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The paper has two goals: the study of the associated graded ring of contracted homogeneous ideals in $K[x, y]$ and the study of the Gröbner fan of the ideal P of the rational normal curve in $\mathbb{P}^d$. These two problems are, quite surprisingly, very tightly related. We completely classify the contracted ideals with Cohen–Macaulay associated graded ring in terms of the numerical invariants arising from Zariski's factorization. We determine explicitly the initial ideals (monomial or not) of P, that are Cohen–Macaulay.

1. Introduction

The goal of the paper is twofold:

(1) to describe the Cohen–Macaulay initial ideals of the defining ideal P of the rational normal curve in $\mathbb{P}^d$ in its standard coordinate system and for every positive integer d, and

(2) to identify the homogeneous contracted ideals in $K[x, y]$ whose associated graded ring is Cohen–Macaulay.

The two problems are closely related. Indeed they are essentially equivalent, as we proceed to explain. Let K be a field, $R = K[x, y]$ and I be a homogeneous ideal of R with $\sqrt{I} = \mathbf{m} = (x, y)$. Denote by $\mathrm{gr}_I(R)$ the associated graded ring

$$\bigoplus_k I^k/I^{k+1}$$

of I. The ideal I is said to be contracted if it is contracted from a quadratic extension, that is, if there exists a linear form z in R such that

$$I = IR[\mathbf{m}/z] \cap R.$$

Contracted ideals have been introduced by Zariski in his studies on the factorization property of integrally closed ideals; see [Zariski and Samuel 1960, App. 5]

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or [Huneke and Swanson 2006, Chap. 14]. Every integrally closed ideal I is contracted and has a Cohen–Macaulay associated graded ring $\mathrm{gr}_I(R)$; see [Lipman and Teissier 1981; Huneke 1989]. On the contrary, the associated graded ring of a contracted ideal does not need to be Cohen–Macaulay. Zariski proved a factorization theorem for contracted ideals, asserting that every contracted ideal I has a factorization $I = L_1 \cdots L_s$ where L_i are themselves contracted but of a very special kind. In the homogeneous case and with K assumed to be algebraically closed, each L_i is a lex-segment monomial ideal in a specific coordinate system depending on i. Recall that a monomial ideal L in R is a lex-segment ideal if that $x^a y^b \in L$ with $b > 0$ implies that $x^{a+1} y^{b-1} \in L$ also. In [Conca et al. 2005, Corollary 3.14] it is shown that the Cohen–Macaulayness of $\mathrm{gr}_I(R)$ is equivalent to the Cohen–Macaulayness of $\mathrm{gr}_{L_i}(R)$ for every $i = 1, \dots, s$. Therefore to answer (2) one has to characterize the lex-segment ideals L with Cohen–Macaulay associated graded ring. Any lex-segment ideal L of initial degree d can be encoded by a vector $a = (a_0, a_1, \dots, a_d)$ with increasing integral coordinates and $a_0 = 0$. Given L associated to a, we show that $\mathrm{gr}_L(R)$ is Cohen–Macaulay if and only if $\mathrm{in}_a(P)$ defines a Cohen–Macaulay ring. Here $\mathrm{in}_a(P)$ denotes the ideal of the initial forms of P with respect to the vector a. Therefore (1) and (2) are indeed equivalent problems. In Section 4 we solve problem (1) by showing first that P has exactly 2^{d-1} Cohen–Macaulay initial monomial ideals; see Theorem 4.11. Then we show that every Cohen–Macaulay initial ideal of P has a Cohen–Macaulay monomial initial ideal; see Theorem 4.13. In terms of the Gröbner fan of P, Theorem 4.13 can be rephrased as that $\mathrm{in}_a(P)$ is Cohen–Macaulay if and only if a belongs to the union of 2^{d-1} maximal closed cones. These cones are explicitly described by linear homogeneous inequalities. The fact that P has exactly 2^{d-1} Cohen–Macaulay monomial initial ideals can be derived by combining the results of Hoşten and Thomas [2003] and those of O'Shea and Thomas [2005]; see Remark 4.20.

In Section 5 we give an explicit characterization, in terms of the numerical invariants arising from Zariski's factorization, of the Cohen–Macaulay property of the associated graded ring of a contracted homogeneous ideal in $K[x, y]$. In Section 6 we describe the relationship between the Hilbert series of $\mathrm{gr}_L(R)$ and the multigraded Hilbert series of $\mathrm{in}_a(P)$. We discuss also how the formulas for the Hilbert series and the polynomials of $\mathrm{gr}_L(R)$ change by varying the corresponding cones of the Gröbner fan of P. This has a conjectural relation with the hypergeometric Gröbner fan introduced by Saito, Sturmfels and Takayama in [Saito et al. 2000]. In Section 7 we show that the union of a certain subfamily of the 2^{d-1} Cohen–Macaulay cones is itself a cone. We call it the big Cohen–Macaulay cone. Indeed, the big Cohen–Macaulay cone is the union of f_d Cohen–Macaulay cones of the Gröbner fan of P, where f_d denotes the $(d+1)$-th Fibonacci number. In Section 8 we present some examples.

2. Notation and preliminaries

Let S be a polynomial ring over a field K with maximal homogeneous ideal $\mathbf{m}$. For a homogeneous ideal I of S we denote by $\mathrm{gr}_I(S)$, $\mathrm{Rees}(I)$ and $F(I)$ respectively the associated graded ring $\bigoplus_{k\in\mathbb{N}} I^k/I^{k+1}$, the Rees algebra $\bigoplus_{k\in\mathbb{N}} I^k$ and the fiber cone $\bigoplus_{k\in\mathbb{N}} I^k/\mathbf{m}\, I^k$ of I. By the very definition $F(I)$ is a standard graded K-algebra. Furthermore $\mathrm{Rees}(I)$ can be identified with the S-subalgebra of the polynomial ring $S[t]$ generated by ft with $f \in I$.

Let $I \subset S = K[x_1, \dots, x_n]$ be a homogeneous ideal. We may consider the (standard) Hilbert function, Hilbert polynomial and Hilbert series of S/I. The Hilbert series of S/I is $\sum_{i\geq 0} \dim_K(S/I)_i z^i$ and we denote it by $H_{S/I}(z)$. The series $H_{S/I}(z)$ has a rational expression $h(z)/(1-z)^d$ where $h(z) \in \mathbb{Z}[z]$ and d is the Krull dimension of S/I. The polynomial $h(z)$ is called the (standard) h-polynomial of S/I. In particular, $h(0) = 1$ and $h(1)$ is the ordinary multiplicity of S/I, denoted by $e(S/I)$.

If I is $\mathbf{m}$-primary, we may consider also the (local) Hilbert functions, Hilbert polynomials and Hilbert series of I (or of $\mathrm{gr}_I(S)$). There are two Hilbert functions associated with I in this context. We denote them by $H(I, k)$ and $H^1(I, k)$ and they are defined by

$$H(I, k) = \dim_K(I^k/I^{k+1}) \quad \text{and} \quad H^1(I, k) = \dim_K(S/I^{k+1}).$$

The corresponding Hilbert series are

$$H_I(z) = \sum_{k\geq 0} H(I, k) z^k \quad \text{and} \quad H_I^1(z) = \sum_{k\geq 0} H^1(I, k) z^k.$$

Obviously, $H_I(z) = (1 - z)H_I^1(z)$. The series $H_I^1(z)$ has a rational expression

$$H_I^1(z) = \frac{h(z)}{(1-z)^{n+1}}$$

where $h(z)$ is a polynomial with integral coefficients and is called the (local) h-polynomial of I or of $\mathrm{gr}_I(S)$. The Hilbert functions $H(I, k)$ and $H^1(I, k)$ agree for large k with polynomials $P_I(z)$ and $P_I^1(z)$ at $z = k$. The polynomials $P_I(z)$ and $P_I^1(z)$ are called the Hilbert polynomials of I. Their coefficients, with respect to an appropriate binomial basis, are integers and are called Hilbert coefficients of I and are denoted by $e_i(I)$. Precisely,

$$P_I^1(z) = \sum_{i=0}^{n} (-1)^i e_i(I) \binom{n-i+z}{n-i}.$$

In particular, $h(0) = \dim_K S/I$ and $h(1) = e_0(I)$ that is the multiplicity of I.

Definition 2.1. Let $I \subset S = K[x_1, \dots, x_n]$ be a homogeneous ideal of codimension c and not containing linear forms. Then

(1) S/I has minimal multiplicity if $e(S/I) = c+1$, and

(2) S/I has a short h-vector if its h-polynomial is $1+cz$, that is, if the Hilbert series of S/I is $(1+cz)/(1-z)^{n-c}$.

We denote the Castelnuovo–Mumford regularity of a graded S-module M by $\operatorname{reg}(M)$. For results on the Castelnuovo–Mumford regularity and the minimal multiplicity we refer the readers to [Eisenbud and Goto 1984]. We just recall that if S/I has a short h-vector, then it has minimal multiplicity. On the other hand, if S/I is Cohen–Macaulay with minimal multiplicity, then it has a short h-vector. We will need the next lemma whose easy proof follows from the standard facts.

Lemma 2.2. *Let $I \subset S$ be a homogeneous ideal. Assume S/I has a short h-vector. Then S/I is Cohen–Macaulay if and only if* $\operatorname{reg}(I) = 2$.

Every vector $a = (a_0, \dots, a_d) \in \mathbb{Q}^{d+1}_{\geq 0}$ induces a graded structure on the polynomial ring $K[t_0, \dots, t_d]$ by letting $\deg t_i = a_i$. Every monomial t^α is then homogeneous of degree

$$\deg_a t^\alpha = a\alpha = \sum_{i=0}^{d} a_i \alpha_i .$$

For every nonzero polynomial $f = \sum_{i=1}^{k} \lambda_i t^{\alpha_i}$ we set

$$\deg_a f = \max\{a\alpha_i : \lambda_i \neq 0\} \quad \text{and} \quad \operatorname{in}_a(f) = \sum_{a\alpha_i = \deg_a f} \lambda_i t^{\alpha_i}.$$

Then for every ideal I one defines the initial ideal $\operatorname{in}_a(I)$ of I with respect to a to be

$$\operatorname{in}_a(I) = \big(\operatorname{in}_a(f) : f \in I, \quad f \neq 0\big).$$

Similarly, given a term order τ, we denote by $\operatorname{in}_\tau(I)$ the ideal of the initial monomials of elements of I. Given $a \in \mathbb{Q}^{d+1}_{\geq 0}$ the term order defined by

$$t^\alpha \geq t^\beta \text{ if and only if } a\alpha > a\beta \text{ or } \big(a\alpha = a\beta \text{ and } t^\alpha \geq t^\beta \text{ with respect to } \tau\big)$$

is denoted by τa.

One easily shows that $\operatorname{in}_\tau(\operatorname{in}_a(I)) = \operatorname{in}_{\tau a}(I)$. Hence $\operatorname{in}_a(I)$ and I have a common monomial initial ideal. This shows (1) of the following lemma.

Lemma 2.3. *Let I be a homogeneous ideal with respect to the ordinary grading* $\deg t_i = 1$ *and let $a \in \mathbb{Q}^{d+1}_{\geq 0}$. Then*

(1) *S/I and $S/\operatorname{in}_a(I)$ have the same Hilbert function, and*

(2) $\operatorname{depth} S/\operatorname{in}_a(I) \leq \operatorname{depth} S/I$.

Part (2) follows from the standard one-parameter flat family argument; for details see [Eisenbud 1995, Chap. 15] or [Bruns and Conca 2003].

Definition 2.4. Let P be the ideal of the rational normal curve of $\mathbb{P}^d$ in its standard embedding. Namely, P is the kernel of the K-algebra map

$$S = K[t_0, t_1, \dots, t_d] \to K[x, y]$$

sending t_i to $x^{d-i} y^i$.

