Algebra & Number Theory

Volume 1

2007

No. 1

A topological quantum field theory of intersection numbers on moduli spaces of admissible covers

Renzo Cavalieri

mathematical sciences publishers

ALGEBRA AND NUMBER THEORY 1:1(2007)

A topological quantum field theory of intersection numbers on moduli spaces of admissible covers

Renzo Cavalieri

We construct a two-level weighted topological quantum field theory whose structure coefficients are equivariant intersection numbers on moduli spaces of admissible covers. Such a structure is parallel (and strictly related) to the local Gromov–Witten theory of curves of Bryan and Pandharipande. We compute explicitly the theory using techniques of localization on moduli spaces of admissible covers of a parametrized $\mathbb{P}^1$. The Frobenius algebras we obtain are one-parameter deformations of the class algebra of the symmetric group S_d. In certain special cases we are able to produce explicit closed formulas for such deformations in terms of the representation theory of S_d.

Introduction

We study a large class of (equivariant) intersection numbers on moduli spaces of admissible covers. For a smooth algebraic curve X, ramified covers of X of a given degree by smooth curves of a given genus are parametrized by moduli spaces called Hurwitz schemes. A smooth compactification of a Hurwitz scheme can be obtained by allowing suitable degenerations, called admissible covers.

Moduli spaces of admissible covers were introduced in [Harris and Mumford 1982]. Intersection theory on these spaces was for a long time hard and mysterious, mostly because they are in general not normal, even if the normalization is always smooth. It was only recently that Abramovich, Corti and Vistoli [Abramovich et al. 2003] exhibited this normalization as the stack of balanced stable maps of degree 0 from twisted curves to the classifying stack $\mathcal{B}S_d$. This way they attained both the smoothness of the stack and a nice moduli-theoretic interpretation of it. We abuse terminology and refer to Abramovich–Corti–Vistoli spaces as *admissible covers*.

At about the same time, Ionel [2002] developed a parallel theory in the symplectic category and used push-pull techniques on admissible covers to produce new relations in the tautological ring of $M_{g,n}$. (See also [Ionel 2005].)

MSC2000: 14N35.

Keywords: TQFT, topological quantum field theory, admissible covers, Gromov–Witten Invariants.

In [Graber and Vakil 2003b], admissible cover loci within the boundary of moduli spaces of stable maps play a key role in establishing the result that the degree $3g-3$ part of the tautological ring of $\overline{M}_g$ has dimension 1, providing further evidence for a conjecture by Faber, stating that $R(\overline{M}_g)$ is a Gorenstein algebra with socle in degree $3g-3$.

Bryan, Graber and Pandharipande have shown in [Bryan et al. 2005] that the orbifold Gromov–Witten potential of a Gorenstein orbifold can be computed in terms of intersection theory on moduli spaces of admissible covers. With a subtle use of WDVV techniques, they are able to explicitly compute the Gromov–Witten potential for the orbifold $[\mathbb{C}^2/\mathbb{Z}_3]$. Such a computation provides evidence for the crepant resolution conjecture [Bryan and Graber $\geq$ 2008].

We give a few basic definitions and a working description of moduli spaces of admissible covers in Section 1.

For all choices of:

- an r-pointed curve $(X, p_1, \dots, p_n)$;
- a rank two vector bundle $N = L_1 \oplus L_2$ on X, endowed with a natural $\mathbb{C}^* \times \mathbb{C}^*$ action (page 46);
- a vector of partitions $\underline{\eta} = (\eta_1, \dots, \eta_n)$ of a fixed integer d,

we describe the invariants

$$A_d^h(N) := \int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} X, (\eta_1 p_1, \dots, \eta_r p_r))} e^{\mathrm{eq}}(-R^\bullet \pi_* f^*(L_1 \oplus L_2)).$$

The motivation for studying these invariants is twofold. They are natural and interesting intersection numbers on their own, giving rise to a beautiful structure. Secondly, in the context of Gromov–Witten theory, invariants of this form are known as "local" invariants: roughly speaking, they represent the contribution to the Gromov–Witten invariants of a threefold given by rigid curves.

Theorem 3.1. (See page 48.) *The invariants $A_d^h(N)$ can be organized to be the structure coefficients of a $2-$level, semisimple, weighted topological quantum field theory (TQFT).*

Section 2 is dedicated to presenting these structures to the unfamiliar reader, while in Section 3 the specific TQFT $\mathcal{U}$ is constructed.

The generators for the TQFT are explicitly computed in Section 4. The techniques involved are basic dimension counting, reduction to classical intersection theory on moduli spaces of curves, and Atiyah–Bott localization on moduli spaces of admissible covers of a parametrized $\mathbb{P}^1$.

An interesting feature of this theory is that the degree 0 part is constructed from Hurwitz numbers. The embedded (see page 44) Frobenius algebras induced on the Hilbert space by $\mathcal{U}$ are one-parameter deformations of the class algebra of the symmetric group, whose TQFT-theoretic description in terms of Hurwitz numbers was studied in the 1990s in [Dijkgraaf and Witten 1990] and [Freed and Quinn 1993]. An explicit description of such deformations is in general quite complicated. By specializing to the antidiagonal action of $\mathbb{C}^*$ inside $\mathbb{C}^* \times \mathbb{C}^*$, it is possible to diagonalize the theory: closed formulas for our invariants and for the deformation are described in Section 5 in terms of the representation theory of the symmetric group S_d (Theorem 5.2).

This work is closely connected to and follows recent work of Jim Bryan and Rahul Pandharipande [2004; 2005], describing the local Gromov–Witten theory of curves.

There, analogous intersection numbers on moduli spaces of (relative) stable maps are organized in a TQFT that we denote by $\mathcal{BP}$. Theorem 4.1 shows that the two theories coincide in level $(0, 0)$. In all other levels, $\mathcal{U}$ is a normalization of $\mathcal{BP}$ via appropriate powers of a universal generating function factor, which should be understood as the contribution of maps containing contracting components to the Gromov–Witten invariants.

This result, the most technical in this paper, is established by computing the genus 0, one-pointed invariants via localization, together with the use of some beautiful Hodge integral computations from [Faber and Pandharipande 2000; Ekedahl et al. 2001; Graber and Vakil 2003a]. The explicit statement is this:

Theorem 4.3 (See page 56.). *The coefficients for the one-pointed invariants of* $\mathcal{U}$ *in level* $(0, -1)$ *are given by the generating functions*

$$A_d(0|0,-1)_\eta = (-1)^{d-\ell(\eta)} \frac{\left(2\sin\frac{u}{2}\right)^d}{s_1^{\ell(\eta)}\mathfrak{z}(\eta)\prod 2\sin\frac{\eta_i u}{2}},$$

Notation. Here and throughout the paper $\ell(\eta)$ denotes the length r of a partition $\eta = (\eta_1, \ldots, \eta_r)$.

A direct check in the one-pointed case, together with the semisimplicity of both theories, yields:

Corollary 0.1. *The coefficients of the theories* $\mathcal{U}$ *and* $\mathcal{BP}$ *are related by*

$$A_d(g \mid k_1, k_2)_{\underline{\eta}} = (d!)^{k_1+k_2} s_1^{dk_2} s_2^{dk_1} \mathcal{BP}_d(g, \mid k_1, k_2)_{\underline{\eta}} \mathcal{BP}_d(0 \mid 0, -1)^{k_1+k_2}_{(1,\ldots,1)}.$$

This close proximity to Gromov–Witten theory reinforces our interest in moduli spaces of admissible covers, as it anticipates the possibility of a fertile exchange of information between the two contexts. In particular, embedded in the theory $\mathcal{U}^\circ$ (the circle superscript indicates we are restricting our attention to connected covers) we rediscover the classical result:

Aspinwall–Morrison formula.

$$\int_{[\overline{M}_{0,0}(\mathbb{P}^1,d)]} R^1\pi_* f^*(\mathbb{O}(-1)\oplus\mathbb{O}(-1)) = \left(\frac{A_d^{\circ,0}(0\mid -1,-1)}{u^{2d-2}}\right)_{|u=0} = \frac{1}{d^3}$$

The technique of Atiyah–Bott localization suits very well the spaces of admissible covers of a parametrized $\mathbb{P}^1$; the fact that these spaces are smooth (as DM stacks) requires no need for a virtual fundamental class in order to do intersection theory on them. The modularity of the boundary-fixed loci naturally produces topological recursions that live completely within the realm of admissible covers.

1. Admissible covers

Moduli spaces of admissible covers are a "natural" compactification of the Hurwitz schemes, parametrizing ramified covers of smooth Riemann Surfaces. The fundamental idea is that, in order to understand limit covers, we allow the base curve to degenerate together with the cover. Branch points are not allowed to come together; as two or more branch points tend to collide, a new component of the base curve sprouts from the point of collision, and the points transfer onto it. Similarly, upstairs the cover splits into a nodal cover.

More formally: let $(X, p_1, \dots, p_r)$ be an r-pointed nodal curve of genus g.

Definition 1.1. An *admissible cover* $\pi : E \to X$ of degree d is a finite morphism satisfying the following:

(1) E is a nodal curve.

(2) Every node of E maps to a node of X.

(3) The restriction of $\pi : E \to X$ to $X \setminus (p_1, \dots, p_r)$ is étale of constant degree d.

(4) Nodes can be smoothed. This means that given an admissible cover $\pi : E \to X$, and a node of E, we can find a family of admissible covers $\pi' : E' \to X'$ such that $\pi : E \to X$ is the central fiber of the family, and there are local analytic coordinates and a positive integer $n \le d$ such that X', E' and π' are described by

$$E : e_1e_2 = a, \quad X : x_1x_2 = a^n, \quad \pi : x_1 = e_1^n,\ x_2 = e_2^n.$$

We recall here the notation we use in this paper, and refer the reader to [Cavalieri 2005] for a more extensive discussion.

Let $(X, p_1, \dots, p_r)$ be as before, and let $\eta_1, \dots, \eta_r$ be partitions of the fixed integer d. We denote by

$$\overline{\mathrm{Adm}}(h \xrightarrow{d} X, (\mu_1 p_1, \dots, \mu_r p_r))$$

the stack of *possibly disconnected*, degree d admissible covers of the curve X by curves of genus h, such that

- the ramification profile over the base point p_i is described by the partition η_i;
- all other ramification is simple (and is not marked).

The following variations are also used:

Connected admissible covers: We add the superscript $\circ$ to restrict our attention to admissible covers by connected curves. *Admissible covers of a genus g curve:* We denote by

$$\overline{\mathrm{Adm}}(h \xrightarrow{d} g)$$

the stack of admissible covers of a curve of genus g. This means that also the base curve is allowed to vary in families.

Admissible covers of a parametrized $\mathbb{P}^1$*:* When we intend to fix a parametrization on the base $\mathbb{P}^1$, we write

$$\overline{\mathrm{Adm}}(h \xrightarrow{d} \mathbb{P}^1).$$

Moduli spaces of admissible covers admit forgetful maps to (quotients of) configuration spaces of points on the base curve (or to $\overline{M}_{g,n}$ in the case of admissible covers of a genus g curve), recording the information about branch points that are free to move. Tautological ψ classes on admissible covers are defined by pulling back the ψ classes downstairs via these maps.

