. Bruner and J. P. C. Greenlees, *Connective real K -theory of finite groups*, Mathematical Surveys and Monographs **169**, American Mathematical Society, Providence, RI, 2010. MR Zbl
- [Higson and Kasparov 2001] N. Higson and G. Kasparov, “ E -theory and KK -theory for groups which act properly and isometrically on Hilbert space”, *Invent. Math.* **144**:1 (2001), 23–74. MR Zbl
- [Higson and Roe 2010] N. Higson and J. Roe, “ K -homology, assembly and rigidity theorems for relative eta invariants”, *Pure Appl. Math. Q.* **6**:2 (2010), 555–601. MR Zbl
- [Matthey 2004] M. Matthey, “The Baum–Connes assembly map, delocalization and the Chern character”, *Adv. Math.* **183**:2 (2004), 316–379. MR Zbl
- [Piazza and Schick 2007] P. Piazza and T. Schick, “Groups with torsion, bordism and rho invariants”, *Pacific J. Math.* **232**:2 (2007), 355–378. MR Zbl
- [Piazza and Schick 2014] P. Piazza and T. Schick, “Rho-classes, index theory and Stolz’ positive scalar curvature sequence”, *J. Topol.* **7**:4 (2014), 965–1004. MR Zbl
- [Rosenberg and Stolz 2001] J. Rosenberg and S. Stolz, “Metrics of positive scalar curvature and connections with surgery”, pp. 353–386 in *Surveys on surgery theory*, vol. 2, edited by S. Cappell et al., Ann. of Math. Stud. **149**, Princeton University Press, 2001. MR Zbl
- [Weinberger and Yu 2015] S. Weinberger and G. Yu, “Finite part of operator K -theory for groups finitely embeddable into Hilbert space and the degree of nonrigidity of manifolds”, *Geom. Topol.* **19**:5 (2015), 2767–2799. MR Zbl
- [Xie and Yu 2014] Z. Xie and G. Yu, “Positive scalar curvature, higher rho invariants and localization algebras”, *Adv. Math.* **262** (2014), 823–866. MR Zbl

- [Xie and Yu 2017] Z. Xie and G. Yu, “Higher rho invariants and the moduli space of positive scalar curvature metrics”, *Adv. Math.* **307** (2017), 1046–1069. MR Zbl
- [Zeidler 2016a] R. Zeidler, “Positive scalar curvature and product formulas for secondary index invariants”, *J. Topol.* **9**:3 (2016), 687–724. MR Zbl
- [Zeidler 2016b] R. Zeidler, *Secondary large-scale index theory and positive scalar curvature*, Ph.D. thesis, Georg-August-Universität Göttingen, 2016, <https://tinyurl.com/Zeidler-thesis>.
- [Zenobi 2017] V. F. Zenobi, “Mapping the surgery exact sequence for topological manifolds to analysis”, *J. Topol. Anal.* **9**:2 (2017), 329–361. MR Zbl

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