The ideal P is minimally generated by the 2-minors of the matrix

$$T_d = \begin{pmatrix} t_0 & t_1 & t_2 & \dots & \dots & t_{d-1} \\ t_1 & t_2 & \dots & \dots & t_{d-1} & t_d \end{pmatrix}$$

and it contains the binomials of the form $t_{i_1} t_{i_2} \cdots t_{i_k} - t_{j_1} t_{j_2} \cdots t_{j_k}$ with

$$0 \le i_v, j_v \le d \quad \text{and} \quad i_1 + i_2 + \dots + i_k = j_1 + j_2 + \dots + j_k.$$

The Hilbert series of S/P is $(1 + (d-1)z)/(1-z)^2$.

3. Contracted ideals in dimension 2

We briefly recall from [Zariski and Samuel 1960, App. 5], [Huneke and Swanson 2006, Chap. 14] and [Conca et al. 2005] a few facts about contracted ideals. As we deal only with homogeneous ideals, we will state the results in the graded setting.

Assume K is an algebraically closed field. Let $R = K[x, y]$ and denote by $\mathbf{m}$ its maximal homogeneous ideal. An $\mathbf{m}$-primary homogeneous ideal I of R is said to be contracted if it is contracted from a quadratic extension, that is, if there exists a linear form $z \in R$ such that $I = IS \cap R$, where $S = R[\mathbf{m}/z]$. The property of being contracted can be described in several ways; for instance see [Conca et al. 2005, Prop. 3.3]. To a contracted ideal I one associates a form, the characteristic form of I, defined as $\mathrm{GCD}(I_d)$ where d is the initial degree of I. For our goals, it is important to recall the following definition and theorem.

Definition 3.1. Let I be a homogeneous $\mathbf{m}$-primary ideal in R and let $Q = IR_{\mathbf{m}}$. Let $J \subset R_{\mathbf{m}}$ be a minimal reduction of Q. The deviation of I is the length of Q^2/JQ. It will be denoted by $V(I)$.

Theorem 3.2. *Let I be a homogeneous $\mathbf{m}$-primary ideal in R. One has*

(1) $\mathrm{gr}_I(R)$ *is Cohen–Macaulay if and only if* $V(I) = 0$, *and*

(2) $V(I) = e_0(I) - \dim_K(R/I^2) + 2\dim_K(R/I)$.

See [Huckaba and Marley 1993, Prop. 2.6, Thm. A] for a proof of (1) and [Valla 1979, Lemma 1] for a proof of (2). Similar results are proved also in [Verma 1991].

We recall now Zariski's factorization theorem for contracted ideals and a related statement, [Conca et al. 2005, Cor. 3.14], concerning associated graded rings. In our setting they can be stated as follows.

Theorem 3.3. (1) *Any contracted ideal I has a factorization $I = L_1 \cdots L_s$ where L_i are homogeneous* **m***-primary contracted ideals with characteristic form of type $\ell_i^{\alpha_i}$ for pairwise linearly independent linear forms $\ell_1, \dots, \ell_s$.*

(2) *With respect to the factorization in (1) one has*

$$\operatorname{depth} \operatorname{gr}_I(R) = \min\{\operatorname{depth} \operatorname{gr}_{L_i}(R) : i = 1, \dots, s\}.$$

Lemma 3.4. *The fiber cone $F(I)$ of a contracted ideal I has a short h-vector. Its Hilbert series is $(1+(d-1)z)/(1-z)^2$, where d is the initial degree of I.*

Proof. A contracted ideal of initial degree d is minimally generated by $d+1$ elements and products of contracted ideals are contracted. The initial degree of I^k is kd. Hence I^k has $kd+1$ minimal generators. It follows that the Hilbert series of $F(I)$ is $(1+(d-1)z)/(1-z)^2$. $\square$

A monomial **m**-primary ideal I of $R = K[x, y]$ can be encoded in various ways. We use the following. Let $d \in \mathbb{N}$ be such that $x^d \in I$ and, for $i = 0, \dots, d$, set $a_i(I) = \min\{j : x^{d-i}y^j \in I\}$. Then we have $0 = a_0(I) \le a_1(I) \le \cdots \le a_d(I)$. Obviously, the map

$$I \to a = (a_0(I), \dots, a_d(I))$$

establishes a bijective correspondence between the set of **m**-primary monomial ideals containing x^d and the set of weakly increasing sequences $a = (a_0, \dots, a_d)$ of nonnegative integers with $a_0 = 0$. The inverse map is

$$a = (0 = a_0 \le a_1 \cdots \le a_d) \to (x^{d-i}y^{a_i} : i = 0, \dots, d).$$

It is easy to see that if a corresponds to I, then $\dim_K R/I = \sum_{i=0}^{d} a_i$. Furthermore, if a' corresponds to J, then the sequence associated to the product IJ is $(c_0, c_1, c_2, \dots)$ where $c_i = \min\{a_j + a'_k : j + k = i\}$. In particular:

Lemma 3.5. *Let I be a monomial ideal and $a = (a_0, \dots, a_d)$ be the corresponding sequence. Then the Hilbert function of I is given by*

$$H^1(I, k) = \sum_{i=0}^{(k+1)d} \min\{a_{j_1} + \cdots + a_{j_{k+1}} : j_1 + \cdots + j_{k+1} = i\}.$$

We set $b_i(I) = a_i(I) - a_{i-1}(I)$ for $i = 1, \dots, d$ and observe that the ideal I can be as well described via the sequence $b_1(I), \dots, b_d(I)$ of nonnegative integers.

A monomial ideal I is a lex-segment ideal if $x^i y^j \in I$ for some $j > 0$ implies $x^{i+1}y^{j-1} \in I$. The **m**-primary lex-segment ideals are contracted and correspond exactly to strictly increasing a-sequences (equivalently, positive b-sequences) in the above correspondence, provided one takes $d = \min\{j \in \mathbb{N} : x^j \in I\}$.

Remark 3.6. With respect to suitable coordinate systems the ideals L_i in Theorem 3.3 are lex-segment ideals.

It follows from Theorem 3.3 and Remark 3.6 that the study of the depth of the associated graded ring of contracted ideals boils down to the study of the depth of $\mathrm{gr}_L(R)$ for a lex-segment ideal L. One has $\operatorname{depth}\mathrm{gr}_L(R) = \operatorname{depth}\mathrm{Rees}(L) - 1$ since R is regular; see [Huckaba and Marley 1993, Cor. 2.7]. Therefore we can as well study the depth of $\mathrm{Rees}(L)$ for lex-segment ideals L. Trung and Hoa gave in [Trung and Hoa 1986] a characterization of the Cohen–Macaulay property of affine semigroup rings. Since $\mathrm{Rees}(L)$ is an affine semigroup ring one could hope to use their results to describe the lex-segment ideals L such that $\mathrm{Rees}(L)$ is Cohen–Macaulay. In practice, however, we have not been able to follow this idea.

Let L be the lex-segment ideal with the associated a-sequence $a = (a_0, \dots, a_d)$ and b-sequence $(b_1, \dots, b_d)$. We present $\mathrm{Rees}(L)$ as a quotient of $R[t_0, \dots, t_d]$ by the R-algebra map

$$\psi : R[t_0, \dots, t_d] \to \mathrm{Rees}(L) \subset R[t]$$

obtained by sending $t_i \mapsto x^{d-i} y^{a_i} t$. Set

$$\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^{d+1}, \quad \text{and} \quad \mathbf{d} = (d, d-1, d-2, \dots, 0) \in \mathbb{N}^{d+1}.$$

Lemma 3.7. *With the above notation*, $\ker\psi$ *is generated by the binomials*

(1) $x t_i - y^{b_i} t_{i-1}$ *with* $i = 1, \dots, d$, *and*

(2) $t^\alpha - y^u t^\beta$ *where* $\alpha, \beta \in \mathbb{N}^{d+1}$ *satisfy*

$$\mathbf{1}(\alpha - \beta) = 0, \quad \mathbf{d}(\alpha - \beta) = 0, \quad u = a(\alpha - \beta) \geq 0.$$

Proof. Let J be the ideal generated by the binomials of type (1) and (2). Obviously $J \subseteq \ker\psi$. Since $\ker\psi$ is generated by the binomials it contains, it is enough to show that every binomial $M_1 - M_2 \in \ker\psi$ with $\mathrm{GCD}(M_1, M_2) = 1$ belongs to J. Up to multiples of elements of type (1), we may assume that if x divides one of the M_i, say M_1, then $M_1 = x^i y^j t_0^k$. But this clearly contradicts the fact that $M_1 - M_2 \in \ker\psi$. In other words, every binomial in $\ker\psi$ is, up to multiples of elements of type (1), a multiple of an element of type (2). □

Lemma 3.8. *Let L be a lex-segment ideal and $a = (a_0, \dots, a_d)$ its associated a-sequence*, *then we have*

(1) $\mathrm{Rees}(L)/(y)\,\mathrm{Rees}(L) = K[x, t_0, \dots, t_d]/x(t_1, \dots, t_d) + \mathrm{in}_a(P)$,

(2) $F(L) = K[t_0, \dots, t_d]/\,\mathrm{in}_a(P)$, *and*

(3) $\operatorname{depth}\operatorname{gr}_L(R) = \operatorname{depth}\operatorname{Rees}(L) - 1 = \operatorname{depth} F(L)$,

where P is the ideal introduced in Definition 2.4.

Proof. Set $F = F(L)$, $G = \operatorname{gr}_L(R)$ and $\mathcal{R} = \operatorname{Rees}(L)$. First note that (2) follows from (1) since $F = \mathcal{R}/(x, y)\mathcal{R}$. To prove (1) we have to show that

$$\ker\psi + (y) = x(t_1, \dots, t_d) + \operatorname{in}_a(P) + (y).$$

For the inclusion $\subseteq$ we show that the generators of $\ker\psi$ of type (1) and (2) in Lemma 3.7 belong to the ideal on the right hand side. This is obvious for those of type (1). For those of type (2), note that for any such $t^\alpha - y^u t^\beta$ one has $t^\alpha - t^\beta \in P$ and $\operatorname{in}_a(t^\alpha - t^\beta) = t^\alpha$ if $u > 0$ and $\operatorname{in}_a(t^\alpha - t^\beta) = t^\alpha - t^\beta$ if $u = 0$.

The inclusion $\supseteq$ for the elements of $x(t_1, \dots, t_d)$ is obvious. Further, since P is generated by binomials, one knows that $\operatorname{in}_a(P)$ is generated by $\operatorname{in}_a(t^\alpha - t^\beta)$ with $t^\alpha - t^\beta \in P$; see [Sturmfels 1996, Chap. 1]. If $a(\alpha - \beta) = 0$, then $\operatorname{in}_a(t^\alpha - t^\beta) = t^\alpha - t^\beta$ and $t^\alpha - t^\beta \in \ker\psi$. If instead $u = a(\alpha - \beta) > 0$, then $\operatorname{in}_a(t^\alpha - t^\beta) = t^\alpha$ and $t^\alpha - y^u t^\beta \in \ker\psi$ so $t^\alpha \in \ker\psi + (y)$.

To prove (3) note that $P \subseteq (t_1, \dots, t_d)$ and hence $\operatorname{in}_a(P) \subseteq (t_1, \dots, t_d)$. It follows that

$$(t_1, \dots, t_d) \subseteq \big(x(t_1, \dots, t_d) + \operatorname{in}_a(P)\big) : x \subseteq (t_1, \dots, t_d) : x = (t_1, \dots, t_d).$$

Hence

$$(t_1, \dots, t_d) = \big(x(t_1, \dots, t_d) + \operatorname{in}_a(P)\big) : x$$

and we get a short exact sequence

$$0 \to K[x, t_0](-1) \to \mathcal{R}/(y)\mathcal{R} \to F \to 0.$$

By Lemma 3.4 the ring F is 2-dimensional with short h-vector. It follows that the same is true for $\mathcal{R}/(y)\mathcal{R}$ with respect to the standard grading. Using the depth formula for short exact sequences [Bruns and Herzog 1993, Prop. 1.2.9], we have that if $\operatorname{depth}\mathcal{R}/(y)\mathcal{R}$ is 0 or 1, then $\operatorname{depth} F = \operatorname{depth}\mathcal{R}/(y)\mathcal{R}$. Finally, if $\mathcal{R}/(y)\mathcal{R}$ is Cohen–Macaulay then $\operatorname{reg}(\mathcal{R}/(y)\mathcal{R}) = 1$ and it follows that $\operatorname{reg} F = 1$. Then from Lemma 2.2 we can conclude that F is Cohen–Macaulay.