There is also a natural map from a moduli space of admissible covers of genus h to the corresponding moduli space of curves $\overline{M}_h$, obtained by forgetting the cover map and only remembering the source curve. Tautological λ classes on admissible covers are defined by pulling back λ classes (the Chern classes of the Hodge bundle on the moduli space of stable curves) via these maps.

Admissible covers of a nodal curve. Admissible covers of a nodal curve can be described combinatorially in terms of admissible covers of the irreducible components of the curve. This is extremely useful because it opens the way to the use of degeneration techniques and induction. Crucial to this work are the following identities [Li 2002] taking place in the Chow ring with rational coefficients.

Reducible nodal curve: Let

$$X = X_1 \bigcup_{x_1 = x_2} X_2$$

be a nodal curve of genus g, obtained by attaching at a point two irreducible curves of genus g_1 and g_2. Then

$$[\overline{\mathrm{Adm}}(h \xrightarrow{d} X)] = \sum_{\eta, h_1, h_2} \mathfrak{z}(\eta)[\overline{\mathrm{Adm}}(h_1 \xrightarrow{d} X_1, (\eta))] \times [\overline{\mathrm{Adm}}(h_2 \xrightarrow{d} X_2, (\eta))], \quad (1\text{-}1)$$

where $\eta = ((\eta^1)^{m_1}, \dots, (\eta^k)^{m_k})$ runs over all partitions of d, the numbers h_1 and h_2 satisfy $h_1 + h_2 + \ell(\eta) - 1 = h$, and we have defined the combinatorial factor

$$\mathfrak{z}(\eta) := \prod m_i!(\eta^i)^{m_i}. \tag{1-2}$$

In particular, $\mathfrak{z}(\eta)$ is the order of the centralizer in S_d of any group element in the conjugacy class of η.

Note. If we are dealing with admissible cover spaces with also a prescribed vector of ramification conditions $\underline{\mu}$, analogous formulas hold; the μ_i need to be distributed on the two twigs X_1 and X_2 in all possible ways.

Irreducible nodal curve: Let

$$X = X'/\{x_1 = x_2\}$$

be a nodal curve of genus g, obtained by gluing two distinct points of an irreducible curve X' of genus $g-1$. As an element in the Chow ring with rational coefficients, we can then express

$$[\overline{\mathrm{Adm}}(h \xrightarrow{d} X)] = \sum_{\eta} \mathfrak{z}(\eta)[\overline{\mathrm{Adm}}(h' \xrightarrow{d} X', (\eta, \eta))], \tag{1-3}$$

where the sum is over all partitions η of d, and h' is determined by

$$h' + \ell(\eta) = h.$$

2. Topological quantum field theories

As an excellent and elementary reference for two-dimensional topological quantum field theories in mathematics we mention [Kock 2004].

Definition 2.1. A $(1+1)$*-dimensional topological quantum field theory* is a functor of tensor categories:

$$\mathcal{T} : \mathsf{2Cob} \longrightarrow \mathsf{Free\ Rmod}.$$

On the right-hand side is the category of free modules over a commutative ring R, and on the left is the category 2Cob described thus:

- The *objects* are one-dimensional oriented closed manifolds (finite disjoint unions of oriented circles).
- The *morphisms* are (equivalence classes of) oriented cobordisms between two objects. We can think of them as oriented topological surfaces with oriented boundary components.
- We *compose* two morphisms by concatenation; equivalently, we glue negatively oriented boundary components of one surface to positively oriented boundary components of the other.

– The *tensor operation* is the disjoint union.

The free module $H := \mathcal{T}(S^1)$ is called the *Hilbert space* of the TQFT.

All topological surfaces can be decomposed into discs, annuli, and pairs of pants. Therefore, the structure of a TQFT is completely determined if it is described on these basic building blocks.

Tensor notation. It is convenient, for explicit computations, to use tensor notation for TQFTs. We choose a basis $e_1, \dots, e_r$ for the Hilbert space H, and denote the dual basis by $e^1, \dots, e^r$. Let $W_m^n(g)$ be a genus-g cobordism from m to n circles. The map

$$\mathcal{T}(W_m^n(g)) : H^{\otimes m} \to H^{\otimes n}$$

can be thought of as a vector in $(H^*)^{\otimes m} \otimes H^{\otimes n}$. We denote by

$$\Gamma(W_m^n(g))_{i_1,\dots,i_m}^{j_1,\dots,j_n}$$

the coefficient of $\mathcal{T}(W_m^n(g))$ in the direction of the basis element $e^{i_1} \otimes \cdots \otimes e^{i_m} \otimes e_{j_1} \otimes \cdots \otimes e_{j_n}$ (see Figure 1):

$$\mathcal{T}(W_m^n(g)) = \sum \Gamma(W_m^n(g))_{i_1,\dots,i_m}^{j_1,\dots,j_n} e^{i_1} \otimes \cdots \otimes e^{i_m} \otimes e_{j_1} \otimes \cdots \otimes e_{j_n}.$$

Frobenius algebras. A TQFT gives the Hilbert space H the structure of a commutative Frobenius algebra. This means it defines an associative and commutative multiplication $\cdot$ and an inner product (also called the metric of the TQFT) $\langle\ ,\ \rangle$ on H such that

$$\langle h_1 \cdot h_2, h_3 \rangle = \langle h_1, h_2 \cdot h_3 \rangle \tag{2-1}$$

for all h_1, h_2, h_3 in the Hilbert space H. It is easy to see how the structure is induced: multiplication is the map associated to the $(-,-,+)$ pair of pants, the inner product is the scalar map associated to the $(-,-)$ annulus. As a consequence, we see immediately that the cap with positively oriented boundary corresponds

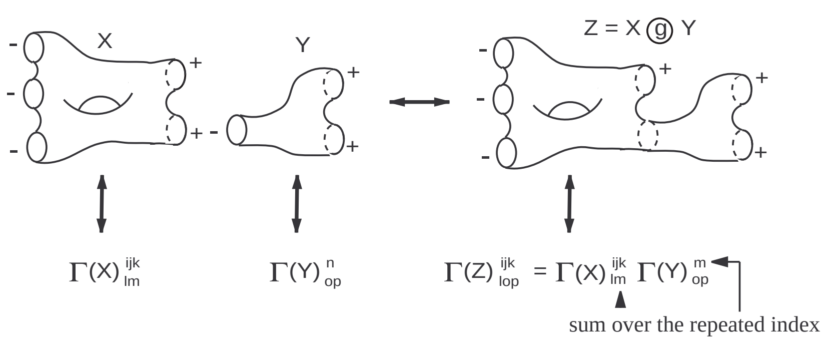


Figure 1. Gluing in tensor notation.

to the unit vector for the multiplication map just defined, whereas the $(-)$ cap corresponds to the counit operator in the Frobenius algebra.

Definition 2.2. A TQFT $\mathcal{T}$ is *semisimple* if the Frobenius algebra induced on the Hilbert space H is semisimple; equivalently, if there is an orthonormal basis $e_1, \ldots e_r$ for H such that

$$e_i \cdot e_j = \delta_{ij} e_i.$$

Yet another equivalent condition is that $\mathcal{T}$ be a direct sum

$$\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_r,$$

where all $\mathcal{T}_i$ are TQFTs with Hilbert space equal to the ground ring.

Let $e_1, \ldots, e_r$ be a semisimple basis for H. We can think of each e_i as the identity vector for the space H_i. Let $e^1, \ldots, e^r$ be the dual basis. Then semisimplicity is equivalent to asking all nondiagonal coefficients to vanish:

$$\Gamma_{i_1,\ldots,i_n}^{j_1,\ldots,j_m}(W_m^n(g)) = 0,$$

unless $i_1 = i_2 = \cdots = i_n = j_1 = \cdots = j_m$.

There are now r universal constants $\lambda_1, \ldots, \lambda_r$ that govern the structure of the TQFT. They can be defined in many equivalent ways. Here are two equivalent descriptions that we will be using later on:

(1) $1/\lambda_i$ is the image of the basis vector e_i via the counit operator.

(2) λ_i is the i-th eigenvalue of the genus adding operator (this is the linear map associated to the torus with a negative and a positive puncture, represented in Figure 3).

Structure Theorem 2.3. *Let $\mathcal{T}$ be a semisimple TQFT, and all notation as above. Denote by $W_m^n(g)$ a genus g surface with m input and n output holes. Then*

$$\mathcal{T}(W_m^n(g)) = \sum_{i=1}^{r} \lambda_i^{g+n-1} \underbrace{e^i \otimes \cdots \otimes e^i}_{m \text{ times}} \otimes \underbrace{e_i \otimes \cdots \otimes e_i}_{n \text{ times}}.$$

In particular,

$$\mathcal{T}(W_0^0(g)) = \sum_{i=1}^{r} \lambda_i^{g-1}. \tag{2-2}$$

The TQFT of Hurwitz numbers. Dijkgraaf and Witten [1990] used the TQFT approach to give a beautiful and elegant solution to a classical mathematical problem: counting ramified and unramified covers of a topological surface, as follows.

Let $(X, p_1, \ldots, p_r, q_1, \ldots, q_s)$ be an $(r+s)$-marked smooth topological surface. Let $\underline{\eta} = (\eta_1, \ldots, \eta_r)$ be a vector of partitions of the integer d. We define the

Hurwitz number

$$H_d^{h,X}(\underline{\eta})$$

as the weighted number of degree d covers $C \xrightarrow{\pi} X$ such that

- C is a surface of genus h;
- π is unramified over $X \setminus \{p_1, \dots, p_r, q_1, \dots, q_s\}$;
- π ramifies with profile η_i over p_i;
- π has simple ramification over q_i.

The weight is the number of automorphisms of such covers.

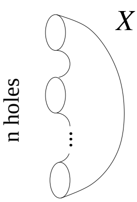
For a Hurwitz number to be nonzero, s, h and $\underline{\eta}$ must satisfy the Riemann–Hurwitz formula. This is why we omit s from the notation. In particular, if we require $s = 0$, then (at most one value of) h is determined by $\underline{\eta}$. We denote by $H_d^X(\underline{\eta})$ the corresponding Hurwitz number.

We define the TQFT $\mathfrak{D}$ as follows:

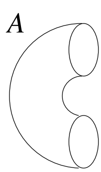
(1) the ground field is $\mathbb{C}$;

(2) the Hilbert space is $H = \bigoplus_{\eta \vdash d} \mathbb{C}e_\eta$, where $\eta \vdash d$ means that η is a partition of d;

(3) morphisms are assigned according to the prescription



$$\xmapsto{\mathfrak{D}} \quad \mathfrak{D}(X) : \begin{array}{ccc} H^{\otimes n} & \to & \mathbb{C} \\ e_{\eta_1} \otimes \cdots \otimes e_{\eta_n} & \mapsto & H_d^X(\underline{\eta}). \end{array}$$



$$\xmapsto{\mathfrak{D}} \quad \mathfrak{D}(A) = \sum \mathfrak{z}(\eta) e_\eta \otimes e_\eta.$$

Fact 2.4 (Dijkgraaf, Witten/Freed, Quinn). *The assignment above defines a semi-simple TQFT $\mathfrak{D}$. Let η be a partition of d, representing a conjugacy class of the symmetric group, and let h be an element in this conjugacy class. Via the identification*

$$e_\eta = \frac{1}{d!} \sum_{g \in S_d} g^{-1} h g,$$

the Hilbert space is isomorphic, as a Frobenius algebra, to the class algebra of the symmetric group in d letters, $\mathfrak{Z}(\mathbb{C}[S_d])$.

A semisimple basis is indexed by irreducible representations ρ of S_d. If ρ is such a representation and $\mathcal{X}_\rho$ its character function, then

$$e_\rho = (\dim \rho) \sum_{\eta \vdash d} \mathcal{X}_\rho(\eta) e_\eta. \tag{2-3}$$

This allows one to recover the classical Burnside formula expressing the number of unramified covers of a genus g curve:

$$\boxed{H_d^{gd-d+1,g}(\phi) = \sum_\rho \left(\frac{d!}{\dim \rho}\right)^{2g-2}.} \tag{2-4}$$

Weighted TQFTs. A weighted TQFT contains some extra structure with respect to an ordinary TQFT. Every cobordism comes equipped with a sequence of weights, or levels. When you concatenate two cobordisms, you add the levels componentwise. We are in particular interested in the theory with 2 levels.

Define the category $2\mathrm{Cob}^{k_1,k_2}$ as follows:

(1) Objects and tensor structure are the same as in 2Cob.

(2) Morphisms are given by triples (W, k_1, k_2), where W is an oriented cobordism as in 2Cob and k_1, k_2 are two integers called levels.

(3) Composition of morphisms consists in concatenating cobordisms and adding levels componentwise.

Definition 2.5. A weighted TQFT is a functor of tensor categories

$$\mathcal{WT} : 2\mathrm{Cob}^{k_1,k_2} \longrightarrow \mathrm{FRMod}.$$

Clearly, if we restrict our attention to cobordisms with weight (0, 0), we obtain an ordinary TQFT. More generally, there exists a $\mathbb{Z} \times \mathbb{Z}$ worth of ordinary TQFTs embedded in a weighted TQFT. Denote by $\mathcal{X}$ the Euler characteristic of a cobordism W. For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, restricting the weighted TQFT to cobordisms with level

$$(a\mathcal{X}, b\mathcal{X})$$

yields an ordinary TQFT.

Generation results. There are several possible ways to generate a weighted TQFT. A natural one consists in generating the level (0, 0) TQFT, and then giving natural operators that allow one to shift the levels. These elements can be chosen to be, for example, the cylinders with weight $(\pm 1, 0)$ and $(0, \pm 1)$. These operators change the levels of the cobordisms without altering its topology. An equivalent, and equally natural choice, is given by the caps, as illustrated in Figure 2.

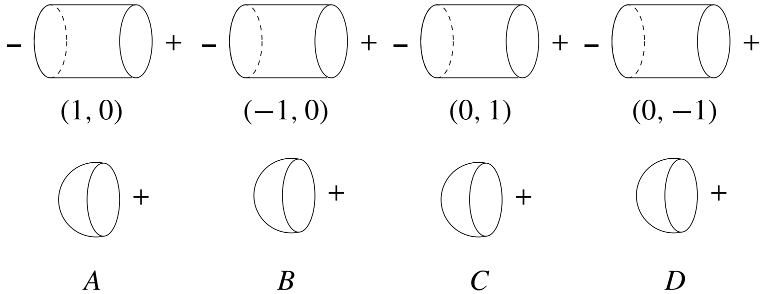
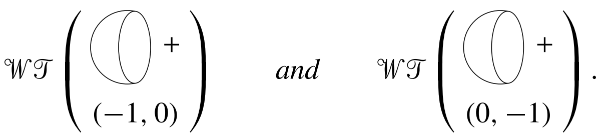


Figure 2. Level-changing objects.

In particular, A is the inverse of B and C is the inverse of D in the level $(0,0)$ Frobenius algebra. Hence the following generation result.

Fact 2.6 [Bryan and Pandharipande 2004, 4.1]. *A weighted TQFT $\mathscr{WT}$ is uniquely determined by a commutative Frobenius algebra over k for the level $(0,0)$ theory and by two distinguished invertible elements in the Frobenius algebra*:



Semisimple weighted TQFTs. A weighted TQFT of rank r is semisimple if there is a basis for the Hilbert space such that all the nonzero tensors in the theory are diagonal. This is equivalent to requiring all embedded ordinary TQFTs to be semisimple (possibly with rescaled semisimple bases). Let $\lambda_1, \dots, \lambda_r$ be the eigenvalues of the level $(0,0)$ genus adding operator. Let $\mu_1, \dots, \mu_r$ be the eigenvalues of the level $(-1,0)$ annulus, and $\bar{\mu}_1, \dots, \bar{\mu}_r$ be the eigenvalues for the level $(0,-1)$ annulus, as illustrated in Figure 3.

Fact 2.7 [Bryan and Pandharipande 2004, 5.2]. *Let $\mathscr{WT}$ be a semisimple TQFT. Denote by $W_m^n(g|k_1,k_2)$ a cobordism of genus g between m input and n output*

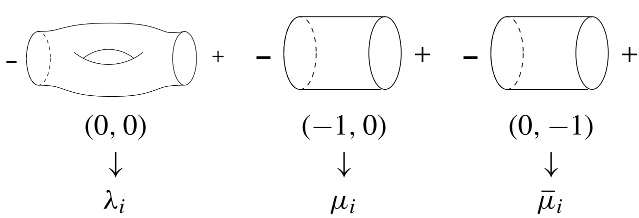


Figure 3. The genus-adding and level-changing operators.

holes, of level (k_1, k_2). *Then*

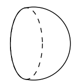
$$\mathcal{T}(W_m^n(g|k_1,k_2)) = \sum_{i=1}^{r} \lambda_i^{g+n-1} \mu_i^{-k_1} \bar{\mu}_i^{-k_2} \underbrace{e^i \otimes \cdots \otimes e^i}_{m \text{ times}} \otimes \underbrace{e_i \otimes \cdots \otimes e_i}_{n \text{ times}}.$$

In particular,

$$\mathcal{T}(W_0^0(g|k_1,k_2)) = \sum_{i=1}^{r} \lambda_i^{g-1} \mu_i^{-k_1} \bar{\mu}_i^{-k_2}.$$

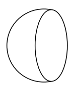
Note. Equivalent definitions can be given for the quantities λ_i, μ_i and $\bar{\mu}_i$. Denote by $e_1, \ldots, e_r$ the vectors of a semisimple basis for the weighted TQFT $\mathcal{WT}$:

- λ_i^{-1} is the value of the level (0, 0) counit on e_i:



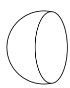
$$\mathcal{WT}\left(- \quad (0,0) \right)(e_i) = \lambda_i^{-1}.$$

- μ_i is the coefficient of e_i in the level $(-1, 0)$ +disc vector:



$$\mathcal{WT}\left(+ \quad (-1,0) \right) = \sum \mu_i e_i.$$

- $\bar{\mu}_i$ is the coefficient of e_i in the level $(0, -1)$ +disc vector:



$$\mathcal{WT}\left(+ \quad (0,-1) \right) = \sum \bar{\mu}_i e_i.$$

3. Construction of the theory

The admissible covers invariants. Let $(X, p_1, \ldots, p_r)$ be a smooth, irreducible, projective curve of genus g with r distinct marked points, and let $N = L_1 \oplus L_2$ be a rank-2 vector bundle on X. The torus $T = \mathbb{C}^* \times \mathbb{C}^*$ acts naturally on N: the first coordinate scales (with weight one) the fiber of L_1, the second coordinate scales the fiber of L_2.

The T-equivariant cohomology of a point is a polynomial ring in two indeterminates, which we denote by

$$H_T^*(pt) = \mathbb{C}[s_1, s_2].$$

We are interested in the following class of intersection numbers:

$$A_d^h(N) := \int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} X, (\eta_1 p_1, \ldots, \eta_r p_r))} e^{\mathrm{eq}}(-R^\bullet \pi_* f^*(L_1 \oplus L_2)),$$

where

- $\overline{\mathrm{Adm}}(h \xrightarrow{d} X(\eta_1 p_1, \dots, \eta_r p_r))$ is as defined on page 38;
- e^{eq} is the equivariant Euler class of the virtual bundle in question;
- π is the universal family over the space of admissible covers;
- f is the universal cover map followed by the canonical contraction map to X.

By [Bryan and Pandharipande 2001], this integral only depends on the genus g of the curve X and on the degrees k_1 and k_2 of the line bundles L_1 and L_2. In the TQFT formulation about to be given it will be useful to emphasize this fact, so we choose to denote these invariants by

$$A^h_d(N)_{\underline{\eta}} = A^h_d(g|k_1,k_2)_{\underline{\eta}}.$$

We consider these invariants for all genera h, and organize them in generating function form:

$$A_d(g|k_1,k_2)_{\underline{\eta}} := \sum_{h\in\mathbb{Z}} u^{\star(h)} A^h_d(g|k_1,k_2)_{\underline{\eta}}, \tag{3-1}$$

where the exponent for the generating function is defined by

$$\star(h) = \dim(\overline{\mathrm{Adm}}(h \xrightarrow{d} X, (\eta_1 p_1, \dots, \eta_r p_r))) = 2h-2+d(2-2g-r)+\sum_{i=1}^{r}\ell(\eta_i).$$

By expanding the equivariant Euler class in terms of ordinary Chern classes and equivariant parameters, we can express these invariants in terms of nonequivariant integrals. Let $h \in \mathbb{Z}\cup\phi$ be a function of b_1, b_2 determined by the equation

$$b_1+b_2 = \dim(\overline{\mathrm{Adm}}(h \xrightarrow{d} X, (\eta_1 x_1, \dots, \eta_r x_r))) = 2h-2+d(2-2g-r)+\sum_{i=1}^{r}\ell(\eta_i).$$

Define

$$A^{b_1,b_2}_d(g|k_1,k_2)_{\underline{\eta}} := \int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} X, (\eta_1 x_1,\dots,\eta_r x_r))} c_{b_1}(-R^\bullet\pi_* f^*(L_1))c_{b_2}(-R^\bullet\pi_* f^*(L_2)).$$

Then the relative invariants are

$$A_d(g|k_1,k_2)_{\underline{\eta}} := \sum_{b_1+b_2=0}^{\infty} u^{b_1+b_2} i s_1^{r_1-b_1} s_2^{r_2-b_2} A^{b_1,b_2}_d(g|k_1,k_2)_{\underline{\eta}}. \tag{3-2}$$

This shows that the partition function for our invariants is a Taylor series in u, whose coefficients are rational functions in s_1 and s_2. The degree of these rational functions is independent of h. It is equal to

$$r_1+r_2-b_1-b_2 = d(2g-2-r) - \sum_{i=1}^{r}\ell(\eta_i).$$

The weighted TQFT $\mathcal{U}$**.** We construct a weighted TQFT $\mathcal{U}$, whose structure coefficients encode the invariants just presented.