We have shown that $\operatorname{depth}\mathcal{R} - 1 = \operatorname{depth}\mathcal{R}/(y)\mathcal{R} = \operatorname{depth} F$. Since by [Huckaba and Marley 1993, Cor. 2.7] $\operatorname{depth} G = \operatorname{depth}\mathcal{R} - 1$, the proof of (3) is complete. □

Summing up, we have shown:

Proposition 3.9. *Let L be a lex-segment ideal in $R = K[x, y]$ with associated a-sequence $a = (a_0, \dots, a_d)$. Then*

$$\operatorname{depth}\operatorname{gr}_L(R) = \operatorname{depth} K[t_0, \dots, t_d]/\operatorname{in}_a(P)$$

where P is the ideal in Definition 2.4.

4. Cohen–Macaulay initial ideals of the ideal of the rational normal curve

The results of the previous sections show that the study of the contracted ideals of $K[x, y]$ whose associated graded ring is Cohen–Macaulay is equivalent to the study of the initial ideals of P defining Cohen–Macaulay rings. In this section we describe the initial ideals of P (with respect to vectors and in the given coordinates) defining Cohen–Macaulay rings. We will say that an ideal I is Cohen–Macaulay if the quotient ring defined by I is Cohen–Macaulay. The steps of the classification are

(1) to classify the 2-dimensional Cohen–Macaulay monomial ideals with minimal multiplicity,

(2) to identify those of the form $\mathrm{in}_\tau(P)$ for some term order τ among the ideals of (1), and

(3) to identify the vectors $a \in \mathbb{Q}_{\geq 0}^{d+1}$ such that $\mathrm{in}_a(P)$ is Cohen–Macaulay (monomial or not).

We start by classifying the 2-dimensional Cohen–Macaulay square-free monomial ideals with minimal multiplicity. Square-free monomial ideals are in bijective correspondence with simplicial complexes. In particular, square-free monomial ideals defining algebras of Krull dimension 2 are in bijective correspondence with simplicial complexes of dimension 1, that is, graphs. The correspondence goes like this: to any graph with vertex set V and edge set E one associates the monomial ideal on variables V, whose generators are the products xy such that $\{x, y\}$ is not in E and the square-free monomials of degree 3.

Recall that a graph G with n vertices and e edges is a tree if it satisfies the following equivalent conditions:

(1) For every distinct vertices x and y, there exists exactly one path in G connecting x and y;

(2) G is connected and $n - e = 1$;

(3) G is connected and if we remove any edge the resulting graph is disconnected.

Lemma 4.1. *The* 2*-dimensional Cohen–Macaulay square-free monomial ideals with minimal multiplicity correspond to trees.*

Proof. Let G be a graph with n vertices and e edges. Let A be the corresponding quotient ring. The Hilbert series of A is given by

$$1 + \frac{nz}{(1-z)} + \frac{ez^2}{(1-z)^2};$$

see [Bruns and Herzog 1993, 5.1.7]. This implies immediately that A has minimal multiplicity if and only if $n-e=1$. The Cohen–Macaulay property of A corresponds to the connectedness of G; see [Bruns and Herzog 1993, 5.1.26]. So we are considering connected graphs with $n-e=1$, that is, trees. □

Next we extend our characterization from the square-free monomial ideals to general monomial ideals. We will make use of the following lemma, whose easy proof belongs to the folklore of the subject.

Lemma 4.2. *Let I be a monomial ideal generated in degree 2 and with linear syzygies. Let x, y, z be variables. We have*

(1) *if $x^2, y^2 \in I$ then $xy \in I$, and*

(2) *if $x^2, yz \in I$ then either $xy \in I$ or $xz \in I$.*

Proof. Say I is generated by monomials $m_1 = x^2$, $m_2 = y^2$ and other monomials $m_3, \dots, m_s$. Take a free module F with basis $e_1, e_2, \dots e_s$ and map e_j to m_j. The syzygy module $\operatorname{Syz}(I)$ is generated by the reduced Koszul relations $ae_i - be_j$ with $a = m_j / \operatorname{GCD}(m_i, m_j)$ and $b = m_i / \operatorname{GCD}(m_i, m_j)$. By assumption we know that $\operatorname{Syz}(I)$ is generated by the elements $ae_i - be_j$ with $\deg a = 1$. Call this set G. Now $y^2 e_1 - x^2 e_2$ is in $\operatorname{Syz}(I)$ and therefore can be written as $\sum v_{ij}(ae_i - be_j)$ where the sum is extended to the elements $ae_i - be_j \in G$. It follows that there must be in G an element of the form $ye_1 - be_i$. We deduce that $m_i / \operatorname{GCD}(m_i, x^2) = y$, forcing m_i to be xy. Similarly one proves (2). □

The crucial inductive step is encoded in the following lemma.

Lemma 4.3. *Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal such that S/I is Cohen–Macaulay of dimension 2 with minimal multiplicity. Let x_i be a variable such that $x_i^2 \in I$. Set $S' = K[x_j : 1 \le j \le n, j \ne i]$.*

(1) *$I : (x_i)$ is generated by exactly $n-2$ variables.*

(2) *Write $I + (x_i) = J + (x_i)$ where J is a monomial ideal of S'. Then S'/J is a 2-dimensional Cohen–Macaulay ring with minimal multiplicity.*

Proof. First we show that $I : (x_i)$ is generated by variables. Let m be one of the generators of I. We have to show that if x_i does not divide m then there exists a variable x_j such that $x_j | m$ and $x_i x_j \in I$. Let V be the set of the variables whose square is not in I and let Q be the set of the variables whose square is in I. If m is divisible by a variable in Q, then we are done by Lemma 4.2 (1). Otherwise, if m is not divisible by a variable in Q, then $m = x_j x_k$ with $x_j, x_k \in V$, $j \ne k$. By Lemma 4.2 (2) we have that either $x_i x_j$ or $x_i x_k$ is in I. Knowing that $I : (x_i)$ is generated by variables we deduce that $I : (x)$ is a prime ideal, hence an associated prime of the Cohen–Macaulay ideal I. Thus the codimension of $I : (x_i)$ is $n-2$. This proves (1). The standard short exact sequence $0 \to S/I : (x_i)(-1) \to S/I \to S'/J \to 0$ shows that the Hilbert series of S'/J is $1 + (n-3)z/(1-z)^2$. Hence S'/J has a short h-vector. Furthermore, the exact sequence implies that the regularity of S'/J is 1, that is, $\mathrm{reg}(J) = 2$. By Lemma 2.2 we conclude that S'/J is Cohen–Macaulay. □

Corollary 4.4. *With the assumptions in Lemma 4.3 and the notation of its proof, there exist $x_j, x_k \in V$ such that $I : (x_i) = (x_v : 1 \leq v \leq n$ and $v \notin \{j,k\})$ and $x_j x_k \notin I$.*

Proof. We know by Lemma 4.3 that there are variables x_j and x_k so that

$$I : (x_i) = (x_v : 1 \leq v \leq n \text{ and } v \notin \{j,k\}).$$

If $x_j x_k \in I$, then $x_j x_k \in I : (x_i)$. This is a contradiction. □

Definition 4.5. Let V and Q be disjoint sets of variables. Let G be a tree with vertex set V and edge set E. Let $\phi : Q \to E$ be a map. Let J be the square-free monomial ideal associated with G. Let $H = (Q)^2 + (xy : x \in Q, y \in V \text{ and } y \notin \phi(x))$. We define

$$I(G,\phi) = J + H.$$

Example 4.6. Let $V = \{v_1, v_2, v_3, v_4\}$ and $Q = \{q_1, q_2, q_3, q_4, q_5\}$. Let G be the tree on V with edges $E = \{e_1, e_2, e_3\}$ where

$$e_1 = \{v_1, v_2\},\ e_2 = \{v_1, v_3\},\ e_3 = \{v_3, v_4\}.$$

Let $\phi : Q \to E$ be the map sending q_1, q_2, q_3, q_4, q_5 to e_2, e_1, e_1, e_3, e_2, respectively. Then $J = (v_1 v_4, v_2 v_3, v_2 v_4)$ and

$$\begin{aligned} H =& (q_1, q_2, q_3, q_4, q_5)^2 + q_1(v_2, v_4) + q_2(v_3, v_4) \\ &+ q_3(v_3, v_4) + q_4(v_1, v_2) + q_5(v_2, v_4). \end{aligned}$$

In the next proposition we achieve the first step of the classification.

Proposition 4.7. *Let $I \subset S$ be a monomial ideal. The following conditions are equivalent*:

(1) *there exist a tree G and a map $\phi : Q \to E$ such that $I = I(G,\phi)$*;

(2) *S/I is a 2-dimensional Cohen–Macaulay ring with minimal multiplicity.*

Proof. First we show that every ideal of type $I(G,\phi)$ defines a Cohen–Macaulay, 2-dimensional ring with minimal multiplicity. We proceed by induction on the cardinality of Q. If Q is empty, then the ideal I is 2-dimensional Cohen–Macaulay with minimal multiplicity by Lemma 4.1. Now assume Q is not empty and pick $q \in Q$. By construction $I(G,\phi) : (q)$ is generated by $Q \cup V \setminus \phi(q)$ and

$$I(G,\phi) + (q) = I(G,\phi') + (q)$$

where ϕ' is the restriction of ϕ to $Q' = Q \setminus \{q\}$. By induction we know that $I(G, \phi') \subset K[V, Q']$ is 2-dimensional Cohen–Macaulay with minimal multiplicity. The short exact sequence

$$0 \to K[V, Q]/I(G, \phi) : (q)(-1) \to K[V, Q]/I(G, \phi) \to K[V, Q']/I(G, \phi') \to 0$$

shows that $I(G, \phi)$ is 2-dimensional with minimal multiplicity and since both $K[V, Q]/I(G, \phi) : (q)$ and $K[V, Q']/I(G, \phi')$ are 2-dimensional and Cohen–Macaulay, $K[V, Q]/I(G, \phi)$ is Cohen–Macaulay.

We show now that every 2-dimensional Cohen–Macaulay monomial ideal I with minimal multiplicity is of the form $I(G, \phi)$. Let Q be the set of variables whose square is in I and V the remaining variables. We argue by induction on the cardinality of Q. If Q is empty, then I is square-free and we know that I is associated to a tree. If Q is not empty, let $q \in Q$. Write $I + (q) = J + (q)$ with J a monomial ideal not involving the variable q. By Lemma 4.3 we know that $J \subset K[V, Q \setminus \{q\}]$ is 2-dimensional Cohen–Macaulay with minimal multiplicity. Therefore there exists a tree G with vertices V and edges E, and a map

$$\phi' : Q \setminus \{q\} \to E$$

so that $J = I(G, \phi')$. On the other hand $I = J + q(I : q)$ and by Corollary 4.4 there are $x, y \in V$ so that $I : (q) = (Q \cup V \setminus \{x, y\})$ with $xy \notin I$. Hence $\{x, y\}$ belongs to E and we extend ϕ' to Q by sending q to $\{x, y\}$. Call the resulting map ϕ. By construction, $I(G, \phi) = I$. □

Now we come to the second step of our classification: to describe the Cohen–Macaulay monomial initial ideals of P. Let I be a monomial Cohen–Macaulay initial ideal of P. Since P is 2-dimensional with a short h-vector, so is I. By what we have shown above, there exist G and ϕ so that $I = I(G, \phi)$. We want to describe which pairs (G, ϕ) arise in this way.