The ground ring is defined to be $R = \mathbb{C}[[u]](s_1, s_2)$.

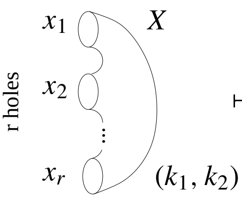
The Hilbert space of the theory is a free R-module of rank equal to the number of partitions of the integer d. A privileged basis is indexed by such partitions η:

$$\mathcal{H} = \bigoplus_{\eta \vdash d} R e_\eta.$$

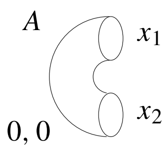
We denote the dual space by $\mathcal{H}^*$, and the dual basis vectors by e^η.

To construct our TQFT we reason topologically. We think of the marked points on a curve $(X, x_1, \dots, x_{r+s})$ as of punctures that we can enlarge into loops. We can assign positive or negative orientation to such loops, and arrange the negatively oriented loops $x_1, \dots, x_r$ to the left, the positively oriented ones to the right (after relabelling $x_{r+i} = y_i$). We now have an oriented cobordism.

To completely determine the structure of the theory we define the scalar maps associated to arbitrary cobordisms into the empty set, and the coproduct, which allows us to move boundary components from left to right:



$$\overset{\mathcal{U}}{\longmapsto} \mathcal{U}(X) : \begin{array}{ccc} H^{\otimes r} & \to & \mathbb{C}[[u]](s_1, s_2) \\ e_{\eta_1} \otimes \cdots \otimes e_{\eta_r} & \mapsto & A_d(g|k_1, k_2)_{\underline{\eta}}. \end{array}$$



$$\overset{\mathcal{U}}{\longmapsto} \mathcal{U}(A) = \sum_{\eta \vdash d} \mathfrak{z}(\eta)(s_1 s_2)^{\ell(\eta)} e_\eta \otimes e_\eta.$$

The combinatorial factor $\mathfrak{z}(\eta)$ is defined in (1-2).

Theorem 3.1. *The structure* $\mathcal{U}$ *defined in the previous paragraph is a two-level, weighted semisimple TQFT.*

In practical terms, it is convenient to adopt the conventional tensor notation of riemannian geometry. If $X = (X, x_1, \dots, x_r, y_1, \dots, y_s | k_1, k_2)$ represents a cobordism of genus g and level k_1, k_2 from r circles to s circles, then $\mathcal{U}(X)$ is an element of $(\mathcal{H}^*)^{\otimes r} \otimes \mathcal{H}^{\otimes s}$. We denote by

$$A_d(g|k_1, k_2)^{\mu_1, \dots, \mu_s}_{\eta_1, \dots, \eta_r}$$

the coordinate of $\mathcal{U}(X)$ in the direction of the basis element

$$e^{\eta_1} \otimes \cdots \otimes e^{\eta_r} \otimes e_{\mu_1} \otimes \cdots \otimes e_{\mu_s}.$$

With this notation, the coproduct gives the following formula for raising and lowering indices:

$$A_d(g|k_1,k_2)^{\mu_1,\dots,\mu_s}_{\eta_1,\dots,\eta_r} = \left(\prod_{i=1}^{s} \mathfrak{z}(\mu_i)(s_1s_2)^{\ell(\mu_i)}\right) A_d(g|k_1,k_2)_{\eta_1,\dots,\eta_r,\mu_1,\dots,\mu_s}. \tag{3-3}$$

Proof of Theorem 3.1. Proving that $\mathcal{U}$ is indeed a weighted TQFT amounts to verifying three statements:

(*Identity*) The tensor associated to the level (0, 0) trivial cobordism from the circle to the circle is the identity morphism of the Hilbert space $\mathcal{H}$.

(*Gluing two curves*) For any two vectors $\underline{\eta}$, $\underline{\mu}$ of partitions of d, and integers satisfying $g = g' + g''$, $k_1 = k_1' + k_1''$, $k_2 = k_2' + k_2''$,

$$A_d(g|k_1,k_2)^{\mu_1,\dots,\mu_s}_{\eta_1,\dots,\eta_r} = \sum_{\nu \vdash d} A_d(g'|k_1',k_2')^{\nu}_{\eta_1,\dots,\eta_r} A_d(g''|k_1'',k_2'')^{\mu_1,\dots,\mu_s}_{\nu}. \tag{3-4}$$

(*Self-gluing*) For any vector of partitions $\underline{\eta}$ and integers g, k_1, k_2,

$$A_d(g+1|k_1,k_2)_{\eta_1,\dots,\eta_r} = \sum_{\nu \vdash d} A_d(g|k_1,k_2)^{\nu}_{\eta_1,\dots,\eta_r,\nu}. \tag{3-5}$$

Identity. This fact is easily proven. One very clever way to do it, pursued in [Bryan and Pandharipande 2005], is to notice that the degree-0 coefficients in our TQFT agree with the classical TQFT of Hurwitz numbers constructed in [Dijkgraaf and Witten 1990] and recalled on page 43. The vanishing of all higher-degree terms can be obtained as a straightforward consequence of the gluing laws, or simply by showing that the dimensions of the moduli spaces in question exceed the maximum degree of a nonequivariant class in the integrand.

Gluing two curves. To minimize bookkeeping, we prove the result when $r = s = 0$ (that is, when the resulting glued curve is not marked). In the general case, the proof follows exactly the same steps, and all the extra indices are simply carried along for the ride.

Consider a one-parameter family of genus g curves W, and the corresponding map to the moduli space,

$$\begin{array}{c} W \\ \downarrow \\ \varphi : \mathbb{A}^1 \longrightarrow \overline{M}_g, \end{array}$$

such that all fibers are smooth curves of genus g, apart from the central fiber

$$W_0 = X_1 \bigcup_{b_1 = b_2} X_2,$$

which is a nodal curve obtained by attaching at a point two smooth curves of genus g' and g'' (with $g' + g'' = g$).

Consider the moduli space $\overline{\mathrm{Adm}}(h \xrightarrow{d} g)$ of admissible covers of a genus g curve by a genus h curve, all ramification simple. By [Abramovich et al. 2003], there is a flat morphism

$$\overline{\mathrm{Adm}}(h \xrightarrow{d} g) \to \overline{M}_g,$$

We can construct the cartesian diagram

$$\begin{array}{ccccc} \mathcal{A}_s = \overline{\mathrm{Adm}}(h \xrightarrow{d} W_s) & \hookrightarrow & \mathcal{A} & \longrightarrow & \overline{\mathrm{Adm}}(h \xrightarrow{d} g) \\ \downarrow & & \downarrow & & \downarrow \\ \{s\} & \hookrightarrow & \mathbb{A}^1 & \longrightarrow & \overline{M}_g \end{array} \tag{3-6}$$

The stack $\mathcal{A}$ must be thought of as the stack of relative admissible covers of the family W. For $s \neq 0$, we obtain admissible covers of a smooth genus g curve; for $s = 0$, we recover admissible covers of the nodal curve W_0.

It is possible to construct two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on W with the following properties:

(1) $\mathcal{L}_i$ restricted to any fiber W_s is a line bundle $L_{i,s}$ of degree k_i.

(2) Over the central fiber W_0, $\mathcal{L}_i$ restricts to a line bundle $L'_{i,s}$ of degree k'_i on X_1, and restricts to a line bundle $L''_{i,s}$ of degree k''_i on X_2.

(3) $\mathbb{C}^*$ acts naturally on $\mathcal{L}_i$ by scaling the fibers (with weight one).

Consider the diagram

$$\begin{array}{ccccc} \mathcal{U}_{\mathcal{A}} & \xrightarrow{f} & \mathcal{W} & \longrightarrow & W \\ \pi \downarrow & \swarrow & & & \\ \mathcal{A} & & & & \end{array}$$

where $\mathcal{U}_{\mathcal{A}}$ is the universal family of the moduli space $\mathcal{A}$, $\mathcal{W}$ is the universal target and f the universal admissible cover map.

The pull-push

$$\mathcal{I} = -R^{\bullet}\pi_* f^*(\mathcal{L}_1 \oplus \mathcal{L}_2)$$

is a virtual bundle of virtual rank $r = 2g - 2 - d(k_1 + k_2)$.

By the flatness of the family $\mathcal{A}$ over $\mathbb{A}^1$, the integral of the top Chern class $c_r(\mathcal{I})$ restricted to a fiber $\mathcal{A}_s$ is independent of the fiber. For $s \neq 0$, we obtain

$$\int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} W_s)} c_r(\mathcal{I}\,|_s) = A_d^h(g|k_1, k_2).$$

We want to evaluate the same expression restricted to $s = 0$, and show it equals the right-hand side of (3-4). We choose to show the equality at the generic genus

h degree of the generating function, to emphasize the geometric nature of the construction. We hence need to establish the following claim, which consists of expanding the genus h term in Equation (3-4), and lowering indices as in (3-3).

Claim 3.2.

$$\int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} W_0)} c_r(\mathcal{I}\,|_0) = \sum_{\nu \vdash d} \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1, h_2} A_d^{h_1}(g'|k_1', k_2')_\nu A_d^{h_2}(g''|k_1'', k_2'')_\nu,$$

where the second sum is over pairs of indices such that $h_1 + h_2 + \ell(\nu) - 1 = h$.

Proof. Recall that, by (1-1),

$$[\overline{\mathrm{Adm}}(h \xrightarrow{d} W_0)] = \sum_{\nu \vdash d} \mathfrak{z}(\nu) \sum_{h_1, h_2} [\overline{\mathrm{Adm}}(h_1 \xrightarrow{d} X_1, (\nu b_1))] \times [\overline{\mathrm{Adm}}(h_2 \xrightarrow{d} X_2, (\nu b_2))],$$

where $h_1 + h_2 + \ell(\nu) - 1 = h$ and

$$\dim(\overline{\mathrm{Adm}}(h_1 \xrightarrow{d} X_1, (\nu b_1))) + \dim(\overline{\mathrm{Adm}}(h_2 \xrightarrow{d} X_2, (\nu b_2))) = \dim(\overline{\mathrm{Adm}}(h \xrightarrow{d} W_0)).$$

Consider the pullback of the normalization sequence associated to the restriction of $\mathcal{L}_i$ to W_0:

$$0 \to f^*(L_{i,0}) \to f^*(L'_{i,0}) \oplus f^*(L''_{i,0}) \to f^*(L_{i,0})\,|_{X_1 \cap X_2} \to 0.$$

This sequence yields a long exact sequence of higher direct image sheaves

$$\begin{aligned} 0 \to R^0\pi_* f^*(L_{i,0}) \to R^0\pi_* f^*(L'_{i,0}) \oplus R^0\pi_* f^*(L''_{i,0}) \to R^0\pi_* f^*(L_{i,0})\,|_{X_1 \cap X_2} \\ \to R^1\pi_* f^*(L_{i,0}) \to R^1\pi_* f^*(L'_{i,0}) \oplus R^1\pi_* f^*(L''_{i,0}) \to 0. \end{aligned}$$

Notice that $(L_{i,0})\,|_{X_1 \cap X_2}$ is a skyscraper sheaf $\mathbb{C}_b$, on which $\mathbb{C}^*$ acts with weight 1.