Lemma 4.8. *Let I be a monomial initial ideal of P generated in degree* 2.

(1) *For every v, $0 \le v \le 2d$, there exists exactly one monomial $t_i t_j$ such that $i + j = v$ and $t_i t_j \notin I$. In particular, t_0^2 and t_d^2 are not in I.*

(2) *Let $0 \le i < j < k \le d$. Assume that t_i^2, t_j^2 and t_k^2 are not in I. Then $t_i t_k \in I$.*

(3) *Let $0 \le i < j < k \le d$. Assume that $t_i^2 \notin I$ and $t_s t_j \notin I$ for every s with $i \le s \le j$. Then $t_i t_k \in I$.*

Proof. (1) By definition, $K[t_0, \dots, t_d]/P$ is bigraded by setting $\deg t_i = (d - i, i)$ and 1-dimensional in each bihomogeneous component. Therefore the ring $K[t_0, \dots, t_d]/I$ is also bigraded and 1-dimensional in each bihomogeneous component. This proves the assertion. (2) Since $t_i^{k-j} t_k^{j-i} - t_j^{k-i} \in P$, we have that either $t_i^{k-j} t_k^{j-i}$ or t_j^{k-i} belongs to I. If t_j^{k-i} belongs to I, since I is generated in degree 2, then t_j^2 belongs to I; a contradiction. So we have $t_i^{k-j}t_k^{j-i} \in I$. Since t_i^2 and t_k^2 are not in I we conclude that $t_i t_k \in I$. (3) We show that there exist a positive integer a and $i \leq s \leq j$ so that $t_i^a t_k - t_s t_j^a \in P$. Then since, by assumption, the factors of degree 2 of $t_s t_j^a$ are not in I, we have that a factor of degree 2 of $t_i^a t_k$ is in I. Since $t_i^2 \notin I$, it follows that $t_i t_k \in I$. To show that a and s as above exist, note that the condition $t_i^a t_k - t_s t_j^a \in P$ translates into $s = k - a(j-i)$. So we have to show that there exists a positive integer a such that

$$i \leq k - a(j-i) \leq j,$$

or equivalently

$$\frac{k-j}{j-i} \leq a \leq \frac{k-i}{j-i}.$$

But this interval has length 1 so it contains an integer which is positive since $i < j < k$. □

The matrix T_d appearing in the following theorem is defined in Section 2.

Theorem 4.9. *Let I be a monomial ideal of $K[t_0, \dots, t_d]$. The following conditions are equivalent.*

(1) *There exists a sequence $0 = i_0 < i_1 < i_2 \cdots < i_k = d$ such that $I(G, \phi) = I$ where $V = \{t_{i_0}, t_{i_1}, \dots, t_{i_k}\}$, $Q = \{t_0, \dots, t_d\} \setminus V$, G is the tree (a line) with vertex set V and with edge set $E = \{\{t_{i_j}, t_{i_{j+1}}\} : 0 \leq j \leq k-1\}$ and the map $\phi : Q \to E$ sends t_s to $\{t_{i_j}, t_{i_{j+1}}\}$ where $i_j < s < i_{j+1}$.*

(2) *There exists a sequence $0 = i_0 < i_1 < i_2 \cdots < i_k = d$ such that I is generated by*
   (a) *the main diagonals $t_{v-1}t_r$ of the 2-minors of the matrix T_d with column indices v, r such that $v \leq i_j < r$ for some j, and*
   (b) *the antidiagonals $t_v t_{r-1}$ of the 2-minors of the matrix T_d with column indices v, r such that $i_j < v < r \leq i_{j+1}$ for some j.*

(3) *I is a Cohen–Macaulay initial ideal of P.*

Proof. To prove that (1) and (2) are equivalent is just a direct check. To prove that (1) and (2) imply (3), it is enough to describe a term order τ such that

$$\mathrm{in}_\tau(P) = I(G, \phi).$$

Indeed, the inclusion $\mathrm{in}_\tau(P) \supseteq I(G, \phi)$ is enough because the two ideals have the same Hilbert function. To do so, consider a vector $b = (b_1, b_2, \dots, b_d)$ such that $b_r > b_v$ if $v \leq i_j < r$ for some j and $b_v > b_r$ if $i_j < v < r \leq i_{j+1}$ for some j. In Remark 4.10 we will show a canonical way to construct such a vector. Define $a = (a_0, a_1, \dots, a_d)$ by setting $a_0 = 0$ and $a_i = \sum_{j=1}^{i} b_j$. Consider a term order τ refining (no matter how) the order defined by the vector a. We claim that the initial term with respect to τ of the minor with column indices v, r $(v < r)$ of the matrix T_d is the one prescribed by (2). The minor with those column indices is $t_{v-1}t_r - t_v t_{r-1}$. The weights with respect to a of the two terms are $a_{v-1} + a_r$ and $a_v + a_{r-1}$. Hence $a_{v-1} + a_r > a_v + a_{r-1}$ if and only if $b_r > b_v$ and $a_{v-1} + a_r < a_v + a_{r-1}$ if and only if $b_r < b_v$. Therefore, by construction, the initial forms of the minors of T_d with respect to a are exactly the monomials prescribed by (2).

We prove now that (3) implies (1). Assume that I is a Cohen–Macaulay initial ideal of P. Then I has regularity 2 and, in particular, is generated by elements of degree 2. Set $V = \{t_i : t_i^2 \notin I\}$ and $Q = \{t_i : t_i^2 \in I\}$. We know by Lemma 4.8 (1) that t_0 and t_d are in V. So with $0 = i_0 < i_1 < i_2 \cdots < i_k = d$ we may write $V = \{t_{i_0}, t_{i_1}, \dots, t_{i_k}\}$. Since I is 2-dimensional Cohen-Macaulay with minimal multiplicity, it has the form $I = I(G', \phi')$. We prove that $G' = G$ and $\phi' = \phi$ where G and ϕ are those described in (1). Note that, by Lemma 4.8 (2), we have that $t_{i_v}t_{i_r} \in I$ whenever $r - v > 1$ so that $\{t_v, t_r\}$ is not an edge of G'. This implies that the underlying tree G' is exactly the line G. It remains to prove that $\phi = \phi'$. In other words, we have to prove that for every $t_j \in Q$, say $i_r < j < i_{r+1}$, one has $t_j t_{i_r} \notin I$ and $t_j t_{i_{r+1}} \notin I$. By contradiction, let j be the smallest element of Q that does not satisfy the required condition, say $i_r < j < i_{r+1}$.

Claim. If $0 \le v < r$ and $q \ge j$ then $t_{i_v}t_q \in I$.

To prove the claim, note that, by the choice of j, we know that $t_s t_{i_{v+1}} \notin I$ for every $i_v \le s \le i_{v+1}$ and $i_{v+1} \le i_r < j$. Therefore we may apply Lemma 4.8 (3) to the indices i_v, i_{v+1}, q and we conclude that $t_{i_v}t_q \in I$. In particular, for $q = j$ the claim says that $t_{i_v}t_j \in I$ for every $v < r$. So $\phi'(t_j)$ must be an edge of the form $\{t_{i_u}, t_{i_{u+1}}\}$ with $u > r$. It follows that $t_{i_r}t_j \in I$. But, according to Lemma 4.8 (1), there exists (exactly one but we do not need this) a monomial $t_a t_b$ (say $a \le b$) such that $a + b = i_r + j$ and $t_a t_b \notin I$. We distinguish now three cases.

Case 1. $i_r < a$ and $b < j$, so that a and b are both in Q. A contradiction (the square of Q is contained in I).

Case 2. $a < i_r$, $b > j$ and $a \in V$. The claim above says that $t_a t_b \in I$. A contradiction.

Case 3. $a < i_r$, $b > j$ and $a \in Q$. Say $i_u < a < i_{u+1}$. By induction we know that $\{t_v : t_v t_a \notin I\} = \{t_{i_u}, t_{i_{u+1}}\}$ and therefore $b > j > i_r \ge i_{u+1}$. So $t_a t_b \in I$. A contradiction.

This concludes the proof. □

Remark 4.10. Given $0 = i_0 < i_1 < i_2 \cdots < i_k = d$, in the proof of Theorem 4.9 we have used a vector $b = (b_1, b_2, \dots, b_d) \in \mathbb{Q}^d_{\ge 0}$ with the property that $b_r > b_v$ if $v \le i_j < r$ for some j and $b_v > b_r$ if $i_j < v < r \le i_{j+1}$ for some j. Of course there are many vectors with this property. But there is just one "permutation" vector, whose entries are the numbers $1, \dots, d$ permuted in some way, with this property. It arises as follows: for $j = 1, \ldots, k$, consider the vector $c_j = (i_j, i_j - 1, \ldots, i_{j-1} + 1)$ and define the vector b to be the concatenation of $c_1, c_2 \ldots, c_k$.

The following theorem summarizes what we have proved so far.

Theorem 4.11. (1) *The ideal P has exactly 2^{d-1} distinct Cohen–Macaulay monomial initial ideals.*

(2) *They are in bijective correspondence with the sequences $0 = i_0 < i_1 < \cdots < i_k = d$, namely with their radical.*

(3) *Each of them can be obtained with a term order associated to a vector $a = (a_0, a_1, \ldots, a_d) \in \mathbb{N}^{d+1}$ with $0 = a_0 < a_1 < \cdots < a_d$.*

(4) *Each of them is obtained by taking the appropriate initial terms of the 2-minors of the matrix T_d.*

(5) *The reduced Gröbner basis of P giving the Cohen–Macaulay initial ideal corresponding to the sequence $0 = i_0 < i_1 < i_2 \cdots < i_k = d$ is the set of polynomials*

$$\begin{aligned} \underline{t_s t_r} - t_{i_v} t_{s+r-i_v}, &\quad \textit{if} \quad 2i_v \le s + r \le i_v + i_{v+1}, \\ \underline{t_s t_r} - t_{i_{v+1}} t_{s+r-i_{v+1}}, &\quad \textit{if} \quad i_v + i_{v+1} \le s + r \le 2i_{v+1}, \end{aligned}$$

where the initial terms are underlined.

Definition 4.12. For every sequence $\mathbf{i} = (i_0, i_1, \ldots, i_k)$ with $0 = i_0 < i_1 < \cdots < i_k = d$ we denote by $C(\mathbf{i})$ the open cone in $\mathbb{Q}^{d+1}_{\ge 0}$ of the points $(a_0, \ldots, a_d)$ satisfying the inequalities

$$\begin{aligned} a_s + a_r > a_{i_v} + a_{s+r-i_v}, &\quad \text{if} \quad 2i_v \le s + r \le i_v + i_{v+1}, \\ a_s + a_r > a_{i_{v+1}} + a_{s+r-i_{v+1}}, &\quad \text{if} \quad i_v + i_{v+1} \le s + r \le 2i_{v+1}, \end{aligned}$$

and by $\overline{C(\mathbf{i})}$ the corresponding closed cone, that is, the subset of $\mathbb{Q}^{d+1}_{\ge 0}$ described by the inequalities above where $>$ is replaced throughout by $\ge$.

We are ready to state and prove the main theorem of this section.

Theorem 4.13. *Let $a \in \mathbb{Q}^{d+1}_{\ge 0}$. Then $\mathrm{in}_a(P)$ is Cohen–Macaulay if and only if $\mathrm{in}_a(P)$ has a Cohen–Macaulay initial monomial ideal. In other words,*

$$\left\{a \in \mathbb{Q}^{d+1}_{\ge 0} : \mathrm{in}_a(P) \textit{ is Cohen–Macaulay}\right\} = \bigcup_{\mathbf{i}} \overline{C(\mathbf{i})}$$

where the union is indexed by the 2^{d-1} sequences $\mathbf{i} = (0 = i_0 < i_1 < \ldots < i_k = d)$.