We now restrict our attention to a connected component of $\mathcal{A}_0$ on which the covers split as two smooth covers of genus h_1 and h_2, with ramification profile ν over the shadows of the node. Here, $f^*(L_{i,0})\,|_{X_1 \cap X_2}$ is a trivial vector bundle of rank $\ell(\nu)$, endowed with a natural $\mathbb{C}^*$ action. The preceding exact sequence then leads to

$$c_{r_i}(-R^\bullet\pi_* f^*(L_{i,0})) = s_i^{\ell(\nu)} c_{r_i'}(-R^\bullet\pi_* f^*(L'_{i,0})) c_{r_i''}(-R^\bullet\pi_* f^*(L''_{i,0})),$$

and finally

$$c_r(\mathcal{I}\,|_0) = (s_1 s_2)^{\ell(\nu)} c_{r'}(\mathcal{I}\,|_0') c_{r''}(\mathcal{I}\,|_0'').$$

Putting everything together yields the claim:

$$\int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} W_0)} c_r(\mathcal{I}\,|_0)$$

$$= \sum_\nu \mathfrak{z}(\nu) \sum_{h_1,h_2} \int_{\overline{\mathrm{Adm}}(h_1 \xrightarrow{d} X_1,(\nu b_1)) \times \overline{\mathrm{Adm}}(h_2 \xrightarrow{d} X_2,(\nu b_2))} c_r(\mathcal{I}|_0)$$

$$= \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1,h_2} \int_{\overline{\mathrm{Adm}}(h_1 \xrightarrow{d} X_1,(\nu b_1))} c_{r'}(\mathcal{I}|_0') \int_{\overline{\mathrm{Adm}}(h_2 \xrightarrow{d} X_2,(\nu b_2))} c_{r''}(\mathcal{I}|_0'')$$

$$= \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1,h_2} A_d^{h_1}(g'|k_1', k_2')_\nu A_d^{h_2}(g''|k_1'', k_2'')_\nu. \qquad \square$$

Self-gluing. The structure of the proof is very similar to the previous case. Again, we simplify the notation by assuming $r = 0$. Consider a one-parameter family of genus g curves W, and the corresponding map into the moduli space,

$$\begin{array}{c} W \\ \downarrow \\ \varphi : \mathbb{A}^1 \longrightarrow \overline{M}_g, \end{array}$$

such that all fibers are smooth curves of genus g, apart from the central fiber

$$W_0 = X/\{b_1 = b_2\},$$

which is a nodal curve obtained by identifying two distinct points on an irreducible smooth curve X of genus $g-1$.

As before, we construct a cartesian diagram of the form (3-6) and two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on W with properties (1) and (3) from page 50, plus

(2) Over the central fiber W_0, $\mathcal{L}_i$ pulls back to a line bundle $L'_{i,s}$ of degree k_i on the normalization X.

We now consider the equivariant top Chern class of the pull-push

$$\mathcal{I} = -R^\bullet \pi_* f^*(\mathcal{L}_1 \oplus \mathcal{L}_2).$$

For $s \neq 0$,

$$\int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} W_s)} c_r(\mathcal{I}\,|_s) = A_d^h(g|k_1, k_2).$$

Again, we can show that the corresponding integral over the central fiber yields exactly the genus h expansion of the right-hand side of Equation (3-5).

Claim 3.3. $\displaystyle\int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} W_0,)} c_r(\mathcal{I}\,|_0) = \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} A_d^{h'}(g-1|k_1, k_2)_{\nu,\nu},$ *where* $h' + \ell(\nu) = h$.

Proof. By (1-3), we have $[\overline{\mathrm{Adm}}(h \xrightarrow{d} W_0)] = \sum_{\nu \vdash d} \mathfrak{z}(\nu)[\overline{\mathrm{Adm}}(h' \xrightarrow{d} X, (\nu b_1, \nu b_2))]$, with $h' + \ell(\nu) = h$.

As in the previous claim, after chasing the normalization sequence for the curve W_0 we obtain, over a connected component of $\mathscr{A}_0$ characterized by covers with ramification profile ν over the shadows of the node, the following decomposition:

$$c_r(\mathscr{I}\,|_0) = (s_1 s_2)^{\ell(\nu)} c_{r'}(\mathscr{I}\,|'_0). \tag{3-7}$$

With this in hand, it is easy to obtain the claim and so conclude the proof of Theorem 3.1:

$$\begin{aligned}
\int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} W_0)} c_r(\mathscr{I}\,|_0) &= \sum_{\nu} \mathfrak{z}(\nu) \int_{\overline{\mathrm{Adm}}(h' \xrightarrow{d} X, (\nu b_1, \nu b_2))} c_r(\mathscr{I}\,|_0) \\
&= \sum_{\nu} \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \int_{\overline{\mathrm{Adm}}(h' \xrightarrow{d} X, (\nu b_1, \nu b_2))} c_{r'}(\mathscr{I}\,|'_0) \\
&= \sum_{\nu} \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} A_d^{h'}(g-1|k'_1, k'_2)_{\nu,\nu}.
\end{aligned}$$

□

4. Computing the theory

In order to determine the whole weighted TQFT it is sufficient to compute a small number of invariants, as seen in Fact 2.6. Among the many possible choices for a set of generators, we choose as the generators for the level $(0, 0)$ TQFT

(1) the coefficients $A_d(0|0, 0)_\eta$ of the open $(-)$ disc,

(2) the coefficients $A_d(0|0, 0)^{\eta,\mu}$ of the $(+, +)$ annulus, and

(3) the coefficients $A_d(0|0, 0)_{\eta,\mu,\nu}$ associated to the $(-, -, -)$ pair of pants,

and as the generators for level shifting

(4) the coefficients of the Calabi–Yau caps $A_d(0|-1, 0)_\eta$ and $A_d(0|0, -1)_\eta$.

Theorem 4.1. *The level* $(0, 0)$ *TQFT coincides with the level* $(0, 0)$ *theory of* [Bryan and Pandharipande 2004].

The significant difference in the theories lies in the Calabi–Yau caps, which we will compute (starting on page 55) by localization on the moduli spaces of admissible covers.

Proof of Theorem 4.1. It is simple to compute independently the coefficients for the cap. Dimension counts show they are degenerate, in the sense that only the constant term of the series is nonzero. The coefficients for the $(+, +)$ cylinder agree by definition. We will conclude the proof by showing that the coefficients for the pair of pants are the same.

The level* (0, 0) *pair of pants. The invariants $A^{\circ}_d(0|0,0)_{\eta,\nu,\mu}$ of the level $(0,0)$ pair of pants are computed by the integrals:

$$\int_{\overline{\mathrm{Adm}}^{\circ}(h\xrightarrow{d}\mathbb{P}^1,(\eta 0,\mu 1,\nu\infty))} c^{\mathrm{eq}}_{2h-2}(-R^{\bullet}\pi_* f^*(\mathbb{O}_{\mathbb{P}^1}\oplus\mathbb{O}_{\mathbb{P}^1})).$$

The dimension of the moduli space in question is

$$2h-d-2+\ell(\eta)+\ell(\mu)+\ell(\nu).$$

Hence, if $\ell(\eta)+\ell(\mu)+\ell(\nu) > d+2$, the relative connected integrals vanish. The disconnected integrals are then obtained inductively from invariants of lower degree d.

All other invariants have contributions from connected components, and hence need to be computed directly.

In [Bryan and Pandharipande 2004, Appendix] it is shown that all invariants can be recursively determined from $A_d(0|0,0)_{(d),(d),(2)}$, the invariant corresponding to full ramification over two points, and a simple transposition over the third point. Their proof uses only TQFT formalism; hence it suffices to prove the following statement.

Lemma 4.2. *For $d \geq 2$,*

$$A_d(0|0,0)_{(d),(d),(2)} = \frac{s_1+s_2}{2s_1s_2}\left(d\cot\frac{du}{2}-\cot\frac{u}{2}\right).$$

(This result differs from the analogous one in [Bryan and Pandharipande 2004] by a factor of $-i$, reflecting a normalization in their generating function conventions that we have not adopted.)

Proof. The full ramification conditions force our covers to be connected. Hence the connected and disconnected invariants coincide.

According to (3-2), we have

$$A_d(0|0,0)_{(d),(d),(2)} = \sum_{b_1+b_2=0}^{\infty} u^{b_1+b_2} s_1^{h-1-b_1} s_2^{h-1-b_2} \int_{\overline{\mathrm{Adm}}(h\xrightarrow{d}\mathbb{P}^1,((d)0,(d)1,(2)\infty))} c_{b_1}(\mathbb{E}^*)c_{b_2}(\mathbb{E}^*),$$

with b_1+b_2 equal to the dimension of the moduli space, which is

$$\dim(\overline{\mathrm{Adm}}(h\xrightarrow{d}\mathbb{P}^1,((d)0,(d)1,(2)\infty))) = 2h-1.$$

For a given value of h, the only nonvanishing terms in the expression above are those where $(b_1,b_2)=(h,h-1)$ or $(b_1,b_2)=(h-1,h)$. Adding the two, we obtain

$$A^h_d(0|0,0)_{(d),(d),(2)} = \frac{s_1+s_2}{s_1s_2}\int_{\overline{\mathrm{Adm}}(h\xrightarrow{d}\mathbb{P}^1,((d)0,(d)1,(2)\infty))} -\lambda_h\lambda_{h-1}$$

and consequently, the generating function

$$A_d(0|0,0)_{(d),(d),(2)} = \frac{s_1+s_2}{s_1 s_2} \sum_{h=0}^{\infty} u^{2h-1} \int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} \mathbb{P}^1, ((d)0,(d)1,(2)\infty))} -\lambda_h \lambda_{h-1},$$

where λ_k denotes the k-th Chern class of the (pullback of the) Hodge bundle $\mathbb{E}$.

Recall that we defined the λ classes on moduli spaces of admissible covers simply by pulling them back from the appropriate moduli spaces of stable curves. In particular we have the diagram

$$\begin{array}{ccc} \overline{\mathrm{Adm}}(h \xrightarrow{d} \mathbb{P}^1, ((d)0, (d)1, (2)\infty)) & \xrightarrow{\rho} & \overline{M}_{h,2} \\ & \searrow & \downarrow \pi \\ & & \overline{M}_h \end{array}$$

The map ρ is defined by marking on the admissible covers the unique preimages of the branch points 0 and 1. The Hodge bundle on $\overline{M}_h$ pulls back to the Hodge bundle on $\overline{M}_{h,2}$, hence we can think of the λ classes on the moduli space of admissible covers as pulled back from $\overline{M}_{h,2}$.

Denote by $H_d \subset M_{h,2}$ the locus of curves admitting a degree d map to $\mathbb{P}^1$ which is totally ramified at the marked points. Let

$$\overline{H}_d \subset \overline{M}_{h,2}$$

be the closure of H_d, consisting of possibly nodal curves admitting a degree d map to a tree of rational curves, fully ramified over the two marked points. The image of the map

$$\rho : \overline{\mathrm{Adm}}(h \xrightarrow{d} \mathbb{P}^1, ((d)0, (d)1, (2)\infty)) \longrightarrow \overline{M}_{h,2}$$

is precisely $\overline{H}_d$, and ρ is a degree $2h$ map onto its image.

From this we conclude that

$$\int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} \mathbb{P}^1, ((d)0,(d)1,(2)\infty))} -\lambda_h \lambda_{h-1} = 2h \int_{[\overline{H}_d]} -\lambda_h \lambda_{h-1}.$$

This is the integral computed in [Bryan and Pandharipande 2004, pages 28–29]; hence the result follows. This proves Lemma 4.2 and therefore Theorem 4.1. □

The Calabi–Yau cap. We can obtain $A_d(0|-1,0)_\eta$ from $A_d(0|0,-1)_\eta$ by simply interchanging the roles of s_1 and s_2. Further:

Theorem 4.3. *Let d be a positive integer, and $\eta = (\eta_1, \dots, \eta_{\ell(\eta)})$ a partition of d. The degree-d Calabi–Yau invariants are*

$$A_d(0|0,-1)_\eta = (-1)^{d-\ell(\eta)} \frac{\left(2\sin\frac{u}{2}\right)^d}{(s_1)^{\ell(\eta)}\mathfrak{z}(\eta)\prod 2\sin\frac{\eta_i u}{2}}.$$

In [Cavalieri 2004], this formula is computed via localization on moduli spaces of (connected) admissible covers in degree 1, 2, 3. The result is obtained by finding relations between the Calabi–Yau cap invariants and generating functions for simple Hurwitz numbers. Two types of obstructions arise in degrees beyond 3. First, fixed loci inside moduli spaces of connected admissible covers are in principle easily described as finite products and quotients of moduli spaces of connected admissible covers, but the combinatorial complexity grows fast. Second, generating functions for simple Hurwitz numbers are not readily available beyond degree 3.

To circumvent the first problem we interpret the fixed loci in the localization as simpler products of disconnected admissible cover spaces. Then all possible Calabi–Yau invariants, not only the fully ramified ones, appear in the recursions. There is one subtlety to be aware of: Calabi–Yau cap invariants are defined as intersection numbers on moduli spaces of admissible covers of a parametrized $\mathbb{P}^1$, whereas the fixed loci are in terms of admissible covers of unparametrized projective lines. Another localization computation, with an appropriate choice of linearizations for the bundles, gives an expression for the invariants in terms of the unparametrized $\mathbb{P}^1$ admissible covers.

To deal with the lack of explicit generating functions for general simple Hurwitz numbers, we notice that the recursive relation that we need to prove is in fact determined by a virtual localization computation on moduli spaces of stable maps. This is yet more evidence of how intimately related this theory and Gromov–Witten theory are.

Proof of Theorem 4.3. We prove the following formula for the connected Calabi–Yau cap invariants:

$$A^\circ_d(0|0,-1)_\eta = \begin{cases} \dfrac{(-1)^{d-1}}{s_1}\dfrac{1}{d}\dfrac{\left(2\sin\frac{u}{2}\right)^d}{2\sin\frac{du}{2}} & \text{for } \eta=(d), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.3 follows from this via exponentiation.

The vanishing of the connected invariants for all partitions but (d) is a dimension count. By (3-1) and (3-2), the genus-h contribution to the connected Calabi–Yau invariants is

$$A^{\circ h}_{d}(0|0,-1)_\eta = \int_{\overline{\mathrm{Adm}}^\circ(h \xrightarrow{d} \mathbb{P}^1,(\eta\infty))} c^{\mathrm{eq}}_{2h+d-1}(-R^\bullet\pi_* f^*(\mathbb{O}_{\mathbb{P}^1} \oplus \mathbb{O}_{\mathbb{P}^1}(-1)))$$
$$= \sum_{b_1,b_2} s_1^{r_1-b_1} s_2^{r_2-b_2} \int_{\overline{\mathrm{Adm}}^\circ(h \xrightarrow{d} \mathbb{P}^1,(\eta\infty))} c_{b_1}(\mathbb{E}^*)c_{b_2}(R^1\pi_* f^*(\mathbb{O}_{\mathbb{P}^1}(-1))),$$

where

- $b_1 + b_2 = \dim(\overline{\mathrm{Adm}}(h \xrightarrow{d} \mathbb{P}^1, (\eta\infty))) = 2h + d + \ell(\eta) - 2$;
- $r_1 = h - 1$ is the virtual rank of the virtual bundle $-R^\bullet\pi_* f^*(\mathbb{O}_{\mathbb{P}^1})$;
- $r_2 = h + d - 1$ is the virtual rank of the virtual bundle $-R^\bullet\pi_* f^*(\mathbb{O}_{\mathbb{P}^1}(-1))$.

Since $-R^\bullet\pi_* f^*(\mathbb{O}_{\mathbb{P}^1}(-1)) = R^1\pi_* f^*(\mathbb{O}_{\mathbb{P}^1}(-1))$ is in fact a vector bundle of rank $h + d - 1$, we also have the constraint

$$b_1 + b_2 \le 2h + d - 1.$$

The only possibly nonvanishing integrals occur when $\ell(\eta) = 1$, i.e. when $\eta = (d)$. The indices b_1 and b_2 are forced to be, respectively, h and $h + d - 1$.

Note. The full ramification condition forces all covers to be connected; the fully ramified connected and disconnected invariants coincide, thus allowing us to drop the superscript $\circ$.

Finally, our task is to prove:

$$\frac{1}{s_1}\sum_{h=0}^{\infty} u^{2h+d-1} \int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} \mathbb{P}^1,((d)\infty))} c_h(\mathbb{E}^*)c_{h+d-1}(R^1\pi_* f^*(\mathbb{O}_{\mathbb{P}^1}(-1))) = \frac{(-1)^{d-1}}{s_1}\frac{1}{d}\frac{\left(2\sin\frac{u}{2}\right)^d}{2\sin\frac{du}{2}}.$$

Calabi–Yau cap invariants: parametrized to unparametrized. We evaluate via localization the Calabi–Yau cap invariant $A^h_d(0|0,-1)_\eta$, for a general partition η.

We linearize the ($\mathbb{C}^*$ action on the) two bundles by assigning to $\mathbb{O}_{\mathbb{P}^1}$ weight 0 over both 0 and ∞, and assigning $\mathbb{O}_{\mathbb{P}^1}(-1)$ weight 0 over 0 and weight 1 over ∞:

weight	over 0	over ∞
$\mathbb{O}_{\mathbb{P}^1}(-1)$	0	1
$\mathbb{O}_{\mathbb{P}^1}$	0	0

There are a priori many fixed loci in the localization computation. However it is possible to rule out a vast majority of them using either dimension counts or linearization considerations (see [Cavalieri 2004] or [Bryan and Pandharipande 2005] for a discussion of these standard localization tricks).

Ultimately, the only possibly contributing fixed loci are those whose general element consists of $\ell(\eta)$ spheres S_i, mapping to the main $\mathbb{P}^1$ with degree η_i, all fully ramified over 0 and ∞. A genus 0 twig sprouts from the point ∞ on the main $\mathbb{P}^1$, covered by $\ell(\eta)$ curves C_i of genus h_i. The curve C_i is attached to S_i at a fully ramified point. The h_i are such that

$$h_1+\ldots+h_{\ell(\eta)}=h+\ell(\eta)-1.$$

Finally, if we denote by $F_{\eta,h}$ the disjoint union of all such fixed loci as the h_i vary, and by N the normal bundle to such fixed loci, we obtain from localization:

$$A_d^h(0|0,-1)_\eta=\int_{F_{\eta,h}}\frac{e^{\mathrm{eq}}(-R^\bullet\pi_*f^*(\mathbb{O}_{\mathbb{P}^1}\oplus\mathbb{O}_{\mathbb{P}^1}(-1)))\,|_{F_{h,\eta}}}{e^{\mathrm{eq}}(N)}.$$

Recursion via localization on admissible covers. We now suppose $d>1$ and consider the auxiliary integral

$$I^h=\int_{\overline{\mathrm{Adm}}^\circ(h\xrightarrow{d}\mathbb{P}^1)}e^{\mathrm{eq}}(-R^\bullet\pi_*f^*(\mathbb{O}_{\mathbb{P}^1}\oplus\mathbb{O}_{\mathbb{P}^1}(-1))),\tag{4-1}$$

computed on the space of connected admissible covers. It vanishes for dimension reasons: we are integrating a class whose highest nonequivariant factor has codimension $(2h+d-1)$ on a space of dimension $2h+2d-2$.

On the other hand, if we evaluate the integral via localization we get a relation among Calabi–Yau cap invariants. We let a one-dimensional torus act naturally on the moduli space and denote the equivariant parameter by s. We choose to linearize the two bundles with the following weights:

weight	over 0	over ∞
$\mathbb{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathbb{O}_{\mathbb{P}^1}$	1	1

The possibly contributing fixed loci E_{η,h_0,h_∞} are represented by connected localization graphs such that any vertex over ∞ has valence 1; see [Cavalieri 2004]. They can be indexed by triples (η,h_0,h_∞), where

- $\eta=(d_1,\ldots,d_{\ell(\eta)})$ is a partition of d representing the configuration of the spheres over the main $\mathbb{P}^1$;
- h_0 is the genus of the curve lying over 0;
- h_∞ is the genus of the curve lying over ∞ (considered as a disconnected curve);
- $h_0+h_\infty=h-\ell(\eta)+1$.

We recognize that a general element in the fixed locus E_{η,h_0,h_∞} is obtained by gluing together an element in the fixed locus F_{η,h_∞} with a connected admissible covers of a genus 0 curve, with a special point of ramification η. Keeping in account the stacky contribution from the gluing, our integral I on E_{η,h_0,h_∞} reduces to

$$\begin{aligned} I^h_{\eta,h_0,h_\infty} &= \mathfrak{z}(\eta)\int_{\overline{\mathrm{Adm}}^\circ(h_0\xrightarrow{d}\mathbb{P}^1,\eta)\times F_{\eta,h_\infty}} \frac{e^{\mathrm{eq}}(-R^\bullet\pi_* f^*(\mathbb{O}_{\mathbb{P}^1}\oplus\mathbb{O}_{\mathbb{P}^1}(-1)))\mid_{\overline{\mathrm{Adm}}^\circ(h\xrightarrow{d}\mathbb{P}^1,\eta)\times F_{\eta,h_\infty}}}{e^{\mathrm{eq}}(N)} \\ &= \mathfrak{z}(\eta)s^{2\ell(\eta)}A_d^{h_\infty}(0|0,-1)_\eta\int_{\overline{\mathrm{Adm}}^\circ(h_0\xrightarrow{d}\mathbb{P}^1,\eta)}\frac{c_{h_0}(\mathbb{E}^*\otimes\mathbb{C}_1)c_{h_0}(\mathbb{E}^*\otimes\mathbb{C}_{-1})}{s(s-\psi_\eta)}, \end{aligned}$$

where $\mathbb{C}_a$ is a trivial line bundle where the torus acts on the fibers with weight a.