Proof. First we prove the inclusion $\supseteq$. Let $a \in \overline{C(\mathbf{i})}$. To see that $\mathrm{in}_a(P)$ is Cohen–Macaulay it is enough to prove that it has a Cohen–Macaulay initial ideal. Just take a' in the open cone $C(\mathbf{i})$ and check that $\mathrm{in}_{a'}(\mathrm{in}_a(P)) = \mathrm{in}_{a'}(P)$. This is easy since it is enough to check that $\mathrm{in}_{a'}(\mathrm{in}_a(P)) \supseteq \mathrm{in}_{a'}(P)$.

In order to prove the opposite inclusion, let $a \in \mathbb{Q}^{d+1}_{\geq 0}$ be such that $\mathrm{in}_a(P)$ is Cohen–Macaulay. We have to show that $a \in \overline{C(\mathbf{i})}$ for some $\mathbf{i}$. We know already that if $\mathrm{in}_a(P)$ is monomial then $a \in C(\mathbf{i})$ where $\mathbf{i}$ is the sequence of the indices i such that no power of t_i belongs to $\mathrm{in}_a(P)$. So we are left with the case $\mathrm{in}_a(P)$ is nonmonomial. To treat this case we first note that, without loss of generality, we may assume that $a = (a_0, \dots, a_d)$ with $a_i \in \mathbb{N}$, $a_0 = 0$, and $a_i < a_{i+1}$. This is because cleaning denominators and adding to a multiples of the vectors $(1, \dots, 1)$ and $(0, 1, 2, \dots, d)$ change neither $\mathrm{in}_a(P)$ nor the membership in the cones. Then we may associate to a a lex-segment ideal L in $R = K[x, y]$ as described in Section 3. We compute the deviation $V(L)$ of L in terms of the a_i's. It is well-know, see for instance [Delfino et al. 2003], that the multiplicity $e_0(L)$ of L is twice the area of the region $\mathbb{R}^2_+ \setminus \mathrm{New}(L)$ where $\mathrm{New}(L)$ is the Newton polytope of L, that is the convex hull of the set of elements $(a, b) \in \mathbb{N}^2$ such that $x^a y^b \in L$. To determine $e_0(L)$ we describe the vertices of $\mathrm{New}(L)$. The generators of L are the elements $x^{d-i} y^{a_i}$. Set $i_0 = 0$ and assume that $i_t < d$ is already defined. Then we set

$$m = \min\big\{(a_j - a_{i_t})/(j - i_t) : j = i_t + 1, \dots, d\big\},$$
$$i_{t+1} = \max\big\{j : i_t + 1 < j \leq d \text{ and } (a_j - a_{i_t})/(j - i_t) = m\big\}.$$

The procedure stops when we have reached, say after k steps, $i_k = d$. By construction the points $(d - c, a_c)$ with $c \in \{i_0, i_1, \dots, i_k\}$ are the vertices of $\mathrm{New}(L)$. Taking into account that twice the area of the triangle with vertices $(0, 0)$, $(d - i_t, a_{i_t})$, $(d - i_{t+1}, a_{i_{t+1}})$ is $a_{i_{t+1}}(d - i_t) - a_{i_t}(d - i_{t+1})$ and that $a_0 = 0$, $i_0 = 0$ we obtain

$$e_0(L) = a_0(i_1 - i_0) + \sum_{t=1}^{k-1} a_{i_t}(i_{t+1} - i_{t-1}) + a_{i_k}(i_k - i_{k-1}).$$

For $j = 0, \dots, 2d$ set $\alpha_j = \min\{a_s + a_r : s + r = j\}$ and

$$\beta_j = \begin{cases} a_{i_t} + a_{j-i_t} & \text{if } \ 2i_t \leq j \leq i_t + i_{t+1}, \\ a_{i_{t+1}} + a_{j-i_{t+1}} & \text{if } \ i_t + i_{t+1} \leq j \leq 2i_{t+1}. \end{cases}$$

Since $\dim_K(R/L) = \sum_{i=0}^{d} a_i$ and $\dim_K(R/L^2) = \sum_{i=0}^{2d} \alpha_i$ we have

$$\begin{aligned} V(L) &= e_0(L) - \sum_{i=0}^{2d} \alpha_i + 2\sum_{i=0}^{d} a_i \\ &= e_0(L) - \sum_{i=0}^{2d} \beta_i + 2\sum_{i=0}^{d} a_i + \sum_{i=0}^{2d} (\beta_i - \alpha_i) \\ &= Z + \sum_{i=0}^{2d} (\beta_i - \alpha_i) \end{aligned}$$

where

$$Z = a_0(i_1 - i_0) + \sum_{t=1}^{k-1} a_{i_t}(i_{t+1} - i_{t-1}) + a_{i_k}(i_k - i_{k-1}) - \sum_{i=0}^{2d} \beta_i + 2\sum_{i=0}^{d} a_i.$$

We claim that $Z = 0$ identically as a linear form in the a_i's. This can be checked directly. That $Z = 0$ follows also from the fact that Z, as a linear function in the a_i's, computes the deviation $V(H)$ where H is any lex-segment with associated vector a in the cone $C(\mathbf{i})$. Since every such lex-segment ideal has a Cohen–Macaulay associated graded ring, we have $V(H) = 0$. Therefore Z vanishes when evaluated at the points of $C(\mathbf{i}) \cap \{a \in \mathbb{N}^{d+1} : 0 = a_0 < a_1 < \cdots < a_d\}$ and so it vanishes identically. Summing up, we have

$$V(L) = \sum_{i=0}^{2d} (\beta_i - \alpha_i).$$

Now, by assumption $\mathrm{in}_a(P)$ is Cohen–Macaulay, thus by Proposition 3.9 $\mathrm{gr}_L(R)$ is Cohen–Macaulay and by Theorem 3.2 $V(L) = 0$. Since $\beta_i \geq \alpha_i$ for every i, it follows that $\beta_i = \alpha_i$ for every i, which in turn implies that $a \in \overline{C(\mathbf{i})}$. □

Remark 4.14. Let $a = (a_0, \ldots, a_d)$ be the vector associated to a lex-segment ideal L. Denote by Y the convex hull of $\{(i, j) : x^i y^j \in L\}$, by V the set of the vertices of Y and by V' the set of the elements $(d - i, a_i)$ belonging to the lower boundary of Y. Clearly $V \subseteq V'$. Assume that $\mathrm{gr}_L(R)$ is Cohen–Macaulay. The proof above shows that $a \in \overline{C(\mathbf{i})}$ where $\{(d - j, a_j) : j \in \mathbf{i}\} = V$. The same argument shows also that $a \in \overline{C(\mathbf{i})}$ for every $\mathbf{i}$ such that $V \subseteq \{(d - j, a_j) : j \in \mathbf{i}\} \subseteq V'$. In particular, a belongs to 2^u of the cones $\overline{C(\mathbf{i})}$ where $u = \#V' - \#V$.

The next example illustrates the remark above.

Example 4.15. Let $a = (0, 2, 4, a_3, a_4, a_5)$ with

$$4 < a_3 < a_4 < a_5, \qquad a_3 > 4 + (a_5 - 4)/3, \qquad a_4 > 4 + 2(a_5 - 4)/3.$$

Then $V = \{(5, 0), (3, 4), (0, a_5)\}$ and $V' = V \cup \{(4, 2)\}$. By the remark above we have that if $\mathrm{in}_a(P)$ is Cohen–Macaulay then $a \in \overline{C(\mathbf{i})} \cap \overline{C(\mathbf{j})}$ with $\mathbf{i} = (0, 1, 2, 5)$ and $\mathbf{j} = (0, 2, 5)$. In this case the Cohen–Macaulay property is equivalent to the inequalities

$$a_4 \geq a_3 + 2, \quad a_5 \geq a_4 + 2, \quad 2a_3 \geq a_4 + 4, \quad 2a_4 \geq a_3 + a_5.$$

Definition 4.16. Let $\sigma \in S_d$ be a permutation. We may associate to σ a cone

$$C_\sigma = \left\{ a \in \mathbb{Q}_{\geq 0}^{d+1} : b_{\sigma^{-1}(1)} < \cdots < b_{\sigma^{-1}(d)} \right\}$$

where $b_i = a_i - a_{i-1}$. We call C_σ the permutation cone associated to σ.

Remark 4.17. As shown in the proof of Theorem 4.9 and Remark 4.10 each cone $C(\mathbf{i})$ contains a specific permutation cone C_σ. The permutations involved in the construction are indeed permutations avoiding the patterns "231" and "312". More precisely, there is a bijective correspondence between the permutations $\sigma \in S_d$ avoiding the patterns "231" and "312" and the cones $C(\mathbf{i})$ so that $C(\mathbf{i}) \supseteq C_\sigma$. However, as we will see, the inclusion $C_\sigma \subseteq C(\mathbf{i})$ can be strict in general. The study of permutation patterns is an important subject in combinatorics; see for instance [Wilf 2002].

The following example illustrates Theorem 4.11.

Example 4.18. Suppose $d = 6$ and take the sequence $\mathbf{i}{=}(i_0{=}0, i_1{=}3, i_2{=}4, i_3{=}6)$. The corresponding Cohen–Macaulay initial ideal I of P is obtained by dividing the matrix T_6 into blocks (from column $i_v + 1$ to i_{v+1})

$$T_6 = \begin{pmatrix} t_0 & t_1 & t_2 & | & t_3 & | & t_4 & t_5 \\ t_1 & t_2 & t_3 & | & t_4 & | & t_5 & t_6 \end{pmatrix}$$

and then taking the antidiagonals of minors whose columns belong to the same block:

$$t_1^2,\ t_1t_2,\ t_2^2,\ t_5^2,$$

and the main diagonals from minors whose columns belong to different blocks:

$$t_0t_4,\ t_0t_5,\ t_0t_6,\ t_1t_4,\ t_1t_5,\ t_1t_6,\ t_2t_4,\ t_2t_5,\ t_2t_6,\ t_3t_5,\ t_3t_6.$$

The ideal I is the initial ideal of P with respect to every term order refining the weight $a = (0, 3, 5, 6, 10, 16, 21)$ obtained from the "permutation" vector

$$\sigma = (3, 2, 1|4|6, 5) \in S_6$$

by setting $a_0 = 0$ and $a_i = \sum_{j=1}^{i} \sigma_j$. With respect to this term order the 2-minors of T_6 are a Gröbner basis of P but not the reduced Gröbner basis. The corresponding reduced Gröbner basis is

$$\begin{array}{lllll}
\underline{t_1^2} - t_0t_2, & \underline{t_1t_2} - t_0t_3, & \underline{t_2^2} - t_1t_3, & \underline{t_0t_4} - t_1t_3 & \underline{t_0t_5} - t_2t_3, \\
\underline{t_1t_4} - t_2t_3, & \underline{t_0t_6} - t_3^2, & \underline{t_1t_5} - t_3^2, & \underline{t_2t_4} - t_3^2, & \underline{t_1t_6} - t_3t_4, \\
\underline{t_2t_5} - t_3t_4, & \underline{t_2t_6} - t_4^2, & \underline{t_3t_5} - t_4^2, & \underline{t_3t_6} - t_4t_5, & \underline{t_5^2} - t_4t_6.
\end{array}$$

So for every vector $a = (a_0, a_1, \dots, a_6) \in \mathbb{Q}^7_{\geq 0}$ satisfying the system of linear inequalities

$$\begin{array}{llll}
2a_1 > a_0 + a_2 *, & a_1 + a_2 > a_0 + a_3, & 2a_2 > a_1 + a_3 *, & a_0 + a_4 > a_1 + a_3 *, \\
a_0 + a_5 > a_2 + a_3, & a_1 + a_4 > a_2 + a_3, & a_0 + a_6 > 2a_3, & a_1 + a_5 > 2a_3, \\
a_2 + a_4 > 2a_3, & a_1 + a_6 > a_3 + a_4, & a_2 + a_5 > a_3 + a_4, & a_2 + a_6 > 2a_4, \\
a_3 + a_5 > 2a_4, & a_3 + a_6 > a_4 + a_5 *, & 2a_5 > a_4 + a_6 *, &
\end{array}$$

we have $\mathrm{in}_a(P) = I$. The 15 linear homogeneous inequalities above define the open Cohen–Macaulay cone $C(\mathbf{i})$. The description is however far from being minimal. The inequalities marked with $*$ are indeed sufficient to describe $C(\mathbf{i})$. In terms of $b_i = a_i - a_{i-1}$ the inequalities can be described by $b_3 < b_2 < b_1 < b_4 < b_6 < b_5$, that is, $C(\mathbf{i}) = C_\sigma$.