After expanding and using Mumford's relation [1983] saying that $c(\mathbb{E})c(\mathbb{E}^*)$ equals 1, we obtain

$$\begin{aligned} I^h_{\eta,h_0,h_\infty} &= \mathfrak{z}(\eta)s^{\ell(\eta)+2-d}A_d^{h_\infty}(0|0,-1)_\eta\int_{\overline{\mathrm{Adm}}^\circ(h_0\xrightarrow{d}\mathbb{P}^1,\eta)}(-1)^{h_0}\psi_\eta^{2h_0+d+\ell(\eta)-4} \\ &= \mathfrak{z}(\eta)s^{\ell(\eta)+2-d}A_d^{h_\infty}(0|0,-1)_\eta\frac{(-1)^{h_0}H_d^{h_0}(\eta)}{(2h_0+d+\ell(\eta)-2)!}. \end{aligned}$$

The quantity $H_d^{h_0}(\eta)$ is a simple Hurwitz number, as defined on page 43.

The integral I is evaluated by adding up the contributions from all fixed loci E_{η,h_0,h_∞}:

$$0 = I^h = \sum_{\eta\vdash d}\sum_{h_0+h_\infty=h-\ell(\eta)+1} I^h_{\eta,h_0,h_\infty}. \tag{4-2}$$

This holds for all genera h, and can be expressed in a very compact form in the language of generating functions. Define

$$\mathscr{H}_{d,\eta(u)} := \sum\frac{(-1)^h H_d^h(\eta)}{(2h+d+\ell(\eta)-2)!}u^{(2h+d+\ell(\eta)-2)}.$$

Then formulas (4-2), for all genera h, are encoded in the relation

$$0 = \sum_{\eta\vdash d}\mathfrak{z}(\eta)s^{\ell(\eta)+2-d}A_d(0|0,-1)_\eta(u)\mathscr{H}_{d,\eta}(u). \tag{4-3}$$

This relation determines $A_d(0|0,-1)_{(d)}$ in terms of generating functions for simple Hurwitz numbers and of the invariants $A_d(0|0,-1)_\eta$, for $\ell(\eta)\geq 2$, which can be inductively determined via exponentiation if we assume the theory up to degree $d-1$. The theory has been explicitly computed up to degree 3 in [Cavalieri 2004]; hence the induction can start.

To prove Theorem 4.3 it therefore suffices to show that (4-3) holds for the conjectured values of the Calabi–Yau invariants. After substituting and simplifying, this amounts to proving that

$$0 = \sum_{\eta \vdash d} (-1)^{\ell(\eta)} \frac{\mathcal{H}_{d,\eta}(u)}{\prod_{\eta_i \in \eta} 2 \sin \frac{\eta_i u}{2}}. \tag{4-4}$$

Virtual localization on stable maps. Relation (4-4) is the result of explicitly evaluating via virtual localization the auxiliary integrals

$$J^h = \int_{\overline{M}_h(\mathbb{P}^1, d)} e^{\mathrm{eq}}(-R^\bullet \pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))).$$

Again dimension reasons grant us the vanishing of this integral. We proceed to linearize the bundles by assigning weights as follows:

weight	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathcal{O}_{\mathbb{P}^1}$	1	1

The analysis of the possibly contributing fixed loci is parallel to the previous section. The contribution by the fixed locus E_{η,h_0,h_∞} is

$$\sum_{h_1+\ldots+h_{\ell(\eta)}=h_\infty+\ell(\eta)-1} J_{\eta,h_0,h_1,\ldots,h_{\ell(\eta)}},$$

with

$$J_{\eta,h_0,h_1,\ldots,h_{\ell(\eta)}} = \frac{1}{\mathfrak{z}(\eta)} \int_{\overline{M}_{h_0,\ell(\eta)}} \frac{c_{h_0}(\mathbb{E}^* \otimes \mathbb{C}_1) c_{h_0}(\mathbb{E}^* \otimes \mathbb{C}_1) c_{h_0}(\mathbb{E}^* \otimes \mathbb{C}_{-1})}{\prod \left(\eta_i{}^{-1} - \psi_i\right)} \times \prod_{i=1}^{\ell(\eta)} \frac{\eta_i^{\eta_i}}{\eta_i!} \int_{\overline{M}_{h_i,1}} \frac{c_{h_i}(\mathbb{E}^* \otimes \mathbb{C}_1) c_{h_i}(\mathbb{E}^* \otimes \mathbb{C}_1) c_{h_i}(\mathbb{E}^*)}{-\eta_i{}^{-1} - \psi_1}.$$

(See [Hori et al. 2003, Chapter 27] for a clear and detailed explanation of how to compute these terms, or [Bryan and Pandharipande 2004, proof of Theorem 5.1] for a very similar computation.)

After simplifying via Mumford's relation and rearranging things, the preceding formula becomes

$$\frac{(-1)^{h_0}}{\mathrm{Aut}(\eta)} \prod_{i=1}^{\ell(\eta)} \frac{\eta_i^{\eta_i}}{\eta_i!} \int_{\overline{M}_{h_0,\ell(\eta)}} \frac{1 - \lambda_1 + \ldots \pm \lambda_{h_0}}{\prod(1 - \eta_i \psi_i)} \prod_{i=1}^{\ell(\eta)} -\eta_i^{2h_i - 1} \int_{\overline{M}_{h_i,1}} \lambda_{h_i} \psi_1^{2h_i - 2}. \tag{4-5}$$

We recognize in formula (4-5) two famous results in the field. The first is the *ELSV formula*, establishes the connection between Hurwitz numbers and Hodge integrals [Ekedahl et al. 2001; Graber and Vakil 2003a]:

$$H^h_d(\eta) = \frac{(2h+d+\ell(\eta)-2)!}{\mathrm{Aut}(\eta)} \prod_{i=1}^{\ell(\eta)} \frac{\eta_i^{\eta_i}}{\eta_i!} \int_{\overline{M}_{h,\ell(\eta)}} \frac{1-\lambda_1+\ldots\pm\lambda_h}{\prod(1-\eta_i\psi_i)}.$$

The second is Faber and Pandharipande's formula [2000], expressing in generating function form the following class of integrals:

$$\mathcal{L}(u) := \sum u^{2h-1} \int_{\overline{M}_{h,1}} \lambda_h \psi_1^{2h-2} = \frac{1}{2\sin\frac{u}{2}}.$$

Now it is a matter of careful bookkeeping to translate all this information in the language of generating functions. Doing so concludes the proof of Theorem 4.3 by establishing that

$$\begin{aligned} 0 = \sum_{h\in\mathbb{Z}} J^h u^{2h+2d-2} &= \sum_{\eta\vdash d} (-1)^{\ell(\eta)} \mathcal{H}_{d,\eta}(u) \prod_{\eta_i\in\eta} \mathcal{L}(\eta_i u) \\ &= \sum_{\eta\vdash d} (-1)^{\ell(\eta)} \frac{\mathcal{H}_{d,\eta}(u)}{\prod_{\eta_i\in\eta} 2\sin\frac{\eta_i u}{2}}. \end{aligned}$$

□

5. A specialization of the theory

We now discuss a specialization of the theory, obtained by embedding a one-dimensional torus inside the two-dimensional torus T, and considering the theory as depending from one equivariant parameter instead of two.

We specialize to the antidiagonal action, and notice that the coefficients for the product simplify dramatically. It is possible to obtain nice closed formulas for our theory, and to view our TQFT as a one-parameter deformation of the classical TQFT of Hurwitz numbers studied in [Dijkgraaf and Witten 1990; Freed and Quinn 1993]. Our formulas show connections to the representation theory of the symmetric group S_d.

The antidiagonal action. Embed $\mathbb{C}^*$ in the two-dimensional torus T via the map

$$\alpha \mapsto \left(\alpha, \frac{1}{\alpha}\right).$$

$\mathbb{C}^*$ acts on N by composing this embedding with the natural action of T constructed in page 46. If we set

$$H^*_{\mathbb{C}^*}(pt) = \mathbb{C}[s],$$

the one-parameter theory obtained with this action corresponds to setting

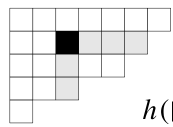
$$s = s_1 = -s_2.$$

The Q-dimension of an irreducible representation. Let ρ be an irreducible representation of the symmetric group on d letters S_d. Classically, a partition of d, and hence a Young diagram, can be canonically associated to ρ; see [Fulton and Harris 1991, Chapter 4], for example. Recall that the *hook length* $h(\square)$ of a cell $\square$ in a Young diagram is the number of cells in the L-shaped strip, or "hook", having the given cell as its northwest corner (see figure on the next page). We now define the Q-dimension of the representation ρ by setting

$$\frac{\dim_Q \rho}{d!} := \prod_{\square \in \rho} \frac{1-Q}{1-Q^{h(\square)}} = \prod_{\square \in \rho} \frac{1}{1+Q+\cdots+Q^{h(\square)-1}} \tag{5-1}$$

The classical hook-length formula says that

$$\dim \rho = d! \Big/ \prod_{\square \in \rho} h(\square).$$



$h(\blacksquare) = 6$

Thus formula (5-1) specializes to the ordinary dimension of ρ when $Q = 1$.

The level $(0,0)$ *TQFT.* The main result is that the level $(0,0)$ TQFT completely collapses to the Dijkgraaf TQFT $\mathfrak{D}$. In particular, we have explicit formulas for the semisimple basis of the Frobenius algebra. The basis vectors are indexed by irreducible representations of the symmetric group S_d.

Lemma 5.1. *For the antidiagonal action, the level* $(0,0)$ *series have no nonzero terms of positive degree in* u.

Proof. (Essentially by Bryan and Pandharipande.) Endow $\mathbb{C}$ with the $\mathbb{C}^*$ action

$$\alpha \cdot z = \alpha^n z.$$

This corresponds to considering $\mathbb{C}$ as an equivariant line bundle over a point, whose first equivariant Chern class is ns. We denote such an equivariant line bundle by $\mathbb{C}_{ns}$.

The level $(0,0)$ partition functions are, up to some pure weight factor, constructed from integrals of the form

$$\int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} X, (\eta^1 x_1, \ldots, \eta^r x_r))} e^{\mathrm{eq}}(\mathbb{E}^* \otimes \mathbb{C}_s) e^{\mathrm{eq}}(\mathbb{E}^* \otimes \mathbb{C}_{-s}) \\ \int_{\overline{\mathrm{Adm}}(h \xrightarrow{d} X, (\eta^1 x_1, \ldots, \eta^r x_r))} (-1)^h e^{\mathrm{eq}}((\mathbb{E}^* \oplus \mathbb{E}) \otimes \mathbb{C}_s).$$

Equivariant Chern classes of a bundle also are products of ordinary Chern classes times the appropriate factor of s. But by Mumford's relation $c(\mathbb{E})c(\mathbb{E}^*) = 1$, all Chern classes (but the 0-th) of the bundle $\mathbb{E}^* \oplus \mathbb{E}$ vanish. Hence the only possibly nonvanishing integrals occur when the dimension of the moduli space is 0, which then constitutes the degree 0 term in our generating functions. $\square$

Thus we have already found a semisimple basis for the corresponding Frobenius algebra in (2-3). All we need to do is adjust for the equivariant parameter. Let ρ be an irreducible representation of the symmetric group S_d, with character function χ_ρ; a *semisimple basis* for the level (0, 0) TQFT is given by the vectors

$$e_\rho = \frac{\dim \rho}{d!} \sum_{\eta \vdash d} (s)^{\ell(\eta)-d} \chi_\rho(\eta) e_\eta.$$

Notation. If $\eta = (\eta_1, \ldots, \eta_r)$ is a partition, we define

$$n(\eta) := 0\eta_1 + 1\eta_2 + \cdots + (r-1)\eta_r.$$

Theorem 5.2. *The partition functions corresponding to surfaces without boundary in the weighted TQFT are given in closed form by*

$$\boxed{A_d(g|k_1, k_2) = (-1)^a s^b \sum_\rho \left(\frac{d!}{\dim \rho}\right)^{2g-2} \left(\frac{\dim \rho}{\dim_Q \rho}\right)^{k_1+k_2} Q^{n(\rho)k_1+n(\rho')k_2},}$$

where $a := d(g-1-k_2)$, $b := d(2g-2-k_1-k_2)$, $Q := e^{iu}$, and ρ is an irreducible representation of the symmetric group S_d, with dual representation ρ'.

Note. By setting $Q = 1$, which corresponds to $u = 0$, we recover the classical formula (2-4) counting unramified covers of a genus g topological surface. Thus any TQFT naturally embedded in our weighted TQFT constitutes a one-parameter deformation of the Dijkgraaf TQFT.

Proof of Theorem 5.2. By Fact 2.7, to completely describe the structure of a semisimple weighted TQFT it suffices to evaluate the following quantities:

- the e_ρ-eigenvalue λ_ρ of the genus-adding operator, or, equivalently, the inverse of the counit evaluated on e_ρ;
- the e_ρ-eigenvalue μ_ρ of the left level-subtracting operator, or, equivalently, the coefficient of e_ρ in the $(0, -1)$ Calabi–Yau cap;
- the e_ρ-eigenvalue $\bar{\mu}_\rho$ of the right level-subtracting operator, or, equivalently, the coefficient of e_ρ in the $(-1, 0)$ Calabi–Yau cap.

The computation of λ_ρ coincides exactly with the one in [Bryan and Pandharipande 2004]. We therefore omit it.

To compute μ_ρ and $\bar{\mu}_\rho$ we first observe that the tensors associated to the Calabi–Yau caps in our theory are scalar multiples of the tensors in Bryan and Pandharipande's theory:

$$\mathcal{U}(CYcap) = 2^d \left(\sin\frac{u}{2}\right)^d \mathcal{BP}(CYcap) = \frac{(1-Q)^d}{Q^{(d/2)}(-i)^d} \mathcal{BP}(CYcap).$$

This, together with the formulas in [Bryan and Pandharipande 2004, page 36], implies that

$$\mu_\rho = s^d \frac{d!}{\dim\rho}(1-Q)^d s_\rho(Q), \qquad \bar{\mu}_\rho = (-s)^d \frac{d!}{\dim\rho}(1-Q)^d s_{\rho'}(Q),$$

where s_ρ denotes the Schur function of the representation ρ, and is defined to be (see [Macdonald 1995])

$$s_\rho(Q) := Q^{n(\rho)} \prod_{\square\in\rho} \frac{1}{1-Q^{h(\square)}}.$$

Plugging this in, we obtain

$$\begin{aligned}
\mu_\rho &= s^d\left(\frac{d!}{\dim\rho}\right)(1-Q)^d Q^{n(\rho)} \prod_{\square\in\rho}\frac{1}{1-Q^{h(\square)}}\\
&= s^d\left(\frac{d!}{\dim\rho}\right) Q^{n(\rho)} \prod_{\square\in\rho}\frac{1-Q}{1-Q^{h(\square)}} = s^d\left(\frac{\dim_Q\rho}{\dim\rho}\right)Q^{n(\rho)},\\
\bar{\mu}_\rho &= (-s)^d\left(\frac{d!}{\dim\rho}\right)(1-Q)^d Q^{n(\rho')} \prod_{\square\in\rho'}\frac{1}{1-Q^{h(\square)}}\\
&= s^d\left(\frac{d!}{\dim\rho}\right) Q^{n(\rho')} \prod_{\square\in\rho'}\frac{1-Q}{1-Q^{h(\square)}} = s^d\left(\frac{\dim_Q\rho}{\dim\rho}\right)Q^{n(\rho')}.
\end{aligned}$$

The theorem is then obtained by using these coefficients in the formula given by Fact 2.7. □

Acknowledgements. This paper owes a lot to Jim Bryan and Rahul Pandharipande, whose work provided amazing motivation and a useful roadmap. In particular I thank Jim Bryan for suggesting to me to look at moduli spaces of admissible covers, and for graciously answering all my pestering emails. I am grateful to my advisor, Aaron Bertram, for his constant support, motivation, and expert guidance. I also thank Alastair Craw, Bill Fulton, Y. P. Lee and Ravi Vakil for very useful conversations and feedback.

References

[Abramovich et al. 2003] D. Abramovich, A. Corti, and A. Vistoli, "Twisted bundles and admissible covers", *Comm. Algebra* **31**:8 (2003), 3547–3618. Special issue in honor of Steven L. Kleiman. MR 2005b:14049 Zbl 1077.14034

[Bryan and Graber ≥ 2008] J. Bryan and T. Graber, "The crepant resolution conjecture". In preparation.

[Bryan and Pandharipande 2001] J. Bryan and R. Pandharipande, "BPS states of curves in Calabi-Yau 3-folds", *Geom. Topol.* **5** (2001), 287–318. MR 2002h:14092 Zbl 1063.14068

[Bryan and Pandharipande 2004] J. Bryan and R. Pandharipande, "The local Gromov–Witten theory of curves", preprint, 2004. math.AG/0411037

[Bryan and Pandharipande 2005] J. Bryan and R. Pandharipande, "Curves in Calabi–Yau threefolds and topological quantum field theory", *Duke Math. J.* **126**:2 (2005), 369–396. MR 2005k:14117 Zbl 1084.14053

[Bryan et al. 2005] J. Bryan, T. Graber, and R. Pandharipande, "The orbifold quantum cohomology of $\mathbb{C}^2/\mathbb{Z}_3$ and Hurwitz–Hodge integrals", preprint, 2005. math.AG/0510335

[Cavalieri 2004] R. Cavalieri, "Hodge-type integrals on moduli spaces of admissible covers", preprint, 2004. math.AG/0411500

[Cavalieri 2005] R. Cavalieri, *A topological quantum field theory of intersection numbers for moduli spaces of admissible covers*, Ph.D. thesis, Univ. of Utah, 2005, http://www.math.lsa.umich.edu/~crenzo/thesis.pdf.

[Dijkgraaf and Witten 1990] R. Dijkgraaf and E. Witten, "Topological gauge theories and group cohomology", *Comm. Math. Phys.* **129**:2 (1990), 393–429. MR 91g:81133 Zbl 0703.58011

[Ekedahl et al. 2001] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, "Hurwitz numbers and intersections on moduli spaces of curves", *Invent. Math.* **146**:2 (2001), 297–327. MR 2002j:14034 Zbl 1073.14041

[Faber and Pandharipande 2000] C. Faber and R. Pandharipande, "Hodge integrals and Gromov–Witten theory", *Invent. Math.* **139**:1 (2000), 173–199. MR 2000m:14057 Zbl 0960.14031

[Freed and Quinn 1993] D. S. Freed and F. Quinn, "Chern-Simons theory with finite gauge group", *Comm. Math. Phys.* **156**:3 (1993), 435–472. MR 94k:58023 Zbl 0788.58013

[Fulton and Harris 1991] W. Fulton and J. Harris, *Representation theory: a first course*, vol. 129, Graduate Texts in Mathematics, Springer-Verlag, New York, 1991. A first course, Readings in Mathematics. MR 93a:20069 Zbl 0744.22001

[Graber and Vakil 2003a] T. Graber and R. Vakil, "Hodge integrals and Hurwitz numbers via virtual localization", *Compositio Math.* **135**:1 (2003), 25–36. MR 2004c:14108 Zbl 1063.14032

[Graber and Vakil 2003b] T. Graber and R. Vakil, "Relative virtual localization and vanishing of tautological classes on moduli spaces of curves", preprint, 2003. math.AG/0309227

[Harris and Mumford 1982] J. Harris and D. Mumford, "On the Kodaira dimension of the moduli space of curves", *Invent. Math.* **67**:1 (1982), 23–88. With an appendix by William Fulton. MR 83i:14018 Zbl 0506.14016

[Hori et al. 2003] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, Clay Mathematics Monographs **1**, American Math. Soc., Providence, 2003. MR 2004g:14042 Zbl 1044.14018

[Ionel 2002] E.-N. Ionel, "Topological recursive relations in $H^{2g}(\mathcal{M}_{g,n})$", *Inventiones Math.* **148**:3 (2002), 627–658. MR 2003d:14065 Zbl 1056.14076

[Ionel 2005] E.-N. Ionel, "Relations in the tautological ring of $\mathcal{M}_g$", *Duke Math. J.* **129**:1 (2005), 157–186. MR 2006c:14040 Zbl 1086.14023

[Kock 2004] J. Kock, *Frobenius algebras and 2D topological quantum field theories*, London Math. Soc. Student Texts **59**, Cambridge Univ. Press, 2004. MR 2005a:57028 Zbl 1046.57001

[Li 2002] J. Li, "A degeneration formula of GW-invariants", *J. Differential Geom.* **60**:2 (2002), 199–293. MR 2004k:14096 Zbl 1063.14069

[Macdonald 1995] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford Univ. Press, New York, 1995. MR 96h:05207 Zbl 0824.05059

[Mumford 1983] D. Mumford, "Towards an enumerative geometry of the moduli space of curves", pp. 271–328 in *Arithmetic and geometry*, vol. II, edited by M. Artin and J. Tate, Progr. Math. **36**, Birkhäuser, Boston, 1983. MR 85j:14046 Zbl 0554.14008

Communicated by Andrei Okounkov
Received 2007-02-10 Accepted 2007-05-13

crenzo@umich.edu *University of Michigan, Department of Mathematics, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, United States*