Remark 4.19. (1) There exist Cohen–Macaulay ideals of dimension 2 with minimal multiplicity and without Cohen–Macaulay initial monomial ideals in the given coordinates. For instance, let J be the ideal of $K[t_0, \dots, t_4]$ generated by the 2-minors of the matrix

$$\begin{pmatrix} t_0 & t_2 & t_4 - t_0 & 0 \\ t_1 & t_3 & 0 & t_4 + t_0 \end{pmatrix}.$$

Then J has the expected codimension and hence it is 2-dimensional Cohen–Macaulay with minimal multiplicity. No monomial initial ideal of J is quadratic since the degree 2 part of every monomial initial ideal has codimension 2. Hence no monomial initial ideal of J is Cohen–Macaulay. This example shows that Theorem 4.13 does not hold for 2-dimensional binomial Cohen–Macaulay ideals with minimal multiplicity.

(2) $\mathrm{in}_a(P)$ can be monomial, quadratic and non-Cohen–Macaulay. For example, for $d = 4$ the ideal $I = (t_3^2, t_2t_3, t_1t_3, t_0t_4, t_0t_3, t_1^2)$ is a non-Cohen–Macaulay (indeed nonpure) monomial initial ideal of P. The corresponding cone is described in terms of $b_i = a_i - a_{i-1}$ by the inequalities $b_3 > b_1 > b_2 > b_4$ and $b_3 + b_4 > b_1 + b_2$.

(3) $\mathrm{in}_a(P)$ can be quadratic with linear 1-syzygies and not Cohen–Macaulay. For instance, with $d = 7$ the ideal generated by

$$\begin{gathered} t_2^2,\ t_4^2,\ t_6^2,\ t_1t_2,\ t_0t_4,\ t_0t_5,\ t_1t_4,\ t_0t_6,\ t_1t_5,\ t_2t_4,\ t_0t_7,\ t_1t_6,\ t_3t_4, \\ t_1t_7,\ t_2t_6,\ t_3t_6,\ t_3t_7,\ t_4t_6,\ t_1^2 + t_0t_2,\ -t_4t_5 + t_2t_7,\ -t_5t_6 + t_4t_7 \end{gathered}$$

is an initial ideal of P with linear 1-syzygies and a nonlinear 2-syzygy.

(4) We do not know any example as the one in (3) if we further assume that $\mathrm{in}_a(P)$ is a monomial ideal. Note however that 2-dimensional non-Cohen–Macaulay quadratic monomial ideals with a short h-vector and linear 1-syzygies exist, for example $(t_1t_3, t_1t_5, t_0t_2, t_2t_5, t_0t_3, t_2^2, t_2t_4, t_2t_3, t_0t_4, t_4t_5)$.

Remark 4.20. As Rekha Thomas pointed out to us, one can deduce from results in [Hoşten and Thomas 2003; O'Shea and Thomas 2005] that P has exactly one Cohen–Macaulay monomial initial ideal for each regular triangulation of the underlying point configuration A. In [Hoşten and Thomas 2003, Theorem 5.5(ii)] it is proved that for every regular triangulation of A there exists exactly one initial ideal having no embedded primes (they are called Gomory initial ideals in that paper). In [O'Shea and Thomas 2005] it is proved that every Gomory initial ideal coming from a Δ-normal configuration is Cohen–Macaulay. Since every triangulation of A is Δ-normal, one can conclude that the Gomory ideals of P are indeed Cohen–Macaulay. Hence these results imply that P has exactly 2^{d-1} Cohen–Macaulay monomial initial ideals.

5. Contracted ideals whose associated graded ring is Cohen–Macaulay

In this section we use the results of Section 4 to solve the problem (2) mentioned in Section 1. Since the ideal P is homogeneous with respect to the vectors $(1, 1, \dots, 1)$ and $(0, 1, 2, \dots, d)$ of $\mathbb{Q}^{d+1}$, each cone of the Gröbner fan of P is determined by its intersection with

$$W_d = \left\{ (a_0, a_1, \dots, a_d) \in \mathbb{N}^{d+1} : 0 = a_0 < a_1 < \cdots < a_d \right\}.$$

As explained in Section 3, W_d parametrizes the lex-segment ideals of initial degree d. For a given $d \in \mathbb{N}$, $d > 0$, we set

$$CM_d = \left\{ a \in \mathbb{Q}^{d+1}_{\geq 0} : \operatorname{in}_a(P) \text{ is Cohen–Macaulay} \right\}$$

the "Cohen–Macaulay region" of the Gröbner fan of P. According to Theorem 4.13 we have

$$CM_d = \bigcup_{\mathbf{i}} \overline{C(\mathbf{i})}$$

where the union is extended to all of the 2^{d-1} sequences $\mathbf{i} = (0 = i_0 < i_1 < \dots < i_k = d)$.

Theorem 5.1. *Let $d_1, \dots, d_s$ be positive integers and $a_1, \dots, a_s$ be vectors such that $a_i \in W_{d_i}$. Let $\ell_1, \dots, \ell_s, z$ be linear forms in $R = K[x, y]$ such that each pair of them is linearly independent. For every $i = 1, \dots, s$, consider the lex-segment ideals L_i associated to a_i with respect to ℓ_i, z, that is,*

$$L_i = (\ell_i^{d_i - j} z^{a_{ij}} : j = 0, \dots, d_i).$$

Set $I = L_1 \cdots L_s$. We have

(1) *I is contracted and every homogeneous contracted ideal in $R = K[x, y]$ arises in this way, and*

(2) *$\operatorname{gr}_I(R)$ is Cohen–Macaulay if and only if $a_i \in CM_{d_i}$ for all $i = 1, \dots, s$.*

Proof. (1) is a restatement of Zariski's factorization theorem for contracted ideals. (2) follows from Theorem 3.3, Proposition 3.9 and Theorem 4.13. □

Theorem 5.1 can be generalized as follows.

Theorem 5.2. *Let $I \subset K[x, y]$ be a monomial ideal (not necessarily contracted) and let $a = (a_0, \dots, a_d)$ be its associated sequence. Then $\mathrm{gr}_I(R)$ is Cohen–Macaulay if and only if $a \in CM_d$.*

Proof. If a is strictly increasing, then I is a lex-segment ideal. Hence I is contracted and the statement is a special case of Theorem 5.1. If a is not strictly increasing, then we set $a' = a + (0, 1, \dots, d)$ and let L be the monomial ideal associated to a'. Since a' is strictly increasing, L is a lex-segment ideal. The cones $C(\mathbf{i})$ are described by the inequalities that are homogeneous with respect to $(0, 1, \dots, d)$. Therefore a belongs to CM_d if and only if a' does. By construction, I is the quadratic transform of the contracted ideal L in the sense of [Conca et al. 2005, Sect. 3]. Further we know that $\operatorname{depth} \mathrm{gr}_I(R) = \operatorname{depth} \mathrm{gr}_L(R)$ according to [Conca et al. 2005, Thm. 3.12]. In summary, $\mathrm{gr}_I(R)$ is Cohen–Macaulay if and only if $\mathrm{gr}_L(R)$ is Cohen–Macaulay if and only if $a' \in CM_d$ if and only if $a \in CM_d$. □

Remark 5.3. (1) In $K[x, y]$ denote by C the class of contracted ideals, by C_1 the class of the ideals in C with Cohen–Macaulay associated graded ring and by C_2 the class of integrally closed ideals. We have $C \supset C_1 \supset C_2$. One knows that C and C_2 are closed under product. On the other hand C_1 is not: the lex-segment ideals associated to the sequences $(0, 4, 6, 7)$ and $(0, 2)$ belong to C_1 and their product does not. However, C_1 is closed under powers: if $I \in C_1$ then $I^k \in C_1$. This can be seen, for instance, by looking at the Hilbert function of I. Furthermore we will show in Section 7 that a certain subset of C_1 is closed under product.

(2) For a lex-segment ideal L in $K[x, y]$ we have seen that $\mathrm{gr}_L(R)$ and $F(L)$ have the same depth. We believe that $\mathrm{gr}_I(R)$ and $F(I)$ have the same depth for every contracted ideal I. In [D'Cruz et al. 1999, Thm. 3.7, Cor. 3.8] D'Cruz, Raghavan and Verma proved that the Cohen–Macaulayness of $\mathrm{gr}_I(R)$ is equivalent to that of $F(I)$. Note however that for a monomial ideal I the rings $\mathrm{gr}_I(R)$ and $F(I)$ might have different depth. For instance, for the ideal I associated to $(0, 2, 2, 3)$ one has $\operatorname{depth} \mathrm{gr}_I(R) = 1$ and $\operatorname{depth} F(I) = 2$.

Remark 5.4. Two of the cones of the Cohen–Macaulay region CM_d are special as they correspond to opposite extreme selections.

(1) If $\mathbf{i} = (0, 1, 2, \dots, d)$, then the closed cone $\overline{C(\mathbf{i})}$ is described by the inequality system $a_i + a_j \geq a_u + a_v$ with $u = \lfloor (i + j)/2 \rfloor$, $v = \lceil (i + j)/2 \rceil$ for every i, j or, equivalently, by $b_{i+1} \geq b_i$ for every $i = 1, \dots, d - 1$. In other words, $C(\mathbf{i})$ equals its permutation cone C_{id}, where $id \in S_d$ is the identity permutation. In this case the initial ideal of P is $(t_i t_j : j - i > 1)$ and it can be realized by the lex-order $t_0 < t_1 < \cdots < t_d$ or by the lex-order $t_0 > t_1 > \cdots > t_d$. This is the only radical monomial initial ideal of P. The points in $W_d \cap \overline{C(\mathbf{i})}$ correspond

to integrally closed lex-segment ideals. Indeed, they are the products of d complete intersections of type (x, y^u).

(2) If $\mathbf{i} = (0, d)$ then the closed cone $\overline{C(\mathbf{i})}$ is described by the inequality system

$$a_i + a_j \geq a_0 + a_{i+j}, \quad \text{if } i + j \leq d \quad \text{and}$$
$$a_i + a_j \geq a_d + a_{i+j-d}, \quad \text{if } i + j \geq d.$$

It can be realized by the revlex order with $t_0 < t_1 < \cdots < t_d$ or by the revlex order with $t_0 > t_1 > \cdots > t_d$. The corresponding initial ideal of P is $(t_1, \ldots, t_{d-1})^2$. The lex-segment ideals L belonging to the cone are characterized by the fact that $L^2 = (x^d, y^{a_d})L$, that is, they are exactly the lex-segment ideals with a monomial minimal reduction and the reduction number 1. It is not difficult to show that the simple homogeneous integrally closed ideals of $K[x, y]$ are exactly the ideals of the form $\overline{(x^d, y^c)}$, with $\mathrm{GCD}(d, c) = 1$. In other words, $\overline{C(\mathbf{i})}$ contains (the exponent vectors of) all the simple integrally closed ideals of order d. The associated permutation cone is C_σ with $\sigma = (d, d-1, \ldots, 1)$. For $d \leq 3$ one has $C(\mathbf{i}) = C_\sigma$. For $d = 4$ one has $C(\mathbf{i}) \supsetneq C_\sigma$ and $\overline{C(\mathbf{i})} = \overline{C_\sigma} \cup \overline{C_\tau}$ with $\tau = (4, 2, 3, 1)$. For $d > 4$ the cone $\overline{C(\mathbf{i})}$ is not the union of the closure of the permutation cones it contains. For $d = 5$, for example, the cone $\overline{C(\mathbf{i})}$ is described by the inequalities

$$b_1 + b_2 \geq b_3 + b_4, \quad b_2 + b_3 \geq b_4 + b_5, \quad b_1 \geq b_i \geq b_5 \ \text{ with } i = 2, 3, 4,$$

and hence it intersects but it does not contain the cone associated with the permutation $(5, 2, 4, 3, 1)$.

(3) Apart from the example discussed in (1), the other Cohen–Macaulay monomial initial ideals of P arising from lex orders are exactly those associated to sequences $\mathbf{i} = (0, 1, \ldots, \widehat{j}, \ldots, d)$ for some $0 < j < d$. Apart from the example discussed in (2), the other Cohen–Macaulay monomial initial ideals of P arising from revlex orders are exactly those associated to sequences $\mathbf{i} = (0, j, d)$ for some $0 < j < d$. Therefore, starting from $d = 5$, there are Cohen–Macaulay monomial initial ideals of P not coming from lex or revlex orders. For instance, the initial ideal associated to $\mathbf{i} = (0, 1, 4, 5)$ is such an example.

6. Describing the Hilbert series of $\mathrm{gr}_L(R)$

Let L be a lex-segment ideal in $R = K[x, y]$ with associated a-sequence $a = (a_0, a_1, \ldots, a_d)$. We have seen in the proof of Theorem 4.13 that the multiplicity $e_0(L)$ can be expressed as a linear function in a_i's. In terms of initial ideals of P, that assertion can be rephrased as follows. Let I be a monomial initial ideal of P

and let C_I be the corresponding closed cone in the Gröbner fan of P,

$$C_I = \left\{ a \in \mathbb{Q}_{\geq 0}^{d+1} : \operatorname{in}_\tau(\operatorname{in}_a(P)) = I \right\}$$

where τ is a given term order such that $\operatorname{in}_\tau(P) = I$. Let $\mathbf{i}=(0=i_0<i_1<\ldots<i_k=d)$ and let the set of the integers $0 \leq j \leq d$ be such that $t_j \notin \sqrt{I}$. Then

$$\sqrt{I} = (t_j : j \notin \mathbf{i}) + (t_{i_v} t_{i_r} : r - v > 1).$$

Consider the linear form in $\mathbb{Z}[A_0, \ldots, A_d]$ given by

$$e_0^I = A_0(i_1 - i_0) + \sum_{t=1}^{k-1} A_{i_t}(i_{t+1} - i_{t-1}) + A_{i_k}(i_k - i_{k-1}),$$

where the A_i are variables. For every lex-segment ideal L with the associated sequence a belonging to C_I one has that $e_0(L)$ is equal to e_0^I evaluated at $A = a$.

So the "same" formula holds in all the cones of the Gröbner fan associated with the same radical, that is, in all the cones of which the union forms a maximal cone of the secondary fan [Sturmfels 1996, p. 71]. We establish now similar formulas for the Hilbert function $H^1(L, k)$ and the h-polynomial of $\operatorname{gr}_L(R)$.

To this end, consider $S = K[t_0, t_1, \ldots, t_d]$ equipped with its natural $\mathbb{Z}^{d+1}$-graded structure. The quotient S/I is $\mathbb{Z}^{d+1}$-graded and we denote by $H_{S/I}(\underline{t})$ its $\mathbb{Z}^{d+1}$ graded Hilbert series, namely

$$H_{S/I}(\underline{t}) = \sum_{\alpha \in \mathbb{N}^{d+1}} \dim[S/I]_\alpha t^\alpha = \sum_{t^\alpha \notin I} t^\alpha$$

where $t^\alpha = t_0^{\alpha_0} \cdots t_d^{\alpha_d}$. The key observation is contained in the following lemma.

Lemma 6.1. *Let L be a lex-segment ideal with associated vector a belonging to C_I. For $k \in \mathbb{N}$ set $M_k(I) = \{\alpha \in \mathbb{N}^{d+1} : t^\alpha \notin I, |\alpha| = k\}$. Denote by $\sum M_k(I)$ the sum of the vectors in $M_k(I)$. By construction $\sum M_k(I) \in \mathbb{N}^{d+1}$ and*

$$H^1(L, k-1) = a \cdot \sum M_k(I)$$

for all k.

Proof. Set $C_k = a \cdot \sum M_k(I)$. Writing t^α as $t_{j_1} \cdots t_{j_k}$, we may rewrite C_k as the sum $a_{j_1} + \cdots + a_{j_k}$ over all monomials $t_{j_1} \cdots t_{j_k} \notin I$. By construction

$$a_{j_1} + \cdots + a_{j_k} = \min\left\{a_{i_1} + a_{i_2} + \cdots + a_{i_k} : i_1 + i_2 + \cdots + i_k = j_1 + j_2 + \cdots + j_k\right\}$$

if and only if $t_{j_1} \cdots t_{j_k} \notin I$. Therefore C_k is the sum over all v, $0 \leq v \leq kd$ of $\min\{a_{i_1} + a_{i_2} + \cdots + a_{i_k} : i_1 + i_2 + \cdots + i_k = v\}$. But this is exactly $H^1(L, k-1)$; see Lemma 3.5. ☐

In terms of Hilbert series Lemma 6.1 can be rewritten as the follows.

Lemma 6.2. *Let L be a monomial ideal with associated sequence a belonging to C_I. Then*

$$H_L^1(z) = a \cdot \nabla H_{S/I}(\underline{t})_{t_i=z}$$

where $\nabla = (\partial/\partial t_0, \dots, \partial/\partial t_d)$ is the gradient operator.

Remark 6.3. (1) The series $H_{S/I}(\underline{t})$ is rational and it can be described in terms of the multigraded Betti numbers $\beta_{i,\alpha}(S/I) = \dim_K \operatorname{Tor}_i^S(S/I, K)_\alpha$ as

$$H_{S/I}(\underline{t}) = \frac{\sum_{i,\alpha}(-1)^i t^\alpha \beta_{i,\alpha}(S/I)}{\Pi_{i=0}^d (1-t_i)}.$$

(2) A rational expression of $H_{S/I}(\underline{t})$ can be computed also from a Stanley decomposition of S/I. For a monomial ideal I a Stanley decomposition of S/I is a finite set Ω of pairs (σ, τ) where $\sigma \in \mathbb{N}^{d+1}$ and $\tau \subseteq \{0, \dots, d\}$ which induces a decomposition

$$S/I = \bigoplus_{(\sigma,\tau)\in\Omega} t^\sigma K[t_i : i \in \tau]$$

as a K-vector space. Stanley decompositions always exist but they are far from being unique. There are algorithms to compute them; see [Maclagan and Smith 2005] for more. For every Stanley decomposition Ω of S/I clearly one has

$$H_{S/I}(\underline{t}) = \sum_{(\sigma,\tau)\in\Omega} \frac{t^\sigma}{\Pi_{i\in\tau}(1-t_i)}.$$

Combining Lemma 6.1, Remark 6.3 with Lemma 6.2 we obtain:

Corollary 6.4. *For every cone C_I and for a monomial ideal L whose associated sequence a belongs to C_I we have*

$$H_L^1(z) = |a| \frac{1+(d-1)z}{(1-z)^3} + a \cdot \sum_{i\geq 1}(-1)^i \sum_\alpha \beta_{i,\alpha}(S/I) \frac{z^{|\alpha|-1}}{(1-z)^{d+1}} \alpha,$$

where $\beta_{i,\alpha}(S/I)$ are the multigraded Betti numbers of S/I. Moreover

$$H_L^1(z) = a \cdot \sum_{(\sigma,\tau)\in\Omega} \frac{z^{|\sigma|-1}}{(1-z)^{|\tau|+1}} (z\tau + (1-z)\sigma),$$

where Ω is a Stanley decomposition of S/I and we have identified the subset τ with the corresponding $0/1$-vector.

The next proposition summarizes what we have proved so far concerning formulas for h^I and related invariants.

Proposition 6.5. *Given a cone C_I of the Gröbner fan of P there are polynomials $h^I \in \mathbb{Z}[A, z]$ and $Q^I, Q_1^I \in \mathbb{Q}[A, z]$ linear in the variables $A = A_0, \dots, A_d$ and without constant term, such that*

(1) *for every monomial ideal L with associated sequence a belonging to C_I, the polynomial h^I evaluated at $A=a$ equals the h-polynomial of L, Q^I evaluated at $A=a$ equals the Hilbert polynomial P_L of L, and Q_1^I evaluated at $A=a$ equals the Hilbert polynomial P_L^1 of L;*

(2) *h^I, Q^I and Q_1^I can be expressed in terms of the multigraded Betti numbers of I; they can also be expressed in terms of a Stanley decomposition of S/I;*

(3) *in particular,*

$$h^I = |A|(1+(d-1)z) + A\sum_{i\geq 1}(-1)^i \sum_{\alpha} \beta_{i,\alpha}(S/I)\frac{z^{|\alpha|-1}}{(1-z)^{d-2}}\alpha,$$

where $\beta_{i,\alpha}(S/I)$ are the multigraded Betti numbers of S/I, and

$$h^I = A\cdot \sum_{(\sigma,\tau)\in\Omega} z^{|\sigma|-1}(1-z)^{2-|\tau|}\left(z\tau+(1-z)\sigma\right),$$

where Ω is a Stanley decomposition of S/I; explicit expressions for Q^I and Q_1^I can be obtained from that of h^I.

Similarly one has expressions for the Hilbert coefficients e_i^I as a linear function in the variables A. Now we discuss the dependence of the polynomials h^I and Q_1^I on I.

Proposition 6.6. *Let I, J be monomial initial ideals of P. Then*

(1) *$e_0^I = e_0^J$ if and only if $\sqrt{I}=\sqrt{J}$,*

(2) *$h^I = h^J$ if and only if $I=J$, and*

(3) *$Q_1^I = Q_1^J$ if and only if I and J have the same saturation, equivalently, they coincide from a certain degree on.*

Proof. Denote by A the vector of variables $(A_0,\dots,A_d)$. We have discussed already the fact that the formula for the multiplicity e_0^I identifies and it is identified by the radical of I. For statement (2), we have already seen that the coefficient C_k of z^k in the series $h^I/(1-z)^3$ is exactly $A\cdot\sum M_{k+1}(I)$. Hence $h^I=h^J$ holds if and only if $A\cdot\sum M_k(I) = A\cdot\sum M_k(J)$ for all k, that is, $\sum M_k(I)=\sum M_k(J)$ as vectors for every k. By virtue of [Sturmfels 1996, Corollary 2.7], we conclude that $h^I=h^J$ implies $I=J$. For (3) one just applies the same argument to all large degrees. □

For an ideal I of dimension v we denote by I^{top} the component of dimension v of I, that is, the intersection of the primary components of I of dimension v.

Remark 6.7. Let I, J be monomial initial ideals of P. In terms of $M_k(I)$ the condition $Q^I=Q^J$ is equivalent to $\sum M_k(I)-\sum M_{k-1}(I)=\sum M_k(J)-\sum M_{k-1}(J)$ for all $k\gg 0$. There is some computational evidence that $Q^I=Q^J$ could be equivalent to $I^{top} = J^{top}$. This is related with the hypergeometric Gröbner fan of P; see [Saito et al. 2000, Section 3.3]. In particular Example 3.3.7 in [Saito et al. 2000] discusses the secondary fan, the hypergeometric fan and Gröbner fan of P for $d = 4$.

7. The big Cohen–Macaulay cone

Starting from $d = 3$, the Cohen–Macaulay region CM_d is not a cone, that is to say it is not convex; see Section 8 for examples. However, a bunch of the cones $C(\mathbf{i})$ get together to form a big cone.

Proposition 7.1. *Let $B_d = \cup_{\mathbf{i}} \overline{C(\mathbf{i})}$ where the union is extended to all the sequences $\mathbf{i} = \{0 = i_0 < i_1 < \cdots < i_k = d\}$ such that $i_v - i_{v-1} \le 2$ for all $v = 1, \ldots, k$. Then B_d is the closed cone described in terms of the b_i's by the inequalities $b_j \le b_{j+2}$ for all $j = 1, \ldots, d-2$.*

Proof. Let B' be the cone described by the inequalities $b_j \le b_{j+2}$ for all $j = 1, \ldots, d-2$. We have to show that $B_d = B'$. For the inclusion $\subseteq$, let $a \in B_d$ and $b_j = a_j - a_{j-1}$. Then $a \in \overline{C(\mathbf{i})}$ for a sequence $\mathbf{i} = \{0 = i_0 < i_1 < \cdots < i_k = d\}$ such that $i_v - i_{v-1} \le 2$ for all $v = 1, \ldots, k$. For every j, $1 \le j \le d-2$, at least one among j and $j+1$ is in $\mathbf{i}$. We distinguish two cases.

Case 1. $j \in \mathbf{i}$, say $j = i_v$. Then set $s = j-1$ and $r = j+2$. We have $2i_v \le s+r \le i_v + i_{v+1}$ and so, by Theorem 4.11 (5), $a_s + a_r \ge a_{i_v} + a_{s+r-i_v}$ is one of the defining inequalities of $\overline{C(\mathbf{i})}$. Explicitly, $a_{j-1} + a_{j+2} \ge a_j + a_{j+1}$, that is, $b_j \le b_{j+2}$.

Case 2. $j+1 \in \mathbf{i}$, say $j+1 = i_{v+1}$. Then set $s = j-1$ and $r = j+2$. We have $i_v + i_{v+1} \le s+r \le 2i_{v+1}$ and so, by Theorem 4.11 (5), $a_s + a_r \ge a_{i_{v+1}} + a_{s+r-i_{v+1}}$ is one of the defining inequalities of $\overline{C(\mathbf{i})}$. Explicitly, $a_{j-1} + a_{j+2} \ge a_j + a_{j+1}$, that is, $b_j \le b_{j+2}$.

For the inclusion $\supseteq$, let $a \in B'$ and $b_i = a_i - a_{i-1}$. Set

$$U = \{ j : 1 \le j \le d-1, \ b_j > b_{j+1} \}.$$

Since $b_j \le b_{j+2}$ for all j, U does not contain pairs of consecutive numbers. Set $\mathbf{i} = \{0, 1, \ldots, d\} \setminus U = (0 = i_0 < \cdots < i_k = d)$. Note that for all $0 \le r, s \le d$ such that $s - r \ge 2$ one has $b_s + b_{s-1} \ge b_{r+1} + b_r$ and hence $a_s + a_r \ge a_{s-2} + a_{r+2}$. This fact, together with the definition of U, implies that for all $0 \le r, s \le d$ one has

$$a_s + a_r \ge \begin{cases} a_j + a_{j+1}, & \text{if } s+r = 2j+1, \\ 2a_j, & \text{if } s+r = 2j \text{ and } j \in \mathbf{i}, \\ a_{j-1} + a_{j+1}, & \text{if } s+r = 2j \text{ and } j \notin \mathbf{i}. \end{cases}$$

Using this information one proves directly that a satisfies the inequalities in Theorem 4.11 (5) defining $\overline{C(\mathbf{i})}$. □

Remark 7.2. (1) The number f_d of the cones $\overline{C(\mathbf{i})}$ appearing in the description of B_d satisfies the recursion $f_d = f_{d-1} + f_{d-2}$ with $f_1 = 1$ and $f_2 = 2$. Hence f_d is the $(d+1)$-th Fibonacci number.

(2) One also has $B_d = \cup \overline{C_\sigma}$ where $\sigma \in S_d$ satisfies $\sigma(j) < \sigma(j+2)$ for $j = 1, \dots, d-2$. There are $\binom{d}{\lfloor d/2 \rfloor}$ such permutations.

(3) Indeed, each cone $\overline{C(\mathbf{i})}$ appearing in the description of B_d is the union of permutation cones $\overline{C_\sigma}$. Precisely, the permutations involved are those $\sigma \in S_d$ such that $\sigma(j) < \sigma(j+2)$ for all $j = 1, \dots, d-2$ and such that $\sigma(j) > \sigma(j+1)$ if and only if $j \notin \mathbf{i}$. The number of these permutations, say $n(\mathbf{i})$, is a product of Catalan numbers. Recall that the n-th Catalan number is $c(n) = (n+1)^{-1}\binom{2n}{n}$. Decompose $\{1, \dots, d\} \setminus \mathbf{i}$ as a disjoint union $\cup_{i=1}^{t} V_i$ where V_i are of the form $\{a, a+2, \dots\}$ and are maximal. Then $n(\mathbf{i}) = c(|V_1|) \cdots c(|V_t|)$. For instance, if $\mathbf{i} = (0, 2, 3, 5, 7, 9, 10, 12, 14)$ then $\{1, \dots, 14\} \setminus \mathbf{i} = \{1\} \cup \{4, 6, 8\} \cup \{11, 13\}$ and hence $\overline{C(\mathbf{i})}$ is the union of $c(1)c(3)c(2) = 10$ permutation cones.

(4) The family B_d with $d \in \mathbb{N}$ is closed under multiplication, that is, if $a \in B_d$ and $a' \in B_e$ and $\mathbf{c} = a \cdot a'$, then $\mathbf{c} \in B_{d+e}$. Set $\mathbf{c} = (c_0, c_1, \dots, c_{d+e})$. To show that $\mathbf{c} \in B_{d+e}$ one has to prove that $c_{j+2} - c_{j+1} \geq c_j - c_{j-1}$, that is,

$$c_{j+2} + c_{j-1} \geq c_{j+1} + c_j.$$

By definition, $c_{j+2} = a_v + a'_u$ with $v + u = j + 2$ and $c_{j-1} = a_w + a'_z$ with $w + z = j - 1$. Since $(v - w) + (u - z) = (v + u) - (w + z) = 3$ we may assume that $v - w \geq 2$. Then $a_v + a_w \geq a_{v-2} + a_{w+2}$ and hence

$$c_{j+2} + c_{j-1} = a_v + a'_u + a_w + a'_z \geq a_{v-2} + a'_u + a_{w+2} + a'_z \geq c_j + c_{j+1}.$$

8. Examples with small d

In this section we describe, for small d, the Gröbner fan and the Cohen–Macaulay region, and give formulas for the Hilbert series associated to the various cones. For simplicity, the cones will be described in terms of $b_1, \dots, b_d$ where $b_i = a_i - a_{i-1}$.

For $d = 1$, there is not much to say. The ideal P is 0, $CM_1 = \mathbb{Q}^2_{\geq 0}$ and

$$CM_1 \cap W_1 = \{(0, a) \in \mathbb{N}^2 : a > 0\}.$$

For $d = 2$ the Gröbner fan has two maximal cones, both Cohen–Macaulay. The lex cone $C(0, 1, 2)$ is described by $b_1 \leq b_2$ and the revlex cone $C(0, 2)$ is described by $b_1 \geq b_2$.

For $d = 3$ the Gröbner fan has 8 maximal cones. 4 of them are Cohen–Macaulay and 4 have depth 1. We show below the cones. Each table shows

(1) an initial ideal I of P,

(2) the linear inequalities defining the corresponding cone in the Gröbner fan, and

(3) the coefficients of the h-vector of $\mathrm{gr}_L(R)$ for the ideal L corresponding to points (a_0, a_1, a_2, a_3) in the cone.

The expressions of the h-vectors have been computed using Stanley decompositions and the formula in Corollary 6.4. The Stanley decompositions have been computed by the algorithm presented in [Maclagan and Smith 2005]. Cones (a), (b), (c) and (d) are Cohen–Macaulay cones. In particular (a) is the lex cone and (d) is the revlex cone. The union of (a), (b) and (c) is the big cone B_3 and it is defined by $b_1 \le b_3$. The revlex cone (d) is isolated; it intersects B_3 only at $b_1 = b_2 = b_3$. In particular the Cohen–Macaulay region is not a cone.

(a) (t_1t_3, t_0t_3, t_0t_2), $b_1 \le b_2 \le b_3$, (h_0) $a_0+a_1+a_2+a_3$ (h_1) a_1+a_2	(b) (t_1t_3, t_0t_3, t_1^2) $b_2 \le b_1 \le b_3$, (h_0) $a_0+a_1+a_2+a_3$ (h_1) $a_0-a_1+2a_2$
(c) (t_2^2, t_0t_3, t_0t_2), $b_1 \le b_3 \le b_2$, (h_0) $a_0+a_1+a_2+a_3$ (h_1) $2a_1-a_2+a_3$	(d) (t_2^2, t_1t_2, t_1^2), $b_3 \le b_2 \le b_1$, (h_0) $a_0+a_1+a_2+a_3$ (h_1) $2a_0-a_1-a_2+2a_3$

The non-Cohen–Macaulay cones are

(e) $(t_2^2, t_1t_2, t_0t_2, t_0^2t_3)$, $b_3 \le b_1 \le b_2$ and $b_3+b_2 \ge 2b_1$, (h_0) $a_0+a_1+a_2+a_3$ (h_1) $a_0+a_1-2a_2+2a_3$ (h_2) $-a_0+a_1+a_2-a_3$	(f) $(t_1t_3, t_1t_2, t_1^2, t_0t_3^2)$, $b_2 \le b_3 \le b_1$ and $b_1+b_2 \le 2b_3$, (h_0) $a_0+a_1+a_2+a_3$ (h_1) $2a_0-2a_1+a_2+a_3$ (h_2) $-a_0+a_1+a_2-a_3$
(g) $(t_1t_3, t_1t_2, t_1^2, t_2^3)$, $b_2 \le b_3 \le b_1$ and $b_1+b_2 \ge 2b_3$, (h_0) $a_0+a_1+a_2+a_3$ (h_1) $2a_0-2a_1+a_2+a_3$ (h_2) $a_1-2a_2+a_3$	(h) $(t_2^2, t_1t_2, t_0t_2, t_1^3)$, $b_3 \le b_1 \le b_2$ and $b_3+b_2 \le 2b_1$, (h_0) $a_0+a_1+a_2+a_3$ (h_1) $a_0+a_1-2a_2+2a_3$ (h_2) $a_0-2a_1+a_2$

For $d = 4$ there are 42 cones of the Gröbner fan. 10 of them have depth 0, 24 have depth 1 and 8 are Cohen–Macaulay. The big Cohen–Macaulay cone is the union of 5 of the 8 Cohen–Macaulay cones. The remaining 3 are isolated. The following example illustrates Proposition 6.6 and Remark 6.7. The ideals I, J below are non-Cohen–Macaulay initial ideals of P. They satisfy $Q^I = Q^J$ and $Q_1^I \ne Q_1^J$. We display the ideals and the formulas for the coefficients e_0, e_1, e_2

that have been computed via Stanley decompositions.

$I\ \ (t_1t_3, t_1t_2, t_0t_2, t_3^3, t_1^2t_4, t_1^3, t_2t_4, t_2t_3, t_2^2)$	$J\ \ (t_1t_3, t_1t_2, t_1^2, t_3^3, t_2t_4, t_2t_3, t_2^2)$
$(e_0)\ 4a_0 + 4a_4$	$(e_0)\ 4a_0 + 4a_4$
$(e_1)\ 3a_0 - a_1 - 3a_3 + 4a_4$	$(e_1)\ 3a_0 - a_1 - 3a_3 + 4a_4$
$(e_2)\ -a_0 + 2a_1 - 2a_3 + a_4$	$(e_2)\ a_2 - 2a_3 + a_4$

In this case $I^{top} = J^{top} = (t_1t_3, t_2, t_3^3, t_1^2)$ as expected by Remark 6.7 and $J = J^{sat} \neq I^{sat} = (t_2, t_1t_3, t_1^2t_4, t_3^3, t_1^3)$ as proved in Proposition 6.6.

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